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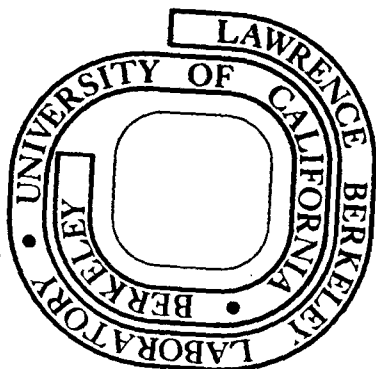
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BEAT HAMILTONIANS AND GENERALIZED PONDEROMOTIVE  
FORCES IN HOT MAGNETIZED PLASMA<sup>†</sup>

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ABSTRACT

A novel approach to the theory of nonlinear mode coupling in hot magnetized plasma is presented. The formulation retains the conceptual simplicity of the familiar ponderomotive-scalar-potential method, but removes the approximations. The essence of the approach is a canonical transformation of the single-particle Hamiltonian, designed to eliminate those interaction terms which are linear in the fields. The new entity (the "oscillation centre") then has no first-order jittering motion, and generalized ponderomotive forces appear as nonlinear terms in the transformed Hamiltonian. This viewpoint is applied to derive a compact symmetric formula for the general three-wave coupling coefficient in hot uniform magnetized plasma, and to extend the conventional ponderomotive-scalar-potential method to the domain of strongly magnetized plasma.

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I. Introduction

The subject of this paper is a generalized concept of "ponderomotive force", the nonlinear force on particles arising from the beating of two high-frequency waves. The generalization leads to a novel and powerful approach to the theory of nonlinear interactions among waves and particles, such as those which occur in problems of parametric instability (Advances in Plasma Physics 1976) and weak plasma turbulence (Davidson 1972; Tsytovich 1977).

The term "ponderomotive", as used in plasma physics, refers to nonlinear low-frequency phenomena induced by high-frequency fields. The notion of a time-averaged ponderomotive force dates back to the radio-frequency confinement schemes of the late 1950's (Motz and Watson 1967). The essential idea was that particles are expelled from regions of higher time-averaged field intensity (Kibble 1966). More recently, ponderomotive-force effects have been studied in connection with profile modification in laser-plasma interaction (Lee et al. 1977). The nonlinear force arising from a single monochromatic field's beating with itself bears a simple and general relation to the linear susceptibility of the plasma medium (Cary and Kaufman 1977).

In the presence of several interacting fields, an extended notion applies. The usual version of the concept is the following: two high-frequency modes beat together to produce a low-frequency scalar potential; this "ponderomotive potential" then drives a

nonlinear current which acts as a source for the self-consistent beat disturbance. This point of view (when applicable) has led to an appealing intuitive understanding of certain parametric-instability processes. Drake and co-authors (1974) used the approach to unify in a simple way the various parametric instabilities associated with laser fusion schemes. Manheimer and Ott (1974) extended the method to the case of weakly magnetized plasma, assuming the electron gyrofrequency to be much less than the high frequencies of the interacting modes. Other authors have also applied the method to magnetized plasmas in various limits (Litvak and Trakhtengerts 1972; Bujarbarua et al. 1974; Berger and Chen 1976; Sanuki and Schmidt 1977). However, a formulation which is valid for arbitrary magnetic-field strength has not yet been derived and justified.

The assumption of only weakly magnetized plasma, if necessary, represents a serious deficiency of the method; in particular, it excludes from consideration all radio-frequency heating schemes. We were therefore led to explore the limits of validity of the ponderomotive-scalar-potential approximation. Our motivation has been the notion that, in considering nonlinear processes in plasma, it should be helpful to think in terms of entities which experience purely nonlinear forces. We wished to extend the usual point of view in several directions: arbitrary ordering of frequencies, kinetic effects for all modes, strongly magnetized plasma. Such a generalized formulation is presented in this paper. It implies velocity dependent ponderomotive forces and, in particular, the notion of a ponderomotive vector potential.

The technique that we employ is based on a canonical transformation of the single-particle Hamiltonian. The transformation is designed to eliminate the first-order interaction of nonresonant particles with the perturbing fields. The new entity (the "oscillation centre") then has no first-order jittering motion, and the "ponderomotive forces" appear as nonlinear terms in the transformed Hamiltonian. Note that no frequency ordering or time averaging is required here; only a transformation to new variables is involved. The Hamiltonian formalism leads to explicit expressions for the required nonlinear currents, which can be decomposed into the current of oscillation centres and the "polarization" corrections. The procedure is quite general in principle.

The oscillation-centre representation is appealing conceptually, and is ideally suited for understanding how the familiar ponderomotive scalar-potential approximation relates to more general theories. The canonical formalism was first developed by Dewar (1973) to establish a rigorous theory of quasilinear diffusion for unmagnetized plasma. It was then used by Johnston (1976) to study induced scattering of waves in magnetized plasma, greatly simplifying the standard Vlasov derivation (Porkolab and Chang 1972). The useful extension of the point of view to other nonlinear processes will be demonstrated here.

The paper is organized as follows. In Section II, the apparatus of the oscillation-centre transformation is developed. We derive there a compact formula for the Hamiltonian of an oscillation-centre in hot magnetized plasma. In Section III, we calculate

the nonlinear perturbation in current density from the oscillation-centre point of view. Then in Section IV, we apply these results to the problem of resonant three-wave interaction, and so obtain a general symmetric formula for the coupling coefficient. Although this compact formula has been derived previously (Larsson 1975; Larsson and Stenflo 1976), our novel formulation gives new insight into the origin of the various terms.

In the remainder of the paper, we then turn our attention to the four-wave problem and to certain limiting cases of our general formulas. In Section V, we investigate beat forces and currents in the cold-plasma approximation. We recover the usual formulas for ponderomotive scalar potential, and introduce the notion of a ponderomotive vector potential. In Section VI, the ponderomotive-scalar-potential method is developed as a consistent limiting case of our more general theory and without restriction on magnetic-field strength. The formulation is then used to tie together several scattered results in the literature. In Section VII, we show that the usual ponderomotive-scalar-potential approximation is generally inadequate in strongly magnetized plasma. We then present the necessary generalization of the method. Our approach and philosophy are discussed further in Section VIII.

## II. Oscillation-Centre Transformation

Consider a physical system described by a Hamiltonian  $H_0(\underline{q}, \underline{p}, t)$  and subjected to a small perturbation  $\delta H(\underline{q}, \underline{p}, t)$  of order  $\epsilon$ . The application we have in mind is to a single particle of a plasma, where  $H_0$  corresponds to the equilibrium fields and  $\delta H$  to perturbing wave fields. Let us perform any near-identity canonical transformation

$$(\underline{q}, \underline{p}, H) \rightarrow (\underline{Q}, \underline{P}, K), \quad K = H_0 + \delta K,$$

characterized by the perturbative generating function  $S(\underline{q}, \underline{P}, t)$  of order  $\epsilon$ . The corresponding transformation equations are then (Goldstein 1950)

$$\begin{aligned} \underline{Q} &= \underline{q} + \partial/\partial \underline{P} S(\underline{q}, \underline{P}, t), \\ \underline{P} &= \underline{p} - \partial/\partial \underline{q} S(\underline{q}, \underline{P}, t), \\ K(\underline{Q}, \underline{P}, t) &= H(\underline{q}, \underline{p}, t) + \partial/\partial t S(\underline{q}, \underline{P}, t). \end{aligned} \quad (1)$$

If we choose to eliminate variables  $(\underline{Q}, \underline{p})$  in favour of variables  $(\underline{q}, \underline{P})$ , then (1) becomes the Hamilton-Jacobi equation

$$\begin{aligned} \partial/\partial t S(\underline{q}, \underline{P}, t) + H(\underline{q}, \underline{P} + \partial S/\partial \underline{q}, t) \\ = K(\underline{q} + \partial S/\partial \underline{P}, \underline{P}, t). \end{aligned} \quad (2)$$

Since  $\underline{p}$  is just a dummy variable, we can now replace it by  $p$  in order to avoid hybrid notation in (2).

The Hamilton-Jacobi equation (2) determines the generating function  $S$  when the nature of the new Hamiltonian  $K$  is specified. To solve it, we expand all quantities in powers of the perturbation parameter  $\epsilon$ , and then arrange that the equation be satisfied order by order. Thus, we substitute into (2) the series expansions

$$H = H_0 + \sum_{n=1}^{\infty} \epsilon^n H^{(n)},$$

$$S = \sum_{n=1}^{\infty} \epsilon^n S^{(n)}, \quad K = H_0 + \sum_{n=1}^{\infty} \epsilon^n K^{(n)}.$$

In this paper, our calculations are correct to second order in  $\epsilon$ . We impose the requirement that  $K^{(1)} = 0$ , and name the resultant transformation the "oscillation-centre transformation." The new entity (the "oscillation-centre") sees only a second-order perturbation; the first-order "jitter" in response to  $H^{(1)}$  has been transformed away. Although the condition  $K^{(1)} = 0$  does not determine  $S^{(2)}$  uniquely, we find the choice  $S^{(2)} = 0$  to be useful in this work. There remains from the expanded Hamilton-Jacobi equation a formula for the new Hamiltonian,

$$K^{(2)} = H^{(2)} + \frac{\partial H^{(1)}}{\partial p} \cdot \frac{\partial S^{(1)}}{\partial q} - \frac{1}{2} \frac{\partial^2 H_0}{\partial q \partial q} : \frac{\partial S^{(1)}}{\partial p} \frac{\partial S^{(1)}}{\partial p} + \frac{1}{2} \frac{\partial^2 H_0}{\partial p \partial p} : \frac{\partial S^{(1)}}{\partial q} \frac{\partial S^{(1)}}{\partial q}, \quad (3)$$

and an equation for the generating function  $S^{(1)}$ ,

$$D_t S^{(1)} = -H^{(1)} \quad (4)$$

where  $D_t$  denotes the convective time derivative following the unperturbed orbit,

$$D_t \equiv \partial_t + \{ \cdot, H_0 \}. \quad (5)$$

The braces in (5) denote the Poisson-bracket operation.

In the case of a "resonant" perturbation, the operator  $D_t$  cannot be inverted in (4), the perturbation procedure breaks down, and a two-time-scale refinement of the transformation becomes necessary (Dewar 1973). In the context of this paper, however, no such refinement will be necessary since resonant-particle effects are ignored.

Let us apply these general relations to the Hamiltonian of a single particle in collisionless Vlasov plasma. We permit the plasma to be magnetized and hot, but assume it to be nonrelativistic and uniform in space. The unperturbed vector potential corresponding to a uniform magnetic field  $B_0 \hat{z}$  can be written

$$\underline{A}_0(x) = z^{-1} B_0 (-y\hat{x} + x\hat{y}). \quad (6)$$

The perturbation will consist of a finite set of coherent linear waves whose amplitudes can vary slowly (compared with their periods)

due to resonant mode coupling (see Appendix A). Thus, the perturbed scalar and vector potentials are of the form

$$\delta\phi(\underline{x}, t) = \sum_{\substack{a=-N \\ a \neq 0}}^N \phi_a \exp(i\mathbf{k}_{-a} \cdot \underline{x} - i\omega_a t), \quad (7)$$

$$\underline{\delta A}(\underline{x}, t) = \sum_{\substack{a=-N \\ a \neq 0}}^N \underline{A}_{-a} \exp(i\mathbf{k}_{-a} \cdot \underline{x} - i\omega_a t)$$

where the reality conditions

$$\phi_{-a} = \phi_a^*, \quad \underline{A}_{-a} = \underline{A}_a^*, \quad \mathbf{k}_{-a} = -\mathbf{k}_a, \quad \omega_{-a} = -\omega_a$$

are implicit. We refrain from specifying a gauge condition here in order that we may test later for gauge invariance.

