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AGGREGATION OF UNCERTAINTY AND MULTIVARIATE DEPENDENCE:

The Value of Pooling of Inventories under Non-Normal Dependent Demand

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Abstract

In this paper we apply emerging theories in probability and statistics to examine the value of pooling of inventories under arbitrary non-normal dependent demand structures. Eppen (1979) showed that inventory costs in a centralized system increase with the correlation between multivariate normal product demands; using multivariate stochastic orders, we generalize this statement to arbitrary distributions. We then describe methods to construct models with arbitrary dependence structure, using the copula of a multivariate distribution to capture the dependence between the components of a random vector. For broad classes of distributions with arbitrary marginals, we confirm that pooling of inventories is more valuable when demands are less positively dependent.

1. INTRODUCTION

Consider a firm having to determine inventory levels for the same product in many retail locations with stochastic demand. If inventory is centralized, as opposed to being kept at the retail outlets, the demands from all locations are pooled, so the company will face lower aggregate demand uncertainty and hence lower costs. Many variations of this "pooling effect," first analyzed by Eppen (1979) in inventory management, exist. Intuitively, this pooling effect becomes less valuable as demands are more positively dependent, but almost all such analysis so far, including Eppen (1979), has had to focus on the multivariate normal case due to the intractability of dealing with multivariate dependence under non-normal distributions. In this paper, we show how Eppen's original results can be generalized to a broad class of non-normal distributions with arbitrary marginals.

We build on recent concepts in probability theory: multivariate stochastic orders and copulae. In the first part of the paper, we discuss multivariate orders and their interrelationships, thus synthesizing the theoretical framework for comparing multivariate random variables with arbitrary dependence structures in terms of the sum-convex order, the order that is the most relevant for studying pooling of inventories with newsboy-type cost functions. We formalize the intuitive notion that inventory costs in a centralized system increase as demands are more positively dependent. In the second part of the paper, we provide examples to illustrate how the theoretical framework can be applied in cases with non-normal distributions with arbitrary marginals and a wide range of dependence structures. We use the multivariate stochastic orders to derive specific results on the effect of dependence under the sum-convex order for two broad classes of bivariate and multivariate random vectors with arbitrary marginals.

This paper is organized as follows. In Section 2, we review relevant literature in the areas of inventory pooling and in probability theory. In Section 3 we formally introduce the inventory pooling problem. In Section 4 we describe the analytical framework required for our analysis. We review several existing multivariate stochastic orders and relate them to the sum-convex order. Section 5 uses these multivariate orders to state a more general version of Eppen's (1979) result. Section 6 provides a bivariate and a multivariate application of this generalization, using copulae to model the dependence structure of a multivariate distribution. Section 7 offers conclusions and future research directions.

2. LITERATURE REVIEW

We first summarize relevant literature related to inventory pooling followed by a review of some recent work in probability theory. A large body of work has grown around various manifestations of Eppen's (1979) notion of pooling of inventories, or Eppen and Schrage's (1981) extension that includes lead times. Tagaras (1992) shows that allowing transshipments between retailers leads to similar results as including a distribution center. Recently, Dong and Rudi (2001) study the impact of correlation on price interactions under transshipment, and Netessine et al. (2000) study the impact of correlation on flexible service capacity under multivariate normal demand. Van Mieghem and Rudi (2002) further extend the analysis of pooling to "newsvendor networks."

Jönsson and Silver (1987) present an exhaustive study of the impact of changing input parameters on system performance; Gerchak and Mossman (1992) show how the order quantity and associated costs depend on the randomness parameter in a simple and highly interpretable manner. Erkip et al. (1990) find that high positive correlation among products and among

successive time periods (around 0.7) results in significantly higher safety stock than the no-correlation case. Alfaro and Corbett (2003) analyze the value of pooling under suboptimal inventory policies, and report on numerical and empirical experiments with non-normal demand data. Federgruen and Zipkin (1984) extend Eppen and Schrage's (1981) model in three important ways: finite horizon, other-than-normal demand distributions, and non-identical retailers.

The benefits of delayed product differentiation or postponement are quite similar to those of the pooling effect, referring to multiple products instead of multiple locations (Garg and Lee, 1999). Collier (1982) and Baker et al. (1986) propose models to minimize aggregate safety stock levels by using component standardization, subject to a service level constraint. More recently, Groenevelt and Rudi (2000) and Rudi (2000) have examined the interactions between the optimal inventory policy, the degree of component commonality, demand variability and correlation under bivariate distributions. Cattani (2000) showed that the benefits of risk pooling may be sufficient to offset the higher costs due to selling a universal product. Various types of postponement are studied in Lee and Tang (1997,1998) and Kapuscinski and Tayur (1999); Ho and Tang (1998) and the references therein contains further discussion of the pooling effect in the context of product variety.

So far, though, this work related to pooling of inventories has generally lacked a framework for assessing the impact of dependence on the value of pooling when demands are non-normal. Whenever dependence has been explicitly included, it has generally been in the context of bivariate or multivariate normal demands; the current paper provides a framework that could be used to generalize much of that work to the non-normal case.

We refer to work on multivariate stochastic orders and on the copula where appropriate. Recent work on multivariate orders includes Scarsini and Shaked (1996), Shaked and

Shanthikumar (1994), Scarsini (1998), Müller and Scarsini (2000), and Müller and Stoyan (2002), while Joe (1997) and Nelsen (1999) provide good overviews of theory and applications of copulae. Clemen and Reilly (1999) introduce the multivariate normal copula and discuss its use in the context of combining expert opinions. Many of the results presented here draw on this body of literature. However, the contribution of the current paper lies in the combination and application of these recent concepts from probability and statistics in order to state, in general terms, that higher positive dependence leads to higher inventory costs in a centralized system, and therefore lower benefits from pooling. In addition, we illustrate how, using the copula, one can construct wide classes of bivariate and multivariate random vectors or demand distributions with arbitrary marginals and demonstrate that higher positive dependence still leads to higher costs.

3. POOLING OF INVENTORIES

A well-known problem in inventory theory is to decide how much inventory to carry when faced with uncertain demand; the decision-maker has to trade off h, the per unit holding costs of excess inventory against p, the per unit shortage costs of not meeting all demand. For a single product i with stochastic demand x_i and associated cumulative demand distribution $F_i(x_i)$, the decision-maker's cost function $TC(q_i)$ depends on his inventory level q_i as follows: $TC(q_i) = E[h(q_i - x_i)^+] + E[p(x_i - q_i)^+]$ where $(z)^+ = \max\{0, z\}$. It is well known that the optimal order quantity is $q_i^* = F_i^{-1}(p/(p + h))$. If the firm sells the product at multiple retail locations, demand is a multivariate random variable $\mathbf{X}$ with corresponding distribution F. It is sufficient to determine the optimal inventory levels for each location independently, regardless of dependence structure, to minimize total expected costs.

