Title
STATISTICAL CALCULATION OF COMPLETE EVENTS IN MEDIUM-ENERGY NUCLEAR COLLISIONS

Author
Randrup, J.

Publication Date
1983-04-01

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Presented at the Nordic Winter School on Nuclear Physics, Hemsedal, Norway, April 10-20, 1983

STATISTICAL CALCULATION OF COMPLETE EVENTS IN MEDIUM-ENERGY NUCLEAR COLLISIONS

J. Randrup

April 1983

NUCLEAR SCIENCE DIVISION

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

LBL-15877

Statistical Calculation of Complete Events in Medium-Energy Nuclear Collisions*

Jørgen Randrup

Nuclear Science Division
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

Notes for a lecture given at the Nordic Winter School on Nuclear Physics, Hemsedal, Norway, April 10-20, 1983

This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

*The work reported in this lecture was done in collaboration with George Fái, Roland Eötvös University, Budapest, Hungary.

Statistical Calculation of Complete Events in Medium-Energy Nuclear Collisions*

Jørgen Randrup
Nuclear Science Division
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

1. Introduction

Several heavy-ion accelerators throughout the world are presently able to deliver beams of heavy nuclei with kinetic energies in the range from tens to hundreds of MeV per nucleon, the so-called medium or intermediate energy range. At such energies a large number of final channels are open, each consisting of many nuclear fragments. The disassembly of the collision system is expected to be a very complicated process and a detailed dynamical description is beyond our present capability. However, by virtue of the complexity of the process, statistical considerations may be useful. A statistical description of the disassembly yields the least biased expectations about the outcome of a collision process and provides a meaningful reference against which more specific dynamical models, as well as the data, can be discussed.

When the interest is focused on inclusive observables the statistical description can be based on the grand canonical approximation.[1-5] However, it is now possible to detect electronically practically all charged fragments emerging from a collision and thus good-quality, nearly exclusive data can be obtained. [One such detection system is the Plastic Ball complex located at the Bevalac in Berkeley.[6]] Therefore theory must also address exclusive quantities. The proper basis is then the microcanonical approximation where the conservation laws are obeyed event by event (and not just on the average as in the grand canonical approximation).

This lecture presents the essential tools for formulating a statistical model for the nuclear disassembly process. We consider the quick disassembly ("explosion") of a hot nuclear system, a so-called source, into multifragment final states, which compete according to their statistical weight. First some useful notation is introduced. Then the expressions for exclusive and inclusive distributions are given and the factorization of an exclusive distribution into inclusive ones is carried out. In turn, the grand canonical approximation for one-fragment inclusive distributions is introduced. Finally, it is outlined how to generate a statistical sample of complete final states.[7] On this basis, a model for statistical simulation of complete events in medium-energy nuclear collisions has been developed.[8]

2. The event set

An ideal exclusive measurement yields complete information on all fragments in the final state. An event f is then characterized by the multiplicity n_α of the various fragment species α:n,p,d,t,... together with their four-momenta:

$$f = \left\{ P_\alpha^{\ell_\alpha} \, , \, \ell_\alpha \in (1,n_\alpha) \right\}_\alpha \tag{1}$$

The n_α fragments of the species α are arbitrarily labeled by $\ell_\alpha \in (1,n_\alpha)$ and their four-momenta are denoted by $P_\alpha^{\ell_\alpha} = (\vec{p}_\alpha^{\,\ell_\alpha}, E_\alpha^{\ell_\alpha})$. Because of the identity of the fragments within a given species the actual labeling is without significance, i.e., f is invariant under arbitrary permutations of the labels ℓ_α.