The Hamiltonian of a nonrelativistic charged particle viewing the fields (6) and (7) is

$$H(\underline{r}, \underline{p}, t) = e\delta\phi(\underline{r}, t) + (2m)^{-1} \left[ \underline{p} - ec^{-1} \underline{A}_0(\underline{r}) - ec^{-1} \underline{\delta A}(\underline{r}, t) \right]^2$$

where  $\underline{r}$  denotes the Cartesian position vector in physical space and  $\underline{p}$  the canonically conjugate momentum. The coefficients in the series expansion in powers of  $\epsilon$  are thus

$$\begin{aligned} H_0(\underline{r}, \underline{p}) &= (2m)^{-1} \left[ \underline{p} - ec^{-1} \underline{A}_0(\underline{r}) \right]^2, \\ H^{(1)}(\underline{r}, \underline{p}, t) &= e\delta\phi(\underline{r}, t) - e(mc)^{-1} \left[ \underline{p} - ec^{-1} \underline{A}_0(\underline{r}) \right] \cdot \underline{\delta A}(\underline{r}, t) \\ H^{(2)}(\underline{r}, \underline{p}, t) &= e^2(2mc^2)^{-1} \left[ \underline{\delta A}(\underline{r}, t) \right]^2, \\ H^{(n)}(\underline{r}, \underline{p}, t) &= 0 \quad \text{for } n \geq 3. \end{aligned} \quad (8)$$

It is helpful to work with the velocity variable

$$\underline{w}(\underline{r}, \underline{p}) \equiv m^{-1} \left[ \underline{p} - ec^{-1} \underline{A}_0(\underline{r}) \right]$$

instead of the unphysical momentum variable  $\underline{p}$ . Since the independent variables in our Hamiltonian formalism are  $\underline{r}$  and  $\underline{p}$ , we must therefore take due account of the chain-rule relations

$$\begin{aligned} \left. \frac{\partial}{\partial \underline{p}} \right|_{\underline{r}} &= \frac{1}{m} \frac{\partial}{\partial \underline{w}}, \\ \left. \frac{\partial}{\partial \underline{r}} \right|_{\underline{p}} &= \left. \frac{\partial}{\partial \underline{r}} \right|_{\underline{w}} + \frac{\Omega}{2} \hat{z} \times \frac{\partial}{\partial \underline{w}}, \end{aligned} \quad (9)$$

where  $\Omega$  denotes the signed gyrofrequency  $e B_0/mc$ .

According to prescription (4), the generating function for the oscillation-center transformation is to be determined from the equation



$$D_t S^{(1)} = -H^{(1)} = ec^{-1} \sum_a (\underline{w} \cdot \underline{A}_a - c\phi_a) \exp(i\mathbf{k}_a \cdot \underline{r} - i\omega_a t). \quad (10)$$

The solution to (10) has the form

$$S^{(1)} = \sum_a S_a^{(1)} = \sum_a S_a(\underline{w}) \exp(i\mathbf{k}_a \cdot \underline{r} - i\omega_a t).$$

Let us introduce cylindrical coordinates in velocity space,

$(w_\perp, w_z, \psi)$ , and expand the functions  $S_a(\underline{w})$  in terms of Fourier-Bessel transforms (Johnston 1976) by writing

$$S_a(\underline{w}) = \exp[i(k_{a\perp} w_\perp / \Omega) \sin(\psi - \psi_a)] \times \sum_{p=-\infty}^{\infty} \tilde{S}_a^{(p)}(w_\perp, w_z) J_p(k_{a\perp} w_\perp / \Omega) \exp[-ip(\psi - \psi_a)] \quad (11)$$

where  $\psi_a$  denotes the cylindrical angle for vector  $\underline{k}_a$ , and  $J_p$  the  $p$ -th order Bessel function. We can then solve (10) to obtain

$$\tilde{S}_a^{(p)}(w_\perp, w_z) = iec^{-1} (\omega_a - k_{az} w_z - p\Omega)^{-1} \times \left\{ (w_z A_{ax} - c\phi_a) + w_\perp \left( \frac{p\Omega}{k_{a\perp} w_\perp} \right) \left( \frac{k_{a\perp} \cdot A_{a\perp}}{k_{a\perp}} \right) + iw_\perp \left[ J_p' \left( \frac{k_{a\perp} w_\perp}{\Omega} \right) / J_p \left( \frac{k_{a\perp} w_\perp}{\Omega} \right) \right] \left( \frac{k_{a\perp} \times A_{a\perp}}{k_{a\perp}} \right)_z \right\} \quad (12)$$

Since our concern in this work is with nonresonant mode coupling, we simply exclude the "resonant" particles for which  $(\omega_a - k_{az} w_z - p\Omega) \lesssim \epsilon$ .

The oscillation-centre Hamiltonian  $K^{(2)}$  consists of bilinear combinations of the fundamental frequencies  $\omega_a$ . From (3), the component at frequency  $(\omega_b + \omega_c)$  is given by the formula

$$K^{(2)}(b,c) = H^{(2)}(b,c) + \frac{\partial H_b^{(1)}}{\partial \underline{p}} \cdot \frac{\partial S_c^{(1)}}{\partial \underline{r}} + \frac{\partial H_c^{(1)}}{\partial \underline{p}} \cdot \frac{\partial S_b^{(1)}}{\partial \underline{r}} - \frac{\partial^2 H_0}{\partial \underline{r} \partial \underline{r}} : \frac{\partial S_b^{(1)}}{\partial \underline{p}} \frac{\partial S_c^{(1)}}{\partial \underline{p}} + \frac{\partial^2 H_0}{\partial \underline{p} \partial \underline{p}} : \frac{\partial S_b^{(1)}}{\partial \underline{r}} \frac{\partial S_c^{(1)}}{\partial \underline{r}} \quad (13)$$

Now, (8) and the chain-rule relations (9) imply

$$\frac{\partial^2 H_0}{\partial \underline{r} \partial \underline{r}} = \frac{m\Omega^2}{4} (1_{\underline{w}\underline{w}} - \hat{z}\hat{z}), \quad \frac{\partial^2 H_0}{\partial \underline{p} \partial \underline{p}} = \frac{1}{m} \frac{1}{\underline{w}\underline{w}}$$

Let us define the operators

$$D_a Q = -i(\omega_a - \underline{k}_a \cdot \underline{w}) Q - \Omega \partial Q / \partial \psi, \\ \tilde{D}_a Q \equiv D_a Q + \Omega \hat{z} \times Q,$$

and introduce the notation  $\partial = \partial / \partial \underline{w}$ . There follows the commutation identity

$$\underline{D}_a \underline{D}_a - D_a \underline{\partial} = i \underline{k}_a + \Omega \hat{z} \times \underline{\partial}, \quad (14)$$

and, from (10), the relation

$$D_a S_a^{(1)} = -H_a^{(1)}.$$

Combining these results, we obtain from (13) the compact formula

$$K_{b,c}^{(2)}(\underline{w}) = (2m)^{-1} (D_b \underline{\partial} S_b) \cdot (\tilde{D}_c \underline{\partial} S_c) + (b \leftrightarrow c). \quad (15)$$

In summary, we have now calculated explicitly the Hamiltonian of an oscillation-centre in terms of the known generating function (11). The forces derivable from  $K^{(2)}$  may be viewed as generalized (i. e., velocity-dependent) ponderomotive forces. Note, however, that we have not ordered frequencies and averaged over time; we have simply performed a canonical transformation. The "corrections" to the representation are stored in the generating function and can be recovered systematically.

### III. Nonlinear Currents

To demonstrate the usefulness of the oscillation-centre viewpoint, let us consider the coupling of three coherent linear waves  $a, b, c$ , which satisfy the resonant matching conditions

$$\omega_a = \omega_b + \omega_c, \quad \underline{k}_a = \underline{k}_b + \underline{k}_c. \quad (16)$$

For simplicity, we specify the radiation gauge condition  $\phi = 0$  for this investigation; the arbitrary gauge will be restored in Sec. V.

According to (A 10), the evolution of the action density in each wave is determined by the nonlinear current density produced by the beating of the other two waves. To evaluate these nonlinear currents, it will be necessary to sum our single-particle formulas of Sec. II over a distribution of particle velocities. The distribution function for particles,  $f(\underline{r}, \underline{p}, t)$ , and for oscillation centres,  $F(\underline{R}, \underline{P}, t)$ , each satisfies its respective Vlasov equation

$$\begin{aligned} \partial_t f(\underline{r}, \underline{p}, t) + \{f, H\} &= 0, \\ \partial_t F(\underline{R}, \underline{P}, t) + \{F, K\} &= 0. \end{aligned} \quad (17)$$

Accordingly, the unperturbed steady-state distribution satisfies

$$D_t f_0(\underline{r}, \underline{p}) = 0$$

with  $D_t$  given by (5), and so must have the form  $f_0(w_1, w_2)$  i. e., uniform in space and independent of cylindrical angle  $\psi$ .

Our approach will be to decompose the true physical current into two parts, namely, the current of oscillation centres and the "polarization" corrections. It is therefore convenient to introduce the "polarization density"  $\Delta(\underline{r}, \underline{p}, t)$  defined by

$$\Delta(\underline{r}, \underline{p}, t) = f(\underline{r}, \underline{p}, t) - F(\underline{r}, \underline{p}, t) \quad (18)$$

A series expansion for  $\Delta$  in powers of  $\epsilon$  is easily obtained (Johnston 1976) by substituting the transformation equations (1) into the relation

$$f(\underline{r}, \underline{p}, t) = F(\underline{R}, \underline{P}, t).$$

The first-order coefficient is

$$\Delta_a^{(1)} = \{f_0, S_a^{(1)}\},$$

which yields

$$\Delta_a^{(1)}(\underline{w}) = -m^{-1} (i \underline{k}_a S_a + \Omega \underline{z} \times \underline{\partial} S_a) \cdot \underline{\partial} f_0 \quad (19)$$

with  $S_a(\underline{w})$  given by (11). In second order, we find the more complicated bilinear expression

$$\begin{aligned} \Delta_{(b,c)}^{(2)} = & - \frac{\partial S_b^{(1)}}{\partial \underline{r}} \cdot \frac{\partial^2 S_c^{(1)}}{\partial \underline{p} \partial \underline{p}} \cdot \frac{\partial f_0}{\partial \underline{r}} + \frac{\partial S_b^{(1)}}{\partial \underline{r}} \cdot \frac{\partial^2 S_c^{(1)}}{\partial \underline{p} \partial \underline{r}} \cdot \frac{\partial f_0}{\partial \underline{p}} \\ & + \frac{1}{2} \frac{\partial S_b^{(1)}}{\partial \underline{p}} \frac{\partial S_c^{(1)}}{\partial \underline{p}} : \frac{\partial^2 f_0}{\partial \underline{r} \partial \underline{r}} + \frac{1}{2} \frac{\partial S_b^{(1)}}{\partial \underline{r}} \frac{\partial S_c^{(1)}}{\partial \underline{r}} : \frac{\partial^2 f_0}{\partial \underline{p} \partial \underline{p}} \\ & - \frac{\partial S_b^{(1)}}{\partial \underline{p}} \frac{\partial S_c^{(1)}}{\partial \underline{r}} : \frac{\partial^2 f_0}{\partial \underline{p} \partial \underline{r}} + (b \leftrightarrow c). \end{aligned}$$

Use of the chain-rule relations (9) and the special form of  $f_0(w_1, w_2)$  then leads to the formula

$$\begin{aligned} \Delta_{b,c}^{(2)}(\underline{w}) = & m^{-2} (i \underline{k}_b S_b + 2^{-1} \Omega \underline{z} \times \underline{\partial} S_b) \\ & \cdot [i \underline{\partial} S_c \underline{k}_c + \Omega \underline{z} (\underline{z} \times \underline{\partial}) S_c] \cdot \underline{\partial} f_0 \quad (20) \\ & + 2m^{-2} (i \underline{k}_b S_b + 2 \Omega \underline{z} \times \underline{\partial} S_b) (i \underline{k}_c S_c + 2^{-1} \Omega \underline{z} \times \underline{\partial} S_c) : \underline{\partial} \underline{\partial} f_0 \\ & - i \Omega (4m^2)^{-1} S_b (\underline{z} \times \underline{\partial} S_c) \underline{k}_b : \underline{\partial} \underline{\partial} f_0 + (b \leftrightarrow c). \end{aligned}$$

In the unmagnetized limit, this formula simplifies considerably, yielding

$$\begin{aligned} \lim_{\Omega \rightarrow 0} \Delta_{b,c}^{(2)}(\underline{w}) = & -m^{-2} S_b (\underline{k}_b \cdot \underline{\partial} S_c) (\underline{k}_c \cdot \underline{\partial} f_0) \\ & - 2m^{-2} S_b S_c \underline{k}_b \underline{k}_c : \underline{\partial} \underline{\partial} f_0 + (b \leftrightarrow c). \end{aligned}$$

The leading perturbation in the oscillation-centre distribution  $F$  is of second order in  $\epsilon$ , since we have arranged that  $K^{(1)} = 0$ . From (17), this perturbation  $F^{(2)}$  must satisfy

$$D_t F^{(2)} = \{K^{(2)}, f_0\},$$

and hence

$$D_a F_{b,c}^{(2)}(\underline{w}) = m^{-1} (i \underline{k}_a K_{b,c}^{(2)} + \Omega \hat{z} \times \underline{\partial} K_{b,c}^{(2)}) \cdot \underline{\partial} f_0, \quad (21)$$

where the beat Hamiltonian  $K_{b,c}^{(2)}$  is given by formula (15). The matching conditions (16) are implicit in (21).