However, the firm can also keep a central inventory instead of local inventories (ignoring transportation lead times), hence aggregating demand from multiple locations into one single random variable, allowing it to exploit "statistical economies of scale." This is referred to as the "pooling effect", first characterized by Eppen (1979). For the decentralized case, the problem is given by $\min_{q_i} E[\sum_{i=1}^{N} TC_D(q_i)] = \sum_{i=1}^{N} E_i[h(q_i - x_i)^+] + E_i[p(x_i - q_i)^+]$, whereas for the centralized case it is $\min_q E[TC_C(q)] = E[h(q - \sum_{i=1}^{N} x_i)^+] + E[p(\sum_{i=1}^{N} x_i - q)^+]$.

If demand follows a normal distribution $F \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with correlations ρ_{ij} between all products $i \neq j$, expected total costs for the decentralized and centralized cases are:

$$E[TC_D] = K \sum_{i=1}^{N} \sigma_i \tag{3.1}$$

$$E[TC_C] = K \sqrt{\sum_{i=1}^{N} \sigma_i^2 + 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sigma_i \sigma_j \rho_{ij}} \tag{3.2}$$

where K is a constant that depends on p and h. We can now examine the effect of dependence on total costs before and after centralization. If $\sigma_i = \sigma \; \forall i$ and $\rho_{ij} = \rho \; \forall i \neq j$, then

$$E[TC_C] = K\sigma \sqrt{N + \rho N(N-1)} \tag{3.3}$$

so the value of pooling $E[TC_D] - E[TC_C]$ is nonnegative and decreasing in ρ. If $\rho = 1$, then $E[TC_C] = E[TC_D]$; if $\rho = 0$, then $E[TC_C] = E[TC_D]/\sqrt{N}$; if $\rho = -1/(N-1)$, then $E[TC_C] = 0$. We can summarize this well-known effect of correlation on total costs as follows:

THEOREM 1 (EPPEN 1979): *When demand follows a normal distribution, total costs after pooling are increasing in all bivariate correlation coefficients ρ_{ij}.*

This is the statement that we will generalize to near-arbitrary distributions with arbitrary marginals in Section 5 using the multivariate stochastic orders from the next section. The idea of pooling has been applied in many settings, as discussed in the literature review. Almost without exception, though, that work has been confined to the multivariate or even bivariate normal case. Using the concepts gathered in this paper we can state a much more general version of this result, confirm the intuition that the benefits of centralization decrease as the individual demands become more positively dependent, and construct examples of multivariate demand distributions with arbitrary marginals and a broad range of dependence structures. We next describe the multivariate stochastic orders that are essential to generalize Theorem 1 to multivariate non-normal distributions with arbitrary dependence structures.

4. MULTIVARIATE STOCHASTIC ORDERS

In this section we introduce several multivariate stochastic orders in order to be able to formalize the notion of "more dependent" product demands. First, let us summarize some well-known univariate orders, following Shaked and Shanthikumar (1994) or Levy (1998). Let $\mathbf{X}$ and $\mathbf{Y}$ be two univariate random variables with distributions F and G respectively, for which the expectations $E[\phi(\mathbf{X})]$ and $E[\phi(\mathbf{Y})]$ exist for the classes of functions ϕ used in the definitions below. We use $\succ$ to denote weak dominance.

DEFINITIONS: $\mathbf{X} \succ_{\text{FSD}} \mathbf{Y}$ if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all increasing functions $\phi : \Re \rightarrow \Re$.

$\mathbf{X} \succ_{\text{SSD}} \mathbf{Y}$ if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all increasing concave functions $\phi : \Re \rightarrow \Re$.

$\mathbf{X} \succ_{\text{cx}} \mathbf{Y}$ if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all convex functions $\phi : \Re \rightarrow \Re$.

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Note that FSD (first-order stochastic dominance) is sufficient but not necessary for SSD (second-order stochastic dominance). One can define a concave order $\succ_{cv}$ analogously to the convex order. The concave order restricted to increasing utility functions yields the definition of SSD. Since it easy to verify that $\mathbf{X} \succ_{cv} \mathbf{Y}$ if and only if $\mathbf{X} \prec_{cx} \mathbf{Y}$, we do not explicitly discuss the concave order. Rather, in this paper, we focus on the convex order applied to the sum of random variables with arbitrary dependence structures; we call this the sum-convex order. We occasionally relate the sum-convex order to SSD, as the latter is widely used for comparing portfolios of risky assets from the perspective of a risk-averse investor.

4.1. *Multivariate Stochastic Orders*

In order to compare multivariate random vectors, we need to define appropriate multivariate stochastic orders. The multivariate version of $\succ_{cx}$ is easy to define using multivariate convex functions, but often of limited use. To study pooling of inventories, we use an order that we call the sum-convex order, closely related to the multivariate extension of SSD. Let $\mathbf{X}_i$ denote component i in random vector $\mathbf{X}$.

DEFINITIONS: let random vector $\mathbf{X}$ and $\mathbf{Y}$ have dimensions N_X and N_Y respectively. Then $\mathbf{X}$ dominates $\mathbf{Y}$ in the *sum-convex order*, written as $\mathbf{X} \succ_{scx} \mathbf{Y}$, if and only if

$$\sum_{i=1}^{N_X} \mathbf{X}_i \succ_{cx} \sum_{i=1}^{N_X} \mathbf{Y}_i .$$

$\mathbf{X}$ dominates $\mathbf{Y}$ in terms of *second-order stochastic dominance (SSD)*, written as $\mathbf{X} \succ_{SSD} \mathbf{Y}$, if and only if $\sum_{i=1}^{N_X} \mathbf{X}_i \succ_{SSD} \sum_{i=1}^{N_X} \mathbf{Y}_i$ in the univariate sense.

The relationship between $\succ_{\text{scx}}$ and $\succ_{\text{SSD}}$ is immediate:

LEMMA 1: $\mathbf{X} \succ_{\text{scx}} \mathbf{Y} \Rightarrow \mathbf{Y} \succ_{\text{SSD}} \mathbf{X}$.

For clarity and continuity of presentation, we defer all proofs of Lemmas and references to existing proofs to the Appendix. One could of course dispense with the sum-convex order by always writing $\sum_{i=1}^{N_X} \mathbf{X}_i \succ_{\text{cx}} \sum_{i=1}^{N_X} \mathbf{Y}_i$; however, it is notationally convenient to define the sum-convex order separately. Moreover, we can relate it to other multivariate orders that have been defined in the probability literature, and show that $\succ_{\text{scx}}$ is strictly weaker (and hence more general) than these existing orders. The ranking of random vectors under $\succ_{\text{scx}}$ depends on the variability of aggregate value, which depends on the variability of the individual components and on the interdependence between them. The multivariate convex order does not deal with variability and dependence in a useful way. Later we explore the notion of dependence of multivariate random variables in more depth; first, we need to place the sum-convex order in a larger context by reviewing several other existing multivariate stochastic orders generated by specific classes of functions ϕ. As described in the next section, this is essential to generalize Theorem 1 to multivariate non-normal distributions with arbitrary dependence, and also provides a basis to generalize other work related to pooling to such distributions. Following Müller and Scarsini (2001), define the difference operator $\Delta_i^\varepsilon \phi(\mathbf{X}) := \phi(\mathbf{X} + \varepsilon \mathbf{\imath}_i) - \phi(\mathbf{X})$ where $\mathbf{\imath}_i$ is the i-th unit vector in $\mathfrak{R}^N$ and $\varepsilon > 0$.