The total multiplicity of the event f is given by $n_f = \sum_\alpha n_\alpha$. Furthermore, its total baryon number, charge, and four-momentum are, respectively,

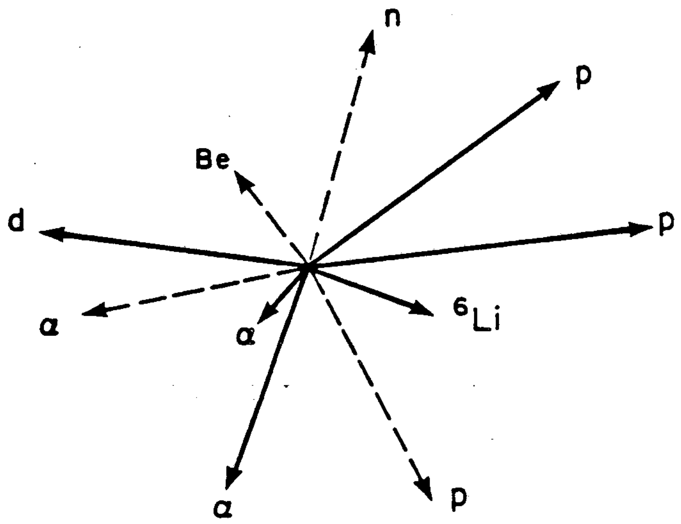
$$A_f = \sum_\alpha n_\alpha A_\alpha$$
$$Z_f = \sum_\alpha n_\alpha Z_\alpha \tag{2}$$
$$P_f = \sum_\alpha \sum_{\ell_\alpha=1}^{n_\alpha} P_\alpha^{\ell_\alpha}$$

where A_α and Z_α denote the baryon number and charge characterizing the fragment species α. It is often convenient to use the brief notation $i[f]$ for $\{A_f Z_f P_f\}$ collectively. Rather than the charge Z, it is often preferable to use the isospin projection $T = A/2 - Z$.

The set of all such events, the <u>event set</u> $\mathfrak{F} = \{f\}$, has certain notationally convenient algebraic properties. Most importantly, it is possible to define an <u>addition</u> in $\mathfrak{F}$. Thus, for any two events $\tilde{f},\bar{f} \in \mathfrak{F}$ we have $f = \tilde{f} + \bar{f}$ iff $n_\alpha = \tilde{n}_\alpha + \bar{n}_\alpha\ \forall\alpha$ and $\left\{P_\alpha^{\ell_\alpha}, \ell_\alpha \in (1,n_\alpha)\right\} = \left\{\tilde{P}_\alpha^{\ell_\alpha}, \ell_\alpha \in (1,\tilde{n}_\alpha), \bar{P}_\alpha^{\ell_\alpha}, \ell_\alpha \in (1,\bar{n}_\alpha)\right\}$, i.e., the sum event f is obtained by simply extending the event $\tilde{f}$ by the event $\bar{f}$, as shown in the figure. The event set $\mathfrak{F}$ is an abelian semi-group with respect to addition and the <u>null event</u>, $f = 0$, which has $n_\alpha = 0\ \forall\alpha$, is the neutral element. When $f = \tilde{f} + \bar{f}$ we may also write $\bar{f} = f - \tilde{f}$, which is often convenient. We note that the function $i[f]$ defined above is additive: $i[\tilde{f} + \bar{f}] = i[\tilde{f}] + i[\bar{f}]$.

It is also possible to introduce a partial <u>ordering</u> in the event set $\mathfrak{F}$. We shall write $\tilde{f} \leq f$ (or equivalently, $f \geq \tilde{f}$), iff $\exists\ \bar{f} \in \mathfrak{F}$: $f = \tilde{f} + \bar{f}$. In words: f encompasses $\tilde{f}$ iff $\tilde{f}$ can be extended to f. Obviously, this order relation is reflexive ($f \leq f\ \forall f$), transitive ($f \leq f' \wedge f' \leq f'' \Rightarrow f \leq f''$) and anti-symmetric ($f \leq f' \wedge f' \leq f \Rightarrow f = f'$), as it should be.

It is easy to see that, with respect to the two binary operations intersection, $\cap$, and union, $\cup$, acting on the sets (1) characterizing the events, $\mathfrak{F}$ has the properties of a complete lattice, i.e., any non-empty subset $\{f,f',\ldots\}$ of $\mathfrak{F}$ has a least upper bound and a greatest lower bound. These are given by



——— $$\tilde{f} = \{\tilde{p}^1_p, \tilde{p}^2_p, \tilde{p}^1_d, \tilde{p}^1_\alpha, \tilde{p}^2_\alpha, \tilde{p}^1_{^6Li}\}$$