Armed with these formulas for the perturbed distribution functions, we turn next to the perturbation in current density. The total current density  $\underline{j}(\underline{x}, t)$  in the plasma can be written in the form

$$\underline{j}(\underline{x}, t) = \sum_s e \int d^3 r \int d^3 p \delta(\underline{x} - \underline{r}) f(\underline{r}, \underline{p}, t) \partial H(\underline{r}, \underline{p}, t) / \partial \underline{p},$$

where the indicated summation is over all species  $s$ . Since  $f = (F + \Delta)$  and  $F^{(1)} = 0$ , the second-order perturbation in  $\underline{j}$  is therefore

$$\underline{j}^{(2)}(\underline{x}, t) = \sum_s e \int d^3 r \int d^3 p \delta(\underline{x} - \underline{r}) \left[ (F^{(2)} + \Delta^{(2)}) \frac{\partial H_0}{\partial \underline{p}} + \Delta^{(1)} \frac{\partial H^{(1)}}{\partial \underline{p}} + f_0 \frac{\partial H^{(2)}}{\partial \underline{p}} \right]. \quad (22)$$

The final term is actually zero by (8) since the plasma is nonrelativistic.

Now according to (A10), the desired coupling coefficient is the interaction energy  $-c^{-1} \underline{A}_a^* \cdot \underline{j}_{b,c}^{(2)}$ . Noting the relation

$$- \frac{e}{c} \underline{\partial} A \cdot \frac{\partial H}{\partial \underline{p}} = \frac{\partial H}{\partial \underline{\epsilon}},$$

we obtain from (22) the result

$$\begin{aligned} -c^{-1} \underline{A}_a^* \cdot \underline{j}_{b,c}^{(2)} &= \sum_s \int d^3 w H_a^{(1)*} F_{b,c}^{(2)} \\ &+ \sum_s \int d^3 w \left( H_a^{(1)*} \Delta_{b,c}^{(2)} + 2H_{a,-c}^{(2)*} \Delta_b^{(1)} + 2H_{a,-b}^{(2)*} \Delta_c^{(1)} \right). \end{aligned} \quad (23)$$

The first term on the right-hand side of (23) may be viewed as the coupling by oscillation centres, and the remaining terms the coupling due to "polarization" corrections. Note the pleasing structural symmetry of the various terms.

#### IV. Kinetic Three-Mode Coupling Coefficient

In this section, we complete the explicit evaluation of the coupling coefficient (23) by inserting our formulas (19), (20), and (21) for the perturbed distribution functions. The result will be compact and manifestly symmetric in the three generating functions.

From (21), the oscillation-centre ( $F^{(2)}$ ) contribution to (23) is

$$\begin{aligned} \text{O.C. term} &= \sum_s m^{-1} \int d^3w H_a^{(1)*} D_a^{-1} \left[ (i\mathbf{k}_a K_{b,c}^{(2)} + \Omega \hat{z} \times \partial K_{b,c}^{(2)}) \cdot \partial f_0 \right] \\ &= - \sum_s m^{-1} \int d^3w \left[ D_a^{-1} H_a^{(1)*} \right] (i\mathbf{k}_a K_{b,c}^{(2)} + \Omega \hat{z} \times \partial K_{b,c}^{(2)}) \cdot \partial f_0 \end{aligned}$$

where  $K_{b,c}^{(2)}$  is given by formula (15). Since  $D_a H_a^{(1)} = -S_a$  we get

$$\text{O. C. term} = \sum_s m^{-1} \int d^3w S_a^* (i\mathbf{k}_a K_{b,c}^{(2)} + \Omega \hat{z} \times \partial K_{b,c}^{(2)}) \cdot \partial f_0.$$

Partial integration then yields the result

$$\begin{aligned} \text{O. C. term} &= \sum_s m^{-1} \int d^3w f_0 \\ &\times \left[ (-i\mathbf{k}_a \cdot \partial S_a^*) K_{b,c}^{(2)} + (-i\mathbf{k}_a S_a^* + \Omega \hat{z} \times \partial S_a^*) \cdot \partial K_{b,c}^{(2)} \right] \end{aligned}$$

Notice that if modes b and c are cold plasma waves, then only the first term survives since  $K_{b,c}^{(2)}$  will be independent of velocity.<sup>†</sup> Finally, upon insertion of (15) for  $K_{b,c}^{(2)}$ , the oscillation-centre contribution to the coupling coefficient becomes

$$\begin{aligned} \text{O. C. term} &= \sum_s (2m^2)^{-1} \int d^3w f_0 \left\{ (i\mathbf{k}_a \cdot \partial S_a^*) (D_b \partial S_b) \cdot (\tilde{D}_c \partial S_c) \right. \\ &\quad \left. + (-i\mathbf{k}_a S_a^* + \Omega \hat{z} \times \partial S_a^*) \cdot \partial \left[ (D_b \partial S_b) \cdot (\tilde{D}_c \partial S_c) \right] + (b \leftrightarrow c) \right\}. \end{aligned} \quad (24)$$

Turning next to the  $\Delta^{(1)}$  terms in (23), we note from (8) that

$$2H_{a,-b}^{(2)} = m^{-1} \partial H_a^{(1)} \cdot \partial H_b^{(1)*} = m^{-1} (\partial D_a S_a) \cdot (\partial D_b^* S_b^*),$$

and from (19) and (14) that

$$\Delta_c^{(1)} = -m^{-1} (\partial D_c S_c - D_c \partial S_c) \cdot \partial f_0.$$

Thus, integrating by parts to extract  $f_0$ , we find

$$\begin{aligned} \Delta^{(1)} \text{ terms} &= \sum_s m^{-2} \int d^3w f_0 \\ &\times \left[ (i\mathbf{k}_b \cdot \partial S_b) (\partial D_c S_c) \cdot (\partial D_a^* S_a^*) + (b \leftrightarrow c) \right]. \end{aligned} \quad (25)$$

<sup>†</sup> This statement is actually incorrect as we shall see in Section V.

There remains the  $\Delta^{(2)}$  term in (23). To evaluate it, we insert formula (20) for  $\Delta_{b,c}^{(2)}$  and again integrate by parts to eliminate all derivatives of  $f_0$ . After some straight forward (though admittedly lengthy) algebra, we obtain

$$\begin{aligned} \Delta^{(2)} \text{ term} &= \sum_{\mathbf{s}} (2m^2)^{-1} \int d^3\mathbf{w} f_0 \\ &\times \left\{ (D_a^* S_a^*) \left[ (i\mathbf{k}_b \cdot \underline{\partial} S_c)(i\mathbf{k}_c \cdot \underline{\partial} S_b) - (i\mathbf{k}_b \cdot \underline{\partial} S_b)(i\mathbf{k}_c \cdot \underline{\partial} S_c) \right] \right. \\ &+ 2(i\mathbf{k}_b \cdot \underline{\partial} S_b)(\underline{\partial} D_a^* S_a^*) \cdot (D_c \underline{\partial} S_c - \underline{\partial} D_c S_c) \\ &+ \Omega \left[ (D_a^* S_a^*)(\underline{\partial} S_b)(i\mathbf{k}_a) - \Omega(\underline{\partial} D_a^* S_a^*)(\hat{z} \times \underline{\partial} S_b) \right] : \underline{\partial}(\hat{z} \times \underline{\partial}) S_c \\ &\left. - \Omega(i\mathbf{k}_b \cdot \underline{\partial} D_a^* S_a^*)(\underline{\partial} S_b) \cdot (\hat{z} \times \underline{\partial} S_c) + (b \leftrightarrow c) \right\}. \end{aligned}$$

Again, this formula simplifies considerably in the unmagnetized limit.

To complete the derivation of the coupling coefficient, we must add the three contributions (24), (25) and (26). With judicious manipulation, the resultant sum can be cast in a form which is manifestly symmetric in the three generating functions. An outline of the algebra is given in Appendix B. We present here just the final result

$$\begin{aligned} -c^{-1} \underline{A}_a^* \cdot \underline{j}_{b,c}^{(2)} &= \sum_{\mathbf{s}} m^{-2} \int d^3\mathbf{w} f_0(\mathbf{w}_1, \mathbf{w}_2) \\ &\times \left\{ (-i\mathbf{k}_a \cdot \underline{\partial} S_a^*)(D_b \underline{\partial} S_b) \cdot (D_c \underline{\partial} S_c) \right. \\ &+ (i\mathbf{k}_b \cdot \underline{\partial} S_b)(D_c \underline{\partial} S_c) \cdot (D_a^* \underline{\partial} S_a^*) \\ &+ (i\mathbf{k}_c \cdot \underline{\partial} S_c)(D_a^* \underline{\partial} S_a^*) \cdot (D_b \underline{\partial} S_b) \\ &\left. - \Omega(\underline{\partial} S_a^*) \cdot \left[ i\mathbf{k}_{bz}(\underline{\partial} S_b) \times (D_c \underline{\partial} S_c) + (b \leftrightarrow c) \right] \right\}. \end{aligned} \quad (27)$$

The generating functions  $S_a(\mathbf{w})$  appearing in (27) were calculated explicitly in Sec. II and are given by (11) and (12). Our formula is compact and symmetric under interchange of the labels (a\*, b, c). This symmetry, together with (A10), implies the Manley-Rowe action-transfer relations

$$\omega_a^{-1} \dot{W}_a = -\omega_b^{-1} \dot{W}_b = -\omega_c^{-1} \dot{W}_c.$$

The symmetric expression (27) has previously been derived [Larsson 1975, Larsson and Stenflo 1976] using different notation and a different method. The present formulation illustrates the oscillation-centre viewpoint and gives new insight into the origin of the various terms in (27). The first term arises from the oscillation-centre contribution (24). The second and third terms can be traced to the polarization contributions (25) and (26). The final term

appears only in magnetized plasma and has hybrid origins; its symmetry is a consequence of the matching conditions (16) and the triple-product vector identity.

An appealing feature of formula (27) is that the coupling coefficient is expressed in terms of just three scalar functions of velocity, namely, the generating functions for each wave. In particular applications, it may be possible to approximate these functions. Stenflo and Larsson (1977) have illustrated the use of the general expression (27) in several particular situations. In the remainder of this paper, we shall also turn our attention to certain limiting cases.

#### V. Beat Forces and Currents in the Cold-Plasma Approximation

Consider again the interaction of a resonant triplet of waves  $a$ ,  $b$ , and  $c$ , which satisfy the matching conditions (16). We have treated the general case in Sections III and IV. In order to simplify our general formulas, let us suppose in this section that two of the three waves (say modes  $a$  and  $c$ ) are adequately described by a cold-fluid model for the magnetized plasma (Stix 1962). No assumption will be made about the "beat mode"  $b$ , however, We shall also restore here the arbitrary gauge of Section II in order to display manifest gauge invariance of certain quantities to be derived. The invariant electric-field vectors  $\underline{E}_a$  are related to the scalar and vector potentials  $\phi_a$  and  $\underline{A}_a$  according to

$$\underline{E}_a = -i\underline{k}_a \phi_a + i\omega_a c^{-1} \underline{A}_a . \quad (28)$$

Since modes  $a$  and  $c$  are to be treated as cold plasma waves, it is appropriate to expand the associated generating functions  $S_a(\underline{w})$  and  $S_c(\underline{w})$  in powers of the velocity  $\underline{w}$ . Accordingly, we write

$$S_a(\underline{w}) = \sum_{n=0}^N S_a^{[n]}(\underline{w}) + O(w^{N+1}), \quad (29)$$

where  $S_a^{[n]}(\underline{w})$  is of order  $w^n$ . From (11) and (12), we find for the leading terms in such an expansion

$$\begin{aligned}
 S_a^{[0]} &= -ie \omega_a^{-1} \phi_a, \\
 S_a^{[1]} &= -e\omega_p^{-2} \left[ \underline{w} \cdot \underline{\chi}_0(\omega_a) \cdot \underline{E}_a \right], \\
 S_a^{[2]} &= -e \omega_a^{-1} \omega_p^{-2} (\underline{w}_z E_{az}) \left[ \underline{w} \cdot \underline{\chi}_0(\omega_a) \cdot \underline{k}_a \right] \\
 &\quad -2^{-1} e\omega_a^{-1} \omega_p^{-2} \sum_{\pm, -} (\underline{w}_x \pm i\underline{w}_y)(E_{ax} \mp iE_{ay}) \\
 &\quad \times \left[ \underline{w} \cdot \underline{\chi}_0(\omega_a \pm \Omega) \cdot \underline{k}_a \right], \tag{30}
 \end{aligned}$$

where  $\omega_p$  denotes the plasma frequency. We have employed here the notation  $\underline{\chi}_0(\omega)$  to denote the familiar cold-plasma linear susceptibility tensor evaluated at frequency  $\omega$ , i. e., (Stix 1962)

$$\underline{\chi}_0(\omega) \equiv -\frac{\omega_p^2}{(\omega^2 - \Omega^2)} \begin{bmatrix} 1 & i\Omega\omega^{-1} & 0 \\ -i\Omega\omega^{-1} & 1 & 0 \\ 0 & 0 & (1 - \Omega^2\omega^{-2}) \end{bmatrix}$$