DEFINITIONS: a function $\phi : \Re^N \to \Re$ is *supermodular* if $\Delta_i^\varepsilon \Delta_j^\delta \phi(\mathbf{X}) \geq 0$ for all $\mathbf{X} \in \Re^N$, $1 \leq i < j \leq N$, $i \neq j$, and all $\varepsilon, \delta \geq 0$.

A function $\phi : \Re^N \to \Re$ is *directionally convex* if $\Delta_i^\varepsilon \Delta_j^\delta \phi(X) \geq 0$ for all $\mathbf{X} \in \Re^N$, $1 \leq i, j \leq N$, and all $\varepsilon, \delta \geq 0$.

A function $\phi : \Re^N \to \Re$ is *componentwise convex* if $\Delta_i^\varepsilon \Delta_i^\delta \phi(X) \geq 0$ for all $\mathbf{X} \in \Re^N$, $1 \leq i \leq N$, and all $\varepsilon, \delta \geq 0$.

Directional convexity does not imply and is not implied by the usual notion of convexity. The following lemma provides the relationship between these three definitions.

LEMMA 2: ϕ *is directionally convex* $\Leftrightarrow \phi$ *is supermodular and componentwise convex.*

We can now define the corresponding stochastic orders:

DEFINITIONS: $\mathbf{X}$ dominates $\mathbf{Y}$ in the *supermodular order*, written as $\mathbf{X} \succ_{sm} \mathbf{Y}$, if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all supermodular functions ϕ.

$\mathbf{X}$ dominates $\mathbf{Y}$ in the *directionally convex order*, written as $\mathbf{X} \succ_{dcx} \mathbf{Y}$, if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all directionally convex functions ϕ.

$\mathbf{X}$ dominates $\mathbf{Y}$ in the *componentwise convex order*, written as $\mathbf{X} \succ_{ccx} \mathbf{Y}$, if and only if $E[\phi(\mathbf{X})] \geq E[\phi(\mathbf{Y})]$ for all componentwise convex functions ϕ.

The following lemmas summarize existing relationships between the orders introduced previously; Figure 1 also summarizes these relationships.

LEMMA 3: $\mathbf{X} \succ_{ccx} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{dcx} \mathbf{Y}$.

LEMMA 4: $\mathbf{X} \succ_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{dcx} \mathbf{Y}$ and $\mathbf{X}_i =_{st} \mathbf{Y}_i \ \forall i$.

LEMMA 5: $\mathbf{X} \succ_{ccx} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{cx} \mathbf{Y}$.

LEMMA 6: $\mathbf{X} \succ_{dcx} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{scx} \mathbf{Y}$.

The following order, from Scarsini (1998), is useful in the context of portfolio optimization.

DEFINITION: $\mathbf{X}$ dominates $\mathbf{Y}$ in the *positive linear convex order*, written as $\mathbf{X} \succ_{plcx} \mathbf{Y}$, if and only if $\mathbf{a}^{\mathrm{T}} \mathbf{X} \succ_{cx} \mathbf{a}^{\mathrm{T}} \mathbf{Y}$ for all $\mathbf{a}^{\mathrm{T}} \in \Re_+^N$.

The following lemmas are immediate:

LEMMA 7: $\mathbf{X} \succ_{plcx} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{scx} \mathbf{Y}$.

LEMMA 8: $\mathbf{X} \succ_{dcx} \mathbf{Y}$ and $\mathbf{X}_i =_{st} \mathbf{Y}_i \Rightarrow \mathbf{X} \succ_{plcx} \mathbf{Y}$.

LEMMA 9: $\mathbf{X} \succ_{cx} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{plcx} \mathbf{Y}$.

In Scarsini's (1998) application, the components of $\mathbf{X}$ are assets an investor can select for inclusion in a portfolio. The investor selects an optimal portfolio by choosing the appropriate vector of weights $\mathbf{a}$, the proportion of his wealth that he invests in each specific asset. In our

application, as in Eppen (1979), the firm cannot change the proportion of total sales generated by each specific product; all elements of the vector of weights $\mathbf{a}$ are equal to 1. That is why the sum-convex order is more appropriate for our purposes than the already existing $\succ_{\mathrm{plcx}}$ order. Example 1 below illustrates that the implication in Lemma 7 is strict, which means that $\mathbf{X} \succ_{\mathrm{scx}} \mathbf{Y}$ is a weaker and hence more general condition than $\mathbf{X} \succ_{\mathrm{plcx}} \mathbf{Y}$, despite being a mathematically trivial variation on the $\succ_{\mathrm{plcx}}$ order (itself a mathematically simple but useful variation of the convex order).

4.2. The sum-convex order is strictly weaker than the $\succ_{\mathrm{plcx}}$ order

The $\succ_{\mathrm{plcx}}$ order is useful for choosing between random vectors of arbitrarily weighted combinations of components. Here we show that the sum-convex order is strictly more useful for choosing between *given* random vectors than the $\succ_{\mathrm{plcx}}$ order, as it is easy to find random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X} \succ_{\mathrm{scx}} \mathbf{Y}$ but not $\mathbf{X} \succ_{\mathrm{plcx}} \mathbf{Y}$. An obvious case is $\mathbf{X}$ and $\mathbf{Y}$ with different dimensionality, as $\succ_{\mathrm{plcx}}$ is then not well defined while $\succ_{\mathrm{scx}}$ is. Example 1 gives a pair of equidimensional multivariate normal random vectors with equal marginals but different dependence structure, which can be compared in the sum-convex order but not in the $\succ_{\mathrm{plcx}}$ order (and therefore not in the supermodular order or other stronger orders).

EXAMPLE 1. Assume we have two random vectors $\mathbf{X}$ and $\mathbf{Y}$, each with $N = 4$ components, following normal distributions F and G respectively, where $F \sim N(\mathbf{\mu}, \mathbf{\Sigma}_X)$ and $G \sim N(\mathbf{\mu}, \mathbf{\Sigma}_Y)$, with $\mathbf{\mu} = \mathbf{\iota}_4$ (the four-dimensional vector with all elements equal to one). Further let:

$$\mathbf{\Sigma}_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma}_Y = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Note that both are well-defined covariance matrices as they are positive semi-definite. The aggregate values of the random vectors, defined as $\mathbf{x} = \sum_{j=1}^{4} \mathbf{X}_j$ and $\mathbf{y} = \sum_{j=1}^{4} \mathbf{Y}_j$ respectively, follow a normal distribution, with $\mathbf{x} \sim N(4,4)$ and $\mathbf{y} \sim N(4,0)$. Clearly, $\mathbf{x} \succ_{cx} \mathbf{y}$, or $\mathbf{X} \succ_{scx} \mathbf{Y}$.

