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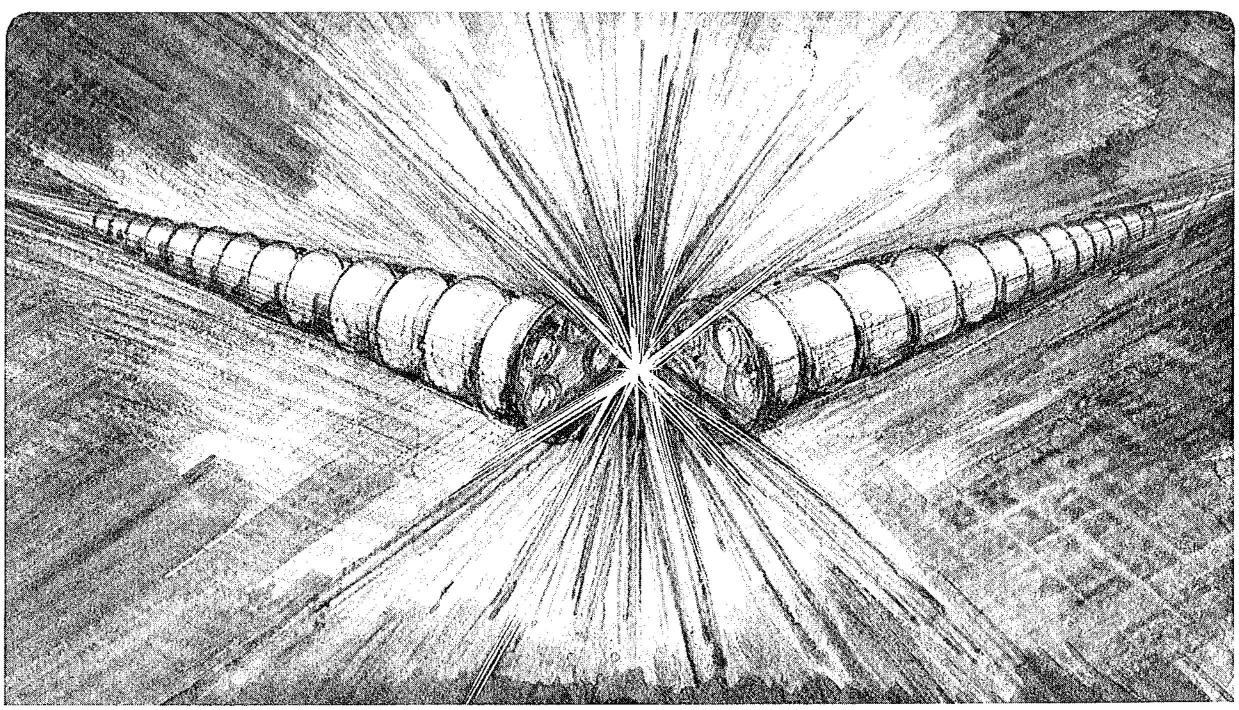
Swapan Chattopadhyay

September 1982



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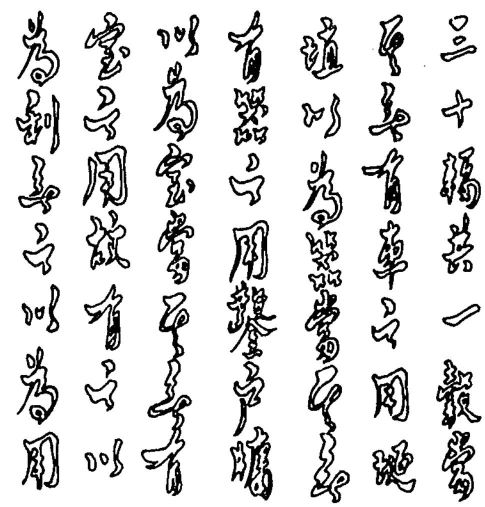
* 

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"Thirty spokes unite at the wheel's hub;
It is the center hole [literally, "from their not being"]
that makes it useful.
Shape clay into a vessel;
It is the space within that makes it useful.
Cut out doors and windows for a room;
It is the holes which make it useful.
Therefore profit comes from what is there;
Usefulness from what is not there."