The oscillation-centre Hamiltonian  $K_{a,c}^{(2)}$  corresponding to the expanded generating functions (29) can be similarly represented in the form

$$K_{a,\pm c}^{(2)}(\underline{w}) = e\phi_{a,\pm c}^{(2)} - ec^{-1} \underline{w} \cdot \underline{A}_{a,\pm c}^{(2)} + O(w^2).$$

Our notation for the leading coefficients in this velocity series expansion is deliberately suggestive, since  $\phi^{(2)}$  and  $\underline{A}^{(2)}$  play the role of scalar and vector potentials experienced by an oscillation centre. Insertion of the expanded generating functions (29) into expression (15) leads to the following gauge-invariant formulas for these potentials:

$$\phi_{a,-c}^{(2)} = -e(2m\omega_p^2)^{-1} \underline{E}_a \cdot \left[ \omega_c \omega_a^{-1} \underline{\chi}_0^*(\omega_c) + (a \leftrightarrow c) \right] \cdot \underline{E}_c^*, \tag{31}$$

$$\phi_{a,c}^{(2)} = e(2m\omega_p^2)^{-1} \underline{E}_a \cdot \left[ \omega_c \omega_a^{-1} \underline{\chi}_0(\omega_c) + (a \leftrightarrow c) \right] \cdot \underline{E}_c,$$

$$\begin{aligned}
 A_{a,-c}^{(2)} &= -ec\omega_a (2m\omega_p^4)^{-1} \left[ \underline{\chi}_0^*(\omega_c) \cdot \underline{E}_c^* \right] \left\{ \underline{k}_c \cdot \left[ \underline{\chi}_0(\omega_a) - \omega_p^2 \omega_a^{-2} \underline{1} \right] \cdot \underline{E} \right\} \\
 &\quad - ic\Omega(2m\omega_a)^{-1} \underline{\partial} \left[ \underline{E}_a \cdot (\hat{z} \times \underline{\partial} S_c^{[2]*}) \right] + (a \leftrightarrow c)^*, \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 A_{a,c}^{(2)} &= ec\omega_a (2m\omega_p^4)^{-1} \left[ \underline{\chi}_0(\omega_c) \cdot \underline{E}_c \right] \left\{ \underline{k}_c \cdot \left[ \underline{\chi}_0(\omega_a) - \omega_p^2 \omega_a^{-2} \underline{1} \right] \cdot \underline{E}_a \right\} \\
 &\quad - ic\Omega(2m\omega_a)^{-1} \underline{\partial} \left[ \underline{E}_a \cdot (\hat{z} \times \underline{\partial} S_c^{[2]}) \right] + (a \leftrightarrow c),
 \end{aligned}$$

where  $S_c^{[2]}$  is given by (30). The associated "pseudo-electric" and "pseudo-magnetic" fields acting on an oscillation centre are



then, in analogy with (28),

$$\underline{E}_{a,\pm c}^{(2)} = -i(\underline{k}_a \pm \underline{k}_c)\phi_{a,\pm c}^{(2)} + ic^{-1}(\omega_a \pm \omega_c)\underline{A}_{a,\pm c}^{(2)},$$

$$\underline{B}_{a,\pm c}^{(2)} = i(\underline{k}_a \pm \underline{k}_c) \times \underline{A}_{a,\pm c}^{(2)}.$$

Our result (31) for the oscillation-centre scalar potential is consistent with the known formula for "ponderomotive potential" in the presence of a magnetic field (Motz and Watson 1967). Note, however, that we have not assumed  $|\omega_a - \omega_c| \ll \omega_a, \omega_c$ , and averaged over a fast time scale; we have simply transformed to new variables. Formulas (31) and (32) simplify greatly in the unmagnetized limit. One obtains, for example,

$$\lim_{\Omega \rightarrow 0} \underline{E}_{a,-c}^{(2)} = -ie(m\omega_a\omega_c)^{-1} \left\{ (\underline{k}_a - \underline{k}_c)(\underline{E}_a \cdot \underline{E}_c^*) + 2(\omega_a - \omega_c) \left[ \omega_c^{-1} \underline{E}_c^*(\underline{k}_c \cdot \underline{E}_a) + \omega_a^{-1} \underline{E}_a(\underline{k}_a \cdot \underline{E}_c^*) \right] \right\}. \quad (33)$$

In the ordered-frequency case  $|\omega_a - \omega_c| \ll \omega_a, \omega_c$ , the last two terms in (33) can be neglected, and one obtains the usual ponderomotive force, derivable from a scalar potential. However, in the general unordered case, the contribution from the oscillation-centre vector potential must be retained, even in this cold-plasma limit.

The contribution to the nonlinear current density due to the response of oscillation centres to the beat pseudo-electric field can be written

$$\underline{j}_{a,-c}^{(2)} = \sum_s \underline{\sigma}(\omega_b) \cdot \underline{E}_{a,-c}^{(2)}, \quad (34)$$

where  $\underline{\sigma}(\omega_b)$  denotes the linear conductivity tensor for the plasma at frequency  $\omega_b$  (not necessarily cold). The linear conductivity tensor  $\underline{\sigma}$  is, of course, related to the susceptibility tensor  $\underline{\chi}$  by

$$\underline{\sigma}(\omega) = i\omega(4\pi)^{-1} \underline{\chi}(\omega).$$

Now, the self-consistent electric field  $\underline{E}_b$  at frequency  $\omega_b$  is driven by the true nonlinear current  $\underline{j}_{a,-c}^{(2)}$  according to (see Appendix A)

$$\underline{D}(b) \cdot \underline{E}_b = 4\pi(i\omega_b)^{-1} \underline{j}_{a,-c}^{(2)}, \quad (35)$$

where  $\underline{D}(b)$  denotes the linear dispersion tensor. Defining the "polarization current"  $\underline{\Delta j}_{a,-c}^{(2)}$  as the difference between the true nonlinear current and the oscillation-centre current (34),

$$\underline{\Delta j}_{a,-c}^{(2)} = \underline{j}_{a,-c}^{(2)} - \underline{j}_{a,-c}^{(2)},$$

we find from (22) that

$$\begin{aligned} \underline{\Delta}_{a,-c}^{j(2)} = & \sum_s e \int d^3w \underline{w} \Delta_{a,-c}^{(2)}(\underline{w}) + \sum_s ne^2(mc)^{-1} \underline{A}_{a,-c}^{(2)} \\ & - \sum_s e^2(mc)^{-1} \int d^3w \left( \underline{A}_a \Delta_c^{(1)*} + \underline{A}_c^* \Delta_a^{(1)} \right). \end{aligned} \quad (36)$$

Let us introduce the perturbed fluid-velocity vectors  $\underline{v}_a$  and  $\underline{v}_c$ , defined by

$$\underline{v}_a = (ne)^{-1} g_0(\omega_a) \cdot \underline{E}_a. \quad (37)$$

Then, inserting the expanded generating functions (29) into our formulas (19) and (20) for the polarization density  $\Delta$ , we obtain from (36) the gauge-invariant result

$$\begin{aligned} \underline{\Delta}_{a,-c}^{j(2)} = & - \sum_s ne(2\omega_a \omega_c)^{-1} \underline{v}_a \cdot (\omega_c \underline{k}_a - \omega_a \underline{k}_c) \cdot \underline{v}_c^* \\ & - \sum_s i\Omega ne(4\omega_a \omega_c)^{-1} (\hat{z} \times \underline{v}_a) \cdot (\underline{v}_c^* \underline{k}_c + \underline{k}_c \underline{v}_c^*) \\ & - \sum_s i\Omega ne(4m\omega_a)^{-1} \left[ \Omega(\hat{z} \times \underline{v}_a \cdot \underline{\partial}) \underline{1} + em^{-1} \underline{\partial} \underline{E}_a \right] \\ & \cdot (\hat{z} \times \underline{\partial} S_c^{[2]*}) + (a \leftrightarrow c)^*, \end{aligned}$$

where  $S_c^{[2]}$  is given by (30). The nonlinear current density to be inserted in (35) is the sum of the two contributions (34) and (38).

The expressions derived in this section have been obtained from our general formulas in Sections II to IV on the basis of a single assumption, namely, that modes a and c can be adequately described by the cold-plasma model (Stix 1962). No assumption was made concerning the beat mode b for which a full kinetic treatment has been implicit. The familiar "ponderomotive-scalar-potential method" (Drake et al 1974; Manheimer and Ott 1974) can be derived from the results of this section by making certain further approximations; we study this limit in the next section. Note, however, that outside its domain of validity, we already have the needed correction terms here at our disposal.

## VI. The Ponderomotive-Scalar-Potential Method

The term "ponderomotive-scalar-potential method" is used in this paper to apply to the resonant interaction between cold-plasma modes ("high frequency") and modes for which kinetic effects are retained ("low frequency"). The method consists of the following prescription for describing such interactions. The high-frequency nonlinear currents are approximated by the product of the low-frequency density perturbation and the high-frequency velocity perturbation. The low-frequency nonlinear current is approximated by the linear response to a scalar ("ponderomotive") potential produced by the beating of the high-frequency modes. Note that in the language of Section V, both the ponderomotive vector potential (32) and the polarization current (38) are omitted. These approximations can be inferred to be consistent, even in strongly magnetized plasma, since they imply the Manley-Rowe relations; we verify this statement in Appendix C. The approach has intuitive appeal and has been found useful by many authors. It has been applied to unmagnetized ( $\Omega = 0$ ) plasma (Litvak and Trakhtengerts 1971; Drake et al 1974; Hasegawa et al. 1976), to weakly magnetized ( $\Omega \ll \omega_0$ ) plasma (Mannheimer and Ott 1974; Bujarbarua et al. 1974), and (uncritically) to strongly magnetized ( $\Omega \sim \omega_0$ ) plasma (Litvak and Trakhtengerts 1972; Sanuki and Schmidt 1977). In this section, we develop the method without restriction on magnetic-field strength and use the formulation to tie together several scattered results in the literature. General criteria for the validity of the method in strongly magnetized plasma will be given in Section VII.

Let us consider, then, a representative four-mode interaction in magnetized plasma involving a "pump" mode  $(\omega_a, \underline{k}_a)$ , two "sideband" modes  $(\omega_c, \underline{k}_c)$ ,  $(\omega_d, \underline{k}_d)$ , and a "beat" mode  $(\omega_b, \underline{k}_b)$ . The associated matching conditions are shown in Figure 1; for example,  $\omega_c = (\omega_a - \omega_b)$  and  $\omega_d = (\omega_a + \omega_b)$ . We assume that the frequency of the beat mode  $b$  is much less than the high frequencies  $\omega_a, \omega_c, \omega_d$ . Kinetic effects are retained for mode  $b$ , but the other modes are described by cold-plasma equations. We therefore need not distinguish among the susceptibility tensors  $\chi_0(\omega_a)$ ,  $\chi_0(\omega_c)$  and  $\chi_0(\omega_d)$ , since they are independent of  $\underline{k}$  and the frequencies are approximately equal; we write

$$\chi_0^{(s)}(\omega_a) \approx \chi_0^{(s)}(\omega_c) \approx \chi_0^{(s)}(\omega_d) = \chi_0^{(s)}.$$

Note that  $\chi_0^{(s)}$  is an Hermitian matrix.