- - - - $$\bar{f} = \{\bar{p}^1_n, \bar{p}^1_p, \bar{p}^1_\alpha, \bar{p}^1_{^9Be}\}$$

$$f = \tilde{f} + \bar{f} = \{p^1_n, p^1_p, p^2_p, p^3_p, p^1_d, p^1_\alpha, p^2_\alpha, p^3_\alpha, p^1_{^6Li}, p^1_{^9Be}\}$$

Figure caption

Illustration of event addition: The events $\tilde{f}$ (containing six fragments) and $\bar{f}$ (containing four fragments) are added to form the event f (containing all ten fragments).

$$\sup\{f,f',\ldots\} = f \cup f' \cup \ldots$$
$$\inf\{f,f',\ldots\} = f \cap f' \cap \ldots \qquad (3)$$

respectively.

It is important to note that the event set has the partition $\mathfrak{F} = \bigcup_{i_0} \mathfrak{F}_{i_0}$, where the disjoint subsets $\mathfrak{F}_{i_0}$ are defined by

$$\mathfrak{F}_{i_0} = \{f \mid i[f] = i_0\} \qquad (4)$$

For given initial conditions, characterized by the quantities $\{A_0 Z_0 P_0\} = i_0$, only events $f \in \mathfrak{F}_{i_0}$ are physically accessible, due to the conservation of baryon number, charge, and four-momentum.

Events with unit total multiplicity are elementary objects in the event set. Any event f can be decomposed in terms of <u>elementary events</u>:

$$\forall f \ \exists \{f_k\} \ : \ f = \sum_{k=1}^{n_f} f_k \quad , \quad n_{f_k} = 1 \quad \forall k \qquad (5)$$

This decomposition is unique (apart from permutations of the labels k).

In the discussion above, an event f is defined in terms of the four-momenta of the fragments immediately after the explosion (see eq. (1)). It is important to recognize that the specification of such an f actually characterizes an entire class of different final states, f = {F}, each final state being of the form

$$F = \left\{ Q_\alpha^{\ell_\alpha} \, P_\alpha^{\ell_\alpha} \quad , \quad \ell_\alpha \in (1, n_\alpha) \right\}_\alpha \qquad (6)$$

where the Lorentz vector $Q_\alpha^{\ell_\alpha}$ denotes the position of the fragment in space-time shortly after the explosion. In our statistical model it is assumed that the disassembly occurs at a definite time within a certain characteristic volume. The space-time information is then given by the spatial positions $\vec{R}_\alpha^{\ell_\alpha}$ of the fragments at the disassembly time. Since any further interaction between the fragments after the explosion is neglected, all final states F differing only in the spatial configuration of their fragments at the time of disassembly emerge with the same set of four-momenta and thus belong to the same event f. In the statistical model all such final states are equally probable*, and the appropriate measure on the event set $\mathfrak{F}$ can therefore be obtained by properly enumerating the different spatial fragment configurations in a given event f.

This task is generally complicated and we resort to the approximation introduced in ref. [7]: the integration over a given fragment's position is approximated by an effective volume:

$$\int d\vec{R}_\alpha^{\ \ell_\alpha} \rightarrow X\Omega_0 \qquad (7)$$

Here Ω_0 is a suitable reference volume, usually equal to A_0/ρ_0 where A_0 is the number of baryons in the source and $\rho_0 \approx 0.17\ \mathrm{fm}^{-3}$ is the standard nuclear matter density. The model parameter X, which is of the order of unity, controls the average effectively available volume and can be related approximately to the "break-up" density as discussed in ref. [8]. Thus it follows that the sum over final states F can be reduced to a sum over events (classes of final states) f:

$$\sum_F \rightarrow \sum_f (X\Omega_0)^{n_f} . \qquad (8)$$

This defines the proper measure on $\mathfrak{F}$.

It follows from the above discussion that an event f may also be thought of as specifying a macrostate of the system, characterized by the quantities $i[f] = \{A_f Z_f P_f\}$, while F enumerates the corresponding microstates.

