Title

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Journal
Automatica, 50(10)

ISSN
0005-1098

Author
O'Reilly, Oliver M

Publication Date
2014-10-01

DOI
10.1016/j.automatica.2014.08.005

Peer reviewed

On the formulation of cost functions for torque-optimized control of rigid bodies ⋆

Oliver M. O'Reilly [a]

[a] *Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA 94720-1740. U.S.A.*

Abstract

In the context of controlling the attitude of a rigid body, this communique uses recent results on representations of torques (moments) to establish cost functions. The resulting cost functions are both properly invariant under whatever choice of Euler angles is used to parameterize the rotation of the rigid body and have transparent physical interpretations. The function is related to existing works in geometric control theory and applications of optimal control theory to biomechanical systems.

Key words: kinetic modeling and control of biological systems, model formulation, modeling for control optimization, rotations, torques.

1 Introduction

The recent work of [8] features a rigid body model for the human head and proposes interesting solutions to the challenging problem of controlling the motion of the head. Of particular interest is the computation of the generalized torque components $(\tau_{\phi_1}, \tau_{\phi_2}, \tau_{\phi_3})$ needed to move the head from one orientation to another while satisfying a constraint on the orientation known as Donder's constraint. The chosen cost function to minimize is (from [8, Eqn. (31)])

$$\mathcal{C} = \int_0^T \left[\tau^2_{\phi_1}(t) + \tau^2_{\phi_2}(t) + \tau^2_{\phi_3}(t)\right] dt. \tag{1}$$

Although the cost function appears at first glance to be the magnitude squared of the moment (torque) acting on the rigid body modeling the head, it is not. Motivated by this example, we seek to find a more appropriate cost function.

The avenue we use to find the cost function features recent works [4–6] on a (non-orthogonal) basis known as the dual Euler basis. The use of this basis in the representation of moments provides a clear correspondence between the components of a generalized torque and the moment vector acting on the rigid body. Alternative formulations of the cost function we propose can be found in a recent paper by Bloch et al. [1] and a work by Ghosh et al. [3] that appeared while the present paper was under review. With help from [7], we are able to provide additional physical insight into the cost function found in [1] and show its equivalence to the cost function being proposed.

2 Background

To facilitate comparisons with the literature, we use a set of 1-2-3 Euler angles (or Tait-Bryan angles) to parameterize the rotation from a fixed left-handed set of Cartesian basis vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ to a set of left-handed orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Using this parameterization, the angular velocity vector $\boldsymbol{\omega}$ has the representation [1]

$$\boldsymbol{\omega} = \dot{\phi}_1\mathbf{g}_1 + \dot{\phi}_2\mathbf{g}_2 + \dot{\phi}_3\mathbf{g}_3. \tag{2}$$

⋆ Corresponding author O. M. O'Reilly. Tel. +510-642-0877.
Email address: `oreilly@berkeley.edu` (Oliver M. O'Reilly).

[1] We are assuming that ϕ_k is a counterclockwise angle of rotation about $\mathbf{g}_k$ and follow the conventions in [5] in defining the sequence of rotations used to define the Euler angle parameterization.

Here, the Euler basis vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ have the representations

$$\begin{aligned}
\mathbf{g}_1 &= \mathbf{E}_1, \\
\mathbf{g}_2 &= \cos(\phi_1)\,\mathbf{E}_2 - \sin(\phi_1)\,\mathbf{E}_3, \\
\mathbf{g}_3 &= -\sin(\phi_2)\,\mathbf{E}_1 + \cos(\phi_2)\left(\sin(\phi_1)\,\mathbf{E}_2 + \cos(\phi_1)\,\mathbf{E}_3\right).
\end{aligned} \tag{3}$$

The dual Euler basis vectors $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$ can be computed using the nine identities $\mathbf{g}_i \cdot \mathbf{g}^k = 1$ if $i = k$ and is otherwise 0:

$$\begin{aligned}
\mathbf{g}^1 &= \mathbf{E}_1 + \tan(\phi_2)\left(\sin(\phi_1)\,\mathbf{E}_2 + \cos(\phi_1)\,\mathbf{E}_3\right), \\
\mathbf{g}^2 &= \cos(\phi_1)\,\mathbf{E}_2 - \sin(\phi_1)\,\mathbf{E}_3, \\
\mathbf{g}^3 &= \sec(\phi_2)\left(\sin(\phi_1)\,\mathbf{E}_2 + \cos(\phi_1)\,\mathbf{E}_3\right).
\end{aligned} \tag{4}$$

As is well known, the Euler basis fails to be a basis when $\phi_2 = \pm\frac{\pi}{2}$. At these singular values of ϕ_2, the dual Euler basis is not defined and we henceforth assume that $\phi_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

It can be shown that Lagrange's equations for the rotational motion of a rigid body are simply related to the balance of angular momentum for a rigid body:

$$\frac{d}{dt}\left(\frac{\partial T_{\text{rot}}}{\partial \dot{\phi}_k}\right) - \frac{\partial T_{\text{rot}}}{\partial \phi_k} = \mathbf{M} \cdot \mathbf{g}_k, \tag{5}$$

where $\mathbf{M}$ is the resultant moment relative to the center of mass acting on the body and T_{rot} is the rotational kinetic energy of the rigid body.