Now, the effective electric field seen by an oscillation centre of species  $s$  at the beat frequency  $\omega_b$  can be written

$$\underline{E}_b^{(s)} = \underline{E}_b + \underline{\mathcal{E}}_b^{(s)}, \quad (39)$$

where  $\underline{E}_b$  denotes the self-consistent electric field in the plasma, and  $\underline{\mathcal{E}}_b^{(s)}$  the field derived from the scalar ponderomotive potentials,

$$\underline{\mathcal{E}}_b^{(s)} \equiv -ik_b (\phi_{a,-c}^{(2)} + \phi_{d,-a}^{(2)}).$$

Inserting formula (31) for  $\phi_{a,-c}^{(2)}$ , we get

$$\underline{\mathcal{E}}_b^{(s)} = ik_b (4\pi m_s e_s)^{-1} \left[ (\underline{E}_c^* \cdot \chi_0^{(s)} \cdot \underline{E}_a) + (\underline{E}_a^* \cdot \chi_0^{(s)} \cdot \underline{E}_d) \right]. \quad (40)$$

According to the prescription given above, the low-frequency equation (35) is to be approximated as

$$\underline{D}^{(b)} \cdot \underline{E}_b = 4\pi(i\omega_b)^{-1} \sum_s \underline{\mathcal{E}}_b^{(s)} \cdot \underline{\mathcal{E}}_b^{(s)}, \quad (41)$$

where  $\underline{\mathcal{G}}_b^{(s)}$  denotes the kinetic conductivity tensor. Using (39), we can recast (41) in the form

$$\underline{D}^{(b)} \cdot \underline{E}_b^{(s)} = \underline{\mathcal{G}}_b^{(s)} + \sum_{s' \neq s} \chi_b^{(s')} \cdot (\underline{\mathcal{E}}_b^{(s)} - \underline{\mathcal{E}}_b^{(s')}). \quad (42)$$

Turning to the corresponding high-frequency equations, we note that the high-frequency velocity perturbation  $v_a^{(s)}$  is defined by (37), and the low-frequency density perturbation  $\delta n_b^{(s)}$  is, from the continuity equation,

$$\delta n_b^{(s)} = -i(4\pi e_s)^{-1} (k_b \cdot \chi_b^{(s)} \cdot \underline{E}_b), \quad (43)$$

with  $\underline{E}_b^{(s)}$  given by (39). According to the stated rules of the method, the high-frequency versions of (35) are then

$$\underline{D}_0^{(a)} \cdot \underline{E}_a = 4\pi(i\omega_a)^{-1} \sum_s e_s (v_c^{(s)} \delta n_b^{(s)} + v_d^{(s)} \delta n_b^{(s)*}),$$

$$\underline{D}_0^{(c)} \cdot \underline{E}_c = 4\pi(i\omega_c)^{-1} \sum_s e_s v_a^{(s)} \delta n_b^{(s)*},$$

$$\underline{D}_0^{(d)} \cdot \underline{E}_d = 4\pi(i\omega_d)^{-1} \sum_s e_s v_a^{(s)} \delta n_b^{(s)}$$

Upon insertion of (37) and (43), we get the coupled equations

$$\begin{aligned} \underline{D}_0^{(a)} \cdot \underline{E}_a = i \sum_s (4\pi m_s e_s)^{-1} \left[ (\chi_0^{(s)} \cdot \underline{E}_c)(k_b \cdot \chi_b^{(s)} \cdot \underline{E}_b^{(s)}) \right. \\ \left. - (\chi_0^{(s)} \cdot \underline{E}_d)(k_b \cdot \chi_b^{(s)*} \cdot \underline{E}_b^{(s)*}) \right], \end{aligned} \quad (44)$$

$$\underline{D}_0^{(c)} \cdot \underline{E}_c = -i \sum_s (4\pi m_s e_s)^{-1} (\chi_0^{(s)} \cdot \underline{E}_a)(k_b \cdot \chi_b^{(s)*} \cdot \underline{E}_b^{(s)*}), \quad (45)$$

$$\underline{D}_0^{(d)} \cdot \underline{E}_d = i \sum_s (4\pi m_s e_s)^{-1} (\chi_0^{(s)} \cdot \underline{E}_a)(k_b \cdot \chi_b^{(s)} \cdot \underline{E}_b^{(s)}). \quad (46)$$

If (45) and (46) are solved for  $\underline{E}_c^*$  and  $\underline{E}_d$ , and the solutions inserted into (40), then (42) leads to the equations

$$\underline{D}(b) \cdot \underline{E}_b^{(s)} = -(4\pi)^{-2} \sum_{s''} (n_{s''} e_{s''})^{-1} \underline{Q}(s, s'') (\underline{k}_b \cdot \underline{\chi}_b^{(s'')}) \cdot \underline{E}_b^{(s'')}, \quad (47)$$

where we have defined

$$\begin{aligned} \underline{Q}(s, s'') \equiv & \underline{k}_b (n_s e_s)^{-1} \left\{ \left[ \underline{D}_0(c)^{-1} \cdot (\underline{\chi}_0^{(s'')})^* \cdot \underline{E}_a^* \right] \cdot (\underline{\chi}_0^{(s)}) \cdot \underline{E}_a \right. \\ & \left. + (c \leftrightarrow d)^* \right\} \\ & + \sum_{s' \neq s} (\underline{\chi}_b^{(s')}) \cdot \underline{k}_b \left\{ \left[ \underline{D}_0(c)^{-1} \cdot (\underline{\chi}_0^{(s'')})^* \cdot \underline{E}_a^* \right] \right. \\ & \left. \cdot \left[ (n_s e_s)^{-1} \underline{\chi}_0^{(s)} - (n_{s'} e_{s'})^{-1} \underline{\chi}_0^{(s')} \right] \cdot \underline{E}_a + (c \leftrightarrow d)^* \right\}. \end{aligned}$$

Suppose there are  $N$  species of charge,  $s = 1, 2, \dots, N$ . Then (47) represents  $3N$  linear algebraic equations for the components of vectors  $\underline{E}_b^{(1)}, \underline{E}_b^{(2)}, \dots, \underline{E}_b^{(N)}$ . The nonlinear dispersion relation is the condition which allows a nontrivial solution of these equations. In general, it is obtained by setting the determinant of the  $3N$ -dimensional matrix of coefficients equal to zero.

Suppose, however, that the beat mode  $b$  is longitudinal, with  $\underline{E}_b^{(s)} = E_b^{(s)} \hat{k}_b$ . The  $3N$  equations (47) can then be reduced to  $N$  equations for  $E_b^{(s)}$  by projecting vectors onto the direction  $\hat{k}_b$ . If the sideband modes  $c$  and  $d$  are also treated as electrostatic, then these  $N$  equations reduce further to the form

$$\begin{aligned} \epsilon_b E_b^{(s)} = & - \sum_{s''} k_b^2 \chi_b^{(s'')} E_b^{(s'')} \\ & \times \left\{ (k_c^2 \epsilon_c)^{-1} \mu_c^{(s'')} \left[ \mu_c^{(s)} + \sum_{s' \neq s} \chi_b^{(s')} (\mu_c^{(s)} - \mu_c^{(s')}) \right] \right. \\ & \left. + (k_d^2 \epsilon_d)^{-1} \mu_d^{(s'')} \left[ \mu_d^{(s)*} + \sum_{s' \neq s} \chi_b^{(s')} (\mu_d^{(s)*} - \mu_d^{(s')*}) \right] \right\}, \quad (48) \end{aligned}$$

where we have defined the parameters

$$\mu_c^{(s)} \equiv (4\pi n_s e_s)^{-1} (\hat{k}_c \cdot \underline{\chi}_0^{(s)} \cdot \underline{E}_a).$$

If, in addition, one treats the pump mode  $a$  in the "dipole approximation" ( $|k_a| \rightarrow 0$ ), one has that  $\underline{k}_d = -\underline{k}_c = \underline{k}_b$  and hence  $\mu_d^{(s)} = -\mu_c^{(s)} \equiv \mu_s$ , where

$$\mu_s = -\frac{e_s}{m_s} \left[ \left( \frac{k_{bz} E_{az}}{\omega_a^2} + \frac{k_{b\perp} \cdot E_{a\perp}}{(\omega_a^2 - \Omega_s^2)} \right) + i \left( \frac{(k_{bx} E_{ay} - k_{by} E_{ax}) \Omega_s}{\omega_a (\omega_a^2 - \Omega_s^2)} \right) \right].$$

Suppose there exists a species  $\bar{s}$  such that

$$|\mu_{\bar{s}}| \gg |\mu_s| \quad \forall s \neq \bar{s}$$

Then choosing  $s = \bar{s}$  in (48), and retaining only the term  $s = \bar{s}$  on the right-hand side, we obtain the nonlinear dispersion relation

$$\epsilon_b = -|\mu_{\bar{s}}|^2 \chi_b^{(\bar{s})} \left( 1 + \sum_{s \neq \bar{s}} \chi_b^{(s)} \right) (\epsilon_c^{-1} + \epsilon_d^{-1}).$$

When  $\bar{s}$  corresponds to electrons, and when  $|\chi_b^{(e)}| \gg 1$ ,  $|\chi_b^{(i)}| \gg 1$ , this dispersion relation agrees with that derived by Porkolab (1974) for lower-hybrid heating [his Eq. (15)].

A different special case obtains when  $N = 2$  with  $\mu_\alpha \sim \mu_\beta$  and  $|\chi_b^{(\alpha)}| \gg 1$ ,  $|\chi_b^{(\beta)}| \gg 1$ . Equation (42) then implies that

$$\chi_b^{(\alpha)} E_b^{(\alpha)} \approx -\chi_b^{(\beta)} E_b^{(\beta)},$$

and one therefore obtains from (48) the dispersion relation

$$\epsilon_b = -|\mu_\alpha - \mu_\beta|^2 \chi_b^{(\alpha)} \chi_b^{(\beta)} (\epsilon_c^{-1} + \epsilon_d^{-1}).$$

In the limit  $\omega_a \gg \Omega_\alpha, \Omega_\beta$ , this result agrees with Eq. (10) of Ott et al. (1973), derived for a plasma containing two ion species with different charge-to-mass ratios.

Let us now withdraw these various approximations and restore our 3N electromagnetic equations (47). That complicated analysis simplifies considerably if we do not insist upon treating all species symmetrically. Suppose the plasma consists of electrons  $e$  and several species of ions  $i$ . Based on the fact that  $m_i \gg m_e$ , we shall neglect  $\underline{\mathcal{E}}_b^{(i)}$  and  $\chi_{\omega_0}^{(i)}$ . Equations (47) therefore reduce to

$$D(b) \cdot \underline{E}_b^{(e)} = \left( 1 + \sum_i \chi_b^{(i)} \right) \cdot \underline{\mathcal{E}}_b^{(e)}, \quad (49)$$

$$D(b) \cdot \underline{E}_b^{(i)} = -\chi_b^{(e)} \cdot \underline{\mathcal{E}}_b^{(e)}, \quad (50)$$

where, from (40), (45) and (46),

$$\begin{aligned} \underline{\mathcal{E}}_b^{(e)} = & - (4m_e)^{-2} \underline{k}_b (\underline{k}_b \cdot \chi_b^{(e)} \cdot \underline{E}_b^{(e)}) \\ & \times \left\{ \left[ D_{\omega_0}(c)^{-1} \cdot (\chi_{\omega_0}^{(e)*} \cdot \underline{E}_a^*) \right] \cdot (\chi_{\omega_0}^{(e)} \cdot \underline{E}_a) \right. \\ & \left. + \left[ D_{\omega_0}(d)^{-1} \cdot (\chi_{\omega_0}^{(e)} \cdot \underline{E}_a) \right] \cdot (\chi_{\omega_0}^{(e)*} \cdot \underline{E}_a^*) \right\}. \end{aligned} \quad (51)$$

We introduce  $\underline{Q}_b$ , cofactor tensor of the transpose of  $\underline{D}(b)$ , defined by

$$\underline{D}(b)^{-1} = \underline{Q}_b / \det \underline{D}(b).$$

Then substitution of (51) into (49) and the inner product of the resultant equation with  $(\underline{k}_b \cdot \underline{X}_b^{(e)})$  lead to the dispersion relation

$$\det \underline{D}(b) = -(4\pi n_e e)^{-2} (\underline{k}_b \cdot \underline{X}_b^{(e)}) \cdot \underline{Q}_b \cdot \left[ (1 + \sum_i \underline{X}_b^{(i)} \cdot \underline{k}_b) \right. \\ \left. \times \left\{ \left[ \underline{D}_c^{(c)} \right]^{-1} \cdot (\underline{X}_c^{(e)*} \cdot \underline{E}_a^*) \cdot (\underline{X}_c^{(e)} \cdot \underline{E}_a) \right. \right. \\ \left. \left. + \left[ \underline{D}_c^{(d)} \right]^{-1} \cdot (\underline{X}_c^{(e)} \cdot \underline{E}_a) \cdot (\underline{X}_c^{(e)*} \cdot \underline{E}_a^*) \right\} \right] \quad (52)$$

This relation generalizes the results of Drake et al. (1974) and of Manheimer and Ott (1974) to the case of strongly magnetized plasma.