3. Exclusive and inclusive distributions

Consider now a disassembling source characterized by the quantities $\{A_0 Z_0 P_0\} = i_0$. In the statistical model all final states compatible with the conservation of these quantities are equally probable (ignoring the additional constraints associated with the center-of-mass position and the total angular momentum.*
Therefore, the relative probability that the system

*In the present treatment conservation of the overall center-of-mass position and the total angular momentum is neglected, since these effects are expected to be rather small and their inclusion would complicate the treatment disproportionately.

disassembles into a final state belonging to a specified event f is given by

$$p(i_0|f) = (\chi\Omega_0)^{n_f}\ \delta(i_0 - i[f])/\mathfrak{I}(i_0) \tag{9}$$

As discussed above, the volume factor expresses the statistical weight of the different spatial configurations of the n_f fragments in f. The normalization constant is determined from the requirement that p be normalized,

$$\mathfrak{I}(i_0) = \sum_f (\chi\Omega_0)^{n_f}\ \delta(i_0 - i[f]) \tag{10}$$

This quantity is often referred to as the <u>phase-space integral</u>.

The distribution $p(i_0|f)$ pertains to the ideal situation where the specification of the event f is complete, corresponding to an exclusive measurement. Hence p is referred to as the <u>exclusive</u> distribution. When only partial specification of the event is made, as is most often the case in practice, the relevant quantity is the corresponding <u>inclusive</u> distribution P. This distribution can be obtained from the exclusive distribution by integrating over the unspecified quantities.

In particular, when the partial specification is such that complete information is given for some of the fragments and none for the rest, the inclusive distribution is given by

$$P(i_0|\tilde{f}) = \sum_{f\geq\tilde{f}} p(i_0|f) = \sum_{\bar{f}} p(i_0|\tilde{f} + \bar{f}) \tag{11}$$

Here the event $\tilde{f}$ characterizes the partial specification and the sum is over all events f that encompass $\tilde{f}$. [For example, if only one fragment is specified we have $\tilde{f} = f_1$, where f_1 is an elementary (i.e. one-fragment) event and the sum is over all f whose decomposition (5) into elementary events contain f_1 as a term.] The second equation in (11) follows by employing $\bar{f} = f - \tilde{f}$ as the independent variable.

It is possible to express inclusive distributions in terms of the phase-space integrals (10):

$$
\begin{aligned}
P(i_o|\tilde{f}) &= \sum_{\bar{f}} p(i_o|\tilde{f} + \bar{f}) \\
&= \sum_{\bar{f}} (\chi\Omega_o)^{n_{\tilde{f}}+n_{\bar{f}}} \delta(i_o - i[\tilde{f}] - i[\bar{f}])/\mathfrak{I}(i_o) \qquad (12) \\
&= (\chi\Omega_o)^{n_{\tilde{f}}} \mathfrak{I}(i_o - i[\tilde{f}])/\mathfrak{I}(i_o)
\end{aligned}
$$

i.e. the reduced inclusive probability $P(i_o|\tilde{f})/(\chi\,\Omega_o)^{n_{\tilde{f}}}$ for obtaining the partial event $\tilde{f}$ is equal to the complementary phase-space integral $\mathfrak{I}(i_o - i[\tilde{f}])$ divided by the total phase space integral $\mathfrak{I}(i_o)$, as one would intuitively expect.

By combination of (9) and (12) it is possible to factorize the exclusive distribution $p(i_o|f)$ into simpler quantities. Thus, for any decomposition $f = \tilde{f} + \bar{f}$, we have

$$
\begin{aligned}
p(i_o|f) &= (\chi\Omega_o)^{n_f} \delta(i_o - i[f])/\mathfrak{I}(i_o) \\
&= (\chi\Omega_o)^{n_{\tilde{f}}+n_{\bar{f}}} \delta(i_o - i[\tilde{f}] - i[\bar{f}])/\mathfrak{I}(i_o) \qquad (13) \\
&= (\chi\Omega_o)^{n_{\tilde{f}}} \mathfrak{I}(i_o - i[\tilde{f}])/\mathfrak{I}(i_o) \\
&\quad (\chi\Omega_o)^{n_{\bar{f}}} \delta(i_o - i[\tilde{f}] - i[\bar{f}])/\mathfrak{I}(i_o - i[\tilde{f}]) \\
&= P(\tilde{f}|i_o)\ p(i_o - i[\tilde{f}]|\bar{f})
\end{aligned}
$$

This relation expresses the fact that the exclusive probability for obtaining the event f is equal to the inclusive probability for obtaining some specified part $\tilde{f} \leq \bar{f}$ of the event f times the exclusive probability for also obtaining the remaining part of the event $\bar{f} = \tilde{f} - f$, given that $\tilde{f}$ has already been obtained.

