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by

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Abstract

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This thesis explores quantitative invariants in symplectic geometry. The first chapter is based on joint work with Julian Chaidez [11], where we construct the first examples of dynamically convex domains that are not symplectomorphic to any convex domain. A key component of this work is a quantitative convexity criterion involving the Ruelle invariant.

In the second chapter, based on the paper [30], we establish a sharp dynamical characterization of convex domains in $\mathbb{R}^4$ that admit a symplectic embedding into a given cylinder. More specifically, we demonstrate that the cylindrical capacity of a convex domain matches the minimum symplectic area of a disk-like global surface of section for the natural Reeb flow on the boundary of the domain. This contributes to progress toward the strong Viterbo conjecture regarding the equivalence of symplectic capacities on convex domains.

The third chapter comprises a joint paper with Michael Hutchings [31]. We show a quantitative C^∞ closing lemma for area preserving surface diffeomorphisms. Our proof relies on spectral invariants arising from periodic Floer homology. We also establish a new Weyl law for these spectral invariants.

Contents

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Last but not least, I would like to thank my family and personal friends for their support and for making this journey enjoyable. Thank you all!

Introduction

This thesis contains three papers written during my time as a graduate student at UC Berkeley. A unifying theme throughout these works is the prominent role played by quantitative invariants of symplectic and dynamical origin, such as the Ruelle invariant, symplectic capacities, and spectral invariants.

The first chapter consists of a joint paper with Julian Chaidez [11], focusing on the role of convexity in symplectic geometry. Understanding domains in Euclidean space up to symplectomorphism is a fundamental question in symplectic geometry. The class of convex domains is known to have numerous special symplectic and dynamical properties, and there are fascinating open conjectures regarding the symplectic geometry of convex domains, such as the Viterbo conjecture. However, from the perspective of symplectic geometry, convexity is a mysterious assumption because it is not preserved by symplectomorphisms. In their seminal paper [55], Hofer-Wysocki-Zehnder introduced the influential notion of dynamical convexity, which is an intrinsic symplectic property. Although it is known that every convex domain is dynamically convex, the question of whether dynamical convexity characterizes convexity up to symplectomorphism has been a longstanding open problem. In our paper [11], we resolve this question in the negative. We construct the first examples of dynamically convex domains which are not symplectomorphic to any convex domain. Our proof relies on a quantitative convexity criterion involving the Ruelle invariant, a classic invariant in dynamics.

The second chapter consists of the paper [30]. Ever since Gromov proved his celebrated non-squeezing theorem, the symplectic embedding problem—i.e. the question of whether a given symplectic manifold symplectically embeds into another—has attracted a significant amount of attention. To systematically approach such questions, Ekeland and Hofer introduced the notion of a symplectic capacity, which can be thought of as a measurement of symplectic size. Two basic examples of symplectic capacities are the ball capacity, which measures the largest size of a ball that can symplectically embed into a given symplectic manifold, and the cylindrical capacity, which measures the smallest size of a cylinder into which the symplectic manifold can embed. To this day, many more symplectic capacities have been constructed using a wide range of different techniques, such as the calculus of variations, Floer theory, pseudo-holomorphic curves, and microlocal sheaf theory. An important open conjecture about symplectic capacities, often referred to as the strong Viterbo conjecture, states that all symplectic capacities agree on convex domains in Euclidean space. The construction of many symplectic capacities is based on Hamiltonian dynamics. While there are many results asserting the equality of various dynamical capacities, very little is known about the relationship between dynamical capacities and the ball and cylindrical capacities, whose definition is based on symplectic embeddings. In [30], we bridge the gap between dynamics and the cylindrical capacity in dimension four. We provide a sharp dynamical characterization of convex domains that admit a symplectic embedding into a given cylinder. This characterization involves global surfaces of section, an important notion in dynamics, originating from the work of Poincaré, which plays a prominent role in modern symplectic dynamics.

The third chapter consists of the joint paper with Michael Hutchings [31]. Pugh's famous C^1 closing lemma states that a recurrent point of a differentiable dynamical system can be turned into a periodic point by an arbitrarily small perturbation of the system. As the name suggests, "small" refers to the C^1 topology on the space of dynamical systems. It is an important question whether analogous results hold in higher regularity—i.e. if the C^1 topology is replaced by the C^r topology for $r > 1$. Breakthroughs on this question were made by Irie [77] and Asaoka-Irie [7], who proved C^∞ closing lemmas for three-dimensional Reeb flows and Hamiltonian surface diffeomorphisms, respectively. In [31], we establish a C^∞ closing lemma for arbitrary area preserving surface diffeomorphisms. A key novelty of our work is that our closing lemma is quantitative. We provide an upper bound on the periods of newly generated periodic orbits in terms of the size of the perturbation. Our mechanism for detecting periodic orbits is based on spectral invariants arising from periodic Floer homology (PFH), a version of Floer theory for area preserving surface maps, originally developed by Hutchings and based on pseudoholomorphic curves and Seiberg-Witten gauge theory. As a byproduct of our work, we prove a new Weyl law for PFH spectral invariants, which, roughly speaking, states that PFH spectral invariants asymptotically recover symplectic volume.

Chapter 1

3D convex contact forms and the Ruelle invariant

1.1 Introduction

A contact manifold (Y, ξ) is an odd dimensional manifold equipped with a hyperplane field $\xi \subset TY$, called the contact structure, that is the kernel of a 1-form α such that

$$\ker(d\alpha) \subset TY \text{ is rank 1} \qquad \text{and} \qquad \alpha|_{\ker(d\alpha)} > 0$$

A 1-form satisfying this condition is called a contact form on (Y, ξ). Every contact form comes equipped with a natural Reeb vector field R, defined by

$$\alpha(R) = 1 \qquad \iota_R d\alpha = 0$$

Note that the Reeb vector-field preserves the 1-form α and the natural volume form $\alpha \wedge d\alpha^{n-1}$, where $\dim(Y) = 2n-1$. The dynamical properties of Reeb vector fields (e.g. the existence of closed orbits and their properties) are the subject of immense interest in symplectic geometry and dynamical systems.

Contact manifolds arise naturally as hypersurfaces in symplectic manifolds satisfying a certain stability condition. In fact, Weinstein introduced contact manifolds in [103] inspired by the following prototypical example of this phenomenon, due to Rabinowitz [89].

Example 1.1.1. We say that a domain $X \subset \mathbb{R}^{2n}$ with smooth boundary Y is *star-shaped* if

$$0 \in \operatorname{int}(X) \qquad \text{and} \qquad \partial_r \text{ is transverse to } Y$$

Let ω and Z denote the standard symplectic form and Liouville vector field on $\mathbb{R}^{2n}$. That is

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \qquad Z = \frac{1}{2}\sum_i x_i\partial_{x_i} + y_i\partial_{y_i} = \frac{1}{2}r\partial_r \tag{1.1.1}$$

Then the restriction $\lambda|_Y$ of the Liouville 1-form $\lambda = \iota_Z\omega$ is a contact form.

Example 1.1.2. The *standard* contact structure ξ on $S^{2n-1} \subset \mathbb{R}^{2n}$ is given by $\xi = \ker(\lambda|_{S^{2n-1}})$.

Every contact form on the standard contact sphere arises as the pullback of λ via a map to a star-shaped boundary Y. Indeed, if $\alpha = f \cdot \lambda|_{S^{2n-1}}$ for $f > 0$ is a contact form for ξ, then

$$\alpha = \phi^*\lambda \qquad \text{where} \qquad \phi(\theta) = (f(\theta)^{1/2}, \theta) \quad \text{in radial coordinates } (0,\infty)_r \times S^{2n-1}_\theta$$

Moreover, every star-shaped boundary Y admits such a map from the sphere. Thus, from the perspective of contact geometry, the study of star-shaped boundaries is equivalent to the study of contact forms on the standard contact sphere.

1.1.1 Convexity

In this paper, we are primarily interested in studying contact forms arising as boundaries of convex domains.

Definition 1.1.3. A contact form α on S^{2n-1} is *convex* if there is a convex star-shaped domain $X \subset \mathbb{R}^{2n}$ with boundary Y and a strict contactomorphism $(S^3, \alpha) \simeq (Y, \lambda|_Y)$.

In contrast to the star-shaped case, not every contact form on S^{2n-1} is convex, and the Reeb flows of convex contact forms possess many special dynamical properties, both proven and conjectural.

In [102], Viterbo proposed a particularly remarkable systolic inequality for Reeb flows on convex boundaries. To state it, let (Y, α) be a closed contact manifold with contact form of dimension $2n-1$, and recall that the volume $\mathrm{vol}(Y, \alpha)$ and systolic ratio $\mathrm{sys}(Y, \alpha)$ are given by

$$\mathrm{vol}(Y,\alpha) = \int_Y \alpha \wedge d\alpha^{n-1} \qquad \text{and} \qquad \mathrm{sys}(Y,\alpha) = \frac{\min\{\text{period } T \text{ of an orbit}\}^n}{\mathrm{vol}(Y,\alpha)} \tag{1.1.2}$$

Note that if Y is the boundary of a star-shaped domain $X \subset \mathbb{R}^{2n}$, then the contact volume of $(Y, \lambda|_Y)$ is related to the volume of X via $\mathrm{vol}(Y, \lambda|_Y) = n!\,\mathrm{vol}(X)$. The weak Viterbo conjecture that originally appeared in [102] can be stated as follows.

Conjecture 1.1.4. *[102] Let α be a convex contact form on S^{2n-1}. Then the systolic ratio is bounded by* 1*.*

$$\mathrm{sys}(S^{2n-1}, \alpha) \le 1$$

There is also a strong Viterbo conjecture (c.f. [48]), stating that all normalized symplectic capacities are equal on convex domains. For other special properties of convex domains, see [55, 102].

Despite the plethora of distinctive properties that convex contact forms possess, a characterization of convexity entirely in terms of contact geometry has remained elusive.

Problem 1.1.5. Give an intrinsic characterization of convexity that does not reference a map to $\mathbb{R}^{2n}$.

1.1.2 Dynamical Convexity

In the seminal paper [55], Hofer-Wysocki-Zehnder provided a candidate answer to Problem 1.1.5.

Definition 1.1.6 (Def. 3.6, [55])**.** A contact form α on S^3 is *dynamically convex* if the Conley-Zehnder index $\mathrm{CZ}(\gamma)$ of any closed Reeb orbit γ is greater than or equal to 3.

The Conley-Zehnder index of a Reeb orbit plays the role of the Morse index in symplectic field theory and other types of Floer homology (see §1.2.2 for a review). Thus, on a naive level, dynamical convexity may be viewed as a type of "Floer-theoretic" convexity. If X is a convex domain whose boundary Y has positive definite second fundamental form, then Y is dynamically convex [55, Thm 3.7]. Note that this condition is open and dense among convex boundaries.

In [55], Hofer-Wysocki-Zehnder proved that the Reeb flow of a dynamically convex contact form admits a surface of section. In the decades since, dynamical convexity has been used as a key hypothesis in many significant works on Reeb dynamics and other topics in contact and symplectic geometry. See the papers of Hryniewicz [57], Zhou [104,105], Abreu-Macarini [4,5], Ginzburg-Gürel [44], Fraunfelder-Van Koert [38] and Hutchings-Nelson [70] for just a few examples. However, the following question has remained stubbornly open (c.f. [38, p. 5]).

Question 1.1.7. Is every dynamically convex contact form on S^3 also convex?

The recent paper [2] of Abbondandolo-Bramham-Hryniewicz-Salomão (ABHS) has suggested that the answer to Question 1.1.7 should be no. They construct dynamically convex contact forms on S^3 with systolic ratio close to 2. There is substantial evidence for the weak Viterbo conjecture (cf. [12]), and so these contact forms are likely *not* convex. However, this was not proven in [2].

Even more recently, Ginzburg-Macarini [43] addressed a version of Question 1.1.7 in higher dimensions that incorporates the assumption of symmetry under the antipod map $S^{2n-1} \to S^{2n-1}$. Their work did not address the general case of Question 1.1.7.

1.1.3 Main Result

The main purpose of this paper is to resolve Question 1.1.7.

Theorem 1.1.8. *There exist dynamically convex contact forms α on S^3 that are not convex.*

Theorem 1.1.8 is an immediate application of Proposition 1.1.9 and 1.1.12, which we will now describe.

1.1.4 Ruelle Bound

For our first result, recall that any closed contact 3-manifold (Y, ξ) with contact form α that satisfies $c_1(\xi) = 0$ and $H^1(Y; \mathbb{Z}) = 0$ has an associated *Ruelle invariant* [92]

$$\mathrm{Ru}(Y, \alpha) \in \mathbb{R}$$

Roughly speaking, the Ruelle invariant is the integral over Y of a time-averaged rotation number that measures the degree to which different Reeb trajectories twist counter-clockwise around each other (see §1.2.4 for a detailed review). Our result is stated most elegantly using the quantity

$$\mathrm{ru}(Y, \alpha) = \frac{\mathrm{Ru}(Y, \alpha)}{\mathrm{vol}(Y, \alpha)^{1/2}}$$

This *Ruelle ratio* is invariant under scaling of the contact form, unlike the Ruelle invariant itself.

In recent work [69] motivated by embedded contact homology, Hutchings investigated the Ruelle invariant of toric domains in $\mathbb{C}^2$. In that paper, the Ruelle invariant of the standard ellipsoid $E = E(a, b) \subset \mathbb{C}^2$ with symplectic radii $0 < a \leq b$ (see §1.3.1) was computed as

$$\mathrm{Ru}(E) = a + b \tag{1.1.3}$$

The systolic ratio $\mathrm{sys}(E)$ and contact volume $\mathrm{vol}(\partial E, \lambda|_{\partial E})$ are well-known to be a/b and ab respectively. Thus we have the following relation between the systolic and Ruelle ratios.

$$\mathrm{ru}(E) = \mathrm{sys}(E)^{1/2} + \mathrm{sys}(E)^{-1/2} \quad \text{and thus} \quad 1 < \mathrm{ru}(E) \cdot \mathrm{sys}(E)^{1/2} = \mathrm{sys}(E) + 1 \leq 2$$

Our first result may be viewed as a generalization of the estimate on the right to arbitrary convex contact forms on S^3.

Proposition 1.1.9 (Prop 1.3.1)**.** *There are constants $C > c > 0$ such that, for any convex contact form α on S^3, the following inequality holds.*

$$c \leq \mathrm{ru}(S^3, \alpha) \cdot \mathrm{sys}(S^3, \alpha)^{1/2} \leq C \tag{1.1.4}$$

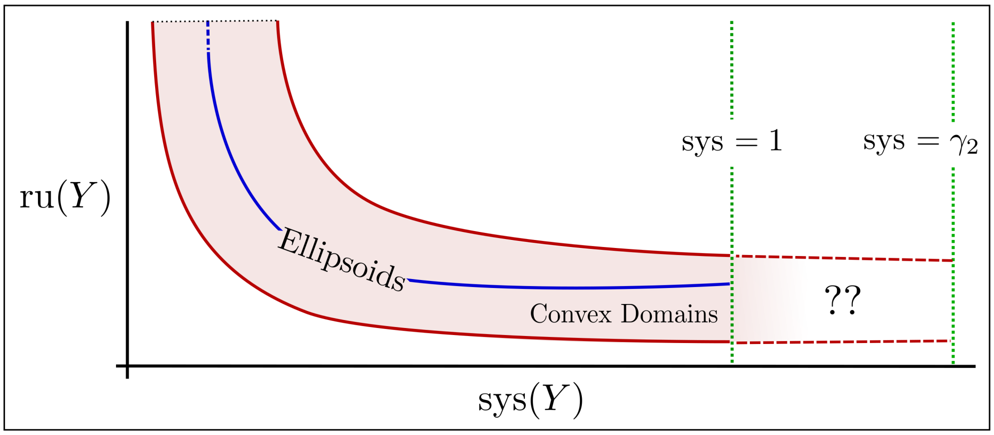
Note that a result of Viterbo [102, Thm 5.1] states that there exists a constant γ_2 such that $\mathrm{sys}(S^3, \alpha) \leq \gamma_2$ for any convex contact form. Thus, Proposition 1.1.9 also implies that

Corollary 1.1.10. *There is a constant $c > 0$ such that, for any convex contact form α on S^3, we have*

$$c \leq \mathrm{ru}(S^3, \alpha) \tag{1.1.5}$$

It is notable that, even for ellipsoids, the systolic ratio can be arbitrarily close to 0 and the Ruelle ratio can be arbitrarily close to ∞. We have included a helpful visualization of Proposition 1.1.9 in the $\mathrm{sys} - \mathrm{ru}$ plane in Figure 1.1.

Figure 1.1: A plot of the region of the sys − ru plane containing convex contact forms, depicted in light red. The blue arc is the region occupied by ellipsoids, and the green lines represent the sys = 1 bound and the sys = γ_2 bound. The Viterbo conjecture states that the region of convex domains with systolic ratio larger than 1 is empty, and so it is partially shaded in this figure.



Let us explain the idea of the proof of Proposition 1.1.9. First, as explained above, the result holds for ellipsoids. By applying John's ellipsoid theorem [78], we can sandwich a given convex domain X between an ellipsoid E and its scaling $4 \cdot E$. After applying an affine symplectomorphism to X and E, we may assume that E is standard. That is

$$E(a,b) \subset X \subset 4 \cdot E(a,b)$$

Note that this symplectomorphism does not change the Ruelle invariant (see §1.3.1). Now note that the minimum length of a closed orbit is monotonic under inclusion of convex domains, since it coincides the the Ekeland-Hofer-Zehnder capacity in the convex setting (cf. [15]). Applying this and the monotonicity of volume, we find that

$$\frac{ab}{2} \leq \mathrm{vol}(X) \leq 2^8 \cdot \frac{ab}{2} \qquad \text{and} \qquad 2^{-8} \cdot \frac{a}{b} \leq \mathrm{sys}(Y) \leq 2^8 \cdot \frac{a}{b} \tag{1.1.6}$$

If the Ruelle invariant were also monotonic, then one could immediately acquire Proposition 1.1.9 from (1.1.6) and (1.1.3). Unfortunately, this is not evidently the case.

The resolution of this issue comes from a beautiful formula (Proposition 1.3.10) relating the second fundmantal form and local rotation of the Reeb flow on a contact hypersurface Y in $\mathbb{R}^4$. This is due originally to Ragazzo-Salomão [90], albeit in different language from this paper. Using this relation (§1.3.2), we derive estimates for the Ruelle invariant in terms of diameter, area and total mean curvature. By standard convexity theory (i.e. the theory of mixed volumes), these quantities are monotonic under inclusion of convex domains. This allows us to compare the Ruelle invariant of X to that of its sandwiching ellipsoids, and thus prove the result.

Remark 1.1.11 (Enhancing Prop 1.1.9)**.** In future work, we plan to investigate optimal constants c and C for Proposition 1.1.9, and to generalize the result to higher dimensions.

1.1.5 A Counterexample

In order to prove Theorem 1.1.8 using Proposition 1.1.9, we construct dynamically convex contact forms that violate both sides of the estimate (1.1.4). This is the subject of our second new result.

Proposition 1.1.12 (Prop 1.4.1)**.** *For every $\varepsilon > 0$, there exists a dynamically convex contact form α on S^3 satisfying*

$$\mathrm{vol}\big(S^3, \alpha\big) = 1 \qquad \mathrm{sys}(S^3, \alpha) \geq 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \alpha) \leq \varepsilon$$

and there exists a dynamically convex contact form β on S^3 satisfying

$$\mathrm{vol}\big(S^3, \beta\big) = 1 \qquad \mathrm{sys}(S^3, \beta) \geq 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \beta) \geq \varepsilon^{-1}$$

The construction of these examples follows the open book methods of Abbondandolo-Bramham-Hryniewicz-Salomão in [2, 3]. Namely, we develop a detailed correspondence between the properties of a Hamiltonian disk map $\phi : \mathbb{D} \to \mathbb{D}$ and the properties of a contact form α on S^3 constructed using ϕ via the open book construction (see Proposition 1.4.8). This includes a new formula relating the Ruelle invariant of ϕ in the sense of [92] and the Ruelle invariant of (S^3, α). We then construct Hamiltonian disk maps ϕ with all of the appropriate properties to produce dynamically convex contact forms on S^3 satisfying the conditions in Proposition 1.1.12.

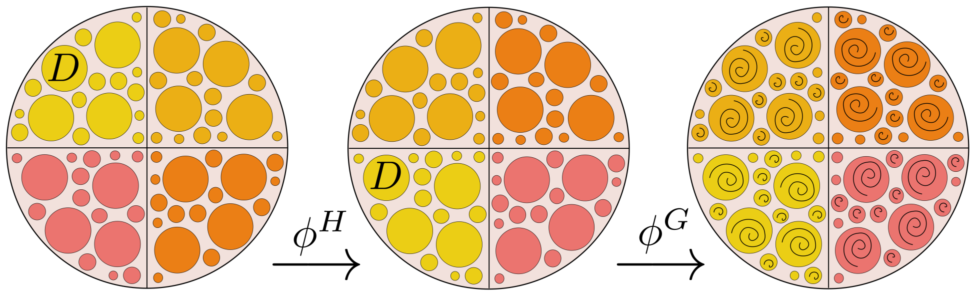
Let us briefly outline the construction in the small Ruelle case, as the large Ruelle case is similar. The special Hamiltonian map ϕ is acquired by composing two maps ϕ^H and ϕ^G. The map ϕ^H is a counter-clockwise rotation by angle $2\pi(1 + 1/n)$ for large n. The map ϕ^G is compactly supported on a disjoint union U of disks D, and rotates (most of) each disk D clockwise about its center by angle slightly less than 4π. See Figure 1.2 for an illustration of this map.

Applying Proposition 1.4.8, we can show that the volume and Ruelle invariant of (S^3, α) are (up to negligible error) proportional to the following quantities.

$$\mathrm{vol}\big(S^3, \alpha\big) \sim \pi^2 - 2\sum_D \mathrm{area}(D)^2 \qquad \mathrm{Ru}(S^3, \alpha) \sim 2\pi - 2\sum_D \mathrm{area}(D)$$

By choosing U to fill most of $\mathbb{D}$ and choosing all of the disks in U to be very small, we can make the Ruelle invariant very small relative to the volume. This process preserves the minimal action of a closed orbit (up to a small error) and dynamical convexity, producing the desired small Ruelle invariant example.

Figure 1.2: The map $\phi = \phi^G \circ \phi^H$ for $n = 4$. Here ϕ^H rotates $\mathbb{D}$ counter-clockwise by 90 degrees and ϕ^G twists each disk D by roughly 720 degrees clockwise.



Remark 1.1.13. Our examples *do not* coincide with the ABHS examples in [2]. However, we believe that improvements of Proposition 1.1.12 may make our analysis applicable to those examples.

Remark 1.1.14. In general, it is possible for the Ruelle invariant of a Reeb flow on S^3 to be negative. However, Proposition 1.1.9 implies (via the lower bound) that the Ruelle invariant of a convex contact form is always positive. In fact, this is a much simpler property to prove than Proposition 1.1.9 itself, using similar methods. However, we were not able to push the construction in §1.4 to yield a dynamically convex contact form with non-positive Ruelle invariant.

Outline

This concludes the introduction **§1.1**. The rest of the paper is organized as follows.

In **§1.2**, we cover basic preliminaries needed in later sections: the rotation number (§1.2.1), the Conley-Zehnder index (§1.2.2), invariants of Reeb orbits (§1.2.3) and the Ruelle invariant (§1.2.4).

In **§1.3**, we prove Proposition 1.1.9. We start by discussing the curvature-rotation formula and some consequences (§1.3.2). We then derive a lower bound for a relevant curvature integral (§1.3.3). We conclude by proving the main bound (§1.3.4).

In **§1.4**, we prove Proposition 1.1.12. We first discuss general preliminaries on Hamiltonian disk maps (§1.4.1), open books (§1.4.2) and radial Hamiltonians (§1.4.3). We then construct a Hamiltonian flow on the disk (§1.4.4) before concluding with the main proof (§1.4.5).

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1.2 Rotation Numbers And Ruelle Invariant

In this section, we review some preliminaries on rotation numbers, Conley-Zehnder indices and the Ruelle invariant, which we will need in later parts of the paper.

Remark 1.2.1. The rotation number, also known as the homogeneous Maslov quasimorphism, and the Conley-Zehnder index were originally introduced by Gelfand-Lidskii [41] albeit using different terminology. For a more contemporary perspective, see Salamon-Zehnder [94].

1.2.1 Rotation Number

Consider the universal cover $\widetilde{\mathrm{Sp}}(2)$ of the symplectic group $\mathrm{Sp}(2)$. We will view a group element Φ as a homotopy class of paths with fixed endpoints

$$\Phi : [0,1] \to \mathrm{Sp}(2) \quad \text{with} \quad \Phi(0) = \mathrm{Id}$$

Recall that a *quasimorphism* $q : G \to \mathbb{R}$ from a group G to the real line is a map such that there exists a $C > 0$ such that

$$|q(gh) - q(g) - q(h)| < C \qquad \text{for all } g, h \in G \tag{1.2.1}$$

A quasimorphism is *homogeneous* if $q(g^k) = k \cdot q(g)$ for any $g \in G$. Finally, two quasimorphisms q and q' are called *equivalent* if the function $|q - q'|$ on G is bounded. Note that any quasimorphism is equivalent to a unique homogeneous one.

The universal cover of the symplectic group possesses a canonical homogeneous quasimorphism, due to the following result of Salamon-Ben Simon [93].

Theorem 1.2.2 ([93], Thm 1)**.** *There exists a unique homogeneous quasimorphism*

$$\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$$

that restricts to the following homomorphism $\rho : \widetilde{\mathrm{U}}(1) \to \mathbb{R}$ on the universal cover of $\mathrm{U}(1)$.

$$\rho(\gamma) = L \qquad \textit{on the path } \gamma : [0,1] \to \mathrm{U}(1) \textit{ with } \gamma(t) = \exp(2\pi i L t) \tag{1.2.2}$$

Definition 1.2.3. The *rotation number* $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ is the quasimorphism in Theorem 1.2.2.

The rotation number is often characterized more explicitly in the literature as a lift of a map to the circle. More precisely, it is characterized as the unique lift

$$\widetilde{\sigma} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R} \qquad \text{of} \qquad \sigma : \mathrm{Sp}(2) \to S^1 \qquad \text{such that} \qquad \widetilde{\sigma}(\mathrm{Id}) = 0 \tag{1.2.3}$$

via the covering map $\mathbb{R} \to S^1 \subset \mathbb{C}$ given by $\theta \mapsto e^{2\pi i\theta}$. Here σ is defined as follows. Let $\Phi \in \mathrm{Sp}(2)$ have real eigenvalues λ, λ^{-1} and let $\Psi \in \mathrm{Sp}(2)$ have complex (unit) eigenvalues $\zeta, \overline{\zeta}$ with $\mathrm{Im}\,\zeta > 0$. Also fix an arbitrary $v \in \mathbb{R}^2 \setminus 0$, identified with an element of $\mathbb{C}$ in the usual way. Then

$$\sigma(\Phi) = \begin{cases} 0 & \text{if } \lambda > 0 \\ 1/2 & \text{if } \lambda < 0 \end{cases} \quad \text{and} \quad \sigma(\Psi) = \begin{cases} \zeta & \text{if } \langle iv, \Phi v\rangle > 0 \\ \overline{\zeta} & \text{if } \langle iv, \Phi v\rangle < 0 \end{cases} \tag{1.2.4}$$

Here iv denotes multiplication of v by $i \in \mathbb{C}$, i.e. the rotation of v by 90 degrees counter-clockwise. All of the elements of Sp(2) fall into one of the two categories above, and so σ is determined everywhere by (1.2.4).

Lemma 1.2.4. *The rotation number* $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *is the lift of* $\sigma : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z}$ *with* $\rho(\mathrm{Id}) = 0$.

Proof. We verify the properties in Theorem 1.2.2. The lift $\widetilde{\sigma}$ is a quasimorphism by Lemmas 1.2.6 and 1.2.7 below. Also note that since eigenvalues and the sign of $\langle iv, \Phi v\rangle$ are invariant under conjugation, σ is as well.

To check that $\widetilde{\sigma}$ is homogeneous, note that if Φ has real eigenvalues of sign $s = \pm 1$, then $\sigma(\Phi^k) = \frac{1}{2}(1 - s^k) = ks \mod 1$. On the otherhand, if Φ has complex unit eigenvalues $\zeta, \bar{\zeta}$ for $\mathrm{Re}(\zeta) > 0$, then it is conjugate to a rotation $\exp(2\pi i\theta) \in U(1) \subset \mathrm{Sp}(2)$ on $\mathbb{C} \simeq \mathbb{R}^2$ and thus

$$\sigma(\Phi^k) = \sigma(\exp(2\pi ik\theta)) = k\theta \mod 1$$

Thus $\sigma(\Phi^k) = k \cdot \sigma(\Phi) \mod 1$ and the lift satisifes $\widetilde{\sigma}(\Phi^k) = k\widetilde{\sigma}(\Phi)$. Finally, if $\gamma : [0, 1] \to \mathrm{Sp}(2)$ is given by $\gamma(t) = \exp(2\pi iLt)$ then

$$\sigma \circ \gamma : [0, 1] \to \mathbb{R}/\mathbb{Z} \quad \text{is given by} \quad \sigma \circ \gamma(t) = Lt \mod 1 \in \mathbb{R}/\mathbb{Z}$$

This implies that the lift is $t \mapsto Lt$, so that $\widetilde{\sigma}(\gamma) = L$. This proves the needed criteria. □

We will also need to utilize several inhomogeneous versions of the rotation number depending on a choice of unit vector. These are defined a follows.

Definition 1.2.5. The *rotation number* $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ *relative to* $s \in S^1$ is the unique lift of the map

$$\sigma_s : \mathrm{Sp}(2) \to S^1 \qquad \Phi \mapsto |\Phi s|^{-1} \cdot \Phi s \in S^1 \subset \mathbb{R}^2$$

via the covering map $\mathbb{R} \to S^1 \subset \mathbb{C}$ given by $\theta \mapsto e^{2\pi i\theta} \cdot s$ such that $\rho_s(\mathrm{Id}) = 0$. Here Φs denotes the application of the matrix $\Phi \in \mathrm{Sp}(2)$ to the unit vector $s \in S^1$.

The rotation numbers relative to $s \in S^1$ and the lift of σ all have bounded difference from one another. Precisely, we have the following lemma.

Lemma 1.2.6. *The maps $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ and the lift $\widetilde{\sigma} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ of σ have bounded difference. More precisely, we have the following bounds.*

$$|\rho_s - \widetilde{\sigma}| \le 1 \qquad \textit{and} \qquad |\rho_s - \rho_t| \le 1 \qquad \textit{for any pair } s, t \in S^1 \tag{1.2.5}$$

Proof. First, assume that $\Phi : [0,1] \to \mathrm{Sp}(2)$ is a path such that $\Phi(t)$ has no negative real eigenvalues for any $t \in [0,1]$. Then

$$\sigma \circ \Phi(t) \neq 1/2 \qquad \text{and} \qquad \sigma_s \circ \Phi(t) \neq -s \in S^1 \qquad \text{for any } s \in S^1 \text{ and } t \in [0,1]$$

It follows that the relevant lifts of $\sigma \circ \Phi$ and $\sigma_s \circ \Phi$ to maps $[0,1] \to \mathbb{R}$ remain in the interval $(-1/2, 1/2)$ for all t. Thus

$$\widetilde{\sigma}(\Phi) \in (-1/2, 1/2) \qquad \text{and} \qquad \rho_s(\Phi) \in (-1/2, 1/2)$$

This clearly implies (1.2.5) since $\Phi(t)$ does not have any negative eigenvalues. Since σ induces an isomorphism $\pi_1(\mathrm{Sp}(2)) \to \pi_1(S^1)$, we know that for any pair $\Phi, \Phi' \in \widetilde{\mathrm{Sp}}(2)$ with the same projection to $\mathrm{Sp}(2)$

$$\widetilde{\sigma}(\Phi) = \widetilde{\sigma}(\Phi') \qquad \text{implies} \qquad \Phi = \Phi'$$

In particular, the above analysis extends to any Φ with $\widetilde{\sigma}(\Phi) \in (-1/2, 1/2)$. In the general case, note that the path $\gamma : [0,1] \to S^1$ given by $\gamma(t) = \exp(\pi i \cdot kt)$ for an integer $k \in \mathbb{Z}$ satisfies

$$\widetilde{\sigma}(\gamma) = \rho_s(\gamma) = k/2 \qquad \widetilde{\sigma}(\Phi\gamma) = \widetilde{\sigma}(\Phi) + \widetilde{\sigma}(\gamma) \qquad \rho_s(\Phi\gamma) = \rho_s(\Phi) + \rho_s(\gamma)$$

Any path Ψ can be decomposed (up to homotopy) as $\Phi\gamma$ where γ is as above and $\Phi : [0,1] \to \mathrm{Sp}(2)$ is a path with $\widetilde{\sigma}(\Phi) \in (-1/2, 1/2)$. This reduces to the special case. □

This can be used to demonstrate that ρ_s is a quasimorphism. As noted in the proof of Lemma 1.2.4, this implies that $\widetilde{\sigma}$ is a quasimorphism as well.

Lemma 1.2.7. *The map $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ is a quasimorphism for any $s \in S^1$. In fact, we have*

$$|\rho_s(\Psi\Phi) - \rho_s(\Psi) - \rho_s(\Phi)| \le 1 \qquad \textit{for any} \qquad s \in S^1 \tag{1.2.6}$$

Proof. Let $\Phi : [0,1] \to \mathrm{Sp}(2)$ and $\Psi : [0,1] \to \mathrm{Sp}(2)$ be two elements of $\widetilde{\mathrm{Sp}}(2)$ viewed as paths in $\mathrm{Sp}(2)$. Consider the product $\Psi\Phi$ in the universal cover of $\mathrm{Sp}(2)$, represented by the path

$$\Phi(2t) \text{ for } t \in [0, 1/2] \qquad \text{and} \qquad \Psi(2t-1)\Phi(1) \text{ for } t \in [1/2, 1]$$

By examining the path $\sigma_s \circ \Psi\Phi : [0,1] \to S^1$ and the lift to $\mathbb{R}$, we deduce the following property.

$$\rho_s(\Psi\Phi) = \rho_{\Phi(s)}(\Psi) + \rho_s(\Phi) \tag{1.2.7}$$

Here $\Phi(s)$ is shorthand for the unit vector $\Phi(1)s/|\Phi(1)s|$. Applying Lemma 1.2.6, we have

$$|\rho_s(\Psi\Phi) - \rho_s(\Psi) - \rho_s(\Phi)| \leq |\rho_{\Phi(s)}(\Psi) - \rho_s(\Psi)| \leq 1$$

This proves the quasimorphism property. □

1.2.2 Conley-Zehnder Index

Let $\mathrm{Sp}_\star(2) \subset \mathrm{Sp}(2)$ denote the subset of elements $\Phi \in \mathrm{Sp}(2)$ such that $\Phi - \mathrm{Id}$ is invertible, and let $\widetilde{\mathrm{Sp}}_\star(2)$ be the inverse image of $\mathrm{Sp}_\star(2)$ under $\pi : \widetilde{\mathrm{Sp}}(2) \to \mathrm{Sp}(2)$.

The *Conley-Zehnder index* is a natural integer invariant of paths in $\mathrm{Sp}_\star(2)$, denoted as follows.

$$\mathrm{CZ} : \widetilde{\mathrm{Sp}}_\star(2) \to \mathbb{Z}$$

This invariant was introduce in [41] (also see [94]). We will use the following formula as our definition throughout this paper.

$$\mathrm{CZ}(\Phi) = \lfloor \rho(\Phi) \rfloor + \lceil \rho(\Phi) \rceil \tag{1.2.8}$$

There are several inequivalent ways to extend the Conley-Zehnder index to the entire symplectic group. We will follow [55, §3] and [2, §2.2], and use the following extension.

Convention 1.2.8. In this paper, the *Conley-Zehnder index* $\mathrm{CZ} : \widetilde{\mathrm{Sp}}(2) \to \mathbb{Z}$ will be the maximal lower semi-continuous extension of the ordinary Conley-Zehnder index.

The extension in Convention 1.2.8 can be bounded from below in terms of the rotation number.

Lemma 1.2.9. *Let $\Phi \in \widetilde{\mathrm{Sp}}(2)$. Then*

$$\mathrm{CZ}(\Phi) \geq 2 \cdot \lceil \rho(\Phi) \rceil - 1 \tag{1.2.9}$$

Proof. For $\Phi \in \widetilde{\mathrm{Sp}}_\star(2)$, (1.2.9) is an immediate consequence of (1.2.8). In the other case, note that the maximal lower semi-continuous extension is defined by the property that

$$\mathrm{CZ}(\Phi) = \liminf_{\Psi \to \Phi} \mathrm{CZ}(\Psi) \qquad \text{for any } \Phi \notin \widetilde{\mathrm{Sp}}_\star(2)$$

Any $\Phi \notin \widetilde{\mathrm{Sp}}_\star(2)$ has eigenvalue 1, and so Lemma 1.2.4 implies that $\rho(\Phi) \in \mathbb{Z}$. Since ρ is continuous, we find that

$$\mathrm{CZ}(\Phi) = \liminf_{\Psi \to \Phi} \lfloor \rho(\Psi) \rfloor + \lceil \rho(\Psi) \rceil \geq \lfloor \rho(\Phi) - 1/2 \rfloor + \lceil \rho(\Phi) - 1/2 \rceil = 2 \cdot \lceil \rho(\Phi) \rceil - 1$$

This proves the lower bound in every case. □

1.2.3 Invariants Of Reeb Orbits

Let (Y, ξ) be a closed contact 3-manifold with $c_1(\xi) = 0$ and let α be a contact 1-form on Y.

Under this hypothesis on the Chern class, ξ is isomorphic as a symplectic vector-bundle to the trivial bundle $\mathbb{R}^2$. A *trivialization* τ of ξ is a bundle isomorphism

$$\tau : \xi \simeq \mathbb{R}^2 \qquad \text{denoted by} \qquad \tau(y) : \xi_y \simeq \mathbb{R}^2 \qquad \text{satisfying} \qquad \tau(y)^*\omega = d\alpha|_\xi$$

Two trivializations are *homotopic* if they are connected by a 1-parameter family of bundle isomorphisms. Given a trivialization τ, we may associate a *linearized* Reeb flow

$$\Phi_\tau : \mathbb{R} \times Y \to \mathrm{Sp}(2) \qquad \text{given by} \qquad \Phi_\tau(T, y) = \tau(\phi_T(y)) \circ d\phi_T(y) \circ \tau^{-1}(y) \tag{1.2.10}$$

Here $\phi : \mathbb{R} \times Y \to Y$ is the Reeb flow, i.e. the flow generated by the Reeb vector field R, and we use the notation $\phi_T(y) = \phi(T, y)$. The linearized flow lifts uniquely to a map

$$\widetilde{\Phi}_\tau : \mathbb{R} \times Y \to \widetilde{\mathrm{Sp}}(2) \qquad \text{with} \qquad \widetilde{\Phi}_\tau|_{0 \times Y} = \mathrm{Id} \in \widetilde{\mathrm{Sp}}(2)$$

We will refer to $\widetilde{\Phi}_\tau$ as the *lifted* linearized Reeb flow. Explicitly, it maps (T, y) to the homotopy class of the path $\Phi_\tau(\cdot, y)|_{[0,T]}$. Note that this lift satisfies the cocyle property

$$\widetilde{\Phi}_\tau(S + T, y) = \widetilde{\Phi}_\tau(T, \phi_S(y)) \cdot \widetilde{\Phi}_\tau(S, y) \tag{1.2.11}$$

Definition 1.2.10. Let $\gamma : \mathbb{R}/L\mathbb{Z} \to Y$ be a closed Reeb orbit of Y. The *action* of γ is given by

$$\mathcal{A}(\gamma) = \int \gamma^*\alpha = L \tag{1.2.12}$$

Likewise, the *rotation number* and *Conley-Zehnder index* of γ with respect to τ are given by

$$\rho(\gamma, \tau) := \rho \circ \widetilde{\Phi}_\tau(L, y) \qquad \mathrm{CZ}(\gamma, \tau) := \mathrm{CZ}(\widetilde{\Phi}_\tau(L, y)) \qquad \text{where } y = \gamma(0) \tag{1.2.13}$$

These invariants depend only on the homotopy class of τ, and if $H^1(Y; \mathbb{Z}) = 0$ (e.g. if Y is the 3-sphere) there is a unique trivialization up to homotopy. In this case, we let

$$\rho(\gamma) := \rho(\gamma, \tau) \qquad \text{and} \qquad \mathrm{CZ}(\gamma) := \mathrm{CZ}(\gamma, \tau) \qquad \text{for any } \tau \tag{1.2.14}$$

In §1.4, we will need the following easy observation, which follows immediately from Lemma 1.2.9 and our way of defining CZ (see Convention 1.2.8).

Lemma 1.2.11. *Let α be a contact form on S^3 with $\rho(\gamma) > 1$ for every closed Reeb orbit. Then α is dynamically convex.*

1.2.4 Ruelle Invariant

Let (Y, ξ) be a closed contact 3-manifold with $c_1(\xi) = 0$ equipped with a contact form α and a homotopy class of trivialization $[\tau]$ of ξ. Here we discuss the *Ruelle invariant*

$$\mathrm{Ru}(Y, \alpha, [\tau]) \in \mathbb{R}$$

associated to the data of Y, α and $[\tau]$.

Remark 1.2.12. This invariant was originally introduced by Ruelle in [92] for area preserving diffeomorphisms of surfaces and volume preserving flows on 3-manifolds. Variants of this construction have also appeared under different names in other settings, e.g. as the asymptotic Maslov index [17, p. 1423].

It will be helpful to describe a more general construction that subsumes that of the Ruelle invariant. For this purpose, we also fix a continuous quasimorphism

$$q : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$$

Pick a representative trivialization τ of $[\tau]$ and let $\widetilde{\Phi}_\tau : \mathbb{R} \times Y \to \widetilde{\mathrm{Sp}}(2)$ be the lifted linearized Reeb flow. We can associate a time-averaged version of q over the space Y, as follows.

Proposition 1.2.13. *The* 1*-parameter family of functions $f_T : Y \to \mathbb{R}$ given by the formula*

$$f_T(y) := \frac{q \circ \widetilde{\Phi}_\tau(T, y)}{T} \tag{1.2.15}$$

converges in $L^1(Y; \mathbb{R})$ and almost everywhere to a function $f(\alpha, q, \tau) : Y \to \mathbb{R}$ with the following properties.

(a) (Quasimorphism) If q and r are equivalent quasimorphisms, i.e. $|q - r|$ is bounded, then

$$f(\alpha, q, \tau) = f(\alpha, r, \tau)$$

(b) (Trivialization) If σ and τ are homotopic trivializations of ξ, then

$$f(\alpha, q, \sigma) = f(\alpha, q, \tau)$$

(c) (Contact Form) The integral $F(\alpha)$ of $f(\alpha, q, \tau)$ over Y is continuous in the C^2-topology on $\Omega^1(Y)$.

In order to prove the existence part of this result, we will need to use a verison of Kingman's subadditivity theorem appearing in [79].

Theorem 1.2.14. *Let Y be a measure space and let $\phi : \mathbb{R} \times Y \to Y$ be a flow with invariant measure μ. Let $g_T : Y \to \mathbb{R}$ for $T \in \mathbb{R}$ be a family of $L^1(Y, \mu)$ functions such that, for some constants $C, D > 0$, we have*

$$g_{S+T} \le g_S + \phi_S^* g_T + C \qquad \int_Y g_T \cdot \mu \ge -D \cdot T \qquad \int_Y \big(\sup_{0 \le S \le 1} |g_S| \big) \cdot \mu < \infty$$

Then the maps $\frac{g_T}{T}$ converge in $L^1(Y, \mu)$ and pointwise almost everywhere as $T \to \infty$.

Remark 1.2.15. There is also a version of Theorem 1.2.14 in [79] for a discrete dynamical system, i.e a map $\phi : Y \to Y$ preserving μ. The statement is directly analogous to Theorem 1.2.14, but the last condition on the integrability of $\sup_{0 \le S \le 1} |g_S|$ is unnecessary. We will use this version in §1.4.1.

Remark 1.2.16. This statement is a slight variation of Theorem 4 in [79], which states the result for general sub-additive processes. Our version follows from the discussion in §1.3 of [79] for the continuous parameter space $\mathbb{R}$. Note that we also weaken sub-additivity by allowing g_T to be sub-additive with respect to T up to an overall constant factor.

Proof. (Proposition 1.2.13) We prove the existence of the limit and the properties (a)-(c) separately.

Convergence. We apply Kingman's ergodic theorem, Theorem 1.2.14. Fix a constant $C > 0$ for the quasimorphism q satisfying (1.2.1). Let g_T denote the function on Y given by

$$g_T := T f_T = q \circ \widetilde{\Phi}_\tau(T, -)$$

Now we verify the properties in Theorem 1.2.14. First, due to the cocycle property (1.2.11) we have

$$g_{S+T} = q \circ \widetilde{\Phi}_\tau(S + T, -) \le q \circ \widetilde{\Phi}_\tau(S, -) + q \circ \widetilde{\Phi}_\tau(T, \phi_S(-)) + C = g_S + \phi_S^* g_T + C \quad (1.2.16)$$

We can analogously show that $g_{S+T} \ge g_S + \phi_S^* g_T - C$. In particular, if $T > 0$ is a sufficiently large time with $T = n + S$ and $S \in [0, 1]$, then

$$\int_Y g_T \cdot \alpha \wedge d\alpha \ge \sum_{k=0}^{n-1} \int_Y \phi_k^* g_1 \cdot \alpha \wedge d\alpha + \int_Y \phi_n^* g_S \cdot \alpha \wedge d\alpha - CT \ge -AT \qquad (1.2.17)$$

Here A is any number larger than C and larger than the quantity

$$-\min \left\{ \int_Y g_S \cdot \alpha \wedge d\alpha \ : \ S \in [0, 1] \right\}$$

Finally, since $q \circ \widetilde{\Phi}_\tau$ is continuous on $\mathbb{R} \times Y$, it is clear that $\sup_{T \in [0,1]} |f_T|$ is continuous and bounded. In particular, it is integrable. Thus g_T satisfies the criteria in Theorem 1.2.14, and we may conclude that $\frac{g_T}{T}$ converges in L^1 and almost everywhere to a map

$$f(\alpha, q, \tau) \in L^1(Y; \mathbb{R})$$

Quasimorphisms. Let q and r be equivalent quasimorphisms, and pick $C > 0$ such that $|q - r| < C$ everywhere. Then

$$\|\frac{q \circ \widetilde{\Phi}_\tau}{T} - \frac{r \circ \widetilde{\Phi}_\tau}{T}\|_{L^1} \leq \frac{C \cdot \mathrm{vol}(Y, \alpha)}{T}$$

Taking the limit as $T \to \infty$ shows that the limiting functions $f(\alpha, q, \tau)$ and $f(\alpha, r, \tau)$ are equal.

Trivializations. Let σ and τ be two trivializations of ξ in the homotopy class $[\tau]$. Then there is a transition map $\Psi : Y \to \mathrm{Sp}(2)$ given by

$$\Psi(y) : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{with} \qquad \Psi(y) = \tau(y) \cdot \sigma(y)^{-1}$$

The linearized flows of σ and τ are related via this transition map, by the following formula.

$$\Phi_\tau(T, y) = \Psi(\phi(T, y)) \cdot \Phi_\sigma(T, y) \cdot \Psi^{-1}(y)$$

Since σ and τ are homotopic, Ψ is homotopic to a constant map. In particular, Ψ lifts to the universal cover of $\mathrm{Sp}(2)$. Thus we may write

$$\widetilde{\Phi}_\tau(T, y) = \widetilde{\Psi}(\phi(T, y)) \cdot \widetilde{\Phi}_\sigma(T, y) \cdot \widetilde{\Psi}^{-1}(y)$$

Here $\widetilde{\Psi} : Y \to \widetilde{\mathrm{Sp}}(2)$ is any lift of Ψ. The quasimorphism property of ρ now implies that

$$\|\frac{q \circ \widetilde{\Phi}_\sigma(T, y)}{T} - \frac{q \circ \widetilde{\Phi}_\tau(T, y)}{T}\|_{L^1} \leq \frac{2C + \sup |q \circ \widetilde{\Psi}| + \sup |q \circ \widetilde{\Psi}^{-1}|}{T} \cdot \mathrm{vol}(Y, \alpha)$$

Taking the limit as $T \to \infty$ shows that $f(\alpha, q, \sigma) = f(\alpha, q, \tau)$.

Contact Form. Fix a contact form α and an $\varepsilon > 0$. Since q is a quasimorphism, there exists a $C > 0$ depending only on q such that

$$|\rho \circ \widetilde{\Phi}_\tau(nT, y) - \sum_{k=0}^{n-1} \rho \circ \widetilde{\Phi}_\tau(T, \phi_T^k(y))| \leq Cn \qquad \text{for any } n, T > 0$$

We can divide by nT and rewrite this estimate in terms of f_T to see that

$$|f_{nT} - \frac{1}{n}\sum_{k=0}^{n-1} f_T \circ \phi_T^k| \leq \frac{C}{T} \qquad \text{for any } n, T > 0$$

We can then integrate over Y and take the limit as $n \to \infty$ to acquire

$$|F(\alpha) - \int_Y f_T \cdot \alpha \wedge d\alpha| = \lim_{n \to \infty} |\int_Y (f_{nT} - f_T) \cdot \alpha \wedge d\alpha| \tag{1.2.18}$$

$$= \lim_{n\to\infty} |\int_Y (f_{nT} - \frac{1}{n}\sum_{k=0}^{n-1} f_T \circ \phi_T^k) \cdot \alpha \wedge d\alpha| \le \frac{C \cdot \mathrm{vol}(Y,\alpha)}{T}$$

We use the fact that ϕ_T preserves $\alpha \wedge d\alpha$ in moving from the first to the second line above.

Next, fix a different contact form β. Let $\widetilde{\Psi}_\tau$ be the lifted linearized flow for β, and let

$$g_T : Y \to \mathbb{R} \quad \text{where} \quad g_T(y) = \frac{q \circ \widetilde{\Psi}_\tau(T,-)}{T}$$

Due to (1.2.18), we can fix a $T > 0$ such that, for all β sufficiently C^0-close to α, we have

$$|F(\alpha) - \int_Y f_T \cdot \alpha \wedge d\alpha| < \frac{\varepsilon}{3} \quad \text{and} \quad |F(\beta) - \int_Y g_T \cdot \beta \wedge d\beta| < \frac{2C\,\mathrm{vol}(Y,\alpha)}{T} < \frac{\varepsilon}{3} \tag{1.2.19}$$

Furthermore, we may bound the integrals of f_T and g_T as follows.

$$|\int_Y f_T \cdot \alpha \wedge d\alpha - \int_Y g_T \cdot \beta \wedge d\beta| \le \int_Y |f_T - g_T| \cdot \alpha \wedge d\alpha + |\int_Y f_T \cdot (\alpha \wedge d\alpha - \beta \wedge d\beta)| \tag{1.2.20}$$

$$\le \|f_T - g_T\|_{C^0(Y)} \cdot \mathrm{vol}(Y,\alpha) + \|g_T\|_{C^0(Y)} \cdot |\,\mathrm{vol}(Y,\alpha) - \mathrm{vol}(Y,\beta)|$$

We can choose β sufficiently close to α in $C^2(Y)$ so that $\widetilde{\Psi}_\tau$ is arbitrarily C^0-close to $\widetilde{\Phi}_\tau$ on $[0,T] \times Y$. Since Y is compact, the image of $\widetilde{\Phi}(T,-)$ is compact in $\widetilde{\mathrm{Sp}}(2)$ for fixed time T. Thus, since q is continuous, $g_T = q \circ \widetilde{\Psi}_\tau(T,-)$ can also be made arbitrarily C^0-close to $f_T = q \circ \widetilde{\Phi}_\tau(T,-)$. In particular, for β sufficiently C^2-close to α we have

$$\|f_T - g_T\|_{C^0(Y)} \cdot \mathrm{vol}(Y,\alpha) + \|g_T\|_{C^0(Y)} \cdot |\,\mathrm{vol}(Y,\alpha) - \mathrm{vol}(Y,\beta)| < \frac{\varepsilon}{3} \tag{1.2.21}$$

Together, (1.2.19), (1.2.20) and (1.2.21) imply that, for β sufficiently C^2-close to α, we have $|F(\alpha) - F(\beta)| < \varepsilon$. This proves continuity.

This concludes the proof of the existence and properties of $f(\alpha, q, \tau)$, and of Proposition 1.2.13. □

Proposition 1.2.13 allows us to introduce the Ruelle invariant as an integral quantity, as follows.

Definition 1.2.17 (Ruelle Invariant)**.** The *local rotation number* rot_τ of a closed contact manifold (Y,α) equipped with a (homotopy class of) trivialization τ is the following limit in L^1.

$$\mathrm{rot}_\tau : Y \to \mathbb{R} \qquad \text{given by} \qquad \mathrm{rot}_\tau := \lim_{T\to\infty} \frac{\rho \circ \widetilde{\Phi}_\tau(T,-)}{T} \tag{1.2.22}$$

Similarly, the *Ruelle invariant* $\mathrm{Ru}(Y,\alpha,\tau)$ is the integral of the local rotation number over Y, i.e.

$$\mathrm{Ru}(Y,\alpha,\tau) = \int_Y \mathrm{rot}_\tau \cdot \alpha \wedge d\alpha = \lim_{T\to\infty} \frac{1}{T} \int_Y \rho \circ \widetilde{\Phi}_\tau \cdot \alpha \wedge d\alpha \tag{1.2.23}$$

We will require an alternative expression for the Ruelle invariant in order to derive estimates later in the paper.

The Reeb flow ϕ on Y preserves the contact structure, and so lifts to a flow on the total space of the contact structure ξ. Since this flow is fiberwise linear, it descends to the (oriented) projectivization $P\xi$. A trivialization τ determines an identification $P\xi \simeq Y \times \mathbb{R}/\mathbb{Z}$, and so a flow

$$\bar{\Phi} : \mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to Y \times \mathbb{R}/\mathbb{Z} \qquad \text{generated by a vector field } \bar{R} \text{ on } Y \times \mathbb{R}/\mathbb{Z} \tag{1.2.24}$$

Let $\theta : Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the tautological projection.

Definition 1.2.18. The *rotation density* $\varrho_\tau : Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the Lie derivative

$$\varrho_\tau := \mathcal{L}_{\bar{R}}(\theta) \tag{1.2.25}$$

Lemma 1.2.19. *The Ruelle invariant* $\mathrm{Ru}(Y, \alpha, \tau)$ *is written using the rotation density* ϱ_τ *as*

$$\mathrm{Ru}(Y, \alpha, \tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\int_Y \bar{\Phi}_t^* \varrho_\tau(-, s) \cdot \alpha \wedge d\alpha \Big) dt \qquad \textit{for any fixed } s \in \mathbb{R}/\mathbb{Z}$$

Proof. By comparing Definition 1.2.5 with the formula (1.2.24), one may verify that

$$\sigma_s \circ \Phi_\tau(T, y) \quad \text{and} \quad \theta \circ \bar{\Phi}(T, y, s) - s \quad \text{are equal in } \mathbb{R}/\mathbb{Z}$$

Therefore, these formulas define a single map $\mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, admitting a unique lift to a map $F : \mathbb{R} \times Y \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ that vanishes on $0 \times Y \times \mathbb{R}/\mathbb{Z}$. The first formula implies that

$$F(T, y, s) = \rho_s \circ \widetilde{\Phi}_\tau(T, y) \tag{1.2.26}$$

On the other hand, let t be the $\mathbb{R}$-variable of F and $\theta \circ \bar{\Phi}$. Then the t-derivative of F is

$$\frac{dF}{dt}|_T = \frac{d}{dt}(\theta \circ \bar{\Phi})|_T = \bar{\Phi}_T^*(\mathcal{L}_{\bar{R}}(\theta)) = \bar{\Phi}_T^* \varrho_\tau$$

Integrating this identity and combining it with (1.2.26), we acquire the formula

$$\rho_s \circ \widetilde{\Phi}_\tau(T, y) = F(T, y, s) = \int_0^T \bar{\Phi}_t^* \varrho_\tau(y, s) \cdot dt \tag{1.2.27}$$

Now, since ρ_s and ρ are equivalent by Lemma 1.2.6, we can apply Proposition 1.2.13(a) to see that

$$\mathrm{Ru}(Y, \alpha, \tau) = \lim_{T\to\infty} \int_Y \frac{\rho_s \circ \widetilde{\Phi}_\tau(T, -)}{T} \cdot \alpha \wedge d\alpha \tag{1.2.28}$$

We then apply (1.2.27) and Fubini's theorem to see that the righthand side is given by

$$\lim_{T\to\infty} \frac{1}{T} \int_Y \Big(\int_0^T \bar{\Phi}_t^* \varrho_\tau(-, s) dt \Big) \alpha \wedge d\alpha = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\int_Y \bar{\Phi}_t^* \varrho_\tau(-, s) \cdot \alpha \wedge d\alpha \Big) dt \tag{1.2.29}$$

This concludes the proof. □

1.3 Bounding The Ruelle Invariant

Let $X \subset \mathbb{R}^4$ be a convex, star-shaped domain with smooth contact boundary (Y, λ). In this section, we derive the following estimate for the Ruelle ratio.

Proposition 1.3.1. *There exist positive constants c and C independent of Y such that*

$$c \leq \mathrm{ru}(Y, \lambda) \cdot \mathrm{sys}(Y, \lambda)^{1/2} \leq C$$

The proof follows the outline discussed in the introduction.

We begin (§1.3.1) with a review of the geometry of standard ellipsoids $E(a, b)$ in $\mathbb{C}^4$, including a variant of John's theorem (Corollary 1.3.6). We then present the key curvature-rotation formula (§1.3.2) and use it to bound the Ruelle invariant between two curvature integrals (Lemma 1.3.11). We then prove several bounds for one of these curvature integrals in terms of diameter, area and total mean curvature (§1.3.3). We collect this analysis together in the final proof (§1.3.4).

Notation 1.3.2. We will require the following notation throughout this section.

(a) g is the standard metric on $\mathbb{R}^4$ with connection ∇, and $\mathrm{dvol}_g = \frac{1}{2}\omega^2$ is the corresponding volume form. We also use $\langle u, v\rangle$ to denote the inner product of two vectors $u, v \in \mathbb{R}^4$.

(b) ν is the outward normal vector field to Y and ν^* is the dual 1-form with respect to g.

(c) σ is the restriction of g to Y and dvol_σ is the corresponding metric volume form. Furthermore, $\mathrm{area}(Y)$ denotes the surface area of X, i.e. the volume $\mathrm{vol}_\sigma(Y)$ of Y with respect to σ. Note that $\lambda \wedge d\lambda$ and dvol_σ are related (via the Liouville vector field Z of $\mathbb{R}^4$) by

$$\lambda \wedge d\lambda = \iota_Z(\frac{\omega^2}{2}|_Y) = \iota_Z(\mathrm{dvol}_g|_Y) = \iota_Z(\nu^* \wedge \mathrm{dvol}_\sigma) = \langle Z, \nu\rangle \, \mathrm{dvol}_\sigma \tag{1.3.1}$$

(d) S is the second fundamental form of Y, i.e. the bilinear form given on any $u, w \in TY$ by

$$S(u, w) := \langle \nabla_u \nu, w\rangle$$

(e) H is the mean curvature of Y. It is given by

$$H := \frac{1}{3}\,\mathrm{trace}\, S$$

Note that, in this section, we will slightly abuse notation and use λ to denote both the Liouville form $\iota_Z\omega$ and the contact form on Y induced by restriction.

1.3.1 Standard Ellipsoids

Recall that a *standard ellipsoid* $E(a_1, \dots, a_n) \subset \mathbb{C}^n$ with parameters $a_i > 0$ for $i = 1, \dots, n$ is defined as follows.

$$E(a_1, \dots, a_n) := \Big\{ z = (z_i) \in \mathbb{C}^n \ : \ \sum_i \frac{\pi |z_i|^2}{a_i} \le 1 \Big\} \tag{1.3.2}$$

For example, $E(a) \subset \mathbb{C}$ is the disk of area a, and $E(a, \dots, a) \subset \mathbb{C}^n$ is the ball of radius $(a/\pi)^{1/2}$.

We beginn this section with a discussion of the Riemannian and symplectic geometry of standard ellipsoids in $\mathbb{C}^2$. All of the relevant geometric quantities for this section can be computed explicitly in this setting. Let us record the outcome of these calculations.