Indeed, consider the limit of a weak magnetic field. When  $\Omega_i \ll \omega_b$ , the ion susceptibilities can be taken to be scalars. Furthermore, when  $\Omega_e \ll \omega_a$ , the high-frequency modes can be treated as unaffected by the magnetic field, and so we have in particular

$$\underline{X}_0^{(e)} \approx -\omega_{pe}^2 \omega_a^{-2} \underline{1},$$

$$\underline{D}_0^{(c)} \approx D_c^l \hat{k}_c \hat{k}_c + D_c^t \left( \frac{1}{v} - \hat{k}_c \hat{k}_c \right).$$

The dispersion relation (52) thus reduces to

$$\det \underline{D}(b) = -(4\pi n_e e)^{-2} \omega_{pe}^2 \omega_a^{-2} \left( 1 + \sum_i \underline{X}_b^{(i)} \right) (\underline{k}_b \cdot \underline{X}_b^{(e)}) \cdot \underline{Q}_b \cdot \underline{k}_b \\ \times \left\{ \frac{|\hat{k}_c \cdot \underline{E}_a|^2}{D_c^l} + \frac{|\hat{k}_c \times \underline{E}_a|^2}{D_c^t} + (c \rightarrow d) \right\},$$

in agreement with Eq. (13) of Manheimer and Ott (1974).

In summary, we have developed the ponderomotive-scalar-potential method as a limiting case of our more general theory. The oscillation-centre viewpoint is especially well-suited for understanding how this familiar limiting theory relates to the general case. The approximations in this section are shown to be internally consistent in Appendix C. The domain of validity of the method and generalizations beyond that domain are the subject of the next section.

VII. Validity of the Ponderomotive-Scalar-Potential Approximation  
in Magnetized Plasma

In this section, we examine the validity of the approximations in Section VI, and then develop a more general theory by reinstating certain neglected terms. This generalized theory is a continuation of our work in Section V and assumes only that (in three-wave interaction) two of the three waves are cold-plasma modes.

The ponderomotive-scalar-potential method approximates the true low-frequency nonlinear current

$$\underline{j}_{a,-c}^{(2)} = \underline{j}_{a,-c}^{(2)} + \underline{\Delta j}_{a,-c}^{(2)}$$

by (41), i. e., by just the scalar-potential portion of  $\underline{j}_{a,-c}^{(2)}$

$$\underline{j}_{a,-c}^{(2)} \approx \sum_s \frac{\sigma_s^{(s)}}{\omega_b} \cdot (-i\underline{k}_b) \phi_{a,-c}^{(2)}$$

Contributions from the ponderomotive vector potential  $\underline{A}_{a,-c}^{(2)}$  and the polarization current  $\underline{\Delta j}_{a,-c}^{(2)}$  are omitted. Collecting our results in Section V, we find that the retained and omitted contributions to  $\underline{j}_{a,-c}^{(2)}$  are, in terms of the perturbed fluid-velocity vectors (37),

$$\begin{aligned} \text{Retained} &= - \sum_s n_s e_s \omega_b \omega_{ps}^{-2} (\chi_b^{(s)} \cdot \underline{k}_b) \\ &\times \left[ (\underline{v}_a \cdot \underline{v}_c^*) + i\Omega_s (\omega_a + \omega_c) (2\omega_a \omega_c)^{-1} (\hat{z} \times \underline{v}_a) \cdot \underline{v}_c^* \right], \\ \text{Omitted} &= \sum_s \left\{ (4\pi c)^{-1} \omega_{ps}^2 \left( \frac{1}{\omega} + \omega_b^2 \omega_{ps}^{-2} \chi_b^{(s)} \right) \cdot \underline{A}_{a,-c}^{(2)} \right. \\ &+ n_s e_s (\omega_a \omega_c)^{-1} \left[ \omega_a \underline{v}_a (\underline{k}_c \cdot \underline{v}_c^*) + \omega_c \underline{v}_c (\underline{k}_a \cdot \underline{v}_a) \right. \\ &\quad \left. \left. - i\Omega_s 2^{-1} (\underline{k}_a + \underline{k}_c) (\hat{z} \times \underline{v}_a) \cdot \underline{v}_c^* \right] \right. \\ &\left. + \Omega_s^2 n_s e_s (2m_s \omega_a \omega_c)^{-1} \left[ i\omega_a (\hat{z} \times \underline{v}_c^*) \cdot \underline{\partial} (\hat{z} \times \underline{\partial}) S_a^{[2]} + (a \leftrightarrow c)^* \right] \right\} \end{aligned} \quad (53)$$

where, from (32) and (30),  $\underline{A}_{a,-c}^{(2)}$  and  $S_a^{[2]}$  are given by

$$\begin{aligned} \underline{A}_{a,-c}^{(2)} &= -m_s c (e_s \omega_c)^{-1} \underline{v}_c^* \underline{k}_c \cdot \left[ \underline{v}_a + i\Omega_s (2\omega_a)^{-1} (\hat{z} \times \underline{v}_a) \right] \\ &- \Omega_s c (2e_s)^{-1} \underline{\partial} \left\{ \underline{v}_a + i\Omega_s \omega_a^{-1} (\hat{z} \times \underline{v}_a) \right\} \cdot (\hat{z} \times \underline{\partial} S_c^{[2]*}) \\ &+ (a \leftrightarrow c)^*, \end{aligned}$$

$$S_a^{[2]}(\underline{w}) = im_s \omega_a^{-1} \underline{w} \underline{w} : \underline{T}_a^{(s)}$$

$$\omega_{ps}^2 \underline{T}_a^{(s)} = \omega_a \underline{v}_a \underline{v}_a \left[ \chi_{a0}^{(s)}(\omega_a) \cdot \underline{k}_a \right]$$

$$+ 2^{-1} \sum_{+,-} (\omega_a \pm \Omega_s) \left[ \underline{v}_a \pm i(\hat{z} \times \underline{v}_a) \right] \left[ \chi_{a0}^{(s)}(\omega_a \pm \Omega_s) \cdot \underline{k}_a \right]$$



Thus, the general criteria for the validity of the ponderomotive-scalar-potential method in magnetized plasma are, first, that the plasma be cold with respect to the high-frequency modes, and, second, that the projection of the omitted current (54) on  $\underline{E}_b^*$  be negligible compared with the projection of the retained current (53). This last criterion is clearly very complicated in the most general case. It has been shown by Manheimer and Ott (1974) that a sufficient condition is  $\Omega_s \ll \omega_a$ , i. e., a "weakly magnetized" plasma. For stronger magnetic fields, we see from the above formulas that contributions from the omitted polarization current and ponderomotive vector potential tend to become important.

Let us, then, reinstate the omitted current (54) in order to develop a generalized theory which can be safely applied in strongly magnetized plasma. We consider again the resonant triplet of waves of Section V; kinetic effects are retained for mode b, but modes a and c are described by cold-plasma equations. Our only other assumption will be that there are no particles resonant with mode b, and hence that the susceptibility tensor  $\chi_{ab}^{(s)}$  is Hermitian.

It is helpful to define the resistivity tensor  $\eta_a^{(s)}$

$$\eta_a^{(s)} \equiv \frac{\sigma_a^{(s)-1}}{\omega_a}, \quad \eta_{0a}^{(s)} \equiv \eta_a^{(s)} \Big|_{\text{cold}},$$

and to introduce the notation

$$\begin{aligned} \underline{w}_a &\equiv \underline{v}_a - ie_s (m_s \omega_a)^{-1} \underline{E}_a, \\ \underline{u}_a &\equiv -i\omega_{ps}^2 (4\pi\omega_a)^{-1} (\underline{J}_a^{(s)} - \eta_{0a}^{(s)}) \cdot \underline{v}_a, \\ \underline{M}_{wa}^{(s)} &\equiv im_s (2\omega_a)^{-1} (\underline{k}_a \underline{v}_a + \underline{v}_a \underline{k}_a) + \omega_a \underline{\partial} \underline{\partial} S_a^{[2]} \\ &\quad + i\Omega_s \underline{\partial} (\hat{z} \times \underline{\partial}) S_a^{[2]} + i\Omega_s (\hat{z} \times \underline{\partial}) \underline{\partial} S_a^{[2]}. \end{aligned} \quad (56)$$

Now, by (A 10), the desired low-frequency coupling coefficient is the interaction energy  $(\phi_b^* \rho_{a,-c}^{(2)} - c^{-1} \underline{A}_b^* \cdot \underline{j}_{a,-c}^{(2)})$ . Adding (53) and (54), we find after some straightforward algebra

$$\begin{aligned} (\phi_b^* \rho_{a,-c}^{(2)} - c^{-1} \underline{A}_b^* \cdot \underline{j}_{a,-c}^{(2)}) &= \sum_s n_s m_s \left[ \omega_a^{-1} (\underline{k}_a \cdot \underline{v}_a) (\underline{v}_b^* \cdot \underline{v}_c^*) \right. \\ &\quad + \omega_b^{-1} (\underline{k}_b \cdot \underline{v}_b^*) (\underline{v}_c^* \cdot \underline{v}_a) + \omega_c^{-1} (\underline{k}_c \cdot \underline{v}_c^*) (\underline{v}_a \cdot \underline{v}_b^*) \\ &\quad \left. - i\Omega_s \omega_b^{-1} (\omega_c^{-1} k_{cz} - \omega_a^{-1} k_{az}) \underline{v}_b^* \cdot (\underline{v}_a \times \underline{v}_c^*) \right] + Q, \end{aligned} \quad (57)$$

where Q represents the additional terms

$$Q = \sum_s \left\{ n_s m_s (\omega_c^{-1} \underline{k}_c - \omega_a^{-1} \underline{k}_a) \cdot (\underline{v}_a \underline{v}_c^* - \underline{v}_c \underline{v}_a^*) \cdot \underline{u}_b^* \right. \\ \left. + 2^{-1} n_s \Omega_s \omega_b \left[ \omega_c^{-1} (\hat{z} \times \underline{v}_c^*) \cdot \underline{\partial} \underline{\partial} S_a^{[2]} - \omega_a^{-1} (\hat{z} \times \underline{v}_a) \cdot \underline{\partial} \underline{\partial} S_c^{[2]} \right] \cdot \underline{u}_b^* \right. \\ \left. - 2^{-1} n_s \Omega_s \left[ \omega_c^{-1} (\hat{z} \times \underline{v}_c^*) \cdot \underline{M}_{wa}^{(s)} \cdot \underline{v}_b^* + \omega_a^{-1} (\hat{z} \times \underline{v}_a) \cdot \underline{M}_{wc}^{(s)*} \cdot \underline{w}_b^* \right] \right\}$$

It can be shown, by substitution of formula (55) into definition (56), that  $\underline{M}_{wa}^{(s)}$  and  $\underline{M}_{wc}^{(s)}$  are each identically zero. Thus, the low-frequency coupling coefficient (57) can be rewritten in the form

$$(\phi_b^* \rho_{-a,-c}^{(2)} - c^{-1} \underline{A}_{-b}^* \cdot \underline{j}_{a,-c}^{(2)}) = \sum_s n_s m_s \left[ \omega_a^{-1} (\underline{k}_a \cdot \underline{v}_a) (\underline{v}_b^* \cdot \underline{v}_c^*) \right. \\ \left. + \omega_b^{-1} (\underline{k}_b \cdot \underline{v}_b^*) (\underline{v}_c^* \cdot \underline{v}_a) + \omega_c^{-1} (\underline{k}_c \cdot \underline{v}_c) (\underline{v}_a \cdot \underline{v}_b^*) \right. \\ \left. - i \Omega_s \omega_b^{-1} (\omega_c^{-1} \underline{k}_{cz} - \omega_a^{-1} \underline{k}_{az}) \underline{v}_b^* \cdot (\underline{v}_a \times \underline{v}_c^*) + \underline{\Lambda}_{a,-c}^{(2)} \cdot \underline{u}_b^* \right], \quad (58)$$

where

$$\underline{\Lambda}_{a,-c}^{(2)} \equiv (\omega_c^{-1} \underline{k}_c - \omega_a^{-1} \underline{k}_a) \cdot (\underline{v}_a \underline{v}_c^* - \underline{v}_c \underline{v}_a^*) \\ + \Omega_s \omega_b (2m_s)^{-1} \left[ \omega_c^{-1} (\hat{z} \times \underline{v}_c^*) \cdot \underline{\partial} \underline{\partial} S_a^{[2]} - \omega_a^{-1} (\hat{z} \times \underline{v}_a) \cdot \underline{\partial} \underline{\partial} S_c^{[2]*} \right]$$

Formula (58) exhibits a pleasing structural symmetry, similar to that of our general result (27). Since  $\underline{u}_a$  and  $\underline{u}_c$  are zero (a and c are cold-plasma modes), it is simple to cast (58) into a manifestly symmetric form by adding the appropriate vanishing quantities. We can immediately deduce the corresponding high-frequency coupling coefficients since we have verified the Manley-Rowe relations for the general case (27); the low-frequency coupling coefficient was just simpler to evaluate in the present limiting case. If wave b is also treated in the cold-plasma approximation, then  $\underline{u}_b^* \rightarrow 0$  and our coupling coefficient (58) reduces to that derived by Stenflo (1973) using a cold-fluid model for the plasma.