By repeated use of the above relation (13), it is possible to factorize the exclusive distribution p into inclusive quantities. In particular, by decomposing the specified multi-fragment event f in terms of elementary events f_k, $p(i_o|f)$ can be factorized into one-fragment inclusive distributions. Thus, for $f = \Sigma f_k$,

$$p(i_o|f = \sum_{k=1}^{n_f} f_k) = P(i_o|f_1)\ p(i_1|\sum_{k=2}^{n_f} f_k) = \ldots$$

$$= \prod_{k=1}^{n_f} P(i_{k-1}|f_k) \quad p(i_{n_f}|0) \tag{14}$$

Here we have defined $i_k \equiv i_{k-1} - i[f_k]$ for $k \in (1,n_f)$. The exclusive factor $p(i_{n_f}|0)$ vanishes unless the quantities specified by i_{n_f} all vanish, thus guaranteeing that the event f is in fact accessible by the disassembling system characterized by i_o.

4. The grand canonical approximation

Consider a system whose macrostate is specified by the (conserved) quantities $i_o = A_oZ_oP_oE_o$. The associated phase space integral is the number of microstates F having $i[F] = i_o$,

$$\mathfrak{I}(i_o) = \sum_F \delta(i[F] - i_o) = \sum_f (\chi\Omega_o)^{n_f}\ \delta(i[f] - i_o) \tag{15}$$

The phase space integrals for systems whose macrostates differ only relatively slightly from the above one can be obtained by a local expansion of $\mathfrak{I}(i_o)$. Thus, for $i \approx i_o$ we have

$$\mathfrak{I}(i) \approx \mathfrak{I}(i_o)\ e^{\lambda(i_o)\cdot(i-i_o)} \tag{16}$$

where

$$\lambda(i) \equiv \frac{\partial \ln \mathfrak{I}(i)}{\partial i} \tag{17}$$

is the multidimensional derivative of the phase space integral with respect to the conserved quantities.

It is therefore possible to obtain a simple approximate expression for the inclusive distribution $P(\tilde{f}|i_o)$ when the specified partial event $\tilde{f}$ forms a small part of the system, i.e., when $i[\tilde{f}] << i_o$. Indeed, we may write

$$P(\tilde{f}|i_o) = (\chi\Omega_o)^{n_{\tilde{f}}}\mathfrak{I}(i_o - i[\tilde{f}])/\mathfrak{I}(i_o) \approx (\chi\Omega_o)^{n_{\tilde{f}}}\ e^{-\lambda(i_o)\cdot i[\tilde{f}]} \tag{18}$$

This result is recognized as the grand canonical approximation.

In order to determine the quantity λ, use can be made of the identity

$$i_o = \sum_{f_1} i[f]\, p(f|i_o) = \sum_{f_1} \sum_{f \geq f_1} i[f_1] p(f|i_o) = \sum_{f_1} i[f_1] P(f_1|i_o) \quad (19)$$

where f_1 is an elementary (i.e. one-fragment) event. Insertion of (18) for the inclusive distribution on the right-hand side then yields an equation for λ involving only elementary events.

$$i_o = \sum_{f_1} i[f_1]\, X\Omega_o\, e^{-\lambda(i_o)\cdot i[f_1]} \quad (20)$$

Due to Lorentz invariance the phase space integral of the system depends only on the total four-momentum through the combination $E_f^2 - \vec{P}_f^2 c^2 = (M_f c^2)^2$. Consequently, the Lagrange multipliers for energy and momentum are related:

$$\lambda_E = \frac{\partial \ln \mathfrak{I}}{\partial E} = \frac{\partial(Mc^2)}{\partial E} \frac{\partial \ln \mathfrak{I}}{\partial(Mc^2)} = \frac{E}{Mc^2}\, \beta$$

$$\vec{\lambda}_P = \frac{\partial \ln \mathfrak{I}}{\partial \vec{P}} = \frac{\partial(Mc^2)}{\partial \vec{P}} \frac{\partial \ln \mathfrak{I}}{\partial(Mc^2)} = -\frac{\vec{P}}{M}\, \beta$$

where $\beta \equiv \partial \ln \mathfrak{I}/\partial(Mc^2)$, and $\mathfrak{I}(Mc^2)$ is the proper level density of the system, i.e., the level density evaluated in the rest frame. We note that $\lambda_E^2 - \vec{\lambda}_P^2/c^2 = \beta^2$.