However, take $\phi(\mathbf{Z}) = (\mathbf{Z}_1 + \mathbf{Z}_2 - 2)^2$, a convex function of $\mathbf{Z}$: $E[\phi(\mathbf{X})] < E[\phi(\mathbf{Y})]$ so that $\mathbf{X} \succ_{plcx} \mathbf{Y}$ cannot be true. (One can show that $\mathbf{X} \prec_{plcx} \mathbf{Y}$ is also not true.) The fact that the sum-convex order allows many more pairs of random vectors to be compared, even with unequal dimensionality, than existing orders, is what makes it useful for many applications in operations research and decision theory.

4.3. The sum-convex order and the normal distribution

Using these orders, results in the literature allow us to state two sufficient conditions for $\mathbf{X} \succ_{scx} \mathbf{Y}$ with multivariate normal distributions; we use these to obtain statements for more general distributions in Section 6.

LEMMA 10: *let $\mathbf{X}$ and $\mathbf{Y}$ be N-dimensional normal random variables with distributions F and G respectively, where $F \sim N(\mathbf{\mu}_X, \mathbf{\Sigma}_X)$ and $G \sim N(\mathbf{\mu}_Y, \mathbf{\Sigma}_Y)$. Then $\mathbf{X} \succ_{scx} \mathbf{Y}$ if $\mathbf{\mu}_X = \mathbf{\mu}_Y$ and $\mathbf{\Sigma}_X - \mathbf{\Sigma}_Y$ is positive semidefinite.*

LEMMA 11: *let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be N-dimensional normal random variables with distributions* F *and* G *respectively, where* $F \sim N(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$ *and* $G \sim N(\boldsymbol{\mu}_Y, \boldsymbol{\Sigma}_Y)$. *Denote the elements of* $\boldsymbol{\Sigma}_X$ *by* $\sigma_{X,ij}$ *and those of* $\boldsymbol{\Sigma}_Y$ *by* $\sigma_{Y,ij}$. *Then* $\mathbf{X} \succ_{scx} \mathbf{Y}$ *if* $\mathbf{X}_i =_{st} \mathbf{Y}_i$ *and* $\sigma_{X,ij} \geq \sigma_{Y,ij} \ \forall i,j$.

In fact, for the sum-convex order, a tighter necessary and sufficient condition can be formulated. We first need to recall the following lemma to summarize the relationship between the variance of a normal distribution and the univariate convex order.

LEMMA 12: *let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be univariate normal random variables. Then* $\mathbf{X} \succ_{cx} \mathbf{Y} \Leftrightarrow \sigma_\mathbf{X} \geq \sigma_\mathbf{Y}$ *and* $\mu_\mathbf{X} = \mu_\mathbf{Y}$.

Using this result, we can relate the sum-convex order to the mean-variance framework for the multivariate normal distribution:

LEMMA 13: *let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be normal random variables with* N_X *and* N_Y *dimensions and with distributions* F *and* G *respectively, where* $F \sim N(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$ *and* $G \sim N(\boldsymbol{\mu}_Y, \boldsymbol{\Sigma}_Y)$. *Denote the elements of* $\boldsymbol{\Sigma}_X$ *by* $\sigma_{X,ij}$ *and those of* $\boldsymbol{\Sigma}_Y$ *by* $\sigma_{Y,ij}$. *Then*

$$\mathbf{X} \succ_{scx} \mathbf{Y} \Leftrightarrow \sum_{i,j=1}^{N_X} \boldsymbol{\mu}_{X,i} = \sum_{i,j=1}^{N_Y} \boldsymbol{\mu}_{Y,i} \text{ and } \sum_{i,j=1}^{N_X} \sigma_{X,ij} \geq \sum_{i,j=1}^{N_Y} \sigma_{Y,ij}. \tag{4.1}$$

The characterizations in Lemma 10 and 11, based on results for the convex order and the supermodular order respectively, are clearly more restrictive than that in Lemma 13, but we will see below how Lemma 11 can be generalized to non-normal distributions, unlike Lemma 13.

Recall that Example 1 showed that $\succ_{plcx}$ is a strictly stronger condition than $\succ_{scx}$; for the normal case, we can formalize this fact as follows. Following Debreu (1959, pp. 7-8), define a preordering as a binary relation $x \succ y$ which is reflexive and transitive, and a complete preordering as a preordering in which either $x \succ y$ or $y \succ x$ must be true for all pairs (x, y). Lemma 13 and Example 1 then imply:

COROLLARY 1: *the $\succ_{scx}$ order is a complete pre-ordering on the space of multivariate normal distributions* $\mathbf{X}$ *with arbitrary dimensionality that satisfy* $E[\sum_{i=1}^{N_X} \mathbf{X}_i] = \mu$ *for some given* μ, *while the $\succ_{plcx}$ and the $\succ_{sm}$ orders are only partial pre-orderings on the space of multivariate normal distributions with equal dimensionality.*

4.4. *Multivariate Positive Dependence Orders*

In the preceding example, we focused on the multivariate normal distribution. Our objective, though, is to examine the value of pooling under arbitrary non-normal demand with arbitrary dependence structure. For non-normal distributions, dependence is far more general than just a covariance matrix, which only captures bivariate linear dependence relations. Fortunately, there exist orders that rank random variables exclusively based on their dependence. We show how such multivariate positive dependence orders relate to the orders already introduced. In Section 6 we discuss how all this can be applied to pooling of inventories. First, we need to review some definitions of dependence.

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DEFINITIONS: the *survival function* $\overline{F}$ corresponding to the multivariate distribution function F of a random vector $\mathbf{Z}$ is defined by $\overline{F}(z) = \Pr\{\mathbf{Z}_i > z_i \ \forall i = 1, \ldots, N\}$.

$\mathbf{X}$ is *more positive lower orthant dependent* (PLOD) than $\mathbf{Y}$, written as $\mathbf{X} \succ_{\text{plo}} \mathbf{Y}$, if and only if $F(z) \geq G(z) \ \forall z \in \mathfrak{R}^N$.

$\mathbf{X}$ is *more positive upper orthant dependent* (PUOD) than $\mathbf{Y}$, written as $\mathbf{X} \succ_{\text{puo}} \mathbf{Y}$, if and only if $\overline{F}(z) \geq \overline{G}(z) \ \forall z \in \mathfrak{R}^N$.

$\mathbf{X}$ is *more concordant* than $\mathbf{Y}$, written as $\mathbf{X} \succ_c \mathbf{Y}$, if and only if $\mathbf{X} \succ_{\text{plo}} \mathbf{Y}$ and $\mathbf{X} \succ_{\text{puo}} \mathbf{Y}$.