In summary, we have shown the usual ponderomotive-scalar-potential approximation to be generally inadequate in strongly magnetized plasma, and have developed the necessary generalization of the method for the case of resonant three-wave interaction.

### VIII. Discussion

In this paper, we have shown that the oscillation-centre representation provides a systematic and intuitive framework in which to study problems of nonlinear mode coupling. The approach represents a natural extension of the familiar ponderomotive-scalar-potential approximation, and has allowed us to generalize that method to the domain of strongly magnetized plasma. Since our formulation is Hamiltonian, it is therefore quite general in principle. Indeed, we have used the approach to derive a Poisson-bracket formula for the general three-mode coupling coefficient in nonuniform relativistic magnetized plasma (Johnston and Kaufman 1977, 1978).

Our work in this paper has been restricted to the nonlinear coupling of discrete coherent waves. It is appropriate to mention here some recent complementary work of Dewar (1976, 1977a, 1977b) concerning the use of the oscillation-centre picture for statistical spectra. Dewar (1976) has constructed a "renormalized oscillation-centre transformation" which removes the coherent oscillatory motion of a particle in a stochastic potential and thereby isolates the "purely stochastic" part of the motion. Dewar (1977a) obtains an exact nonperturbative solution of the Hamilton-Jacobi equation for the generating function, applicable to resonant trapped particles in a stationary one-dimensional potential. He suggests that if such solutions also exist for a random, time-dependent potential, then

the oscillation-centre approach might lead to Markovian kinetic equations. Finally, Dewar (1977b) has studied the motion of a particle in an ensemble of monochromatic waves of random phase, such as arises in narrow-bandwidth plasma turbulence. By evaluating the averaged propagator in oscillation-centre variables, he finds that the momentum-space operators in the problem simplify greatly, leading to a remarkable factorization of the wave-particle collision operator. Reflecting on Dewar's work cited here together with our own work, we conclude that the oscillation-centre viewpoint is one of very wide applicability and usefulness.

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Appendix A: Multiple-time-scale derivation of the action-transfer equations for resonant three-wave interaction

In this appendix, we derive the action-transfer equations governing the resonant interaction among three coherent waves in a uniform plasma. We begin by combining the two Maxwell curl equations to obtain

$$\partial_t^2 \underline{E} + c^2 \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -4\pi \partial_t \underline{j} \{ \underline{E} \} . \quad (A1)$$

The current density  $\underline{j}$  in the plasma is a functional of the electric field  $\underline{E}$  and can be written formally as

$$\underline{j} \{ \underline{E} \} = \underline{\sigma} \underline{E} + \underline{B} \underline{E} \underline{E} + \underline{T} \underline{E} \underline{E} \underline{E} + \dots ,$$

where  $\underline{\sigma}$ ,  $\underline{B}$  and  $\underline{T}$  denote linear, bilinear and trilinear differential operators, respectively. Relation (A1) can thus be rewritten in the form

$$\underline{L} \underline{E} = -4\pi \partial_t (\underline{B} \underline{E} \underline{E} + \underline{T} \underline{E} \underline{E} \underline{E} + \dots) , \quad (A2)$$

where we have defined the linear operator

$$\underline{L} \underline{E} = \partial_t^2 \underline{E} + c^2 \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) + 4\pi \partial_t \underline{\sigma} \underline{E} .$$

Following Verheest (1976), we seek a nonsecular series solution of (A2) by developing both the operators and the electric field in powers of a small ordering parameter  $\epsilon$ :

$$\underline{E} = \epsilon \underline{E}^{(1)} + \epsilon^2 \underline{E}^{(2)} + \epsilon^3 \underline{E}^{(3)} + \dots ,$$

$$\underline{L} = \underline{L}^{(0)} + \epsilon \underline{L}^{(1)} + \epsilon^2 \underline{L}^{(2)} + \dots ,$$

$$\underline{B} = \underline{B}^{(0)} + \epsilon \underline{B}^{(1)} + \dots , \quad \underline{T} = \underline{T}^{(0)} + \dots .$$

Equating equal powers of  $\epsilon$  in (A2), we obtain, correct to third order,

$$\underline{L}^{(0)} \underline{E}^{(1)} = 0 ,$$

$$\underline{L}^{(0)} \underline{E}^{(2)} = -4\pi \partial_t (\underline{B}^{(0)} \underline{E}^{(1)} \underline{E}^{(1)}) - \underline{L}^{(1)} \underline{E}^{(1)} \quad (A3)$$

$$\underline{L}^{(0)} \underline{E}^{(3)} = -4\pi \partial_t (\underline{B}^{(0)} \underline{E}^{(1)} \underline{E}^{(2)} + \underline{B}^{(0)} \underline{E}^{(2)} \underline{E}^{(1)} +$$

$$\underline{B}^{(1)} \underline{E}^{(1)} \underline{E}^{(1)} + \underline{T}^{(0)} \underline{E}^{(1)} \underline{E}^{(1)} \underline{E}^{(1)})$$

$$- \underline{L}^{(1)} \underline{E}^{(2)} - \underline{L}^{(2)} \underline{E}^{(1)} .$$

We now introduce a multiple-time-scale formalism (Frieman 1963; Sandri 1963, 1965) by writing  $\tau_0 = t$  to denote the fast oscillation time scale for the waves in  $\underline{E}^{(1)}$ , and  $\tau_1 = \epsilon t$ ,  $\tau_2 = \epsilon^2 t$ , etc., to denote successively slower interaction time scales. The freedom represented by the additional time variables is used to remove secularities order by order in the perturbation solution. We proceed by making the replacements (Verheest 1976)

$$f(t) = A(t) \exp(-i\omega t) \rightarrow f(\tau_0, \tau_1, \tau_2, \dots) = A(\tau_1, \tau_2, \dots) \exp(-i\omega \tau_0)$$

$$\partial_t \rightarrow \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1} + \epsilon^2 \frac{\partial}{\partial \tau_2} + \dots,$$

$$\underline{L}^{(0)}(\partial_t) \rightarrow \underline{L}^{(0)}\left(\frac{\partial}{\partial \tau_0}\right), \quad \underline{L}^{(1)}(\partial_t) \rightarrow \underline{M}^{(1)}\left(\frac{\partial}{\partial \tau_0}\right) \frac{\partial}{\partial \tau_1},$$

$$\underline{B}^{(0)}(\partial_t, \partial_t) \rightarrow \underline{B}^{(0)}\left(\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial \tau_0}\right).$$

For a set of  $N$  coherent waves in a uniform plasma the field  $\underline{E}^{(1)}$  takes the form

$$\underline{E}^{(1)} = \sum_{\substack{a=-N \\ a \neq 0}}^N \underline{E}_a(\tau_1, \tau_2, \dots) \exp(i\mathbf{k}_a \cdot \underline{x} - i\omega_a \tau_0). \quad (A5)$$

The reality condition

$$\underline{E}_{-a} = \underline{E}_a^*, \quad \mathbf{k}_{-a} = -\mathbf{k}_a, \quad \omega_{-a} = -\omega_a,$$

are implicit in (A5). Collecting our results (A3) to (A5) we obtain a chain of equations, the first two of which are

$$\sum_a \underline{L}_a(a) \underline{E}_a \exp(i\mathbf{k}_a \cdot \underline{x} - i\omega_a \tau_0) = 0, \quad (A6)$$

$$\begin{aligned} \underline{L}_a^{(0)} \underline{E}_a^{(2)} = & -4\pi \frac{\partial}{\partial \tau_0} \sum_b \sum_c \underline{B}(b,c) \underline{E}_b \underline{E}_c \exp\left[i(\mathbf{k}_b + \mathbf{k}_c) \cdot \underline{x} \right. \\ & \left. - i(\omega_b + \omega_c)\tau_0\right] - \sum_a \underline{M}(a) \frac{\partial \underline{E}_a}{\partial \tau_1} \exp\left[i(\mathbf{k}_a \cdot \underline{x} - \omega_a \tau_0)\right] \end{aligned} \quad (A7)$$

The operator  $\underline{L}(a)$  is defined by the relation

$$\begin{aligned} \underline{L}_a^{(0)}\left(\nabla, \frac{\partial}{\partial \tau_0}\right) \exp(i\mathbf{k}_a \cdot \underline{x} - i\omega_a \tau_0) \\ = \underline{L}_a^{(0)}(i\mathbf{k}_a, -i\omega_a) \exp(i\mathbf{k}_a \cdot \underline{x} - i\omega_a \tau_0) \\ \equiv \underline{L}(a) \exp(i\mathbf{k}_a \cdot \underline{x} - i\omega_a \tau_0), \end{aligned}$$

and the operators  $\underline{B}(b,c)$  and  $\underline{M}(a)$  are similarly related to  $\underline{B}^{(0)}$  and  $\underline{M}^{(1)}$ , respectively. Note that, from (A4),

$$\underline{M}(\underline{a}) = i \partial/\partial\omega_a \underline{L}(\underline{a}).$$

From (A6) and the linear independence of the exponential functions, it follows that

$$\underline{L}(\underline{a}) \underline{E}_a = 0, \quad \underline{a} = \pm 1, \pm 2, \dots, \pm N.$$

Thus, the waves in  $\underline{E}^{(1)}$  must separately satisfy the linear dispersion relation

$$\det \left[ \underline{L}(\underline{a}) \right] = 0.$$

Proceeding to (A7), we note that if secularities are to be avoided in  $\underline{E}^{(2)}$ , then either  $\partial \underline{E}_a / \partial \tau_1$  must vanish or else a balancing must occur between the two sums on the right-hand-side of (A7).

The second possibility requires matching conditions of the form

$$\omega_a = \omega_b + \omega_c, \quad \underline{k}_a = \underline{k}_b + \underline{k}_c,$$

and leads to the relation

$$\underline{M}(\underline{a}) \partial \underline{E}_a / \partial \tau_1 = \sum_{\underline{b}+\underline{c}=\underline{a}} 4\pi i (\omega_b + \omega_c) \underline{B}(\underline{b}, \underline{c}) \underline{E}_b \underline{E}_c \quad (A8)$$

Introducing the beat current density  $\underline{j}_{\underline{b}, \underline{c}}^{(2)}$  and the usual linear dispersion tensor  $\underline{D}(\underline{a})$  through the relations

$$\underline{j}_{\underline{b}, \underline{c}}^{(2)} = \underline{B}(\underline{b}, \underline{c}) \underline{E}_b \underline{E}_c,$$

$$\underline{D}(\underline{a}) = (-i\omega_a)^{-2} \underline{L}(\underline{a}),$$

we can rewrite (A8) in the form

$$\frac{\partial}{\partial \omega_a} \left[ \omega_a^2 \underline{D}(\underline{a}) \right] \cdot \frac{\partial \underline{E}_a}{\partial \tau_1} = -4\pi \omega_a \underline{j}_{\underline{b}, \underline{c}}^{(2)}. \quad (A9)$$

Now, the energy density in mode  $\underline{a}$  is

$$W_a = (4\pi)^{-1} \underline{E}_a^* \cdot \left\{ \omega_a^{-1} \partial/\partial \omega_a \left[ \omega_a^2 \underline{D}(\underline{a}) \right] \right\} \cdot \underline{E}_a.$$

Thus (A9) implies that

$$\partial W_a / \partial \tau_1 = -2 \operatorname{Re}(\underline{j}_{\underline{b}, \underline{c}}^{(2)} \cdot \underline{E}_a^*)$$

Let us express  $\underline{E}_a$  in terms of scalar and vector potentials  $\phi_a$  and  $\underline{A}_a$ ,

$$\underline{E}_a = -i\mathbf{k}_a \phi_a + i\omega_a c^{-1} \underline{A}_a,$$

and invoke the charge-continuity relation

$$i\mathbf{k}_a \cdot \underline{j}_{b,c}^{(2)} = i\omega_a \rho_{b,c}^{(2)}$$

where  $\rho_{b,c}^{(2)}$  denotes the beat charge density. We thereby obtain the promised action-evolution equation

$$\omega_a^{-1} \partial W_a / \partial \tau_1 = 2 \operatorname{Im}(\rho_{b,c}^{(2)} \phi_a^* - c^{-1} \underline{j}_{b,c}^{(2)} \cdot \underline{A}_a^*). \quad (\text{A10})$$

### Appendix B: Outline of some algebra for Section IV

In Section III, the kinetic three-wave coupling coefficient was decomposed into an oscillation-centre contribution and polarization contributions [see Eq. (23)]. In Section IV, these contributions were each evaluated; the results are given in (24), (25), and (26). The purpose of this appendix is to indicate how the sum of these contributions can be manipulated to yield the compact symmetric formula (27).