It is typical to refer to the components of $\mathbf{M}$ as the generalized torques:

$$\tau_{\phi_k} = \mathbf{M} \cdot \mathbf{g}_k. \tag{6}$$

Using the dual Euler basis, it is straightforward to show that

$$\mathbf{M} = \tau_{\phi_1}\mathbf{g}^1 + \tau_{\phi_2}\mathbf{g}^2 + \tau_{\phi_3}\mathbf{g}^3. \tag{7}$$

This representation has parallels to the representation of force vectors using a contravariant basis acting on a particle when a curvilinear coordinate system is used to parameterize the position vector of the particle.

3 An Alternative Cost Function

The cost function we propose is

$$\mathcal{V} = \int_0^T \|\mathbf{M}\|^2\, dt. \tag{8}$$

If a set of 1-2-3 Euler angles is used to parameterize the rotation of the rigid body, then it can be demonstrated from (7) with the help of (4) that $\|\mathbf{M}\|^2 = \mathbf{M} \cdot \mathbf{M}$ in (8) can be replaced by the explicit representation

$$\begin{aligned}
\|\mathbf{M}\|^2 = \tau_{\phi_1}^2 + \tau_{\phi_2}^2 + \tau_{\phi_3}^2 + \big\{&\tan^2(\phi_2)\left(\tau_{\phi_1}^2 + \tau_{\phi_3}^2\right) \\
&+ 2\sec(\phi_2)\tan(\phi_2)\,\tau_{\phi_1}\tau_{\phi_3}\big\}.
\end{aligned} \tag{9}$$

When $\phi_2 = \pm\frac{\pi}{2}$, $\|\mathbf{M}\|$ cannot be defined using a 1-2-3 set of Euler angles and an alternative set must be used to parameterize the rotation at these singular values. It should be clear from (9) that the integrand in the cost function (1) does not represent the magnitude of the moment $\mathbf{M}$.

In contrast to (1) the results produced by the cost function (8) have the attractive feature that they will be independent of the set of Euler angles used to parameterize the rotation (provided singular values in the second Euler angle are avoided).

4 A Geometric Treatment

In a recent work Bloch et al. [1] examined the optimal control of the rotations of a rigid body which minimized the cost function

$$J = \frac{1}{2}\int_0^T \langle\langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle\rangle_*\, dt, \tag{10}$$

where $\boldsymbol{\tau} \in \mathfrak{so}^*(3)$ is a control torque. It is of interest to see how J and $\mathcal{V}$ are equivalent even though the formulations of these functions appear at first glance to be very different. To establish the forthcoming results, we make extensive use of results from Žefran and Kumar [7] and follow the notation of [1].

When a set of Euler angles is used to parameterize the group of rotations $SO(3)$, then the angular velocity $\omega \in \mathfrak{so}(3)$ has the representation

$$\omega = \dot{\phi}_1\frac{\partial}{\partial \phi_1} + \dot{\phi}_2\frac{\partial}{\partial \phi_2} + \dot{\phi}_3\frac{\partial}{\partial \phi_3}, \tag{11}$$

where $\frac{\partial}{\partial \phi_k}$ are basis vectors for $\mathfrak{so}(3)$. In addition, the torque $\boldsymbol{\tau}$ has the representation

$$\boldsymbol{\tau} = \tau_{\phi_1} d\phi^1 + \tau_{\phi_2} d\phi^2 + \tau_{\phi_3} d\phi^3, \tag{12}$$

where $d\phi_k$ are one-forms which form a basis for $\mathfrak{so}^*(3)$. With evident parallels to the Euler and dual Euler basis vectors, the inner-product of $d\phi_k$ and $\frac{\partial}{\partial \phi_j}$ is the Kronecker delta: $\left\langle d\phi_k, \frac{\partial}{\partial \phi_j}\right\rangle = \delta_j^k$.

We define the fixed basis $\left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_1}\right\}$ for $\mathfrak{so}(3)$ and its dual $\left\{dX^1, dX^2, dX^3\right\}$ for $\mathfrak{so}^*(3)$. Then, for a 1-2-3 set of Euler angles we have the following representations:

$$
\begin{aligned}
\frac{\partial}{\partial \phi_1} &= \frac{\partial}{\partial X_1}, \\
\frac{\partial}{\partial \phi_2} &= \cos(\phi_1)\frac{\partial}{\partial X_2} - \sin(\phi_1)\frac{\partial}{\partial X_3}, \\
\frac{\partial}{\partial \phi_3} &= -\sin(\phi_2)\frac{\partial}{\partial X_1} + \cos(\phi_2)\sin(\phi_1)\frac{\partial}{\partial X_2} \\
&\quad + \cos(\phi_2)\cos(\phi_1)\frac{\partial}{\partial X_3},
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
d\phi^1 &= dX^1 + \tan(\phi_2)\left(\sin(\phi_1)\,dX^2 + \cos(\phi_1)\,dX^3\right), \\
d\phi^2 &= \cos(\phi_1)\,dX^2 - \sin(\phi_1)\,dX^3, \\
d\phi^3 &= \sec(\phi_2)\left(\sin(\phi_1)\,dX^2 + \cos(\phi_1)\,dX^3\right).
\end{aligned}
\tag{14}
$$