Lemma 1.3.3 (Ellipsoid Quantities)**.** *Let $E = E(a, b)$ be a standard ellipsoid with $0 < a < b$. Then*

(a) The diameter, surface area and volume of E are given by

$$\operatorname{diam}(E) = \frac{2}{\pi^{1/2}} \cdot b^{1/2} \qquad \operatorname{area}(\partial E) = \frac{4\pi^{1/2}}{3} \cdot \frac{b^2 a^{1/2} - b^{1/2} a^2}{b - a} \qquad \operatorname{vol}(E) = \frac{ab}{2}$$

(b) The total mean curvature of ∂E (i.e. the integral of the mean curvature over ∂E) is given by

$$\int_{\partial E} H \cdot \mathrm{dvol}_\sigma = \frac{2\pi}{3} \cdot \left(b + a + \frac{ab}{b - a} \cdot \log(b/a)\right)$$

(c) The minimum action of a closed orbit on ∂E and the systolic ratio of ∂E are given by

$$c(\partial E) = a \qquad \operatorname{sys}(\partial E) = \frac{a}{b}$$

(d) The Ruelle invariant of ∂E is given by

$$\operatorname{Ru}(\partial E) = a + b$$

Proof. The Ruelle invariant is computed in [69, Lem 2.1 and 2.2], while the minimum period of a closed orbit is computed in [47, §2.1]. The diameter is immediate from (1.3.2). Thus, we will calculate the volume, surface area and total mean curvature.

First assume that $E(a, b) = E(1, b)$ with $b \ge 1$. Let $z_j = x_j + iy_j$ be the standard coordinates on $\mathbb{C}^2 \simeq \mathbb{R}^4$. We will do all of our calculations in the following radial or toric coordinates.

$$r_i = |z_i| \qquad \theta_i = \arg(z_i) \qquad \mu_i = \pi r_i^2$$

The differential $d\mu_i$ and vector-field ∂_{μ_i} are given by

$$d\mu_i = 2\pi r_i dr_i \qquad \text{and} \qquad \partial_{\mu_i} = \frac{1}{2\pi r_i}\partial_{r_i}$$

In the $(\mu_1, \theta_1, \mu_2, \theta_2)$-coordinates, the standard metric g on $\mathbb{C}^2$ is given by

$$\sum_i dr_i^2 + r_i^2 d\theta_i = \sum_i \frac{1}{4\pi\mu_i} d\mu_i^2 + \frac{\mu_i}{\pi} d\theta_i^2$$

The ellipsoid $E(1,b)$ can be described as the sub-level set $F^{-1}(-\infty, 1]$ of the map

$$F : \mathbb{C}^2 \to \mathbb{R} \quad F(z_1, z_2) := \pi|z_1|^2 + \frac{\pi|z_2|^2}{b} = \mu_1 + \frac{\mu_2}{b}$$

The gradient vector-field ∇F of F with respect to g is given by

$$\nabla F = 2\pi r_1 \partial_{r_1} + \frac{2\pi r_2}{b}\partial_{r_2} = 4\pi(\mu_1\partial_{\mu_1} + \frac{\mu_2}{b}\partial_{\mu_2})$$

Note that the normal vector-field $\nu = \nabla F/|\nabla F|$ to $\partial E(1,b)$ can be calculated via this formula. Finally, the complement U of $(\mathbb{C} \times 0) \cup (0 \times \mathbb{C})$ in $E(1,b)$ admits the following parametrization.

$$\phi : (0, \infty) \times S^1 \times S^1 \to \mathbb{C}^4 \qquad \phi(\mu_1, \theta_1, \theta_2) = (\mu_1, \theta_1, b(1-\mu_1), \theta_2)$$

Now we calculate the desired quantities for $E(1,b)$. The metric volume form dvol_g is given by

$$\operatorname{dvol}_g = d(\frac{1}{2\pi}\mu_1) \wedge d\theta_1 \wedge d(\frac{1}{2\pi}\mu_2) \wedge d\theta_2$$

Therefore, the volume may be calculated as the integral

$$\operatorname{vol}(E(1,b)) = \int_{E(1,b)} \frac{1}{4\pi^2} \cdot d\mu_1 \wedge d\theta_1 \wedge d\mu_2 \wedge d\theta_2 = \int_0^1 \int_0^{b(1-\mu_1)} d\mu_1 \wedge d\mu_2 = \frac{b}{2}$$

The area form $\operatorname{dvol}_\sigma$ on $\partial E(1,b)$ is given by $\iota_\nu \operatorname{dvol}_g$, which is simply

$$\operatorname{dvol}_\sigma = \iota_\nu \operatorname{dvol}_g = \frac{1}{|\nabla F|} \cdot \iota_{\nabla F} \operatorname{dvol}_g$$

$$= \frac{1}{\sqrt{4\pi(\mu_1 + \mu_2/b)}} \cdot (\frac{4\pi\mu_1}{4\pi^2} \cdot d\theta_1 \wedge d\mu_2 \wedge d\theta_2 + \frac{4\pi\mu_2}{4\pi^2 b} \cdot d\mu_1 \wedge d\theta_1 \wedge d\theta_2)$$

The pullback of $\operatorname{dvol}_\sigma$ via the map ϕ is given by

$$\phi^* \operatorname{dvol}_\sigma = \frac{1}{2\pi^{3/2}}(\mu_1 + \frac{b(1-\mu_1)}{b^2})^{-1/2} \cdot (\mu_1 \cdot d\theta_1 \wedge d(b(1-\mu_1)) \wedge d\theta_2 + \frac{b}{b}(1-\mu_1) \cdot d\mu_1 \wedge d\theta_1 \wedge d\theta_2)$$

$$= \frac{b^{1/2}}{2\pi^{3/2}} \cdot (1 + (b-1)\mu_1)^{1/2} \cdot d\mu_1 \wedge d\theta_1 \wedge d\theta_2$$

Computing the surface area as the integral of $\phi^* \operatorname{dvol}_\sigma$, we have

$$\operatorname{area}(\partial E(1,b)) = \frac{b^{1/2}}{2\pi^{3/2}} \cdot \int_0^1 \int_0^{2\pi} \int_0^{2\pi} (1 + (b-1)\mu_1)^{1/2} \cdot d\theta_1 \wedge d\theta_2 \wedge d\mu_1$$

$$= \frac{b^{1/2}}{2\pi^{3/2}} \cdot 4\pi^2 \cdot \int_0^1 (1 + (b-1)\mu_1)^{1/2} d\mu_1 = (4\pi b)^{1/2} \cdot \frac{2}{3(b-1)} \cdot (1 + (b-1)\mu_1)^{3/2}|_0^1$$

$$= \frac{4\pi^{1/2}}{3} \cdot \frac{b^2 - b^{1/2}}{b-1}$$

Finally, the mean curvature H is given by

$$H = \frac{1}{3|\nabla F|^3} \cdot (|\nabla F|^2 \cdot \operatorname{tr}(\operatorname{Hess}_F) - \operatorname{Hess}_F(\nabla F, \nabla F)) =$$

$$\frac{4\pi(\mu_1 + \frac{\mu_2}{b^2}) \cdot 4\pi(1 + \frac{1}{b}) - 8\pi^2(\mu_1 + \frac{\mu_2}{b^3})}{3 \cdot (4\pi)^{3/2} \cdot (\mu_1 + \frac{\mu_2}{b^2})^{3/2}} = \frac{\sqrt{\pi}}{3} \cdot \frac{(1 + \frac{2}{b}) \cdot \mu_1 + (\frac{2}{b^2} + \frac{1}{b^3})\mu_2}{(\mu_1 + \frac{\mu_2}{b^2})^{3/2}}$$

The pullback of H by ϕ is given by

$$\phi^* H = \frac{\sqrt{\pi}}{3} \cdot \frac{(1 + \frac{2}{b}) \cdot \mu_1 + (\frac{2}{b^2} + \frac{1}{b^3}) \cdot b(1-\mu_1)}{(\mu_1 + \frac{b(1-\mu_1)}{b^2})^{3/2}}$$

$$= \frac{\sqrt{\pi}}{3} \cdot \frac{\frac{2b+1}{b^2} + \frac{b^2-1}{b^2}\mu_1}{(1 + (1 - \frac{1}{b})\mu_1)^{3/2}} = \frac{\pi^{1/2}}{3b^{1/2}} \cdot \frac{(2b+1) + (b^2-1)\mu_1}{(1 + (b-1)\mu_1)^{3/2}}$$

Computing the mean curvature as the integral of $\phi^*(H \cdot \operatorname{dvol}_\sigma)$, we have

$$\int_{\partial E(1,b)} H \operatorname{dvol}_\sigma = \frac{1}{6\pi} \cdot \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{(2b+1) + (b^2-1)\mu_1}{1 + (b-1)\mu_1} \cdot d\theta_1 \wedge d\theta_2 \wedge d\mu_1$$

$$= \frac{2\pi}{3} \cdot \int_0^1 \frac{(2b+1) + (b^2-1)\mu_1}{1 + (b-1)\mu_1} \cdot d\mu_1 = \frac{2\pi}{3}(b + 1 + \frac{b}{b-1} \cdot \log(b))$$

To deduce the general case of the computation from this special case note that, if U is any smooth domain, then

$$\operatorname{vol}(\lambda \cdot U) = \lambda^4 \cdot \operatorname{vol}(U) \quad \operatorname{area}(\lambda \cdot U) = \lambda^3 \cdot \operatorname{area}(U) \quad \int_{\lambda \cdot \partial U} H \cdot \operatorname{dvol}_\sigma = \lambda^2 \cdot \int_{\partial U} H \cdot \operatorname{dvol}_\sigma \quad (1.3.3)$$

Any ellipsoid $E(a,b)$ can be scale so to an ellipsoid with $a = 1$, since

$$\lambda \cdot E(a,b) = E(\lambda^2 a, \lambda^2 b) \quad \text{and thus} \quad E(a,b) = a^{1/2} \cdot E(1, b/a)$$

The general case now follows from the special case and the scaling properties (1.3.3). □

Any convex boundary in $\mathbb{R}^{2n}$ can be sandwiched between a standard ellipsoid and a scaling of that ellipsoid by a factor of $2n$, after the application of an affine symplectomorphism. To see this, first recall the following well-known result of John.

Theorem 1.3.4 (John Ellipsoid). *[78] Let $K \subset \mathbb{R}^n$ be a convex domain. Then there exists a unique ellipsoid E of maximal volume in K. Furthermore, if $c \in X$ is the center of E then*

$$E \subset K \subset c + n(E - c)$$

Any ellipsoid E is carried to a standard ellipsoid $E(a, b)$ by some affine symplectomorphism T. Furthermore, note that we have the following elementary result, which can be demonstrated using a Moser argument.

Lemma 1.3.5. *Let $\phi : (Y, \lambda) \to (Y', \lambda')$ be a diffeomorphism such that $\phi^*\lambda' = \lambda + df$. Then ϕ is isotopic to a strict contactomorphism.*

Since $\mathbb{R}^{2n}$ is contractible, $T^*\lambda = \lambda + df$ automatically on $\mathbb{R}^{2n}$. Thus, T carries any star-shaped hypersurface $Y = \partial X$ to a strictly contactomorphic $T(Y)$ by Lemma 1.3.5, and we conclude the following result.

Corollary 1.3.6. *Let $X \subset \mathbb{R}^{2n}$ be a convex star-shaped domain with boundary Y. Then Y is strictly contactomorphic to the boundary ∂K of a convex domain K with $E(a_1, \dots, a_n) \subset K \subset 4 \cdot E(a_1, \dots, a_n)$.*

When a convex domain in $\mathbb{R}^4$ is squeezed between an ellipsoid and its scaling, we can estimate many important geometric quantities of X in terms of the ellipsoid itself.

Lemma 1.3.7. *Let $X \subset \mathbb{R}^4$ be a convex domain with smooth boundary Y such that*

$$E(a, b) \subset X \subset c \cdot E(a, b) \qquad \text{for some } b \geq a > 0 \text{ and } c \geq 0 \tag{1.3.4}$$

Then there is a constant $C > 0$ dependent only on c such that

$$b^{1/2} \leq \operatorname{diam}(X) \leq C \cdot b^{1/2} \qquad ba^{1/2} \leq \operatorname{area}(Y) \leq C \cdot ba^{1/2} \tag{1.3.5}$$

$$b \leq \int_Y H \cdot \operatorname{dvol}_\sigma \leq C \cdot b \qquad \frac{ab}{2} \leq \operatorname{vol}(X) \leq C \cdot ab \tag{1.3.6}$$

$$a \leq c(X) \leq C \cdot a \qquad C^{-1} \cdot \frac{a}{b} \leq \operatorname{sys}(Y) \leq C \cdot \frac{a}{b} \tag{1.3.7}$$

Remark 1.3.8. The optimal constants in the estimates (1.3.5)-(1.3.7) are not important to the arguments below. They could be explicitly computed in the following proof.

Proof. First, note that $c \cdot E(a,b)$ is also a standard ellipsoid. More precisely, we know that

$$c \cdot E(a,b) = E(c^2 \cdot a, c^2 \cdot b)$$

We now derive the desired estimates from Lemma 1.3.3 and the monotonicity of the relevant quantities under inclusion of convex domains.

The diameter $\operatorname{diam}(X)$ and volume $\operatorname{vol}(X)$ are monotonic with respect to inclusion of arbitrary open subsets, and so from Lemma 1.3.3(a) we acquire

$$b^{1/2} \le \operatorname{diam}(X) \le \frac{2c}{\pi^{1/2}} \cdot b^{1/2} \qquad \text{and} \qquad \frac{ab}{2} \le \operatorname{vol}(X) \le \frac{c^4}{2} \cdot ab$$

The surface area and total mean curvature are monotonic with respect to inclusion of convex domains, since

$$\int_Y H \, \mathrm{dvol}_\sigma = 4 \cdot V_2(X) \qquad \text{and} \qquad \operatorname{area}(Y) = 4 \cdot V_3(X)$$

Here $V_i(X)$ is the *ith cross-sectional measure* [9, §19.3], which is monotonic with respect to inclusions of convex domains by [9, p.138, Equation 13]. Furthermore, when $0 < a < b$ (and in the limit as $b \to a$), one may verify that

$$ba^{1/2} \le \frac{b^2 a^{1/2} - b^{1/2} a^2}{b-a} \le \frac{3}{2} \cdot ba^{1/2} \qquad \text{and} \qquad b \le b + a + \frac{ab}{b-a} \cdot \log(b/a) \le 3b \tag{1.3.8}$$

Thus, by applying the monotonicity property, (1.3.8) and Lemma 1.3.3(a)-(b), we have

$$\frac{4\pi^{1/2}}{3} \cdot ba^{1/2} \le \operatorname{area}(Y) \le \frac{4\pi^{1/2}}{3} c^3 \cdot (\frac{3}{2} ba^{1/2}) \qquad \text{and} \qquad \frac{2\pi}{3} \cdot b \le \int_Y H \cdot \mathrm{dvol}_\sigma \le \frac{2\pi}{3} c^2 \cdot 3b$$

Finally, the minimum orbit length $c(X)$ coincides with the 1st Hofer-Zehnder capacity $c_1^{HZ}(X)$ on convex domains, and is thus monotonic with respect to symplectic embeddings. Thus by Lemma 1.3.3(a) and (c), we have

$$a \le c(X) \le c^2 \cdot a \qquad \text{and} \qquad c^{-4} \cdot \frac{a}{b} \le \frac{c(X)^2}{2 \operatorname{vol}(X)} = \operatorname{sys}(Y) \le c^4 \cdot \frac{a}{b}$$

This concludes the proof, after choosing C larger than the constants appearing above. □

1.3.2 Curvature-Rotation Formula

Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}^1$ via

$$\mathbb{R}^4 \ni (x_1, y_1, x_2, y_2) \mapsto x_1 + y_1 I + x_2 J + y_2 K \in \mathbb{H}^1$$

This equips $\mathbb{R}^4$ with a triple of complex structures.

$$I : T\mathbb{R}^4 \to T\mathbb{R}^4 \qquad J : T\mathbb{R}^4 \to T\mathbb{R}^4 \qquad K : T\mathbb{R}^4 \to T\mathbb{R}^4$$

The Reeb vector-field R of the contact form on a star-shaped hypersurface Y is parallel to I applied to the normal vector-field ν to Y. Precisely, we have

$$R = \frac{I\nu}{\langle Z, \nu\rangle} \tag{1.3.9}$$

We can utilize these structures to formulate an explicit representative of the standard homotopy class of trivialization $\tau : \xi \simeq \mathbb{R}^2$ on the contact structure ξ on the boundary Y of the convex star-shaped domain X (or more generally, on any star-shaped boundary).

Definition 1.3.9. The *quaternion trivialization* $\tau : \xi \simeq Y \times \mathbb{C}$ is the symplectic trivialization given by

$$\tau : \xi \xrightarrow{\pi} Q \xrightarrow{q^{-1}} Y \times \mathbb{C}$$

Here $Q \subset TY$ is the symplectic sub-bundle $\operatorname{span}(J\nu, K\nu)$, $\pi : \xi \to Q$ is the projection map from ξ to Q along the Reeb direction, and $q : Y \times \mathbb{C} \to Q$ is the bundle map given on $z = a + ib$ by

$$q_p(z) := z \cdot J\nu_p = (a + Ib) \cdot J\nu_p = aJ\nu_p + bK\nu_p \tag{1.3.10}$$

The key property of the quaternion trivialization is the following relation of the rotation density (see Definition 1.2.18) to extrinsic curvature, originally due to Ragazzo-Salomão (c.f. [90]).

Proposition 1.3.10 (Curvature-Rotation)**.** *[12, Prop 4.7] Let τ be the quaternion trivialization on the contact structure ξ of $Y \subset \mathbb{R}^4$. Then*

$$\varrho_\tau(y, s) = \frac{1}{2\pi \cdot \langle Z_y, \nu_y\rangle}(S(I\nu_y, I\nu_y) + S(e^{2\pi i s} \cdot J\nu_y, e^{2\pi i s} \cdot J\nu_y)) \tag{1.3.11}$$

Note that this result holds for any star-shaped boundary, not only convex ones.

As an easy consequence of (1.3.11), we have the following bound on the Ruelle invariant of Y.

Lemma 1.3.11. *The Ruelle invariant* $\operatorname{Ru}(Y)$ *is bounded by the following curvature integrals.*

$$\frac{1}{2\pi} \cdot \int_Y S(I\nu, I\nu)\, \mathrm{dvol}_\sigma \le \operatorname{Ru}(Y) \le \frac{3}{2\pi} \cdot \int_Y H\, \mathrm{dvol}_\sigma \tag{1.3.12}$$

Proof. By Lemma 1.2.19, we have the following integral formula for the Ruelle invariant.

$$\operatorname{Ru}(Y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_Y [\bar{\Phi}_t^* \varrho_\tau](-, s) \cdot \lambda \wedge d\lambda \right) dt \tag{1.3.13}$$

By the curvature-rotation formula in Proposition 1.3.10, we can write the integrand as

$$[\bar{\Phi}_t^* \varrho_\tau](-, s) = \bar{\Phi}_t^* \Big(\frac{1}{2\pi \cdot \langle Z, \nu \rangle} (S(I\nu, I\nu) + S(e^{2\pi i s} \cdot J\nu, e^{2\pi i s} \cdot J\nu)) \Big) \tag{1.3.14}$$

To bound the righthand side of (1.3.14), note that $I\nu, e^{2\pi i s} \cdot J\nu$ and $e^{2\pi i s} \cdot K\nu$ form an orthonormal basis of TY with respect to the restricted metric $g|_Y$, so that

$$S(I\nu, I\nu) + S(e^{2\pi i s} \cdot J\nu, e^{2\pi i s} \cdot J\nu) + S(e^{2\pi i s} \cdot K\nu, e^{2\pi i s} \cdot K\nu) = \text{trace}(S) = 3H$$

Furthermore, since Y is convex, the second fundamental form S is positive semi-definite. Therefore by (1.3.14), we have the following lower and upper bound.

$$\bar{\Phi}_t^* \Big(\frac{S(I\nu, I\nu)}{\langle Z, \nu \rangle} \Big) \le [\bar{\Phi}_t^* \varrho_\tau](-, s) \le 3 \cdot \bar{\Phi}_t^* \Big(\frac{H}{\langle Z, \nu \rangle} \Big) \tag{1.3.15}$$

It is key here that the lower and upper bounds in (1.3.15) are independent of s. To simplify the two sides of (1.3.15), let $F : Y \times S^1 \to \mathbb{R}$ be any map pulled back from a map $F : Y \to \mathbb{R}$. Since the flow $\bar{\Phi}_t$ on $Y \times S^1$ lifts the Reeb flow ϕ_t on Y, and ϕ_t preserves λ, we have

$$\bar{\Phi}_t^* \Big(\frac{F}{\langle Z, \nu \rangle} \Big) \cdot \lambda \wedge d\lambda = \phi_t^* \Big(\frac{F}{\langle Z, \nu \rangle} \Big) \cdot \lambda \wedge d\lambda = \phi_t^* \Big(F \cdot \frac{\lambda \wedge d\lambda}{\langle Z, \nu \rangle} \Big) = \phi_t^* \Big(F \cdot \text{dvol}_\sigma \Big)$$

Since the integral of $\phi_t^*(F \cdot \text{dvol}_\sigma)$ over Y is independent of t, we have

$$\frac{1}{T} \int_0^T \Big(\int_Y \bar{\Phi}_t^* \Big(\frac{F}{\langle Z, \nu \rangle} \Big) \cdot \lambda \wedge d\lambda \Big) dt = \frac{1}{T} \int_0^T \Big(\int_Y F \cdot \text{dvol}_\sigma \Big) dt = \int_Y F \cdot \text{dvol}_\sigma \tag{1.3.16}$$

By plugging in the estimate (1.3.15) to the integral formula (1.3.13) and applying (1.3.16) to the functions $S(I\nu, I\nu)$ and H on Y, we acquire the desired bound (1.3.12). □

1.3.3 Bounding Curvature Integrals

We now further simplify the lower bound of the Ruelle invariant in Lemma 1.3.11 by estimating (from below) the integral

$$\int_Y S(I\nu, I\nu) \cdot \text{dvol}_\sigma$$

using the geometric quantities (e.g. area and diameter) appearing in §1.3.1. This will help us to leverage the sandwich estimates in Lemma 1.3.7 in the proof of the Ruelle invariant bound in §1.3.4.

Recall that $X \subset \mathbb{R}^4$ denotes a convex domain with smooth boundary Y. Let $\psi : \mathbb{R} \times Y \to Y$ be the flow by $I\nu$. Let S_T and H_T denote the time-averaged versions of $S(I\nu, I\nu)$ and H, respectively.

$$S_T := \frac{1}{T} \int_0^T S(I\nu, I\nu) \circ \psi_t dt \qquad H_T := \frac{1}{T} \int_0^T H \circ \psi_t dt \tag{1.3.17}$$

We will also need to consider a time-averaged acceleration function A_T on Y. Namely, let $\gamma : \mathbb{R} \to Y$ be a trajectory of $I\nu$ with $\gamma(0) = x$. Then we define

$$A_T := \frac{1}{T}\int_0^T |\nabla_{I\nu} I\nu| \circ \psi_t dt \qquad \text{or equivalently} \qquad A_T(x) = \frac{1}{T}\int_0^T |\ddot{\gamma}| dt \tag{1.3.18}$$

The first ingredient to the bounds in this section is the following estimate relating these three time-averaged functions.

Lemma 1.3.12. *For any $T > 0$, the functions A_T, H_T and S_T satisfy $A_T^2 \le 3 \cdot H_T \cdot S_T$ pointwise.*

Proof. In fact, the non-time-averaged version of this estimate holds. We will now show that

$$|\nabla_{I\nu} I\nu|^2 \le 3H \cdot S(I\nu, I\nu) \tag{1.3.19}$$

To start, we need a formula for $\nabla_{I\nu} I\nu$ in terms of the second fundamental form, as follows.

$$\nabla_{I\nu} I\nu = \langle \nu, \nabla_{I\nu} I\nu\rangle \nu + \langle I\nu, \nabla_{I\nu} I\nu\rangle I\nu + \langle J\nu, \nabla_{I\nu} I\nu\rangle J\nu + \langle K\nu, \nabla_{I\nu} I\nu\rangle K\nu$$
$$= -\langle I\nu, \nabla_{I\nu}\nu\rangle \nu - \langle I^2\nu, \nabla_{I\nu}\nu\rangle I\nu - \langle IJ\nu, \nabla_{I\nu}\nu\rangle J\nu - \langle IK\nu, \nabla_{I\nu}\nu\rangle K\nu$$

Applying the quaternionic relations $I^2 = -1$, $IJ = K$ and $IK = -J$, we can rewrite this as

$$-\langle I\nu, \nabla_{I\nu}\nu\rangle \nu + \langle \nu, \nabla_{I\nu}\nu\rangle I\nu - \langle K\nu, \nabla_{I\nu}\nu\rangle J\nu + \langle J\nu, \nabla_{I\nu}\nu\rangle K\nu$$

Finally, applying the definition of the second fundamental form we find that

$$\nabla_{I\nu} I\nu = -S(I\nu, I\nu)\nu - S(I\nu, K\nu)J\nu + S(I\nu, J\nu)K\nu$$

To estimate the righthand side, we note that $S(u,v)^2 \le S(u,u)S(v,v)$ for any vectorfields u and v by Cauchy-Schwarz, since S is positive semi-definite. Thus we have

$$|\nabla_{I\nu} I\nu|^2 \le S(I\nu, I\nu)^2 + S(I\nu, I\nu)S(J\nu, J\nu) + S(I\nu, I\nu)S(K\nu, K\nu) = 3H \cdot S(I\nu, I\nu)$$

This proves (1.3.19) and the desired estimate follows immediately by Cauchy-Schwarz.

$$A_T^2 = \Big(\frac{1}{T}\int_0^T |\nabla_{I\nu} I\nu| \circ \psi_t dt\Big)^2 \le 3 \cdot \frac{1}{T}\int_Y H \circ \psi_t dt \cdot \frac{1}{T}\int_Y S(I\nu, I\nu) \circ \psi_t dt = 3H_T \cdot S_T \tag{1.3.20}$$

This concludes the proof of the lemma. □

As a consequence, we get the following estimate for the curvature integral of interest in terms of area, total mean curvature and the time-averaged acceleration A_T.

Lemma 1.3.13. *Let $\Sigma \subset Y$ be an open subset of Y and let $T > 0$. Then*

$$\int_Y S(I\nu, I\nu) \cdot \mathrm{dvol}_\sigma \ge \frac{\mathrm{area}(\Sigma)^2}{3 \cdot \int_Y H \, \mathrm{dvol}_\sigma} \cdot \min{}_\Sigma(A_T)^2 \tag{1.3.21}$$

Proof. We first note that $I\nu$ preserves the volume form dvol_σ, since

$$\mathcal{L}_{I\nu}(\mathrm{dvol}_\sigma) = d\iota_{I\nu}\,\mathrm{dvol}_\sigma = d\iota_R(\lambda \wedge d\lambda) = d^2\lambda = 0$$

Here R is the Reeb vector-field on Y, and the above equalities follow from (1.3.9) and (1.3.1). Thus, time-averaging leaves the integral over Y unchanged.

$$\int_Y H_T\,\mathrm{dvol}_\sigma = \int_Y H\,\mathrm{dvol}_\sigma \qquad \text{and} \qquad \int_Y S_T\,\mathrm{dvol}_\sigma = \int_Y S(I\nu, I\nu)\,\mathrm{dvol}_\sigma$$

We can thus integrate the estimate $A_T^2 \le 3H_T \cdot S_T$ to see that

$$\min(A_T)^2 \cdot \mathrm{area}(\Sigma)^2 \le \Big(\int_\Sigma A_T \cdot \mathrm{dvol}_\sigma\Big)^2 \le \Big(\sqrt{3} \cdot \int_\Sigma H_T^{1/2} \cdot S_T^{1/2} \cdot \mathrm{dvol}_\sigma\Big)^2$$

$$\le 3 \cdot \int_\Sigma H_T \cdot \mathrm{dvol}_\sigma \cdot \int_\Sigma S_T \cdot \mathrm{dvol}_\sigma \le 3 \cdot \int_Y H \cdot \mathrm{dvol}_\sigma \cdot \int_Y S(I\nu, I\nu) \cdot \mathrm{dvol}_\sigma$$

After some rearrangement, this is the desired estimate. □

Every quantity on the righthand side of (1.3.21) can be controlled using the estimates in Lemma 1.3.7, with the exception of the term involving the time-averaged acceleration A_T. However, we can bound A_T in terms of $\mathrm{diam}(X)^{-1}$, using the following general fact about curves of unit speed.

Lemma 1.3.14. *Let $\gamma : [0, \infty) \to Y$ be a curve with $|\dot\gamma| = 1$ and let C satisfy $0 < C < 1$. Then*

$$\frac{1}{T}\int_0^T |\ddot\gamma| dt \ge \frac{C}{\mathrm{diam}(X)} \qquad \textit{for all } T \gg 0$$

Proof. Let T satisfy $T > CT + 2 \cdot \mathrm{diam}(Y)$. Then by Cauchy-Schwarz, we have

$$\mathrm{diam}(X)\int_0^T |\ddot\gamma| dt \ge \int_0^T |\gamma| \cdot |\ddot\gamma| dt \ge \int_0^T |\langle \ddot\gamma, \gamma\rangle| dt \ge \Big|\int_0^T \langle \ddot\gamma, \gamma\rangle dt\Big| \tag{1.3.22}$$

On the other hand, by integration by parts we acquire

$$\Big|\int_0^T \langle \ddot\gamma, \gamma\rangle dt\Big| \ge \Big|\int_0^T |\dot\gamma|^2 dt - \langle \gamma, \dot\gamma\rangle|_0^T\Big| \ge T - 2\,\mathrm{diam}(X) \ge CT \tag{1.3.23}$$

Combining the estimates (1.3.22) and (1.3.23) yields the claimed bound. □

In particular, Lemma 1.3.14 implies that $A_T \ge C \cdot \mathrm{diam}(X)^{-1}$ for all $C < 1$ and sufficiently large T. Combining this with Lemma 1.3.13 and taking $C \to 1$, we acquire the following corollary.

Corollary 1.3.15. *Let $X \subset \mathbb{R}^4$ be a convex star-shaped domain with boundary Y. Then*

$$\int_Y S(I\nu, I\nu)\, \mathrm{dvol}_\sigma \geq \frac{\mathrm{area}(Y)^2}{3 \cdot \mathrm{diam}(X)^2 \cdot \int_Y H\, \mathrm{dvol}_\sigma} \tag{1.3.24}$$

At this point, we can already apply Lemma 1.3.7 to derive a uniform lower bound for $\mathrm{ru}(Y) \cdot \mathrm{sys}(Y)^{-1/2}$. However, this inequality does not have the desired exponent for sys. In order to fix this, we must derive a different estimate similar to Corollary 1.3.15 when $\mathrm{sys}(Y)$ is near 0. This is the objective of the rest of this part.

We will also need a less crude estimate on the time-averaged acceleration that uses the geometry of vector-field $I\nu$, but requires the hypothesis that X has small systolic ratio.

Lemma 1.3.16. *Suppose that X satisfies $E(a,b) \subset X \subset 4 \cdot E(a,b)$ and let $\Sigma \subset Y$ be the open subset*

$$\Sigma = Y \cap \mathbb{C} \times \mathrm{int}(E(b/2))$$

Then there is an $\varepsilon > 0$ and a $C > 0$ independent of a, b and X such that, if $a/b < \varepsilon$ and $T = b^{1/2}$, then

$$A_T \geq C \cdot a^{-1/2} \quad \text{on} \quad \Sigma \qquad \text{and} \qquad \mathrm{area}(\Sigma) \geq C \cdot \mathrm{area}(Y)$$

Proof. To bound A_T, the strategy is to show that the projection of $I\nu$ to the 2nd $\mathbb{C}$-factor is bounded along Σ by $(a/b)^{1/2}$. Thus, a length $T = b^{1/2}$ trajectory γ of $I\nu$ stays within a ball of diameter roughly $a^{1/2}$, and a variation of Lemma 1.3.14 implies the desired bound.

To bound $\mathrm{area}(\Sigma)$, the strategy is (essentially) to use the monotonicity of area under the inclusion $E(a,b) \subset X$ to reduce to the case of an ellipsoid. We can then use the estimates in Lemmas 1.3.3 and 1.3.7 to deduce the result.

Projection Bound. Let $\pi_j : \mathbb{R}^4 \simeq \mathbb{C}^2 \to \mathbb{C}$ denote the projections to each $\mathbb{C}$-factor for $j = 1, 2$. We begin by noting that there is an $A > 0$ independent of X, a and b such that

$$|\pi_2 \circ I\nu(x)| = |\pi_2 \circ \nu(x)| < A \cdot (a/b)^{1/2} \qquad \text{if} \qquad x \in Y \quad \text{and} \quad \pi_2(x) \in E(3b/4) \tag{1.3.25}$$

To deduce (1.3.25), assume that $x \in Y$ satisfies $\pi_2(x) \in E(3b/4)$ and that $\pi_2 \circ \nu(x) \neq 0$. Let $z \in 0 \times \partial E(b)$ be the unique vector such that $\pi_2(z - x)$ is a positive scaling of $\pi_2(\nu(x))$. Note that $z \in X$ since

$$0 \times E(b) \subset E(a,b) \subset X$$

Furthermore, since X is convex, we know that $\langle \nu(x), w - x \rangle \leq 0$ for any $w \in X$. Therefore

$$0 \geq \langle \nu(x), z - x \rangle = |\pi_2 \circ \nu(x))| \cdot |\pi_2(z - x)| + \langle \pi_1 \circ \nu(x), \pi_1(z - x) \rangle \tag{1.3.26}$$

Now note that since $\pi_2(x) \in E(3b/4)$ and $\pi_2(z) \in \partial E(b)$, we know that

$$|\pi_2(z - x)| \geq \frac{1 - (3/4)^{1/2}}{\pi^{1/2}} \cdot b^{1/2}$$

Likewise, $\pi_1(X) \subset 4 \cdot E(a)$ so that $|\pi_1(z-x)| \le 4a^{1/2}/\pi^{1/2}$. Finally, $|\pi_1 \circ \nu(x)| \le |\nu(x)| = 1$. Thus, we can conclude that

$$|\pi_2 \circ \nu(x)| \le \frac{|\pi_1 \circ \nu(x)| \cdot |\pi_1(z-x)|}{|\pi_2(z-x)|} \le \frac{4}{1-(3/4)^{1/2}} \cdot (a/b)^{1/2}$$

Acceleration Bound. Now let $T = b^{1/2}$ and let $\gamma : [0,T] \to Y$ be a trajectory of $I\nu$ with $\gamma(0) \in \Sigma$. Since $\pi_2(\gamma(0)) \in E(b/2)$, we know that there is an interval $[0,S] \subset [0,T]$ where $\pi_2 \circ \gamma([0,S]) \subset E(3b/4)$. Thus, by (1.3.25), we know that for $t \in [0,S]$ we have

$$|\pi_2(\gamma(t)-\gamma(0))| \le \int_0^t |\pi_2 \circ I\nu \circ \gamma| dt \le A \cdot (a/b)^{1/2} \cdot t \le A \cdot a^{1/2} \tag{1.3.27}$$

By picking $\varepsilon > 0$ small enough so that a/b is small, we can ensure the following inequality.

$$Aa^{1/2} \le (\frac{3b}{4\pi})^{1/2} - (\frac{b}{2\pi})^{1/2} \tag{1.3.28}$$

With this choice of ε, (1.3.27) and (1.3.28) imply that $\pi_2(\gamma(t)-\gamma(0)) \in E(3b/4)$ if $0 \le t \le T$. In fact, (1.3.27) implies that γ is inside of a ball, i.e.

$$\gamma(t) \in E(16a) \times E(\pi A^2 \cdot a) + p \subset B \cdot E(a,a) + p \quad \text{where} \quad p := 0 \times \pi_2(\gamma(0))$$

Here $B := (16 + \pi A^2)^{1/2}$. The diameter of the ball $B \cdot E(a,a)$ is $2B \cdot (a/\pi)^{1/2}$. Therefore, by applying (1.3.22) and (1.3.23) we see that

$$\frac{2Ba^{1/2}}{\pi^{1/2}} \cdot A_T(x) = \frac{\operatorname{diam}(B \cdot E(a,a))}{T} \cdot \int_0^T |\ddot{\gamma}| dt \ge 1 - \frac{2\operatorname{diam}(B \cdot E(a,a))}{T} = 1 - \frac{4B}{\pi^{1/2}} \cdot (a/b)^{1/2}$$

We now choose $C > 0$ and $\varepsilon > 0$ independent of a, b and X, such that

$$A_T(x) \ge (\frac{\pi^{1/2}}{2B} - 2 \cdot (a/b)^{1/2}) \cdot a^{-1/2} \ge Ca^{-1/2} \quad \text{if} \quad a/b \le \varepsilon$$

This proves the desired bound on time-averaged acceleration.

Area Bound. Let U denote the convex domain given by the intersection $X \cap (\mathbb{C} \times E(b/2))$. Note that we have the following inclusion.

$$E(a/2, b/2) \subset E(a,b) \cap (\mathbb{C} \times E(b/2)) \subset U$$

Furthermore, the boundary of U decomposes as follows.

$$\partial U = \Sigma \cup \Sigma' \qquad \text{where} \qquad \Sigma' := X \cap (\mathbb{C} \times \partial E(b/2))$$

Since $X \subset 4 \cdot E(a,b)$, we have $\Sigma' \subset R$ where R is the hypersurface

$$R := 4 \cdot E(a,b) \cap (\mathbb{C} \times \partial E(b/2)) = E(31a/2) \times \partial E(b/2)$$

Combining the above facts and applying the monotonicity of surface area under inclusion of convex domains, we find that

$$\text{area}(\Sigma) = \text{area}(\partial U) - \text{area}(\Sigma') \geq \text{area}(\partial E(a/2, b/2)) - \text{area}(R)$$

By Lemma 1.3.7 and direct calculation, we compute the areas of $\partial E(a/2, b/2)$ and R to be

$$\text{area}(\partial E(a/2, b/2)) \geq 2^{-3/2} \cdot ba^{1/2} \qquad \text{area}(R) = \frac{31a}{2} \cdot (2\pi b)^{1/2} = 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2} \cdot ba^{1/2}$$

Now let $B < 2^{-5/2}$ and choose $\varepsilon > 0$ small enough to that if $a/b < \varepsilon$ then

$$2^{-3/2} - 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2} > B$$

By applying this inequality and the upper bound for area in Lemma 1.3.7, we find that for some $C > 0$ independent of X, a and b and an $\varepsilon > 0$ as above, we have

$$\text{area}(\Sigma) \geq (2^{-3/2} - 31 \cdot (\pi/2)^{1/2} \cdot (a/b)^{1/2}) \cdot ba^{1/2} \geq C \cdot ba^{1/2} \geq \text{area}(Y)$$

This yields the desired area bound and concludes the proof of the lemma. □

By plugging the bounds for A_T and area(Σ) from Lemma 1.3.16 into Lemma 1.3.13, we acquire the following variation of Corollary 1.3.15.

Corollary 1.3.17. *Let X be a convex domain with smooth boundary Y, such that $E(a, b) \subset X \subset 4 \cdot E(a, b)$. Then there exists a $C > 0$ and $\varepsilon > 0$ independent of X, a and b such that*

$$\int_Y S(I\nu, I\nu) \cdot \text{dvol}_\sigma \geq C \cdot \frac{\text{area}(Y)^2}{a \cdot \int_Y H \, \text{dvol}_\sigma} \quad \textit{if} \quad a/b < \varepsilon$$

1.3.4 Proof Of Main Bound

We now combine the results of §1.3.1-1.3.3 to prove Proposition 1.3.1.

Proof. (Proposition 1.3.1) By Lemma 1.3.6, we may assume that X is sandwiched between standard ellipsoid $E(a, b)$ with $0 < a \leq b$ and a scaling.

$$E(a, b) \subset X \subset 4 \cdot E(a, b)$$

We begin by proving the lower bound, under this assumption. By Lemma 1.3.11, we have

$$\text{Ru}(Y) \geq \frac{1}{2\pi} \cdot \int_Y S(I\nu, I\nu) \, \text{dvol}_\sigma \tag{1.3.29}$$

By applying the lower bound in Corollary 1.3.15 and using the estimates for diameter, area, total curvature, volume and systolic ratio in Lemma 1.3.7, we see that for constants $B, C > 0$, we have

$$\int_Y S(I\nu, I\nu) \cdot \mathrm{dvol}_\sigma \geq \frac{\mathrm{area}(Y)^2}{3 \cdot \mathrm{diam}(Y)^2 \cdot \int_Y H \, \mathrm{dvol}_\sigma} \geq B \cdot a \geq C \cdot \mathrm{vol}(X)^{1/2} \cdot \mathrm{sys}(Y)^{1/2} \quad (1.3.30)$$

On the other hand, suppose that $\frac{a}{b} \ll 1$. Due to Lemma 1.3.7, this is equivalent to $\mathrm{sys}(Y) \ll 1$. By Corollary 1.3.17 and the estimates in Lemma 1.3.7, there are constants $A, B, C > 0$ with

$$\int_Y S(I\nu, I\nu) \, \mathrm{dvol}_\sigma \geq A \cdot \frac{\mathrm{area}(Y)^2}{a \cdot \int_Y H \, \mathrm{dvol}_\sigma} \geq B \cdot b \geq C \cdot \mathrm{vol}(X)^{1/2} \cdot \mathrm{sys}(Y)^{-1/2} \quad (1.3.31)$$

By assembling the estimate (1.3.29) with the two estimates (1.3.30) and (1.3.31), we deduce the following lower bound for some $C > 0$.

$$\mathrm{Ru}(Y) \geq C \cdot \mathrm{vol}(X)^{1/2} \cdot \mathrm{sys}(Y)^{-1/2} \quad (1.3.32)$$

After some rearrangement, this is the desired lower bound.

The second inequality is easier to show. By using the upper bound in Lemma 1.3.11 and the estimate for the mean curvature in Lemma 1.3.7, we see that for some $A, C > 0$ we have

$$\mathrm{Ru}(Y) \leq \int_Y H \, \mathrm{dvol}_\sigma \leq A \cdot b \leq C \cdot \mathrm{vol}(X)^{1/2} \cdot \mathrm{sys}(Y)^{-1/2} \quad (1.3.33)$$

This implies the desired upper bound, and concludes the proof. □

1.4 Non-Convex, Dynamically Convex Contact Forms

In this section, we use the methods of [2] to construct dynamically convex contact forms with systolic ratio and volume close to 1, and arbitrarily small and arbitrarily large Ruelle invariant.

Proposition 1.4.1. *For every $\varepsilon > 0$, there exists a dynamically convex contact form α on S^3 satisfying*

$$\mathrm{vol}\big(S^3, \alpha\big) = 1 \qquad \mathrm{sys}(S^3, \alpha) \geq 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \alpha) \leq \varepsilon \quad (1.4.1)$$

and there exists a dynamically convex contact form β on S^3 satisfying

$$\mathrm{vol}\big(S^3, \beta\big) = 1 \qquad \mathrm{sys}(S^3, \beta) \geq 1 - \varepsilon \qquad \mathrm{Ru}(S^3, \beta) \geq \varepsilon^{-1} \quad (1.4.2)$$

1.4.1 Hamiltonian Disk Maps

We begin with some notation and preliminaries on Hamiltonian maps of the disk that we will need for the rest of the section.

Let $\mathbb{D} \subset \mathbb{R}^2$ denote the unit disk in the plane with ordinary coordinates (x, y) and polar coordinates (r, θ). We use λ and ω to denote the standard Liouville form and symplectic form.

$$\lambda := \frac{1}{2}r^2 d\theta = \frac{1}{2}(xdy - ydx) \qquad \text{and} \qquad \omega := rdr \wedge d\theta = dx \wedge dy$$

Let $\phi : [0,1]\times\mathbb{D} \to \mathbb{D}$ be a the Hamiltonian flow (for $t \in [0,1]$) generated by a time-dependent Hamiltonian on $\mathbb{D}$ vanishing on the boundary, i.e.

$$H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R} \qquad \text{with} \qquad H|_{\mathbb{R}/\mathbb{Z}\times\partial\mathbb{D}} = 0$$

We let X_H denote the Hamiltonian vector field and adopt the convention that $\iota_{X_H}\omega = dH$. Since H is constant on the boundary $\partial\mathbb{D}$, the Hamiltonian vector field X_H is tangent to $\partial\mathbb{D}$. Thus ϕ is a well-defined flow on $\mathbb{D}$. The differential of ϕ defines a map $\Phi : \mathbb{R} \times \mathbb{D} \to \mathrm{Sp}(2)$ with $\Phi|_{0\times\mathbb{D}} = \mathrm{Id}$, which lifts uniquely to a map

$$\widetilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2) \qquad \text{satisfying} \qquad \widetilde{\Phi}(S+T, z) = \widetilde{\Phi}(T, \phi_S(z))\widetilde{\Phi}(S, z) \tag{1.4.3}$$

There are two key functions on $\mathbb{D}$ associated to the family of Hamiltonian diffeomorphisms ϕ. First, there is the action and the associated Calabi invariant.

Definition 1.4.2. The *action* $\sigma_\phi : \mathbb{D} \to \mathbb{R}$ and *Calabi invariant* $\mathrm{Cal}(\mathbb{D}, \phi) \in \mathbb{R}$ of ϕ are defined by

$$\sigma_\phi = \int_0^1 \phi_t^*(\iota_{X_H}\lambda + H) \cdot dt \qquad \text{and} \qquad \mathrm{Cal}(\mathbb{D}, \phi) = \int_{\mathbb{D}} \sigma \cdot \omega \tag{1.4.4}$$

The action measures the failure of ϕ to preserve λ, as captured by the following formula.

$$\phi_1^*\lambda - \lambda = d\sigma_\phi \tag{1.4.5}$$

Next, there is the rotation map and the associated Ruelle invariant. To discuss these quantities, we use the following lemma.

Lemma 1.4.3. *Let $\phi : [0,1] \times \mathbb{D} \to \mathbb{D}$ be the flow of a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{D}$ with $\sigma_\phi > 0$. Then the sequences $r_n : \mathbb{D} \to \mathbb{R}$ and $s_n : \mathbb{D} \to \mathbb{R}$ given by*

$$r_n(z) := \frac{1}{n}\rho \circ \widetilde{\Phi}(n, z) \qquad \text{and} \qquad s_n(z) := \frac{1}{n}\sum_{k=0}^{n-1} \sigma_\phi \circ \phi^k(z)$$

converge in $L^1(\mathbb{D})$ to r_ϕ and s_ϕ, respectively. The map s_k^{-1} also converges to s_ϕ^{-1} in $L^1(\mathbb{D})$.

Proof. We apply Kingman's sub-additive ergodic theorem (see [79] and Remark 1.2.15) to the map $g_n = r_n + C$ for sufficiently large $C > 0$. Applying (1.4.3) and the quasimorphism property of ρ, we find that

$$g_{m+n} \le g_m + g_n \circ \phi^m$$

By Kingman's ergodic theorem, this implies that $\frac{g_n}{n}$ has a limit r_∞ in $L^1(\mathbb{D})$. Since $\|g_n - r_n\|_{L^1}$ is bounded, we acquire the same result for r_n.

By Birkhoff's ergodic theorem, s_n converges to a limit $s_\infty \in L^1(\mathbb{D})$. Note that for some $c > 0$, we have

$$c^{-1} \le \sigma_\phi \le c \quad \text{and therefore} \quad c^{-1} \le s_n \le c$$

Thus $s_\infty > 0$ pointwise almost everywhere and s_∞^{-1} is well-defined almost everywhere. Since $|s_n|^{-1} < c$, we can apply the dominated convergence theorem to conclude that s_∞^{-1} is integrable and $s_n^{-1} \to s_\infty^{-1}$ in L^1. A similar argument applies to r_n/s_n, which converges to r_∞/s_∞. □

Definition 1.4.4. The *rotation* $r_\phi : \mathbb{D} \to \mathbb{R}$ and *Ruelle invariant* $\mathrm{Ru}(\mathbb{D}, \phi) \in \mathbb{R}$ of ϕ are defined by

$$r_\phi := \lim_{n\to\infty} r_n \qquad \text{and} \qquad \mathrm{Ru}(\mathbb{D}, \phi) = \int_{\mathbb{D}} r_\phi \cdot \omega \tag{1.4.6}$$

Remark 1.4.5. Our Ruelle invariant $\mathrm{Ru}(\mathbb{D}, \phi)$ of a symplectomorphism of the disk agrees with the two-dimensional version of the invariant introduced by Ruelle in [92].

The action, rotation, Calabi invariant and Ruelle invariant depend only on the homotopy class of ϕ relative to the endpoints, or equivalently the element in the universal cover of $\mathrm{Ham}(\mathbb{D}, \omega)$.

We conclude this review with a discussion of periodic points and their invariants.

Definition 1.4.6. A *periodic point* p of $\phi : \mathbb{D} \to \mathbb{D}$ is a point such that $\phi^k(p) = p$ for some $k \ge 1$. The period $\mathcal{L}(p)$, action $\mathcal{A}(p)$ and rotation number $\rho(p)$ of p are given, respectively, by

$$\mathcal{L}(p) := \min\{j > 0 | \phi^j(p) = p\} \qquad \mathcal{A}(p) = \sum_{i=0}^{\mathcal{L}(p)-1} \sigma_\phi \circ \phi^i(p) \qquad \rho(p) := \rho \circ \widetilde{\Phi}(\mathcal{L}(p), p) \tag{1.4.7}$$

Note that the rotation number can also be written as $\rho(p) = \mathcal{L}(p) \cdot r_\phi(p)$.

1.4.2 Open Books Of Disk Maps

We next review the construction of contact forms on S^3 from symplectomorphisms of the disk, using open books.

Construction 1.4.7. Let $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ be a Hamiltonian with flow $\phi : [0, 1] \times \mathbb{D} \to \mathbb{R}$ such that

(i) Near $\partial\mathbb{D}$, H is of the form $H(t, r, \theta) = B \cdot \pi(1 - r^2)$ for some $B > 0$.

(ii) The action function σ_ϕ of the Hamiltonian is positive everywhere.

We now construct the *open book* contact form α on S^3 associated to $(\mathbb{D}, \phi)$. We proceed by producing two contact manifolds (U, α) and (V, β), then gluing them by a strict contactomorphism.

To construct U, we consider the contact form $dt + \lambda$ on $\mathbb{R} \times \mathbb{D}$. Due to the identity $d\sigma_\phi = \phi_1^*\lambda - \lambda$ in (1.4.5), the map f defined by

$$f : \mathbb{R} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D} \qquad f(t, z) = (t - \sigma_\phi(z), \phi_1(z))$$

is a strict contactomorphism. We form the following quotient space.

$$U = \mathbb{R} \times \mathbb{D}/\sim \qquad \text{defined by } (t, z) \sim f(t, z)$$

Since σ_ϕ is strictly positive by assumption (ii) in Construction 1.4.7, this quotient is a smooth manifold. The contact form $dt + \lambda$ descends to a contact form α on U because f is a strict contactomorphism. Note that a fundamental domain of this quotient is given by

$$\Omega = \{(t, z) | 0 \le t \le \sigma_\phi(z)\}$$

We observe that the Reeb vector field is simply given by $R = \partial_t$. Thus the disk $0 \times \mathbb{D} \subset U$ is a surface of section for the Reeb flow on U with first return map ϕ_1.

To construct V, we choose a small $\varepsilon > 0$ and let

$$V := \mathbb{R}/\pi\mathbb{Z} \times \mathbb{D}(\varepsilon) \qquad \beta := (1 - r^2)dt + \frac{B}{2} r^2 d\theta$$

Here $\mathbb{D}(\varepsilon) \subset \mathbb{C}$ is the disk of radius ε, t is the $\mathbb{R}/\pi\mathbb{Z}$ coordinate and (r, θ) are radial coordinates on $\mathbb{D}(\varepsilon)$. There is a strict contactomorphism Ψ identifying subsets of U and V, given by

$$\Psi : V \setminus (\mathbb{R}/\pi\mathbb{Z} \times 0) \to U \qquad \text{with} \qquad \Psi(t, r, \theta) := (\frac{B}{2} \cdot \theta, \sqrt{1 - r^2}, 2t - B\theta)$$

We now define $Y = \text{int}(U) \cup_\Psi V$ as the gluing of the interior of U and V via Φ, and α as the inherited contact form. Since ϕ is Hamiltonian isotopic to the identity, the resulting contact manifold $(Y, \ker \alpha)$ is contactomorphic to standard contact S^3.

Proposition 1.4.8 (Open Book)**.** *Let H and ϕ be as in Construction 1.4.7. Then there exists a contact form α on S^3 with the following properties.*

(a) (Surface Of Section) There is an embedding $\iota : \mathbb{D} \to S^3$ such that $\iota(\mathbb{D})$ is a surface of section with return map ϕ_1 and first return time σ, and such that $\omega = \iota^* d\alpha$.

(b) *(Volume) The volume of* (S^3, α) *is given by the Calabi invariant of* $(\mathbb{D}, \phi)$*, i.e.*

$$\mathrm{vol}\big(S^3, \alpha\big) = \mathrm{Cal}(\mathbb{D}, \phi)$$

(c) *(Ruelle) The Ruelle invariant of* (S^3, α) *is given by a shift of the Ruelle invariant of* $(\mathbb{D}, \phi)$*.*

$$\mathrm{Ru}(S^3, \alpha) = \mathrm{Ru}(\mathbb{D}, \phi) + \pi$$

(d) *(Binding) The binding* $b = \iota(\partial\mathbb{D})$ *is a Reeb orbit of action* π *and rotation number* $1 + 1/B$*.*

(e) *(Orbits) Every simple orbit* $\gamma \subset S^3 \setminus b$ *corresponds to a periodic point* p *of* ϕ *that satisfies*

$$\mathrm{lk}(\gamma, b) = \mathcal{L}(p) \qquad \mathcal{A}(\gamma) = \mathcal{A}(p) \qquad \rho(\gamma) = \rho(p) + \mathcal{L}(p)$$

In order to relate various invariants associated to (S^3, α) and its Reeb orbits to corresponding structures for $(\mathbb{D}, \phi)$, we need to introduce a certain trivialization of ξ over U.

Construction 1.4.9. Let $(U, \xi|_U)$ be as in Construction 1.4.7. We let τ denote the continuous trivialization of $\xi|_U$ defined as follows. On the fundamental domain Ω, we let

$$\tau : \Omega \to \mathrm{Hom}(\xi|_U, \mathbb{R}^2) \qquad \text{given by} \qquad \tau(t, z) := \exp(2\pi i t/\sigma_\phi(z)) \circ \Phi(t/\sigma_\phi(z), z) \circ \Pi_{\mathbb{D}} \tag{1.4.8}$$

Here $\Phi : [0, 1] \times \mathbb{D} \to \mathbb{D}$ is the differential $d\phi$ of the flow $\phi : [0, 1] \times \mathbb{D} \to \mathbb{D}$ and $\Pi_{\mathbb{D}} : \xi \to T\mathbb{D}$ denotes projection to the (canonically trivial) tangent bundle $T\mathbb{D}$ of $\mathbb{D}$. Note also that $\circ$ denotes composition of bundle maps.

To check that τ descends to a well-defined trivialization on U, we must check that it is compatible with the quotient map $f : \mathbb{R} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D}$. Indeed, we have

$$\tau(\sigma_\phi(z), z) = \Phi(1, z) \circ \Pi_{\mathbb{D}} = \tau(0, \phi_1(z)) \circ df_{(\sigma_\phi(z), z)}$$

This precisely states that projection commutes with the isomorphism identifying tangent spaces in the quotient, so τ descends from Ω to U.

Lemma 1.4.10. *Let* $\tau : \xi|_U \to \mathbb{R}^2$ *be the trivialization in Construction 1.4.9. Then*

(a) *The restriction* $\tau|_K$ *of* τ *to any compact subset* $K \subset \mathrm{int}(U)$ *of the interior of* U *is the restriction of a global trivialization of* ξ *on* S^3*.*

(b) *The local rotation number* $\mathrm{rot}_\tau : U \to \mathbb{R}$ *of* $(U, \alpha|_U)$ *with respect to* τ *agrees with the restriction of the local rotation number* $\mathrm{rot} : S^3 \to \mathbb{R}$ *of* (S^3, α) *with respect to the global trivialization.*

Proof. Let $V = \mathbb{R}/\pi\mathbb{Z} \times \mathbb{D}(\varepsilon)$ and Ψ be as in Construction 1.4.7. For any $\delta < \varepsilon$, we let $V(\delta) \subset V$ and $U(\delta) \subset U$ denote

$$V(\delta) := \mathbb{R}/\pi\mathbb{Z} \times \mathbb{D}(\delta) \subset V \qquad \text{and} \qquad U(\delta) := \operatorname{int}(U) \setminus \operatorname{int}(\Psi(V(\delta)))$$

The sets $U(\delta)$ are an exhaustion of $\operatorname{int}(U)$ by compact, Reeb-invariant contact submanifolds.

To show (a), we assume that $K = U(\delta)$. The homotopy classes of trivializations $\mathfrak{T}$ of ξ over $U(\delta)$ are in bijection with $H^1(U(\delta);\mathbb{Z}) \simeq \mathbb{Z}$. A map to $\mathbb{Z}$ classifying elements of $\mathfrak{T}$ is given by

$$\mathfrak{T} \to \mathbb{Z} \qquad \text{given by} \qquad \sigma \mapsto \operatorname{sl}(\gamma, \sigma) \tag{1.4.9}$$

Here $\operatorname{sl}(\gamma,\sigma)$ is the self-linking number (in the trivialization σ) of the following transverse knot.

$$\gamma : \mathbb{R}/2\pi\mathbb{Z} \to U(\delta) \qquad \gamma(\theta) = \Psi(0,\varepsilon,\theta) = (\frac{B\theta}{2}, \sqrt{1-\varepsilon^2}, -B\theta)$$

The knot γ bounds a Seifert disk $\Sigma = 0 \times \mathbb{D}(\varepsilon)$ in $V \subset S^3$. The line field $\xi \cap \Sigma$ has a single positive elliptic singularity, so the self-linking number of the boundary γ with respect to the global trivialization is $\operatorname{sl}(\gamma) = -1$.

To compute $\operatorname{sl}(\gamma,\tau)$, we push γ into Σ along a collar neighborhood to acquire a nowhere zero section $\eta : \mathbb{R}/2\pi\mathbb{Z} \to \xi$ and then compose with τ to acquire a map $\tau\circ\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2\backslash 0$. Up to isotopy through nowhere zero sections, we can compute that

$$\tau \circ \eta(\theta) = e^{i\theta} \in \mathbb{C} = \mathbb{R}^2$$

On the other hand, the self-linking number can be computed as the negative of the winding number of this map.

$$\operatorname{sl}(\gamma,\tau) = -\operatorname{wind}(\tau\circ\eta) = -1$$

Since the map (1.4.9) classifies elements of $\mathfrak{T}$, this proves that on $U(\delta)$ the trivialization τ agrees with the restriction of a global trivialization.

To show (b), note that since $U(\delta)$ is compact, we can choose a global trivialization of ξ on S^3

$$\sigma : \xi \simeq \mathbb{R}^2 \qquad \text{such that} \qquad \sigma|_{U(\delta)} = \tau|_{U(\delta)}$$

By Proposition 1.2.13(c), $\operatorname{rot}_\sigma = \operatorname{rot}$ on S^3 and so the local rotation numbers satisfy

$$\operatorname{rot}|_{U(\delta)} = \operatorname{rot}_\sigma|_{U(\delta)} = \operatorname{rot}_\tau|_{U(\delta)}$$

Since this holds for any δ, this shows (b) on all of $\operatorname{int}(U)$. □

Remark 1.4.11. It is possible to define the trivialization τ of $\xi|_U$ in such a way that it is not only homotopic but actually equal to the restriction of a global trivialization of the contact structure on S^3. We did not do this in order to avoid complicated formulas in the definition of τ.

The following lemma relates the rotation number of the lifted linearized Reeb flow $\widetilde{\Phi}_\tau : \mathbb{R}\times U \to \widetilde{\mathrm{Sp}}(2)$ on (U,α) to the rotation number of the lifted linearized flow $\widetilde{\Phi} : \mathbb{R}\times\mathbb{D} \to \widetilde{\mathrm{Sp}}(2)$ of the Hamiltonian flow ϕ on $\mathbb{D}$.