The inclusive probability for a fragment with energy e_1 and momentum $\vec{p}_1$ is then

$$P(f_1|i_o) = X\Omega_o\, e^{-\lambda_E e_1 - \vec{\lambda}_P \cdot \vec{p}_1 - \lambda_A A_1 - \lambda_T T_1}$$

$$= X\Omega_o\, e^{-\beta(E_o e_1 - \vec{P}_o c \cdot \vec{p}_1 c)/Mc^2 - \lambda_A A_1 - \lambda_T T_1}$$

$$= X\Omega_o\, e^{-\beta\, (E_1 - \mu A_1 - \nu T_1)}$$

Here we have used the fact that the invariant quantity $(E_o e_1 - \vec{P}_o c \cdot \vec{p}_1 c)/Mc^2$ is equal to the fragment energy E_1 as seen in the CM frame, where $\vec{P}_o$ vanishes. We have also introduced the chemical potentials $\mu = -\lambda_A \tau$ and $\nu = -\lambda_T \tau$ where $\tau = 1/\beta$ is the temperature.

The four constraint equations (20) for the total four-momentum

$$E_o = \sum_{f_1} e_1 \, P(f_1|i_o)$$

$$\vec{P}_o = \sum_{f_1} \vec{p}_1 \, P(f_1|i_o)$$

can be combined to a single equation for β:

$$\begin{aligned} M_o c^2 &= (E_o^2 - \vec{P}_o^2 c^2)/M_o c^2 \\ &= \frac{1}{M_o c^2} \sum_{f_1} (E_o e_1 - \vec{P}_o c \cdot \vec{p}_1 c) \, P(i_o|f_1) \\ &= \sum_{f_1} E_1 \, X\Omega_o \, e^{-\beta(E_1 - \mu A_1 - \nu T_1)} \end{aligned} \tag{21}$$

where $(M_o c^2)^2 = E_o^2 - \vec{P}_o^2 c^2$. Thus, Lorentz invariance reduces the number of Lagrange multipliers associated with the four-momentum to one single multiplier β associated with the rest energy of the system, and the standard form of the grand canonical approximation holds when the four-momenta are referred to the rest frame.

The remaining two constraint equations (20) referring to the total baryon number and charge (or isospin projection rather) read

$$\begin{aligned} A_o &= \sum_{f_1} A_1 \, X\Omega_o \, e^{-\beta(E_1 - \mu A_1 - \nu T_1)} \\ T_o &= \sum_{f_1} T_1 \, X\Omega_o \, e^{-\beta(E_1 - \mu A_1 - \nu T_1)} \end{aligned} \tag{22}$$

For given initial conditions, as specified by $\{M_o A_o T_o\}$, the three equations (21) and (22) are to be solved for the quantities

β,μ,ν. Then eq. (18) yields the inclusive probability for a single fragment $\{E_1A_1T_1\}$,

$$P(i_o|f_1) = \chi\Omega_o\ e^{-\beta(E_1-\mu A_1-\nu T_1)} \tag{23}$$

5. Statistical event generation

The factorization (14) of the exclusive multi-fragment distribution p into inclusive one-fragment distributions is particularly convenient when one seeks to generate a statistical representation of p, i.e., a sample {f} of multi-fragment events that are statistically distributed in the event subset $\mathfrak{F}_{i_o}$ according to the probability density $p(f|i_o)$. To accomplish this task, one may proceed as follows.