For the bivariate case, $\overline{F}(z) = 1 - F(z)$, so the PLOD, PUOD and concordance orders are all equivalent. The concordance order means that if $\mathbf{X} \succ_c \mathbf{Y}$, then the components of $\mathbf{X}$ are more likely than the components of $\mathbf{Y}$ to take on low (or high) values simultaneously. Though the concordance order is intuitively appealing, it is often difficult to verify analytically. However, any order that satisfies the nine axioms summarized in Joe (1997) and included here in the Appendix is called a multivariate positive dependence order (MPDO). The lower orthant, upper orthant and concordance orders all satisfy all nine axioms (Joe 1997, p. 39), as does the supermodular order (Müller and Scarsini 2000). For many purposes involving dependence structure, the supermodular order has proven to be the most useful. When we construct examples of random vectors with various dependence structures in Section 6, we will use the following existing relationships between the supermodular and concordance orders:

LEMMA 14: $\mathbf{X} \succ_{\text{sm}} \mathbf{Y} \Rightarrow \mathbf{X} \succ_c \mathbf{Y}$.

LEMMA 15: $\mathbf{X} \succ_c \mathbf{Y} \Rightarrow \mathbf{X} \succ_{\text{sm}} \mathbf{Y}$ only for $N = 2$.

So far, we have defined a set of multivariate stochastic orders, enabling comparisons of aggregate product demands and comparisons of dependence between non-normal multivariate distributions. We now have the results we need to formulate, in the next section, the more general version of Eppen's (1979) result (stated in Theorem 1), to multivariate non-normal distributions with arbitrary dependence structures. After that, we illustrate how one can construct such non-normal distributions with more general dependence structures in Section 6.

5. DEPENDENCE AND THE VALUE OF POOLING OF INVENTORIES

Using the concepts gathered in this paper we can generalize Theorem 1 (paraphrased from Eppen, 1979), stating that increased dependence reduces the value of pooling, to much broader classes of distributions:

THEOREM 2: *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be multivariate random variables with* N_X *and* N_Y *dimensions. Then* $\mathbf{X} \succ_{\text{scx}} \mathbf{Y}$ *implies* $\min_q E[TC_C(\mathbf{X};q)] \geq \min_q E[TC_C(\mathbf{Y};q)]$, *i.e., the cost after pooling is greater under demand* $\mathbf{X}$ *than under* $\mathbf{Y}$.

PROOF: it is easy to verify that the centralized objective function $E[TC_C(\mathbf{X};q)]$ is convex in $\sum_{i=1}^{N} \mathbf{X}_i$ for given q, so $\mathbf{X} \succ_{\text{scx}} \mathbf{Y}$ implies $E[TC_C(\mathbf{X};q)] \geq E[TC_C(\mathbf{Y};q)]$ for any q, so also $\min_q E[TC_C(\mathbf{X};q)] \geq \min_q E[TC_C(\mathbf{Y};q)]$. □

If $\mathbf{X}$ is more positively dependent than $\mathbf{Y}$ under any dependence order which implies $\mathbf{X} \succ_{scx} \mathbf{Y}$, such as the supermodular order or (for bivariate cases) the concordance order, we can use Theorem 2 and Figure 1 to conclude that the costs after pooling are greater under $\mathbf{X}$ than under $\mathbf{Y}$, which generalizes Theorem 1 to multivariate non-normal distributions with arbitrary dependence structures. Figure 1 summarizes sufficient conditions for $\mathbf{X} \succ_{scx} \mathbf{Y}$ to hold, and hence for costs after pooling to be greater under $\mathbf{X}$ than under $\mathbf{Y}$. As more orders are defined and more links between them established, more sufficient conditions can be added to the framework in Figure 1. In addition, Example 1 (in Section 4.2) shows that using $\mathbf{X} \succ_{scx} \mathbf{Y}$ is a strictly weaker (and hence more general) condition than using $\mathbf{X} \succ_{plcx} \mathbf{Y}$.

In itself, Theorem 2 is almost tautological; it immediately raises the question "when does $\mathbf{X} \succ_{scx} \mathbf{Y}$ hold for any given situation?" However, the framework provided by Theorem 2 and by the links between stochastic orders as summarized in Figure 1 allow us to examine the impact of dependence on costs in a centralized system for far more general distributions. Analogously, one can now return to other existing work on pooling of inventories, postponement of differentiation, etc., and verify which of the orders in Figure 1 apply to the objective function considered in that work. This will then show that many of those existing results can also be generalized to non-normal dependent distributions. In the next section, we define broad classes of multivariate distributions and show how Theorem 2 can be used to demonstrate that higher dependence leads to higher costs using a bivariate and a multivariate example. To do so, we need to model dependence in arbitrary non-normal multivariate distributions, for which we use the copula.

6. EXAMPLES: MULTIVARIATE DEPENDENCE AND POOLING

A relatively recent tool for capturing dependence in arbitrary multivariate distributions is the "copula". This will be useful for us in two ways. First, in constructing stochastic models, the copula allows us to combine arbitrary marginals with an arbitrary dependence structure, rather than limiting us to the few distributions with tractable dependence structures. Second, comparing two multivariate random variables with the same marginals is clearly equivalent to comparing their copulae. We illustrate both these uses with examples at the end of this section; first, we introduce the basic ideas behind the copula (see, for instance, Joe 1997).

DEFINITIONS: let $\widetilde{F}$ denote the *Fréchet class* given a set of marginal distributions; e.g., $\widetilde{F}(F_1,\ldots,F_N)$ is the class of multivariate distributions with given marginals $F_1,\ldots,F_N$.

For any multivariate distribution $F \in \widetilde{F}(F_1,\ldots,F_N)$, the *copula associated with F* is a distribution function $C:[0,1]^N \rightarrow [0,1]$ that satisfies $F(\mathbf{x}) = C\big((F_1(x_1),\ldots,F_N(x_N)\big)$ $\mathbf{x} \in \mathfrak{R}^N$. The copula $C(u_1,\ldots,u_N)$ itself is a joint distribution with uniform marginals.

Let $\mathbf{U}$ and $\mathbf{V}$ be multivariate uniform random variables with distributions C_U and C_V respectively; we will interchangeably write $\mathbf{U} \succ \mathbf{V}$ and $C_U \succ C_V$. Sklar's theorem (see for instance Clemen and Reilly 1999) guarantees that a copula always exists:

SKLAR'S THEOREM: *for any multivariate distribution $F \in \widetilde{F}(F_1,\ldots,F_N)$, the copula as defined above exists. If the F_i are all continuous, then C is unique; otherwise, C is uniquely determined on $\prod_{i=1}^{N}\mathrm{Ran}(F_i)$, where $\mathrm{Ran}(F_i)$ is the range of F_i.*

Clemen and Reilly (1999) show that, for differentiable F_i and C, the joint density can be written as $f(x_1,\dots,x_N) = \prod_{i=1}^{N} f_i(x_i)\, c\big((F_1(x_1),\dots,F_N(x_N))\big)$ where the $f_i(x_i)$ are the densities of the marginals F_i and $c = \partial^N C /[\partial F_1(x_1)\cdots\partial F_N(x_N)]$, the copula density. From the definition, it is clear that the copula is entirely general and fully captures the dependence structure inherent in any multivariate distribution F. Using the notion of the copula, we can now state results for comparisons between random vectors with equal marginals but different dependence structures.