Lemma 1.4.12. *Let $\iota : \mathbb{D} \to U$ denote the inclusion of the disk $0 \times \mathbb{D} \subset U$. Let $p \in \mathbb{D}$ and consider the Reeb trajectory $\gamma : \mathbb{R} \to U$ satisfying $\gamma(0) = \iota(p)$. Let $0 = T_0 < T_1 < T_2 < \dots$ denote the non-negative times at which the trajectory γ intersects the disk $\iota(\mathbb{D})$. Then*

$$\rho \circ \widetilde{\Phi}_\tau(T_k, \iota(p)) = \rho \circ \widetilde{\Phi}(k,p) + k$$

for all non-negative integers k.

Proof. We abbreviate

$$p_i = \phi^i(p) \qquad y_i = \iota(p_i) = \gamma(T_i) \qquad L_i = T_{i+1} - T_i = \sigma_\phi(p_i)$$

Note that the lifted linearized Reeb flow with respect to τ at time T_k can be written as

$$\widetilde{\Phi}_\tau(T_k, y_0) = \widetilde{\Phi}_\tau(L_{k-1}, y_{k-1})\widetilde{\Phi}_\tau(L_{k-2}, y_{k-2}) \dots \widetilde{\Phi}_\tau(L_0, y_0) \tag{1.4.10}$$

The linearized Reeb flow $\widetilde{\Phi}_\tau(L_i, y_i)$ takes place along a trajectory connecting $(0, p_i)$ to $(\sigma_\phi(p_i), p_i)$ in the fundamental domain Ω. We may be directly compute from (1.4.8) that

$$\Phi_\tau(t, y_i) = \exp(2\pi i t/\sigma_\phi(p_i)) \circ \Phi(t/\sigma_\phi(z), p_i) \quad \text{and so} \quad \widetilde{\Phi}_\tau(L_i, y_i) = \widetilde{\Xi} \cdot \widetilde{\Phi}(1, p_i) \tag{1.4.11}$$

Here $\widetilde{\Xi}$ is the unique lift of $\mathrm{Id} \in \mathrm{Sp}(2)$ with $\rho(\widetilde{\Xi}) = 1$. This is a central element of $\widetilde{\mathrm{Sp}}(2)$, so combining (1.4.10) and (1.4.11) we have

$$\widetilde{\Phi}_\tau(T_k, y_0) = \widetilde{\Xi}^k \cdot \widetilde{\Phi}(1, \phi^{k-1}(p)) \cdot \widetilde{\Phi}(1, \phi^{k-2}(p)) \cdots \widetilde{\Phi}(1, p) = \widetilde{\Xi}^k \cdot \widetilde{\Phi}(k, p)$$

Since $\rho(\widetilde{\Xi} \cdot \widetilde{\Psi}) = 1 + \rho(\widetilde{\Psi})$ for any $\widetilde{\Psi} \in \widetilde{\mathrm{Sp}}(2)$, we can conclude that

$$\rho \circ \widetilde{\Phi}_\tau(T_k, \iota(p)) = \rho \circ \widetilde{\Phi}(k,p) + k$$

□

Proof of Proposition 1.4.8. We prove each of the properties (a)-(e) separately.

Surface Of Section. Define the inclusion $\iota : \mathbb{D} \to S^3$ as the following composition.

$$\iota : \mathbb{D} = 0 \times \mathbb{D} \to \mathbb{R} \times \mathbb{D} \xrightarrow{\pi} Y \simeq S^3$$

The surface $0 \times \mathbb{D}$ is transverse to the Reeb vector field ∂_t of $\mathbb{R} \times \mathbb{D}$ and intersects every flowline $\mathbb{R} \times z$. Also, $(\mathbb{R} \times z) \cap \Omega$ has action $\sigma_\phi(z)$ and ends on $(\sigma_\phi(z), z) \sim (0, \phi_1(z))$. Thus $\iota(\mathbb{D}) = \pi(0 \times \mathbb{D})$ is a surface of section with return time σ_ϕ and monodromy ϕ_1. Finally, note that

$$\iota^*(d\alpha) = d(dt + \lambda)|_{0\times\mathbb{D}} = \omega$$

This verifies all of the properties of $\iota : \mathbb{D} \to Y \simeq S^3$ listed in (a).

Calabi Invariant. This property follows from a simple calculation of the volume using the fundamental domain Ω.

$$\mathrm{vol}(Y,\alpha) = \int_Y \alpha \wedge d\alpha = \int_\Omega dt \wedge d\lambda = \int_{\mathbb{D}} \sigma_\phi \cdot \omega = \mathrm{Cal}(\mathbb{D},\phi)$$

Ruelle Invariant. Let $\mathrm{rot} : S^3 \to \mathbb{R}$ be the local rotation number of (S^3,α). By Lemma 1.4.10, the restriction of rot to the (open) fundamental domain $\Omega \subset S^3$ coincides with rot_τ. Since $S^3 \setminus \Omega$ is measure 0 in S^3, we thus have

$$\mathrm{Ru}(S^3,\alpha) = \int_{S^3} \mathrm{rot} \cdot \alpha \wedge d\alpha = \int_\Omega \mathrm{rot}_\tau \cdot dt \wedge \omega = \int_{\mathbb{D}} \iota^* \mathrm{rot}_\tau \cdot \sigma_\phi \omega \tag{1.4.12}$$

Here $\iota^* \mathrm{rot}_\tau$ denotes the pullback of rot_τ via the map $\iota : \mathbb{D} \to S^3$ from (a). We have used the fact that rot_τ is constant along Reeb trajectories. This follows directly from the definition of rot_τ as a time average.

To apply this alternative formula for $\mathrm{Ru}(S^3,\alpha)$, let T_k denote the kth positive time that the Reeb trajectory $\gamma : [0,\infty) \to S^3$ intersects the surface of section $\iota(\mathbb{D})$. Then

$$\iota^* \mathrm{rot}_\tau = \lim_{k\to\infty} \frac{\rho \circ \widetilde{\Phi}_\tau(T_k,-)}{T_k} = \lim_{k\to\infty} \frac{\rho \circ \widetilde{\Phi}(k,-) + k}{\sum_{i=0}^{k-1} \sigma_\phi \circ \phi^i} = \frac{r_\phi + 1}{s_\phi}$$

Here the second equality is a consequence of Lemma 1.4.12. The maps r_ϕ and s_ϕ are the averaged rotation and action maps constructed in Lemma 1.4.3. By construction, these maps are invariant under pullback by ϕ. Thus

$$\int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \omega = \frac{1}{n}\sum_{k=0}^{n-1} \int_{\mathbb{D}} [\phi^k]^* (\frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \omega) = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot s_n \omega \quad \text{where} \quad s_n = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_\phi \circ \phi^k$$

By Lemma 1.4.3, we know that $s_n \to s_\phi$ in $L^1(\mathbb{D})$. Thus, by combining the above formula in the $n \to \infty$ limit with (1.4.12), we acquire the desired property.

$$\mathrm{Ru}(S^3,\alpha) = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot \sigma_\phi \cdot \omega = \int_{\mathbb{D}} \frac{r_\phi + 1}{s_\phi} \cdot s_\phi \cdot \omega = \int_{\mathbb{D}} (r_\phi + 1) \cdot \omega = \mathrm{Ru}(\mathbb{D},\phi) + \pi$$

Binding. Let $b = \iota(\partial\mathbb{D})$ be the binding which coincides with $\mathbb{R}/\pi\mathbb{Z} \times 0$ in V. First note that the Reeb vector field is given on (V,β) by the following formula.

$$R_\beta = \partial_t + \frac{2}{B}\partial_\theta \tag{1.4.13}$$

Thus b is a Reeb orbit. Since b bounds a symplectic disk $\iota(\mathbb{D}) \subset S^3$ of area π, the action is π. To compute $\rho(b)$, note that there is a natural trivialization of $\xi|_V = \ker(\beta)$ given by

$$\nu : \xi|_V \subset TV \xrightarrow{\pi} T\mathbb{D}(\varepsilon) = \mathbb{R}^2$$

The Reeb flow $\phi : \mathbb{R} \times V \to V$ and the linearized Reeb flow $\Phi_\nu : \mathbb{R} \times V \to \mathrm{Sp}(2)$ with respect to ν can be calculated from (1.4.13), as follows.

$$\phi_t(s, z) = (s + t, e^{2it/B} \cdot z) \qquad \Phi_\nu(t, s, z) = e^{2it/B}$$

Thus the rotation number $\rho(b, \nu)$ of b in the trivialization ν is $1/B$. Finally, to compute the rotation number $\rho(b) = \rho(b, \tau)$ with respect to the global trivialization τ on ξ, we note that

$$\rho(b, \tau) - \rho(b, \nu) = \mu(\tau \circ \nu^{-1}|_b) = c_1(\xi|_{\iota(\mathbb{D})}, \tau) - c_1(\xi|_{\iota(\mathbb{D})}, \nu) = -c_1(\xi|_{\iota(\mathbb{D})}, \nu)$$

Here $\mu : \pi_1(\mathrm{Sp}(2)) \to \mathbb{Z}$ is the Maslov index and $c_1(\xi|_{\iota(\mathbb{D})}, -)$ is the relative Chern class of $\xi|_{\iota(\mathbb{D})}$ with respect to a given trivialization over $\iota(\partial\mathbb{D})$, which vanishes for τ.

On the other hand, the trivialization ν is specified by the section of $\xi|_{\iota(\mathbb{D})}$ given by pushing $\iota(\partial\mathbb{D})$ into $\iota(\mathbb{D})$ along a collar neighborhood. Thus, $-c_1(\xi|_{\iota(\mathbb{D})}, \nu)$ is precisely the self-linking number $\mathrm{sl}(b)$ of b. This number can be calculated as a signed count of singularities of the line field $\xi \cap \iota(\mathbb{D})$, which has 1 elliptic singularity. Thus $\mathrm{sl}(b) = -1$ and $\rho(b) = 1 + 1/B$.

Orbits. An embedded closed orbit $\gamma : \mathbb{R}/L\mathbb{Z} \to Y$ of α that is disjoint from the binding b is equivalent to a closed orbit of $(U, \alpha|_U)$. The orbit γ intersects the surface of section $\iota(\mathbb{D})$ transversely at $n \geq 1$ times $T_0 = 0, T_1, \ldots, T_n = L$. Let

$$p_k \in \mathbb{D} \qquad \text{be such that} \qquad \iota(p_k) = \gamma(T_k) \cap \iota(\mathbb{D})$$

Since $\iota(\mathbb{D})$ is a surface of section, we have $p_{i+1} = \phi(p_i)$ and since γ is closed, $p_n = p_0$. Thus $p = p_0$ is a periodic point of period

$$\mathcal{L}(p) = n = \iota_*[\mathbb{D}] \cdot [\gamma] = \mathrm{lk}(\gamma, b)$$

Next, note that on the interval $[T_i, T_{i+1}]$, γ restricts to a map $[T_i, T_{i+1}] \to \Omega$ given by $\gamma(t) = (t, \iota(p_i))$, from which it follows that

$$\mathcal{A}(\gamma) = \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} \gamma^*(dt + \alpha) = \sum_{k=0}^{n-1} \int_0^{\sigma(p_k)} dt = \sum_{k=0}^{n-1} \sigma \circ \phi^k(p) = \mathcal{A}(p)$$

Finally, due to Lemma 1.4.10 we may use the trivialization τ to compute the rotation number. We have

$$\rho(\gamma) = \rho \circ \widetilde{\Phi}_\tau(L, \gamma(0)) = \rho \circ \widetilde{\Phi}(n, p) + n = \rho(p) + \mathcal{L}(p)$$

Here the second equality uses Lemma 1.4.12. This completes the proof of (e), and the entire proposition. □

1.4.3 Radial Hamiltonians

A Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ that is rotationally invariant will be called *radial*. In other words, H is radial if it can be written as

$$H(t, r, \theta) = h(t, r) \qquad \text{for a map} \qquad h : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathbb{R}$$

We will require a few lemmas regarding radial Hamiltonians.

Lemma 1.4.13. *Let $H : \mathbb{D} \to \mathbb{R}$ be an autonomous, radial Hamiltonian with $H = h \circ r$. Then*

$$\sigma_\phi(r,\theta) = h(r) - \frac{1}{2}rh'(r) \qquad \text{and} \qquad r_\phi(r,\theta) = -\frac{h'(r)}{2\pi r} \tag{1.4.14}$$

Proof. We calculate the Hamiltonian vector field X_H and the action function σ_ϕ as follows.

$$X_H = -\frac{h'}{r} \cdot \partial_\theta \qquad \text{and} \qquad \text{and} \qquad \sigma_\phi(r,\theta) = \int_0^1 \phi_t^*(-\frac{rh'(r)}{2} + h(r)) \cdot dt = h(r) - \frac{1}{2}rh'(r)$$

Here we use the fact that the Hamiltonian flow ϕ preserves any function of r. Next, we note that the differential $\Phi : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ of the flow ϕ is given by

$$\Phi(t,z)v = \exp(\frac{-h'}{r} \cdot it)v + \frac{it(rh'' - h')}{r^2} \cdot \exp(\frac{-h'}{r} \cdot it)z \cdot dr(v)$$

Note that if we use $s = iz/|z|$, then $dr(v) = 0$. Thus, if $\widetilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2)$ denotes the lift of Φ, and ρ_s denotes the rotation number relative to s (see Definition 1.2.5) then

$$\Phi(t,z)s = \exp(\frac{-h'(r)}{r} \cdot it)s \qquad \text{and thus} \qquad \rho_s \circ \widetilde{\Phi}(T,z) = T \cdot \frac{-h'(r)}{2\pi r} \tag{1.4.15}$$

Since $\rho_s : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ and $\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ are equivalent quasimorphisms (Lemma 1.2.6), we have

$$r_\phi = \lim_{T \to \infty} \frac{\rho \circ \widetilde{\Phi}(T,-)}{T} = \lim_{T \to \infty} \frac{\rho_s \circ \widetilde{\Phi}(T,-)}{T} = \frac{-h' \circ r}{2\pi r} \qquad \text{in } L^1(\mathbb{D})$$

This conclues the proof of the lemma. □

More generally, a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ is called *radial around* $p \in \mathbb{D}$ if H is invariant under rotation around p, i.e. if H can be written as

$$H(t,x,y) = h(t,r_p) \qquad \text{for a map} \qquad h : \mathbb{R}/\mathbb{Z} \times [0,1] \to \mathbb{R}$$

Here $r_p : \mathbb{D} \to \mathbb{R}$ be the distance from p, i.e. $r_p(z) = |z - p|$.

Lemma 1.4.14. *Let $H : \mathbb{D} \to \mathbb{R}$ be an autonomous Hamiltonian that is radial around $p = (a,b) \in \mathbb{D}$, with $H = h \circ r_p$, in a neighborhood U of p. Then on U, we have*

$$\sigma_\phi = h(r_p) - \frac{1}{2}r_p h'(r_p) + u_p - \phi_1^* u_p \qquad \text{and} \qquad r_\phi = -\frac{h'(r_p)}{2\pi r_p} \tag{1.4.16}$$

Here the map $u_p : \mathbb{D} \to \mathbb{R}$ is given by $u_p(x,y) = (bx - ay)/2$.

Proof. Let λ_p be the radial Liouville form on $(\mathbb{D},\omega)$ centered at p. That is, λ_p is given by

$$\lambda_p = \frac{1}{2}((x-a)dy - (y-b)dx) = \lambda + du_p$$

Let $\tau : \mathbb{D} \to \mathbb{R}$ be the function decribed in (1.4.16). Then by Lemma 1.4.13, we know that on U we have

$$d\tau = (\phi_1^*\lambda_p - \lambda_p) + (\phi_1^* du_p - u_p) = \phi_1^*\lambda - \lambda = d\sigma_\phi$$

Thus it suffices to check that $\sigma_\phi(p) = \tau(p)$. Since $rh'(p) = 0$ and $u_p(p) = u_p(\phi_1(p)) = 0$, we see that $\tau(p) = h(0) = H(p)$. On the other hand, $X_H(p) = 0$, we see that

$$\sigma_\phi(p) = \int_0^1 \phi_t^*(\lambda(X_H) + H)dt = \int_0^1 h(0)dt = \tau(p)$$

Thus $\sigma_\phi(p) = \tau(p)$. The formula for r_ϕ follows from identical arguments to Lemma 1.4.13. □

1.4.4 Special Hamiltonian Maps

We next construct a special Hamiltonian flow ϕ on the disk, depending on a set of parameters, and establish its basic properties with respect to action, rotation and periodic orbits. The desired contact forms in Proposition 1.4.1 with small and large Ruelle invariant correspond (via Proposition 1.4.8) to ϕ for suitable choices of parameters (see §1.4.5).

The special Hamiltonian flow ϕ is constructed as the product of a pair of simpler flows.

$$\phi = \phi^H \bullet \phi^G$$

Here $\phi^G : [0,1] \times \mathbb{D} \to \mathbb{D}$ and $\phi^H : [0,1] \times \mathbb{D} \to \mathbb{D}$ are autonomous flows generated by G and H, and the product $\bullet$ occurs in the universal cover of the group $\mathrm{Ham}(\mathbb{D},\omega)$ of Hamiltonian diffeomorphisms of $(\mathbb{D},\omega)$. We denote the Hamiltonian generating ϕ by

$$H\#G : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$$

Setup 1.4.15. We will require the following setup in the construction of ϕ. The setup and notation established here will be used for the remainder of of §1.4.4.

(a) Fix an integer $n \geq 10$ and let $\mathbb{S}(n,k) \subset \mathbb{D}$ for $0 \leq k \leq n-1$ be the sector of points

$$\mathbb{S}(n,k) := \{re^{i\theta} \in \mathbb{D} \ : \ 2\pi k/n < \theta < 2\pi(k+1)/n\}$$

(b) Fix a finite union $U \subset \mathbb{D}$ of disjoint disks in $\mathbb{D}$ such that each of the component disks $D \subset U$ is contained in one of the sectors $\mathbb{S}(n,k)$ and such that for every $D \subset U$ the disk $e^{2\pi i/n} \cdot D$ is a component disk of U as well. We let

$$d(U) := \max\{\mathrm{diam}(D) \ : \ D \subset U \text{ is a component disk}\}$$

That is, $d(U)$ is the maximal diameter of a disk in U.

(c) Fix a constant $\delta > 0$ that is much smaller than the radius of each disk D, than the distance between any two of the disks D and D', and than the distance between D and the boundary of any of the sectors $\mathbb{S}(n,k)$. For any subset $S \subset \mathbb{D}$, we use the notation

$$N(S) := \{z \in \mathbb{D} || z - p| \leq \delta \text{ for some } p \in S\}$$

The neighborhoods $N(\partial D)$, $N(D)$, $N(U)$ and $N(\partial U)$ will be of particular importance.

(d) Fix a real number $R \in \mathbb{R}$. For every $s > \delta$ sufficiently large compared to δ, there exists a smooth, monotonic function $g_s : [0, s+\delta] \to \mathbb{R}$ with support contained in $[0, s+\delta)$ satisfying the following conditions.

$$g_s(r) = \pi \cdot R \cdot (s^2 - r^2) \qquad \text{if } r \leq s - \delta \tag{1.4.17}$$

$$|g_s'(r)| \leq 2\pi \cdot |R| \cdot (s - \delta) \qquad \text{if } s - \delta \leq r \leq s + \delta \tag{1.4.18}$$

Choose such a function g_s for every s that arises as the radius of a component disk $D \subset U$.

We now introduce the two Hamiltonians H and G in some detail. The construction of H only depends on the integer n. The construction of G depends on U, δ, R and the choice of g_s.

Construction 1.4.16. We let $H : \mathbb{D} \to \mathbb{R}$ denote the radial Hamiltonian given by the formula

$$H(r,\theta) := \frac{\pi(n+1)}{n} \cdot (1 - r^2) \tag{1.4.19}$$

The Hamiltonian vector field is $X_H = \frac{2\pi(n+1)}{n} \cdot \partial_\theta$ and so the Hamiltonian flow is given by

$$\phi^H : \mathbb{R} \times \mathbb{D} \to \mathbb{D} \qquad \text{with} \qquad \phi^H(t,z) = \exp(\frac{2\pi(n+1)}{n} \cdot it) \cdot z \tag{1.4.20}$$

In particular, the time 1 flow is rotation by $\frac{2\pi}{n}$ and preserves the collection U.

Construction 1.4.17. We let $G : \mathbb{D} \to \mathbb{R}$ denote a Hamiltonian that is invariant under rotation by angle $2\pi/n$ and that vanishes away from $N(U)$. That is

$$G(z) = G(e^{2\pi i/n} \cdot z) \qquad \text{and} \qquad G|_{\mathbb{D}\setminus N(U)} = 0 \tag{1.4.21}$$

Furthermore, let D be a component disk of U that is centered at $p \in \mathbb{D}$ and with radius s. Then we assume that G is given by

$$G|_{N(D)} = g_s \circ r_p \tag{1.4.22}$$

in the neighbourhood $N(D)$ of the disk D. This fully specifies G on all of $\mathbb{D}$.

A crucial fact that we will use later without comment is that ϕ^G and ϕ^H commute as elements of the universal cover of $\mathrm{Ham}(\mathbb{D}, \omega)$, i.e. $\phi^G \bullet \phi^H = \phi^H \bullet \phi^G$. The remainder of this section is devoted to calculating properties of the action, rotation and periodic points of the map ϕ.

Lemma 1.4.18 (Action of ϕ)**.** *The action map* $\sigma_\phi : \mathbb{D} \to \mathbb{R}$ *and Calabi invariant* $\mathrm{Cal}(\mathbb{D}, \phi)$ *satisfy*

$$\sigma_\phi = \pi(1 + \frac{1}{n}) + R \sum_{D \subset U} \mathrm{area}(D) \cdot \chi_D + O(d(U)) \qquad \text{on} \quad \mathbb{D} \setminus N(\partial U) \tag{1.4.23}$$

$$\pi/2 + \min\{0, R\} \cdot 2\pi/n \leq \sigma_\phi \leq 2\pi + \max\{0, R\} \cdot 2\pi/n \qquad \text{on} \quad \mathbb{D} \tag{1.4.24}$$

$$\mathrm{Cal}(\mathbb{D}, \phi) = \pi^2(1 + \frac{1}{n}) + R \sum_{D \subset U} \mathrm{area}(D)^2 + O(d(U)) + O(|R| \cdot \mathrm{area}(N(\partial U))) \tag{1.4.25}$$

Proof. Since ϕ^G and ϕ^H commute, we have $\sigma_G \circ \phi_1^H = \sigma_G$ and therefore

$$\sigma_\phi = \sigma_G \circ \phi_1^H + \sigma_H = \sigma_G + \sigma_H$$

Thus we must compute the action map of G and H. First, we note that H is radial by (1.4.19). Thus we apply Lemma 1.4.13 to see

$$\sigma_H = \pi(1 + \frac{1}{n}) \quad \text{on all of } \mathbb{D} \tag{1.4.26}$$

Next we compute the action map of G. Let D be a component disk of U centered at p and of radius s. We can apply Lemma 1.4.14 to see that

$$\sigma_G = g_s(r_p) - \frac{1}{2} r_p g_s'(r_p) + (u_p - (\phi_1^G)^* u_p) = R\,\mathrm{area}(D) + O(d(U)) \quad \text{on } D \setminus N(\partial D)$$

Here we used expression (1.4.17) for g_s. It follows from the definition of u_p in Lemma 1.4.14 and the fact that $d(U)$ is an upper bound on the diameter of D that $|u_p - (\phi_1^G)^* u_p|$ is bounded above by $d(U)$. Since $\sigma_G = 0$ outside of $N(D)$, we thus acquire the formula

$$\sigma_G = R \sum_{D \subset U} \mathrm{area}(D) \cdot \chi_D + O(d(U)) \quad \text{on } \mathbb{D} \setminus N(\partial U) \tag{1.4.27}$$

Adding (1.4.26) and (1.4.27) yields the desired formula (1.4.23) and implies (1.4.24) away from $N(\partial U)$. We can estimate on the neighbourhood $N(\partial D)$

$$|\sigma_G| \leq |g_s(r_p) - \frac{1}{2} r_p g_s'(r_p)| + |u_p - (\phi_1^G)^* u_p| \leq 2|R|\pi s^2 + |u_p - (\phi_1^G)^* u_p|$$

We observe that $\pi s^2 < \pi/n$ and $|u_p - (\phi_1^G)^* u_p| < 1$. Moreover, $\sigma_G \geq 0$ if $R \geq 0$ and $\sigma_G \leq 0$ if $R \leq 0$. Adding σ_G to the formula (1.4.26) for σ_H thus yields (1.4.24) on $N(\partial D)$. Finally, since $|\sigma_G| = O(|R|)$, the integral of σ_G over $\mathbb{D}$ agrees with the integral over $\mathbb{D} \setminus N(\partial U)$ up to an $O(|R| \cdot \mathrm{area}(N(\partial U)))$ term. This proves (1.4.25). □

Lemma 1.4.19 (Rotation of ϕ)**.** *The rotation map $r_\phi : \mathbb{D} \to \mathbb{R}$ and the Ruelle invariant* $\mathrm{Ru}(\mathbb{D},\phi)$ *satisfy*

$$r_\phi = (1 + \frac{1}{n}) + R \sum_{D \subset U} \chi_D \qquad \textit{on} \quad \mathbb{D} \setminus N(\partial U) \tag{1.4.28}$$

$$1 + \frac{1}{n} + \min\{0, R\} \leq r_\phi \leq 1 + \frac{1}{n} + \max\{0, R\} \qquad \textit{on} \quad \mathbb{D} \tag{1.4.29}$$

$$\mathrm{Ru}(\mathbb{D},\phi) = \pi(1 + \frac{1}{n}) + R \sum_{D \subset U} \mathrm{area}(D) + O(R \cdot \mathrm{area}(N(\partial U))) \tag{1.4.30}$$

Proof. In the universal cover of $\mathrm{Ham}(\mathbb{D},\phi)$, the time k flow ϕ^k of $G\#H$ can be factored in terms of the time 1 flow $\phi^G : [0,1] \times \mathbb{D} \to \mathbb{D}$ of G and the time 1 flow $\phi^H : [0,1] \times \mathbb{D} \to \mathbb{D}$ of H, as follows.

$$\phi^k = (\phi^H \bullet \phi^G)^k = \phi^H \bullet \phi^G \bullet \phi^H \bullet \cdots \bullet \phi^H \bullet \phi^G$$

This factorization is inherited by the lifted differential $\tilde{\Phi} : \mathbb{R} \times \mathbb{D} \to \widetilde{\mathrm{Sp}}(2)$ of $\phi : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ due to the cocycle property of $\widetilde{\Phi}$.

$$\widetilde{\Phi}(k,z) = \widetilde{\Phi}^H(1, \phi^G \circ \phi^{k-1}(z)) \bullet \widetilde{\Phi}^G(1, \phi^{k-1}(z)) \bullet \widetilde{\Phi}^H(1, \phi^G \circ \phi^{k-2}(z)) \bullet \cdots \bullet \widetilde{\Phi}^G(1,z) \tag{1.4.31}$$

To apply this, we note that the differential $\Phi^H : [0,1] \times \mathbb{D} \to \mathrm{Sp}(2)$ of the flow of H is given by

$$\Phi^H(t,z) = \exp(2\pi(1 + 1/n) \cdot it) \quad \text{for any } z \in \mathbb{D} \tag{1.4.32}$$

Likewise, the differential $\Phi^G : [0,1] \times \mathbb{D} \to \mathrm{Sp}(2)$ of the flow of G is given by the formula

$$\Phi^G(t,z) = \exp(2\pi \cdot R \cdot it) \text{ if } z \in U \setminus N(\partial U) \qquad \text{and} \qquad \Phi^G(t,z) = \mathrm{Id} \text{ if } z \in \mathbb{D} \setminus N(D) \tag{1.4.33}$$

By combining (1.4.32) and (1.4.33) with the decomposition (1.4.31), we acquire the following formula.

$$\rho \circ \widetilde{\Phi}(k,z) = k \cdot \big(1 + \frac{1}{n} + R \sum_{D \subset U} \chi_D(z)\big) \qquad \text{if} \quad z \in \mathbb{D} \setminus N(\partial U) \tag{1.4.34}$$

By dividing (1.4.34) by k and taking the limit as $k \to \infty$, we acquire the first formula (1.4.28).

Next, we examine the rotation number in the region $N(\partial D)$. Fix a component disk $D \subset U$ centered at p and a point $z \in N(\partial D)$. Let $S \subset N(\partial D)$ be a circle centered at p with $z \in S$, and let $u \in T_z S$ be a unit tangent vector to S at z. Finally, let

$$S_i = \phi^i(S) \qquad z_i = \phi^i(z) \qquad w_i = \phi^G \circ \phi^i(z) \qquad u_i = \Phi(i,z)u \qquad v_i = \Phi^G(1, \phi^i(z))\Phi(i,z)u$$

Note that these points and vectors satisfy $z_i \in S_i$, $w_i \in S_i$, $u_i \in T_{z_i} S_i$ and $v_i \in T_{w_i} S_i$ for each i. By applying the decomposition (1.4.31) and the additivity property (1.2.7) of ρ_s, we see that

$$\rho_u(\widetilde{\Phi}(k,z)) = \sum_{i=0}^{k-1} \rho_{u_i}(\widetilde{\Phi}^G(1,z_i))) + \sum_{i=0}^{k-1} \rho_{v_i}(\widetilde{\Phi}^H(1,w_i)) \tag{1.4.35}$$

Since ϕ^H is just an orthogonal rotation, we can use (1.4.32) to immediately conclude that

$$\rho_{v_i}(\widetilde{\Phi}^H(1, z_i))) = 1 + \frac{1}{n} \tag{1.4.36}$$

On the other hand, since u_i is tangent to the circle S_i, we may use the formula (1.4.15) to see that

$$\rho_{u_i}(\widetilde{\Phi}^G(1, z_i))) = -\frac{g_s'(r_p(z))}{2\pi r_p(z)} \tag{1.4.37}$$

Here g_s is the function such that $G|_{N(D)} = g_s \circ r_p$. By our hypotheses, we know that

$$\frac{|g_s'(r_p(z))|}{2\pi r_p(z)} \leq |R| \qquad \text{and} \qquad \operatorname{sgn}(-\frac{g_s'(r_p(z))}{2\pi r_p(z)}) = \operatorname{sgn}(R)$$

By plugging in the formulas (1.4.35) and (1.4.36), we can estimate $\rho_u \circ \widetilde{\Phi}(k, z)$ as follows.

$$k \cdot (1 + \frac{1}{n} + \min\{0, R\}) \leq \rho_u \circ \widetilde{\Phi}(k, z) \leq k \cdot (1 + \frac{1}{n} + \max\{0, R\})$$

We can therefore estimate r_ϕ. Since ρ_u and ρ are equivalent (Lemma 1.2.6) we find that

$$r_\phi(z) = \lim_{k \to \infty} \frac{\rho_u \circ \widetilde{\Phi}(k, z)}{k} \qquad \text{and thus} \qquad 1 + \frac{1}{n} + \min\{0, R\} \leq r_\phi(z) \leq 1 + \frac{1}{n} + \max\{0, R\}$$

Finally, the Ruelle invariant agrees with the integral of (1.4.28) over $\mathbb{D} \setminus N(\partial U)$ up to an $O(|R| \cdot \operatorname{area}(N(\partial U)))$ term. This proves (1.4.30). □

Lemma 1.4.20 (Periodic Points of ϕ)**.** *Suppose that $R > -2$. Then the periodic points of $\phi : \mathbb{D} \to \mathbb{D}$ satisfy*

$$\mathcal{A}(p) \geq \pi \qquad \textit{and} \qquad \rho(p) + \mathcal{L}(p) > 1 \tag{1.4.38}$$

Proof. First, consider the center $c = 0 \in \mathbb{D}$, where $\phi = \phi^H$. This periodic point has period $\mathcal{L}(c) = 1$. Thus, due to Lemmas 1.4.18 and 1.4.19, the action and rotation number are given by

$$\mathcal{A}(c) = \sigma_\phi(c) = \pi(1 + \frac{1}{n}) \qquad \rho(c) = r_\phi(c) = 1 + \frac{1}{n}$$

Any other periodic point p of H has period $\mathcal{L}(p) \geq n$, since ϕ rotates the sector $\mathbb{S}(n, k)$ to the section $\mathbb{S}(n, k+1)$. Since $n \geq 10$ and $R > -2$, we have $\sigma_\phi \geq \pi/10$ (by Lemma 1.4.18). The action of p is bounded below as follows.

$$\mathcal{A}(p) = \sum_{i=0}^{\mathcal{L}(p)-1} \sigma_\phi(\phi^i(p)) \geq \frac{\pi}{10} \cdot \mathcal{L}(p) \geq \pi$$

Likewise, we apply Lemma 1.4.19 to see that the rotation number of p is bounded as follows.

$$\rho(p) = \mathcal{L}(p) \cdot r_\phi(p) \geq \mathcal{L}(p) \cdot (1 + \frac{1}{n} + \min\{0, R\}) > \mathcal{L}(p) \cdot (-1 + \frac{1}{n}) \geq -\mathcal{L}(p) + 1$$

In particular, the rotation number satisfies $\rho(p) + \mathcal{L}(p) > 1$. □

1.4.5 Main Construction

We conclude this section by proving Proposition 1.4.1.

Proof. (Proposition 1.4.1) We construct the small Ruelle invariant and large Ruelle invariant contact forms separately. The basic strategy in both cases is to construct the special Hamiltonian flow of §1.4.4 with a specific choice of parameters, apply the open book construction of Proposition 1.4.8 to acquire a contact form and verify the desired properties by computing the relevant invariants (e.g. period, index, Calabi invariant) of ϕ.

Small Ruelle Case. We begin by choosing the parameters n, U, δ and R from Setup 1.4.15. Fix a large positive real number κ. Choose an integer $n > \kappa$ and a union U of disks D that each satisfy

$$\pi - \frac{1}{\kappa} < \sum_{D \subset U} \text{area}(D) < \pi \qquad \text{area}(D) < \frac{1}{\kappa} \qquad \text{diam}(D) < \frac{1}{\kappa}$$

Choose $\delta > 0$ so that $\text{area}(N(\partial U)) < \frac{1}{\kappa}$ and choose $R := -2 + \frac{1}{\kappa}$. These parameters define a special Hamiltonian flow $\phi = \phi^G \circ \phi^H$. By direct calculation and Lemma 1.4.18, we know that

$$G\#H = \pi(1 + \frac{1}{n}) \cdot (1 - r^2) \text{ near } \partial\mathbb{D} \qquad \text{and} \qquad \sigma_\phi > 0$$

Therefore, we can apply Construction 1.4.7 to ϕ to acquire a contact form α on S^3.

We now show that (a scaling of) α has the desired properties. First, by Proposition 1.4.8(b) and Lemma 1.4.18, the volume of (S^3, α) is given by the formula

$$\text{Cal}(\mathbb{D}, \phi) = \pi^2(1 + \frac{1}{n}) + R \sum_{D \subset U} \text{area}(D)^2 + O(d(U)) + O(R \cdot \text{area}(N(\partial U))) = \pi^2 + O(\kappa^{-1})$$

Next, by Proposition 1.4.8(c) and Lemma 1.4.19, the Ruelle invariant of (S^3, α) is given by the formula

$$\text{Ru}(\mathbb{D}, \phi) + \pi = \pi(2 + \frac{1}{n}) + R \sum_{D \subset U} \text{area}(D) + O(R \cdot \text{area}(N(\partial U)) = O(\kappa^{-1})$$

Last, by Proposition 1.4.8(d) the binding $b = \iota(\partial\mathbb{D})$ in S^3 has action and rotation number given by

$$\mathcal{A}(b) = \pi \qquad \rho(b) = 1 + \frac{1}{1 + 1/n} > 1$$

Due to Proposition 1.4.8(e) and Lemma 1.4.20, every periodic orbit of (S^3, α) other than b satisfies

$$\mathcal{A}(\gamma) \geq \pi \qquad \rho(\gamma) > 1$$

In particular, α is a dynamically convex contact form. Finally, rescale α by $\text{vol}(S^3,\alpha)^{-1/2}$. Then for any $\varepsilon > 0$, we may choose a κ sufficiently large so that

$$\text{vol}(S^3,\alpha) = 1 \qquad \text{Ru}(S^3,\alpha) = O(\kappa^{-1}) < \varepsilon \qquad \text{sys}(S^3,\alpha) > \frac{\pi^2}{\pi^2 + O(\kappa^{-1})} > 1 - \varepsilon$$

This is precisely the list of properties (1.4.1), and so we have constructed the desired small Ruelle invariant contact form.

Large Ruelle Case. Again, we choose parameters n, U, δ and R from Setup 1.4.15. Fix a large positive real number κ. Choose an integer $n > \kappa$ and a union U of disks D that each satisfy

$$\pi - \frac{1}{\kappa} < \sum_{D \subset U} \text{area}(D) < \pi \qquad \text{area}(D) < \frac{1}{\kappa^2} \qquad \text{diam}(D) < \frac{1}{\kappa}$$

Choose $\delta > 0$ such that $\text{area}(N(\partial U)) < \frac{1}{\kappa^2}$ and set $R = \kappa$. These parameters define a special Hamiltonian flow $\psi = \psi^G \circ \psi^H$. By direct calculation and Lemma 1.4.18, we know that

$$G\#H = \pi(1 + \frac{1}{n}) \cdot (1 - r^2) \text{ near } \partial\mathbb{D} \qquad \text{and} \qquad \sigma_\phi > 0$$

Therefore, we can apply Construction 1.4.7 to ψ to acquire a contact form β on S^3.

Now we show that (a scaling of) β has all of the desired properties. First, by Proposition 1.4.8(b) and Lemma 1.4.18, the volume of (S^3,β) is given by the formula

$$\text{Cal}(\mathbb{D},\psi) = \pi^2(1 + \frac{1}{n}) + R\sum_{D \subset U} \text{area}(D)^2 + O(d(U)) + O(R \cdot \text{area}(N(\partial U))) = \pi^2 + O(\kappa^{-1})$$

Next, by Proposition 1.4.8(c) and Lemma 1.4.19, the Ruelle invariant of (S^3,β) is given by the formula

$$\text{Ru}(\mathbb{D},\phi) + \pi = \pi(2 + \frac{1}{n}) + R\sum_{D \subset U} \text{area}(D) + O(R \cdot \text{area}(N(\partial U)) = \pi \cdot \kappa + O(1)$$

Last, by Proposition 1.4.8(d) the binding $b = \iota(\partial\mathbb{D})$ in S^3 has action and rotation number given by

$$\mathcal{A}(b) = \pi \qquad \rho(b) = 1 + \frac{1}{1 + 1/n} > 1$$

Due to Proposition 1.4.8(e) and Lemma 1.4.20, every periodic orbit of (S^3,β) other than b satisfies

$$\mathcal{A}(\gamma) \geq \pi \qquad \rho(\gamma) > 1$$

In particular, β is a dynamically convex contact form. Finally, rescale β by $\text{vol}(S^3,\beta)^{-1/2}$. Then for any $\varepsilon > 0$, we may choose a κ sufficiently large so that

$$\text{vol}(S^3,\alpha) = 1 \qquad \text{Ru}(S^3,\alpha) = O(\kappa) > \varepsilon^{-1} \qquad \text{sys}(S^3,\alpha) > \frac{\pi^2}{\pi^2 + O(\kappa^{-1})} > 1 - \varepsilon$$

This is precisely the list of properties (1.4.2), and so we have constructed the desired large Ruelle invariant contact form. □

Chapter 2

Disk-like surfaces of section and symplectic capacities

2.1 Introduction

2.1.1 Symplectic capacities

A *symplectic capacity* is a function c that assigns numbers $c(X,\omega) \in [0,\infty]$ to symplectic manifolds (X,ω) of a certain dimension $2n$. Symplectic capacities are required to be monotonic under symplectic embeddings and behave linearly with respect to scalings of the symplectic form. More precisely, one requires:

- (Monotonicity) If (X,ω) symplectically embeds into (X',ω'), then $c(X,\omega) \leq c(X',\omega')$.
- (Conformality) For every $r > 0$, we have $c(X, r\omega) = rc(X,\omega)$.

We will be mainly concerned with symplectic capacities of domains in Euclidean space $\mathbb{R}^{2n} = \mathbb{C}^n$ equipped with the standard symplectic form

$$\omega_0 := \sum_{j=1}^{n} dx_j \wedge dy_j.$$

We define the ball $B(a)$ and cylinder $Z(a)$ of symplectic width $a > 0$ to be the sets

$$B(a) := \left\{ z \in \mathbb{C}^n \mid \pi|z|^2 \leq a \right\} \qquad \text{and} \qquad Z(a) := \left\{ z \in \mathbb{C}^n \mid \pi|z_1|^2 \leq a \right\}.$$

A symplectic capacity is called *normalized* if it satisfies

- (Normalization) $c(B(\pi)) = c(Z(\pi)) = \pi$.

Two examples of normalized symplectic capacities which are easy to define are the *Gromov width* $\mathrm{c_G}$ and the *cylindrical capacity* $\mathrm{c_Z}$. We use the notation $A \overset{s}{\hookrightarrow} B$ to indicate that there exists a *symplectic embedding* of A into B, i.e. a smooth embedding preserving the symplectic structure. Gromov width and cylindrical capacity are given by

$$\mathrm{c_G}(X) \coloneqq \sup\{a \mid B(a) \overset{s}{\hookrightarrow} X\} \qquad \text{and} \qquad \mathrm{c_Z}(X) \coloneqq \inf\{a \mid X \overset{s}{\hookrightarrow} Z(a)\}.$$

Note, however, that it is highly non-trivial to show that $\mathrm{c_G}$ and $\mathrm{c_Z}$ are indeed normalized capacities. In fact, this is equivalent to the celebrated Gromov non-squeezing theorem [46]. There is a whole collection of symplectic capacities whose definition involves Hamiltonian dynamics. Examples of normalized capacities arising this way are the Hofer-Zehnder capacity $\mathrm{c_{HZ}}$ introduced in [56] and the Viterbo capacity $\mathrm{c_{SH}}$ defined in [101] using symplectic homology. Other capacities come in families parametrized by positive integers. Examples are the Ekeland-Hofer capacities $\mathrm{c_k^{EH}}$ defined in [33] and [34] and the equivariant capacities $\mathrm{c_k^{CH}}$ constructed by Gutt and Hutchings in [47] from S^1-equivariant symplectic homology. The first capacities $\mathrm{c_1^{EH}}$ and $\mathrm{c_1^{CH}}$ in these families are normalized. In dimension 4, there exists a sequence of capacities $\mathrm{c_k^{ECH}}$ defined by Hutchings in [65] using embedded contact homology. Again, the first capacity $\mathrm{c_1^{ECH}}$ is normalized. For more information on symplectic capacities, we refer the reader to Cieliebak-Hofer-Latschev-Schlenk [15] and the references therein.

Recall that a contact form on an odd dimensional manifold is a nowhere vanishing 1-form α such that the restriction of $d\alpha$ to the hyperplane field $\xi \coloneqq \ker \alpha$ is non-degenerate. A contact form α induces a Reeb vector field $R = R_\alpha$ characterized by the equations

$$\iota_R \alpha = 1 \qquad \text{and} \qquad \iota_R d\alpha = 0.$$

Studying the dynamical properties of Reeb flows, such as the existence of periodic orbits, is a topic of great interest in symplectic geometry. Contact forms naturally arise on the boundaries of convex or, more generally, star-shaped domains $X \subset \mathbb{R}^{2n}$. We equip $\mathbb{R}^{2n}$ with the radial Liouville vector field Z_0 and the associated Liouville 1-form λ_0

$$Z_0 = \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}) = \frac{1}{2} r \partial_r \qquad \text{and} \qquad \lambda_0 = \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j). \tag{2.1.1}$$

They are related to the symplectic form ω_0 via $d\lambda_0 = \omega_0$ and $\iota_{Z_0}\omega_0 = \lambda_0$. Consider a closed, connected hypersurface $Y \subset \mathbb{R}^{2n}$. The restriction of λ_0 to Y is a contact form if and only if the Liouville vector field Z_0 is transverse to Y. If Y has this property, we call it a star-shaped hypersurface and the domain bounded by Y a star-shaped domain. Note that all star-shaped hypersurfaces are contactomorphic to the sphere S^{2n-1} equipped with its standard contact structure. Moreover, any contact form on S^{2n-1} defining the standard contact structure is strictly contactomorphic to the restriction of λ_0 to some star-shaped hypersurface. Thus studying star-shaped hypersurfaces is equivalent to studying contact forms on S^{2n-1} defining the standard contact structure.

It was proved by Rabinowitz in [88] that there exists a periodic Reeb orbit on the boundary of any star-shaped domain $X \subset \mathbb{R}^{2n}$. If γ is a periodic orbit on ∂X, we define its action $\mathcal{A}(\gamma)$ to be

$$\mathcal{A}(\gamma) \coloneqq \int_\gamma \lambda_0.$$

The capacities $\mathrm{c_{HZ}}$, $\mathrm{c_{SH}}$, $\mathrm{c_k^{EH}}$ and $\mathrm{c_k^{CH}}$ have the following important property: Their value on a star-shaped domain $X \subset \mathbb{R}^{2n}$ is equal to the action $\mathcal{A}(\gamma)$ of some (possibly multiply covered) periodic orbit γ on ∂X. The capacities $\mathrm{c_k^{ECH}}$ have a similar property. Their values can be represented as the sum of the actions of finitely many periodic orbits.

2.1.2 Viterbo's conjecture

In [102], Viterbo stated the following fascinating conjecture concerning normalized symplectic capacities.

Conjecture 2.1.1 (Viterbo conjecture)**.** *Let $X \subset \mathbb{R}^{2n}$ be a convex domain. Then any normalized symplectic capacity c satisfies the inequality*

$$c(X) \leq (n! \operatorname{vol}(X))^{\frac{1}{n}}. \tag{2.1.2}$$

Note that inequality (2.1.2) holds for the Gromov width $\mathrm{c_G}$. This is an easy consequence of the fact that symplectomorphisms are volume preserving. For all other normalized capacities introduced above, Conjecture 2.1.1 is open. It was proved by Artstein-Avidan-Karasev-Ostrover [6] that Conjecture 2.1.1 implies Mahler's conjecture, an old conjecture in convex geometry. This is one of the reasons for the recent increase in interest in Viterbo's conjecture. There is a stronger version of Viterbo's conjecture.

Conjecture 2.1.2 (Strong Viterbo conjecture)**.** *Let $X \subset \mathbb{R}^{2n}$ be a convex domain. Then all normalized symplectic capacities agree on X.*

The strong Viterbo conjecture together with the above observation that Conjecture 2.1.1 holds for $\mathrm{c_G}$ immediately implies that Conjecture 2.1.1 is true for all normalized symplectic capacities. It is an easy consequence of the definitions that any normalized symplectic capacity c satisfies $\mathrm{c_G} \leq c \leq \mathrm{c_Z}$. Thus Conjecture 2.1.2 is equivalent to saying that Gromov width $\mathrm{c_G}$ and cylindrical capacity $\mathrm{c_Z}$ agree on convex domains. The convexity assumption in Viterbo's conjectures is essential. Even within the class of star-shaped domains there exist domains X with arbitrarily small volume such that the cylindrical capacity satisfies $\mathrm{c_Z}(X) \geq 1$ (see Hermann's paper [52]). We refer to Gutt-Hutchings-Ramos [48] for a recent account on Viterbo's conjectures.

2.1.3 Embeddings into cylinders

Except for the Gromov width $\mathrm{c_G}$ and the cylindrical capacity $\mathrm{c_Z}$, the construction of most, if not all, known normalized capacities is based on Hamiltonian dynamics. There has been a significant amount of work showing that many of these dynamical capacities agree. For example, it follows from work of Ekeland, Hofer, Zehnder, Abbondandolo, Kang and Irie that the values of the capacities c_{HZ}, c_{SH}, c_1^{EH} and c_1^{CH} on a convex domain $X \subset \mathbb{R}^{2n}$ all agree with $A_{\min}(X)$, the minimal action $\mathcal{A}(\gamma)$ of a periodic orbit γ on ∂X. We refer to Theorem 1.12 in [48] for a summary. On the other hand, except for the obvious inequalities $c_G \leq c \leq c_Z$, almost nothing is known about the relationship between the dynamical capacities and the embedding capacities c_G and c_Z. The purpose of our work is to bridge the gap between dynamics and symplectic embeddings.

While it is a well established strategy to use dynamics to obstruct symplectic embeddings, in this paper we go in the opposite direction and use dynamical information to construct symplectic embeddings. The dynamical information is given in terms of *global surfaces of section*, an important concept in dynamics going back to Poincaré. Let α be a contact form on a closed 3-manifold Y. We call an embedded surface (with boundary) $\Sigma \subset Y$ a global surface of section for the Reeb flow if the boundary $\partial\Sigma$ is embedded and consists of closed, simple Reeb orbits, the Reeb vector field R is transverse to the interior $\text{int}(\Sigma)$, and every trajectory not contained in $\partial\Sigma$ meets $\text{int}(\Sigma)$ infinitely often forward and backward in time. Surfaces of section are an extremely useful tool in three dimensional Reeb dynamics. For example, they have been used to show that every (non-degenerate) Reeb flow on a closed contact 3-manifold must have either two or infinitely many periodic orbits (see [55], [23] and [16]). In this paper, we will be concerned with disk-like global surfaces of section, i.e. the case that Σ is diffeomorphic to the 2-dimensional closed disk. This implies that the underlying contact manifold must be the 3-sphere S^3 with its unique tight contact structure. For more details on surfaces of section we refer to section 2.2.2.

Let us begin by stating the following general dynamical criterion guaranteeing the existence of symplectic embeddings into a cylinder.

Theorem 2.1.3. *Let $X \subset \mathbb{R}^4$ be a star-shaped domain. Let $\Sigma \subset \partial X$ be a ∂-strong (see Definition 2.2.3) disk-like global surface of section of the natural Reeb flow on ∂X of symplectic area*

$$a := \int_\Sigma \omega_0 = \mathcal{A}(\partial\Sigma).$$

Then there exists a symplectic embedding $X \overset{s}{\hookrightarrow} Z(a)$. In particular, we have $c_Z(X) \leq a$.

The boundary of a general star-shaped domain need not possess a disk-like global surface of section (see van Koert's paper [100]). In this case, Theorem 2.1.3 is vacuous. However, Theorem 2.1.3 is particularly useful when applied to the important class of *dynamically convex* domains because for such domains there are general existence theorems for disk-like global surfaces of section due to Hofer-Wysocki-Zehnder [55], Hryniewicy-Salomão [60] and Hryniewicz [57]. Ever since the notion of dynamical convexity was first introduced in [55], it has played a significant role in numerous papers on Reeb dynamics (see e.g. [57], [4], [5], [44], [104], [70], just to name a few). We recall the definition of dynamical convexity from [55].

Definition 2.1.4 ([55, Definition 1.2])**.** A contact form α on S^3 defining the unique tight contact structure is called *dynamically convex* if every periodic Reeb orbit γ of α has Conley-Zehnder index $\mathrm{CZ}(\gamma)$ at least 3. A star-shaped domain $X \subset \mathbb{R}^4$ is called dynamically convex if the restriction of the standard Liouville 1-form λ_0 (see equation (2.1.1)) to the boundary ∂X is dynamically convex.

Remark 2.1.5. The Conley-Zehnder index $\mathrm{CZ}(\gamma)$ of a periodic Reeb orbit depends on the choice (up to homotopy) of a symplectic trivialization of the contact structure along the orbit. On S^3 every contact structure admits a unique global trivialization up to homotopy. This is the trivialization used in Definition 2.1.4. Let us also point out that usually the Conley-Zehnder index is only defined for non-degenerate orbits. If γ is degenerate, then $\mathrm{CZ}(\gamma)$ in Definition 2.1.4 refers to the lower semicontinuous extension of the Conley-Zehnder index.

It is proved in [55] that every convex domain $X \subset \mathbb{R}^4$ whose boundary has positive definite second fundamental form is dynamically convex.

Let us introduce the following terminology.

Definition 2.1.6. A simple closed orbit of a tight Reeb flow on S^3 is called a *Hopf orbit* if it is unknotted and has self-linking number equal to -1 when viewed as a transverse knot.

The reason for this terminology is that the fibers of the Hopf fibration on S^3 are unknotted and have self-linking number -1. It is shown by Hofer-Wysocki-Zehnder [54] that the boundary of every star-shaped domain carries at least one Hopf orbit. Given a star-shaped $X \subset \mathbb{R}^4$, let us therefore define

$$\mathrm{A}_{\mathrm{Hopf}}(X) := \inf\{\mathcal{A}(\gamma) \mid \gamma \text{ is Hopf orbit on } \partial X\} \in (0,\infty). \tag{2.1.3}$$

The significance of Hopf orbits to our discussion is the following. By work of Hryniewicz-Salomão [60] and Hryniewicz [57], a simple periodic orbit of a dynamically convex Reeb flow on S^3 bounds a disk-like global surface of section if and only if it is a Hopf orbit.

Our main result provides a dynamical characterization of the cylindrical capacity of 4-dimensional dynamically convex domains. It can be thought of as a generalization of the Gromov non-squeezing theorem from the ball to arbitrary dynamically convex domains.

Theorem 2.1.7. *Let $X \subset \mathbb{R}^4$ be a dynamically convex domain and let $a > 0$. Then there exists a symplectic embedding $X \overset{s}{\hookrightarrow} Z(a)$ if and only if $a \geq \mathrm{A}_{\mathrm{Hopf}}(X)$. In particular, this implies that*

$$\mathrm{c_Z}(X) = \mathrm{A}_{\mathrm{Hopf}}(X).$$

Moreover, the infimum in the definition of $\mathrm{c_Z}(X)$ *is attained, i.e. there exists an optimal embedding of* X *into a smallest cylinder.*

Let us point out that sharp symplectic embedding results such as Theorem 2.1.7 are rather rare. Moreover, most known results concern highly symmetric toric domains with integrable flows on their boundaries. In contrast to this, the Reeb dynamics of dynamically convex domains can be extremely rich. It is also worth mentioning that while the cylindrical capacity $\mathrm{c_Z}$ is apriori rather elusive, the quantity $\mathrm{A_{Hopf}}$ can in principle be computed numerically given an explicit domain.

Hryniewicz-Hutchings-Ramos show in [58] that $\mathrm{A_{Hopf}}(X)$ agrees with the first embedded contact homology capacity $\mathrm{c_1^{ECH}}(X)$ for all dynamically convex domains $X \subset \mathbb{R}^4$. We obtain the following corollary.

Corollary 2.1.8. *For all dynamically convex domains* $X \subset \mathbb{R}^4$ *we have*

$$\mathrm{c_Z}(X) = \mathrm{c_1^{ECH}}(X).$$

2.1.4 The local strong Viterbo conjecture

Abbondandolo-Bramham-Hryniewicz-Salomão [3] proved that for all domains $X \subset \mathbb{R}^4$ whose boundary ∂X is smooth and sufficiently close to the unit sphere with respect to the C^3-topology, the minimal action $\mathrm{A_{min}}(X)$ satisfies inequality (2.1.2). This result was generalized to small perturbations of more general 3-dimensional Zoll Reeb flows by Benedetti-Kang [8] and to arbitrary dimension by Abbondandolo-Benedetti in [1]. A consequence of these works is that the capacities $\mathrm{c_{HZ}}$, $\mathrm{c_{SH}}$, $\mathrm{c_1^{EH}}$ and $\mathrm{c_1^{CH}}$ satisfy Conjecture 2.1.1 in a C^3-neighbourhood of the round ball. Our second main result significantly strengthens this in the 4-dimensional case. We prove the full strong Viterbo conjecture (Conjecture 2.1.2) near the ball.

Theorem 2.1.9. *Let* $X \subset \mathbb{R}^4$ *be a convex domain. If* ∂X *is sufficiently close to the unit sphere* $S^3 \subset \mathbb{R}^4$ *with respect to the* C^3*-topology, then all normalized symplectic capacities agree on* X*.*

2.1.5 Systoles and Hopf orbits

The following question was first raised by Hofer-Wysocki-Zehnder in [55].

Question 2.1.10. Let $X \subset \mathbb{R}^4$ be a (dynamically) convex domain. Must a systole of X, i.e. a Reeb orbit on ∂X of least action, be a Hopf orbit and therefore bound a disk-like global surface of section?

This question is particularly interesting in view of Theorem 2.1.7. An affirmative answer would imply that $\mathrm{A_{min}}(X) = \mathrm{c_Z}(X)$ for all (dynamically) convex domains. This would force all normalized capacities which are bounded from below by $\mathrm{A}_{\min}(X)$ to be equal to the cylindrical capacity. The number of known normalized capacities would be cut down to just two: the Gromov width and the cylindrical capacity.

Equality of $\mathrm{A}_{\min}$ and $\mathrm{A}_{\mathrm{Hopf}}$ was proved by Hainz [49] (see also [50]) under certain curvature assumptions.

Theorem 2.1.11 (Hainz)**.** *Let $X \subset \mathbb{R}^4$ be a strictly convex domain. Assume that the principal curvatures $a \geq b \geq c$ of the boundary ∂X satisfy the pointwise pinching condition $a \leq b + c$. Then any periodic Reeb orbit γ on ∂X of Conley-Zehnder index 3 is a Hopf orbit.*

It follows from Ekeland's book [32] (see in particular Theorem 3 and Proposition 9 in chapter V) that for convex domains X with strictly positively curved boundary a Reeb orbit of minimal action has Conley-Zehnder index 3. Thus we have $\mathrm{A}_{\min}(X) = \mathrm{A}_{\mathrm{Hopf}}(X)$ if X satisfies the curvature assumptions in Theorem 2.1.11.

2.1.6 Overview of the proofs

Let us explain the main ideas. Consider the unit disk $\mathbb{D} \subset \mathbb{C}$ equipped with the standard symplectic form $\omega_0 = dx \wedge dy$. Let

$$H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$$

be a 1-periodic Hamiltonian vanishing on the boundary $\partial\mathbb{D}$. Consider *time-energy extended phase space*

$$\widetilde{\mathbb{D}} := \mathbb{R}_s \times (\mathbb{R}/\mathbb{Z})_t \times \mathbb{D}$$

equipped with the symplectic form

$$\widetilde{\omega_0} = ds \wedge dt + \omega_0.$$

Let

$$\Gamma(H) := \{(H(t,z), t, z) \in \widetilde{\mathbb{D}} \mid (t,z) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}\}$$

be the graph of H. This is a hypersurface in $\widetilde{\mathbb{D}}$ of codimension 1. Hence the symplectic form $\widetilde{\omega_0}$ induces a characteristic foliation on $\Gamma(H)$. It is an easy computation (see Lemma 2.3.3) that the vector field

$$R := X_{H_t}(z) + \partial_t + \partial_t H(z,t) \cdot \partial_s$$

is tangent to the characteristic foliation on $\Gamma(H)$. Observe that the projection of the flow of R to the disk $\mathbb{D}$ agrees with the Hamiltonian flow ϕ_H^t on $\mathbb{D}$ induced by H. In particular, we see that the image of the map

$$f : \mathbb{D} \to \Gamma(H) \qquad z \mapsto (H(0,z), 0, z)$$

is a disk-like surface of section of the flow on $\Gamma(H)$ and that the first return map is given by ϕ_H^1.

Main construction. Assume that the Hamiltonian H is strictly positive in the interior $\mathrm{int}(\mathbb{D})$ of the disk and vanishes on the boundary $\partial\mathbb{D}$. We abbreviate

$$\widetilde{\mathbb{D}}_+ := \mathbb{R}_{\geq 0} \times \mathbb{R}/\mathbb{Z} \times \mathbb{D} \qquad \text{and} \qquad \widetilde{\mathbb{D}}_0 := \{0\} \times \mathbb{R}/\mathbb{Z} \times \mathbb{D}.$$

Consider the map

$$\Phi : \widetilde{\mathbb{D}}_+ \to \mathbb{C}^2 \qquad \Phi(s,t,z) := \left(z \ , \ \sqrt{\frac{s}{\pi}} \cdot e^{2\pi i t} \right).$$

Note that the image of Φ is precisely the cylinder $Z(\pi)$. We observe that Φ restricts to a symplectomorphism

$$\Phi : (\widetilde{\mathbb{D}}_+ \setminus \widetilde{\mathbb{D}}_0, \widetilde{\omega_0}) \to (Z(\pi) \setminus (\mathbb{D} \times \{0\}), \omega_0).$$

The image $\Phi(\Gamma(H)) \subset \mathbb{C}^2$ is a smooth hypersurface away from the circle $\partial\mathbb{D} \times \{0\}$. Under suitable assumptions on the boundary behaviour of H, it is smooth everywhere. In order to keep the introduction simple, let us ignore this issue for now. Let $A(H)$ denote the domain bounded by $\Phi(\Gamma(H))$. Since Φ restricts to a symplectomorphism on $\widetilde{\mathbb{D}}_+ \setminus \widetilde{\mathbb{D}}_0$, it maps the characteristic foliation on $\Gamma(H)$ to the characteristic foliation on $\partial A(H)$. Thus $\Phi \circ f$ parametrizes a disk-like surface of section of the characteristic foliation on $\partial A(H)$. The first return map is given by ϕ_H^1. Note that $\partial A(H)$ need not be star-shaped or even of contact type.