- Lao-tzu

*Calligraphy done by Dr. Tai-sen Wang at the request of the author.

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ON STOCHASTIC COOLING OF BUNCHED BEAMS FROM FLUCTUATION AND KINETIC THEORY

Swapan Chattopadhyay

ABSTRACT

A theoretical formalism for stochastic phase-space cooling of bunched beams in storage rings is developed on the dual basis of classical fluctuation theory and kinetic theory of many-body systems in phase-space. The physics is that of a collection of three-dimensional oscillators coupled via retarded nonconservative interactions determined by an electronic feedback loop. At the heart of the formulation is the existence of several disparate time-scales characterizing the cooling process. Both theoretical approaches describe the cooling process in the form of a Fokker-Planck transport equation in phase-space valid up to second order in the strength and first order in the auto-correlation of the cooling signal. With neglect of the collective correlations induced by the feedback loop, identical expressions are obtained in both cases for the coherent damping and Schottky noise diffusion coefficients. These are expressed in terms of Fourier coefficients in a harmonic decomposition in angle of the generalized nonconservative cooling force written in canonical action-angle variables of the particles in six-dimensional phase-space. The formulation includes nonlinear pick-ups and kickers, multi-dimensional cooling with coupled degrees of freedom and intrinsic electronic noise of the feedback system. The effect of dynamic signal suppression arising from feedback loop induced collective correlations manifests naturally in a consistent solution of kinetic theoretic hierarchy for simple cases. For general situations, the existence of disparate time-scales allows one to use simple fluctuation theoretic results but with transport coefficients dynamically suppressed by factors determined independently from the well-known Vlasov theory. The general coupled-mode matrix for the longitudinal and transverse signal suppression for bunched beams is derived and solved in the limit of no synchrotron band overlap. The distinctive feature of synchrotron band overlap in the bunched beam Schottky signal for a higher bandwidth-system is discussed. The signal suppression matrix describing the tensorial collective response of a coasting beam with coupled transverse cooling is also derived. Comparison of analytic results to a numerical simulation study with 90 pseudo-particles in a model cooling system is presented. Estimates

of transverse cooling rates for bunches in a prototype high-energy storage ring with typical large bandwidth feedback systems are provided.

1. PROLOGUE

Stochastic cooling, invented by Simon van der Meer of CERN, Switzerland in 1968 [100], is the technique of increasing the phase-space density of charged particle beams in storage rings by an electronic feedback system that can resolve and affect small microscopic samples of the phase-space of the beam. Intense particle beam sources are important research tools in general. The particular motivation that led to the conception of stochastic cooling was the desire to produce intense antiproton ($\bar{p}$) sources. Such sources allow for proton-antiproton colliding beam physics experiments with sufficient luminosity and center-of-mass energy to cross the threshold for the creation and laboratory manifestiation of the much anticipated massive (80-90 GeV) Intermediate Vector Bosons ($Z^0, W^{\pm}$). These bosons are believed to mediate the weak interaction between particles and to be the source of the "weak neutral current," discovered at CERN, Switzerland and Fermilab, U.S.A. in 1973 [44].

Productive experiments with opposed beams of matter and antimatter in a storage ring require both beams to be dense enough to ensure a large number of collisions or a high event rate. For p-$\bar{p}$ physics, this implies that one has to accumulate a dense enough bunch of antiprotons. Unlike protons, antiprotons are not readily available from any natural source; they must themselves be created in high-energy collisions. Typically antiprotons are created by colliding a beam of high-energy protons against a metal target and then steered magnetically into a specially designed storage ring, called the Accumulator Ring. The production process is extremely inefficient; on the average every million or so high-energy protons striking a target produces one relatively low-energy antiproton. According to a simple estimate for CERN ([36], [92], [96]), one must collect bunches of antiprotons (and protons) each made up of at least 10^{11} particles in order to obtain a useful number of proton-antiproton collisions in the colliding beam machine at CERN. Collection and "stacking" of successive bunches of antiprotons every 2.4 seconds leads to an accumulation rate of 5×10^{11} $\bar{p}$/day. Thanks to the relatively long (at least 32 hours in its rest frame) lifetime of the antiprotons ([2], [20]), it is thus feasible to gather enough antiprotons to do effective p-$\bar{p}$ physics, provided one is willing to wait about a day in accumulating the $\bar{p}$'s.