6.1. *Comparing random vectors with equal marginals but different dependence structures*

There is an immediate link between multivariate dependence orders and the copula following Remark 5.6 in Scarsini and Shaked (1996). Let $T : \mathbf{X} \to T(\mathbf{X})$ be any transform of a multivariate random variable $\mathbf{X}$, where $T(\mathbf{X})$ has the same dimensionality as $\mathbf{X}$ and where each component $T_i(\mathbf{X})$ is an increasing transform of the marginal $\mathbf{X}_i$. A multivariate stochastic order $\succ$ is said to be invariant under increasing transforms if $\mathbf{X} \succ \mathbf{Y}$ implies $T(\mathbf{X}) \succ T(\mathbf{Y})$ for all such T.

LEMMA 16: *let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two multivariate random variables such that* $\mathbf{X}_i =_{st} \mathbf{Y}_i\ \forall i$, *with distributions F and G and corresponding copula's* C_X *and* C_Y *respectively. Then for all orders* $\succ$ *which are invariant under increasing transforms,* $\mathbf{X} \succ \mathbf{Y}$ *if and only if* $C_X \succ C_Y$.

The condition of Lemma 16 follows from Axioms 7 and 8 for multivariate positive dependence orders in the Appendix, which yields:

CorOLLARY 2: *the conditions of Lemma 16 are satisfied for all multivariate positive dependence orders that satisfy Axioms 1-9, so for all MPDOs one can interchangeably compare the distributions or their copulae.*

The supermodular order is an MPDO, but orders such as the convex order are *not* dependence orders, and it is easy to find examples in which $C_X \succ_{cx} C_Y$ but not $\mathbf{X} \succ_{cx} \mathbf{Y}$. In light of Lemma 16, one can model the dependence structure of a random vector using its copula and use multivariate dependence orders to assess the effect of increasing dependence. We illustrate this procedure in Section 6.2.

6.2 Constructing Multivariate Distributions with Arbitrary Marginals

Here we present two examples of a copula and its relationship to the multivariate orders discussed so far. Let the $\mathbf{X}_i$ follow arbitrary univariate distributions F_i respectively; the F_i need not come from the same family of distributions. To model dependence, we first use bivariate Archimedean copulae and then the multivariate normal copula.

THEOREM 3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be arbitrary bivariate random variables with distributions* $F, G \in \widetilde{F}(F_1, F_2)$ *and with copulae* C_X *and* C_Y *respectively. Then* $C_X \succ_c C_Y$ *implies* $\mathbf{X} \succ_{sm} \mathbf{Y}$ *and* $\mathbf{X} \succ_{scx} \mathbf{Y}$.

PROOF: since the concordance ordering is an MPDO (Joe 1997, p. 39), $C_X \succ_c C_Y$ implies that $\mathbf{X} \succ_c \mathbf{Y}$, by Lemma 16. Using the result of Tchen (1980), and Lemmas 4, 6 and 14, we have

$\mathbf{X} \succ_c \mathbf{Y} \Rightarrow \mathbf{X} \succ_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \succ_{scx} \mathbf{Y}$ for bivariate distributions. □

This means that, presented with any two bivariate demand distributions with equal marginals, the one with the more concordant copula will lead to higher costs in a centralized system. To see how Theorem 3 can be applied, consider the class of bivariate Archimedean copulae, which is broad (Nelsen 1999 lists 22 families on pp. 94-97) and useful for several reasons: they can be constructed easily, a wide variety of families of copulae belong to this class, and they possess many nice properties. We will not define the class in general, but consider, for instance, the following specific family (our example will work with many others):

$$C_\theta(u_1, u_2) = \frac{u_1 u_2}{(1 - (1 - u_1^\theta)(1 - u_2^\theta))^{1/\theta}}, \quad 0 < \theta \leq 1 \tag{6.1}$$

The joint distribution is given by $F(X_1, X_2) = C_\theta(F_1(X_1), F_2(X_2))$; moreover, taking the limit $\lim_{\theta \downarrow 0} C_\theta(u_1, u_2) = u_1 u_2$, the product copula, so $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent.

THEOREM 4. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be bivariate random variables with distributions* $F, G \in \widetilde{F}(F_1, F_2)$ *and with copulae* $C_{\theta_1}(u_1, u_2)$ *and* $C_{\theta_2}(u_1, u_2)$, *as defined in (6.1), respectively. Then* $\theta_1 \geq \theta_2$ *implies* $\min_q E[TC_C(\mathbf{X}; q)] \geq \min_q E[TC_C(\mathbf{Y}; q)]$, *i.e., the cost after pooling is greater under demand* $\mathbf{X}$ *than under* $\mathbf{Y}$.

PROOF: Observe that $\partial C_\theta(a,b)/\partial\theta \geq 0$. Thus, $\theta_1 \geq \theta_2$ also implies that $C_{\theta_1}(u_1,u_2) \geq C_{\theta_2}(u_1,u_2)$ $\forall u_1,u_2 \in [0,1]$. By the definition of the concordance order (Nelsen 1999, p. 181), $C_{\theta_1}(u_1,u_2) \geq C_{\theta_2}(u_1,u_2)$ $\forall u_1,u_2 \in [0,1]$ implies that $C_{\theta_1} \succ_c C_{\theta_2}$. Theorem 3 then gives $\mathbf{X} \succ_{sm} \mathbf{Y}$ and $\mathbf{X} \succ_{scx} \mathbf{Y}$ for any copula with $\partial C_\theta(a,b)/\partial\theta \geq 0$. By Theorem 2, $\mathbf{X} \succ_{scx} \mathbf{Y}$ implies that the cost after pooling is greater under $\mathbf{X}$ than under $\mathbf{Y}$, so this example illustrates how higher dependence leads to higher costs after pooling for bivariate distributions with arbitrary marginals and a range of Archimedean copulae. □

For multivariate distributions, the $\succ_c$ order does not imply the $\succ_{sm}$ order, so the construction above does not work. However, we can still construct a broad class of multivariate random variables using the normal copula, discussed in Clemen and Reilly (1999); it again allows arbitrary marginals, but captures dependence exactly as the multivariate normal distribution does, using only pairwise correlations. In other words, the multivariate distribution is fully defined by the marginals F_i and the covariance matrix $\mathbf{\Sigma}$.