Embeddings into the cylinder. Now suppose that $X \subset \mathbb{C}^2$ is a star-shaped domain and that the boundary ∂X admits a disk-like surface of section $\Sigma \subset \partial X$. After scaling, we can always assume that the symplectic area of Σ is equal to π. Suppose that $g : \mathbb{D} \to \Sigma$ is a parametrization of Σ such that $g^*\omega_0 = \omega_0$. Here ω_0 denotes the standard symplectic form on both $\mathbb{C}^2$ and $\mathbb{C}$. Let $\phi \in \mathrm{Ham}(\mathbb{D}, \omega_0)$ be the first return map (see equation (2.2.4)). In section 2.2.2, we explain how to lift ϕ to an element $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(\mathbb{D}, \omega_0)$ of the universal cover. Such a lift depends on a choice of trivialization which, roughly speaking, is an identification of $\partial X \setminus \partial\Sigma$ with the solid torus. In section 2.2.2 we classify isotopy classes of such trivializations via an integer-valued function called degree. Let $\widetilde{\phi}$ denote the lift of ϕ with respect to a trivialization of degree 0. Suppose that $\widetilde{\phi}$ can be generated by a 1-periodic Hamiltonian H which vanishes on the boundary and is strictly positive in the interior. The above discussion shows that ∂X and $\partial A(H)$ admit disk-like surfaces of section whose first return maps agree. In fact, one can show more: The lifts of the first return maps to $\widetilde{\mathrm{Ham}}(\mathbb{D}, \omega_0)$ (with respect to trivializations of degree 0) agree as well. We use a well-known result of Gromov and McDuff (Theorem 2.3.9) to show that this in fact implies that the domains X and $A(H)$ must be symplectomorphic. By definition, $A(H)$ is contained in the cylinder $Z(\pi)$. Therefore, we obtain an embedding of X into the cylinder $Z(\pi)$. We make these arguments precise in

Theorem 2.3.1. Unfortunately, we do not know whether the lift of the first return map of a disk-like surface of section with respect to a trivialization of degree 0 can always be generated by a Hamiltonian which vanishes on the boundary and is strictly positive in the interior. In order to resolve this issue, let us observe that if H and G are Hamiltonians vanishing on the boundary $\partial\mathbb{D}$ and strictly positive in the interior $\mathrm{int}(\mathbb{D})$, then the inequality $H \leq G$ implies the inclusion $A(H) \subset A(G)$. Roughly speaking, this says that we can increase the Hamiltonian generating the first return map by making the domain bigger. More precisely, in Proposition 2.3.2 we prove that any star-shaped domain with a disk-like surface of section in its boundary can be symplectically embedded into a bigger star-shaped domain whose boundary admits a disk-like surface of section of the same area and with the property that the degree 0 lift of the first return map can be generated by a positive Hamiltonian. Theorem 2.1.3 is an easy consequence of Theorem 2.3.1 and Proposition 2.3.2.

Ball embeddings. Suppose that the degree 0 lift of the first return map of the surface of section $\Sigma \subset \partial X$ can be generated by a Hamiltonian H which satisfies the inequality

$$H(t,z) \geq \pi(1-|z|^2). \tag{2.1.4}$$

Then the domain $A(H)$ is squeezed between the ball $B(\pi)$ and the cylinder $Z(\pi)$, i.e.

$$B(\pi) \subset A(H) \subset Z(\pi).$$

Since X is symplectomorphic to $A(H)$, this implies that $\mathrm{c_G}(X) = \mathrm{c_Z}(X)$. The strategy of the proof of Theorem 2.1.9 is to show that if ∂X is sufficiently close to the round sphere, then the shortest Reeb orbit on ∂X must bound a disk-like surface of section with the property that the degree 0 lift of the first return map can be generated by a Hamiltonian satisfying (2.1.4). This is the subject of section 2.4 and makes use of generalized generating functions as introduced in [3]. Let us sketch the main ideas in a special case. We assume that $g : \mathbb{D} \to \Sigma \subset \partial X$ parametrizes a surface of section whose boundary orbit $\partial\Sigma$ has minimal action among all closed Reeb orbits on ∂X. Moreover, assume that $g^*\omega_0 = \omega_0$. Let $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(\mathbb{D},\omega_0)$ be the degree 0 lift of the first return map ϕ to the universal cover. The periodic orbits on ∂X different from the boundary orbit $\partial\Sigma$ correspond to the periodic points of ϕ. As explained in section 2.2.1, any fixed point p of ϕ has a well-defined action $\sigma_{\widetilde{\phi}}(p)$ depending on the lift $\widetilde{\phi}$ to the universal cover. Lifts with respect to a trivialization of degree 0 have the property that $\sigma_{\widetilde{\phi}}(p)$ is equal to the action of the corresponding closed Reeb orbit on ∂X (see Lemma 2.2.5). Since $\partial\Sigma$ is assumed to have minimal action, this implies that

$$\sigma_{\widetilde{\phi}}(p) \geq \mathcal{A}(\partial\Sigma) = \pi \tag{2.1.5}$$

for all fixed points p of ϕ. If ∂X is the unit sphere S^3, then the degree 0 lift of the first return map $\widetilde{\phi}$ is equal to the counter-clockwise rotation by angle 2π. Let us denote this rotation by $\widetilde{\rho} \in \widetilde{\mathrm{Ham}}(\mathbb{D},\omega_0)$. If ∂X is sufficiently close to S^3 with respect to the C^3-topology, then $\widetilde{\phi}$ is C^1-close to $\widetilde{\rho}$. This is proved in [3] and explained in section 2.5. In order to simplify the discussion, let us assume that $\widetilde{\phi}$ is actually equal to $\widetilde{\rho}$ in a small neighbourhood of the boundary $\partial\mathbb{D}$. Therefore, we can regard

$$\psi := \widetilde{\rho}^{-1} \circ \widetilde{\phi}$$

as an element of $\mathrm{Ham}_c(\mathbb{D}, \omega_0)$, the group of Hamiltonian diffeomorphisms compactly supported in the interior $\mathrm{int}(\mathbb{D})$. It is C^1-close to the identity and it follows from (2.1.5) that the action $\sigma_\psi(p)$ is non-negative for all fixed points p. The following result is a special case of Corollary 2.4.3 in section 2.4.

Proposition 2.1.12 (Special case of Corollary 2.4.3)**.** *Let $\psi \in \mathrm{Ham}_c(\mathbb{D}, \omega_0)$ be a Hamiltonian diffeomorphism compactly supported in the interior* $\mathrm{int}(\mathbb{D})$*. Suppose that all fixed points of ψ have non-negative action and that ψ is close to the identity with respect to the C^1-topology. Then ψ can be generated by a non-negative Hamiltonian H with support contained in* $\mathrm{int}(\mathbb{D})$*.*

We apply Proposition 2.1.12 to the Hamiltonian diffeomorphism $\psi = \widetilde{\rho}^{-1} \circ \widetilde{\phi}$. Let G denote the resulting Hamiltonian. We may assume that G_t vanishes for time t close to 0 or 1. Let us define the Hamiltonian K by the formula

$$K(t, z) := \pi(1 - |z|^2).$$

This Hamiltonian generates the rotation $\widetilde{\rho}$. Now set

$$H_t := (K \# G)_t := K_t + G_t \circ (\phi_K^t)^{-1}.$$

This defines a 1-periodic Hamiltonian. Its time-1-flow represents $\widetilde{\phi}$. Since G is non-negative, H satisfies inequality (2.1.4). As explained above, this implies that $B(\pi) \subset A(H) \subset Z(\pi)$ and hence $\mathrm{c_G}(X) = \mathrm{c_Z}(X)$.

Existence of non-negative Hamiltonians. Let us sketch the proof of Proposition 2.1.12. It follows the same basic idea as the proof of Corollary 2.4.3. The advantage of our simplified setting here is that we can work with standard generating functions (see e.g. chapter 9 in [84]) and do not have to appeal to the generalized ones from [3]. Let $\psi = (X, Y)$ denote the components of ψ. There exists a unique generating function $W : \mathbb{D} \to \mathbb{R}$, compactly supported in $\mathrm{int}(\mathbb{D})$, such that

$$\begin{cases} X - x = \ \ \partial_2 W(X, y) \\ Y - y = -\partial_1 W(X, y) \end{cases}$$

The fixed points of ψ are precisely the critical points of W. Moreover, the action of a fixed point is equal to the value of W at the fixed point. Since all fixed points are assumed to have non-negative action, this implies that W takes non-negative values at all its critical points. In particular, this implies that W is non-negative. For $t \in [0, 1]$, let us define the generating function $W_t := t \cdot W$. Let ψ_t denote the compactly supported symplectomorphism generated by W_t. This defines an arc in $\mathrm{Ham}_c(\mathbb{D}, \omega_0)$ from the identity to ψ. Let H be the unique compactly supported Hamiltonian generating the arc ψ_t. Our goal is to show that H is non-negative. A direct computation shows that H_0, the Hamiltonian H at time 0, is equal to W and in particular non-negative. The Hamiltonian H need not be autonomous. However, the following is true. For every fixed $t \in [0,1]$, the set of critical points of H_t is equal to the set of critical points of W. Moreover, W and H_t agree on this set. Hence H_t takes non-negative values on its critical points. Therefore, the Hamiltonian H must be non-negative.

Approximation results. In general, the first return map of a disk-like surface of section need not be equal to the identity in any neighbourhood of $\partial\mathbb{D}$. Nevertheless, it will be convenient to assume that the Reeb flow in a small neighbourhood of the boundary orbit $\partial\Sigma$ has a specific simple form. More precisely, we want to assume that the local first return map of a small disk transverse to the orbit $\partial\Sigma$ is smoothly conjugated to a rotation. The main purpose of section 2.5 is to prove that we may approximate a given contact form with contact forms having this property. This is slightly subtle because we need to keep track of a certain number of higher order derivatives of the Reeb vector field in order to be able to apply the results from section 2.4 to the first return map.

Organization. The rest of the paper is structured as follows:

In §2.2 we review some preliminary material on area preserving disk maps (§2.2.1) and global surfaces of section (§2.2.2).

The main results of section 2.3, namely the embedding result Theorem 2.3.1 and Proposition 2.3.2 on modifications of star-shaped domains, are stated in §2.3.1. The construction of the domain $A(H)$ is explained in §2.3.2. Proofs are given in §2.3.3 and §2.3.4. Note that the reader only interested in Theorems 2.1.3 and 2.1.7 on the cylindrical embedding capacity and not in the local version of the strong Viterbo conjecture (Theorem 2.1.9) may skip §2.4 and §2.5 and directly move on to §2.6, where we prove our main results.

The main results of §2.4 are Theorem 2.4.2 and Corollary 2.4.3 guaranteeing the existence of non-negative Hamiltonians generating certain Hamiltonian diffeomorphisms. They are stated in §2.4.1. In §2.4.2 and §2.4.3 we review material from [3] on generalized generating functions. The only result that is not also explicitly explained in [3] is Proposition 2.4.10. The proofs of the main results of §2.4 are given in §2.4.4 and §2.4.5.

§2.5 is slightly technical in nature. The main result that is needed outside of this section is Proposition 2.5.1 on certain approximations of contact forms.

In §2.6 we give proofs of the main results of our paper.

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2.2 Preliminaries

2.2.1 Area preserving maps of the disk

In this section, we recall some basic concepts and results concerning area preserving diffeomorphisms of the disk. Most of the material is taken from Abbondandolo-Bramham-Hryniewicz-Salomão [3, sections 2.1 and 2.2]. Let ω be a smooth 2-form on the closed unit disk $\mathbb{D} \subset \mathbb{C}$. We assume that ω is positive in the interior $\mathrm{int}(\mathbb{D})$. On the boundary, ω is allowed to vanish. We let $\mathrm{Diff}^+(\mathbb{D})$ denote the group of orientation preserving diffeomorphisms of $\mathbb{D}$. Let

$$\pi : \widetilde{\mathrm{Diff}}(\mathbb{D}) \to \mathrm{Diff}^+(\mathbb{D}) \qquad \widetilde{\phi} \mapsto \phi$$

be the universal cover. We define $\mathrm{Diff}(\mathbb{D}, \omega) \subset \mathrm{Diff}^+(\mathbb{D})$ to be the subgroup of all diffeomorphisms preserving ω. Let $\widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ denote the preimage of $\mathrm{Diff}(\mathbb{D}, \omega)$ under the universal covering map π. If ω is nowhere vanishing on the boundary $\partial\mathbb{D}$, then this agrees with the actual universal cover of $\mathrm{Diff}(\mathbb{D}, \omega)$. However, in general it need not agree with the universal cover (see Remark 2.1 in [3]). Elements $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ can be represented by arcs $(\phi_t)_{t\in[0,1]}$ in $\mathrm{Diff}^+(\mathbb{D})$ which start at the identity and end at $\phi_1 = \pi(\widetilde{\phi}) \in \mathrm{Diff}(\mathbb{D}, \omega)$. Two such arcs are equivalent in $\widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ if they are isotopic in $\mathrm{Diff}^+(\mathbb{D})$ with fixed end points.

Consider a primitive λ of ω and an element $\widetilde{\phi} = [(\phi_t)_{t\in[0,1]}] \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$. Then there exists a unique smooth function $\sigma_{\widetilde{\phi},\lambda} \in C^\infty(\mathbb{D}, \mathbb{R})$ such that

$$\phi^*\lambda - \lambda = d\sigma_{\widetilde{\phi},\lambda} \tag{2.2.1}$$

and

$$\sigma_{\widetilde{\phi},\lambda}(z) = \int_{\{t\mapsto\phi_t(z)\}} \lambda \tag{2.2.2}$$

for all $z \in \partial\mathbb{D}$ (see [3, section 2.1]). We call $\sigma_{\widetilde{\phi},\lambda}$ the *action* of $\widetilde{\phi}$ with respect to λ. We recall the following basic result [3, Lemma 2.2].

Lemma 2.2.1. *Let $\widetilde{\phi}, \widetilde{\psi} \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$. Let λ be a primitive of ω and let u be a smooth real-valued function on $\mathbb{D}$. Then:*

1. $\sigma_{\widetilde{\phi},\lambda+du} = \sigma_{\widetilde{\phi},\lambda} + u \circ \phi - u$
2. $\sigma_{\widetilde{\psi}\circ\widetilde{\phi},\lambda} = \sigma_{\widetilde{\psi},\lambda} \circ \phi + \sigma_{\widetilde{\phi},\lambda}$
3. $\sigma_{\widetilde{\phi}^{-1},\lambda} = -\sigma_{\widetilde{\phi},\lambda} \circ \phi^{-1}$

In particular, item (1) in Lemma 2.2.1 implies that the value $\sigma_{\widetilde{\phi},\lambda}(p)$ at a fixed point p of ϕ is independent of the choice of primitive λ and we will occasionally denote this value by $\sigma_{\widetilde{\phi}}(p)$.

The *Calabi invariant* $\mathrm{Cal}(\widetilde{\phi})$ is defined to be the integral

$$\mathrm{Cal}(\widetilde{\phi}) = \int_{\mathbb{D}} \sigma_{\widetilde{\phi},\lambda} \cdot \omega.$$

It follows from item (1) in Lemma 2.2.1 that this is independent of the choice of primitive λ and from item (2) that

$$\mathrm{Cal} : \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega) \to \mathbb{R}$$

is a group homomorphism.

Let $(\phi_t)_{t\in[0,1]}$ be an arc in $\mathrm{Diff}(\mathbb{D}, \omega)$. Let X_t be the vector field generating ϕ_t. Since ϕ_t preserves ω, the interior product $\iota_{X_t}\omega$ is a closed 1-form. Since $\mathbb{D}$ is simply connected, there exists a smooth function H_t on $\mathbb{D}$, unique up to addition of a constant, such that $dH_t = \iota_{X_t}\omega$. The vector field X_t is tangent to the boundary $\partial\mathbb{D}$. This implies that dH_t vanishes on tangent vectors of $\partial\mathbb{D}$. Thus H_t is constant on the boundary. We will always use the normalization $H_t|_{\partial\mathbb{D}} = 0$. This uniquely specifies H_t. Conversely, if we are given a family of smooth functions H_t vanishing on the boundary, there exists a unique vector field X_{H_t} in the interior $\mathrm{int}(\mathbb{D})$ satisfying $\iota_{X_{H_t}}\omega = dH_t$. If ω does not vanish on $\partial\mathbb{D}$, then X_{H_t} smoothly extends to a vector field on the closed disk which is tangent to the boundary. Note that this is not necessarily true if ω vanishes on the boundary. So while every arc in $\mathrm{Diff}(\mathbb{D}, \omega)$ is generated by a family of Hamiltonians vanishing on the boundary $\partial\mathbb{D}$, not every family of such Hamiltonians generates an arc in $\mathrm{Diff}(\mathbb{D}, \omega)$. The following result [3, Proposition 2.6] expresses the action of $\widetilde{\phi} = [(\phi_t)_{t\in[0,1]}]$ in terms the Hamiltonian H_t.

Lemma 2.2.2. *Suppose that $(\phi_t)_{t\in[0,1]}$ is an arc in $\mathrm{Diff}(\mathbb{D}, \omega)$ generated by a family of Hamiltonians H_t vanishing on the boundary $\partial\mathbb{D}$. Let $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ be the element represented by the arc $(\phi_t)_{t\in[0,1]}$. Then*

$$\sigma_{\widetilde{\phi},\lambda}(z) = \int_{\{t\mapsto\phi_t(z)\}} \lambda + \int_0^1 H_t(\phi_t(z))dt$$

for all $z \in \mathbb{D}$.

2.2.2 Global surfaces of section

Let Y^3 be a closed oriented 3-manifold equipped with a nowhere vanishing vector field R. Let ϕ^t denote the flow generated by R. Let $\Sigma \subset Y$ be an embedded compact surface, possibly with boundary, which we also assume to be embedded. We call Σ a *global surface of section* for the flow ϕ^t if the boundary $\partial\Sigma$ consists of simple periodic orbits of ϕ^t, the vector field R is transverse to $\mathrm{int}(\Sigma)$ and every trajectory of ϕ^t which is not contained in $\partial\Sigma$ meets $\mathrm{int}(\Sigma)$ infinitely often forward and backward in time. We will always orient surfaces of section such that R is positively transverse to Σ, i.e. the orientation of R followed by the orientation of

Σ agrees with the orientation of Y. Consider a boundary orbit γ of Σ. We call γ *positive* if the orientation of γ given by R agrees with the boundary orientation of Σ and *negative* otherwise. We define the *first return time* and *first return map* by

$$\sigma : \mathrm{int}(\Sigma) \to \mathbb{R}_{>0} \qquad \sigma(p) := \inf\{t > 0 \mid \phi^t(p) \in \Sigma\} \tag{2.2.3}$$

and

$$\phi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma) \qquad \phi(p) := \phi^{\sigma(p)}(p). \tag{2.2.4}$$

Studying the dynamics of the flow ϕ^t is equivalent to studying the discrete dynamics of the diffeomorphism ϕ. Let Σ' be a second global surface of section with the same boundary orbits as Σ, i.e. $\partial\Sigma' = \partial\Sigma$. Then the respective first return maps ϕ and ϕ' are smoothly conjugated. To see this, we define a transfer map $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma')$ as follows. Let $z_0 \in \mathrm{int}(\Sigma)$ and let $\tau(z_0)$ denote a real number such that $\phi^{\tau(z_0)}(z_0) \in \mathrm{int}(\Sigma')$. Then there exists a unique smooth extension of τ to a real-valued function on $\mathrm{int}(\Sigma)$ such that $\phi^{\tau(z)}(z) \in \mathrm{int}(\Sigma')$ for all $z \in \mathrm{int}(\Sigma)$. We define $\psi(z) := \phi^{\tau(z)}(z)$. This is a diffeomorphism. The first return maps of Σ and Σ' are related via $\phi = \psi^{-1} \circ \phi' \circ \psi$.
In general, the first return time σ and map ϕ need not smoothly extend to the boundary $\partial\Sigma$. In order to describe the boundary behaviour, we recall a blow-up construction due to Fried [39]. Our exposition follows Florio-Hryniewicz [37]. We define the vector bundle $\xi := TY/\langle R\rangle$ on Y, where $\langle R\rangle$ is the subbundle of TY spanned by R. Moreover, we define the circle bundle $\mathbb{P}_+\xi := (\xi \setminus 0)/\mathbb{R}_+$. The linearization of ϕ^t induces a lift $d\phi^t$ of the flow ϕ^t to ξ. This lift $d\phi^t$ descends to the bundle $\mathbb{P}_+\xi$. Now consider a simple closed orbit γ of ϕ^t. Then the torus $\mathbb{T}_\gamma := \mathbb{P}_+\xi|_\gamma$ is invariant under the projective linearized flow $d\phi^t$. As a set, the blow-up of Y at γ is equal to the disjoint union $\overline{Y} := (Y \setminus \gamma) \sqcup \mathbb{T}_\gamma$. It carries the structure of a compact smooth manifold with boundary $\mathbb{T}_\gamma$. The natural projection $\pi : \overline{Y} \to Y$ is smooth. The pullback of the restriction of the vector field R to $Y \setminus \gamma$ is a smooth vector field $\overline{R}$ on the interior of $\overline{Y}$. It smoothly extends to all of $\overline{Y}$ (see e.g. [37, Lemma A.1]). The resulting flow $\overline{\phi}^t$ on $\overline{Y}$ lifts the flow ϕ^t and its restriction to the boundary $\mathbb{T}_\gamma$ agrees with the projective linearized flow $d\phi^t$. Consider a surface of section $\Sigma \subset Y$. Let $\overline{Y}$ be the simultaneous blow-up of Y at all the boundary orbits of Σ. The surface Σ lifts to a properly embedded surface $\overline{\Sigma} \subset \overline{Y}$ with boundary $\partial\overline{\Sigma}$ contained in $\partial\overline{Y}$. We recall the following definition from [37].

Definition 2.2.3. The global surface of section Σ is called *∂-strong* if the lifted surface $\overline{\Sigma} \subset \overline{Y}$ is a global surface of section for the lifted flow $\overline{\phi}^t$, i.e. if $\overline{R}$ is transverse to $\overline{\Sigma}$ and all trajectories of $\overline{\phi}^t$ meet $\overline{\Sigma}$ infinitely often forward and backward in time.

Since Σ is a surface of section, $\overline{R}$ is clearly transverse to $\overline{\Sigma}$ in the interior of $\overline{Y}$. Moreover, all trajectories in the interior meet $\overline{\Sigma}$ forward and backward in time. Thus the condition for being ∂-strong is equivalent to requiring that $\overline{\Sigma} \cap \mathbb{T}_\gamma$ is a surface of section for the projective linearized flow $d\phi^t$ on $\mathbb{T}_\gamma$ for all boundary orbits γ of Σ.

Lemma 2.2.4. *Suppose that $\Sigma \subset Y$ is a ∂-strong global surface of section. Then the first return time extends to a smooth function $\sigma : \Sigma \to \mathbb{R}_{>0}$ and the first return map extends to a* *diffeomorphism* $\phi : \Sigma \to \Sigma$*. If* Σ' *is a second* ∂*-strong global surface of section with the same boundary orbits, then any transfer map extends to a diffeomorphism* $\psi : \Sigma \to \Sigma'$*.*

Proof. We simply observe that Σ being ∂-strong implies that the first return time and map of the lifted surface $\overline{\Sigma}$ are also defined on the boundary $\partial\overline{\Sigma}$ and smooth. The same argument applies to a transfer map ψ. □

In this paper we will be mainly concerned with *disk-like global surfaces of section*, i.e. the case that Σ is diffeomorphic to the closed unit disk $\mathbb{D}$. The manifold Y is then necessarily diffeomorphic to S^3. Suppose that Σ is a ∂-strong disk-like global surface of section. It will be useful to lift the first return map $\phi : \Sigma \to \Sigma$ to an element $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\Sigma)$ of the universal cover of the space $\mathrm{Diff}^+(\Sigma)$ of orientation preserving diffeomorphisms of Σ. Such a lift depends on a choice of trivialization. Let $\pi : \overline{Y} \to Y$ be the blow-up of Y at the boundary orbit of Σ. A *trivialization* of $\overline{Y}$ is a diffeomorphism $\tau : \mathbb{R}/\mathbb{Z} \times \Sigma \to \overline{Y}$ such that the composition

$$\Sigma \cong 0 \times \Sigma \subset \mathbb{R}/\mathbb{Z} \times \Sigma \xrightarrow{\tau} \overline{Y} \xrightarrow{\pi} Y$$

is simply the inclusion of Σ. Moreover, we require that $\iota_{\tau^*\overline{R}} dt > 0$, where t denotes the coordinate on $\mathbb{R}/\mathbb{Z}$. Since $\mathrm{Diff}^+(\Sigma)$ is connected, the space of trivializations is non-empty. Let $\mathcal{T}$ denote the set of isotopy classes of trivializations of $\overline{Y}$. It is an affine space over $\pi_1(\mathrm{Diff}^+(\Sigma)) \cong \mathbb{Z}$. We exhibit an explicit bijection $\deg : \mathcal{T} \to \mathbb{Z}$ as follows. Let τ be a trivialization and $p \in \partial\Sigma$ a point in the boundary. Then the degree d of the map

$$S^1 \cong \mathbb{R}/\mathbb{Z} \to \partial\Sigma \cong S^1 \qquad t \mapsto \pi(\tau(t,p))$$

is independent of the choice of p and only depends on the isotopy class of τ. Here $\partial\Sigma$ is oriented as the boundary of Σ. We define the *degree* of τ to be $\deg(\tau) := d$. Given a trivialization τ, there is a natural lift $\widetilde{\phi}$ of ϕ to $\widetilde{\mathrm{Diff}}(\Sigma)$ constructed as follows. Let X denote the unique (positive) rescaling of the pullback vector field $\tau^*\overline{R}$ on $\mathbb{R}/\mathbb{Z} \times \Sigma$ such that $\iota_X dt = 1$. The flow of X yields an arc in $\mathrm{Diff}^+(\Sigma)$ from the identity to ϕ. Clearly, the element $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\Sigma)$ represented by this arc only depends on the isotopy class of τ. Let us explain the dependence of the lift on the choice of trivialization. Consider integers d and e and let $\widetilde{\phi}_d$ and $\widetilde{\phi}_e$ denote the lifts of ϕ with respect to trivializations of degrees d and e, respectively. Let $\widetilde{\rho} \in \widetilde{\mathrm{Diff}}(\Sigma)$ be one full positive rotation of Σ. Then the lifts $\widetilde{\phi}_d$ and $\widetilde{\phi}_e$ are related by the identity

$$\widetilde{\rho}^{e-d} \circ \widetilde{\phi}_e = \widetilde{\phi}_d. \tag{2.2.5}$$

Let us now specialize our discussion of global surfaces of section to Reeb flows. Let α be a contact form on Y and let R be the induced Reeb vector field. We abbreviate $\omega := d\alpha|_\Sigma$. This is a closed 2-form on Σ. It vanishes on the boundary $\partial\Sigma$ and is a positive area form in the interior $\mathrm{int}(\Sigma)$. Note that by Stokes' theorem Σ must possess at least one positive boundary orbit. In particular, if Σ is a disk, then its boundary orbit must be positive. Let λ denote the restriction of α to Σ. This defines a primitive of ω. The first return time σ and map ϕ satisfy the identity

$$\phi^*\lambda - \lambda = d\sigma. \tag{2.2.6}$$

This implies that ϕ preserves the area form ω. Similarly, one can show that a transfer map ψ between two global surfaces of section Σ and Σ' with the same boundary orbits is area preserving.

Lemma 2.2.5. *Suppose that Σ is a ∂-strong disk-like global surface of section and let $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\Sigma, \omega)$ denote the lift of the first return map ϕ with respect to a trivialization of degree 0. Then the action $\sigma_{\widetilde{\phi},\lambda}$ agrees with the first return time σ.*

Proof. We need to check that the first return time σ satisfies (2.2.1) and (2.2.2). The first identity is true by (2.2.6). Let $(\phi_t)_{t\in[0,1]}$ be any arc in $\mathrm{Diff}^+(\Sigma)$ representing $\widetilde{\phi}$. Let $z \in \partial\Sigma$ be a point in the boundary and let $\gamma : [0,1] \to \partial\Sigma$ be the path defined by $\gamma(t) := \phi_t(z)$. Let $\delta : [0, \sigma(z)] \to \overline{Y}$ be the trajectory of $\overline{\phi}^t$ starting at $z \in \partial\overline{\Sigma} \cong \partial\Sigma$. We can express $\sigma_{\widetilde{\phi},\lambda}(z)$ and $\sigma(z)$ as

$$\sigma_{\widetilde{\phi},\lambda}(z) = \int_\gamma \lambda \qquad \text{and} \qquad \sigma(z) = \int_\delta \pi^*\alpha = \int_{\tau^{-1}\circ\delta} \tau^*\pi^*\alpha.$$

In order to see that these two numbers agree, we regard γ as a path in $\mathbb{R}/\mathbb{Z} \times \Sigma$ via the inclusion $\Sigma \cong 0 \times \Sigma \subset \mathbb{R}/\mathbb{Z} \times \Sigma$ and form the concatenation $\varepsilon := (\tau^{-1} \circ \delta)\#\overline{\gamma}$. This defines a loop in $\mathbb{R}/\mathbb{Z}\times\partial\Sigma$ which is homotopic to the loop $\mathbb{R}/\mathbb{Z}\times z$. The restriction $\beta := (\tau^*\pi^*\alpha)|_{\mathbb{R}/\mathbb{Z}\times\partial\Sigma}$ is the pullback of the restriction of α to $\partial\Sigma$. Hence β is a closed 1-form. Since τ has degree 0, the loop $\mathbb{R}/\mathbb{Z} \times z$ is mapped to a contractible loop in $\partial\Sigma$ by $\pi \circ \tau$. Thus the integral of β over the loop $\mathbb{R}/\mathbb{Z} \times z$ vanishes. Since ε is homotopic to $\mathbb{R}/\mathbb{Z} \times z$, we obtain

$$0 = \int_\varepsilon \beta = \int_{\tau^{-1}\circ\delta} \tau^*\pi^*\alpha - \int_\gamma \lambda$$

where we have used that the restriction of β to $0 \times \partial\Sigma$ is equal to λ. This concludes the proof. □

2.3 From disk-like surfaces of section to symplectic embeddings

2.3.1 Embedding results

The following theorem says, roughly speaking, that if the boundary of a star-shaped domain $X \subset \mathbb{R}^4$ admits a disk-like global surface of section of symplectic area a such that the lift of the first return map with respect to a trivialization of degree 0 can be generated by a positive Hamiltonian, then the domain X can be symplectically embedded into the cylinder $Z(a)$. The second part of the theorem states that if the lift of the first return map with respect to a trivialization of degree 1 can still be generated by a positive Hamiltonian, then the ball $B(a)$ embeds into X.

Theorem 2.3.1. *Let $X \subset \mathbb{R}^4$ be a star-shaped domain. Let $\Sigma \subset \partial X$ be a ∂-strong disk-like global surface of section of the natural Reeb flow on ∂X. Assume that the local first return map of a small disk transverse to the boundary orbit $\partial\Sigma$ is smoothly conjugated to a rotation. Set $\omega := \omega_0|_\Sigma$ and let*

$$a := \int_\Sigma \omega$$

be the symplectic area of the surface of section.

1. *Let $\widetilde{\phi}_0 \in \widetilde{\mathrm{Diff}}(\Sigma, \omega)$ be the lift of the first return map with respect to a trivialization of degree 0. Suppose that there exists a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$ with the following properties:*

   a) H is strictly positive in the interior $\mathrm{int}(\Sigma)$ and vanishes on the boundary $\partial\Sigma$.

   b) H is autonomous in some neighbourhood of $\partial\Sigma$.

   c) The Hamiltonian vector field X_{H_t} defined by $\iota_{X_{H_t}}\omega = dH_t$ in the interior $\mathrm{int}(\Sigma)$ smoothly extends to the closed disk Σ and is tangent to $\partial\Sigma$.

   d) The arc $(\phi_H^t)_{t\in[0,1]}$ represents $\widetilde{\phi}_0$.

   Then $X \overset{s}{\hookrightarrow} Z(a)$.

2. *Let $\widetilde{\phi}_1 \in \widetilde{\mathrm{Diff}}(\Sigma, \omega)$ be the lift of the first return map with respect to a trivialization of degree 1. Assume that there exists a Hamiltonian $G : \mathbb{R}/\mathbb{Z}\times\Sigma \to \mathbb{R}$ satisfying properties (a)-(c) above such that the arc $(\phi_G^t)_{t\in[0,1]}$ represents $\widetilde{\phi}_1$. Then $B(a) \overset{s}{\hookrightarrow} X \overset{s}{\hookrightarrow} Z(a)$.*

Given a star-shaped domain X and a disk-like surface of section $\Sigma \subset \partial X$, we do not know whether it is always possible to generate the lift $\widetilde{\phi}_0$ by a Hamiltonian which vanishes on the boundary $\partial\Sigma$ and is positive in the interior. The following Proposition says that we may always symplectically embed X into a bigger domain satisfying the hypotheses of Theorem 2.3.1.

Proposition 2.3.2. *Let $X \subset \mathbb{R}^4$ be a star-shaped domain. Let $\Sigma \subset \partial X$ be a ∂-strong disk-like surface of section of the natural Reeb flow on ∂X. Then there exist a star-shaped domain X' and a ∂-strong disk-like surface of section Σ' of the natural Reeb flow on the boundary $\partial X'$ such that Σ and Σ' have the same symplectic areas, X symplectically embeds into X' and the tuple (X', Σ') satisfies all hypotheses of the first assertion of Theorem 2.3.1.*

2.3.2 Main construction

Given a 1-periodic Hamiltonian

$$H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$$

which is positive in the interior $\mathrm{int}(\mathbb{D})$ and vanishes on the boundary $\partial\mathbb{D}$, we construct a domain

$$A(H) \subset \mathbb{C}^2.$$

We show that the characteristic foliation on the boundary $\partial A(H)$ possesses a disk-like surface of section with first return map given by ϕ_H^1.

Lemma 2.3.3. *Let (M,ω) be a symplectic manifold. Let $\widetilde{M} := \mathbb{R}_s \times (\mathbb{R}/\mathbb{Z})_t \times M$ denote time-energy extended phase space equipped with the symplectic form $\widetilde{\omega} := ds \wedge dt + \omega$. Consider a periodic Hamiltonian*

$$H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$$

and let

$$\Gamma(H) := \{(H(t,p),t,p) \mid (t,p) \in \mathbb{R}/\mathbb{Z} \times M\} \subset \widetilde{M}$$

denote its graph inside time-energy extended phase space $\widetilde{M}$. Then the characteristic foliation on $\Gamma(H)$ induced by the symplectic form $\widetilde{\omega}$ is spanned by the vector field

$$X_{H_t}(p) + \partial_t + \partial_t H(t,p) \cdot \partial_s.$$

Proof. We define the autonomous Hamiltonian $\widetilde{H}$ on time-energy extended phase space by

$$\widetilde{H} : \widetilde{M} \to \mathbb{R} \qquad \widetilde{H}(s,t,p) := H(t,p) - s.$$

The graph $\Gamma(H)$ is given by the regular level set $\widetilde{H}^{-1}(0)$. Thus the characteristic foliation on $\Gamma(H)$ is spanned by the restriction of the Hamiltonian vector field $X_{\widetilde{H}}$ to $\Gamma(H)$. We compute

$$d\widetilde{H}(s,t,p) = dH_t(p) + \partial_t H(t,p) \cdot dt - ds = \iota_{X_{H_t} + \partial_t + \partial_t H \cdot \partial_s}(ds \wedge dt + \omega)$$

and conclude that

$$X_{\widetilde{H}} = X_{H_t} + \partial_t + \partial_t H \cdot \partial_s.$$

□

Construction 2.3.4. Consider $\mathbb{C}$ equipped with the standard symplectic form $\omega_0 = dx \wedge dy$. Let

$$\widetilde{\mathbb{C}} := \mathbb{R}_s \times (\mathbb{R}/\mathbb{Z})_t \times \mathbb{C} \qquad \widetilde{\omega_0} := ds \wedge dt + \omega_0$$

denote time-energy extended phase space and abbreviate

$$\widetilde{\mathbb{C}}_+ := \mathbb{R}_{\geq 0} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C} \qquad \text{and} \qquad \widetilde{\mathbb{C}}_0 := \{0\} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}.$$

Consider the map

$$\Phi : \widetilde{\mathbb{C}}_+ \to \mathbb{C}^2 \qquad \Phi(s,t,z) := \left(z \ , \ \sqrt{\frac{s}{\pi}} \cdot e^{2\pi i t} \right).$$

Φ restricts to a diffeomorphism between $\widetilde{\mathbb{C}}_+ \setminus \widetilde{\mathbb{C}}_0$ and $\mathbb{C}^2 \setminus (\mathbb{C} \times 0)$. Moreover $\Phi^*\omega_0 = \widetilde{\omega_0}$. For $a > 0$, let $B^2(a) \subset \mathbb{C}$ denote the closed 2-dimensional disk of area a. Let

$$H : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \mathbb{R}$$

be a smooth function. Assume that:

1. H is strictly positive in the interior $\mathrm{int}(B^2(a))$.
2. There exists a constant $C > 0$ such that in some neighbourhood of $\partial B^2(a)$ the function H is given by
$$H(t,z) = C \cdot (a - \pi|z|^2). \tag{2.3.1}$$

Let
$$\Gamma_-(H) := \{(s,t,z) \in \widetilde{\mathbb{C}}_+ \mid z \in B^2(a) \text{ and } 0 \leq s \leq H(t,z)\}$$
denote the subgraph of H. We define the subset $A(a,H) \subset \mathbb{C}^2$ by
$$A(a,H) := \Phi(\Gamma_-(H)).$$

Lemma 2.3.5 (Basic properties)**.** *The set $A(a,H) \subset \mathbb{C}^2$ defined in Construction 2.3.4 satisfies the following basic properties:*

1. *$A(a,H)$ has smooth boundary and is diffeomorphic to the closed ball D^4.*
2. *$A(a,H) \subset Z(a)$*
3. *If $H(t,z) \geq a - \pi|z|^2$, then $B(a) \subset A(a,H)$.*
4. *The map*
$$f : B^2(a) \to \partial A(a,H) \quad z \mapsto \Phi(H(0,z),0,z)$$
is a parametrization of a disk-like surface of section of the characteristic foliation on $\partial A(a,H)$. We have $f^*\omega_0 = \omega_0$. Consider the lift $\widetilde{\phi}_0$ of the first return map with respect to a trivialization of degree 0. We regard $\widetilde{\phi}_0$ as an element of $\widetilde{\mathrm{Diff}}(B^2(a),\omega_0)$ via the parametrization f. It is represented by $(\phi_H^t)_{t\in[0,1]}$.

Remark 2.3.6. The parametrization f in item (4) of Lemma 2.3.5 is only smooth in the interior $\mathrm{int}(\mathbb{D})$. At the boundary $\partial\mathbb{D}$, the radial derivative $\partial_r f$ blows up. Since H has the special form (2.3.1) and ϕ_H^t is a rotation near the boundary, this does not cause problems.

Proof. Clearly, the boundary $\partial A(a,H)$ is smooth away from the circle $\partial B^2(a) \times \{0\}$. Near this circle, the Hamiltonian H has the special form (2.3.1). Thus $\partial A(a,H)$ can be described by the equation
$$C|z_1|^2 + |z_2|^2 = \frac{Ca}{\pi}$$
near $\partial B^2(a) \times \{0\}$. The solution set of this equation is the boundary of an ellipsoid and in particular smooth. Let G denote the Hamiltonian which is given by formula (2.3.1) on the entire disk $B^2(a)$. The set $A(a,G)$ is an ellipsoid and in particular diffeomorphic to the closed ball D^4. Clearly there exists a diffeomorphism
$$\psi : \Gamma_-(H) \to \Gamma_-(G)$$

between the subgraphs of H and G. In fact, we can choose ψ to agree with the identity map on all points $(s,t,z) \in \Gamma_-(H)$ such that z is close to the boundary $\partial B^2(a)$ or s is close to 0. If ψ has these properties, then $\Phi \circ \psi \circ \Phi^{-1}$ defines a diffeomorphism between $A(a,H)$ and $A(a,G)$. Thus $A(a,H)$ is diffeomorphic to D^4. By construction, $A(a,H)$ is contained in the cylinder $Z(a)$. If $H(t,z) = a - \pi|z|^2$, then $A(a,H)$ is the 4-dimensional ball $B(a)$ of area a. Since $H \leq G$ implies $A(a,H) \subset A(a,G)$, assertion (3) is an immediate consequence. In order to prove assertion (4), let us first observe that Lemma 2.3.3 implies that the map

$$g : B^2(a) \to \Gamma(H) \subset \widetilde{\mathbb{C}}_+ \qquad g(z) := (H(0,z), 0, z)$$

parametrizes a disk-like surface of section of the characteristic foliation on the graph $\Gamma(H)$. The first return map of this surface of section is given by ϕ_H^1. The symplectomorphism Φ in Construction 2.3.4 maps the characteristic foliation on the graph $\Gamma(H)$ to the characteristic foliation on $\partial A(a,H)$. Thus the first return map of the surface of section parametrized by f is equal to ϕ_H^1 as well. In order to show that $(\phi_H^t)_{t\in[0,1]}$ represents the correct lift, simply observe that the composition of the trivialization

$$\tau : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \Gamma(H) \qquad \tau(t,z) := (H(t,z), t, z)$$

of $\Gamma(H)$ with Φ yields a trivialization of $\partial A(a,H)$ of degree 0. This proves assertion (4). □

2.3.3 Proof of Theorem 2.3.1

Throughout this section, we fix the setup of Theorem 2.3.1. We let $X \subset (\mathbb{R}^4, \omega_0)$ denote a star-shaped domain and $\Sigma \subset \partial X$ a ∂-strong disk-like global surface of section of the natural Reeb flow on ∂X induced by the restriction of the standard Liouville 1-form λ_0 defined in (2.1.1). We assume that the local first return map of a small disk transverse to the boundary orbit $\partial\Sigma$ is smoothly conjugated to a rotation. Let $a > 0$ denote the symplectic area of Σ. Our strategy, roughly speaking, is to show that if the degree 0 lift $\widetilde{\phi}_0$ of the first return map can be generated by a Hamiltonian H which is positive in the interior $\operatorname{int}(\Sigma)$ and vanishes on the boundary $\partial\Sigma$, then X is symplectomorphic to the domain $A(a,H)$ constructed in section 2.3.2. We construct a symplectomorphism between X and $A(a,H)$ in two steps. In Proposition 2.3.8 we show that there exists a diffeomorphism $\psi : \partial A(a,H) \to \partial X$ which pulls back $\omega_0|_{\partial X}$ to $\omega_0|_{\partial A(a,H)}$. Then we use a result of Gromov and McDuff (Theorem 2.3.9) to extend ψ to a symplectomorphism between $A(a,H)$ and X. This is done in Corollary 2.3.10.

We begin with the following auxiliary lemma on the existence of a convenient parametrization of a tubular neighbourhood of the boundary orbit.

Lemma 2.3.7. *Let $\varepsilon > 0$ be sufficiently small and let $\mathbb{D}_\varepsilon \subset \mathbb{C}$ denote the disk of radius ε. There exist a ∂-strong disk-like global surface of section $\Sigma' \subset \partial X$ with the same boundary* *orbit as* Σ *and a parametrization* $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X$ *of a tubular neighbourhood of* $\partial\Sigma'$ *such that the following is true.*

$$F^{-1}(\Sigma') = \{(t, re^{i\theta}) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \mid 0 \leq r \leq \varepsilon \text{ and } \theta = 0\} \tag{2.3.2}$$

and

$$F^*\lambda_0 = \frac{1}{2}r^2 \cdot d\theta + (a - \pi b r^2) \cdot dt \tag{2.3.3}$$

where b *is a positive real number.*

Proof. Consider a small disk D transverse to $\partial\Sigma$ whose local first return map is smoothly conjugated to a rotation. We may choose a parametrization $f : \mathbb{D}_\varepsilon \to D$ such that the local first return map, regarded as a diffeomorphism of $\mathbb{D}_\varepsilon$ via f, is a rotation of $\mathbb{D}_\varepsilon$. By an equivariant version of Moser's argument, after modifying the parametrization f we may in addition assume that $f^*\omega_0 = \omega_0$ where ω_0 denotes the standard symplectic form on both $\mathbb{R}^4$ and $\mathbb{D}_\varepsilon$. The primitives $f^*\lambda_0$ and $\lambda := \frac{1}{2}r^2 d\theta$ of the area form ω_0 on $\mathbb{D}_\varepsilon$ differ by an exact 1-form, i.e. $\lambda = f^*\lambda_0 + d\alpha$ for a smooth function α on $\mathbb{D}_\varepsilon$. We may normalize α such that $\alpha(0) = 0$. We define

$$f' : \mathbb{D}_\varepsilon \to \partial X \quad f'(z) := \phi^{\alpha(z)}(f(z))$$

where ϕ^t denotes the Reeb flow on ∂X. This parametrizes a small disk D' transverse to $\partial\Sigma$. A direct computation shows that $f'^*\lambda_0 = f^*\lambda_0 + d\alpha = \lambda$ and the local first return map is still a rotation of $\mathbb{D}_\varepsilon$. Let ρ denote this rotation. Since $\rho^*\lambda = \lambda$, it follows from (2.2.6) that the first return time of D' is constant and equal to a, the action of the orbit $\partial\Sigma$. Let us define the immersion

$$F : \mathbb{R} \times \mathbb{D}_\varepsilon \to \partial X \quad F(t,z) := \phi^t(f'(z)).$$

We have $F^*\lambda_0 = dt + \lambda$ and F is invariant under the diffeomorphism ψ of $\mathbb{R} \times \mathbb{D}_\varepsilon$ defined by $\psi(t,z) := (t - a, \rho(z))$. Thus F descends to a strict contactomorphism between the quotient $(\mathbb{R} \times \mathbb{D}_\varepsilon)/\sim$ of $(\mathbb{R} \times \mathbb{D}_\varepsilon, dt + \lambda)$ by the action of ψ and a tubular neighbourhood of $\partial\Sigma$ in ∂X. It is a direct computation to check that we may choose a diffeomorphism $\tau : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \cong (\mathbb{R} \times \mathbb{D}_\varepsilon)/\sim$ such that the contact form $dt + \lambda$ on $(\mathbb{R} \times \mathbb{D}_\varepsilon)/\sim$ pulls back to a contact form on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon$ of the form (2.3.3) for some real number b. By slight abuse of notation, let $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X$ denote the resulting parametrization of a tubular neighbourhood of $\partial\Sigma$. For appropriate choice of diffeomorphism τ, the preimage $F^{-1}(\Sigma)$ is non-winding, i.e. isotopic to the annulus (2.3.2) in $\mathbb{R} \times \mathbb{D}_\varepsilon$. In fact, since Σ is a ∂-strong surface of section, it is easy to see that we may replace Σ by an isotopic disk-like global surface of section Σ' with the same boundary orbit such that $F^{-1}(\Sigma')$ is equal to the annulus (2.3.2). It remains to show that the constant b must be positive. This is a consequence of the fact that the boundary orbit $\partial\Sigma'$ is positive, i.e. the boundary orientation of Σ' agrees with the orientation induced by the Reeb vector field. Since the Reeb vector field of (2.3.3) is simply given by $\frac{1}{a}(\partial_t + 2\pi b\partial_\theta)$, this means that b must be positive. □

The surface of section Σ' constructed in Lemma 2.3.7 still satisfies the assumptions of Theorem 2.3.1 because we can simply use a transfer map $\psi : \Sigma \to \Sigma'$ transport the positive

Hamiltonians H and G generating the lifts $\widetilde{\phi}_0$ and $\widetilde{\phi}_1$ of the first return map of Σ to obtain positive Hamiltonians on Σ'. Thus we may replace Σ by Σ' and assume in addition that there exists a parametrization F of a tubular neighbourhood of $\partial\Sigma$ satisfying (2.3.2) and (2.3.3).
Our next step is to construct a special parametrization $f : B^2(a) \to \Sigma$ such that $f^*\omega = \omega_0$ where ω_0 denotes the standard symplectic form on $B^2(a)$. For $re^{i\theta}$ near the boundary $\partial B^2(a)$ we define

$$f(re^{i\theta}) = F\left(\frac{\theta}{2\pi} \ , \ \sqrt{\frac{1}{b\pi}(a - \pi r^2)}\right).$$

A direct computation involving (2.3.3) shows that this pulls back ω to ω_0. Since both $(B^2(a), \omega_0)$ and (Σ, a) have area a, we can use a Moser type argument to extend f to an area preserving map $f : (B^2(a), \omega_0) \to (\Sigma, \omega)$.
Consider the degree 0 lift $\widetilde{\phi}_0$ of the first return map of Σ. Via the parametrization f we can regard $\widetilde{\phi}_0$ as an element of $\widetilde{\operatorname{Diff}}(B^2(a), \omega_0)$. It follows from (2.3.3) and a short computation that $\widetilde{\phi}_0$ is a rotation by angle $2\pi/b$ near the boundary $\partial B^2(a)$. By the assumptions in the first assertion of Theorem 2.3.1, there exists a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \mathbb{R}$ which vanishes on the boundary, is positive in the interior, is autonomous near the boundary and generates $\widetilde{\phi}_0$. We argue that we may in addition assume that

$$H(t, z) = \frac{1}{b}(a - \pi|z|^2) \qquad \text{for } z \text{ sufficiently close to } \partial B^2(a). \tag{2.3.4}$$

Since H is autonomous near the boundary, it is invariant under ϕ_H^1, which is a rotation by $2\pi/b$. If b is irrational, then this implies that H is invariant under arbitrary rotations and it is not hard to see that H must in fact be given by (2.3.4). If b is rational, then H need not be invariant under arbitrary rotations near the boundary. We show that we may replace H by a Hamiltonian which is rotation invariant. There exists a symplectomorphism g of $(B^2(a), \omega_0)$ supported in a small neighbourhood of $\partial B^2(a)$ which commutes with the rotation by angle $2\pi/b$ such that the level sets of $H \circ g$ near the boundary are circles centred at the centre of $B^2(a)$. The time-1-map of $H \circ g$ is given by $g^{-1} \circ \phi_H^1 \circ g$. Away from a neighbourhood of $\partial B^2(a)$ this agrees with ϕ_H^1 because g is equal to the identity. Near the boundary, ϕ_H is a rotation by angle $2\pi/b$ and thus commutes with g. Hence $g^{-1} \circ \phi_H^1 \circ g$ agrees with ϕ_H^1 on all of $B^2(a)$. Now simply replace H by $H \circ g$. Then H is rotation invariant near the boundary and again it follows that it must be given by (2.3.4).

Proposition 2.3.8. *There exists a diffeomorphism $\psi : \partial A(a, H) \to \partial X$ such that $\psi^*(\omega_0|_{\partial X}) = \omega_0|_{\partial A(a,H)}$.*

Proof. After possibly shrinking ε, we may define

$$F' : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial A(a, H) \quad F'(t, z) := \left(\sqrt{\frac{a}{\pi} - b|z|^2} \cdot e^{2\pi i t} \ , \ z\right).$$

It follows from (2.3.4) and the definition of $A(a,H)$ that the image of F' is contained in $\partial A(a,H)$ for ε sufficiently small. Moreover, we define

$$f' : B^2(a) \to \partial A(a,H) \quad f'(z) := \left(z, \sqrt{\frac{H(0,z)}{\pi}} \right).$$

A direct computation shows that the following two diffeomorphisms between submanifolds of $\partial A(a,H)$ and ∂X agree on their overlap and pull back $\omega_0|_{\partial X}$ to $\omega_0|_{\partial A(a,H)}$:

$$F \circ F'^{-1} : \mathrm{im}(F') \to \mathrm{im}(F)$$

and

$$f \circ f'^{-1} : \mathrm{im}(f') \to \mathrm{im}(f)$$

On $\mathrm{im}(F') \cup \mathrm{im}(f')$ we may therefore define ψ to agree with $F \circ F'^{-1}$ and $f \circ f'^{-1}$, respectively. Next, we explain how to extend to a diffeomorphism between $\partial A(a,H)$ and ∂X. We pull back the Reeb vector field R on ∂X via $F \circ F'^{-1}$ to obtain a vector field R' on $\mathrm{im}(F')$ which is tangent to the characteristic foliation. We smoothly extend to a vector field on $\partial A(a,H)$, still denoted by R', which is everywhere tangent to the characteristic foliation. The embedding f' parametrizes a surface of section of the flow generated by R'. By Lemma 2.3.5, the lift of the first return map with respect to a trivialization of degree 0 is represented by $(\phi_H^t)_{t\in[0,1]}$, which also represents the degree 0 lift of the first return map of the surface of section of ∂X parametrized by f. After replacing R' by a positive scaling $\chi \cdot R'$ for a suitable smooth function $\chi : \partial A(a,H) \to \mathbb{R}_{>0}$, we may assume that the first return times of the two surfaces of section f and f' agree as well. This allows us to extend ψ to a diffeomorphism $\psi : \partial A(a,H) \to \partial X$ by requiring that ψ intertwines the flows on $\partial A(a,H)$ and ∂X generated by R' and R, respectively. Set $\omega := \psi^*(\omega_0|_{\partial X})$. We need to show that $\omega = \omega_0|_{\partial A(a,H)}$. By construction of ψ, the pull-back of the characteristic foliation on ∂X via ψ is equal to the characteristic foliation on $A(a,H)$. Thus ω_0 and ω induce the same characteristic foliations on $\partial A(a,H)$. Moreover, the restrictions of ω_0 and ω to $\mathrm{im}(F')$ and $\mathrm{im}(f')$ agree. Cartan's formula implies

$$\mathcal{L}_{R'}\omega_0 = \iota_{R'} d\omega_0 + d\iota_{R'}\omega_0 = 0$$

where we use that ω_0 is closed and R' is contained in its kernel. Similarly, $\mathcal{L}_{R'}\omega = 0$. Let $p \in \partial A(a,H) \setminus \mathrm{im}(F')$ and let $v, w \in T_p \partial A(a,H)$. Our goal is to show that $\omega_0(v,w) = \omega(v,w)$. The trajectory of R' through p intersects the surface of section $\mathrm{im}(f')$ after finite time. Let $\widetilde{p}$ be the first intersection point. We transport v and w via the flow of R' to obtain vectors $\widetilde{v}, \widetilde{w} \in T_{\widetilde{p}} \partial A(a,H)$. Since $\mathcal{L}_{R'}\omega_0$ and $\mathcal{L}_{R'}\omega$ vanish, we have $\omega_0(v,w) = \omega_0(\widetilde{v},\widetilde{w})$ and $\omega(v,w) = \omega(\widetilde{v},\widetilde{w})$. After replacing (p,v,w) by $(\widetilde{p},\widetilde{v},\widetilde{w})$, we can therefore assume w.l.o.g. that p is contained in the surface of section $\mathrm{im}(f')$. In addition we may assume that v and w are tangent to $\mathrm{im}(f')$. Indeed, replacing v and w by their projections onto $T_p \,\mathrm{im}(f')$ along the characteristic foliation does not change $\omega_0(v,w)$ and $\omega(v,w)$ because the kernels of ω_0 and ω are tangent to the characteristic foliation. Now we simply use that the restrictions $\omega_0|_{\mathrm{im}(f')}$ and $\omega|_{\mathrm{im}(f')}$ agree by construction of ψ. □

We recall the following well-known theorem due to Gromov and McDuff (see Theorem 9.4.2 in [83]).

Theorem 2.3.9 (Gromov-McDuff)**.** *Let (M, ω) be a connected symplectic 4-manifold and $K \subset M$ be a compact subset such that the following holds.*

1. *There is no symplectically embedded 2-sphere $S \subset M$ with self-intersection number $S \cdot S = -1$.*
2. *There exists a symplectomorphism $\psi : \mathbb{R}^4 \setminus V \to M \setminus K$, where $V \subset \mathbb{R}^4$ is a star-shaped compact set.*

Then (M, ω) is symplectomorphic to $(\mathbb{R}^4, \omega_0)$. Moreover, for every open neighbourhood $U \subset M$ of K, the symplectomorphism can be chosen equal to ψ^{-1} on $M \setminus U$.