Each event f is considered as a sum of elementary one-fragment events, $f = \Sigma f_k$. To generate an event f, one first makes a random selection of the term f_1 on the basis of the inclusive probability distribution $P(f_1|i_o)$. Once f_1 has been selected, the remaining part of the event is known to be characterized by the quantities $i_1 = i_o - i[f_1]$. The next term f_2 is subsequently selected on the basis of $P(f_2|i_1)$, and the further reduced residual event can be characterized. This procedure is repeated until no residual system remains. [That this is guaranteed to happen at some point follows from the fact that only actually accessible events are constructed by this procedure: In eq. (11) the inclusive probability P is nonvanishing only if in fact the specified event $\tilde{f}$ is part of an actually accessible event f, and therefore $\tilde{f}$ has a counterpart $\bar{f}$ such that $\tilde{f} + \bar{f} = f$.] In this way a single event f is constructed. By employing the procedure repeatedly, a statistical sample $\{f\} \in \mathfrak{F}_{i_o}$ of desired size can be generated.

The procedure outlined above is a mathematically valid way of generating a representative sample of the exact many-fragment distribution $p(f|i_o)$. However, the method requires the exact one-fragment inclusive distributions, which are cumbersome to calculate, particularly when several excitable fragment species are included.

Some degree of approximation is therefore necessary. Fortunately, one-fragment distributions, which are the only ones required in the procedure, are much easier to approximate than more exclusive quantities. It is therefore possible to turn the mathematical procedure into a practical method. The key is to employ the grand canonical approximation separately for each of the inclusive factors in (14). The grand canonical approximation is accurate for one-fragment distributions as long as the fragment considered is only a small part of the system (cf. section 4). This condition is reasonably well fulfilled for most of the factors in the product (14), although it is substantially violated for the last few factors. References [7,8] describe how this idea is implemented and discusses the quantitative validity of the approximation, in the context of medium-energy nuclear collisions.

This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

*The work reported in this lecture was done in collaboration with George Fái, Roland Eötvös University, Budapest, Hungary.

References

1. A. Mekjian, Phys. Rev. C17 (1978) 1051.
2. J. Gosset, J.I. Kapusta and G.D. Westfall, Phys. Rev. C18 (1978) 844.
3. R. Bond, P.J. Johansen, S.E. Koonin and S. Garpman, Phys. Lett. 71B (1977) 43.
4. J. Randrup and S.E. Koonin, Nucl. Phys. A356 (1981) 223.
5. G. Fái and J. Randrup, Nucl. Phys. A381 (1982) 557.
6. A. Baden, H.H. Gutbrod, H. Löhner, M.R. Maier, A.M. Poskanzer, T. Renner, H. Riedesel, H.G. Ritter, H. Spieler, A. Warwick, F. Weik and H. Wieman, GSI-82-5 report.
7. J. Randrup and G. Fái, Phys. Lett. 115B (1982) 281.
8. G. Fái and J. Randrup, LBL-14248 (1983), Nucl. Phys. A (in press).

<u>Problem</u>

Derive the grand partition function $\mathcal{Z}$ by summing $\exp\{-\beta[E_F - \mu A_F - \nu T_F]\}$ over all final states F of the form (6).

<u>Solution</u>

Let the sequence $(n_1 n_2 \ldots)$ denote that particular final channel which contains n_α fragments of the species α. Each such fragment has position $\vec{r}^{(\alpha)}_{i\alpha}$ and momentum $\vec{p}^{(\alpha)}_{i\alpha}$ where $i_\alpha = 1,..,n_\alpha$. Its total energy is $(\vec{P}^\alpha_{i_\alpha})^2/2M_\alpha + M_\alpha c^2 + \epsilon^\alpha_{i_\alpha}$ where $\epsilon^\alpha_{i_\alpha}$ is the intrinsic excitation energy. The associated baryon number is A_α. (For simplicity the isospin degree of freedom is ignored, since it is straightforward to generalize the result afterwards. We also assume that non-relativistic kinematics can be used so that the kinetic energy is $p^2/2M$.) The grand partition function is then obtained by adding up $\exp\{-\beta[E_F-\mu A_F]\}$ for all such distinct final states F:

$$\mathcal{Z}(\beta,\mu) = \sum_F e^{-\beta[E_F-\mu A_F]}$$

$$= \sum_{n_1\geq 0} \frac{1}{n_1!}\int \frac{d\vec{r}^{(1)}_1 d\vec{p}^{(1)}_1}{h^3}\, \rho^{(1)}(\epsilon^{(1)}_1)d\epsilon^{(1)}_1 \ldots \int \frac{d\vec{r}^{(1)}_{n_1} d\vec{p}^{(1)}_{n_1}}{h^3}\, \rho^{(1)}\, (\epsilon^{(1)}_{n_1})\, d\epsilon^{(1)}_{n_1}$$