THEOREM 5. *Let $\mathbf{X}$ and $\mathbf{Y}$ be arbitrary multivariate random variables with distributions $F, G \in \widetilde{F}(F_1, \ldots, F_N)$ and with normal copulae C_X and C_Y, characterized by covariance matrices $\mathbf{\Sigma}_X$ and $\mathbf{\Sigma}_Y$ with elements $\sigma_{X,ij}$ and $\sigma_{Y,ij}$, respectively. Then $\sigma_{X,ij} \geq \sigma_{Y,ij}$ $\forall i,j$ implies $\mathbf{X} \succ_{sm} \mathbf{Y}$ and $\mathbf{X} \succ_{scx} \mathbf{Y}$.*

PROOF: let $\mathbf{X}$ and $\mathbf{X'}$ be multivariate random variables with copula C_X, and let $\mathbf{Y}$ and $\mathbf{Y'}$ have copula C_Y. Then $\mathbf{X'} \succ_{sm} \mathbf{Y'} \Leftrightarrow C_X \succ_{sm} C_Y \Leftrightarrow \mathbf{X} \succ_{sm} \mathbf{Y}$ because the supermodular order is a

dependence order. Now let $\mathbf{X'}$ and $\mathbf{Y'}$ be multivariate normal random variables; then Theorem 4.2 in Müller and Scarsini (2000, p. 117), used in Lemma 11 above, shows that the conditions in the theorem imply $\mathbf{X'} \succ_{sm} \mathbf{Y'}$, so also $\mathbf{X} \succ_{sm} \mathbf{Y}$, from which the rest follows.   □

As above, $\mathbf{X} \succ_{scx} \mathbf{Y}$ implies that the cost after pooling is greater under $\mathbf{X}$ than under $\mathbf{Y}$, so this example illustrates how higher dependence leads to higher costs after pooling for multivariate distributions with arbitrary marginals and a normal copula. Comparing Theorem 5 with expression (3.2) for normal distributions clearly highlights the trade-off inherent in assuming normal marginals. In both cases, the copula is normal: if the marginals are also normal, $\sigma_{X,ij} \geq \sigma_{Y,ij}$ for *any* i, j is sufficient for the costs after pooling to increase, while if the marginals are not assumed to be normal, we must have $\sigma_{X,ij} \geq \sigma_{Y,ij}$ for *all* i, j to be able to show that costs after pooling increase. It is possible that tighter conditions than this can be found, though we are not aware of any. For the bivariate case with normal copula but arbitrary marginals, Theorem 5 reduces to the statement that inventory costs in a centralized system with arbitrary marginals and a normal copula are increasing in the correlation coefficient ρ, as one would expect.

7. CONCLUSIONS

In this paper we have generalized Eppen's (1979) result, on how inventory costs after pooling increase with dependence between the individual demands, to near-arbitrary multivariate dependent demand distributions, and we have also illustrated how to construct such distributions. In doing so, we have provided a basis to extend the large literature that has sprung from that principle to more general demand distributions. Altogether, this framework allows one to address

problems of pooling of inventories without needing to resort to assumptions of independence or multivariate normality. There are many other potential areas of application of these concepts in decision theory, risk assessment, reliability, portfolio comparison and inventory theory; we hope that this paper will stimulate more work in these areas.

APPENDIX: AXIOMS FOR MULTIVARIATE POSITIVE DEPENDENCE ORDERS

AXIOM 1 (bivariate concordance): $\mathbf{X} \succ \mathbf{Y}$ implies that for any pair $1 \leq i < j \leq N$, $F_{ij}(z) \geq G_{ij}(z) \ \forall z \in \Re^2$ where F_{ij} and G_{ij} are the bivariate margins.

AXIOM 2 (transitivity): $\mathbf{X} \succ \mathbf{Y}$ and $\mathbf{Y} \succ \mathbf{Z}$ imply $\mathbf{X} \succ \mathbf{Z}$.

AXIOM 3 (reflexivity): $\mathbf{X} \succ \mathbf{X}$.

AXIOM 4 (equivalence): $\mathbf{X} \succ \mathbf{Y}$ and $\mathbf{Y} \succ \mathbf{X}$ imply $\mathbf{X} =_{st} \mathbf{Y}$.

AXIOM 5 (upper bound): $\mathbf{X}_U \succ \mathbf{X}$ for all random vectors $\mathbf{X}$, where $\mathbf{X}_U$ is the Fréchet upper bound (defined in Section 7).

AXIOM 6 (invariance to limit in distribution): $\mathbf{X}^n \succ \mathbf{Y}^n$ for $n = 1,2,\ldots$, and $\mathbf{X}^n \to_d \mathbf{X}$ and $\mathbf{Y}^n \to_d \mathbf{Y}$ as $n \to \infty$ imply that $\mathbf{X} \succ \mathbf{Y}$.

AXIOM 7 (invariance to order of indices): $(\mathbf{X}_1,\ldots,\mathbf{X}_n) \succ (\mathbf{Y}_1,\ldots,\mathbf{Y}_n)$ implies $(\mathbf{X}_{\pi(1)},\ldots,\mathbf{X}_{\pi(N)}) \succ (\mathbf{Y}_{\pi(1)},\ldots,\mathbf{Y}_{\pi(N)})$ for all permutations π of $(1,\ldots,N)$.

AXIOM 8 (invariance to increasing transforms): $(\mathbf{X}_1,\ldots,\mathbf{X}_n) \succ (\mathbf{Y}_1,\ldots,\mathbf{Y}_n)$ implies $(a(\mathbf{X}_1),\ldots,\mathbf{X}_n) \succ (a(\mathbf{Y}_1),\ldots,\mathbf{Y}_n)$ for all strictly increasing functions a.

AXIOM 9 (closure under marginals): $(\mathbf{X}_1,\ldots,\mathbf{X}_n) \succ (\mathbf{Y}_1,\ldots,\mathbf{Y}_n)$ implies $(\mathbf{X}_{\pi(1)},\ldots,\mathbf{X}_{\pi(n)}) \succ (\mathbf{Y}_{\pi(1)},\ldots,\mathbf{Y}_{\pi(n)})$ for all $n < N$.

APPENDIX: PROOFS OF LEMMAS

PROOF OF LEMMA 1: $\mathbf{X} \succ_{scx} \mathbf{Y} \Rightarrow \mathbf{Y} \succ_{cv} \mathbf{X} \Rightarrow \mathbf{Y} \succ_{icv} \mathbf{X} \Rightarrow \mathbf{Y} \succ_{SSD} \mathbf{X}$, where $\succ_{icv}$ is the "increasing concave order" defined by Scarsini (1998).

PROOF OF LEMMA 2: follows from the definitions of supermodular, componentwise convex and directionally convex functions.

PROOF OF LEMMA 3: see Müller and Scarsini (2001, p. 728).

PROOF OF LEMMA 4: follows from combining Theorem 2.5 in Müller and Scarsini (2001, p. 727) with Lemma 2.2 in Bäuerle (1997, p. 186).

PROOF OF LEMMA 5: follows from Theorem 5.A.13 in Shaked and Shanthikumar (1994, p.159).

PROOF OF LEMMA 6: it is easy to verify that the function $\phi(\sum_{i=1}^{N} \mathbf{X}_i)$ is supermodular and componentwise convex for all convex functions ϕ. The equivalence in Lemma 2 then implies it must be directionally convex, which immediately gives the desired result.

PROOF OF LEMMA 7: follows by restricting $\mathbf{a}$ to ι, the vector with all elements equal to one in the definition of the $\succ_{plcx}$ order.

PROOF OF LEMMA 8: see Theorem 5, Scarsini (1998, p. 101).

PROOF OF LEMMA 9: follows from Scarsini (1998), expression (1) on p. 96.