Corollary 2.3.10. *For $j \in \{1, 2\}$, let $A_j \subset \mathbb{R}^4$ be a compact submanifold diffeomorphic to the closed disk D^4. Assume that there exists a diffeomorphism $\psi : \partial A_1 \to \partial A_2$ such that $\psi^*(\omega_0|_{\partial A_2}) = \omega_0|_{\partial A_1}$. Then the boundary ∂A_1 is of contact type if and only if ∂A_2 is of contact type. In this case, ψ extends to a symplectomorphism*

$$\psi : (A_1, \omega_0|_{A_1}) \to (A_2, \omega_0|_{A_2}). \tag{2.3.5}$$

Proof. By the uniqueness part of the coisotropic neighbourhood theorem in [45], there exist open neighbourhoods U_j of ∂A_j such that ψ extends to a symplectomorphism

$$\psi : (U_1, \omega_0|_{U_1}) \to (U_2, \omega_0|_{U_2}).$$

Being of contact type is a property that only depends on a small neighbourhood of a hypersurface. Thus ∂A_1 is if contact type if and only if ∂A_2 is. Suppose now that this is the case. After possibly shrinking U_1, we can find a Liouville vector field Z_1 defined on U_1 and transverse to ∂A_1. Let λ_1 denote the associated Liouville 1-form defined by $\lambda_1 = \iota_{Z_1}\omega_0$. Let Z_2 and λ_2 denote the push-forwards via ψ. For $j \in \{1, 2\}$ let $(\widehat{A_j}, \omega_j)$ be the symplectic completion of $(A_j, \omega_0|_{A_j})$ obtained by attaching a cylindrical end using the Liouville vector field Z_j. Let $\widehat{U_j}$ denote the union of U_j with the cylindrical end attached to A_j. Clearly, ψ extends to a symplectomorphism

$$\psi : (\widehat{U_1}, \omega_1) \to (\widehat{U_2}, \omega_2).$$

The contact manifold $(\partial A_1, \ker \lambda_1)$ is fillable. Hence it follows from Eliashberg's paper [35] that it is contactomorphic to S^3 equipped with the standard tight contact structure. Thus we can find a star-shaped domain $V \subset \mathbb{R}^4$ and a strict contactomorphism from $(\partial V, \lambda_0)$ to $(\partial A_1, \lambda_1)$, where λ_0 denotes the standard Liouville 1-form on $\mathbb{R}^4$. There exists an open neighbourhood U_0 of $\mathbb{R}^4 \setminus V$ such that this strict contactomorphism extends to a symplectomorphism

$$\phi : (U_0, \omega_0) \to (\widehat{U_1}, \omega_1).$$

By Theorem 2.3.9, there exists a symplectomorphism

$$\phi_1 : (\mathbb{R}^4, \omega_0) \to (\widehat{A_1}, \omega_1)$$

which agrees with ϕ on the complement of V. Similarly, applying Theorem 2.3.9 to the composition

$$\psi \circ \phi : (U_0, \omega_0) \to (\widehat{U_2}, \omega_2)$$

we obtain a symplectomorphism

$$\phi_2 : (\mathbb{R}^4, \omega_0) \to (\widehat{A_2}, \omega_2)$$

agreeing with $\psi \circ \phi$ on the complement of V. The composition $\phi_2 \circ \phi_1^{-1}$ restricts to a symplectomorphism $(A_1, \omega_0) \to (A_2, \omega_0)$ extending the given diffeomorphism $\psi : \partial A_1 \to \partial A_2$. □

Proof of Theorem 2.3.1. We prove the first assertion. By Proposition 2.3.8 and Corollary 2.3.10, X is symplectomorphic to $A(a, H)$. By the second item of Lemma 2.3.5 we have $A(a, H) \subset Z(a)$. Thus $X \overset{s}{\hookrightarrow} Z(a)$.
We prove the second assertion. We define the Hamiltonian

$$K : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \mathbb{R} \quad K(t, z) := a - \pi|z|^2.$$

This Hamiltonian generates $\widetilde{\rho}$, the full positive rotation of $B^2(a)$ by angle 2π. Our goal is to show that we may assume that the Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \mathbb{R}$ generating $\widetilde{\phi}_0$ satisfies in addition $K \leq H$. Then it follows from Proposition 2.3.8, Corollary 2.3.10 and the third item in Lemma 2.3.5 that $B(a) \hookrightarrow X \hookrightarrow Z(a)$. We regard the lift $\widetilde{\phi}_1$ as an element of $\widetilde{\mathrm{Diff}}(B^2(a), \omega_0)$ via the parametrization f. It follows from (2.2.5) that $\widetilde{\phi}_1 = \widetilde{\rho}^{-1} \circ \widetilde{\phi}_0$. In particular, $\widetilde{\phi}_1$ is a rotation by angle $2\pi(1/b - 1)$ near the boundary. By assumption, there exists a Hamiltonian $G : \mathbb{R}/\mathbb{Z} \times B^2(a) \to \mathbb{R}$ generating $\widetilde{\phi}_1$ which is positive in the interior, vanishes on the boundary and is autonomous near the boundary. It follows as in the case of the Hamiltonian H generating $\widetilde{\phi}_0$ that we can assume that G is given by

$$G(z, t) = (\frac{1}{b} - 1) \cdot (a - \pi|z|^2)$$

near the boundary. Define

$$H_t := (K \# G)_t = K_t + G_t \circ (\phi_K^t)^{-1}.$$

This defines a 1-periodic Hamiltonian generating $\widetilde{\phi}_0$ which has the special form (2.3.4) near $\partial B^2(a)$ and is bounded below by $K_t(z) = a - \pi|z|^2$. □

2.3.4 Proof of Proposition 2.3.2

We begin with some auxiliary lemmas. The first lemma concerns *positive paths* in the linear symplectic group (see Lalonde-McDuff [81]). Let $\mathrm{Sp}(2n)$ denote the group of all linear symplectomorphisms of $\mathbb{R}^{2n}$. Moreover, let J_0 denote the matrix representing the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We recall that, for every path $S : [0,1] \to \mathbb{R}^{2n\times 2n}$ of symmetric matrices, the solution to the initial value problem

$$\dot{\Phi}(t) = J_0 S(t)\Phi(t) \qquad \text{and} \qquad \Phi(0) = \mathrm{id}$$

is an arc Φ in $\mathrm{Sp}(2n)$ starting at the identity. Conversely, every such arc Φ arises this way for a unique path S of symmetric matrices. A path Φ in $\mathrm{Sp}(2n)$ is called *positive* if $S(t)$ is positive definite for all t. Lemma 2.3.11 below characterizes the elements $\widetilde{\Phi}$ of the universal cover $\widetilde{\mathrm{Sp}}(2)$ which can be represented by positive arcs. This characterization is given in terms of the rotation number

$$\rho : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}.$$

We give two equivalent definitions of ρ. The first one involves eigenvalues of symplectic matrices. We begin by defining a function $\overline{\rho} : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z}$. The complex eigenvalues of a matrix $A \in \mathrm{Sp}(2)$ are either given by λ, λ^{-1} for $\lambda \in \mathbb{R} \setminus \{0\}$ or by $e^{\pm 2\pi i\theta}$ for $\theta \in (0, 1/2)$. In the former case, we define

$$\overline{\rho}(A) := \begin{cases} 0 & \text{if } \lambda > 0 \\ 1/2 & \text{if } \lambda < 0 \end{cases}$$

In the latter case, we fix an arbitrary vector $v \in \mathbb{R}^2 \setminus \{0\}$ and define

$$\overline{\rho}(A) := \begin{cases} \theta & \text{if } \langle J_0 v, Av\rangle > 0 \\ -\theta & \text{if } \langle J_0 v, Av\rangle < 0 \end{cases}$$

The rotation number ρ is defined to be the unique lift of $\overline{\rho}$ to the universal cover $\widetilde{\mathrm{Sp}}(2)$ satisfying $\rho(\mathrm{id}) = 0$.
For our second definition of ρ, we fix $v \in \mathbb{R}^2 \setminus \{0\}$ and define $\overline{\rho}_v : \mathrm{Sp}(2) \to \mathbb{R}/\mathbb{Z}$ to be the auxiliary function characterized by

$$Av \in \mathbb{R}_{>0} \cdot e^{2\pi i \overline{\rho}_v(A)} \cdot v$$

We let $\rho_v : \widetilde{\mathrm{Sp}}(2) \to \mathbb{R}$ denote the unique lift of $\overline{\rho}_v$ to the universal cover satisfying $\rho_v(\mathrm{id}) = 0$ and define

$$\rho(\widetilde{\Phi}) := \lim_{k\to\infty} \frac{\rho_v(\widetilde{\Phi}^n)}{n}.$$

We refer to [12, appendix A] for a proof that these two definitions of ρ coincide.

Lemma 2.3.11. *Let $\widetilde{\Phi} \in \widetilde{\mathrm{Sp}}(2)$. Then $\widetilde{\Phi}$ can be represented by a positive arc in $\mathrm{Sp}(2)$ if and only if the rotation number $\rho(\widetilde{\Phi})$ is strictly positive.*

Proof. Suppose that $\widetilde{\Phi}$ is represented by a positive arc $(\Phi(t))_{t\in[0,1]}$ in $\mathrm{Sp}(2)$ starting at the identity. Let $S(t)$ denote the path of positive definite matrices generating $\Phi(t)$. We may choose $\varepsilon > 0$ such that $\langle z, S(t)z\rangle \geq \varepsilon$ for all $t \in [0,1]$ and all unit vectors $z \in \mathbb{R}^2$. Fix $v \in \mathbb{R}^2 \setminus \{0\}$. A direct computation shows that

$$\frac{d}{dt}\overline{\rho}_v(\Phi(t)) = \frac{\langle \Phi(t)v, S(t)\Phi(t)v\rangle}{|\Phi(t)v|^2} \geq \varepsilon.$$

It is immediate from our second definition of ρ that this implies that $\rho(\widetilde{\Phi}) \geq \varepsilon > 0$.
Conversely, suppose that $\rho(\widetilde{\Phi}) > 0$. Our goal is to construct a positive arc in $\mathrm{Sp}(2)$ representing $\widetilde{\Phi}$. We claim that we may reduce ourselves to the case $\rho(\widetilde{\Phi}) \in (0,1)$. Indeed, for $\tau \in \mathbb{R}$, let $\widetilde{R}_\tau \in \widetilde{\mathrm{Sp}}(2)$ denote the element represented by the arc $(e^{it\tau})_{t\in[0,1]}$ in $\mathrm{U}(1) \subset \mathrm{Sp}(2)$. Consider the function $\tau \mapsto \rho((\widetilde{R}_\tau)^{-1} \circ \widetilde{\Phi})$. This function is continuous and it is clear from our second definition of ρ that it is decreasing and that it diverges to $-\infty$ as $\tau \to +\infty$. Thus we may pick $\tau > 0$ such that $\rho((\widetilde{R}_\tau)^{-1} \circ \widetilde{\Phi}) \in (0,1)$. Since $\widetilde{R}_\tau$ is represented by a positive arc by definition, it suffices to show that the same is true for $(\widetilde{R}_\tau)^{-1} \circ \widetilde{\Phi}$. Hence, after replacing $\widetilde{\Phi}$ by $(\widetilde{R}_\tau)^{-1} \circ \widetilde{\Phi}$, we may assume w.l.o.g. that $\rho(\widetilde{\Phi}) \in (0,1)$.
Let $\Phi \in \mathrm{Sp}(2)$ denote the projection of $\widetilde{\Phi}$ to $\mathrm{Sp}(2)$. Since $\rho(\widetilde{\Phi}) \notin \mathbb{Z}$, the spectrum of Φ does not contain positive real numbers. Thus Theorem 1.2 in [81] implies that there exists a positive arc $(\Phi(t))_{t\in[0,1]}$ in $\mathrm{Sp}(2)$ starting at the identity and ending at $\Phi(1) = \Phi$ such that $\Phi(t)$ does not have any positive real eigenvalues for any $t > 0$. Let $[(\Phi(t))_t]$ denote the element of the universal cover represented by the arc $(\Phi(t))_t$. Our goal is to show that $\widetilde{\Phi} = [(\Phi(t))_t]$. Since the projections of these to elements to $\mathrm{Sp}(2)$ agree, it is enough to show that the rotation numbers $\rho(\widetilde{\Phi})$ and $\rho([(\Phi(t))_t])$ coincide. These rotation numbers must agree in $\mathbb{R}/\mathbb{Z}$, so it is actually enough to show that $\rho([(\Phi(t))_t]) \in (0,1)$. Positivity of the arc $(\Phi(t))_t$ implies that $\rho([(\Phi(t))_t]) > 0$. This follows from the implication of Lemma 2.3.11 already proved above. Since the spectrum of $\Phi(t)$ does not contain positive real numbers for any $t > 0$, we have $\overline{\rho}(\Phi(t)) \neq 0$ for all $t > 0$. We deduce that $\rho([(\Phi(t))_t]) < 1$. This concludes our proof that $\widetilde{\Phi}$ can be represented by a positive arc. □

Lemma 2.3.12. *Let $X \subset \mathbb{R}^4$ be a star-shaped domain and let $\Sigma \subset \partial X$ be a ∂-strong disk-like global surface of section. Let $\varepsilon > 0$ be sufficiently small and let $\mathbb{D}_\varepsilon \subset \mathbb{C}$ denote the disk of radius ε. Then there exist a ∂-strong disk-like global surface of section Σ' with the same boundary orbit as Σ and a parametrization $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X$ of a tubular neighbourhood of Σ such that the following is true:*

$$F^{-1}(\Sigma') = \{(t, re^{i\theta}) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \mid 0 \leq r \leq \varepsilon \text{ and } \theta = 0\} \tag{2.3.6}$$

and

$$F^*\lambda_0 = \frac{1}{2}r^2 d\theta + H dt \tag{2.3.7}$$

where $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \mathbb{R}$ is a Hamiltonian such that $H_t(0) = \int_{\partial\Sigma} \lambda_0$ and the differential $dH_t(0)$ vanishes. Moreover, the Hessian $\nabla^2 H_t(0)$ is negative definite.

Proof. Let $a := \int_{\partial\Sigma} \lambda_0$ be the action of the orbit $\partial\Sigma$. Let ξ denote the contact structure on ∂X. Let $\tau : \xi|_{\partial\Sigma} \cong \mathbb{R}^2$ be a symplectic trivialization of $\xi|_{\partial\Sigma}$ with the property that Σ does not wind with respect to τ. Via the trivialization τ, the linearized Reeb flow $d\phi^t$ along $\partial\Sigma$ induces an arc $\Phi : [0, a] \to \mathrm{Sp}(2)$ representing an element $\widetilde{\Phi}$ of the universal cover $\widetilde{\mathrm{Sp}}(2)$. Since $\partial\Sigma$ is a positive boundary orbit of Σ, the linearized Reeb flow along $\partial\Sigma$ winds non-negatively with respect to the surface of section Σ, i.e. $\rho(\widetilde{\Phi}) \geq 0$. The fact that Σ is ∂-strong actually implies that $\rho(\widetilde{\Phi})$ is strictly positive. Hence it follows from Lemma 2.3.11 that $\widetilde{\Phi}$ is represented by a positive arc. We may therefore choose a loop $(S(t))_{t\in\mathbb{R}/a\mathbb{Z}}$ of symmetric positive definite matrices generating an arc $\Psi : [0, a] \to \mathrm{Sp}(2)$ which represents $\widetilde{\Phi}$. After replacing τ by an isotopic trivialization, we can assume that the arc Ψ is induced by the linearized Reeb flow. Using the trivialization τ, we can choose a parametrization $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X$ of a tubular neighbourhood of $\partial\Sigma$ such that

1. The pullback $F^*\lambda_0$ is given by $a \cdot dt$ on the circle $\mathbb{R}/\mathbb{Z} \times 0$.
2. The pullback $F^*\omega_0$ agrees with ω_0 on the the circle $\mathbb{R}/\mathbb{Z} \times 0$. Here ω_0 denotes the standard symplectic form on $\mathbb{R}^4$ and also the 2-form on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon$ whose restriction to fibres $t \times \mathbb{D}_\varepsilon$ agrees with the standard symplectic form on $\mathbb{D}_\varepsilon$ and which vanishes on the vector field ∂_t.
3. The linearized Reeb flow of $F^*\lambda_0$ along the orbit $\mathbb{R}/\mathbb{Z} \times 0$ is given by the arc Ψ.

The remaining argument proceeds exactly as the proof of Lemma 5.2 in [59]. The result is a modification of the parametrization F such that properties (1)-(3) above still hold and such that $F^*\lambda_0$ is of the form (2.3.7) for some Hamiltonian H which satisfies $H_t(0) = a$ and $dH_t(0) = 0$. It follows from the fact that the linearized flow Ψ is generated by symmetric positive definite matrices and our sign conventions that $\nabla^2 H_t(0)$ is negative definite. Finally, since Σ does not wind with respect to the parametrization F, we can achieve (2.3.6) by isotoping Σ and possibly shrinking the tubular neighbourhood. □

Lemma 2.3.13. *Let D be a closed 2-dimensional disk. Let λ be a Liouville 1-form on D, i.e. $\omega := d\lambda$ is a symplectic form and the Liouville vector field W characterized by $\iota_W\omega = \lambda$ is transverse to ∂D. Let $I \subset \mathbb{R}$ be a closed interval and endow $I \times D$ with the contact form $dt + \lambda$. Here t denotes the coordinate on I. Set $M := \mathbb{R}_s \times I \times D$. We can regard M as the symplectization of $I \times D$ and equip it with the symplectic form $\omega_M := d(e^s(dt + \lambda))$. We can also regard M as time-energy extended phase space of D and endow it with the symplectic form $\widetilde{\omega} := ds \wedge dt + \omega$. We abbreviate $M_+ := \mathbb{R}_{\geq 0} \times I \times D$ and $M_0 := 0 \times I \times D$. There exists a symplectic embedding*

$$G : (M_+, \widetilde{\omega}) \to (M_+, \omega_M)$$

which restricts to the identity on M_0.

Proof. We assume w.l.o.g. that 0 is contained in the interior of I. We define the vector field Y on M by

$$Y := \partial_s - W - t \cdot \partial t.$$

Since W is outward pointing at ∂D and $t \cdot \partial_t$ is outward pointing at ∂I, the flow of Y is defined for all positive times. We define G to be the embedding which is uniquely determined by requiring $G(0,t,z) = (0,t,z)$ for all $(t,z) \in I \times D$ and $G^*Y = \partial_s$. Let us check that

$$G^*\omega_M = \widetilde{\omega}.$$

We compute

$$\begin{aligned} \mathcal{L}_Y\omega_M &= \mathcal{L}_Y d(e^s(\lambda + dt)) \\ &= d((\mathcal{L}_Y e^s)(\lambda + dt) + e^s\mathcal{L}_Y(\lambda + dt)) \\ &= d(e^s(\lambda + dt - d\iota_W\lambda - \iota_W d\lambda - d\iota_{t\partial_t}dt)) \\ &= 0. \end{aligned}$$

Clearly $\mathcal{L}_{\partial_s}\widetilde{\omega} = 0$ as well. Thus is suffices to check that $G^*\omega_M$ and $\widetilde{\omega}$ agree on the set M_0. This is equivalent to showing that the pullbacks to M_0 of $G^*\omega_M$ and $\widetilde{\omega}$ and of $\iota_{\partial_s}G^*\omega_M$ and $\iota_{\partial_s}\widetilde{\omega}$ agree. Since the restriction of G to M_0 is the identity, the pullback of $G^*\omega_M$ to M_0 is equal to ω. This agrees with the pullback of $\widetilde{\omega}$. We compute

$$\begin{aligned} \iota_{\partial_s}G^*\omega_M &= \iota_{\partial_s}G^*d(e^s(\lambda + dt)) \\ &= G^*\iota_Y e^s(\omega + ds \wedge \lambda + ds \wedge dt) \\ &= G^*e^s(-\lambda + \lambda + dt + tds) \\ &= G^*e^s(dt + tds). \end{aligned}$$

The pullback of this form to M_0 is simply dt. This agrees with the pullback of $\iota_s\widetilde{\omega}$. This establishes $G^*\omega_M = \widetilde{\omega}$ □

Proof of Proposition 2.3.2. Our first step is to construct a star-shaped domain $X' \subset \mathbb{R}^4$ which contains X, agrees with X outside a small neighbourhood of $\partial\Sigma$ and has a ∂-strong disk-like surface of section $\Sigma' \subset \partial X'$ of the same symplectic area as Σ such that the local first return map of a small disk transverse to the boundary orbit $\partial\Sigma'$ is smoothly conjugated to a rotation. We apply Lemma 2.3.12. After replacing Σ by a disk-like global surface of section with the same boundary orbit, we may choose a parametrization $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X$ of a tubular neighbourhood of $\partial\Sigma$ satisfying (2.3.6) and (2.3.7). Consider time-energy extended phase space $\mathbb{R}_s \times (\mathbb{R}/\mathbb{Z})_t \times \mathbb{D}_\varepsilon$ equipped with the symplectic form $\widetilde{\omega_0} := \omega_0 + ds \wedge dt$ where ω_0 denotes the standard symplectic form on $\mathbb{D}_\varepsilon$. The pullback of $\widetilde{\omega_0}$ via the parametrization

$$G : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \quad G(t,z) := (H_t(z), t, z)$$

of the graph $\Gamma(H)$ is given by $\omega_0 + dH \wedge dt$. It follows from (2.3.7) that this agrees with $F^*\omega_0$ where, by slight abuse of notation, ω_0 also denotes the standard symplectic form on $\mathbb{R}^4$. We set $\psi := F \circ G^{-1}$. This defines a diffeomorphism from $\Gamma(H)$ to $\mathrm{im}(F)$ satisfying $\psi^*\omega_0 = \widetilde{\omega_0}|_{\Gamma(H)}$. By the coisotropic neighbourhood theorem in [45], we may extend ψ to a symplectomorphism defined on some open neighbourhood U of the graph $\Gamma(H)$. Note that the push-forward of the vector field ∂_s via ψ is transverse to ∂X and outward pointing. Let H' be a C^1-small perturbation of H supported in a small neighbourhood of $\mathbb{R}/\mathbb{Z} \times 0$ with the following properties:

1. $H' \geq H$
2. Near $\mathbb{R}/\mathbb{Z} \times 0$ the Hamiltonian H' is given by
$$H'_t(z) = H_t(0) - b|z|^2 \tag{2.3.8}$$
for some positive constant b.

This is possible because the Hessian $\nabla^2 H_t(0)$ is negative definite. We define X' to be the star-shaped domain which agrees with X outside $\operatorname{im}(\psi)$ and which satisfies

$$\partial X' \cap \operatorname{im}(\psi) = \psi(\Gamma(H')).$$

The inequality $H' \geq H$ implies that X is contained on X'. By (2.3.8), the orbit $\partial\Sigma$ on ∂X also is an orbit on $\partial X'$. If follows from our construction of X' that there exists an embedding $F' : \mathbb{R}/\mathbb{Z} \times \mathbb{D}_\varepsilon \to \partial X'$ which agrees with F near $\mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}_\varepsilon$ such that

$$F'^*\omega_0 = \omega_0 + dH' \wedge dt.$$

We define $\Sigma' \subset \partial X'$ to agree with Σ outside $\operatorname{im}(F')$ and to be given by

$$\{F'(t,r) \mid 0 \leq r \leq \varepsilon \text{ and } t \in \mathbb{R}/\mathbb{Z}\}$$

inside $\operatorname{im}(F')$. This clearly defines a disk-like surface of section of the Reeb flow on $\partial X'$. Its symplectic area agrees with the symplectic area of Σ because $\partial\Sigma' = \partial\Sigma$. It follows from (2.3.8) that the local return map of the orbit $\partial\Sigma'$ is smoothly conjugated to a rotation.
Let us replace (X, Σ) by (X', Σ'). We may choose a smooth parametrization $f : \mathbb{D} \to \Sigma$ such that near the boundary $\partial\mathbb{D}$ the pullback $\omega := f^*\omega_0$ is rotation invariant and the first return map ϕ is a rotation. Let $\widetilde{\phi}_0$ denote the lift of ϕ with respect to a degree 0 trivialization. Note that by positivity of the constant b in (2.3.8), the rotation angle of $\widetilde{\phi}_0$ near $\partial\mathbb{D}$ must be strictly positive. Thus $\widetilde{\phi}_0$ can be generated by a Hamiltonian H which vanishes on the boundary and is autonomous and strictly positive in some neighbourhood of $\partial\mathbb{D}$. We do not know whether we can choose H to be strictly positive everywhere in the interior. Our strategy is to construct a domain X' which contains X and agrees with X outside an open neighbourhood of Σ such that the degree 0 lift of the first return map of the Reeb flow on $\partial X'$ can be generated by a Hamiltonian H' which agrees with H near $\partial\mathbb{D}$ and is strictly positive in the interior. Let $K : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ be a non-negative Hamiltonian which is compactly supported in the interior $\operatorname{int}(\mathbb{R}/\mathbb{Z} \times \mathbb{D})$. The composition $\widetilde{\phi}_0 \circ (\phi_K^t)_{t\in[0,1]} \in \widetilde{\operatorname{Diff}}(\mathbb{D}, \omega)$ is generated by the Hamiltonian

$$(H\#K)_t := H_t + K \circ (\phi_H^t)^{-1}. \tag{2.3.9}$$

If K is sufficiently large, then this Hamiltonian is strictly positive. The Hamiltonian $H\#K$ need not be 1-periodic. This can be remedied as follows: First note that we can assume that H_t is strictly positive in the interior for t in some open neighbourhood of $0 \in \mathbb{R}/\mathbb{Z}$. Then it suffices to consider K with the property that K_t vanishes for t near 0. In this situation $H\#K$ smoothly extends to a 1-periodic Hamiltonian which is strictly positive in the interior and autonomous and rotation invariant near the boundary. Our goal is to construct X' such that the degree 0 lift of the first return map is given by $\widetilde{\phi}_0 \circ (\phi_K^t)_{t\in[0,1]}$.
Let $D \subset \operatorname{int}(\mathbb{D})$ be a closed disk centred at 0 and containing the support of K. After possibly increasing the radius of D, we can assume that $\lambda := f^*\lambda_0$ is a Liouville 1-form on D whose associated Liouville vector field is transverse to ∂D. Let $\varepsilon > 0$ be sufficiently small and let

$$F : [0,\varepsilon] \times D \to \partial X$$

be the unique embedding such that $F(0,z) = f(z)$ and such that the pullback of the Reeb vector field R on ∂X via F is given by ∂_t where t is the coordinate on $[0,\varepsilon]$. This implies that $F^*\lambda_0 = \lambda + dt$. Let $M := \mathbb{R} \times [0,\varepsilon] \times D$ be the symplectization of $([0,\varepsilon] \times D, \lambda + dt)$ equipped with the symplectic form $\omega_M := d(e^s(\lambda + dt))$. Using the radial Liouville vector field Z_0 on $\mathbb{R}^4$, we can extend F to a symplectic embedding $F : (M, \omega_M) \to (\mathbb{R}^4, \omega_0)$ mapping ∂_s to Z_0. Our modification of X will be supported inside the image of F.
We apply Lemma 2.3.13 and obtain a symplectic embedding $G : (M_+, \widetilde{\omega}) \to (M_+, \omega_M)$. Let us reparametrize K_t such that it is compactly supported in the time interval $(0,\varepsilon)$ and still generates the same time-1-flow. Let $\Gamma_-(K)$ denote the subgraph of K, i.e. the set

$$\Gamma_-(K) = \{(s,t,z) \in M_+ \mid s \le K_t(z)\}.$$

We define X' to be the union

$$X' := X \cup F(G(\Gamma_-(K))).$$

Σ is a disk-like global surface of section of the characteristic foliation on $\partial X'$ and it follows from Lemma 2.3.3 that the degree 0 lift of the first return map is given by $\widetilde{\phi'}_0 = \widetilde{\phi}_0 \circ (\phi_K^t)_{t\in[0,1]}$. By our construction of K, the first return map $\widetilde{\phi'}_0$ can be generated by a Hamiltonian satisfying the hypotheses in Theorem 2.3.1.
The domain X' might not be star-shaped. We argue that X' must be symplectomorphic to a star-shaped domain for appropriate choice of Hamiltonian K. Our strategy is to define a contact form β on $\partial X'$ such that $d\beta = \omega_0|_{X'}$. This contact form must be tight and there exists a star-shaped domain $X'' \subset \mathbb{R}^4$ such that $(\partial X', \beta)$ is strictly contactomorphic to $(\partial X'', \lambda_0|_{\partial X''})$. Corollary 2.3.10 then implies that X'' is symplectomorphic to X'. On the complement of the image of $F : M \to \mathbb{R}^4$ we simply define $\beta := \lambda_0|_{\partial X'}$. We parametrize the intersection $\operatorname{im}(F) \cap \partial X'$ via

$$F' : [0,\varepsilon] \times D \to \operatorname{im}(F) \cap \partial X' \quad (t,z) \mapsto F(G(K_t(z),t,z)).$$

A direct computation shows that $F'^*\omega_0 = \omega + dK_t \wedge dt$. Moreover, we have $F'^*\lambda_0 = dt + \lambda$ near the boundary of $[0,\varepsilon] \times D$. Thus we may extend β to a smooth 1-form on all of $\partial X'$

be requiring that $F'^*\beta = dt + \lambda + K_t dt$. The resulting 1-form clearly satisfies $d\beta = \omega_0|_{\partial X'}$. It remains to check β is indeed a contact form on $\mathrm{im}(F) \cap \partial X'$ for appropriate choice of K. This amounts to showing that $F'^*(\beta \wedge d\beta)$ is a positive volume form on $[0, \varepsilon] \times D$. We compute

$$F'^*(\beta \wedge d\beta) = dt \wedge ((1 + K_t)\omega + \lambda \wedge dK_t).$$

Thus it suffices to construct K such that $(1+K_t)\omega+\lambda\wedge dK_t$ is a positive area form on D. The first term clearly is a positive area form since $K_t \geq 0$. Let W denote the Liouville vector field on D induced by λ. We can guarantee that the second term is non-negative by choosing K_t to be constant outside a small neighbourhood of ∂D and requiring that $dK_t(W) \leq 0$ inside that small neighbourhood. Clearly we still have the freedom to make (2.3.9) positive. □

2.4 A positivity criterion for Hamiltonian diffeomorphisms

The results of this section are inspired by the fixed point theorem stated in [3, Theorem 3]. In fact, our proofs rely on the generalized generating functions introduced in [3, sections 2.3 to 2.6].

2.4.1 Statement of the positivity criterion

The generalized generating function framework we use applies to area-preserving diffeomorphisms of the disk $\mathbb{D}$ which are *radially monotone* in the sense of the following definition.

Definition 2.4.1. A diffeomorphism $\phi \in \mathrm{Diff}^+(\mathbb{D})$ is called *radially monotone* if it fixes the center 0 and if the image of the radial foliation of $\mathbb{D}$ under ϕ is transverse to the foliation of $\mathbb{D}$ by circles centred at 0.

We state the main result of this section.

Theorem 2.4.2. *Let ω be a smooth 2-form on $\mathbb{D}$ which is positive in the interior $\mathrm{int}(\mathbb{D})$. Moreover, assume that ω is rotation invariant near the origin and the boundary $\partial\mathbb{D}$. Let $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ and set $\phi := \pi(\widetilde{\phi})$. Assume that:*

1. *ϕ fixes the origin and is radially monotone.*
2. *The restriction of ϕ to a small disk centred at the origin is a rotation.*
3. *The restriction of ϕ to a small annular neighbourhood of $\partial\mathbb{D}$ is a rotation.*
4. *The action $\sigma_{\widetilde{\phi}}(p)$ is positive for all fixed points p of ϕ.*

Then there exists a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ with the following properties:

1. *H is strictly positive in the interior $\mathrm{int}(\mathbb{D})$ and vanishes on the boundary $\partial\mathbb{D}$.*

2. *H is autonomous and rotation invariant in some neighbourhood of the origin and $\partial\mathbb{D}$.*

3. *The Hamiltonian vector field X_{H_t} defined by $\iota_{X_{H_t}}\omega = dH_t$ in the interior $\operatorname{int}(\mathbb{D})$ smoothly extends to the closed disk $\mathbb{D}$ and is tangent to $\partial\mathbb{D}$.*

4. *The arc $(\phi_H^t)_{t\in[0,1]}$ represents $\widetilde{\phi}$.*

In order to prove Theorem 2.1.9, we need to apply Theorem 2.4.2 to area-preserving diffeomorphisms of the disk which are C^1-close to the identity. Such diffeomorphisms need not be radially monotone. However, they are smoothly conjugated to radially monotone diffeomorphisms (see [3, Proposition 2.24]). We use this observation to deduce the following corollary of Theorem 2.4.2.

Corollary 2.4.3. *Let ω be a smooth 2-form on $\mathbb{D}$ which is positive in the interior $\operatorname{int}(\mathbb{D})$. Let $\widetilde{\phi} \in \widetilde{\operatorname{Diff}}(\mathbb{D},\omega)$ and set $\phi \coloneqq \pi(\widetilde{\phi})$. Assume that:*

1. *$\widetilde{\phi}$ is C^1-close to the identity $\operatorname{id}_{\mathbb{D}}$.*

2. *ϕ is smoothly conjugated to a rotation in some neighbourhood of the boundary $\partial\mathbb{D}$.*

3. *The action $\sigma_{\widetilde{\phi}}(p)$ is positive for all fixed points p of ϕ.*

Then there exists a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ with the following properties:

1. *H is strictly positive in the interior $\operatorname{int}(\mathbb{D})$ and vanishes on the boundary $\partial\mathbb{D}$.*

2. *H is autonomous in some neighbourhood of $\partial\mathbb{D}$.*

3. *The Hamiltonian vector field X_{H_t} defined by $\iota_{X_{H_t}}\omega = dH_t$ in the interior $\operatorname{int}(\mathbb{D})$ smoothly extends to the closed disk $\mathbb{D}$ and is tangent to $\partial\mathbb{D}$.*

4. *The arc $(\phi_H^t)_{t\in[0,1]}$ represents $\widetilde{\phi}$.*

2.4.2 Lifts to the strip

In order to represent area preserving diffeomorphisms of the disk $\mathbb{D}$ by generalized generating functions, it will be convenient to lift them to the strip $S \coloneqq [0,1] \times \mathbb{R}$. This is carefully explained in [3, section 2.3]. Here we summarize the relevant material. Consider the map

$$p : S \to \mathbb{D} \quad (r,\theta) \mapsto r \cdot e^{i\theta}.$$

The restriction of p to $(0,1] \times \mathbb{R}$ is a covering map to $\mathbb{D} \setminus \{0\}$. The translation

$$T : S \to S \quad (r,\theta) \mapsto (r, \theta + 2\pi)$$

generates the group of deck transformations. Any orientation preserving diffeomorphism $\phi \in \mathrm{Diff}^+(\mathbb{D})$ fixing the origin lifts to a diffeomorphism $\Phi \in \mathrm{Diff}^+(S)$ (see [3, Lemma 2.10]). The lift Φ commutes with T, i.e.

$$T \circ \Phi = \Phi \circ T.$$

Any two lifts are related by composition with a deck transformation. A lift $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D})$ of ϕ to the universal cover uniquely specifies a lift $\Phi \in \mathrm{Diff}^+(S)$ as follows: Represent $\widetilde{\phi}$ by a smooth arc $(\phi_t)_{t\in[0,1]}$ in $\mathrm{Diff}^+(\mathbb{D})$ starting at the identity. This arc uniquely lifts to an arc $(\Phi_t)_{t\in[0,1]}$ in $\mathrm{Diff}^+(S)$ starting at the identity. Now simply set $\Phi := \Phi_1$.

If the diffeomorphism $\phi \in \mathrm{Diff}^+(\mathbb{D})$ is radially monotone, then any lift Φ is monotone in the sense of the following definition.

Definition 2.4.4. Let $\Phi \in \mathrm{Diff}^+(S)$ be a diffeomorphism and denote the components of Φ by (R, Θ). We call Φ *monotone* if $\partial_1 R(r, \theta) > 0$ for all $(r, \theta) \in S$.

The following characterization of monotonicity will be useful in later sections.

Lemma 2.4.5. *Let $\Phi \in \mathrm{Diff}^+(S)$ be a diffeomorphism preserving the boundary components of S. Let $\Gamma(\Phi) \subset S \times S$ denote the graph of Φ. Then Φ is monotone if and only if*

$$\pi : \Gamma(\Phi) \subset S \times S \to S \quad (r, \theta, R, \Theta) \mapsto (R, \theta)$$

is a diffeomorphism.

Proof. Let $\Phi(r, \theta) = (R(r, \theta), \Theta(r, \theta))$ denote the components of Φ. The map

$$(\mathrm{id}_S, \Phi) : S \to \Gamma(\Phi) \quad (r, \theta) \mapsto (r, \theta, R(r, \theta), \Theta(r, \theta))$$

is a parametrization of $\Gamma(\Phi)$. Clearly, π is a diffeomorphism if and only if the composition

$$\pi \circ (\mathrm{id}_S, \Phi) : S \to S \quad (r, \theta) \mapsto (R(r, \theta), \theta)$$

is a diffeomorphism. This is the case if and only if

$$R(\cdot, \theta) : [0, 1] \to [0, 1] \tag{2.4.1}$$

is a diffeomorphism for every fixed $\theta \in \mathbb{R}$. By assumption Φ preserves the boundary components of S. Thus $R(0, \theta) = 0$ and $R(1, \theta) = 1$ for every θ. Hence (2.4.1) is a diffeomorphism if and only if $\partial_1 R > 0$, i.e. Φ is monotone. □

Now suppose that $\mathbb{D}$ is equipped with a 2-form ω which is positive in the interior $\mathrm{int}(\mathbb{D})$. Then

$$\Omega := p^*\omega$$

is a 2-form on S which is positive in the interior and invariant under the translation T, i.e.

$$T^*\Omega = \Omega.$$

If $\phi \in \mathrm{Diff}(\mathbb{D}, \omega)$ fixes the origin and preserves ω, then any lift Φ preserves Ω, i.e. $\Phi \in \mathrm{Diff}(S, \Omega)$.

2.4.3 Generalized generating functions on the strip

Throughout this section, let Ω be a 2-form on the strip S which is positive in the interior $\mathrm{int}(S)$ and preserved by T. Moreover, let $\Phi \in \mathrm{Diff}(S,\Omega)$ be a diffeomorphism which preserves Ω and the boundary components of S and which commutes with T. We equip the product $S\times S$ with the closed 2-form $(-\Omega)\oplus\Omega$. The restriction of this 2-form to the interior of $S\times S$ is a symplectic form. Consider a primitive 1-form α of $(-\Omega)\oplus\Omega$ whose restriction to the diagonal $\Delta\subset S\times S$ vanishes. The graph $\Gamma(\Phi)$ is a Lagrangian submanifold of $(S\times S,(-\Omega)\oplus\Omega)$, i.e. the restriction of $(-\Omega)\oplus\Omega$ to $\Gamma(\Phi)$ vanishes. Therefore, the restriction of the primitive α to $\Gamma(\Phi)$ is closed. Since S is simply connected, it is also exact, i.e. it can be written as

$$\alpha|_{\Gamma(\Phi)} = dW_{\Phi,\alpha} \tag{2.4.2}$$

for a function $W_{\Phi,\alpha}:\Gamma(\Phi)\to\mathbb{R}$, which is unique up to addition of a constant. We call $W_{\Phi,\alpha}$ a *generalized generating function* for Φ with respect to α (c.f. [84, Definition 9.3.8]). In Lemma 2.4.6 below, we specify a preferred normalization of $W_{\Phi,\alpha}$ which we will use throughout this paper. For $z\in\{1\}\times\mathbb{R}\subset\partial S$, let γ_z denote the path

$$\gamma_z:[0,1]\to\partial S\times\partial S \qquad \gamma_z(t):=(z,(1-t)z+t\Phi(z)).$$

Lemma 2.4.6. *There exists a unique smooth function $W_{\Phi,\alpha}:\Gamma(\Phi)\to\mathbb{R}$ satisfying*

$$dW_{\Phi,\alpha}=\alpha|_{\Gamma(\Phi)} \tag{2.4.3}$$

and

$$W_{\Phi,\alpha}(z,\Phi(z))=\int_{\gamma_z}\alpha \qquad \textit{for all } z\in\{1\}\times\mathbb{R}. \tag{2.4.4}$$

Proof. Set $z_0:=(1,0)$. Clearly, there exists a unique function $W_{\Phi,\alpha}$ satisfying (2.4.3) such that (2.4.4) holds for z_0. We need to check that this function $W_{\Phi,\alpha}$ satisfies (2.4.4) for all $z\in\{1\}\times\mathbb{R}$. Fix $z\in\{1\}\times\mathbb{R}$. We define two paths δ and ε by

$$\delta:[0,1]\to\partial S\times\partial S \qquad \delta(t)=((1-t)z_0+tz,\Phi((1-t)z_0+tz))$$

and

$$\varepsilon:[0,1]\to\partial S\times\partial S \qquad \varepsilon(t)=((1-t)z_0+tz,(1-t)z_0+tz).$$

The path δ is contained in $\Gamma(\Phi)$ and goes from $(z_0,\Phi(z_0))$ to $(z,\Phi(z))$. The path ε connects (z_0,z_0) to (z,z) and is contained in Δ. The concatenations $\gamma_{z_0}\#\delta$ and $\varepsilon\#\gamma_z$ are homotopic inside $\partial S\times\partial S$ with fixed end points. Since the restriction of $(-\Omega)\oplus\Omega$ to $\partial S\times\partial S$ vanishes, the restriction of α to this subspace is closed. Thus

$$\int_{\gamma_{z_0}\#\delta}\alpha=\int_{\varepsilon\#\gamma_z}\alpha.$$

Using (2.4.3) and the fact that $W_{\Phi,\alpha}$ satisfies (2.4.4) for z_0, the left hand side evaluates to

$$\int_{\gamma_{z_0}\#\delta}\alpha = \int_{\gamma_{z_0}}\alpha + \int_\delta \alpha = W_{\Phi,\alpha}(z_0,\Phi(z_0)) + W_{\Phi,\alpha}(z,\Phi(z)) - W_{\Phi,\alpha}(z_0,\Phi(z_0)) = W_{\Phi,\alpha}(z,\Phi(z)).$$

Since the restriction of α to the diagonal Δ vanishes, the right hand side is given by

$$\int_{\varepsilon\#\gamma_z}\alpha = \int_\varepsilon \alpha + \int_{\gamma_z}\alpha = \int_{\gamma_z}\alpha.$$

This concludes our proof that (2.4.4) holds for all $z \in \{1\}\times\mathbb{R}$. □

Let β be a second primitive 1-form of $(-\Omega)\oplus\Omega$ whose restriction to Δ vanishes. Then the difference $\alpha-\beta$ is exact, i.e. there exists a smooth function u on $S\times S$ such that $\alpha-\beta = du$. Since the restrictions of α and β to the diagonal Δ vanish, the function u must be constant on Δ. Let us normalize u such that $u|_\Delta = 0$. The following lemma relates the generalized generating functions of Φ with respect to α and β.

Lemma 2.4.7. *Let $W_{\Phi,\alpha}$ and $W_{\Phi,\beta}$ be the generalized generating functions of Φ with respect to α and β. Then*

$$W_{\Phi,\alpha} = W_{\Phi,\beta} + u|_{\Gamma(\Phi)}.$$

In particular, since u vanishes on the diagonal Δ, the value of a generalized generating function at a fixed point of Φ is independent of the choice of primitive 1*-form.*

Proof. We set $W := W_{\Phi,\beta} + u|_{\Gamma(\Phi)}$. We need to check that this function satisfies (2.4.3) and (2.4.4). In order to show (2.4.3), we compute

$$dW = dW_{\Phi,\beta} + du|_{\Gamma(\Phi)} = \beta|_{\Gamma(\Phi)} + (\alpha-\beta)|_{\Gamma(\Phi)} = \alpha|_{\Gamma(\Phi)}.$$

Let $z \in \{1\}\times\mathbb{R}$. We have

$$W(z,\Phi(z)) = W_{\Phi,\beta}(z,\Phi(z)) + u(z,\Phi(z)) = \int_{\gamma_z}\beta + \int_{\gamma_z} du = \int_{\gamma_z}\alpha.$$

Here the second equality uses that u vanishes on the diagonal Δ. This shows (2.4.4). □

There are two primitives of $(-\Omega)\oplus\Omega$ whose associated generalized generating functions are of particular importance to our discussion. The first such primitive is given by $(-\Lambda)\oplus\Lambda$ where Λ is a primitive of the area form Ω on S. It will be useful to regard the associated generalized generating function $W_{\Phi,(-\Lambda)\oplus\Lambda}$ as a function on S via the parametrization $(\mathrm{id}_S,\Phi) : S \to \Gamma(\Phi)$ of the graph $\Gamma(\Phi)$. We define

$$\Sigma_{\Phi,\Lambda} := W_{\Phi,(-\Lambda)\oplus\Lambda} \circ (\mathrm{id}_S,\Phi)$$

and call it the *action* of Φ with respect to Λ. The characterizing equations (2.4.3) and (2.4.4) for the generalized generating function $W_{\Phi,(-\Lambda)\oplus\Lambda}$ can be expressed in terms of the action $\Sigma_{\Phi,\Lambda}$ as

$$\Phi^*\Lambda - \Lambda = d\Sigma_{\Phi,\Lambda} \tag{2.4.5}$$

and

$$\Sigma_{\Phi,\Lambda}(1,\theta) = \int_{\delta_\theta} \Lambda \qquad \text{for all } \theta \in \mathbb{R} \tag{2.4.6}$$

where δ_θ denotes the path

$$\delta_\theta : [0,1] \to \partial S \quad \delta_\theta(t) := (1-t)\cdot(1,\theta) + t\cdot\Phi(1,\theta).$$

The following basic properties of the action $\Sigma_{\Phi,\Lambda}$ will be useful later on.

Lemma 2.4.8. *1. Let $(\Phi_t)_{t\in[0,1]}$ be an arc in $\mathrm{Diff}(S,\Omega)$ starting at the identity. Let*

$$H : [0,1]\times S \to \mathbb{R}$$

be a Hamiltonian generating this arc. If we normalize H by $H_t(1,\theta) = 0$, then the action $\Sigma_{\Phi,\Lambda}$ may be computed via

$$\Sigma_{\Phi,\Lambda}(z) = \int_{\{t\mapsto\Phi_t(z)\}} \Lambda + \int_0^1 H_t(\Phi_t(z))dt.$$

2. Suppose that $\Lambda = p^*\lambda$ is the pull-back of a primitive λ of ω and that Φ is the lift of a diffeomorphism $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D},\omega)$ fixing the origin. Then $\Sigma_{\Phi,\Lambda} = \sigma_{\widetilde{\phi},\lambda}\circ p$.

Proof. Statement (1) is the analog of [3, Proposition 2.6], which deals with area-preserving diffeomorphisms of the disk $\mathbb{D}$. The proof given in [3] carries over to the case of the strip almost verbatim and we will not repeat it here.

We prove (2). We compute

$$d(\sigma_{\widetilde{\phi},\lambda}\circ p) = p^*d\sigma_{\widetilde{\phi},\lambda} = p^*(\phi^*\lambda - \lambda) = \Phi^*p^*\lambda - p^*\lambda = \Phi^*\Lambda - \Lambda.$$

Here the third equality uses the identity $p\circ\Phi = \phi\circ p$. Let $(\phi_t)_{t\in[0,1]}$ be an arc in $\mathrm{Diff}^+(\mathbb{D})$ representing $\widetilde{\phi}$. Then

$$\sigma_{\widetilde{\phi},\lambda}\circ p(1,\theta) = \int_{t\mapsto\phi_t(p(1,\theta))} \lambda = \int_{p\circ\delta_\theta} \lambda = \int_{\delta_\theta} p^*\lambda = \int_{\delta_\theta} \Lambda = \Sigma_{\Phi,\Lambda}(1,\theta).$$

Here the second equality uses that the restriction of λ to $\partial\mathbb{D}$ is closed and that $t\mapsto\phi_t(p(1,\theta))$ and $p\circ\delta_\theta$ are homotopic in $\partial\mathbb{D}$ with fixed end points. This shows that $\sigma_{\widetilde{\phi},\lambda}\circ p$ satisfies (2.4.5) and (2.4.6). Thus $\sigma_{\widetilde{\phi},\lambda}\circ p = \Sigma_{\Phi,\Lambda}$. □

Let us define a second special primitive of $(-\Omega) \oplus \Omega$. We write

$$\Omega = F(r, \theta) \cdot dr \wedge d\theta$$

where F is a smooth function on S which is positive in the interior and invariant under T. Next, we define functions A and B on S by

$$A(r, \theta) := \int_0^r F(s, \theta) ds \qquad \text{and} \qquad B(r, \theta) := \int_0^\theta F(r, \vartheta) d\vartheta.$$

We let (r, θ, R, Θ) denote coordinates on $S \times S$ and define

$$\Xi := (A(R, \theta) - A(r, \theta)) \cdot d\theta + (B(R, \theta) - B(R, \Theta)) \cdot dR.$$

A direct computation shows that $d\Xi = (-\Omega) \oplus \Omega$ and that the restriction of Ξ to the diagonal $\Delta \subset S \times S$ vanishes. The resulting generalized generating function $W_{\Phi,\Xi}$ is particularly useful if the diffeomorphism $\Phi \in \mathrm{Diff}(S, \Omega)$ is monotone in the sense of Definition 2.4.4. From now on, let us assume that this is the case. Consider the projection

$$\pi_\Delta : \Gamma(\Phi) \to S \quad (r, \theta, R, \Theta) \mapsto (R, \theta). \tag{2.4.7}$$

By Lemma 2.4.5, π_Δ is a diffeomorphism. It will be convenient to view $W_{\Phi,\Xi}$ as a function on S via the diffeomorphism π_Δ. We abbreviate

$$W := W_{\Phi,\Xi} \circ \pi_\Delta^{-1}.$$

Equation (2.4.3) can be rewritten in terms of W as

$$\begin{cases} \partial_1 W(R, \theta) = B(R, \theta) - B(R, \Theta) \\ \partial_2 W(R, \theta) = A(R, \theta) - A(r, \theta) \end{cases} \qquad \text{for all } (r, \theta, R, \Theta) \in \Gamma(\Phi). \tag{2.4.8}$$

The normalization (2.4.4) simply becomes

$$W|_{\{1\} \times \mathbb{R}} = 0 \tag{2.4.9}$$

because the restriction of the primitive Ξ to $(\{1\} \times \mathbb{R}) \times (\{1\} \times \mathbb{R})$ vanishes. We summarize the relevant properties of W in the following Proposition (see Proposition 2.15, Lemma 2.16 and Proposition 2.17 in [3]).

Proposition 2.4.9. *Suppose that $\Phi \in \mathrm{Diff}(S, \Omega)$ is monotone and commutes with T. Then there exists a unique generating function $W : S \to \mathbb{R}$ satisfying equations* (2.4.8) *and the normalization* (2.4.9)*. The function W is invariant under T and is constant on the boundary components of S. The interior critical points of W are precisely the interior fixed points of Φ. We have $W(p) = \Sigma_{\Phi,\Lambda}(p)$ for all fixed points p of Φ and any primitive Λ of Ω. If the restriction of Λ to $\{0\} \times \mathbb{R}$ vanishes, then W agrees with $\Sigma_{\Phi,\Lambda}$ on $\{0\} \times \mathbb{R}$.*

Proof. Existence and uniqueness of W follow from Proposition 2.15 in [3]. Moreover, this proposition asserts that W is invariant under T and constant on the boundary components of S and that the interior critical points of W are precisely the interior fixed points of W. The remaining assertions are proved in [3, Lemma 2.16 and Proposition 2.17] in the special case that Φ is the lift of a diffeomorphism $\widetilde{\phi} \in \widetilde{\mathrm{Diff}}(\mathbb{D},\omega)$ and $\Lambda = p^*\lambda$ for a primitive λ of ω. Since the statements in Proposition 2.4.9 are slightly more general, we provide independent proofs. It is a direct consequence of Lemma 2.4.7 that $W(p) = \Sigma_{\Phi,\Lambda}(p)$ for all fixed points p of Φ. Suppose that the restriction of Λ to $\{0\}\times\mathbb{R}$ vanishes. This implies that the restriction of $(-\Lambda)\oplus\Lambda$ to $(\{0\}\times\mathbb{R})^2$ vanishes. Similarly, the restriction of the primitive Ξ to this subspace vanishes. It is a direct consequence of (2.4.3) that the generating functions $W_{\Phi,\Xi}$ and $W_{\Phi,(-\Lambda)\oplus\Lambda}$ are both constant on $\Gamma(\Phi)\cap(\{0\}\times\mathbb{R})^2$. Let u be the unique smooth function on $S\times S$ whose restriction to the diagonal Δ vanishes and which satisfies $\Xi = (-\Lambda)\oplus\Lambda + du$. Since both Ξ and $(-\Lambda)\oplus\Lambda$ restrict to zero on $(\{0\}\times\mathbb{R})^2$, the function u vanishes on this set. By Lemma 2.4.7, this implies that $W_{\Phi,\Xi}$ and $W_{\Phi,(-\Lambda)\oplus\Lambda}$ agree on $\Gamma(\Phi)\cap(\{0\}\times\mathbb{R})^2$. We conclude that W and $\Sigma_{\Phi,\Lambda}$ agree and are constant on $\{0\}\times\mathbb{R}$. □

In order to avoid technicalities involving the behaviour of Φ and W near ∂S, let us now assume in addition that Ω is translation invariant in some small neighbourhood of ∂S. In other words, $F(r,\theta)$ does not depend on θ for r sufficiently close to 0 or 1. Moreover, we will restrict our attention to diffeomorphisms Φ whose restrictions to neighbourhoods of the two boundary components of S are translations. More precisely, we will assume that there exist constants θ_0 and θ_1 such that for $j\in\{0,1\}$

$$\Phi(r,\theta) = (r,\theta+\theta_j) \qquad \text{if } r \text{ is sufficiently close to } j. \tag{2.4.10}$$

Since the following result is not explicitly stated in [3], we provide a proof.

Proposition 2.4.10. *There exists a bijective correspondence between the set of all diffeomorphisms $\Phi \in \mathrm{Diff}(S,\Omega)$ which are monotone, commute with T and satisfy* (2.4.10) *and the set of all smooth functions $W : S \to \mathbb{R}$ satisfying*

1. $0 < A(r,\theta) - \partial_2 W(r,\theta) < A(1,\theta)$ *for all* $(r,\theta)\in \mathrm{int}(S)$
2. $\partial_{12}W(r,\theta) < F(r,\theta)$ *for all* $(r,\theta)\in\mathrm{int}(S)$
3. $W\circ T = W$
4. *There exist constants c_1, c_2 and c_3 such that*

$$\begin{cases} W(r,\theta) = c_1 + c_2\cdot\int_0^r F(s,\theta)ds & \text{if } r \text{ is sufficiently close to } 0\\ W(r,\theta) = c_3\cdot\int_r^1 F(s,\theta)ds & \text{if } r \text{ is sufficiently close to } 1. \end{cases}$$

Φ and W correspond to each other under this bijection if and only if equations (2.4.8) *hold.*

Proof. Let $\Phi \in \mathrm{Diff}(S,\Omega)$ be a diffeomorphism which is monotone, commutes with T and satisfies (2.4.10). Let W be the associated generating function satisfying (2.4.8) and the normalization (2.4.9). We verify that W satisfies properties (1)-(4). Property (3) actually is a consequence of Proposition 2.4.9. It follows from (2.4.10) that any point $(r,\theta,R,\Theta) \in \Gamma(\Phi)$ sufficiently close to the boundary satisfies $r = R$. Thus equation (2.4.8) implies that

$$\partial_2 W(R,\theta) = A(R,\theta) - A(r,\theta) = 0$$

near ∂S. Hence W is independent of θ in a neighbourhood of the boundary. Near ∂S we also have

$$B(r,\theta) = \theta \cdot F(r).$$

Here we use that $F(r,\theta) = F(r)$ does not depend on θ near ∂S. Thus (2.4.8) yields

$$\partial_1 W(R,\theta) = B(R,\theta) - B(R,\Theta) = (\theta - \Theta) \cdot F(R).$$

Using (2.4.10) we obtain

$$\partial_1 W(R,\theta) = -\theta_j \cdot F(R)$$

for R close to $j \in \{0,1\}$. Property (4) is an immediate consequence. Next we check property (1). Rearranging the second equation in (2.4.8) gives

$$A(R,\theta) - \partial_2 W(R,\theta) = A(r,\theta).$$

Now we simply use that $A(0,\theta) = 0$ and that $A(r,\theta)$ is strictly monotonic in r for fixed θ. It remains to verify (2). By Lemma 2.4.5, monotonicity of Φ implies that the projection π_Δ defined in equation (2.4.7) is a diffeomorphism. Let us now parametrize $\Gamma(\Phi)$ by the inverse of this diffeomorphism

$$\pi_\Delta^{-1} : S \to \Gamma(\Phi) \subset S \times S \quad (R,\theta) \mapsto \begin{pmatrix} r(R,\theta) \\ \theta \\ R \\ \Theta(R,\theta) \end{pmatrix}.$$

For $j \in \{1,2\}$ let $\pi_j : S \times S \to S$ be the projection onto the j-th factor. The restriction of π_j to $\Gamma(\Phi)$ is a diffeomorphism onto S. This shows that

$$(R,\theta) \mapsto (r(R,\theta),\theta) \qquad \text{and} \qquad (R,\theta) \mapsto (R,\Theta(R,\theta))$$

are both diffeomorphisms of S. In fact, these diffeomorphisms are orientation preserving. Thus the linearizations

$$\begin{pmatrix} \partial_1 r(R,\theta) & \partial_2 r(R,\theta) \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ \partial_1\Theta(R,\theta) & \partial_2\Theta(R,\theta) \end{pmatrix}$$

both have positive determinant. This implies $\partial_1 r(R,\theta) > 0$ and $\partial_2\Theta(R,\theta) > 0$. Let us view both sides of the first equation in (2.4.8) as functions of R and θ. Then differentiating with respect to θ yields

$$\partial_{12}W(R,\theta) = F(R,\theta) - F(R,\Theta(R,\theta)) \cdot \partial_2\Theta(R,\theta).$$

In the interior of S, both $F(R,\Theta)$ and $\partial_2\Theta$ are strictly positive. Thus

$$\partial_{12}W(R,\theta) < F(R,\theta)$$

proving (2).

Let us now prove the converse direction of Proposition 2.4.10. We start with a generating function W satisfying properties (1)-(4). Let us first show that we may solve equations (2.4.8) for r and Θ and obtain a smooth map

$$\operatorname{int}(S) \ni (R,\theta) \mapsto (r(R,\theta), \Theta(R,\theta)) \in \operatorname{int}(S).$$

The function $B(R,\cdot)$ is a diffeomorphism of $\mathbb{R}$ for fixed $R \in (0,1)$. Thus we may apply the implicit function theorem and solve the first equation in (2.4.8) for $\Theta(R,\theta)$ in the interior $\operatorname{int}(S)$. For fixed θ, the function $A(\cdot,\theta)$ is a diffeomorphism from $(0,1)$ onto $(0, A(1,\theta))$. By property (1), $\partial_2 W(R,\theta)$ is contained in the image of $A(R,\theta) - A(\cdot,\theta)$. Again we invoke the implicit funciton theorem to solve the second equation in (2.4.8) for $r(R,\theta)$. The map

$$\iota : \operatorname{int}(S) \to S \times S \quad (R,\theta) \mapsto \begin{pmatrix} r(R,\theta) \\ \theta \\ R \\ \Theta(R,\theta) \end{pmatrix} \tag{2.4.11}$$

parametrizes a smooth submanifold of $S \times S$. We show that this submanifold is in fact the graph of a diffeomorphism of $\operatorname{int}(S)$. Differentiating the first equation in (2.4.8) with respect to θ and the second equation with respect to R yields:

$$\begin{cases} \partial_{12}W(R,\theta) = F(R,\theta) - F(R,\Theta(R,\theta)) \cdot \partial_2\Theta(R,\theta) \\ \partial_{12}W(R,\theta) = F(R,\theta) - F(r(R,\theta),\theta) \cdot \partial_1 r(R,\theta) \end{cases}$$

Using property (2), we conclude that $\partial_1 r > 0$ and $\partial_2\Theta > 0$. Hence the composition $\pi_j \circ \iota$ is a diffeomorphism onto its image for $j \in \{1,2\}$. We show that this image actually is all of $\operatorname{int}(S)$. By property (4), we have $\partial_2 W = 0$ near ∂S. Thus the second equation in (2.4.8) implies that $r(R,\theta) = R$ for R near 0 or 1. This shows that $r(\cdot,\theta)$ is a diffeomorphism of $(0,1)$ for fixed θ. Therefore the image of $\pi_1 \circ \iota$ is $\operatorname{int}(S)$. The function $\partial_1 W$ is bounded. For fixed $R \in (0,1)$, the function $B(R,\cdot)$ is an orientation preserving diffeomorphism of $\mathbb{R}$. Thus the first equation in (2.4.8) implies that $\lim_{\theta\to\pm\infty} \Theta(R,\theta) = \pm\infty$. Hence the image of $\pi_2 \circ \iota$ is $\operatorname{int}(S)$. This shows that the image of ι is the graph of a diffeomorphism $\Phi \in \operatorname{Diff}(\operatorname{int}(S))$.

Equations (2.4.8) imply that the pull-back of Ξ via ι is closed. Therefore Φ preserves Ω. Property (4) implies that in a neighbourhood of ∂S equations (2.4.8) become:

$$\begin{cases} c \cdot F(R) = (\theta - \Theta) \cdot F(R) \\ 0 = A(R,\theta) - A(r,\theta) \end{cases}$$

We conclude that Φ satisfies (2.4.10) near ∂S. Therefore Φ smoothly extends to the closed strip and we have $\Phi \in \mathrm{Diff}(S,\Omega)$. In order to check that Φ commutes with T, first note that $A(R,\theta)$ is invariant under T and that $B(R,\theta+2\pi) = B(R,\theta) + B(R,2\pi)$. In combination with invariance of W under T, this implies that $r(R,\theta)$ is invariant under T and that $\Theta(R,\theta+2\pi) = \Theta(R,\theta) + 2\pi$. Hence the image of ι is invariant under the diagonal action of T on $S \times S$, which implies that Φ commutes with T. The composition of $\pi_\Delta \circ \iota$ is equal to id_S. Thus π_Δ is a diffeomorphism, which is equivalent to monotonicity of Φ by Lemma 2.4.5. □

2.4.4 Proof of the positivity criterion for radially monotone diffeomorphisms

Proof of Theorem 2.4.2. Let $\Phi \in \mathrm{Diff}(S,\Omega)$ be the lift of $\widetilde{\phi}$ to the strip. Φ is monotone and commutes with T. Since ϕ is a rotation near the origin and the boundary, Φ is a translation near the two boundary components of S, i.e. Φ satisfies (2.4.10). It follows from rotation invariance of ω near 0 and $\partial\mathbb{D}$ that Ω is translation invariant near ∂S. Let W be the unique generating function satisfying properties (1)-(4) in Proposition 2.4.10. For $t \in [0,1]$, we define $W_t := t \cdot W$. The functions W_t satisfy conditions (1)-(4) for all t. Indeed, the set of functions satisfying conditions (1)-(4) is convex and both W and the zero function satisfy these conditions. Proposition 2.4.10 therefore yields an isotopy $\Phi_t \in \mathrm{Diff}(S,\Omega)$ starting at the identity and ending at Φ. Let $H : [0,1] \times S \to \mathbb{R}$ be the unique Hamiltonian generating Φ_t which is normalized by $H_t(1,\theta) = 0$. Since Φ_t commutes with T for all t, the Hamiltonian H_t is invariant under T. Near the boundary components of S, the isotopy Φ_t is a translation at constant speed. Hence H is autonomous and translation invariant near the boundary.