$$\sum_{n_2\geq 0} \frac{1}{n_2!}\int \frac{d\vec{r}^{(2)}_1 d\vec{p}^{(2)}_1}{h^3}\, \rho^{(2)}(\epsilon^{(2)}_1)d\epsilon^{(2)}_1 \ldots \int \frac{d\vec{r}^{(2)}_{n_2} d\vec{p}^{(2)}_{n_2}}{h^3}\, \rho^{(2)}\, (\epsilon^{(2)}_{n_2})\, d\epsilon^{(2)}_{n_2}$$

$$\ldots$$

$$\exp\left\{-\beta\left[\frac{(\vec{p}^{(1)}_1)^2}{2M_1} + \ldots + \frac{(\vec{p}^{(1)}_{n_1})^2}{2M_1} + n_1 M_1 c^2 + \epsilon^{(1)}_1 + \ldots + \epsilon^{(1)}_{n_1} - \mu n_1 A_1\right.\right.$$

$$+ \frac{(\vec{p}^{(2)}_1)^2}{2M_2} + \ldots + \frac{(\vec{p}^{(2)}_{n_2})^2}{2M_2} + n_2 M_2 c^2 + \epsilon^{(2)}_1 + \ldots + \epsilon^{(2)}_{n_2} - \mu n_2 A_2$$

$$\left.\left. + \ldots\right]\right\}$$

The division by $n_\alpha!$ takes account of the indistinguishability of fragments of the same species. Since the exponent is additive, each phase space integral over $\left(\vec{r}_{i_\alpha}^{(\alpha)}\ \vec{p}_{i_\alpha}^{(\alpha)}\ \epsilon_{i_\alpha}^{(\alpha)}\right)$ for a given species α yields the same result, namely

$$\int \frac{d\vec{r}_{i_\alpha}^{(\alpha)} d\vec{p}_{i_\alpha}^{(\alpha)}}{h^3} \rho^{(\alpha)}(\epsilon_{i_\alpha}^{(\alpha)})\ d\epsilon_{i_\alpha}^{(\alpha)}$$

$$\exp\left\{-\beta\left[\frac{(\vec{p}_{i_\alpha}^{(\alpha)})^2}{2M_\alpha} + M_\alpha c^2 + \epsilon_{i_\alpha}^{(\alpha)} - \mu A_\alpha\right]\right\}$$

$$= \chi \frac{4\pi}{3} r_0^3 A_0 \left(\frac{2\pi m\tau}{h^2}\right)^{3/2} \zeta_\alpha\ e^{-\beta[M_\alpha c^2 - \mu A_\alpha]} \equiv A_0 \omega_\alpha(\beta,\mu)$$

Here $\tau = 1/\beta$ is the ensemble temperature. It has been assumed that the spatial integrations are independent so that each may be replaced by the effective volume $\chi \frac{4\pi}{3} r_0^3 A_0$. Finally, $\zeta_\alpha(\beta)$ denotes the intrinsic partition function for the species α,

$$\zeta_\alpha(\beta) = \int \rho_\alpha(\epsilon) d\epsilon\ e^{-\beta\epsilon}$$

Therefore,

$$\mathcal{Z} = \sum_{n_1 \geq 0} \frac{1}{n_1!} (A_0\omega_1)^{n_1} \sum_{n_2 \geq 0} \frac{1}{n_2!} (A_0\omega_2)^{n_2} \ldots$$

$$= e^{A_0\omega_1} e^{A_0\omega_2} \ldots = e^{A_0\omega_1 + A_0\omega_2 + \ldots}$$

so that

$$\ell n\, \mathcal{Z} = A_0 \sum_\alpha \omega_\alpha$$

Thus, replacing the species index α by AT, we have

$$\ell n\, \mathcal{Z} = A_0 \sum_{AT} \chi \frac{4\pi}{3} r_0^3 \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2} \zeta_{AT}\ e^{-\beta[M_{AT}c^2 - \mu A - \nu T]}$$

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720