With the help of (9), (12), (14), and the identity $\left\langle\left\langle dX^j, dX^k\right\rangle\right\rangle_* = \delta_k^j$ it is now straightforward to establish a one-to-one correspondence between $\boldsymbol{\tau}$ and $\mathbf{M}$ and to show that

$$
\|\mathbf{M}\|^2 = \langle\langle \boldsymbol{\tau}, \boldsymbol{\tau}\rangle\rangle_* . \tag{15}
$$

The equivalence (15) also enables a straightforward physical interpretation of the components of $\boldsymbol{\tau}$. For completeness we note that the mechanical power of $\mathbf{M}$ has the representations

$$
\mathbf{M}\cdot\boldsymbol{\omega} = \langle \boldsymbol{\tau}, \omega\rangle = \tau_{\phi_1}\dot{\phi}_1 + \tau_{\phi_2}\dot{\phi}_2 + \tau_{\phi_3}\dot{\phi}_3. \tag{16}
$$

5 Movements of the Human Head, Donder's Law, Clinical Moments

In [8] the motion of the head is assumed to be subject to a rotational constraint known as Donder's law. This constraint is expressed in the form (from [8, Eqn. (10)]) $\Delta = 0$, where the function

$$
\Delta(\phi_1, \phi_2, \phi_3) = t\tan^2\left(\frac{\phi_3}{2}\right) + s\tan\left(\frac{\phi_3}{2}\right) + r, \tag{17}
$$

and t, r and s are functions of ϕ_1 and ϕ_2. Taking the derivative of this constraint with respect to time, we find that (17) implies that

$$
\boldsymbol{\omega}\cdot\mathbf{d} = 0 \text{ where } \mathbf{d} = \sum_{k=1}^{3}\frac{\partial\Delta}{\partial\phi_k}\mathbf{g}^k. \tag{18}
$$

The constraint moment $\mathbf{M}_c$ acting on the head which enforces Donder's constraint has the representation

$$
\mathbf{M}_c = \mu\mathbf{d}, \tag{19}
$$

where μ is a Lagrange multiplier. Consequently, Lagrange's equations of motion (5) are replaced by

$$
\frac{d}{dt}\left(\frac{\partial T_{\mathrm{rot}}}{\partial\dot{\phi}_k}\right) - \frac{\partial T_{\mathrm{rot}}}{\partial\phi_k} = \mathbf{M}_o\cdot\mathbf{g}_k - \mu\frac{\partial\Delta}{\partial\phi_k}, \tag{20}
$$

where $\mathbf{M}_o = \tau_{\phi_1}\mathbf{g}^1 + \tau_{\phi_2}\mathbf{g}^2 + \tau_{\phi_3}\mathbf{g}^3$ is now the optimal control moment vector and the resultant moment $\mathbf{M} = \mathbf{M}_c + \mathbf{M}_o$ The equations (20) should be supplemented by (17) and (18).[2] The appropriate cost function for this problem is to replace $\mathbf{M}$ with $\mathbf{M}_o$ in the cost function (8):

$$
\mathcal{V} = \int_0^\tau \|\mathbf{M}_o\|^2\,dt. \tag{21}
$$

An additional benefit of the representations (7) and (19) is that they can be used after the optimal control problem has been solved to give clear clinical interpretations of the control $\mathbf{M}_o$ and constraint moment $\mathbf{M}_c$. Such interpretations are established by relating $\mathbf{M} = \mathbf{M}_o + \mathbf{M}_c$ to forces provided by the various muscle groups acting on the head.

6 Closing Remarks

It would be interesting to see how the optimal torques for a cost function $\int_0^\tau \|\mathbf{M}_o\|^2\,dt$ compare to those computed by the authors of [8] using (1). Indeed some recent results on this comparison can be found in Ghosh et al. [3] which appeared while the present work was in review. By using the material presented in Section 5 on $\mathbf{M}$, the authors of [8] could also determine if $\mu = 0$ for motions which satisfy Donder's law. Such a result would parallel a closely related result in a symmetric rigid body model of the human eye subject to Listing's law that can be found in [2, Appendix IV].

We close by emphasizing the importance of the correspondence between the components of $\mathbf{M}$ and $\boldsymbol{\tau}$ that was noted in Section 4 (cf. (7) and (12)):

$$
\frac{\partial}{\partial\phi_k} \leftrightarrow \mathbf{g}_k, \quad d\phi^i \leftrightarrow \mathbf{g}^i, \quad \omega \leftrightarrow \boldsymbol{\omega}, \quad \boldsymbol{\tau} \leftrightarrow \mathbf{M}. \tag{22}
$$

These results enables a straightforward physical interpretation of the components of $\boldsymbol{\tau}$ and shows how the geometric treatment proposed in [1] is relevant to other works such as [2,8].

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