Our goal is to show that the restriction of H to the complement of the boundary component $\{1\} \times \mathbb{R}$ is strictly positive. Our strategy is the following: If we can show that H_t is strictly positive on all its interior critical points and on the boundary component $\{0\} \times \mathbb{R}$, then it follows that H_t must be strictly positive on the complement of $\{1\} \times \mathbb{R}$. We begin by showing that for every t, the set of interior critical points of H_t is equal to the set of interior critical points of W. In other words, if the velocity $\partial_t\Phi_t(z)$ vanishes for some $t \in [0,1]$, then z must be a critical point of W and is fixed by the entire isotopy Φ_t. Let (R_t, Θ_t) denote the components of Φ_t. The defining equations for the generating function W_t read:

$$\begin{cases} \partial_1 W_t(R_t,\theta) = B(R_t,\theta) - B(R_t,\Theta_t) \\ \partial_2 W_t(R_t,\theta) = A(R_t,\theta) - A(r,\theta) \end{cases}$$

Differentiating with respect to t yields:

$$\begin{cases} \partial_1 W(R_t,\theta) + \partial_{11} W_t(R_t,\theta)\cdot\partial_t R_t = \partial_1 B(R_t,\theta)\cdot\partial_t R_t - \partial_1 B(R_t,\Theta_t)\cdot\partial_t R_t - \partial_2 B(R_t,\Theta_t)\cdot\partial_t\Theta_t \\ \partial_2 W(R_t,\theta) + \partial_{12} W_t(R_t,\theta)\cdot\partial_t R_t = \partial_1 A(R_t,\theta)\cdot\partial_t R_t \end{cases}$$

If $z=(r,\theta)$ is a point satisfying $\partial_t\Phi_t(z)=0$ for some t, then these equations yield:

$$\begin{cases} \partial_1 W(R_t(r,\theta),\theta) = 0 \\ \partial_2 W(R_t(r,\theta),\theta) = 0 \end{cases} \tag{2.4.12}$$

This implies that z is a critical point of W. Indeed, if $t=0$, then $R_t(r,\theta)=r$ and (2.4.12) says that (r,θ) is a critical point of W. If $t>0$, then (2.4.12) implies that $(R_t(r,\theta),\theta)$ is a critical point of W_t. Hence z is fixed by Φ_t and therefore a critical point of W_t. Since $t>0$, this implies that z is a critical point of W. Hence we have verified that every interior critical point of H_t is a critical point of W. Conversely, suppose that z is a critical points of W. Then z is a critical point of W_t for all t. Thus z is fixed by the isotopy Φ_t and hence a critical point of H_t.

Let z be an interior critical point of H_t. By the above discussion, z is a fixed point of Φ. We show that $H_t(z)=\Sigma_\Phi(z)$ for all t. This implies that $H_t(z)>0$. Indeed, all fixed points of ϕ are assumed to have strictly positive action and the same is true for Φ by item (3) in Lemma 2.4.8. Let λ be a primitive of ω and let Λ denote the pull-back to S. For every $\tau\in[0,1]$, we can compute the action $\Sigma_{\Phi_\tau}(z)$ via item (2) in Lemma 2.4.8

$$\Sigma_{\Phi_\tau}(z) = \Sigma_{\Phi_\tau,\Lambda}(z) = \int_{\{[0,\tau]\ni t\mapsto\Phi_t(z)\}}\Lambda + \int_0^\tau H_t(\Phi_t(z))dt = \int_0^\tau H_t(z)dt.$$

Here the last equality uses that z is fixed by the isotopy Φ_t. By Proposition 2.4.9, the action $\Sigma_{\Phi_\tau}(z)$ agrees with $W_\tau(z)$. We obtain

$$\tau\cdot W(z) = \int_0^\tau H_t(z)dt.$$

Differentiating with respect to τ yields $H_\tau(z)=W(z)=\Sigma_\Phi(z)>0$.

Next we show that H_t is positive on the boundary component $\{0\}\times\mathbb{R}$. Since Λ is given by the pull-back $p^*\lambda$ and p maps the entire boundary component $\{0\}\times\mathbb{R}$ to the origin 0 of $\mathbb{D}$, the restriction of Λ to $\{0\}\times\mathbb{R}$ vanishes. Thus item (2) in Lemma 2.4.8 yields

$$\Sigma_{\Phi_\tau,\Lambda}(z) = \int_0^\tau H_t(\Phi_t(z))dt = \int_0^\tau H_t(z)dt$$

for all $z\in\{0\}\times\mathbb{R}$. Here the second equality uses that H_t is translation invariant near the boundary and in particular constant on $\{0\}\times\mathbb{R}$. By Proposition 2.4.9, the action $\Sigma_{\Phi_\tau,\Lambda}$

agrees with W_τ on $\{0\} \times \mathbb{R}$. Thus

$$\tau \cdot W(z) = \int_0^\tau H_t(z) dt$$

for all $z \in \{0\} \times \mathbb{R}$. Differentiating with respect to τ yields $H_\tau(z) = W(z) = \Sigma_{\Phi,\Lambda}(z)$. By item (3) in Lemma 2.4.8, the action $\Sigma_{\Phi,\Lambda}(z)$ is equal to the action $\sigma_{\widetilde{\phi}}(0) > 0$. Hence H_τ is strictly positive on $\{0\} \times \mathbb{R}$ for all τ. This completes the proof that H_t is strictly positive on the complement of $\{1\} \times \mathbb{R}$.

The Hamiltonian H is invariant under T and constant on $\{0\} \times \mathbb{R}$. Thus it descends to a continuous function $H : [0,1] \times \mathbb{D} \to \mathbb{R}$ which is smooth away from the origin. The Hamiltonian flow of H, which is defined on the complement of the origin, is a rotation in some small neighbourhood of the origin. Thus the flow extends to a smooth flow on the entire disk $\mathbb{D}$. This implies that H is actually smooth everywhere. Clearly H satisfies properties (1)-(4) in Theorem 2.4.2.

There is one detail remaining: We actually want the Hamiltonian H to be 1-periodic in time. Here is how to fix this. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth cut-off function which vanishes in an open neighbourhood of $(-\infty, 0]$ and is equal to 1 in an open neighbourhood of $[1, \infty)$. For $\varepsilon > 0$ we define

$$\eta^\varepsilon(t) := \eta\left(\frac{t - 1 + \varepsilon}{\varepsilon}\right).$$

The function η^ε vanishes in a neighbourhood of $(-\infty, 1-\varepsilon]$ and is equal to 1 in a neighbourhood of $[1, \infty)$. Now define

$$G^\varepsilon : [0,1] \times \mathbb{D} \to \mathbb{R} \quad G^\varepsilon(t,z) := (1 - \eta^\varepsilon(t)) \cdot H(t,z) + \eta^\varepsilon(t) \cdot H(0,z).$$

This actually extends to a smooth 1-periodic Hamiltonian $G^\varepsilon : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$. In an open neighbourhood of 0 and $\partial\mathbb{D}$, the Hamiltonian G^ε_t agrees with H_t for all $t \in [0,1]$ because H is autonomous in this region. If $t \in [0, 1-\varepsilon]$, then G^ε_t agrees with H_t on the entire disk $\mathbb{D}$. Moreover, G^ε is strictly positive in the interior of $\mathbb{D}$. The time-1-map $\phi^1_{G^\varepsilon}$ agrees with ϕ in a neighbourhood of 0 and $\partial\mathbb{D}$, but it need not agree with ϕ on the entire disk. As ε approaches 0, the diffeomorphism $(\phi^1_{G^\varepsilon})^{-1} \circ \phi$, which is compactly supported in the complement of 0 and $\partial\mathbb{D}$, converges to the identity in the C^1-topology. Using Lemma 2.5.4, we can therefore find a Hamiltonian $K^\varepsilon : [0,1] \times \mathbb{D} \to \mathbb{R}$, compactly supported in the complement of 0 and $\partial\mathbb{D}$ and vanishing for t close to 0 or 1, such that $\phi^1_{K^\varepsilon} = (\phi^1_{G^\varepsilon})^{-1} \circ \widetilde{\phi}$ and such that $\|X_{K^\varepsilon}\|_{C^0}$ converges to 0 as ε approaches 0. Now define

$$H^\varepsilon_t := (G^\varepsilon \# K^\varepsilon)_t = G^\varepsilon_t + K^\varepsilon_t \circ (\phi^t_{G^\varepsilon})^{-1}$$

for $t \in [0,1]$. This extends to a smooth 1-periodic Hamiltonian. It agrees with H in a neighbourhood of 0 and $\partial\mathbb{D}$ and its time-1-map is $\phi^1_{H^\varepsilon}$ is equal to $\widetilde{\phi}$. For $\varepsilon > 0$ sufficiently small, H^ε is strictly positive in the interior $\operatorname{int}(\mathbb{D})$. Thus we have constructed a 1-periodic Hamiltonian satisfying properties (1)-(4). □

2.4.5 Proof of the positivity criterion for diffeomorphisms close to the identity

The goal of this section is to deduce Corollary 2.4.3 from Theorem 2.4.2. Key ingredient is the following lemma.

Lemma 2.4.11. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a diffeomorphism. Assume that:*

1. *ϕ is sufficiently C^1-close to the identity $\mathrm{id}_{\mathbb{D}}$.*
2. *ϕ is smoothly conjugated to a rotation near $\partial\mathbb{D}$.*
3. *There exists a fixed point $p \in \mathrm{int}(\mathbb{D})$ such that ϕ is smoothly conjugated to a rotation in a neighbourhood of p.*

Then there exists a diffeomorphism $\psi : \mathbb{D} \to \mathbb{D}$ such that:

1. *$\psi(0) = p$*
2. *$\psi^{-1} \circ \phi \circ \psi$ is radially monotone.*
3. *$\psi^{-1} \circ \phi \circ \psi$ is a rotation near 0 and $\partial\mathbb{D}$.*

Proof. We proceed in four steps.
Step 1: Let $\psi : \mathbb{D} \to \mathbb{D}$ be a Möbius transformation such that $\psi(0) = p$. By Proposition 2.24 in [3], the diffeomorphism $\psi^{-1} \circ \phi \circ \psi$ fixes the origin and is radially monotone. After replacing ϕ by $\psi^{-1} \circ \phi \circ \psi$, we can therefore assume that ϕ is radially monotone and smoothly conjugated to rotations near 0 and $\partial\mathbb{D}$. Note, however, that we can no longer guarantee that ϕ is C^1-close to the identity.
Step 2: We show that we may further reduce to the case that ϕ agrees with the linearization $d\phi(0)$ in an entire open neighbourhood of 0. We choose a ϕ-invariant neighbourhood U of 0 and an orientation preserving diffeomorphism $f : (\mathbb{D}, 0) \to (U, 0)$ such that $\rho := f^{-1} \circ \phi \circ f$ is a rotation around the center of $\mathbb{D}$. We may approximate f with respect to the C^1-topology by a diffeomorphism $\widetilde{f} : (\mathbb{D}, 0) \to (U, 0)$ which agrees with its linearization $d\widetilde{f}(0)$ near 0 and with f outside some arbitrarily small neighbourhood of 0. We set $\psi := f \circ \widetilde{f}^{-1}$. This defines a compactly supported diffeomorphism of U. We may smoothly extend it to a diffeomorphism of $\mathbb{D}$ by setting ψ to be equal to the identity outside U. The resulting diffeomorphism $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ is C^1-close to the identity and supported in a small neighbourhood of 0. Near 0 we have

$$\psi^{-1} \circ \phi \circ \psi = (f \circ \widetilde{f}^{-1})^{-1} \circ \phi \circ (f \circ \widetilde{f}^{-1}) = \widetilde{f} \circ (f^{-1} \circ \phi \circ f) \circ \widetilde{f}^{-1} = d\widetilde{f}(0) \circ \rho \circ d\widetilde{f}(0)^{-1}.$$

This shows that $\psi^{-1} \circ \phi \circ \psi$ is linear in some neighbourhood of 0. Since ψ is C^1-close to the identity and fixes the origin 0, radial monotonicity is preserved by conjugation by ψ. We can therefore replace ϕ by $\psi^{-1} \circ \phi \circ \psi$ and assume in addition that ϕ agrees with $d\phi(0)$ near

0.
Step 3: Since ϕ agrees with its linearization near 0 after performing Step 2, we may choose a linear orientation preserving diffeomorphism $g : (\mathbb{D}, 0) \to (U, 0)$ onto a ϕ-invariant neighbourhood of 0 such that $\rho := g^{-1} \circ \phi \circ g$ is a rotation of $\mathbb{D}$. We construct a (non-linear) diffeomorphism $\widetilde{g} : (\mathbb{D}, 0) \to (U, 0)$ satisfying properties (1)-(4) below. Let (R, Θ) and $(\widetilde{R}, \widetilde{\Theta})$ denote the components of g and $\widetilde{g}$ in polar coordinates, respectively.

1. $\widetilde{g}(r, \theta)$ agrees with $g(r, \theta)$ for $r > \frac{3}{4}$.
2. $\widetilde{\Theta}(r, \theta)$ agrees with $\Theta(r, \theta)$ for $r > \frac{1}{2}$.
3. There exists a constant $C > 0$ such that $\widetilde{R}(r, \theta) = C \cdot r$ for $r < \frac{1}{2}$.
4. $\widetilde{\Theta}(r, \theta) = \theta$ for $r < \frac{1}{4}$.

Let us first choose $C > 0$ such that the ball $B_C(0)$ is contained in the image $g(B_{\frac{3}{4}}(0))$. We may choose a smooth function $\widetilde{R}(r, \theta)$ which agrees with $C \cdot r$ for $r < \frac{1}{2}$ and with $R(r, \theta)$ for $r > \frac{3}{4}$ and which satisfies $\partial_1 \widetilde{R}(r, \theta) > 0$. Since g is linear, the function $\Theta(r, \theta)$ is actually independent of r and we denote it by $\Theta(\theta)$. The function $\Theta(\theta)$ is an orientation preserving diffeomorphism of the circle $\mathbb{R}/2\pi\mathbb{Z}$. Hence there exists a smooth isotopy from Θ to the identity. Using such an isotopy, we may define a function $\widetilde{\Theta}(r, \theta)$ such that $\widetilde{\Theta}(r, \theta) = \Theta(\theta)$ for $r > \frac{1}{2}$ and $\widetilde{\Theta}(r, \theta) = \theta$ for $r < \frac{1}{4}$. It is immediate from the construction that $\widetilde{g} = (\widetilde{R}, \widetilde{\Theta})$ is a diffeomorphism satisfying properties (1)-(4) above. As before, we define a diffeomorphism $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ which is compactly supported inside U and agrees with $g \circ \widetilde{g}^{-1}$ inside U. We have

$$\psi^{-1} \circ \phi \circ \psi = (g \circ \widetilde{g}^{-1})^{-1} \circ \phi \circ (g \circ \widetilde{g}^{-1}) = \widetilde{g} \circ (g^{-1} \circ \phi \circ g) \circ \widetilde{g}^{-1} = \widetilde{g} \circ \rho \circ \widetilde{g}^{-1}$$

inside U. Since $\widetilde{g}$ is simply multiplication by C near 0, this is an actual rotation near 0. We need to check that $\psi^{-1} \circ \phi \circ \psi$ is radially monotone. Since this diffeomorphism agrees with ϕ outside U, we only need to check radial monotonicity of the restriction to U, which is given by $\widetilde{g} \circ \rho \circ \widetilde{g}^{-1}$ by the above computation. For $r > \frac{1}{2}$, the function $\widetilde{\Theta}(r, \theta)$ agrees with $\Theta(r, \theta)$ and is therefore independent of r. This implies that the restriction of $\widetilde{g}$ to $\mathbb{D} \setminus B_{\frac{1}{2}}(0)$ preserves the foliations by radial rays. Since ρ is a rotation, the same is true for the restriction of $\widetilde{g} \circ \rho \circ \widetilde{g}^{-1}$ to $\widetilde{g}(\mathbb{D} \setminus B_{\frac{1}{2}}(0))$. This implies radial monotonicity of $\psi^{-1} \circ \phi \circ \psi$ on the set $\widetilde{g}(\mathbb{D} \setminus B_{\frac{1}{2}}(0))$. For $r < \frac{1}{2}$, the function $\widetilde{R}$ has the special form $\widetilde{R}(r, \theta) = C \cdot r$. Thus the restriction of $\widetilde{g}$ to $B_{\frac{1}{2}}(0)$ preserves the foliation by circles centered at the origin. Since ρ is a rotation, the same continues to hold for the restriction of $\widetilde{g} \circ \rho \circ \widetilde{g}^{-1}$ to $\widetilde{g}(B_{\frac{1}{2}}(0))$. Radial monotonicity of $\psi^{-1} \circ \phi \circ \psi$ on the set $\widetilde{g}(B_{\frac{1}{2}}(0))$ is a direct consequence. After replacing ϕ by $\psi^{-1} \circ \phi \circ \psi$, we can hence assume in addition that ϕ is an actual rotation near the origin.
Step 4: It remains to perform a similar construction to turn ϕ into a rotation near $\partial\mathbb{D}$ while preserving radial monotonicity. Let $\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\}$ denote the punctured disk. Let V

be a ϕ-invariant neighbourhood of $\partial\mathbb{D}$ and $f = (R, \Theta) : \dot{\mathbb{D}} \to V$ an orientation preserving diffeomorphism such that $\rho := f^{-1} \circ \phi \circ f$ is a rotation of $\dot{\mathbb{D}}$ around 0. Our goal is to define a diffeomorphism $\widetilde{f} = (\widetilde{R}, \widetilde{\Theta}) : \dot{\mathbb{D}} \to V$ agreeing with f in a small neighbourhood of 0 such that $\widetilde{f} \circ \rho \circ \widetilde{f}^{-1}$ is radially monotone on V and an actual rotation near $\partial\mathbb{D}$. Once we have such $\widetilde{f}$, we may set $\psi := f \circ \widetilde{f}^{-1}$ and extend to a diffeomorphism of $\mathbb{D}$ by setting ψ to be equal to the identity outside V. We have

$$\psi^{-1} \circ \phi \circ \psi = (f \circ \widetilde{f}^{-1})^{-1} \circ \phi \circ (f \circ \widetilde{f}^{-1}) = \widetilde{f} \circ (f^{-1} \circ \phi \circ f) \circ \widetilde{f}^{-1} = \widetilde{f} \circ \rho \circ \widetilde{f}^{-1}$$

which implies that $\psi^{-1} \circ \phi \circ \psi$ is radially monotone and a rotation near $\partial\mathbb{D}$. Here is how we construct $\widetilde{f}$. After shrinking V if necessary, we may assume that the image of $f(r, \cdot)$ is C^1-close to $\partial\mathbb{D}$ for all r. In particular, $\Theta(r, \cdot)$ is a diffeomorphism of $\mathbb{R}/2\pi\mathbb{Z}$ for fixed r. Moreover, we can assume that $\partial_1 R(r, \theta) > 0$. Let $\eta(r)$ be a smoothing of the function $\min(r, \frac{1}{5})$. Assume that both $\eta(r)$ and $\eta'(r)$ are monotonic and that $\eta(r)$ agrees with $\min(r, \frac{1}{5})$ outside $(\frac{1}{5} - \varepsilon, \frac{1}{5} + \varepsilon)$ for some small $\varepsilon > 0$. For $r < \frac{3}{5}$ we set $\widetilde{\Theta}(r, \theta) := \Theta(\eta(r), \theta)$. For $r > \frac{4}{5}$ we set $\widetilde{\Theta}(r, \theta) := \theta$. Choose a smooth isotopy from $\Theta(\frac{1}{5}, \cdot)$ to $\mathrm{id}_{\mathbb{R}/2\pi\mathbb{Z}}$ and use it to define $\widetilde{\Theta}$ in the interval $\frac{3}{5} < r < \frac{4}{5}$. We set $\widetilde{R}(r, \theta)$ to be equal to $R(r, \theta)$ for $r < \frac{2}{5}$. We extend $\widetilde{R}$ is such a way that $\partial_1 \widetilde{R}(r, \theta) > 0$. Moreover, we require that there exists $C > 0$ such that $\widetilde{R}(r, \theta) = 1 + C \cdot (r - 1)$ for $r > \frac{3}{5}$. This finishes the construction of $\widetilde{f}$. Clearly, $\widetilde{f} \circ \rho \circ \widetilde{f}^{-1}$ is an actual rotation near $\partial\mathbb{D}$. We need to check radial monotonicity. On the set $\widetilde{f}(\{r > \frac{3}{5}\})$, radial monotonicity follows from the special form $\widetilde{R}(r, \theta) = 1 + C \cdot (r - 1)$. Inside $\widetilde{f}(\{\frac{1}{5} + \varepsilon < r < \frac{3}{5}\})$, radial monotonicity follows from the fact that $\widetilde{\Theta}(r, \theta)$ is independent of r. For $r < \frac{1}{5} - \varepsilon$ the diffeomorphisms $\widetilde{f}$ and f agree, which implies radial monotonicity in $\widetilde{f}(\{r < \frac{1}{5} - \varepsilon\})$. It remains to verify radial monotonicity inside $\widetilde{f}(\{\frac{1}{5} - \varepsilon < r < \frac{1}{5} + \varepsilon\})$. A direct computation shows that for $r < \frac{2}{5}$

$$dr\left(\partial_1(f \circ \rho \circ f^{-1})(f(r, \theta))\right) = \frac{1}{\det df(r, \theta)} \cdot dR(\rho(r, \theta)) \begin{pmatrix} \partial_2\Theta(r, \theta) \\ -\partial_1\Theta(r, \theta) \end{pmatrix} \tag{2.4.13}$$

and

$$dr\left(\partial_1(\widetilde{f} \circ \rho \circ \widetilde{f}^{-1})(\widetilde{f}(r, \theta))\right) = \frac{1}{\det d\widetilde{f}(r, \theta)} \cdot dR(\rho(r, \theta)) \begin{pmatrix} \partial_2\Theta(\eta(r), \theta) \\ -\partial_1\Theta(\eta(r), \theta) \cdot \eta'(r) \end{pmatrix}. \tag{2.4.14}$$

Note that radial monotonicity of $f \circ \rho \circ f^{-1}$ is precisely saying that the left hand side of equation (2.4.13) is positive. Thus (2.4.13) implies that

$$dR(\rho(r, \theta)) \begin{pmatrix} \partial_2\Theta(r, \theta) \\ -\partial_1\Theta(r, \theta) \end{pmatrix} > 0.$$

By choosing $\varepsilon > 0$ sufficiently small, we can guarantee that

$$dR(\rho(r, \theta)) \begin{pmatrix} \partial_2\Theta(\eta(r), \theta) \\ -\partial_1\Theta(\eta(r), \theta) \end{pmatrix} > 0$$

whenever $r \in (\frac{1}{5} - \varepsilon, \frac{1}{5} + \varepsilon)$. Using the fact that $\partial_1 R > 0$ and $\partial_2 \Theta > 0$, we see that

$$dR(\rho(r,\theta)) \begin{pmatrix} \partial_2\Theta(\eta(r),\theta) \\ 0 \end{pmatrix} > 0.$$

Therefore, the linear functional $dR(\rho(r,\theta))$ is positive on any convex linear combination of the vectors $(\partial_2\Theta(\eta(r),\theta), -\partial_1\Theta(\eta(r),\theta))$ and $(\partial_2\Theta(\eta(r),\theta), 0)$. Using $\eta'(r) \in [0,1]$, we can hence deduce that

$$dR(\rho(r,\theta)) \begin{pmatrix} \partial_2\Theta(\eta(r),\theta) \\ -\partial_1\Theta(\eta(r),\theta)\cdot\eta'(r) \end{pmatrix} > 0.$$

Together with (2.4.14) this implies radial monotonicity of $\widetilde{f}\circ\rho\circ\widetilde{f}^{-1}$ in the annulus $\widetilde{f}(\{\frac{1}{5}-\varepsilon < r < \frac{1}{5}+\varepsilon\})$. □

Proof of Corollary 2.4.3. Let us first prove the corollary under the additional assumption that ϕ possesses a fixed point p such that ϕ is smoothly conjugated to a rotation in some neighbourhood of p. In this situation we may apply Lemma 2.4.11. Let ψ be the resulting diffeomorphism of $\mathbb{D}$. The 2-form $\psi^*\omega$ and the diffeomorphism $\psi^{-1}\circ\widetilde{\phi}\circ\psi \in \widetilde{\mathrm{Diff}}(\mathbb{D},\psi^*\omega)$ satisfy all assumptions in Theorem 2.4.2, except for possibly rotation invariance of $\psi^*\omega$ near 0 and $\partial\mathbb{D}$. Using an equivariant version of Moser's argument, we may construct a diffeomorphism of $\mathbb{D}$ which is supported near 0 and $\partial\mathbb{D}$, commutes with $\psi^{-1}\circ\phi\circ\psi$ and pulls back $\psi^*\omega$ to a 2-form which is rotation invariant near 0 and $\partial\mathbb{D}$. After replacing ψ by its composition with this diffeomorphism, we may assume w.l.o.g. that $\psi^*\omega$ is rotation invariant near the origin and the boundary and apply Theorem 2.4.2 to the tuple $(\psi^*\omega, \psi^{-1}\circ\widetilde{\phi}\circ\psi)$. Let H denote the resulting Hamiltonian. Then the Hamiltonian $H\circ\psi^{-1}$ satisfies all assertions of Corollary 2.4.3.

Let us now consider the general case. We claim that ϕ must possess an interior fixed point which is either degenerate or elliptic. This clearly is the case if ϕ is equal to the identity near the boundary $\partial\mathbb{D}$. So assume that ϕ is not equal to the identity near the boundary. Since ϕ is conjugated to a rotation near the boundary, this implies that there are no fixed points at all near the boundary. Let us assume that all fixed points of ϕ are non-degenerate. Then there are only finitely many fixed points and their signed count equals the Euler characteristic of $\mathbb{D}$, which is equal to 1. The Lefschetz sign of positive hyperbolic fixed points is -1 and the Lefschetz sign of negative hyperbolic and elliptic fixed points 1. Thus there must exist a fixed point which is negative hyperbolic or elliptic. Since $\widetilde{\phi}$ is assumed to be C^1-close to the identity, there are no negative hyperbolic fixed points, which implies that there must exist an elliptic one. This concludes the proof that there must be a degenerate or elliptic interior fixed point. We choose such a fixed point p. By Lemma 2.5.5 there exists a Hamiltonian G, supported in an arbitrarily small neighbourhood of p and with arbitrarily small C^2-norm $\|X_G\|_{C^2}$, such that $\phi' := \phi\circ\phi_G^1$ is smoothly conjugated to an rotation in a neighbourhood of p. We define the lift $\widetilde{\phi}'$ of ϕ' by $\widetilde{\phi}' := \widetilde{\phi}\circ\phi_G$ where $\phi_G \in \widetilde{\mathrm{Diff}}(\mathbb{D},\omega)$ is represented by the arc $(\phi_G^t)_{t\in[0,1]}$. After shrinking $\|X_G\|_{C^2}$ if necessary, we can assume that the action $\sigma_{\widetilde{\phi}'}$ is positive on all fixed points of ϕ'. Thus Corollary 2.4.3 holds for $\widetilde{\phi}'$ by the above discussion. Let H' be a Hamiltonian generating $\widetilde{\phi}'$ and satisfying all assertions in Corollary 2.4.3. In fact, it follows from assertion (2) in Theorem 2.4.2 that H' can be chosen to satisfy $dH'_t(p) = 0$ and $H'_t(p) = \sigma_{\widetilde{\phi}'}(p)$ for all t. The action $\sigma_{\widetilde{\phi}'}(p)$ is close to $\sigma_{\widetilde{\phi}}(p) > 0$. Thus we can bound $H'_t(p)$ from below by a positive constant which can be chosen uniform among all sufficiently small G. After reparametrizing the Hamiltonian flow of G, we can assume w.l.o.g. that G vanishes for t near 0 and 1. We define H by

$$H_t := (H'\#\overline{G})_t = H'_t - G_t \circ \phi^t_G \circ (\phi^t_{H'})^{-1} \tag{2.4.15}$$

for $t \in [0,1]$. This smoothly extends to a 1-periodic Hamiltonian and generates $\widetilde{\phi}$. Since H agrees with H' outside a small neighbourhood of p, it satisfies assertions (2) and (3) of Corollary 2.4.3. If we choose G with sufficiently small norm $\|X_G\|_{C^2}$, we can also guarantee that H is strictly positive in the interior of $\mathbb{D}$. This follows from (2.4.15) and the fact that we have a strictly positive lower bound on $H'_t(p)$ independent of G. Thus H has all desired properties. □

2.5 From Reeb flows to disk-like surfaces of section and approximation results

Let α_0 denote the restriction of the standard Liouville 1-form λ_0 on $\mathbb{R}^4$ defined in (2.1.1) to the unit sphere S^3. We will refer to α_0 as the standard contact form. The goal of this section is to prove the following result.

Proposition 2.5.1. *Every tight contact form α on S^3 which is sufficiently C^3-close to the standard contact form α_0 can be C^2-approximated by contact forms α' with the following properties: There exists a unique Reeb orbit γ of minimal action. The local first return map of a small disk transversely intersecting γ is smoothly conjugated to an irrational rotation. There exists a smooth embedding $f : \mathbb{D} \to S^3$ parametrizing a ∂-strong disk-like global surface of section with boundary orbit γ such that the 2-form $\omega := f^*d\alpha'$, the first return map $\phi \in \mathrm{Diff}(\mathbb{D}, \omega)$ and the lift $\widetilde{\phi}_1 \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ of ϕ with respect to a trivialization of degree 1 satisfy assertions (1)-(3) below.*

1. *$\widetilde{\phi}_1$ is C^1-close to the identity $\mathrm{id}_{\mathbb{D}}$.*
2. *ϕ is smoothly conjugated to an irrational rotation in some neighbourhood of the boundary $\partial\mathbb{D}$.*
3. *All fixed points p of ϕ have positive action $\sigma_{\widetilde{\phi}_1}(p)$.*

2.5.1 Approximation results

The next result says that we may perturb the contact form near an elliptic orbit such that the local first return map of the perturbed Reeb flow is smoothly conjugated to a rotation. We can choose the perturbation such that both the contact form and Reeb vector field are close to the original contact form and Reeb vector field with respect to the C^2-topology.

Proposition 2.5.2. *Let α be a contact form on a 3-manifold Y. Let γ be an elliptic, simple, closed Reeb orbit of α. For every open neighbourhood V of γ and every $\varepsilon > 0$ there exists a contact form α' on Y such that:*

1. *α' agrees with α outside V.*
2. $\|\alpha' - \alpha\|_{C^2} < \varepsilon$
3. $\|R_{\alpha'} - R_\alpha\|_{C^2} < \varepsilon$
4. *Up to reparametrization, γ is a simple closed Reeb orbit of α'. The local first return map of a small disk transversely intersecting γ is smoothly conjugated to an irrational rotation.*

Our proof of Proposition 2.5.2 requires some preparation. Consider $\mathbb{R}^2$ equipped with the standard symplectic form ω_0. Any compactly supported symplectomorphism ϕ which is sufficiently C^1-close to the identity can be represented by a unique compactly supported generating function W (see chapter 9 in [84]). If (X, Y) denote the components of ϕ, the defining equations for the generating function are:

$$\begin{cases} X - x = \ \ \partial_2 W(X, y) \\ Y - y = -\partial_1 W(X, y) \end{cases}$$

Conversely, any compactly supported function W which is sufficiently C^2-close to the identity uniquely determines a compactly supported symplectomorphism ϕ.

Lemma 2.5.3. *Let $k \geq 0$. We equip the space of compactly supported diffeomorphisms with the C^k-topology and the space of compactly supported generating functions with the topology induced by the norm $\|\nabla \cdot\|_{C^k}$. The correspondence between symplectomorphisms and generating functions is continuous in both directions with respect to these topologies.*

Proof. If ϕ is a symplectomorphism with corresponding generating function W, then the graph $\Gamma(\phi)$ of ϕ inside $\mathbb{R}^2 \times \mathbb{R}^2$ can be parametrized via

$$\iota : \mathbb{R}^2 \ni (X, y) \mapsto \begin{pmatrix} X - \partial_2 W(X, y) \\ y \\ X \\ y - \partial_1 W(X, y) \end{pmatrix} \in \Gamma(\phi) \subset \mathbb{R}^2 \times \mathbb{R}^2.$$

For $j \in \{1,2\}$ let $\pi_j : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the projection onto the j-th factor. Then ϕ can be written as

$$\phi = (\pi_2 \circ \iota) \circ (\pi_1 \circ \iota)^{-1}.$$

Clearly, the assignment $W \mapsto \pi_j \circ \iota$ is continuous with respect to the topologies on the spaces of functions and diffeomorphisms specified above. Since composition and taking inverses of diffeomorphisms are continuous operations with respect to the C^k-topology, this shows that the symplectomorphism ϕ depends continuously on W. Conversely, given ϕ we define the diffeomorphism $\psi(x,y) := (X(x,y),y)$. The assignment $\phi \mapsto \psi$ is continuous with respect to the C^k-topology. Define the function

$$V(x,y) := (y - Y(x,y), X(x,y) - x).$$

Then ∇W is given by the composition $\nabla W = V \circ \psi^{-1}$. Both V and ψ^{-1} depend continuously on ϕ with respect to the C^k-topology. Hence the same is true for ∇W. □

Lemma 2.5.4. *There exist a C^1-open neighbourhood $\mathcal{U} \subset \mathrm{Symp}_c(\mathbb{R}^2, \omega_0)$ of the identity and a map*

$$\mathcal{U} \to C_c^\infty([0,1] \times \mathbb{R}^2) \qquad \phi \mapsto H_\phi$$

such that:

1. *H_ϕ generates ϕ.*
2. *$H_{\mathrm{id}} = 0$*
3. *For every integer $k \geq 1$ the following is true: Equip $\mathcal{U}$ with the C^k-topology and $C_c^\infty([0,1] \times \mathbb{R}^2)$ with the topology induced by the norm $H \mapsto \|X_H\|_{C^{k-1}}$ where we view X_H as an element of $C_c^\infty([0,1] \times \mathbb{R}^2, \mathbb{R}^2)$. The map $\phi \mapsto H_\phi$ is continuous with respect to these topologies.*

Proof. Let ϕ be a compactly supported symplectomorphism sufficiently C^1-close to the identity and let W be the associated generating function. For $t \in [0,1]$, let ϕ_t be the symplectomorphism associated to the generating function $t \cdot W$. Let $H_\phi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be the unique compactly supported Hamiltonian generating the flow $(\phi_t)_{t \in [0,1]}$. This yields a map $\phi \mapsto H_\phi$ defined on a C^1-open neighbourhood of the identity. Clearly, assertions (1) and (2) hold. It remains to check continuity. Let $k \geq 1$. Consider the parametrization

$$\iota_t : \mathbb{R}^2 \ni (X,y) \mapsto \begin{pmatrix} X - t \cdot \partial_2 W(X,y) \\ y \\ X \\ y - t \cdot \partial_1 W(X,y) \end{pmatrix} \in \Gamma(\phi_t) \subset \mathbb{R}^2 \times \mathbb{R}^2$$

of $\Gamma(\phi_t)$. We have

$$\phi_t = (\pi_2 \circ \iota_t) \circ (\pi_1 \circ \iota_t)^{-1}$$

where $\pi_j : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ denotes the projection onto the j-th factor. Clearly, the map

$$(C_c^\infty(\mathbb{R}^2), \|\nabla \cdot\|_{C^k}) \ \ni \ W \mapsto \pi_j \circ \iota_t \ \in \ (C^\infty([0,1] \times \mathbb{R}^2, \mathbb{R}^2), \|\cdot\|_{C^k})$$

is continuous. Since composition and inversion of diffeomorphisms are continuous operations with respect to the C^k-topology, this implies that

$$(C_c^\infty(\mathbb{R}^2), \|\nabla \cdot\|_{C^k}) \ \ni \ W \mapsto \phi_t \ \in \ (C^\infty([0,1] \times \mathbb{R}^2, \mathbb{R}^2), \|\cdot\|_{C^k}) \tag{2.5.1}$$

is continuous. We have $X_{H_\phi} = (\partial_t \phi_t) \circ \phi_t^{-1}$. Combining Lemma 2.5.3 with continuity of (2.5.1), we obtain that $\phi \mapsto X_{H_\phi}$ is continuous with respect to the C^k-topology on symplectomorphisms and the C^{k-1}-topology on vector fields. □

Lemma 2.5.5. *Let ω be an area form and ϕ a symplectomorphism defined near the origin of $\mathbb{R}^2$. Assume that 0 is a fixed point of ϕ and that it is either elliptic or degenerate. Then there exists a Hamiltonian $H \in C_c^\infty([0,1] \times \mathbb{R}^2, \mathbb{R})$ supported inside an arbitrarily small open neighbourhood of 0 and with arbitrarily small norm $\|X_H\|_{C^2}$ such that 0 is a fixed point of $\phi \circ \phi_H^1$ and such that $\phi \circ \phi_H^1$ is smoothly conjugated to an irrational rotation in some neighbourhood of 0.*

Proof. After a change of coordinates, we can assume w.l.o.g. that $\omega = \omega_0$. Since 0 is elliptic or degenerate as a fixed point of ϕ, we may choose a C^∞-small Hamiltonian H supported in a small neighbourhood of 0 such that 0 is an elliptic fixed point of $\phi \circ \phi_H^1$ with rotation number an irrational multiple of 2π and such that $\phi \circ \phi_H^1$ is real analytic in some open neighbourhood of 0. It follows from [85, Chapter 23, p. 172-173] that there exists a symplectomorphism ψ defined in an open neighbourhood of 0 and fixing 0 such that

$$\psi^{-1} \circ \phi \circ \phi_H^1 \circ \psi(x,y) = \begin{pmatrix} \cos(\theta(x,y)) & -\sin(\theta(x,y)) \\ \sin(\theta(x,y)) & \cos(\theta(x,y)) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + O_4(x,y)$$

where $\theta(x,y) = \theta_0 + \theta_1(x^2+y^2)$ for real constants θ_0 and θ_1 and $O_4(x,y)$ is a real analytic map vanishing up to order 3 at the origin. Since $d(\phi \circ \phi_H^1)(0)$ is conjugated to an irrational rotation, the constant θ_0 is an irrational multiple of 2π. There exists a symplectomorphism ξ arbitrarily C^3-close to the identity and supported in an arbitrarily small neighbourhood of 0 such that

$$\psi^{-1} \circ \phi \circ \phi_H^1 \circ \psi \circ \xi(x,y) = \begin{pmatrix} \cos(\theta(x,y)) & -\sin(\theta(x,y)) \\ \sin(\theta(x,y)) & \cos(\theta(x,y)) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

near 0. Lemma 2.5.4 yields a Hamiltonian G supported in a small neighbourhood of 0 such that $\phi_G^1 = \xi$ and such that $\|X_G\|_{C^2}$ is controlled by $\|\xi - \mathrm{id}\|_{C^3}$. We may choose an autonomous Hamiltonian K such that $\|K\|_{C^3}$ is arbitrarily small, K is supported in an arbitrarily small neighbourhood of 0 and $K(x,y) = \frac{\theta_1}{4}(x^2+y^2)^2$ near 0. The time-1-map ϕ_K^1 is given by

$$\phi_K^1(x,y) = \begin{pmatrix} \cos(-\theta_1(x^2+y^2)) & -\sin(-\theta_1(x^2+y^2)) \\ \sin(-\theta_1(x^2+y^2)) & \cos(-\theta_1(x^2+y^2)) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

in a neighbourhood of 0. Thus we have

$$\psi^{-1} \circ \phi \circ \phi_H^1 \circ \psi \circ \phi_G^1 \circ \phi_K^1(x,y) = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence

$$\phi \circ \phi_H^1 \circ \psi \circ \phi_G^1 \circ \phi_K^1 \circ \psi^{-1} = \phi \circ \phi_H^1 \circ \phi_{G\circ\psi^{-1}}^1 \circ \phi_{K\circ\psi^{-1}}^1$$

is smoothly conjugated to an irrational rotation. Let $\eta : [0,1] \to [0,1]$ be a smooth function such that $\eta(t) = 0$ for t near 0 and $\eta(t) = 1$ for t near 1 and $\eta' \geq 0$. Define the Hamiltonian F by

$$F : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \qquad F(t,z) := \begin{cases} 3 \cdot \eta'(3t) \cdot K(\eta(3t), \psi^{-1}(z)) & \text{for } 0 \leq t \leq \frac{1}{3} \\ 3 \cdot \eta'(3t-1) \cdot G(\eta(3t-1), \psi^{-1}(z)) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3} \\ 3 \cdot \eta'(3t-2) \cdot H(\eta(3t-2), z) & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

F is compactly supported, vanishes for t near 0 and 1 and generates the symplectomorphism $\phi_H^1 \circ \phi_{G\circ\psi^{-1}}^1 \circ \phi_{K\circ\psi^{-1}}^1$. Thus $\phi \circ \phi_F^1$ is smoothly conjugated to an irrational rotation. By shrinking the supports of H, ξ and K and the norms $\|H\|_{C^\infty}$, $\|\xi - \mathrm{id}\|_{C^3}$ and $\|K\|_{C^3}$, we can make the support of F and the norm $\|X_F\|_{C^2}$ arbitrarily small. □

Lemma 2.5.6. *Let α be a contact form on a 3-manifold Y and let γ be a simple closed Reeb orbit. Let p be a point on γ and let D be a small disk intersecting γ transversely in p. We denote $\omega := d\alpha|_D$. Let $\phi : (U,\omega) \to (D,\omega)$ be the local first-return-map of the Reeb flow, defined in some open neighbourhood $U \subset D$. There exist a C^1-open neighbourhood $\mathfrak{U}$ of zero inside $C_c^\infty([0,1] \times U, \mathbb{R})$ and a map*

$$\mathfrak{U} \to \Omega^1(Y) \qquad H \mapsto \alpha_H$$

such that:

1. *α_H is a contact form and agrees with α outside a small neighbourhood of γ.*

2. *The local first return map of the Reeb flow of α_H is given by $\phi \circ \phi_H^1$.*

3. *$\alpha_0 = \alpha$*

4. *For every interger $k \geq 0$ the following is true: Equip $\mathfrak{U}$ with the topology induced by the norm $\|X_H\|_{C^k}$. Equip $\Omega^1(Y)$ and $\mathrm{Vect}(Y)$ with the C^k-topologies. Then the maps $H \mapsto \alpha_H$ and $H \mapsto R_{\alpha_H}$ are continuous with respect to these topologies.*

Proof. Denote $\lambda := \alpha|_D$. For $T > 0$ sufficiently small, there exists an embedding

$$F : [0,T] \times D \to Y$$

such that the restriction of F to $\{0\} \times D$ is the inclusion of D and such that $F^*\alpha = dt + \lambda$. Let $\eta : [0,T] \to [0,1]$ be a smooth function which is equal to 0 near 0, equal to 1 near T and satisfies $\eta' \geq 0$. Given a Hamiltonian $H \in \mathfrak{U}$, we define H' by

$$H'(z,t) := \eta'(t) \cdot H(z, \eta(t)).$$

H' vanishes for t near 0 and T and its time-T-map agrees with the time-1-map of H. The map $H \mapsto H'$ is continuous with respect to the topology induced by the norm $\|X_H\|_{C^k}$. We define α_H by

$$\alpha_H := (1 + H')dt + \lambda \tag{2.5.2}$$

in the coordinate chart F. We extend α_H to all of Y by setting it equal to α outside $\mathrm{im}(F)$. If H is sufficiently C^1-small, then α_H is a contact form. The Reeb vector field inside the coordinate chart F is given by

$$R_{\alpha_H} = \frac{1}{1 + H' + \lambda(X_{H'})} \cdot (\partial_t + X_{H'}). \tag{2.5.3}$$

This is positively proportional to $\partial_t + X_{H'}$. Thus the local first return map of the disk D induced by the Reeb flow of α_H is given by $\phi \circ \phi_H^1$. It is immediate from formulas (2.5.2) and (2.5.3) that α_H and R_{α_H} depend continuously on H with respect to the topologies specified in assertion (4). □

We are finally ready to prove Proposition 2.5.2.

Proof of Proposition 2.5.2. Let D be a small disk transversely intersecting γ in a point p. Denote $\omega := d\alpha|_D$. For a sufficiently small open neighbourhood $p \in U \subset D$ we have a well-defined local first return map $\phi : (U, \omega) \to (D, \omega)$. The map ϕ has an elliptic fixed point at p. By Lemma 2.5.5, there exists a Hamiltonian $H : [0,1] \times U \to \mathbb{R}$, supported in an arbitrarily small neighbourhood of p and with arbitrarily small norm $\|X_H\|_{C^2}$, such that $\phi \circ \phi_H^1$ is smoothly conjugated to an irrational rotation in some neighbourhood of p. Lemma 2.5.6 yields a contact form α_H on Y which agrees with α outside a small neighbourhood of γ such that the local first return map of the Reeb flow of α_H is given by $\phi \circ \phi_H^1$. By assertion (4) in Lemma 2.5.6, we can make $\|\alpha_H - \alpha\|_{C^2}$ and $\|R_{\alpha_H} - R_\alpha\|_{C^2}$ arbitrarily small by shrinking $\|X_H\|_{C^2}$. □

2.5.2 Proof of Proposition 2.5.1

This section is devoted to the proof of Proposition 2.5.1.

Lemma 2.5.7. *There exists $\varepsilon > 0$ with the following property: Let α be a tight contact form on S^3 satisfying the following conditions:*

1. $\|R_\alpha - R_{\alpha_0}\|_{C^2} < \varepsilon$
2. $R_\alpha = c \cdot R_{\alpha_0}$ *on the great circle* $\Gamma := \{(z_1, 0) \mid |z_1| = 1\}$ *for some constant* $c > 0$.

3. *Γ is the unique shortest Reeb orbit of α.*

4. *The local return map of a small disk transversely intersecting Γ is smoothly conjugated to an irrational rotation.*

Then there exists a smooth embedding $f : \mathbb{D} \to S^3$ parametrizing a ∂-strong disk-like surface of section with boundary orbit Γ such that the 2-form $\omega := f^*d\alpha$ and the lift of the first return map $\widetilde{\phi}_1 \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ with respect to a trivialization of degree 1 satisfy assertions (1)-(3) in Proposition 2.5.1.

Proof. Our proof is based on Proposition 3.6 in [3]. We define

$$f : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to S^3 \quad f(t, e^{i\theta}) := \left(\sin\left(\frac{\pi}{2}r\right) e^{i(\theta+2\pi t)}, \cos\left(\frac{\pi}{2}r\right) e^{2\pi i t}\right).$$

Up to replacing $\mathbb{R}/\pi\mathbb{Z}$ by $\mathbb{R}/\mathbb{Z}$, this agrees with the map f defined in [3]. By assertion (iii) in [3, Proposition 3.6], the pull-back of the Reeb vector field R_α via $f|_{\mathbb{R}/\mathbb{Z}\times\mathrm{int}(\mathbb{D})}$ extends to a smooth vector field R on the closed solid torus $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$. Moreover, the C^1-norm $\|R - \partial_t\|_{C^1}$ is controlled by the C^2-norm $\|R_\alpha - R_{\alpha_0}\|_{C^2}$. This shows that if R_α is sufficiently C^2-close to R_{α_0}, then R is positively transverse to the fibres $t \times \mathbb{D}$ of the solid torus. In particular, the restriction of f to $0 \times \mathbb{D}$ parametrizes a ∂-strong disk-like global surface of section of the Reeb flow of α. Let z be a point in the boundary $\partial\mathbb{D}$. The map

$$\mathbb{R}/\mathbb{Z} \to \Gamma \quad t \mapsto f(t, z)$$

has degree 1. Thus the flow of R on $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$ induces the lift $\widetilde{\phi}_1$ of the first return map of $f|_{0\times\mathbb{D}}$ with respect to a trivialization of degree 1. We see that the C^1-distance between $\widetilde{\phi}_1$ and the identity $\mathrm{id}_\mathbb{D}$ is controlled by the C^2-distance between R_α and R_{α_0}, which yields assertion (1) of Proposition 2.5.1. The hypothesis that the local first return map of the orbit Γ is smoothly conjugated to an irrational rotation implies that the global first return map ϕ is smoothly conjugated to an irrational rotation near the boundary. Let p be a fixed point of ϕ corresponding to a closed Reeb orbit γ of α. By assumption, Γ is the unique shortest Reeb orbit of α. Thus $\int_\gamma \alpha > \int_\Gamma \alpha$. Let $\widetilde{\phi}_0$ denote the lift of ϕ with respect to a trivialization of degree 0. It follows from Lemma 2.2.5 that $\int_\gamma \alpha = \sigma_{\widetilde{\phi}_0}(p)$. The actions of $\widetilde{\phi}_0$ and $\widetilde{\phi}_1$ are related via $\sigma_{\widetilde{\phi}_0}(p) = \sigma_{\widetilde{\phi}_1}(p) + \int_\Gamma \alpha$. Thus we can conclude that $\sigma_{\widetilde{\phi}_1}(p)$ is positive. □

Lemma 2.5.8. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all tight contact forms α on S^3 which satisfy $\|R_\alpha - R_{\alpha_0}\|_{C^2} < \delta$ and for all simple closed Reeb orbits γ of action less than $\frac{3}{2} \cdot \pi$ there exists a diffeomorphism ψ of S^3 such that:*

1. *$R_{\psi^*\alpha} = c \cdot R_{\alpha_0}$ on the great circle $\Gamma := \{(z_1, 0) \mid |z_1| = 1\}$ for some constant $c > 0$.*

2. *$\psi(\Gamma) = \mathrm{im}(\gamma)$*

3. *$\|R_{\psi^*\alpha} - R_{\alpha_0}\|_{C^2} < \varepsilon$*

Proof. This is immediate from Proposition 3.10 in [3]. □

Proof of Proposition 2.5.1. Choose $\varepsilon > 0$ as in Lemma 2.5.7. Choose corresponding $\delta > 0$ as in Lemma 2.5.8. Let $\mathcal{U}$ be a small C^3-open neighbourhood of α_0 such that all $\alpha \in \mathcal{U}$ satisfy $\|R_\alpha - R_{\alpha_0}\|_{C^2} < \delta$. We also demand that (S^3, α) is strictly contactomorphic to the boundary of a strictly positively curved domain. We prove that the Proposition holds for all $\alpha \in \mathcal{U}$. It suffices to consider C^∞-generic α. In particular we can assume that all periodic orbits are non-degenerate and that there exists a unique orbit γ of minimal action. Since (S^3, α) is strictly contactomorphic to the boundary of a strictly positively curved domain, it follows from [32] (see in particular Theorem 3 and Proposition 9 in chapter V) that γ must have Conley-Zehnder index 3 with respect to a global trivialization of the contact structure. This implies that γ must be elliptic or negative hyperbolic. By shrinking $\mathcal{U}$ we can guarantee the linearized return map of γ to be arbitrarily close to the identity. Hence we can guarantee that γ is elliptic. We apply Proposition 2.5.2. This yields a contact form α' approximating α in the C^2-topology and agreeing with α outside a small neighbourhood of γ such that the local return map of γ generated by the Reeb flow of α' is smoothly conjugated to an irrational rotation. We can also demand $\|R_{\alpha'} - R_\alpha\|_{C^2}$ to be arbitrarily small. In particular we can guarantee that $\|R_{\alpha'} - R_{\alpha_0}\|_{C^2} < \delta$. We apply Lemma 2.5.8 to the contact form α' and the Reeb orbit γ of minimal action. Let ψ be a diffeomorphism of S^3 satisfying properties (1)-(3) in Lemma 2.5.8. Then the contact form $\psi^*\alpha'$ satisfies all assumptions in Lemma 2.5.7. Hence there exists a smooth embedding $f : \mathbb{D} \to S^3$ parametrizing a ∂-strong disk-like surface of section of the Reeb flow of $\psi^*\alpha'$ with boundary orbit Γ such that the 2-form $\omega := f^*d(\psi^*\alpha')$ and the first return map $\widetilde{\phi}_1 \in \widetilde{\mathrm{Diff}}(\mathbb{D}, \omega)$ satisfy assertions (1)-(3) in Proposition 2.5.1. Since $\psi^*\alpha'$ and α' are strictly contactomorphic, the same is true for α'. □

2.6 Proofs of the main results

Proof of Theorem 2.1.9. Let $X \subset \mathbb{R}^4$ be a convex domain such that ∂X is C^3-close to the unit sphere S^3. Let $g : S^3 \to \mathbb{R}_{>0}$ be the unique function such that

$$\partial X = \{\sqrt{g(x)} \cdot x \mid x \in S^3\}.$$

The function g is C^3-close to the constant function 1. The pull-back of the contact form $\lambda_0|_{\partial X}$ via the radial map

$$S^3 \to \partial X \qquad x \mapsto \sqrt{g(x)} \cdot x$$

is given by $\alpha := g \cdot \alpha_0$ and is C^3-close to α_0. Let α' be a contact form which is C^2-close to α and satisfies all assertions of Proposition 2.5.1. We claim that there exists a star-shaped domain X' whose boundary $\partial X'$ is C^1-close to ∂X and strictly contactomorphic to (S^3, α'). Indeed, arguing as in the proof of Proposition 3.11 in [3], we conclude that there exists a C^1-open neighbourhood $\mathcal{U} \subset \Omega^1(S^3)$ of α and a map

$$\mathcal{U} \to C^\infty(S^3) \quad \beta \mapsto g_\beta$$

which is continuous with respect to the C^{k+1} topology on $\mathcal{U}$ and the C^k-topology on $C^\infty(S^3)$, maps α to the constant function 1 and has the property that every $\beta \in \mathcal{U}$ is a contact form strictly contactomorphic to $g_\beta \cdot \alpha$. Since α' is C^2-close to α, the function $g_{\alpha'}$ is C^1-close to the constant function 1. We define X' to be the star-shaped domain with boundary

$$\partial X' = \{\sqrt{g_{\alpha'}(x) \cdot g(x)} \cdot x \mid x \in S^3\}.$$

$\partial X'$ is C^1-close to ∂X. The pull-back to S^3 of the contact form $\lambda_0|_{\partial X'}$ via the radial projection is given by $g_{\alpha'} \cdot g \cdot \alpha_0 = g_{\alpha'} \cdot \alpha$. This is strictly contactomorphic to α'. We claim that $\mathrm{c_G}(X') = \mathrm{c_Z}(X')$. Let $f : \mathbb{D} \to \partial X'$ be a surface of section satisfying the assertions of Proposition 2.5.1. This means that we may apply Corollary 2.4.3 to $\widetilde{\phi}_1$, the lift of the first return map with respect to a trivialization of degree 1. Let $H : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \to \mathbb{R}$ denote a Hamiltonian satisfying all assertions of Corollary 2.4.3. This Hamiltonian satisfies all hypotheses for the second part of Theorem 2.3.1. We conclude that $B(a) \overset{s}{\hookrightarrow} X' \overset{s}{\hookrightarrow} Z(a)$ where $a > 0$ is the symplectic area of the surface of section. In particular, this implies that $\mathrm{c_G}(X') = \mathrm{c_Z}(X')$. We can make the C^1-distance between ∂X and $\partial X'$ arbitrarily small by letting the C^2-distance between α and α' go to zero. This shows that X may be approximated in the C^1-topology by star-shaped domains X' whose Gromov width and cylindrical embedding capacity agree. It is an easy consequence of the monotonicity and conformality of symplectic capacities that any symplectic capacity is continuous on the space of all star-shaped domains with respect to the C^0-topology. Therefore $\mathrm{c_G}(X) = \mathrm{c_Z}(X)$. □

Proof of Theorem 2.1.3. We apply Proposition 2.3.2. This yields a star-shaped domain X' such that $X \overset{s}{\hookrightarrow} X'$ and a ∂-strong disk-like global surface of section $\Sigma' \subset \partial X'$ such that Σ' has the same symplectic area as Σ and such that (X', Σ') satisfies all hypotheses in the first part of Theorem 2.3.1. Let $a > 0$ denote the symplectic area of Σ. By Theorem 2.3.1, there exist symplectic embeddings $X \overset{s}{\hookrightarrow} X' \overset{s}{\hookrightarrow} Z(a)$. In particular, we have $\mathrm{c_Z}(X) \leq a$. □

Proof of Theorem 2.1.7. It was proved by Hofer-Wysocki-Zehnder in [54] that every star-shaped domain X possesses a Hopf orbit, i.e. that $\mathrm{A_{Hopf}}(X) < \infty$. Although not explicitly stated in this form, the proof of the existence result of Hopf orbits in [54] shows more, namely that if X symplectically embeds into the cylinder $Z(a)$, then $\mathrm{A_{Hopf}}(X) \leq a$. The reason is the following. Since X embeds into $Z(a)$, it also embeds into the product $M := S^2(a+\varepsilon) \times T^2(b)$. Here $S^2(a+\varepsilon)$ is the 2-sphere equipped with an area form of total area $a+\varepsilon$ for an arbitrarily small $\varepsilon > 0$ and $T^2(b)$ is the 2-torus of total area b for some sufficiently large b. For an arbitrary compatible almost complex structure J, the symplectic manifold M is foliated by J-holomorphic spheres in the homology class $[S^2 \times *]$. Hofer-Wysocki-Zehnder produce a Hopf orbit by neck-stretching J-holomorphic spheres along the boundary of X. Carrying out this procedure with J-holomorphic spheres of symplectic area $a + \varepsilon$ yields a Hopf orbit with action at most $a + \varepsilon$. Therefore $\mathrm{A_{Hopf}}(X) \leq a + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that $\mathrm{A_{Hopf}}(X) \leq a$.

Let us now assume that X is dynamically convex. It is proved by Hryniewicz-Hutchings-Ramos [58] that the infimum in the definition of $\mathrm{A_{Hopf}}(X)$ is attained in this case. This means that there exists a Hopf orbit γ with action $\mathcal{A}(\gamma) = \mathrm{A}_{\mathrm{Hopf}}(X)$. As mentioned in the introduction, it follows from work of Hryniewicy-Salomão [60] and Hryniewicz [57] that γ bounds a disk-like global surface of section. In fact, Florio-Hryniewicz proved in [37, Proposition 2.8] that γ bounds a ∂-strong disk-like surface of section. Thus it is a direct consequence of Theorem 2.1.3 that X symplectically embeds into $Z(\mathrm{A}_{\mathrm{Hopf}}(X))$. □

Chapter 3

PFH spectral invariants and C^∞ closing lemmas

3.1 Introduction

Throughout this paper, let Σ be a closed connected surface of genus g, and let ω be a symplectic (area) form on Σ. We are interested in (orientation-preserving) area-preserving diffeomorphisms $\phi : (\Sigma, \omega) \to (\Sigma, \omega)$. We are also interested in Hamiltonian isotopy classes in the set of all area-preserving diffeomorphisms of (Σ, ω); we denote the Hamiltonian isotopy class of ϕ by $[\phi]$.

3.1.1 Closing lemmas

Our convention is that a *periodic orbit* of ϕ with period k is a set of k distinct points in Σ that are cyclically permuted by ϕ. A *periodic point* is a point in a periodic orbit.

Definition 3.1.1. Let Φ be a Hamiltonian isotopy class of area-preserving diffeomorphisms of (Σ, ω). We say that Φ satisfies the C^∞ *generic density property* if for a C^∞-generic area-preserving diffeomorphism $\phi \in \Phi$, the set of periodic points of ϕ is dense in Σ.

It was proved by Asaoka-Irie [7] that the Hamiltonian isotopy class of the identity satisfies the C^∞ generic density property. It is natural to ask which other Hamiltonian isotopy classes satisfy this property.

Remark 3.1.2. The C^∞ generic density property fails for some Hamiltonian isotopy classes of area-preserving diffeomorphisms of T^2. In fact, it follows from a result of Herman [51, Annexe, Thm. 2.2] that if ϕ is a Diophantine rotation of T^2, then there is a neighborhood of ϕ in the C^∞ topology in the space of maps Hamiltonian isotopic to ϕ such that any map ϕ' in this neighborhood is smoothly conjugate to ϕ, and hence has no periodic orbits. (Thanks to V. Humilière for this reference.)

One approach to proving the C^∞ generic density property is to create periodic orbits through a given region by local perturbations:

Definition 3.1.3. Let Φ be a Hamiltonian isotopy class of area-preserving diffeomorphisms of (Σ, ω). We say that Φ satisfies the C^∞ *closing property* if for every map $\phi \in \Phi$ and for every nonempty open set $\mathcal{U} \subset \Sigma$, there exists a C^∞-small Hamiltonian isotopy supported in $\mathcal{U}$ from ϕ to ϕ' such that ϕ' has a periodic orbit intersecting $\mathcal{U}$.