The first step after adding the quantities (24), (25) and (26) is to separate out the desired result (27). The final triple-product terms in (27) occur in consequence of the identity

$$\begin{aligned} & (\underline{k} \cdot \underline{A})(\underline{B} \cdot \underline{z} \times \underline{C}) + (\underline{k} \cdot \underline{B})(\underline{C} \cdot \underline{z} \times \underline{A}) + (\underline{k} \cdot \underline{C})(\underline{A} \cdot \underline{z} \times \underline{B}) \\ & = k_z \underline{C} \cdot (\underline{B} \times \underline{A}), \end{aligned}$$

which holds for arbitrary vectors  $\underline{k}$ ,  $\underline{A}$ ,  $\underline{B}$ , and  $\underline{C}$ . It remains, then, to show that the excess terms in the sum vanish identically. We can distinguish two categories of excess terms. The terms in the first group are those not explicitly proportional to the gyrofrequency  $\Omega$  and which persist in an unmagnetized plasma. Collecting these terms, we find

$$\begin{aligned}
 \text{Excess terms I} &= \sum_{\mathbf{s}} m^{-2} \int d^3 w f_0(w_{\mathbf{1}}, w_{\mathbf{z}}) \\
 &\times \left\{ 2^{-1} (D_a^* S_a^*) \left[ (i\mathbf{k}_b \cdot \underline{\partial} S_c)(i\mathbf{k}_c \cdot \underline{\partial} S_b) - (i\mathbf{k}_b \cdot \underline{\partial} S_b)(i\mathbf{k}_c \cdot \underline{\partial} S_c) \right] \right. \\
 &+ (i\mathbf{k}_b \cdot D_c \underline{\partial} S_c)(S_a^* i\mathbf{k}_a \cdot \underline{\partial} S_b) \\
 &\left. - (i\mathbf{k}_b \cdot \underline{\partial} S_b)(S_a^* i\mathbf{k}_a \cdot D_c \underline{\partial} S_c) + (b \leftrightarrow c) \right\}. \quad (B1)
 \end{aligned}$$

It is straightforward to show that the terms in (B1) add to zero; use the definition of the operators  $D_a, D_b, D_c$ , invoke the matching conditions (16), and integrate by parts on  $\psi$  (using the fact that  $f_0$  is independent of  $\psi$ ).

The remaining excess terms are those explicitly proportional to  $\Omega$ . They can be combined and written compactly in the form

$$\begin{aligned}
 \text{Excess terms II} &= \sum_{\mathbf{s}} \Omega (2m^2)^{-1} \int d^3 w f_0(w_{\mathbf{1}}, w_{\mathbf{z}}) \\
 &\times \left[ (i\mathbf{k}_a S_a^* - \Omega \hat{\mathbf{z}} \times \underline{\partial} S_a^*) \cdot \underline{M}_b \cdot (\hat{\mathbf{z}} \times \underline{\partial} S_c) + (b \leftrightarrow c) \right], \quad (B2)
 \end{aligned}$$

where we have defined the tensor

$$\begin{aligned}
 \underline{M}_b &\equiv i\mathbf{k}_b (\underline{\partial} S_b) + (\underline{\partial} S_b) i\mathbf{k}_b + \Omega \underline{\partial} (\hat{\mathbf{z}} \times \underline{\partial}) S_b \\
 &+ \Omega (\hat{\mathbf{z}} \times \underline{\partial}) \underline{\partial} S_b + D_b \underline{\partial} \underline{\partial} S_b.
 \end{aligned}$$

It is then simple to use the commutation identity (14) to prove that  $\underline{M}_b$  and  $\underline{M}_c$  are each identically zero, and hence that the remaining excess terms (B2) all vanish.

Appendix C: Internal consistency of the ponderomotive-scalar-potential method

In this appendix, we show that the ponderomotive-scalar-potential approximation in magnetized plasma (Section VI) preserves the Manley-Rowe action-transfer relations (Sturrock 1960). We verify this claim for the particular cases of three-wave interaction and induced scattering of two waves by resonant particles.

1. Three-wave interaction

Consider three waves  $a, b$ , and  $c$ , which satisfy the resonant matching conditions (16). We proceed from the coupled-mode equations (41), (44) and (45), which were derived in the ponderomotive-scalar-potential approximation (here, we let  $\underline{E}_d \rightarrow 0$ ). These equations can be represented schematically as

$$\underline{D}(k) \cdot \underline{E}_k = 4\pi(i\omega_k)^{-1} \underline{j}^{(2)}(k), \quad k = a, b, c.$$

Let us introduce the symbols  $\hat{\mathbf{e}}_k, D(k), D'(k)$ , defined by

$$\hat{\mathbf{e}}_k \equiv \underline{E}_k / E_k, \quad D(k) \equiv \hat{\mathbf{e}}_k^* \cdot \underline{D}(k) \cdot \hat{\mathbf{e}}_k,$$

$$D'(k) \equiv \hat{\mathbf{e}}_k^* \cdot \left[ \frac{\partial \underline{D}(k)}{\partial \omega_k} \right] \cdot \hat{\mathbf{e}}_k.$$



The inner product of the low-frequency equation (41) with  $\hat{e}_b^*$  then leads at once to

$$i D'(b) \dot{E}_b = -i \Gamma_b E_c^* E_a, \quad (C1)$$

where we have defined the coefficient

$$\Gamma_b = k_b (4\pi)^{-1} \sum_s (n_s e_s)^{-1} (\hat{e}_b^* \cdot \chi_{wb}^{(s)} \cdot \hat{e}_b) (\hat{e}_c^* \cdot \chi_{wo}^{(s)} \cdot \hat{e}_a). \quad (C2)$$

Since we are considering the case  $D(b) \rightarrow 0$ , we note from (41) that the self-consistent field  $\underline{E}_b$  dominates the ponderomotive fields  $\underline{E}_b^{(s)}$ . We are therefore justified in approximating  $\underline{E}_b^{(s)}$  [defined in (39)] by  $\underline{E}_b$  where it appears in the high-frequency currents. Thus, the inner products of (44) and (45) with  $\hat{e}_a^*$  and  $\hat{e}_c^*$ , respectively, lead to the relations

$$i D'(a) \dot{E}_a = -i \Gamma_a E_b E_c, \quad i D'(c) \dot{E}_c = -i \Gamma_c E_a E_b, \quad (C3)$$

where the complex coupling coefficients are

$$\Gamma_a = -k_b (4\pi)^{-1} \sum_s (n_s e_s)^{-1} (\hat{e}_b \cdot \chi_{wb}^{(s)} \cdot \hat{e}_b) (\hat{e}_a^* \cdot \chi_{wo}^{(s)} \cdot \hat{e}_c), \quad (C4)$$

$$\Gamma_c = k_b (4\pi)^{-1} \sum_s (n_s e_s)^{-1} (\hat{e}_b \cdot \chi_{wb}^{(s)*} \cdot \hat{e}_b^*) (\hat{e}_c^* \cdot \chi_{wo}^{(s)} \cdot \hat{e}_a). \quad (C5)$$

We now introduce the action density  $J_k$  of each wave, defined by

$$J_k = D'(k) |E_k|^2.$$

The equations of action transfer to be tested are

$$\dot{J}_a = -\dot{J}_b = -\dot{J}_c.$$

It follows from the structure of (C1) and (C3) that these relations are satisfied provided that

$$\Gamma_a = -\Gamma_b^* = -\Gamma_c^*.$$

Inspection of our results (C2), (C4) and (C5) verifies these conditions since, in the absence of resonant particles, the susceptibility tensor  $\chi_{wb}^{(s)}$  is Hermitian.

## 2. Induced scattering

This case differs from case 1 in that the low-frequency beat mode  $b$  is no longer a normal mode of the plasma. We follow Litvak and Trakhtengerts (1972) (who assume the Manley-Rowe relations) by treating mode  $b$  as longitudinal and by neglecting the high-frequency ion susceptibility. We therefore invoke the low-frequency equations (49) and (50), writing

$$E_b^{(e)} = \epsilon_b^{-1} (1 + \chi_b^{(i)}) \mathcal{E}_b^{(e)},$$

$$E_b^{(i)} = i\epsilon_b^{-1} \chi_b^{(e)} \mathcal{E}_b^{(e)},$$

where

$$\epsilon_b = \hat{k}_b \cdot D(b) \cdot \hat{k}_b$$

$$\mathcal{E}_b^{(e)} = -ik_b (4\pi n_e |e|)^{-1} (\hat{e}_c^* \cdot \chi_{\omega_0}^{(e)} \cdot \hat{e}_a) E_c^* E_a.$$

These relations are to be substituted into the high-frequency equations (44) and (45). We again neglect the ion susceptibility  $\chi_{\omega_0}^{(i)}$  and take inner products with  $\hat{e}_a^*$  and  $\hat{e}_c^*$ , respectively, to obtain

$$i D'(a) \dot{E}_a = -k_b^2 (4\pi n_e e)^{-2} |\hat{e}_c^* \cdot \chi_{\omega_0}^{(e)} \cdot \hat{e}_a|^2 \\ \times \epsilon_b^{-1} \chi_b^{(e)} (1 + \chi_b^{(i)}) |E_c|^2 E_a,$$

$$i D'(c) \dot{E}_c = -k_b^2 (4\pi n_e e)^{-2} |\hat{e}_c^* \cdot \chi_{\omega_0}^{(e)} \cdot \hat{e}_a|^2 \\ \times \epsilon_b^{-1*} \chi_b^{(e)*} (1 + \chi_b^{(i)*}) |E_a|^2 E_c.$$

The corresponding action-transfer relations are therefore

$$\dot{J}_c = -\dot{J}_a = 2 k_b^2 (4\pi n_e e)^{-2} |\hat{e}_c^* \cdot \chi_{\omega_0}^{(e)} \cdot \hat{e}_a|^2 \\ \times \text{Im} \left[ \epsilon_b^{-1} \chi_b^{(e)} (1 + \chi_b^{(i)}) \right] |E_a|^2 |E_c|^2, \quad (66)$$

and so the total action in waves  $a$  and  $c$  remains conserved by the induced-scattering process. Both resonant electrons and resonant ions contribute to (66).

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FIGURE CAPTIONS

Fig. 1: Matching-condition diagram for four-mode interaction.

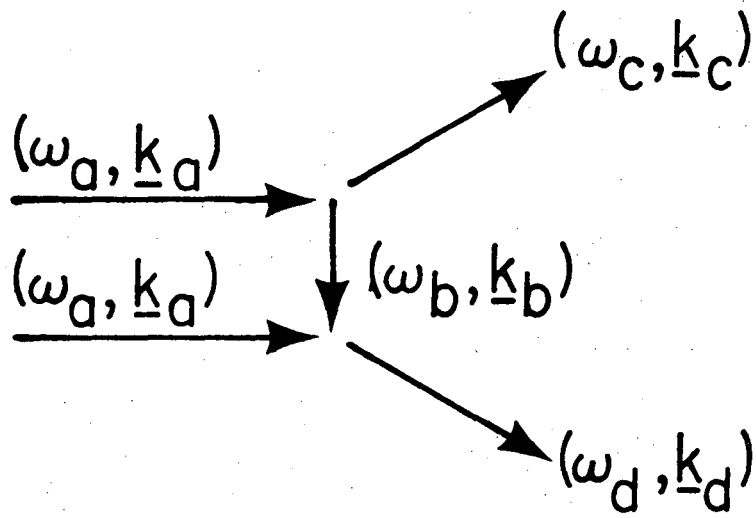


Fig. 1

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