PROOF OF LEMMA 10: Theorem 4 in Scarsini (1998, p. 99) states that the conditions of the theorem are true if and only if $\mathbf{X} \succ_{cx} \mathbf{Y}$, from which the result follows by Lemmas 7 and 9.

PROOF OF LEMMA 11: by Theorem 4.2 in Müller and Scarsini (2000, p. 117), the conditions of the theorem are true if and only if $\mathbf{X} \succ_{sm} \mathbf{Y}$, from which the result follows trivially by Lemmas 4 and 6.

PROOF OF LEMMA 12: the definition of the convex order immediately implies $\mathbf{X} \succ_{cx} \mathbf{Y} \Rightarrow \sigma_{\mathbf{X}} \geq \sigma_{\mathbf{Y}}$, and choosing the convex functions $\phi(z) = z$ and $\phi(z) = -z$ in the definition of the convex order gives $E[\mathbf{X}] \geq E[\mathbf{Y}]$ and $-E[\mathbf{X}] \geq -E[\mathbf{Y}]$ respectively, proving the left-to-right implication. For the converse, consider the function $S^{-}(f - g)$, defined as the number of sign changes of the difference between the normal density functions f and g of $\mathbf{X}$ and $\mathbf{Y}$ respectively. Inspection shows that $f - g$ has exactly two finite zeroes whenever $\sigma_{\mathbf{X}} \geq \sigma_{\mathbf{Y}}$ and $\mu_{\mathbf{X}} = \mu_{\mathbf{Y}}$ and that the sign sequence is $\{+,-,+\}$, so by Theorem 2.A.17 in Shaked and Shantikumar (1994), $\sigma_{\mathbf{X}} \geq \sigma_{\mathbf{Y}}$ and $\mu_{\mathbf{X}} = \mu_{\mathbf{Y}} \Rightarrow \mathbf{X} \succ_{cx} \mathbf{Y}$ for the case of normal distributions. The result on SSD is Theorem 6.2 in Levy (1998, p. 192).

PROOF OF LEMMA 13: write σ_{X}^{2} for the variance of $\sum_{j=1}^{N_X} \mathbf{X}_j$ and σ_{Y}^{2} for the variance of $\sum_{j=1}^{N_Y} \mathbf{Y}_j$. For normal distributions, we know that $\sigma_{X}^{2} = \sum_{i,j=1}^{N_X} \sigma_{X,ij}$, so

$\sigma_X^2 \geq \sigma_Y^2 \Leftrightarrow \sum_{i,j=1}^{N_X} \sigma_{X,ij} \geq \sum_{i,j=1}^{N_Y} \sigma_{Y,ij}$. The result then follows immediately from Lemma 12. The result for SSD follows analogously.

PROOF OF LEMMA 14: see Müller and Scarsini (2000, p. 110).

PROOF OF LEMMA 15: see Müller and Scarsini (2000), Theorem 2.6.

PROOF OF LEMMA 16: by definition, the random variable $\mathbf{U} = (F_1(\mathbf{X}_1),\ldots,F_N(\mathbf{X}_N))$ has distribution C_X, so that $\mathbf{U}_i = F_i(\mathbf{X}_i)$. Define the inverse $F_i^{-1}(\mathbf{U}_i)$ appropriately to ensure existence. Then $C_X(\mathbf{U}) = (F_1(F_1^{-1}(\mathbf{U}_1)),\ldots,F_N(F_N^{-1}(\mathbf{U}_N)))$ and $F(\mathbf{X}) = C_X(F_1(\mathbf{X}_1),\ldots,F_N(\mathbf{X}_N))$; analogous relations hold between $\mathbf{Y}$, G, $\mathbf{V}$ and C_Y. Both $\mathbf{U}_i = F_i(\mathbf{X}_i)$ and $F_i^{-1}(\mathbf{U}_i)$ are increasing in their respective arguments, so the result follows from the assumption of invariance under increasing transforms of the marginals. □

PROOF OF LEMMA 17: Theorem 4.5 in Müller and Scarsini (2001) shows that the conditions in the above Proposition imply $\mathbf{X} \succ_{\text{dcx}} \mathbf{Y}$, which by Lemma 6 implies the desired result. □

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$\mathbf{X} \succ_{sm} \mathbf{Y}$ $\Longrightarrow$ $\begin{cases} \mathbf{X} \succ_{dcx} \mathbf{Y} \\ \mathbf{X}_i =_{st} \mathbf{Y}_i \end{cases}$ $\Longrightarrow$ $\mathbf{X} \succ_{plcx} \mathbf{Y}$ $\Longrightarrow$ $\mathbf{X} \succ_{scx} \mathbf{Y}$

for any N

for N=2 only

$\mathbf{X} \succ_{c} \mathbf{Y}$

$\mathbf{X} \succ_{ccx} \mathbf{Y}$ $\Longrightarrow$ $\mathbf{X} \succ_{cx} \mathbf{Y}$

$\begin{cases} \mathbf{X}_i \succ_{cx} \mathbf{Y}_i \\ \mathbf{X}, \mathbf{Y} \text{ have common C.I. copula } C \end{cases}$ $\Longrightarrow$ $\mathbf{X} \succ_{dcx} \mathbf{Y}$
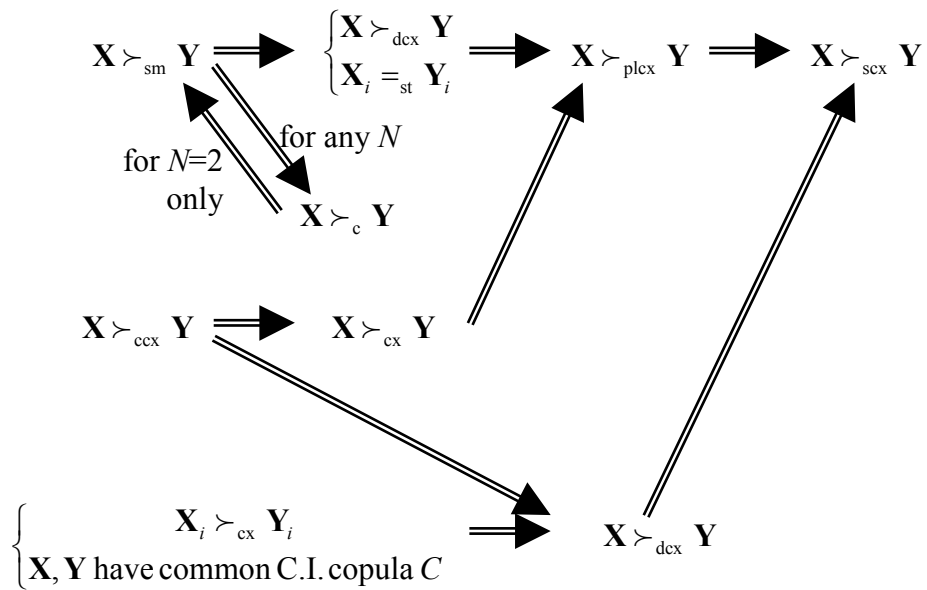
FIGURE 1: Summary of key relationships between multivariate orders

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