Standard arguments, see e.g. [77, §3], show:

Lemma 3.1.4. *If the Hamiltonian isotopy class Φ satisfies the C^∞ closing property, then it satisfies the C^∞ generic density property.*

One can now ask which Hamitonian isotopy classes satisfy the C^∞ closing property. One of our main results is the following:

Theorem 3.1.5. *(proved in §3.7) Let Φ be a Hamiltonian isotopy class of area-preserving diffeomorphisms of (Σ, ω). Suppose that Φ is rational (Definition 3.1.6) and satisfies the U-cycle property (Definition 3.2.23). Then Φ satisfies the C^∞ closing property.*

To explain the rationality hypothesis, we need to introduce a key actor in the story, the *mapping torus* of ϕ. This is a three-manifold defined by

$$Y_\phi = [0,1] \times \Sigma / \sim, \qquad (1, x) \sim (0, \phi(x)). \tag{3.1.1}$$

The mapping torus is a fiber bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ with fiber Σ. If t denotes the $[0,1]$ coordinate on $[0,1] \times \Sigma$, then the vector field ∂_t on $[0,1] \times \Sigma$ descends to a vector field on Y_ϕ, which we also denote by ∂_t. Periodic orbits of the map ϕ of period d correspond to simple periodic orbits of the vector field ∂_t whose projection to S^1 has degree d. Since the map ϕ preserves the symplectic form ω on Σ, this form induces a fiberwise symplectic form ω on Y_ϕ. The latter extends to a closed 2-form ω_ϕ on Y_ϕ, characterized by $\omega_\phi(\partial_t, \cdot) = 0$.

We need to consider how the cohomology class $[\omega_\phi] \in H^2(Y_\phi; \mathbb{R})$ depends on ϕ. Let $\{\phi_s\}_{s \in [0,1]}$ be a smooth isotopy of area-preserving diffeomorphisms of (Σ, ω), and suppose for simplicity that ϕ_s is constant for s close to 0 or 1. (See §3.3 for a more general formalism for Hamiltonian isotopies.) We then obtain a diffeomorphism of mapping tori

$$f : Y_{\phi_0} \xrightarrow{\simeq} Y_{\phi_1}.$$

This is induced by the diffeomorphism of $[0,1] \times \Sigma$ sending

$$(t, x) \longmapsto (t, \phi_t^{-1}(\phi_0(x))). \tag{3.1.2}$$

If the isotopy $\{\phi_s\}$ is Hamiltonian, then $f^*[\omega_{\phi_1}] = [\omega_{\phi_0}] \in H^2(Y_{\phi_0}; \mathbb{R})$.

Definition 3.1.6. The Hamiltonian isotopy class Φ is *rational* if for $\phi \in \Phi$, the cohomology class $[\omega_\phi] \in H^2(Y_\phi; \mathbb{R})$ is a real multiple of a class in the image of $H^2(Y_\phi; \mathbb{Z})$.

Example 3.1.7. Suppose that $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and ω is the restriction of the standard symplectic form on $\mathbb{R}^2$. Any orientation-preserving diffeomorphism of T^2 is isotopic to the diffeomorphism induced by a linear map $A \in \mathrm{SL}_2\,\mathbb{Z}$; see e.g. the introduction to [10]. It follows that any area-preserving diffeomorphism is Hamiltonian isotopic to a map of the form $\phi(x) = Ax + b$ where $A \in \mathrm{SL}_2\,\mathbb{Z}$ and $b \in \mathbb{R}^2/\mathbb{Z}^2$. A computation using the Mayer-Vietoris sequence shows that there is a short exact sequence[1]

$$0 \longrightarrow H_2(\Sigma) \longrightarrow H_2(Y_\phi) \stackrel{h}{\longrightarrow} \mathrm{Ker}(A - I : \mathbb{Z}^2 \circlearrowleft) \longrightarrow 0,$$

where the first arrow is induced by inclusion of the fiber $\{0\} \times \Sigma$, and the map h is given by the homology class in $H_1(T^2)$ of the intersection with $\{0\} \times \Sigma$. If $Z \in H_2(Y_\phi)$, then we have $\int_Z \omega_\phi \equiv \omega(h(Z), b) \mod \mathbb{Z}$. Thus $[\phi]$ is rational if and only if $\omega(v, b) \in \mathbb{Q}$ whenever $v \in \mathrm{Ker}(A - I : \mathbb{Z}^2 \circlearrowleft)$.

In particular, if $A = I$, that is if ϕ is smoothly isotopic to the identity, then $[\phi]$ is rational if and only if $b \in \mathbb{Q}^2/\mathbb{Z}^2$, namely ϕ is a rational rotation. If $A - I$ has rank one, then ϕ is isotopic to a power of a Dehn twist, and $[\phi]$ is rational when $\omega(v, b) \in \mathbb{Q}$ where v is a generator of $\mathrm{Ker}(A - I : \mathbb{Z}^2 \circlearrowleft)$. In all other cases, ϕ is finite order or Anosov and $b_2(Y_\phi) = 1$, so $[\phi]$ is automatically rational.

The U-cycle property is too technical to explain in the introduction, but we will see in Example 3.2.22 and Lemma 3.7.2 below that every rational Hamiltonian isotopy class on S^2 or T^2 satisfies this property. Thus Theorem 3.1.5 implies:

Corollary 3.1.8. *Let Φ be a Hamiltonian isotopy class of area-preserving diffeomorphisms of S^2 or T^2. If Φ is rational, then Φ satisfies the C^∞ closing property.*

Note that the unique Hamiltonian isotopy class of area-preserving diffeomorphisms of S^2 is rational. Any non-rational area-preserving diffeomorphism of T^2 can be perturbed to a rational one by a C^∞-small (non-Hamiltonian) isotopy. Then as in [7, Cor. 1.2], we obtain:

Corollary 3.1.9. *For a C^∞-generic area-preserving diffeomorphism of T^2, the set of periodic points is dense in T^2.*

Remark 3.1.10. After the initial version of this paper was completed, it was shown in [27] that every rational Hamiltonian isotopy class on a surface of any genus satisfies the U-cycle property (see Remark 3.7.3). Thus the U-cycle hypothesis in Theorem 3.1.5 is redundant, and Corollary 3.1.8 also holds for surfaces of higher genus.

Remark 3.1.11. It was shown in [28], which appeared simultaneously with the initial version of this paper, that Corollary 3.1.9 holds for surfaces of higher genus. The argument in [28] also shows that every rational Hamiltonian isotopy class satisfies the C^∞ generic density property.

[1]Our convention is that the homology of a topological space is taken with $\mathbb{Z}$ coefficients by default unless otherwise stated.

3.1.2 Quantitative closing lemmas

Going beyond the C^∞ closing property, our methods also yield *quantitative* closing lemmas, asserting roughly that during a given Hamiltonian isotopy, within time δ a periodic orbit must appear with period $O(\delta^{-1})$. We now give some examples of precise quantitative statements that we can prove.

Definition 3.1.12. Let $\mathcal{U} \subset \Sigma$ be a nonempty open set, let $l \in (0,1)$, and let $a \in (0, \text{area}(\mathcal{U}))$. A $(\mathcal{U}, a, l)$*-admissible Hamiltonian* is a smooth function $H : [0,1] \times \Sigma \to \mathbb{R}$ such that:

- $H(t,x) = 0$ for t close to 0 or 1.
- $H(t,x) = 0$ for $x \notin \mathcal{U}$.
- $H \geq 0$.
- There is an interval $I \subset (0,1)$ of length l and a disk $D \subset \mathcal{U}$ of area a such that $H \geq 1$ on $I \times D$.

Given any Hamiltonian $H : [0,1] \times \Sigma \to \mathbb{R}$ satisfying the first bullet point above, let $\{\varphi_t\}_{t\in[0,1]}$ denote the associated Hamiltonian isotopy (see §3.3 for conventions), and given an area-preserving diffeomorphism ϕ of Σ, write $\phi_H = \phi \circ \varphi_1$.

Theorem 3.1.13. *(proved in §3.7) Let ϕ be an area-preserving diffeomorphism of (S^2, ω), write $A = \int_{S^2} \omega$, let $\mathcal{U} \subset S^2$ be a nonempty open set, and let H be a $(\mathcal{U}, a, l)$-admissible Hamiltonian. If $0 < \delta \leq al^{-1}$, then for some $\tau \in [0, \delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period d satisfying*

$$d \leq \left\lfloor Al^{-1}\delta^{-1} \right\rfloor . \tag{3.1.3}$$

Remark 3.1.14. When $\delta = al^{-1}$ and $Aa^{-1} \notin \mathbb{Z}$, the bound (3.1.3) is sharp! That is, under the hypotheses of Theorem 3.1.13, one cannot prove the existence of a period orbit intersecting $\mathcal{U}$ with period less than $\lfloor Aa^{-1} \rfloor$. The reason is that if $a = \text{area}(\mathcal{U}) - \varepsilon$ for $\varepsilon > 0$ sufficiently small, then the open sets $\phi^i(\mathcal{U})$ for $0 \leq i < \lfloor Aa^{-1} \rfloor$ have total area less than A and thus could be disjoint.

We also obtain a slightly weaker inequality for rational area-preserving diffeomoprhisms of the torus:

Theorem 3.1.15. *(proved in §3.7) Let ϕ be an area-preserving diffeomorphism of (T^2, ω), and write $A = \int_{T^2} \omega$. Suppose that the Hamiltonian isotopy class $[\phi]$ is rational, let $\Omega \in H^2(Y_\phi; \mathbb{Z})$ be an integral cohomology class such that $[\omega_\phi]$ is a positive multiple of the image of Ω, and let $d_0 = \langle \Omega, [T^2] \rangle$. Let $\mathcal{U} \subset \Sigma$ be a nonempty open set, and let H be a $(\mathcal{U}, a, l)$-admissible Hamiltonian. If $0 < \delta \leq al^{-1}$, then for some $\tau \in [0, \delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period d satisfying*

$$d \leq d_0 \left(\left\lfloor Ad_0^{-1} l^{-1} \delta^{-1} \right\rfloor + 1 \right) .$$

Theorems 3.1.13 and 3.1.15 are special cases of a more general statement, Theorem 3.7.4 below, which is also applicable to higher genus surfaces. For some further developments on quantitative closing lemmas, which appeared after the initial version of this paper, see [13, 29, 62].

3.1.3 Background and motivation for the proofs

The main precedent for Theorem 3.1.5 is the proof by Irie [77] of the following C^∞ closing lemma for contact forms on closed three-manifolds. See also the survey [61].

Theorem 3.1.16 (Irie)**.** *Let Y be a closed three-manifold, let λ be a contact form on Y, and let $\mathcal{U} \subset Y$ be a nonempty open set. Then there exists a contact form λ' which is C^∞-close to λ and agrees with λ outside of $\mathcal{U}$, such that the Reeb vector field associated to λ' has a periodic orbit intersecting $\mathcal{U}$.*

The proof uses the *embedded contact homology* of (Y, λ); see [66] for detailed definitions. If λ is nondegenerate, meaning that the periodic orbits of the Reeb vector field are nondegenerate, then the embedded contact homology $ECH(Y, \lambda)$ is the homology of a chain complex generated (over $\mathbb{Z}/2$) by certain finite sets of Reeb orbits with multiplicities, and whose differential counts certain J-holomorphic curves in $\mathbb{R} \times Y$ for a suitable almost complex structure J. Taubes [97] proved that $ECH(Y, \lambda)$ is canonically isomorphic to a version of Seiberg-Witten Floer cohomology of Y as defined by Kronheimer-Mrowka [80], and in particular depends only on the contact structure $\xi = \operatorname{Ker}(\lambda)$. For any contact form λ on Y, possibly degenerate, and for any nonzero class $\sigma \in ECH(Y, \xi)$, there is a "spectral invariant" $c_\sigma(Y, \lambda) \in \mathbb{R}$, which is the total period of a finite set of Reeb orbits with multiplicities homologically selected by ECH. The spectral invariants, unlike ECH, are highly sensitive to the contact form λ.

For the proof of Theorem 3.1.16, we can assume without loss of generality that Y is connected. There is then a well-defined map $U : ECH(Y, \xi) \to ECH(Y, \xi)$, which is induced by a chain map counting certain J-holomorphic curves that are constrained to pass through a base point in $\mathbb{R} \times Y$. Define a *U-sequence* to be a sequence of nonzero classes $\{\sigma_k\}_{k \geq 1}$ in $ECH(Y, \xi)$ such that $U\sigma_{k+1} = \sigma_k$ for all $k \geq 1$. As explained for example in [24, Lem. A.1], results of Kronheimer-Mrowka [80] on Seiberg-Witten Floer homology imply that U-sequences always exist.

The key ingredient now is the following "Weyl law" for ECH spectral invariants proved[2] in [25].

Theorem 3.1.17. *[25] Let Y be a closed connected three-manifold, let λ be a contact form on Y, and let $\{\sigma_k\}_{k \geq 1}$ be a U-sequence in $ECH(Y, \xi)$. Then*

$$\lim_{k \to \infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{k} = 2 \operatorname{vol}(Y, \lambda). \tag{3.1.4}$$

[2]This was earlier proved in a special case in [65], and later given a different proof by Sun [96].

Here the *contact volume* is defined by

$$\operatorname{vol}(Y,\lambda) = \int_Y \lambda \wedge d\lambda.$$

To prove Theorem 3.1.16, one can define a smooth one-parameter family of contact forms $\{\lambda_t\}$ such that $\lambda_0 = \lambda$, outside of $\mathcal{U}$ we have $\lambda_t = \lambda$, and $\frac{d}{dt}\operatorname{vol}(Y,\lambda_t) > 0$. There must then exist Reeb orbits of λ_t passing through $\mathcal{U}$ for arbitrarily small t. Otherwise, there exists $\delta > 0$ such that λ_t has no Reeb orbit passing through $\mathcal{U}$ for $t \in [0,\delta]$. One can deduce that each spectral invariant $c_\sigma(Y,\lambda_t)$ is independent of $t \in [0,\delta]$. It then follows from the Weyl law (3.1.4) that $\operatorname{vol}(Y,\lambda_t)$ is also independent of $t \in [0,\delta]$, which is a contradiction.

Returning to area-preserving surface diffeomorphisms: Asaoka-Irie proved that a C^∞-generic Hamiltonian diffeomorphism of (Σ,ω) has dense periodic points by starting with a Hamiltonian diffeomorphism ϕ and constructing a contact three-manifold with an open book decomposition whose page is Σ with a disk removed, and whose monodromy is a slight modification of ϕ. One can then apply Theorem 3.1.16 to find a C^∞-small perturbation of the contact form with dense Reeb orbits, and translate this back to a C^∞-small perturbation of ϕ with dense periodic orbits.

It is not obvious how to extend the above argument to other Hamiltonian isotopy classes, because there are cohomological obstructions to defining the desired contact form. We will instead work more directly with *periodic Floer homology* (PFH). This is a theory which is defined analogously to ECH, but using periodic orbits of an area-preserving surface diffeomorphism instead of Reeb orbits of a contact form on a three-manifold. PFH is isomorphic to a version of Seiberg-Witten Floer cohomology of the mapping torus Y_ϕ, as shown by Lee-Taubes [82]. Originally, PFH was defined before ECH; see [63, 71]. Since then there have been many applications of ECH to dynamics of Reeb vector fields in three dimensions and symplectic embeddings in four dimensions. Applications of PFH have only recently begun to appear, including the spectacular proof by Cristofaro-Gardiner, Humilière, and Seyfaddini [21] that the group of compactly supported area-preserving homeomorphisms of the disk is not simple, and additional applications of PFH to area-preserving homeomorphisms of the two-sphere [20].

We will prove a PFH analogue of the "Weyl law" (3.1.4) in Theorem 3.8.1 below, replacing the notion of "U-sequence" by a notion of "U-cycle". This Weyl law implies closing lemmas as in Theorem 3.1.5 (under slightly stronger hypotheses on the U-cycles), following Irie's proof of Theorem 3.1.16.

Remark 3.1.18. Cristofaro-Gardiner, Prasad, and Zhang [28] have independently proved a related Weyl law in PFH by different methods. This Weyl law (together with a nonvanishing result for Seiberg-Witten Floer cohomology) is used in [28] to prove the generic density results described in Remark 3.1.11.

A Weyl law is really much stronger than necessary to detect the creation of periodic orbits. Indeed, a Weyl law implies that during a suitable perturbation, infinitely many spectral invariants change, with certain asymptotics; but to detect the creation of a periodic orbit, it suffices to show that a single spectral invariant changes by any nonzero amount. In §3.6 we introduce a refinement of Irie's argument which, instead of a Weyl law, uses bounds on "spectral gaps" coming from ball packings in symplectic cobordisms, to detect the creation of periodic orbits. This method in fact leads to stronger, quantitative closing lemmas as in §3.1.2 above.

The rest of the paper is organized as follows. In §3.2 we review the definition of periodic Floer homology. In §3.3 we discuss invariance of PFH under Hamitonian isotopy; here we find it useful to relate a Hamiltonian isotopy to the graph of a function $Y_\phi \to \mathbb{R}$. In §3.4 we explain how to define the PFH spectral numbers we will use. In §3.5 we prove a key lemma which gives relations between PFH spectral invariants of different maps in the same Hamiltonian isotopy class arising from ball packings in symplectic cobordisms between the graphs of different Hamiltonians. In §3.6 we use this lemma to show how "spectral gaps" in PFH allow one to detect the creation of periodic orbits, with quantitative bounds. In §3.7 we use this machinery to prove all of our theorems stated above. Finally, in §3.8 we state and prove a Weyl law for PFH spectral invariants.

Acknowledgments. We thank the anonymous referees for detailed comments which helped us improve the clarity of the exposition.

3.2 Periodic Floer homology

We now set up the version of PFH that we will be using, which one might call "twisted PFH", in more detail. Most of this material is also explained in [21, 63, 64, 71, 82], although we will use a particular bookkeeping formalism of Novikov rings and reference cycles to keep track of areas of holomorphic curves. Apart from this bookkeeping, PFH is extremely similar to ECH, and we will refer to the lecture notes [66] on ECH for some definitions and basic results that do not differ significantly from the PFH case.

3.2.1 PFH generators and holomorphic curves

Let (Σ, ω) be a closed connected surface of genus g with a symplectic (area) form, and let $\phi : (\Sigma, \omega) \to (\Sigma, \omega)$ be an area-preserving diffeomorphism.

Definition 3.2.1. An *orbit set* is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ such that:

- The α_i are distinct periodic orbits of ϕ.
- The m_i are positive integers.

Let Y_ϕ denote the mapping torus of ϕ as in (3.1.1). For an orbit set as above, regarding the periodic orbits α_i as embedded loops in Y_ϕ, we define the homology class

$$[\alpha] = \sum_i m_i[\alpha_i] \in H_1(Y_\phi).$$

A periodic orbit of ϕ of period k is *nondegenerate* if for $x \in \Sigma$ in the periodic orbit, the derivative $d\phi^k_x : T_x\Sigma \to T_x\Sigma$ does not have 1 as an eigenvalue. A nondegenerate orbit as above is *hyperbolic* if $d\phi^k_x$ has real eigenvalues. We say that ϕ is nondegenerate if all of its periodic orbits (including multiply covered periodic orbits where the points in Σ are not distinct) are nondegenerate; this holds for C^∞ generic ϕ in any Hamiltonian isotopy class Φ.

Definition 3.2.2. Assume that ϕ is nondegenerate. A *PFH generator* is an orbit set $\alpha = \{(\alpha_i, m_i)\}$ such that $m_i = 1$ whenever α_i is hyperbolic.

Remark 3.2.3. The requirement that $m_i = 1$ when α_i is hyperbolic is motivated by the relation with Seiberg-Witten theory, and some such condition is needed to obtain a topological invariant; see [66, §2.7]. This requirement is also used in the proof that the differential ∂_J defined below satisfies $\partial_J^2 = 0$; see [66, §5.4] for explanation in the analogous situation of ECH.

Notation 3.2.4. If γ and γ' are 1-cycles[3] in Y_ϕ with $[\gamma] = [\gamma'] \in H_1(Y_\phi)$, let $H_2(Y_\phi, \gamma, \gamma')$ denote the set of relative homology classes of 2-chains Z in Y_ϕ with $\partial Z = \gamma - \gamma'$. This is an affine space over $H_2(Y_\phi)$.

To define the differential on the chain complex below, we will need to choose a generic almost complex structure J on $\mathbb{R} \times Y_\phi$ satisfying the following conditions. To state them, let $E \to Y_\phi$ denote the vertical tangent bundle of $Y_\phi \to S^1$; this subbundle of TY_ϕ plays an analogous role in PFH to the contact structure ξ in ECH.

Definition 3.2.5. An almost complex structure J on $\mathbb{R} \times Y_\phi$ is *admissible* if:

- $J(\partial_s) = \partial_t$, where s denotes the $\mathbb{R}$ coordinate on $\mathbb{R} \times Y_\phi$.
- J is independent of s, i.e. invariant under translation of the $\mathbb{R}$ factor in $\mathbb{R} \times Y_\phi$.
- J sends E to itself, rotating positively with respect to the fiberwise symplectic form ω. This last condition means that if $v \in E$ and $v \neq 0$, then $\omega(v, Jv) > 0$.

Fix a generic admissible J as above. We consider J-holomorphic curves of the form

$$u : (C, j) \longrightarrow (\mathbb{R} \times Y_\phi, J),$$

[3]In this paper, a "1-cycle" will always be a finite integer linear combination of closed oriented 1-dimensional submanifolds.

where the domain C is a compact connected Riemann surface with finitely many punctures. Here j denotes the (almost) complex structure on C, and the holomorphic curve equation is $J \circ du = du \circ j$. We assume that in a neighborhood of each puncture, the map u is asymptotic to $\mathbb{R}$ cross a periodic orbit of ϕ as $s \to +\infty$ or $s \to -\infty$. We declare two such curves to be equivalent if they differ by composition with a biholomorphism of the domains. The curve u is *multiply covered* if it factors through a branched cover of domains with degree greater than 1; otherwise u is *somewhere injective*. In the latter case, u is an embedding except possibly for finitely many singular points; see [66, §5.1] for explanation in the analogous case of ECH. In the somewhere injective case, we can identify the holomorphic curve u with its image in $\mathbb{R} \times Y_\phi$, which by abuse of notation we also denote by C.

We define a *J-holomorphic current* in $\mathbb{R} \times Y_\phi$ to be a finite formal linear combination

$$\mathcal{C} = \sum_i d_i C_i$$

where the C_i are distinct somewhere injective J-holomorphic curves as above, and the d_i are positive integers. If α and β are orbit sets with $[\alpha] = [\beta]$, let $\mathcal{M}^J(\alpha, \beta)$ denote the moduli space of J-holomorphic currents $\mathcal{C}$ in $\mathbb{R} \times Y_\phi$ which as currents are asymptotic to α as $s \to +\infty$ and to β as $s \to -\infty$. For $Z \in H_2(Y_\phi, \alpha, \beta)$, let $\mathcal{M}^J(\alpha, \beta, Z)$ denote the set of such currents that represent the relative homology class Z. See [66, §3] for more precise definitions in the analogous case of ECH. Note that $\mathbb{R}$ acts on $\mathcal{M}^J(\alpha, \beta, Z)$ by translation of the $\mathbb{R}$ coordinate on $\mathbb{R} \times Y_\phi$.

We note for later use that the admissibility conditions on the almost complex structure J imply the following:

- If η is a simple periodic orbit of ∂_t in Y_ϕ, then the "trivial cylinder" $\mathbb{R} \times \eta$ is an embedded J-holomorphic curve in $\mathbb{R} \times Y_\phi$.
- The restriction of ω_ϕ to any J-holomorphic curve is pointwise nonnegative. Consequently,
$$\mathcal{M}^J(\alpha, \beta, Z) \neq \emptyset \Longrightarrow \int_Z \omega_\phi \geq 0. \tag{3.2.1}$$

Given orbit sets $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ with $[\alpha] = [\beta]$, and given $Z \in H_2(Y_\phi, \alpha, \beta)$, the *ECH index*[4] is defined to be

$$I(\alpha, \beta, Z) = c_\tau(\alpha, \beta, Z) + Q_\tau(\alpha, \beta, Z) + \sum_i \sum_{k=1}^{m_i} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} \mathrm{CZ}_\tau(\beta_j^k) \in \mathbb{Z}. \tag{3.2.2}$$

Here τ is a homotopy class of trivialization of the bundle E over the orbits α_i and β_j, while c_τ denotes the relative first Chern class, Q_τ denotes the relative self-intersection number,

[4] Perhaps here it should be called the "PFH index".

and $\mathrm{CZ}_\tau(\gamma^k)$ denotes the Conley-Zehnder index of the k^{th} iterate of the periodic orbit γ with respect to τ. The ECH index $I(\alpha, \beta, Z)$ does not depend on the choice of trivialization τ, although the individual terms on the right hand side do. For the proof of this fact, and for detailed definitions of the above notions, see [63, §3.3], or [64, §2] for the more general case of stable Hamiltonian structures which includes both PFH and ECH.

Example 3.2.6. If $\alpha = \beta$, then there is a canonical bijection $H_2(Y_\phi, \alpha, \alpha) \simeq H_2(Y_\phi)$, as both sets have the same definition. If $[\Sigma] \in H_2(Y_\phi)$ denotes the homology class of a fiber of $Y_\phi \to S^1$, then we have

$$I(\alpha, \alpha, [\Sigma]) = 2(d - g + 1). \tag{3.2.3}$$

This is because the first term in (3.2.2) here is

$$c_\tau(\alpha, \alpha, [\Sigma]) = \langle c_1(E), [\Sigma]\rangle = \langle c_1(T\Sigma), [\Sigma]\rangle = 2 - 2g.$$

The second term in (3.2.2) is

$$Q_\tau(\alpha, \alpha, [\Sigma]) = 2d.$$

This holds because by [64, Eq. (3.11)], the self-intersection number $Q_\tau(\alpha, \alpha, [\Sigma])$ is twice the algebraic intersection number of $\mathbb{R} \times \alpha$ with a fiber. The third term in (3.2.2) here is zero because the sums are empty.

If $\mathcal{C} \in \mathcal{M}^J(\alpha, \beta, Z)$, we write $I(\mathcal{C}) = I(\alpha, \beta, Z)$. A key property of the ECH index is the following; see e.g. [66, Prop. 3.7] for the ECH case which is completely analogous.

Proposition 3.2.7. *Suppose that J is generic. Let α and β be orbit sets with $[\alpha] = [\beta] \in H_1(Y_\phi)$. Suppose*[5] *that $[\alpha] \cdot [\Sigma] > g$. Then:*

(a) If $I(\mathcal{C}) = 1$, then we can write $\mathcal{C} = \mathcal{C}_0 + C_1$, where $\mathcal{C}_0$ is $\mathbb{R}$-invariant, i.e. a (possibly zero) finite linear combination of trivial cylinders, and C_1 is an embedded holomorphic curve of Fredholm index 1, cut out transversely.

(b) If $I(\mathcal{C}) = 2$ and if α and β are PFH generators, then we can write $\mathcal{C} = \mathcal{C}_0 + C_2$, where $\mathcal{C}_0$ is $\mathbb{R}$-invariant, and C_2 is an embedded holomorphic curve of Fredholm index 2, cut out transversely.

Note that the embedded holomorphic curves C_1 and C_2 above live in moduli spaces of embedded J-holomorphic curves which have dimension one and two, respectively.

3.2.2 The chain complex

To define the PFH of ϕ in general, we need to keep track of some information about relative homology classes of holomorphic curves in $\mathbb{R} \times Y_\phi$. There are different options for how to do this, resulting in different versions of PFH. We will use a version with Novikov ring coefficients, which depends on the following choice:

[5] See Remark 3.2.13 for the reason why we assume that $[\alpha] \cdot [\Sigma] > g$.

Choice 3.2.8. Let $\mathrm{Ker}([\omega_\phi])$ denote the kernel of $\langle[\omega_\phi],\cdot\rangle : H_2(Y_\phi) \to \mathbb{R}$. Below, fix a subgroup $G \subset \mathrm{Ker}([\omega_\phi])$. On a first reading it may be simplest to just consider the case $G = \{0\}$, although later we will find it convenient to choose $G = \mathrm{Ker}([\omega_\phi])$.

Definition 3.2.9. Let q be a formal variable and[6] write $\mathbb{F} = \mathbb{Z}/2$. Let Λ_G denote the Novikov ring consisting of formal sums

$$\sum_{A \in H_2(Y_\phi)/G} p_A q^A$$

where $p_A \in \mathbb{F}$, such that for each $R \in \mathbb{R}$, there are only finitely many A such that $p_A \neq 0$ and $\langle[\omega_\phi], A\rangle > R$.

Definition 3.2.10. A *reference cycle* for ϕ is a 1-cycle γ in Y_ϕ. We define the *degree* $d(\gamma) = [\gamma]\cdot[\Sigma] \in \mathbb{Z}$, where $[\Sigma] \in H_2(Y_\phi)$ denotes the homology class of a fiber of $Y_\phi \to S^1$. Note that if $\alpha = \{(\alpha_i, m_i)\}$ is an orbit set with $[\alpha] = [\gamma] \in H_1(Y_\phi)$, then the total period of the orbits α_i, counted with their multiplicities m_i, must equal the degree $d(\gamma)$, and in particular $d(\gamma) \geq 0$ if such an orbit set exists.

Definition 3.2.11. Fix a subgroup G as above and a reference cycle γ for ϕ. A (G,γ)-*anchored orbit set* is a pair (α, Z), where α is an orbit set with $[\alpha] = [\gamma] \in H_1(Y_\phi)$, and $Z \in H_2(Y_\phi, \alpha, \gamma)/G$.

We define the *symplectic action* of (α, Z) by

$$\mathcal{A}(\alpha, Z) = \int_Z \omega_\phi.$$

This is well defined by our assumption that $G \subset \mathrm{Ker}([\omega_\phi])$.

When ϕ is nondegenerate, a (G,γ)-*anchored PFH generator* is a (G,γ)-anchored orbit set (α, Z) for which α is a PFH generator.

We can now define the periodic Floer homology $HP(\phi,\gamma,G)$, which is a Λ_G-module.

Definition 3.2.12. If ϕ is nondegenerate and γ is a reference cycle, define $CP(\phi,\gamma,G)$ to be the set of (possibly infinite) formal sums

$$\sum_{\alpha,Z} n_{\alpha,Z}(\alpha, Z) \tag{3.2.4}$$

where:

- The sum is over (G,γ)-anchored PFH generators (α, Z).

[6] One can also use coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}/2$; it is explained in [74, §9] how to set up orientations for the differential in ECH, and this carries over to PFH. However the applications in this paper, and all other applications of ECH and PFH so far, only need $\mathbb{Z}/2$ coefficients.

- Each coefficient $n_{\alpha,Z} \in \mathbb{F}$.
- For each $R \in \mathbb{R}$, there are only finitely many (α, Z) such that $n_{\alpha,Z} \neq 0$ and $\mathcal{A}(\alpha, Z) > R$.

Then $CP(\phi, \gamma, G)$ is a Λ_G-module, with the Λ_G-action given by

$$\sum_{A \in H_2(Y_\phi)/G} p_A q^A \cdot \sum_{\alpha,Z} n_{\alpha,Z}(\alpha, Z) = \sum_{\alpha,W} \left(\sum_{A \in H_2(Y_\phi)/G} p_A n_{\alpha,W-A} \right) (\alpha, W).$$

The finiteness conditions imply that the right hand side[7] is a well defined element of $CP(\phi, \gamma, G)$.

Remark 3.2.13. In general, to define the differential when $d(\gamma) \leq g$, one needs to choose a slight perturbation of an admissible almost complex structure, relaxing the last condition in Definition 3.2.5. This is because if $J(E) = E$, then each fiber of $\mathbb{R} \times Y_\phi \to \mathbb{R} \times S^1$ is a J-holomorphic curve, which is not cut out transversely when $g > 0$, and this interferes with compactness arguments to define the PFH differential when $d(\gamma) \leq g$. See e.g. [63, §9.5]. For simplicity, **we assume below that** $d(\gamma) > g$ so that we can stick with admissible almost complex structures. The theory below can be extended to the case $d(\gamma) \leq g$ with some additional work, but this will not be necessary for the applications here.

Definition 3.2.14. For generic admissible J, we define a differential

$$\partial_J : CP(\phi, \gamma, G) \longrightarrow CP(\phi, \gamma, G)$$

by

$$\partial_J \sum_{\alpha,Z} n_{\alpha,Z}(\alpha, Z) = \sum_{\beta,W} \left(\sum_{\alpha} \sum_{\substack{V \in H_2(Y_\phi,\alpha,\beta) \\ I(\alpha,\beta,V)=1}} n_{\alpha,W+V} \# \frac{\mathcal{M}^J(\alpha, \beta, V)}{\mathbb{R}} \right) (\beta, W). \tag{3.2.5}$$

Here the first sum on the right hand side is over (G, γ)-anchored PFH generators (β, W), the second sum on the right hand side is over PFH generators α homologous to γ, and $\#$ denotes the mod 2 count.

Lemma 3.2.15. *∂_J is well defined.*

Proof. Assume that J is generic. By Proposition 3.2.7(a) and the compactness result[8] of [63, Thm. 1.8(a)], given homologous PFH generators α and β and given $R > 0$, the set

$$\bigcup_{\substack{V \in H_2(Y_\phi,\alpha,\beta) \\ I(\alpha,\beta,V)=1 \\ \int_V \omega_\phi < R}} \frac{\mathcal{M}^J(\alpha, \beta, V)}{\mathbb{R}}$$

[7] One can also write the right hand side in more informal notation as $\sum_{A,\alpha,Z} p_A n_{\alpha,Z}(\alpha, Z + A)$.

[8] The compactness result and other results in [63] made an additional hypothesis of "d-admissibility", asserting that (ϕ, J) has a nice form near the periodic orbits of period at most $d(\gamma)$. This hypothesis is no longer needed thanks to the asymptotic analysis of Siefring [95].

is finite. In particular, for a fixed $V \in H_2(Y_\phi, \alpha, \beta)$ with $I(\alpha, \beta, V) = 1$, the set $\mathcal{M}^J(\alpha, \beta, V)/\mathbb{R}$ is finite, so that the mod 2 counts in (3.2.5) are well defined.

To prove that the sum on the right hand side of (3.2.5) is a well defined element of $CP(\phi, \gamma, G)$, fix $\sum_{\alpha, Z} n_{\alpha, Z}(\alpha, Z) \in CP(\phi, \gamma, G)$. We need to show that for each real number R, there are only finitely many (G, γ)-anchored PFH generators (β, W) with $\int_W \omega_\phi > R$ such that the sum in parentheses on the right hand side of (3.2.5) has any nonzero terms; and we need to show that for each such (β, W), there are only finitely many nonzero terms.

By (3.2.1), if $\int_W \omega_\phi > R$ and $\mathcal{M}^J(\alpha, \beta, V)$ is nonempty, then $\int_V \omega_\phi \geq 0$, so $\int_{W+V} \omega_\phi > R$. Write $Z = W + V$. By the definition of $CP(\phi, \gamma, G)$, there are only finitely many (G, γ)-anchored PFH generators (α, Z) with $\int_Z \omega_\phi > R$ and $n_{\alpha, Z} \neq 0$. For each such pair, and for each of the finitely many PFH generators β with $[\beta] = [\gamma]$, it follows from the aforementioned compactness result of [63, Thm. 1.8(a)] that the union over W with $\int_W \omega_\phi > R$ and $I(\alpha, \beta, Z - W) = 1$ of the moduli spaces $\mathcal{M}^J(\alpha, \beta, Z - W)/\mathbb{R}$ is finite. □

It follows from minor modifications of [73, Thm. 7.20] (which applies to ECH) that $\partial_J^2 = 0$.

Definition 3.2.16. We define the *periodic Floer homology* $HP(\phi, \gamma, G)$ to be the homology of the chain complex $(CP(\phi, \gamma, G), \partial_J)$.

The Λ_G-module $HP(\phi, \gamma, G)$ does not depend on the choice of J. That is, if J_1 and J_2 are admissible and generic, then there is a canonical isomorphism

$$\Psi_{J_1, J_2} : H_*((CP(\phi, \gamma, G), \partial_{J_2}) \xrightarrow{\simeq} H_*((CP(\phi, \gamma, G), \partial_{J_1}). \tag{3.2.6}$$

These canonical isomorphisms have the properties that $\Psi_{J_1, J_2} \circ \Psi_{J_2, J_3} = \Psi_{J_1, J_3}$ when J_3 is admissible and generic, and Ψ_{J_1, J_1} is the identity.

We have the canonical isomorphisms (3.2.6) because by a special case[9] of a theorem of Lee-Taubes [82, Thm. 6.2], there is a canonical isomorphism

$$H_*((CP(\phi, \gamma, G), \partial_J)) \simeq HM^{-*}(Y_\phi, \mathfrak{s}_\Gamma, -r[\omega_\phi]; \Lambda_G) \tag{3.2.7}$$

for $r >> 0$. Here the right hand side is a version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [80, Def. 30.2.3], perturbed using the cohomology class $-r[\omega_\phi]$, while $\mathfrak{s}_\Gamma$ is a spin-c structure determined by $\Gamma = [\gamma]$. This version of Seiberg-Witten Floer cohomology is the homology of a chain complex which is generated over Λ_G by solutions to the three-dimensional Seiberg-Witten equations on Y_ϕ, perturbed using the closed 2-form $r\omega_\phi$, modulo gauge transformations $Y_\phi \to S^1$; see [82, §1.2]. The differential counts solutions to similarly perturbed four-dimensional Seiberg-Witten equations on $\mathbb{R} \times Y_\phi$, modulo gauge

[9] The version of PFH that appears in [82, Thm. 6.2] uses different notation and depends on a "$(c_\Gamma, [\omega_\phi])$-complete local system for periodic Floer homology", where $\Gamma = [\gamma] \in H_1(Y_\phi)$. Our version of PFH arises from such a local system, which assigns to each PFH generator α with $[\alpha] = \Gamma$ the Λ_G-module consisting of elements of $CP(\phi, \gamma, G)$ only involving α. If β is another PFH generator with $[\beta] = \Gamma$, then the local system assigns to each class $V \in H_2(Y_\phi, \alpha, \beta)$ the morphism of Λ_G-modules sending $\sum_Z n_Z(\alpha, Z) \mapsto \sum_W n_{W+V}(\beta, W)$. The case $G = 0$, which is called "maximally-twisted coefficients" in [82], is further discussed in [82, Cor. 6.1].

transformations $\mathbb{R} \times Y_\phi \to S^1$ whose homotopy class in $[\mathbb{R} \times Y_\phi, S^1] = H_2(Y_\phi)$ is contained in the subgroup G. The proof of the isomorphism (3.2.7) shows that for $r >> 0$, if we choose the metric in the Seiberg-Witten equations to be determined by ω_ϕ and J, then there is an isomorphism on the chain level. In fact, according to [82, Thm. 6.1], if $r >> 0$ then there is a bijection between the Seiberg-Witten solutions counted by the Seiberg-Witten Floer differential, and the J-holomorphic currents counted by the PFH differential. Modding out these Seiberg-Witten solutions by a restricted set of gauge transformations corresponds to keeping track of relative homology classes of holomorphic currents on the PFH side.

Remark 3.2.17. If γ' is another reference cycle with $[\gamma] = [\gamma']$, and if $Z \in H_2(Y_\phi, \gamma, \gamma')/G$, then it follows from the definition (3.2.5) that there is an isomorphism of chain complexes

$$\psi_Z : (CP(\phi, \gamma, G), \partial_J) \xrightarrow{\simeq} (CP(\phi, \gamma', G), \partial_J)$$

sending

$$\sum_{\alpha, W} n_{\alpha, W}(\alpha, W) \longmapsto \sum_{\alpha, W} n_{\alpha, W+Z}(\alpha, W). \tag{3.2.8}$$

This induces an isomorphism of Λ_G-modules

$$\Psi_Z : HP(\phi, \gamma, G) \xrightarrow{\simeq} HP(\phi, \gamma', G) \tag{3.2.9}$$

depending only on the relative homology class Z. Thus, up to isomorphism, $HP(\phi, \gamma, G)$ depends only on the diffeomorphism ϕ and the homology class $[\gamma]$, and not on the choice of reference cycle γ.

We will see in Proposition 3.3.1 below that the isomorphism class of $HP(\phi, \gamma, G)$ is also invariant under Hamiltonian isotopy of ϕ. Thus for a Hamiltonian isotopy class Φ, a homology class $\Gamma \in H_1(Y_\phi)$, and a subgroup G of $\operatorname{Ker}([\omega_\phi])$, we have a well-defined isomorphism class of Λ_G-modules $HP(\Phi, \Gamma, G)$.

3.2.3 Examples of PFH

Example 3.2.18. Let ϕ be the identity map on Σ. Although ϕ is degenerate, one can define its PFH to be the PFH of a nondegenerate Hamiltonian perturbation; see Remark 3.3.4 below.

The mapping torus is given by $Y_\phi = S^1 \times \Sigma$. We have

$$H_2(Y_\phi) = H_2(\Sigma) \oplus \left(H_1(S^1) \otimes H_1(\Sigma)\right).$$

It follows that the Novikov ring Λ_G consists of formal sums

$$\sum_{k \leq k_0} a_k q^{k[\Sigma]} \tag{3.2.10}$$

where the coefficients a_k are elements of a group ring,

$$a_k \in \mathbb{F}[(H_1(S^1) \otimes H_1(\Sigma))/G].$$

Write $[S^1] = [S^1] \times \{\mathrm{pt}\} \in H_1(Y_\phi)$. If d is a nonnegative integer and if $\Gamma = d[S^1]$, then we can choose the reference cycle γ to consist of d circles of the form $S^1 \times \{x\}$, and there is an isomorphism

$$HP\left(\mathrm{id}_\Sigma, d[S^1] \times \{x\}, G\right) \simeq \mathrm{Sym}^d H_*(\Sigma; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda_G. \tag{3.2.11}$$

Here Sym^d denotes the degree d part of the graded symmetric product; given a homogeneous basis of $H_*(\Sigma; \mathbb{F})$, this is a vector space over $\mathbb{F}$ with a basis consisting of symmetric degree d monomials in basis elements of $H_*(\Sigma; \mathbb{F})$, where basis elements in $H_1(\Sigma; \mathbb{F})$ cannot be repeated[10].

To prove the isomorphism (3.2.11), one fixes d and replaces the identity map with the time 1 flow ϕ of a C^2-small autonomous Hamiltonian $H : \Sigma \to \mathbb{R}$ which is a Morse function. It follows from Definition 3.2.2 that PFH generators in the class $\Gamma = d[S^1]$ correspond to degree d symmetric monomials in critical points of H, where index 1 critical points cannot be repeated. One can choose a metric g_Σ on Σ making the pair (H, g_Σ) Morse-Smale, along with a corresponding almost complex structure J on $\mathbb{R} \times Y_\phi$ for which Morse flow lines give rise to J-holomorphic cylinders. The S^1 symmetry of the mapping torus can be used to show that no other J-holomorphic curves contribute to the PFH differential; this argument is worked out in [36,86] for the very similar problem of computing the ECH of prequantization bundles. In particular, if we choose H to be a perfect Morse function (this means that the Morse homology differential vanishes, or equivalently here that there are exactly $2g + 2$ critical points), then the PFH differential vanishes, and the chain complex agrees with the right hand side of (3.2.11). The isomorphism (3.2.11) depends only on a choice of anchors for the degree d PFH generators.

For a homology class $\Gamma \in H_1(S^1 \times \Sigma)$ which is not of the form $d[S^1]$ for a nonnegative integer d, the PFH is zero, because after a small perturbation of the identity as above, there are no PFH generators in the class Γ.

Some more examples of PFH (more precisely untwisted PFH in the monotone case, see §3.2.6 below) are computed in [71] and [82, Cor. 1.5]. For classes Γ with $d = \Gamma \cdot [\Sigma] = 1$, the PFH is closely related[11] to the Floer homology for symplectic fixed points, which has been computed by Cotton-Clay [18].

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[10] It is also true (a special property of surfaces) that there is a canonical isomorphism $\mathrm{Sym}^d H_*(\Sigma; \mathbb{F}) = H_*(\mathrm{Sym}^d \Sigma; \mathbb{F})$, although we do not need this.

[11] One might expect that $d = 1$ PFH and fixed point Floer homology are the same, both with the differential counting holomorphic cylinders. However when $g(\Sigma) > 0$, in principle the PFH differential may count some additional holomorphic curves, due to the fact that $d \leq g$ here; see Remark 3.2.13.

3.2.4 The U map

There is also a well-defined map

$$U : HP(\phi, \gamma, G) \longrightarrow HP(\phi, \gamma, G). \tag{3.2.12}$$

This is induced by a chain map which is defined analogously to the differential (3.2.5); but instead of counting $I = 1$ holomorphic currents modulo $\mathbb{R}$ translation, it counts $I = 2$ holomorphic currents that are constrained to pass through a base point in $\mathbb{R} \times Y_\phi$.

To be precise, fix $y \in Y_\phi$ which is not on any periodic orbit of the vector field ∂_t. Given homologous PFH generators α and β, and given $Z \in H_2(Y_\phi, \alpha, \beta)$, define

$$\mathcal{M}^J_y(\alpha, \beta, Z) = \left\{ \mathcal{C} \in \mathcal{M}^J(\alpha, \beta, Z) \;\middle|\; (0, y) \in \mathcal{C} \right\}.$$

For a generic admissible J we define a map

$$U_{J,y} : CP(\phi, \gamma, G) \longrightarrow CP(\phi, \gamma, G)$$

by

$$U_{J,y} \sum_{\alpha, Z} n_{\alpha,Z}(\alpha, Z) = \sum_{\beta, W} \left(\sum_{\alpha} \sum_{\substack{V \in H_2(Y_\phi, \alpha, \beta) \\ I(\alpha,\beta,V)=2}} n_{\alpha, W+V} \# \mathcal{M}^J_y(\alpha, \beta, V) \right) (\beta, W).$$

This map is well defined by an argument similar to the proof of Lemma 3.2.15, using Proposition 3.2.7(b). Similarly to the proof that $\partial_J^2 = 0$, the map $U_{J,y}$ is a chain map:

$$\partial_J U_{J,y} = U_{J,y} \partial_J.$$

We define the U map (3.2.12) to be the map on homology induced by the chain map $U_{J,y}$. Since Y_ϕ is connected, the U map (3.2.12) does not depend on the choice of base point y; one can use a path between two choices of base point y to define a chain homotopy between the corresponding chain maps $U_{J,y}$. See [75, §2.5] for details in the analogous case of ECH.

The U map (3.2.12) does not depend on the choice of J either, because under the Lee-Taubes isomorphism (3.2.7), it corresponds to a U map on Seiberg-Witten Floer homology defined by Kronheimer-Mrowka in [80, §25.3]. Taubes proved an analogous statement for ECH in [98, Thm. 1.1], and the same argument works for the PFH case[12].

We will see in Proposition 3.3.1 below that the U map is invariant (in a certain sense) under Hamiltonian isotopy of ϕ.

[12] It should be noted that Kronheimer-Mrowka and Taubes use different but equivalent definitions of the U map on Seiberg-Witten Floer homology. Taubes defines the U map from a chain map which counts Seiberg-Witten solutions for which α vanishes at the base point $(0, y) \in \mathbb{R} \times Y$. Here α denotes the component of the spinor in the $+i$ eigenspace of Clifford multiplication by λ; see [98, §1.b]. Kronheimer-Mrowka define the U map from a chain map counting solutions to Seiberg-Witten moduli spaces after taking the cap product of the moduli spaces with a Cech cocycle representing the cohomology class in the configuration space corresponding to the generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$; see [80, §9.7]. One can choose the Cech cocycle so that the two chain maps agree.

Example 3.2.19. Suppose that $\Sigma = S^2$ and ϕ is Hamiltonian isotopic to the identity. Let d be a positive integer, and set $\gamma = d[S^1] \times \{x\}$ as in Example 3.2.18. Here we must take $G = \{0\}$. Under the identification (3.2.11), denote the generators of $\operatorname{Sym}^d H_*(S^2;\mathbb{F})$, in increasing homological degree, by $e_{d,0}, e_{d,1}, \ldots, e_{d,d}$. Then a calculation as in [66, §4.1] shows that

$$\begin{aligned} Ue_{d,i} &= e_{d,i-1}, \qquad\qquad i = 1, \ldots, d,\\ Ue_{d,0} &= q^{-[S^2]} e_{d,d}. \end{aligned}$$

The above example has an important property which we now formalize.

Definition 3.2.20. Let ϕ be an area-preserving diffeomorphism of (Σ, ω), let γ be a reference cycle for ϕ, and let G be as in Choice 3.2.8. We say that a nonzero element $\sigma \in HP(\phi, \gamma, G)$ is *U-cyclic* if there is a positive integer m such that

$$U^{m(d(\gamma)-g+1)}\sigma = q^{-m[\Sigma]}\sigma. \tag{3.2.13}$$

We say that σ is U-cyclic *of order m* if m is the smallest positive integer with this property. (Here ϕ can be degenerate; see Remark 3.3.4.)

Remark 3.2.21. In general, if $U^k\sigma = q^{-m[\Sigma]}\sigma$ for some k, then we must have $k = m(d(\gamma) - g + 1)$. The reason is that in the nondegenerate case, U counts holomorphic currents with ECH index $I = 2$; while if α is any PFH generator with $\alpha \cdot [\Sigma] = d$, then by Example 3.2.6 we have $I(\alpha, \alpha, [\Sigma]) = 2(d - g + 1)$.

Example 3.2.22. Suppose that $\Sigma = S^2$ and $\gamma = d[S^1] \times \{x\}$ where d is a positive integer. Then it follows from Example 3.2.19 that $e_{d,i}$, and indeed every nonzero element of $HP(\phi, \gamma, \{0\})$, is U-cyclic of order 1.

Definition 3.2.23. We say that the Hamiltonian isotopy class $[\phi]$ has the *U-cycle property* if there exist U-cyclic elements with arbitrarily large degree. That is, we require that for all positive integers d_0, there exist a subgroup $G \subset \operatorname{Ker}([\omega_\phi])$, a reference cycle γ for ϕ with $d(\gamma) \geq d_0$, and a U-cyclic element $\sigma \in HP(\phi, \gamma, G)$. (This condition is invariant under Hamiltonian isotopy of ϕ.)

3.2.5 Filtered PFH

Fix a nondegenerate symplectomorphism ϕ, a reference cycle γ, a group G as in Choice 3.2.8, and a real number L. Define $CP^L(\phi, \gamma, G)$ to be the set of formal sums (3.2.4) in $CP(\phi, \gamma, G)$ such that $\mathcal{A}(\alpha, Z) < L$ whenever $n_{\alpha,Z} \neq 0$. For a generic admissible J, it follows from (3.2.1) that $CP^L(\phi, \gamma, G)$ is a subcomplex of $(CP(\phi, \gamma, G), \partial_J)$.

Definition 3.2.24. We define the *filtered PFH*, denoted by $HP^L(\phi, \gamma, G)$, to be the homology of the subcomplex $(CP^L(\phi, \gamma, G), \partial_J)$.

Inclusion of chain complexes induces a map

$$\imath^L : HP^L(\phi,\gamma,G) \longrightarrow HP(\phi,\gamma,G). \tag{3.2.14}$$

Similarly we have inclusion-induced maps

$$\imath^{L_1,L_2} : HP^{L_1}(\phi,\gamma,G) \longrightarrow HP^{L_2}(\phi,\gamma,G) \tag{3.2.15}$$

for $L_1 \le L_2$. With respect to these maps, $HP(\phi,\gamma,G)$ is the direct limit of $HP^L(\phi,\gamma,G)$ as $L \to \infty$.

The filtered homology $HP^L(\phi,\gamma,G)$, as well as the inclusion-induced maps (3.2.14) and (3.2.15), do not depend on the choice of J. An analogous statement for ECH is proved in [76, Thm. 1.3], and a similar argument applies here.

3.2.6 The monotone case

We now recall two alternate versions of PFH which are defined in the following special situation, which is possible when $[\phi]$ is rational. This is in fact the only case that we need to consider in order to prove our theorems stated in §3.1.

Definition 3.2.25. For $\Gamma \in H_1(Y_\phi)$, we say that the pair (ϕ,Γ) is *monotone* if[13] the cohomology class $[\omega] \in H^2(Y_\phi;\mathbb{R})$ is a real multiple of the image of the class

$$c_1(E) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y_\phi;\mathbb{Z}).$$

In this case, one can define a simpler, "untwisted" version[14] of PFH, which we denote here by $\overline{HP}(\phi,\Gamma)$. When ϕ is nondegenerate, this is the homology of a chain complex $\overline{CP}(\phi,\Gamma)$ which is freely generated over $\mathbb{F}$ by the PFH generators in the homology class Γ. For a generic admissible almost complex structure J, the differential is defined by

$$\partial_J\alpha = \sum_\beta \sum_{\substack{V\in H_2(Y_\phi,\alpha,\beta)\\ I(\alpha,\beta,V)=1}} \#\frac{\mathcal{M}^J(\alpha,\beta,V)}{\mathbb{R}}\beta. \tag{3.2.16}$$

In the monotone case one can also define a twisted version of PFH without using a Novikov ring[15], which we denote here by $\widetilde{HP}(\phi,\gamma,G)$, where γ is a reference cycle and G is as in Choice 3.2.8. This version of PFH is a module over the group ring $\mathbb{F}[H_2(Y_\phi)/G]$.

[13]This condition is an analogue of the following. In the context of Hamiltonian Floer homology, one says that a symplectic manifold (X,ω) is "monotone" if $[\omega]$ is a real multiple (sometimes assumed to be nonnegative) of $c_1(TX)$ on $\pi_2(X)$. For both PFH and Hamiltonian Floer homology, monotonicity allows one to bound the symplectic area of holomorphic curves in terms of their index, which is a step towards obtaining finite counts. See e.g. [53], which introduced the use of Novikov rings to define Hamiltonian Floer homology in some non-monotone cases.

[14]This is the original version of PFH from [63, 71].

[15]This is analogous to the twisted ECH introduced in [72, §11.2].

Again assuming that ϕ is nondegenerate, it is the homology of a chain complex $\widetilde{CP}(\phi,\gamma,G)$ which is freely generated over $\mathbb{F}$ by (G,γ)-anchored PFH generators, and whose differential is defined by

$$\partial_J(\alpha,Z)=\sum_{\beta}\sum_{\substack{V\in H_2(Y_\phi,\alpha,\beta)\\ I(\alpha,\beta,V)=1}}\#\frac{\mathcal{M}^J(\alpha,\beta,V)}{\mathbb{R}}(\beta,Z-V). \tag{3.2.17}$$

The differentials (3.2.16) and (3.2.17) are well defined because when computing the differential of a generator, the monotonicity hypothesis implies that there is an upper bound on the integral of ω_ϕ over all holomorphic currents that one needs to count, so that one obtains a finite count; compare Lemma 3.2.15.

Suppose now that the reference cycle γ is positively transverse to the fibers of $Y_\phi\to S^1$. A framing τ of γ then induces a $\mathbb{Z}$-grading on $\widetilde{HP}(\phi,\gamma,G)$. The grading of a generator (α,Z) is defined by

$$|(\alpha,Z)|=I(\alpha,\gamma,Z) \tag{3.2.18}$$

where the right hand side is defined as in (3.2.2), but with no Conley-Zehnder terms for γ. The grading (3.2.18) descends to a $\mathbb{Z}/N$ grading on $\overline{HP}(\phi,\Gamma)$, where N denotes the divisibility of $c_1(E)+2\,\mathrm{PD}(\Gamma)$ in $\mathrm{Hom}(H_2(Y_\phi),\mathbb{Z})$; note that N is an even integer.

The definition of the U map carries over to $\overline{HP}$ and $\widetilde{HP}$, and with respect to the above gradings, it has degree -2.

Even in the monotone case, we will need to use a twisted version of PFH with reference cycles in order to define spectral invariants. We will later need the following relation between twisted and untwisted versions:

Lemma 3.2.26. *Suppose that ϕ is nondegenerate and (ϕ,Γ) is monotone, let γ be a reference cycle with $[\gamma]=\Gamma$, and choose $G=\mathrm{Ker}([\omega_\phi])$. Then there is a noncanonical isomorphism of Λ_G-modules*

$$HP(\phi,\gamma,G)\simeq\overline{HP}(\phi,\Gamma)\otimes_F\Lambda_G. \tag{3.2.19}$$

Under the above isomorphism, if $d=d(\Gamma)$, then

$$U^{d-g+1}\longleftrightarrow U^{d-g+1}\otimes q^{-[\Sigma]}. \tag{3.2.20}$$

Proof. Let $A\in H_2(Y_\phi)/G$ be the unique class such that $\langle[\omega_\phi],A\rangle$ is positive and minimal. Then the Novikov ring Λ_G is canonically identified with $\mathbb{F}((q^{-A}))$, namely the ring of Laurent series in q^{-A} with coefficients in $\mathbb{F}$.

As in Remark 3.2.17, we can assume without loss of generality that γ is positively transverse to E. Choose a framing τ of γ as needed to define a $\mathbb{Z}$-grading on $\widetilde{HP}(\phi,\gamma,G)$ and a $\mathbb{Z}/N$ grading on $\overline{HP}(\phi,\Gamma)$. It follows from the definitions that there is a canonical isomorphism

$$\widetilde{HP}_i(\phi,\gamma,G)=\overline{HP}_{i\bmod N}(\phi,\Gamma). \tag{3.2.21}$$

On the left hand side, multiplication by q^{-A} shifts the grading down by N. It follows that $HP(\phi,\gamma,G)$ is canonically identified with the set of sequences $(\sigma_i)_{i\in\mathbb{Z}}$ where $\sigma_i\in$

$\widetilde{HP}_i(\phi, \gamma, G)$ and $\sigma_i = 0$ if i is sufficiently large. If we choose a right inverse of the projection $\mathbb{Z} \to \mathbb{Z}/N$, then together with (3.2.21) this defines an identification of the above set of sequences with $\overline{HP}(\phi, \Gamma) \otimes_F \Lambda_G$. This gives an isomorphism (3.2.19).

To prove (3.2.20), we observe that under the isomorphism (3.2.19) constructed above,

$$U^{N/2} \longleftrightarrow U^{N/2} \otimes q^{-A}.$$

The positive[16] integer $2(d - g + 1)$ must be divisible by N, since $c_1(E) + 2\,\mathrm{PD}(\Gamma)$ evaluates to $2(d - g + 1)$ on $[\Sigma]$. It follows that

$$U^{d-g+1} \longleftrightarrow U^{d-g+1} \otimes q^{-(2(d-g+1)/N)A}. \tag{3.2.22}$$

By monotonicity, we have $\langle c_1(E) + 2\,\mathrm{PD}(\Gamma), A\rangle = N$, and it follows that $(2(d-g+1)/N)A = [\Sigma]$ in $H_2(Y_\phi)/G$. Putting this into (3.2.22) proves (3.2.20). □

3.3 Invariance of PFH under Hamiltonian isotopy

We now work out how PFH and the additional structure on it defined above behave under Hamiltonian isotopy of ϕ.

It is useful for our purposes to define a Hamiltonian isotopy of ϕ via a smooth map

$$H : Y_\phi \longrightarrow \mathbb{R}.$$

Under the projection $[0, 1] \times \Sigma \to Y_\phi$, the map H pulls back to a map $\widetilde{H} : [0, 1] \times \Sigma \to \mathbb{R}$ satisfying $\widetilde{H}(1, x) = \widetilde{H}(0, \phi(x))$. For $t \in [0, 1]$, let $H_t = \widetilde{H}(t, \cdot) : \Sigma \to \mathbb{R}$, and let X_{H_t} denote the associated Hamiltonian vector field on Σ; we use the sign convention $\omega(X_{H_t}, \cdot) = dH_t$. Let $\{\varphi_t\}_{t\in[0,1]}$ denote the Hamiltonian isotopy defined by $\varphi_0 = \mathrm{id}_\Sigma$ and $\partial_t\varphi_t = X_{H_t} \circ \varphi_t$. We define $\phi_H = \phi \circ \varphi_1$.

As in (3.1.2), we have a diffeomorphism

$$f_H : Y_\phi \xrightarrow{\simeq} Y_{\phi_H}$$

defined by the diffeomorphism of $[0, 1] \times \Sigma$ sending

$$(t, x) \longmapsto (t, \varphi_t^{-1}(x)).$$

If γ is a reference cycle in Y_ϕ, let γ_H denote its pushforward $(f_H)_\#\gamma$ in Y_{ϕ_H}.

Proposition 3.3.1. *Let ϕ be a (possibly degenerate) area-preserving diffeomorphism of (Σ, ω), let $\gamma \subset Y_\phi$ be a reference cycle, and fix G as in Choice 3.2.8. For $H_1, H_2 : Y_\phi \to \mathbb{R}$ with $H_1 < H_2$ and ϕ_{H_1}, ϕ_{H_2} nondegenerate, there is a canonical isomorphism*

$$\Psi_{H_1,H_2} : HP(\phi_{H_2}, \gamma_{H_2}, G) \longrightarrow HP(\phi_{H_1}, \gamma_{H_1}, G) \tag{3.3.1}$$

with the following properties:

[16] Recall from Remark 3.2.13 that we are making the standing assumption that $d > g$.

(a) If $H_2 < H_3$ and if ϕ_{H_3} is also nondegenerate, then

$$\Psi_{H_1,H_3} = \Psi_{H_1,H_2} \circ \Psi_{H_2,H_3} : HP(\phi_{H_3}, \gamma_{H_3}, G) \longrightarrow HP(\phi_{H_1}, \gamma_{H_1}, G).$$

(b) $U \circ \Psi_{H_1,H_2} = \Psi_{H_1,H_2} \circ U$.

(c) The isomorphism (3.3.1) *is the direct limit as $L \to \infty$ of canonical maps*

$$\Psi^L_{H_1,H_2} : HP^L(\phi_{H_2}, \gamma_{H_2}, G) \longrightarrow HP^{L+\Delta}(\phi_{H_1}, \gamma_{H_1}, G) \tag{3.3.2}$$

where

$$\Delta = \int_\gamma (H_2 - H_1)dt. \tag{3.3.3}$$

(d) If $H_2 - H_1$ is a constant $C > 0$, so that $\phi_{H_1} = \phi_{H_2}$ and $\gamma_{H_1} = \gamma_{H_2}$, then $\Psi^L_{H_1,H_2} = \imath^{L,L+dC}$ where $d = [\gamma] \cdot [\Sigma]$, see (3.2.15). *In particular, Ψ_{H_1,H_2} is the identity map.*

Proof. We proceed in 6 steps.

Step 1. To prepare to define the map Ψ_{H_1,H_2}, we construct a "strong symplectic cobordism of stable Hamiltonian structures" between $(Y_{\phi_{H_1}}, \omega_{\phi_{H_1}})$ and $(Y_{\phi_{H_2}}, \omega_{\phi_{H_2}})$ as follows.

Consider the "symplectization" of the mapping torus defined by

$$X = \mathbb{R} \times Y_\phi$$

with the symplectic form

$$\omega_X = ds \wedge dt + \omega_\phi.$$

Here s denotes the $\mathbb{R}$ coordinate on $\mathbb{R} \times Y_\phi$.

Given $H : Y_\phi \to \mathbb{R}$, define an inclusion

$$\begin{aligned} \imath_H : Y_\phi &\longrightarrow \mathbb{R} \times Y_\phi, \\ z &\longmapsto (H(z), z). \end{aligned}$$

We can then identify the mapping torus Y_{ϕ_H} with a hypersurface in $\mathbb{R} \times Y_\phi$ via the inclusion

$$\imath_H \circ f_H^{-1} : Y_{\phi_H} \longrightarrow \mathbb{R} \times Y_\phi. \tag{3.3.4}$$

Note that there is a symplectomorphism

$$\mathbb{R} \times Y_{\phi_H} \longrightarrow \mathbb{R} \times Y_\phi \tag{3.3.5}$$

induced by the symplectomorphism of $(\mathbb{R} \times [0,1] \times \Sigma, ds \wedge dt + \omega)$ sending

$$(s, t, x) \longmapsto (s + H(t, x), t, \varphi_t(x)).$$

The inclusion (3.3.4) is the restriction of the symplectomorphism (3.3.5) to $\{0\} \times Y_{\phi_H}$. It follows that

$$(\imath_H \circ f_H^{-1})^* \omega_X = \omega_{\phi_H}. \tag{3.3.6}$$

We note also that under the inclusion (3.3.4), the reference cycle γ_H corresponds to the graph of H on γ in $\mathbb{R} \times Y_\phi$.

Now if $H_1 < H_2$, define

$$M = \{(s, z) \in \mathbb{R} \times Y_\phi \mid H_1(z) \leq s \leq H_2(z)\} \tag{3.3.7}$$

with the symplectic form $\omega_M = (\omega_X)|_M$. It follows from the above calculations that the boundary components of M have neighborhoods in M symplectomorphic to $[0, \varepsilon) \times Y_{\phi_{H_1}}$ and $(-\varepsilon, 0] \times Y_{\phi_{H_2}}$, where the latter manifolds are equipped with the restrictions of the symplectic forms on the symplectizations of $Y_{\phi_{H_1}}$ and $Y_{\phi_{H_2}}$. Using these neighborhood identifications, we can glue to form the "symplectization completion" of M, which is a symplectic four-manifold

$$\overline{M} = \left((-\infty, 0] \times Y_{\phi_{H_1}}\right) \bigcup_{Y_{\phi_{H_1}}} M \bigcup_{Y_{\phi_{H_2}}} \left([0, \infty) \times Y_{\phi_{H_2}}\right). \tag{3.3.8}$$

We note that there is a canonical symplectomorphism

$$\overline{M} \simeq \mathbb{R} \times Y_\phi \tag{3.3.9}$$

which is the inclusion on M, and which on the rest of (3.3.8) is defined using the restrictions of the symplectomorphisms (3.3.5) for H_1 and H_2.

Step 2. Suppose now that ϕ_{H_1} and ϕ_{H_2} are nondegenerate. Observe that $S = (\mathbb{R} \times \gamma) \cap M$ defines a 2-chain in the cobordism M with $\partial S = \gamma_{H_2} - \gamma_{H_1}$. The cobordism M, together with the 2-chain S, induces the desired map Ψ_{H_1,H_2} in (3.3.1), as a special case of a general construction of cobordism maps[17] on PFH by Chen [14, Thm. 1]. Chen's cobordism map in this case is defined by composing the Lee-Taubes isomorphism (3.2.7) on both sides with a Seiberg-Witten cobordism map

$$HM^{-*}(Y_{\phi_{H_2}}, \mathfrak{s}_\Gamma, -r[\omega_{\phi_{H_2}}]; \Lambda_G) \longrightarrow HM^{-*}(Y_{\phi_{H_1}}, \mathfrak{s}_\Gamma, -r[\omega_{\phi_{H_1}}]; \Lambda_G). \tag{3.3.10}$$

Here $r >> 0$ and $\Gamma = [\gamma] \in H_1(Y_\phi)$. The map (3.3.10) is induced by a chain map which counts solutions to the Seiberg-Witten equations on $\overline{M}$ perturbed using r times the symplectic form on $\overline{M}$, similarly to the way the differential on the right hand side of (3.2.7) counts solutions to perturbed Seiberg-Witten equations on $\mathbb{R} \times Y_\phi$. The PFH cobordism maps in [14, Thm. 1] satisfy a composition property which implies assertion (a), and they commute with the U maps, giving assertion (b); these properties follow from corresponding properties of Seiberg-Witten cobordism maps.

Step 3. To prove assertion (c), we will use the fact from [14, Thm. 1] that the map Ψ_{H_1,H_2} satisfies a crucial "holomorphic curve axiom". We now state this property.

[17] This is related to the construction of cobordism maps on ECH in [76, Thm. 1.9].

Let J_1 and J_2 be almost complex structures on $\mathbb{R} \times Y_{\phi_{H_1}}$ and $\mathbb{R} \times Y_{\phi_{H_2}}$ as needed to define differentials ∂_{J_1} on $CP(\phi_{H_1}, \gamma_{H_1}, G)$ and ∂_{J_2} on $CP(\phi_{H_2}, \gamma_{H_2}, G)$. We can extend J_1 and J_2 to an almost complex structure J on $\overline{M}$ whose restriction to M is compatible with the symplectic form ω_M.

Let α be an orbit set for ϕ_{H_2} and let β be an orbit set for ϕ_{H_1}. Define a *broken J-holomorphic current* in $\overline{M}$ from α to β to be a tuple $(\mathcal{C}_{k_+}, \mathcal{C}_{k_+-1}, \ldots, \mathcal{C}_{k_-})$ where $k_+ \geq 0 \geq k_-$, and there are orbit sets $\alpha = \alpha(k+), \alpha(k_+ - 1), \ldots, \alpha(0)$ for ϕ_{H_2} and orbit sets $\beta(0), \beta(-1), \ldots, \beta(k_-) = \beta$ for ϕ_{H_1}, such that:

- $\mathcal{C}_i \in \mathcal{M}^{J_2}(\alpha(i), \alpha(i-1))/\mathbb{R}$ for $i > 0$.
- $\mathcal{C}_0 \in \mathcal{M}^J(\alpha(0), \beta(0))$. That is, $\mathcal{C}_0$ is a J-holomorphic current in $\overline{M}$ which as a current is asymptotic to $\alpha(0)$ as $s \to \infty$ on $[0, \infty) \times Y_{\phi_{H_2}}$, and asymptotic to $\beta(0)$ as $s \to -\infty$ on $(-\infty, 0] \times Y_{\phi_{H_1}}$.
- $\mathcal{C}_i \in \mathcal{M}^{J_1}(\beta(i+1), \beta(i))/\mathbb{R}$ for $i < 0$.

The holomorphic curves axiom now states that the map Ψ is induced by a chain map

$$\psi : (CP(\phi_{H_2}, \gamma_{H_2}, G), \partial_{J_2}) \longrightarrow (CP(\phi_{H_1}, \gamma_{H_1}, G), \partial_{J_1})$$

with the following property. Similarly to (3.2.5), we can write ψ in the form

$$\psi \sum_{\alpha, Z} n_{\alpha, Z}(\alpha, Z) = \sum_{\beta, W} \left(\sum_{\alpha} \sum_{\substack{V \in H_2(Y_\phi, \alpha, \beta) \\ I(\alpha, \beta, V) = 0}} n_{\alpha, W+V} m_{\alpha, \beta, V} \right) (\beta, W). \tag{3.3.11}$$

Here the first sum on the right hand side is over (G, γ_{H_1})-anchored PFH generators (β, W) for ϕ_{H_1}, the second sum on the right hand side is over PFH generators α for ϕ_{H_2} in the homology class $[\gamma_{H_2}]$, and $m_{\alpha, \beta, V} \in \mathbb{F}$. The key property is now:

(*) If the coefficient $m_{\alpha, \beta, V} \neq 0$, then there is a broken J-holomorphic current[18] in $\overline{M}$ from α to β which, under the identification (3.3.9), represents the relative homology class V.

Note that the chain map ψ is not canonical; see Remark 3.3.2 below.

Step 4. We claim now that if (β, W) is a (G, γ_{H_1})-anchored PFH generator for ϕ_{H_1}, if $(\alpha, W + V)$ is a (G, γ_{H_2})-anchored PFH generator for ϕ_{H_2}, and if there exists a broken J-holomorphic curent in $\overline{M}$ from α to β in the relative homology class V, then

$$\int_W \omega_{\phi_{H_1}} \leq \int_{W+V} \omega_{\phi_{H_2}} + \Delta. \tag{3.3.12}$$

[18]The "holomorphic curves axiom" as stated in [14, Thm. 1] implies a slightly weaker statement than (*), namely that for fixed α and β, if any of the coefficients $m_{\alpha, \beta, Z}$ is nonzero, then there exists a broken J-holomorphic current from α to β. The property (*) follows from the same argument, keeping track of the relative homology classes of the holomorphic currents.

To prove this, by (3.2.1) we can assume without loss of generality that the J-holomorphic current has $k_+ = k_- = 0$, and thus consists of a single J-holomorphic current $\mathcal{C} \in \mathcal{M}^J(\alpha, \beta, V)$. We can also assume without loss of generality (by a slight modification of the cobordism) that $\mathcal{C}$ is transverse to ∂M. Under the decomposition (3.3.8), we can divide $\mathcal{C}$ into three pieces: let

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{C} \cap \left((-\infty, 0] \times Y_{\phi_{H_1}}\right), \\ \mathcal{C}_0 &= \mathcal{C} \cap M, \\ \mathcal{C}_2 &= \mathcal{C} \cap \left([0, \infty) \times Y_{\phi_{H_2}}\right). \end{aligned}$$

Since the almost complex structures J_1 and J_2 are admissible, as in (3.2.1) we have

$$\int_{\mathcal{C}_1} \omega_{\phi_{H_1}} \geq 0, \tag{3.3.13}$$

$$\int_{\mathcal{C}_2} \omega_{\phi_{H_2}} \geq 0. \tag{3.3.14}$$

Also, since J is ω_M-compatible, we have

$$\int_{\mathcal{C}_0} \omega_M \geq 0. \tag{3.3.15}$$

We now deduce (3.3.12) by applying Stokes's theorem. To start, write $\eta_1 = \mathcal{C}_1 \cap (\{0\} \times Y_{\phi_{H_1}})$ and $\eta_2 = \mathcal{C}_2 \cap (\{0\} \times Y_{\phi_{H_2}})$. Then $\mathcal{C}_1$ projects, via the projection $(-\infty, 0] \times Y_{\phi_{H_1}} \to Y_{\phi_{H_1}}$, to a relative homology class of 2-chain $[\mathcal{C}_1] \in H_2(Y_{\phi_{H_1}}, \eta_1, \beta)$. Likewise, $\mathcal{C}_2$ projects to a relative homology class of 2-chain $[\mathcal{C}_2] \in H_2(Y_{\phi_{H_2}}, \alpha, \eta_2)$.

It follows from the homological assumption on $\mathcal{C}$ that in M, the 2-cycle

$$(\imath_{H_1} \circ f_{H_1}^{-1})_\#([\mathcal{C}_1] + W) + (\imath_{H_2} \circ f_{H_2}^{-1})_\#([\mathcal{C}_2] - W - V) + \mathcal{C}_0 - S$$

is nullhomologous. Consequently the integral of the closed 2-form ω_M over this 2-cycle vanishes. By (3.3.6), this means that

$$\int_{\mathcal{C}_1 + W} \omega_{\phi_{H_1}} + \int_{\mathcal{C}_2 - W - V} \omega_{\phi_{H_2}} + \int_{\mathcal{C}_0} \omega_M - \Delta = 0. \tag{3.3.16}$$

Combining (3.3.13), (3.3.14), (3.3.15), and (3.3.16) proves (3.3.12).

Step 5. We now prove (c). It follows from Step 4 that the chain map (3.3.11) restricts to a chain map

$$\psi^L : (CP^L(\phi_{H_2}, \gamma_{H_2}, G), \partial_{J_2}) \longrightarrow (CP^{L+\Delta}(\phi_{H_1}, \gamma_{H_1}, G), \partial_{J_1})$$

We define the map $\Psi^L_{H_1,H_2}$ in (3.3.2) to be the map on filtered PFH induced by ψ^L. Although the chain map ψ is defined only up to chain homotopy, the chain homotopies satisfy a version of the “holomorphic curves axiom” which implies that $\Psi^L_{H_1,H_2}$ depends only on J_1 and J_2. See e.g. [67, Prop. 6.2] for an analogous argument in the case of ECH. The map $\Psi^L_{H_1,H_2}$ does not depend on J_1 and J_2 either. An analogous statement for ECH was proved in [76, Thm. 1.9], and this carries over to the case of PFH using the analysis of Chen [14, Thm. 1].

Step 6. Finally, the proof of property (d) follows the proof of the “scaling” property for ECH cobordism maps in [76, Thm. 1.9]. □

Remark 3.3.2. Naively one would like to define the chain map (3.3.11) by taking $m_{\alpha,\beta,V}$ to be a count of $I = 0$ holomorphic currents in $\mathcal{M}^J(\alpha, \beta, V)$. Unfortunately it is not currently known in general[19] how to directly count J-holomorphic currents with $I = 0$ in a completed cobordism, due to transversality difficulties with multiply covered holomorphic curves; see [66, §5.5] for explanation in the case of ECH, where there are similar issues. The actual chain map (3.3.11) is defined instead by counting solutions to the Seiberg-Witten equations on $\overline{M}$, using the metric determined by J and the symplectic form, and perturbed using a large multiple of the symplectic form as in [82]. The chain map (3.3.11) is not canonical, because in cases where transversality fails for holomorphic curves, the chain map depends on additional small perturbations to the Seiberg-Witten equations needed to obtain transversality of the relevant moduli spaces of Seiberg-Witten solutions.

Remark 3.3.3. In Proposition 3.3.1, if we drop the hypothesis that $H_1 < H_2$, then there is still a canonical isomorphism (3.3.1). One can define this isomorphism as $\Psi_{H_1,H_3} \circ \Psi^{-1}_{H_2,H_3}$ where ϕ_{H_3} is nondegenerate and $H_1, H_2 < H_3$. By Proposition 3.3.1(a) and (d), this isomorphism does not depend on the choice of H_3.

Remark 3.3.4. If ϕ is degenerate, then we can define $HP(\phi, \gamma, G)$ by first perturbing ϕ to be nondegenerate via a Hamiltonian isotopy. By Remark 3.3.3, the PFH modules for such perturbations of ϕ are all canonically isomorphic to each other.

3.4 Spectral invariants in PFH

We now define spectral invariants in PFH, analogously to the spectral invariants in ECH defined in [65, Def. 4.1].

Definition 3.4.1. Suppose that ϕ is nondegenerate, let γ be a reference cycle, fix G as in Choice 3.2.8, and let σ be a nonzero class in $HP(\phi, \gamma, G)$. Define the *spectral invariant*

$$c_\sigma(\phi, \gamma) \in \mathbb{R}$$

to be the infimum over $L \in \mathbb{R}$ such that σ is in the image of the inclusion-induced map (3.2.14).

[19] Such a count is possible in some special cases; see e.g. [14, Thm. 2], [42], and [91].

We now establish some properties of the spectral invariants c_σ. We first consider the dependence of c_σ on basic choices. Given a nonzero Novikov ring element $\lambda = \sum_{A\in H_2(Y_\phi)/G} p_A q^A \in \Lambda_G$, define

$$|\lambda| = \max\{\langle[\omega_\phi], A\rangle \mid p_A \neq 0\}. \tag{3.4.1}$$

Note that this maximum is well-defined by the definition of the Novikov ring Λ_G.

Proposition 3.4.2. *Suppose $\sigma \in HP(\phi,\gamma,G)$ is nonzero.*

(a) If $\lambda \in \Lambda_G$ is invertible[20]*, then*

$$c_{\lambda\sigma}(\phi,\gamma) = c_\sigma(\phi,\gamma) + |\lambda|.$$

(b) In the notation of Remark 3.2.17, we have

$$c_{\Psi_Z(\sigma)}(\phi,\gamma') = c_\sigma(\phi,\gamma) - \int_Z \omega.$$

Proof. (a) It follows from the definitions that multiplication by λ induces an isomorphism of chain complexes

$$(CP^L(\phi,\gamma,G),\partial_J) \xrightarrow{\simeq} (CP^{L+|\lambda|}(\phi,\gamma,G),\partial_J).$$

The induced isomorphism on homology

$$HP^L(\phi,\gamma,G) \xrightarrow{\simeq} HP^{L+|\gamma|}(\phi,\gamma,G) \tag{3.4.2}$$

respects the maps (3.2.14), and it follows that

$$c_{\lambda\sigma}(\phi,\gamma) \leq c_\sigma(\phi,\gamma) + |\lambda|.$$

The inverse of the isomorphism (3.4.2) is induced by multiplication of chains by λ^{-1}, and this implies the reverse inequality since $|\lambda^{-1}| = -|\lambda|$.

(b) This follows by a similar argument. □

We now begin to explore how spectral invariants behave under Hamiltonian isotopy, using the notation of Proposition 3.3.1.

Proposition 3.4.3. *Let ϕ be a (possibly degenerate) area-preserving diffeomorphism of (Σ,ω), let γ be a reference cycle for ϕ, let $H_1, H_2 : Y_\phi \to \mathbb{R}$ with $H_1 < H_2$, and suppose that ϕ_{H_1} and ϕ_{H_2} are nondegenerate. Let σ_2 be a nonzero class in $HP(\phi_{H_2},\gamma_{H_2},G)$. Write $\sigma_1 = \Phi_{H_1,H_2}(\sigma_2) \in HP(\phi_{H_1},\gamma_{H_1},G)$. Then*

$$c_{\sigma_1}(\phi_{H_1},\gamma_{H_1}) \leq c_{\sigma_2}(\phi_{H_2},\gamma_{H_2}) + \int_\gamma (H_2 - H_1)dt. \tag{3.4.3}$$

[20]A Novikov ring element $\lambda = \sum_{A\in H_2(Y_\phi)/G} p_A q^A$ is invertible if and only if the maximum in (3.4.1) is realized by a unique class $A \in H_2(Y_\phi)/G$.

Proof. Write $\Delta = \int_\gamma (H_2 - H_1)dt$. By Proposition 3.3.1(c), for each real number L we have a commutative diagram

$$\begin{array}{ccc} HP^L(\phi_{H_2}, \gamma_{H_2}, G) & \xrightarrow{\imath^L} & HP(\phi_{H_2}, \gamma_{H_2}, G) \\ \Psi^L_{H_1,H_2} \downarrow & & \downarrow \Psi_{H_1,H_2} \\ HP^{L+\Delta}(\phi_{H_1}, \gamma_{H_1}, G) & \xrightarrow{\imath^{L+\Delta}} & HP(\phi_{H_1}, \gamma_{H_1}, G). \end{array}$$

It follows that if σ_2 is in the image of the top arrow, then σ_1 is in the image of the bottom arrow. □

Remark 3.4.4. If $H_2 - H_1$ is a constant $C > 0$, then equality holds in (3.4.3):

$$c_{\sigma_1}(\phi_{H_1}, \gamma_{H_1}) = c_{\sigma_2}(\phi_{H_2}, \gamma_{H_2}).$$

This follows from the definitions and Proposition 3.3.1(d).

Corollary 3.4.5. *(a) The definition of $c_\sigma(\phi, \gamma)$ has a unique extension to the case where ϕ is degenerate*[21] *such that the following continuity property holds: Let $\sigma \in HP(\phi, \gamma, G)$, let $H_1, H_2 : Y_\phi \to \mathbb{R}$, and for $i = 1, 2$ let σ_i denote the corresponding class in $HP(\phi_{H_i}, \gamma_{H_i}, G)$. Then*

$$|c_{\sigma_1}(\phi_{H_1}, \gamma_{H_1}) - c_{\sigma_2}(\phi_{H_2}, \gamma_{H_2})| \le d(\gamma) \max_{Y_\phi} |H_2 - H_1|. \tag{3.4.4}$$

(b) The extended spectral invariants satisfy the conclusions of Propositions 3.4.2 and 3.4.3.

Proof. This follows from Proposition 3.4.3 and Remark 3.4.4 using the formal procedure in [65, §3.1] and [22, §2.5]. □

The spectral invariants c_σ have the following "spectrality" property.

Proposition 3.4.6. *Let ϕ be an area-preserving diffeomorphism of (Σ, ω), possibly degenerate, and suppose that $[\phi]$ is rational. Then for any reference cycle γ, any G as in Choice 3.2.8, and any nonzero class $\sigma \in HP(\phi, \gamma, G)$, there exists a (G, γ)-anchored orbit set (α, Z) such that*

$$c_\sigma(\phi, \gamma) = \mathcal{A}(\alpha, Z).$$

Proof. To start, note that since $[\phi]$ is rational, the set of values that $[\omega] \in H^2(Y_\phi; \mathbb{R})$ takes on $H_2(Y_\phi)$ is discrete.

Suppose first that ϕ is nondegenerate. Then there are only finitely many PFH generators in the homology class $[\gamma]$. Let S denote the set of actions of (G, γ)-anchored PFH generators;

[21]Here the PFH of a degenerate map ϕ is defined by Remark 3.3.4.

then the set S is discrete. If $L < L'$ and the interval $[L, L')$ does not intersect S, then the inclusion-induced map

$$\imath^{L,L'} : HP^L(\phi, \gamma, G) \longrightarrow HP^{L'}(\phi, \gamma, G)$$

is an isomorphism, since it is induced by an isomorphism of chain complexes. It then follows from Definition 3.4.1 that $c_\sigma(\phi, \gamma) \in S$.

Suppose now that ϕ is degenerate. Let $\{H_i\}_{i\geq 1}$ be a sequence of Hamiltonians converging to 0 in C^∞ such that each ϕ_{H_i} is nondegenerate. Let σ_i denote the class in $HP(\phi_{H_i}, \gamma_{H_i}, G)$ corresponding to σ under the canonical isomorphism given by Remarks 3.3.3 and 3.3.4. By the continuity in (3.4.4), we have

$$c_\sigma(\phi, \gamma) = \lim_{i\to\infty} c_{\sigma_i}(\phi_{H_i}, \gamma_{H_i}).$$

By the nondegenerate case, for each i there exists a (G, γ_{H_i})-anchored PFH generator $(\alpha(i), Z(i))$ for ϕ_{H_i} such that $c_{\sigma_i}(\phi_{H_i}, \gamma_{H_i}) = \mathcal{A}(\alpha(i), Z(i))$. Since Σ is compact and each periodic orbit in each $\alpha(i)$ has period at most $d(\gamma)$, by the Arzela-Ascoli theorem we can pass to a subsequence so that $\alpha(i)$ converges in C^∞ to an orbit set α for ϕ. Then the distance from the sequence $\mathcal{A}(\alpha(i), Z(i))$ to the set $\{\mathcal{A}(\alpha, Z) \mid Z \in H_2(Y, \alpha, \gamma)/G\}$ limits to zero. Since the latter set is discrete by our rationality hypothesis, the sequence $(\mathcal{A}(\alpha(i), Z(i)))$ converges to an element of it. □

Remark 3.4.7. Without the hypothesis that $[\phi]$ is rational, Proposition 3.4.6 still holds if ϕ is nondegenerate, by [99, Thm. 1.4]; see also [87]. However we do not know whether Proposition 3.4.6 extends to the case where $[\phi]$ is not rational and ϕ is degenerate.

3.5 The ball packing lemma

We now prove a key fact, Lemma 3.5.2 below, which will be needed for the proofs of the main theorems. This lemma gives a relation between the PFH spectral invariants of two different Hamiltonian perturbations of ϕ.

To state the lemma, recall that a four-dimensional *Liouville domain* is a compact symplectic four-manifold (X, ω) with boundary Y such that there exists a 1-form λ on X for which $d\lambda = \omega$ and $\lambda|_Y$ is a contact form on Y. We further require that the orientation of Y given by $\lambda \wedge d\lambda$ agrees with the boundary orientation of X. We allow X to be disconnected.

If (X, ω) is a Liouville domain as above, its *alternative ECH capacities* are a sequence of real numbers

$$0 = c_0^{\mathrm{Alt}}(X, \omega) < c_1^{\mathrm{Alt}}(X, \omega) \leq c_2^{\mathrm{Alt}}(X, \omega) \leq \cdots \leq \infty$$

defined in [68]. To briefly review the definition, when (X, ω) is nondegenerate, meaning that all Reeb orbits of the contact form $\lambda|_Y$ are nondegenerate, we define

$$c_k^{\mathrm{Alt}}(X, \omega) = \sup_{\substack{J\in\mathcal{J}(\overline{X}) \\ x_1,\ldots,x_k \in X \text{ distinct}}} \inf_{u\in\mathcal{M}^J(\overline{X}; x_1,\ldots,x_k)} \mathcal{E}(u). \tag{3.5.1}$$

Here $\overline{X} = X \cup_Y ([0,\infty) \times Y)$, similarly to (3.3.8). The notation $\mathcal{J}(\overline{X})$ indicates the set of almost complex structures J on $\overline{X}$ such that J is ω-compatible on X, and on $[0,\infty) \times Y$, analogously to Definition 3.2.5, J is independent of the $[0,\infty)$ coordinate s, sends ∂_s to the Reeb vector field, and sends the contact structure $\operatorname{Ker}(\lambda)$ to itself, rotating positively with respect to $d\lambda$. Furthermore, $\mathcal{M}^J(\overline{X}; x_1, \ldots, x_k)$ denotes the moduli space of J-holomorphic curves u in $\overline{X}$ for which the domain is a compact (possibly disconnected) Riemann surface with finitely many punctures near which u is asymptotic to Reeb orbits as $s \to \infty$. We require that u is nonconstant on each component of the domain. Finally, $\mathcal{E}(u)$ denotes the sum over the punctures of u of the symplectic action (period) of the corresponding Reeb orbit.

It is shown in [68] that c_k^{Alt} has a unique C^0-continuous extension to degenerate Liouville domains. We will need the following examples. For $r > 0$, define the ball

$$B(r) = \{z \in \mathbb{C}^2 \mid \pi|z|^2 \leq r\}$$

with the restriction of the standard symplectic form on $\mathbb{C}^2 = \mathbb{R}^4$. Note that the Euclidean volume of the ball is given by

$$\operatorname{vol}(B(r)) = \frac{r^2}{2}. \tag{3.5.2}$$

It is shown in [68, Thm. 6] that the capacities of a ball are given by

$$c_k^{\mathrm{Alt}}(B(r)) = dr, \tag{3.5.3}$$

where d is the unique nonnegative integer such that

$$d^2 + d \leq 2k \leq d^2 + 3d.$$

It is also shown in [68, Thm. 6] that the capacities of a disjoint union are given by

$$c_k^{\mathrm{Alt}}\left(\coprod_{i=1}^n (X_i, \omega_i)\right) = \max_{k_1+\cdots+k_n=k} \sum_{i=1}^n c_{k_i}^{\mathrm{Alt}}(X_i, \omega_i). \tag{3.5.4}$$

For the above examples, the alternative ECH capacities c_k^{Alt} agree with the original ECH capacities c_k^{ECH} defined in [65]. The calculation in [65, Prop. 8.4] deduces from (3.5.2), (3.5.3), and (3.5.4) that if X is a finite disjoint union of balls, then[22]

$$\lim_{k\to\infty} \frac{c_k^{\mathrm{Alt}}(X)^2}{k} = 4\operatorname{vol}(X). \tag{3.5.5}$$

Returning to PFH, to simplify notation we use the following convention:

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[22]This calculation enters into the proof of the ECH Weyl law (3.1.4) for the tight contact structure on S^3.

Notation 3.5.1. If $\sigma \in HP(\phi,\gamma,G)$, and if $H : Y_\phi \to \mathbb{R}$, let $\sigma_H \in HP(\phi_H,\gamma_H,G)$ denote the class corresponding to σ under the canonical isomorphism given by Remarks 3.3.3 and 3.3.4. Write

$$c_\sigma(\phi,\gamma,H) = c_{\sigma_H}(\phi_H,\gamma_H) \in \mathbb{R}.$$

Lemma 3.5.2. *Let ϕ be an area-preserving diffeomorphism of (Σ,ω), and let $H_1, H_2 : Y_\phi \to \mathbb{R}$ with $H_1 \le H_2$. Let γ be a reference cycle for ϕ, let $\sigma \in HP(\phi,\gamma,G)$, let k be a nonnegative integer, and suppose that $U^k\sigma \neq 0$. Let (X,ω) be a compact Liouville domain and suppose there exists a symplectic embedding of (X,ω) into the cobordism M from* (3.3.7). *Then*

$$c_{U^k\sigma}(\phi,\gamma,H_1) \le c_\sigma(\phi,\gamma,H_2) + \int_\gamma (H_2 - H_1)dt - c_k^{\mathrm{Alt}}(X,\omega).$$

Remark 3.5.3. When $k=0$, Lemma 3.5.2 reduces to the inequality (3.4.3).

For most of our applications, the only case of Lemma 3.5.2 that we need is where $k=1$ and X is a ball. (Only for the proof of the Weyl law in Theorem 3.8.1 below will we need a more general case where $k>1$ and X is a disjoint union of balls.) This case of the lemma asserts that under the hypotheses of the lemma, if the ball $B(r)$ can be symplectically embedded into M, then

$$c_{U\sigma}(\phi,\gamma,H_1) \le c_\sigma(\phi,\gamma,H_2) + \int_\gamma (H_2 - H_1)dt - r. \tag{3.5.6}$$

Proof of Lemma 3.5.2. By the continuity in Corollary 3.4.5, we can assume without loss of generality, by slightly decreasing H_1 and increasing H_2 if necessary, that $H_1 < H_2$ and that ϕ_{H_1} and ϕ_{H_2} are nondegenerate. We now proceed in three steps.

Step 1. Choose J_1, J_2, and J as in Step 3 of the proof of Proposition 3.3.1. Also, fix k distinct points $z_1,\ldots,z_k \in \overline{M}$.

We claim that the composition

$$U^k \circ \Psi_{H_1,H_2} = \Psi_{H_1,H_2} \circ U^k : HP(\phi_{H_2},\gamma_{H_2},G) \longrightarrow HP(\phi_{H_1},\gamma_{H_1},G) \tag{3.5.7}$$

is induced by a (noncanonical) chain map

$$\psi : (CP(\phi_{H_2},\gamma_{H_2},G),\partial_{J_2}) \longrightarrow (CP(\phi_{H_1},\gamma_{H_1},G),\partial_{J_1})$$

with the following property: Similarly to (3.3.11), we can write ψ in the form

$$\psi \sum_{\alpha,Z} n_{\alpha,Z}(\alpha,Z) = \sum_{\beta,W}\left(\sum_\alpha \sum_{\substack{V\in H_2(Y_\phi,\alpha,\beta)\\ I(\alpha,\beta,V)=2k}} n_{\alpha,W+V}m_{\alpha,\beta,V}\right)(\beta,W), \tag{3.5.8}$$

such that:

(*) If the coefficient $m_{\alpha,\beta,V} \neq 0$, then there is a broken J-holomorphic current in $\overline{M}$ from α to β in the relative homology class V passing through the points $z_1, \ldots, z_k$.

The reason is that in the proof of the "holomorphic curves axiom" in [14], the chain map counts solutions to the Seiberg-Witten equations on $\overline{M}$ perturbed using a large multiple of the symplectic form; and as the perturbation goes to infinity, the zero set α for a sequence of Seiberg-Witten solutions converge to a holomorphic current. Here α denotes the component of the spinor in the positive imaginary eigenspace of Clifford multiplication by the symplectic form on $\overline{M}$. Similarly, as in [98] (see the review in §3.2.4), the map (3.5.7) can be induced by a chain map[23] counting Seiberg-Witten solutions where α is constrained to vanish at the points $z_1, \ldots, z_k$. As the perturbation goes to infinity, the zero sets of α for the Seiberg-Witten solutions converge to a holomorphic current passing through the points $z_1, \ldots, z_k$.

Step 2. Define Δ as in equation (3.3.3). We claim that by making suitable choices of J and $z_1, \ldots, z_k$, we can arrange for the chain map (3.5.8) to have the following property:

(**) If the coefficient $m_{\alpha,\beta,Z} \neq 0$, then the inequality (3.3.12) can be refined to

$$\int_W \omega_{\phi_{H_1}} \leq \int_{W+V} \omega_{\phi_{H_2}} + \Delta - c_k^{\mathrm{Alt}}(X, \omega).$$

To prove this, by the C^0 continuity of c_k^{Alt}, we can assume without loss of generality that (X, ω) is nondegenerate. Write $Y = \partial X$, and let λ denote the contact form on Y. We will identify X with its image under the symplectic embedding into M. We can remove X from $\overline{M}$ and attach symplectization ends to form a new completed cobordism

$$\overline{M}' = ((-\infty, 0] \times Y) \cup_Y \left(\overline{M} \setminus X\right). \tag{3.5.9}$$

Choose an almost complex structure $J_X \in \mathcal{J}(\overline{X})$ as in (3.5.1). We can then choose the almost complex structure J on M so that it glues to J_X in (3.5.9), to give a well-defined almost complex structure J' on $\overline{M}'$. We can further choose a sequence of ω_M-compatible almost structures $J(n)$ on M such that $J(n)$ agrees with J outside of X, and moreover inside of X, the boundary has a neighborhood that can be identified with $(-n, 0] \times \partial Y$ so that $J(n)$ agrees with J'.

By placing the points $z_1, \ldots, z_k$ inside X, and using a compactness argument as in [62, Lem. 3], we can arrange the following: For any sequence of broken $J(n)$-holomorphic currents as in (*), with an upper bound on $\int_{W+V} \omega_{\phi_{H_1}} - \int_W \omega_{\phi_{H_2}}$, after passing to a subsequence the following hold:

[23] If one chooses different points $z_1', \ldots, z_k'$ in $\overline{M}$, one obtains a chain homotopic chain map. One can define a chain homotopy by choosing paths η_i in $\overline{M}$ from z_i to z_i' and counting Seiberg-Witten solutions where α vanishes somewhere on η_i for each $i = 1, \ldots, k$. If one moves the points far up on the positive end of $\overline{M}$, one obtains a chain map corresponding to $\Psi_{H_1,H_2} \circ U^k$; and if one moves the points far down on the negative end of $\overline{M}$, one obtains a chain map corresponding to $U^k \circ \Psi_{H_1,H_2}$.

- The holomorphic currents in $\overline{M}$ converge on compact sets to a J'-holomorphic current in $\overline{M}'$, which on $(-\infty,0]\times\partial B_i$ is asymptotic to a finite multiset α_M of Reeb orbits.
- The homolomorphic currents in X converge on compact sets to a J_X-holomorphic curve $u\in\mathcal{M}^{J_X}(\overline{X};z_1,\ldots,z_k)$, asymptotic to a finite multiset α_X of Reeb orbits.
- If $\alpha_X\neq\alpha_M$, then there is a broken J_X-holomorphic current in $\mathbb{R}\times Y$ from α_M to α_X, as in [66, §5.3]. In particular, in all cases we have

$$\mathcal{E}(u)=\int_{\alpha_X}\lambda\le\int_{\alpha_M}\lambda. \tag{3.5.10}$$

Repeating Step 4 of the proof of Proposition 3.3.1 then shows that

$$\int_{\alpha_M}\lambda+\int_W\omega_{\phi_{H_1}}\le\int_{W+V}\omega_{\phi_{H_2}}+\Delta.$$

By the inequality (3.5.10), it follows that

$$\mathcal{E}(u)+\int_W\omega_{\phi_{H_1}}\le\int_{W+V}\omega_{\phi_{H_2}}+\Delta.$$

Since J_X and $z_1,\ldots,z_k$ were arbitrary, it follows from the definition (3.5.1) that we can choose J_X and $z_1,\ldots,z_k$ so as to replace $\mathcal{E}(u)$ by $c_k^{\mathrm{Alt}}(X,\omega)$ in the above inequality. It then follows that (**) holds with $J=J(n)$ if n is sufficiently large.

Step 3. If we make the choices as in Step 2, then for each $L\in\mathbb{R}$, the chain map (3.5.8) restricts to a chain map

$$\psi:\left(CP^L(\phi_{H_2},\gamma_{H_2},G),\partial_{J_2}\right)\longrightarrow\left(CP^{L+\Delta-c_k^{\mathrm{Alt}}(X,\omega)}(\phi_{H_1},\gamma_{H_1},G),\partial_{J_1}\right)$$

The induced map on homology fits into a commutative diagram

$$\begin{array}{ccc} HP^L(\phi_{H_2},\gamma_{H_2},G) & \xrightarrow{\imath^L} & HP(\phi_{H_2},\gamma_{H_2},G)\\ \big\downarrow & & \big\downarrow{\scriptstyle U^k\circ\Psi_{H_1,H_2}}\\ HP^{L+\Delta-c_k^{\mathrm{Alt}}(X,\omega)}(\phi_{H_1},\gamma_{H_1},G) & \xrightarrow{\imath^{L+\Delta-c_k^{\mathrm{Alt}}(X,\omega)}} & HP(\phi_{H_1},\gamma_{H_1},G).\end{array}$$

We are now done as in the proof of Proposition 3.4.3. □

3.6 From spectral gaps to periodic orbits

We now explain a mechanism for detecting the creation of periodic orbits. The following concept will be useful:

Definition 3.6.1. Let ϕ be an area-preserving diffeomorphism of (Σ, ω) and let d be an integer with $d > g$. Define the *minimal spectral gap*

$$\operatorname{gap}_d(\phi) \in [0, \infty]$$

to be the infimum, over reference cycles γ for ϕ with $d(\gamma) = d$, subgroups $G \subset \operatorname{Ker}([\omega_\phi])$, and classes $\sigma \in HP(\phi, \gamma, G)$ with $U\sigma \neq 0$, of $c_\sigma(\phi, \gamma) - c_{U\sigma}(\phi, \gamma)$.

In the above definition note that $c_\sigma(\phi, \gamma) - c_{U\sigma}(\phi, \gamma) \geq 0$ by (3.2.1). We now have the following relation between spectral gaps and creation of periodic orbits.

Proposition 3.6.2. *Let ϕ be an area-preserving diffeomorphism of (Σ, ω), and suppose that the Hamiltonian isotopy class $[\phi]$ is rational. Let $\mathcal{U} \subset \Sigma$ be a nonempty open set and let H be a $(\mathcal{U}, a, l)$-admissible Hamiltonian as in Definition 3.1.12. Let d be an integer with $d > g$, and suppose that*

$$\operatorname{gap}_d(\phi) < a. \tag{3.6.1}$$

Then for some $\tau \in [0, l^{-1} \operatorname{gap}_d(\phi)]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period $\leq d$.

Proof. We proceed in four steps.

Step 1. We first claim that for any class $\sigma \in HP(\phi, \gamma, G)$ with $U\sigma \neq 0$, and for any $\delta \geq 0$, we have

$$c_{U\sigma}(\phi, \gamma) \leq c_\sigma(\phi, \gamma, \delta H) + \delta \int_\gamma H\, dt - \min(\delta l, a). \tag{3.6.2}$$

Here we are using the convention of Notation 3.5.1 on the right hand side.

To prove (3.6.2), recall from Definition 3.1.12 that there is a disk $D \subset \mathcal{U}$ of area a and an interval $I \subset (0, 1)$ of length l such that $H \geq 1$ on $I \times D$. We can regard H as defined on Y_ϕ as in §3.3. Let M_δ denote the cobordism (3.3.7) between the mapping torus Y_ϕ and the graph of δH. Then we can symplectically embed the polydisk $P(a, \delta l)$, namely the symplectic product of two-disks of areas a and δl, into M_δ. Consequently, we can symplectically embed the ball $B(\min(\delta l, a))$ into M_δ. The inequality (3.6.2) now follows from the $k = 1$ case of Lemma 3.5.2; see Remark 3.5.3.

Step 2. Suppose now that $\delta \geq 0$ and

(*) for all $\tau \in [0, \delta]$, the map $\phi_{\tau H}$ has no periodic orbit intersecting $\mathcal{U}$ with period $\leq d$.

We claim that if γ is a reference cycle for ϕ with $d(\gamma) = d$, and if $\sigma \in HP(\phi, \gamma, G)$ is a nonzero class, then

$$c_\sigma(\phi, \gamma, \delta H) = c_\sigma(\phi, \gamma) - \delta \int_\gamma H\, dt. \tag{3.6.3}$$

To prove (3.6.3), let S denote the set of actions of (G, γ)-anchored orbit sets for ϕ. By the hypothesis (*), if $\tau \in [0, \delta]$, then the set of actions of (G, γ)-anchored orbit sets for $\phi_{\tau H}$

is $S - \tau \int_\gamma H\,dt$. Since we are assuming that $[\phi]$ is rational, it follows from Proposition 3.4.6 that the function

$$\begin{aligned} [0,\delta] &\longrightarrow \mathbb{R}, \\ \tau &\longmapsto c_\sigma(\phi,\gamma,\tau H) + \tau\int_\gamma H\,dt \end{aligned} \tag{3.6.4}$$

takes values in the set S. This function is also continuous by Corollary 3.4.5. However the set S has measure zero as in [77, Lem. 2.2]. It follows that the function (3.6.4) is constant, and this proves (3.6.3).

Step 3. We now show that if $\delta > l^{-1}\operatorname{gap}_d(\phi)$, then for some $\tau \in [0,\delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period $\le d$. Suppose to get a contradiction that (*) holds. Suppose that $d(\gamma) = d$ and that $\sigma \in HP(\phi,\gamma,G)$ satisfies $U\sigma \neq 0$. Then combining (3.6.2) and (3.6.3) gives

$$c_\sigma(\phi,\gamma) - c_{U\sigma}(\phi,\gamma) \ge \min(\delta l, a).$$

It then follows from Definition 3.6.1 that

$$\operatorname{gap}_d(\phi) \ge \min(\delta l, a).$$

Since we assumed that $\delta l > \operatorname{gap}_d(\phi)$, this contradicts the hypothesis (3.6.1).

Step 4. The proposition follows from Step 3 by replacing $\mathcal{U}$ by an open set $\mathcal{V}$ such that $\overline{V} \subset \mathcal{U}$ and H is supported in $[0,1]\times\mathcal{V}$, and using a compactness argument. □

As an example of the significance of Proposition 3.6.2, we have the following corollary, which is a PFH analogue of [26, Lem. 3.1] for ECH:

Corollary 3.6.3. *Let ϕ be an area-preserving diffeomorphism of (Σ,ω), and suppose that the Hamiltonian isotopy class $[\phi]$ is rational. Let d be an integer with $d > g$ and suppose that $\operatorname{gap}_d(\phi) = 0$. Then every point in Σ is contained in a periodic orbit of ϕ with period $\le d$. In particular, ϕ is periodic with period $\le d!$.*

Proof. It follows immediately from Proposition 3.6.2 that every nonempty open set $\mathcal{U} \subset \Sigma$ contains a periodic point of period $\le d$. It then follows from a compactness argument that every point in Σ is periodic with period $\le d$. □

3.7 Proofs of theorems

We now prove all of our theorems stated in §3.1. We begin with the following simple observation:

Lemma 3.7.1. *Suppose that $HP(\phi,\gamma,G)$ contains U-cyclic elements. Write $d = d(\gamma)$ and $A = \int_\Sigma \omega$. Then*

$$\operatorname{gap}_d(\phi) \le \frac{A}{d-g+1}.$$

Proof. We are given that equation (3.2.13) holds for some positive integer m. It follows using Proposition 3.4.2(a) that

$$\begin{aligned} mA &= c_\sigma(\phi,\gamma) - c_{U^{m(d-g+1)}\sigma}(\phi,\gamma) \\ &= \sum_{i=1}^{m(d-g+1)} \big(c_{U^{i-1}\sigma}(\phi,\gamma) - c_{U^i\sigma}(\phi,\gamma)\big). \end{aligned}$$

Since each of the summands on the right hand side is nonnegative, at least one of them must be less than or equal to $A/(d-g+1)$. □

Proof of Theorem 3.1.5. Suppose that $[\phi]$ is rational and satisfies the U-cycle property. Let $\mathcal{U} \subset \Sigma$ be a nonempty open set. We need to show that there is a C^∞ small Hamiltonian perturbation, supported in $\mathcal{U}$, of ϕ to a map having a periodic orbit intersecting $\mathcal{U}$.

Let H be a $(\mathcal{U}, a, l)$-admissible Hamiltonian. It is enough to show that for all $\delta > 0$, there exists $\tau \in [0,\delta]$ such that $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$.

Since $[\phi]$ has the U-cycle property, it follows from Lemma 3.7.1 that

$$\liminf_{d\to\infty} \operatorname{gap}_d(\phi) = 0.$$

Thus we can find $d > g$ such that $\operatorname{gap}_d(\phi) < \min(a, l\delta)$. For such d, since $[\phi]$ is rational, Proposition 3.6.2 implies that for some $\tau \in [0,\delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ of period at most d. □

As noted in §3.1.1, Corollary 3.1.8 follows from Theorem 3.1.5 and the following lemma:

Lemma 3.7.2. *Let ϕ be an area-preserving diffeomorphism of T^2, and suppose that the Hamiltonian isotopy class $[\phi]$ is rational, i.e. $[\omega_\phi]$ is a positive multiple of the image of an integral class $\Omega \in H^2(Y_\phi;\mathbb{Z})$. Then $[\phi]$ has the U-cycle property. In fact, if Γ is a positive integer multiple of* $\operatorname{PD}(\Omega)$*, then $HP(\phi,\Gamma,\operatorname{Ker}([\omega_\phi])) \neq 0$, and every nonzero element of the latter is U-cyclic of order ≤ 6.*

Proof. Let ϕ be an area-preserving diffeomorphism of (Σ,ω), of arbitrary genus for now, and assume that $[\phi]$ is rational. Since the cohomology class $[\omega_\phi] \in H^2(Y_\phi;\mathbb{R})$ is a real multiple of the image of an integral cohomology class $\Omega \in H^2(Y_\phi;\mathbb{Z})$, we can find classes $\Gamma \in H_1(Y_\phi)$ with $d(\Gamma)$ arbitrarily large such that the pair (ϕ,Γ) is monotone as in Definition 3.2.25. (Simply take Γ to the Poincaré dual of $n\Omega - c_1(E)/2$ where n is a large integer.)

For such a Γ, a result of Lee-Taubes [82, Cor. 1.3], tensored with $\mathbb{Z}/2$, asserts that if $g > 0$ and $d(\Gamma) > 2g-2$, then we have the following variant of the isomorphism (3.2.7):

$$\overline{HP}(\phi,\Gamma) \simeq \overline{HM}^{-*}(Y_\phi, \mathfrak{s}_\Gamma, c_b; \mathbb{Z}/2). \tag{3.7.1}$$

Here the left hand side is the untwisted PFH from §3.2.6. The right hand side is an instance of the "bar" version of Seiberg-Witten Floer cohomology, with the "balanced" perturbation, defined by Kronheimer-Mrowka [80, §30]. As with (3.2.7), the isomorphism (3.7.1) intertwines the U maps on both sides, as discussed in §3.2.4.

If $\Sigma = T^2$, then for Γ as above, by computations in [80, §35.3] (see the remark after [82, Cor. 1.3]), we always have $\overline{HP}(\phi, \Gamma) \neq 0$, with the graded pieces $\overline{HP}_i(\phi, \gamma)$ having rank ≤ 2, on which U acts as an isomorphism. In particular, $U^{d(\Gamma)}$ is a permutation of the set of nonzero elements in each graded piece $\overline{HP}_i(\Phi, \Gamma)$. Then there is a positive integer $m \leq 6$ such that $U^{md(\Gamma)}$ is the identity on all such groups $\overline{HP}(\phi, \Gamma)$. We are now done by Lemma 3.2.26. □

Remark 3.7.3. The paper [27], which appeared after the original version of this paper, studies U-cyclic elements in more detail and generality. In particular, the proof of [27, Thm. 1] shows that if R is any coefficient ring then

$$(1 - U^{d(\Gamma)-g+1})^{b_1(Y_\phi)+1}\overline{HM}^{-*}(Y_\phi, \mathfrak{s}_\Gamma, c_b; R) = 0.$$

In our case where $R = \mathbb{Z}/2$, it follows that $U^{m(d(\Gamma)-g+1)}$ equals the identity, where $m = b_1(Y_\phi) + 1$ when $b_1(Y_\phi)$ is odd, and $m = b_1(Y_\phi) + 2$ when $b_1(Y_\phi)$ is even. In addition, it was shown in [28], which appeared simultaneously with the original version of this paper, that

$$\overline{HM}^{-*}(Y_\phi, \mathfrak{s}_\Gamma, c_b; R) \neq 0.$$

As a result, Lemma 3.7.2 can be upgraded to assert the following: Let ϕ be an area-preserving diffeomorphism of Σ in a rational Hamiltonian isotopy class $[\phi]$. Suppose that Γ is monotone, i.e. $[\omega_\phi]$ is a multiple of $c_1(E) + 2\,\mathrm{PD}(\Gamma)$, and that $d(\Gamma) > \max\{2g-2, 0\}$. Then $HP(\phi, \Gamma, \mathrm{Ker}([\omega_\phi])) \neq 0$, and every nonzero element of the latter is U-cyclic of order $\leq m$. Note that for rational ϕ there exist monotone Γ of arbitrarily large degrees $d(\Gamma)$. In particular, every rational ϕ has the U-cycle property.

To prove Theorems 3.1.13 and 3.1.15, we first prove a more general statement:

Theorem 3.7.4. *Let ϕ be an area-preserving diffeomorphism of (Σ, ω) such that the Hamiltonian isotopy class $[\phi]$ is rational. Suppose that there exists a positive integer d_0 such that ϕ has U-cyclic elements of degree d whenever d is a positive multiple of d_0 with $d > g$. Let $\mathcal{U} \subset \Sigma$ be a nonempty open set and write $A = \int_\Sigma \omega$. Let H be a $(\mathcal{U}, a, l)$-admissible Hamiltonian. If $\delta l \leq a$, then for some $\tau \in [0, \delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period at most $d_0 k$, where*

$$k = \left\lfloor \frac{A\delta^{-1}l^{-1} + g - 1}{d_0} \right\rfloor + 1. \tag{3.7.2}$$

Proof. Write $d = kd_0$. It follows from (3.7.2) that

$$d > A\delta^{-1}l^{-1} + g - 1. \tag{3.7.3}$$

In particular, it follows from (3.7.3) that $d > g$, since $\delta l \leq a < A$. Then by Lemma 3.7.1, we have

$$\mathrm{gap}_d(\phi) \leq \frac{A}{d - g + 1}.$$

Also, it follows from the above inequalities that $\operatorname{gap}_d(\phi) < \delta l \le a$. It then follows from Proposition 3.6.2 that for some $\tau \in [0, l^{-1}\operatorname{gap}_d(\phi)] \subset [0,\delta]$, the map $\phi_{\tau H}$ has a periodic orbit intersecting $\mathcal{U}$ with period at most d. □

Proof of Theorem 3.1.13. If $\Sigma = S^2$, then by Example 3.2.22, ϕ has U-cyclic elements of degree d for all positive integers d. Thus Theorem 3.7.4 applies with $d_0 = 1$ to give the result. □

Proof of Theorem 3.1.15. This follows from Theorem 3.7.4 and Lemma 3.7.2. □

3.8 Asymptotics of PFH spectral invariants

To conclude, we now prove the following "Weyl law" for PFH spectral invariants. Under certain hypotheses, it describes how the difference in spectral invariants $c_\sigma(\phi,\gamma,H_2) - c_\sigma(\phi,\gamma,H_1)$ behaves as the degree $d(\gamma) \to \infty$.

Theorem 3.8.1. *Let ϕ be a (possibly degenerate) area-preserving diffeomorphism of (Σ,ω). Let $\{G_i\}_{i\ge1}$ be a sequence of subgroups of $\operatorname{Ker}([\omega_\phi])$, let $\{\gamma_i\}_{i\ge1}$ be a sequence of reference cycles for ϕ, and for each $i \ge 1$, let $\sigma_i \in HP(\phi,\gamma_i,G_i)$ be a nonzero class. Assume that:*

- $\lim_{i\to\infty} d(\gamma_i) = \infty$.
- *There is a positive integer m such that each class σ_i is U-cyclic of order $\le m$.*

Let $H_1, H_2 : Y_\phi \to \mathbb{R}$. Write $A = \int_\Sigma \omega$. Then

$$\boxed{\lim_{i\to\infty} \frac{c_{\sigma_i}(\phi,\gamma_i,H_2) - c_{\sigma_i}(\phi,\gamma_i,H_1) + \int_{\gamma_i}(H_2-H_1)dt}{d(\gamma_i)} = A^{-1}\int_{Y_\phi}(H_2-H_1)\omega_\phi \wedge dt.}$$

Remark 3.8.2. If ϕ is rational, then Theorem 3.8.1 is not vacuous. In this case, as explained in Remark 3.7.3, one can find a sequence of nonzero classes $\sigma_i \in HP(\phi,\gamma_i,\operatorname{Ker}([\omega_\phi]))$ with $d(\gamma_i) \to \infty$, and each σ_i is automatically U-cyclic of order $\le b_1(Y_\phi) + 2$.

Example 3.8.3. Let D be a disk with a symplectic form ω of area 1, and let ϕ be the time 1 map of a Hamiltonian $H : [0,1] \times D \to \mathbb{R}$ which vanishes on $[0,1] \times \{x\}$ when x is near ∂D. Then ϕ defines an area-preserving diffeomorphism of S^2, with a symplectic form of area 1, which is the identity on an open set. Recall from Example 3.2.19 that if $\gamma = d[S^1] \times \{x\}$, where x corresponds to a point on ∂D, then $HP(\phi,\gamma,\{0\})$ is the free Λ-module generated by classes $e_{d,0},\ldots,e_{d,d}$; and each of these classes is U-cyclic of order 1. Note that if ϕ is the identity, then each spectral invariant $c_{e_{d,i}}(\phi,\gamma) = 0$. It then follows from Theorem 3.8.1 that in general, we have

$$\lim_{d\to\infty} \frac{c_{e_{d,0}}(\phi, d[S^1]\times\{x\})}{d} = \int_{[0,1]\times D} H\omega \wedge dt. \tag{3.8.1}$$

Here the right hand side (up to constant factors depending on conventions) is the *Calabi invariant* of ϕ; see e.g. [40].

The special case of (3.8.1) where ϕ is a "monotone twist" was proved[24] by direct calculation in [21, Thm. 5.1], and this result plays a key role in the proof of the simplicity conjecture. It is also noted in [21, §7.4] that (3.8.1) implies that the Calabi invariant extends to a homomorphism defined on the group of compactly supported "hameomorphisms" of the disk. The latter statement was subsequently proved using different methods in [19, Thm. 1.4].

Proof of Theorem 3.8.1. We use a "ball packing" argument similar to [25, §3.2].

Suppose first that $H_1 < H_2$. Let X be a finite disjoint union of balls symplectically embedded in M and write $V = \mathrm{vol}(X)$.

We know that each σ_i is U-cyclic of order m_i where $m_i \le m$. By Proposition 3.4.2(a), if $k_i = m_i n_i(d(\gamma_i) - g + 1)$ where n_i is an integer, then we have

$$c_{U^{k_i}\sigma_i}(\phi, \gamma_i, H_1) = c_{\sigma_i}(\phi, \gamma_i, H_1) - A m_i n_i.$$

Then Lemma 3.5.2 gives

$$c_{\sigma_i}(\phi, \gamma_i, H_2) - c_{\sigma_i}(\phi, \gamma_i, H_1) + \int_{\gamma_i} (H_2 - H_1) dt \ge c_{k_i}^{\mathrm{Alt}}(X) - A m_i n_i. \tag{3.8.2}$$

Now choose

$$n_i = \left\lfloor \frac{d(\gamma_i)^2 V}{m_i A^2 (d(\gamma_i) - g + 1)} \right\rfloor .$$

Then it follows from (3.8.2) that

$$\begin{aligned} \liminf_{i\to\infty} \frac{c_{\sigma_i}(\phi, \gamma_i, H_2) - c_{\sigma_i}(\phi, \gamma_i, H_1) + \int_{\gamma_i}(H_2 - H_1)dt}{d(\gamma_i)} &\ge \liminf_{i\to\infty} \frac{c_{k_i}^{\mathrm{Alt}}(X) - A m_i n_i}{d(\gamma_i)} \\ &= A^{-1} V \end{aligned}$$

Here in the second line we have used (3.5.5) and the hypothesis that $d(\gamma_i) \to \infty$.

Now we can choose X to make V arbitrarily close to

$$\begin{aligned} \mathrm{vol}(M, \omega_M) &= \frac{1}{2}\int_M \omega_M \wedge \omega_M \\ &= \int_{Y_\phi} (H_2 - H_1)\omega_\phi \wedge dt. \end{aligned}$$

[24] The paper [21] writes a slightly different, but equivalent, version of (3.8.1). That paper defines spectral invariants using the variant $\widetilde{HP}(\phi, \gamma, \{0\})$ from §3.2.6, which is possible here since monotonicity holds. Our spectral invariant $c_{e_{d,i}}(\phi, d[S^1] \times \{x\})$ agrees with the spectral invariant denoted in [21] by $c_{d,2i-d}(\phi)$.

Thus we obtain

$$\liminf_{i\to\infty} \frac{c_{\sigma_i}(\phi,\gamma_i,H_2) - c_{\sigma_i}(\phi,\gamma_i,H_1) + \int_{\gamma_i}(H_2-H_1)dt}{d(\gamma_i)} \geq A^{-1}\int_{Y_\phi}(H_2-H_1)\omega_\phi\wedge dt.$$

By Remark 3.4.4, both sides of the above inequality change by the same amount if one adds a constant to H_1 or H_2. Thus the above inequality is true for any H_1 and H_2, without the hypothesis that $H_1 < H_2$. In particular, the above inequality is true with H_1 and H_2 switched, which gives

$$\limsup_{i\to\infty} \frac{c_{\sigma_i}(\phi,\gamma_i,H_2) - c_{\sigma_i}(\phi,\gamma_i,H_1) + \int_{\gamma_i}(H_2-H_1)dt}{d(\gamma_i)} \leq A^{-1}\int_{Y_\phi}(H_2-H_1)\omega_\phi\wedge dt.$$

The above two inequalities imply the theorem. □

Remark 3.8.4. By choosing the ball packings carefully, as in the proof of [69, Thm. 1.1], one can show that the rate of convergence in Theorem 3.8.1 is $O(d(\gamma_i)^{-1/2})$.

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