However, storing a large number of individual $\bar{p}$ pulses into a relatively small phase-space volume, determined by the phase-space acceptance of the storage ring, poses an extremely difficult problem. Antiprotons emerge from the target with a range of velocities and directions. Viewed in their own frame of reference the antiprotons form a gas and their random motions define a certain kinetic temperature. If this temperature is too high, some of the particles will strike the walls of the accelerator and the beam will be depleted. Thus the transverse temperature $T_\perp$ must be reduced. The average transverse temperature of antiprotons produced by proton beams is

$$kT_\perp = \frac{(p_\perp)^2}{2m_p}$$

where

$$\langle p_\perp \rangle \sim 300 \text{ MeV/c}$$

and thus

$$(kT_\perp) \simeq 5 \times 10^6 \text{ eV}.$$

The typical transverse temperature accepted by a high-energy storage ring is [36]

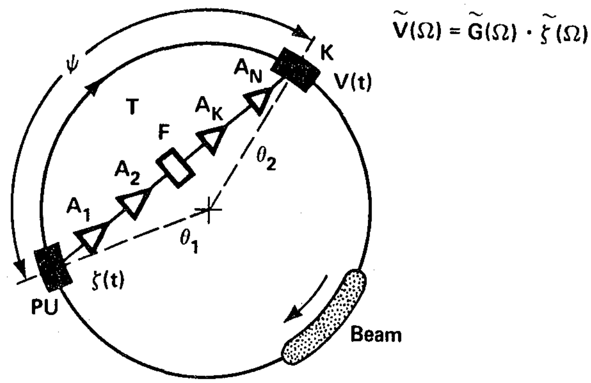
$$(kT_\perp) \simeq 1.2 \times 10^4 \text{ eV}.$$

Moreover, after a few injections of antiprotons, nearly any conceivable storage ring will have its phase-space completely filled. One thus needs to "cool" or compress the antiproton beam in phase-space (i.e. to reduce its random motions) in order to keep it as concentrated as possible before it enters the main accelerator and collider ring. This was the original motivation for phase-space cooling.

Nature provides us with a process of dissipating the extra thermal kinetic energy of charged particle beams circulating in a storage ring in the form of synchrotron radiation, which helps in overcoming beam degradation and increasing the luminosity ([21], [55], [58], [91]). This natural process is extremely efficient for lighter mass particles like electrons and positrons [94]. Significant contributions to particle physics have been made by many electron-positron storage rings around the world and still larger ones are being designed (LEP at CERN). However, by the same token, this kind of device is limited eventually by the rapid increase with energy of the radio frequency power which is needed to compensate synchrotron radiation that slows the particles down in longitudinal directed collisional momentum. Heavy-particle $(p\text{-}\bar{p})$ storage rings at very high energies are not limited by synchrotron radiation power. However, one is then forced to face the problem of inventing an artificial external dissipitative process, to keep the beams highly dense in phase-space, required to ensure a sufficiently high event rate.

One such external dissipative process, designed to increase the phase-space density in heavy-particle beams where there is no significant synchrotron radiation damping, is known as "electron cooling," proposed by G. I. Budker of Novosibirsk, U.S.S.R. in 1966 [23]. In this scheme an electron beam moves parallel to a heavy-particle beam at the same longitudinal speed. Coulomb interactions damp the motion of the heavy particle beam, because the light electrons carry most of the energy away from each Coulomb scattering with a heavy particle. In the language of statistical thermodynamics, the beams can be described by temperature and entropy as well. Thus laws of thermodynamics apply when two beams are brought together. The electron beam has lower longitudinal and transverse temperatures than the heavy particle beam and the latter will be "cooled" as the two beams relax to a temperature equilibrium. Electron cooling is very rapid at proton or antiproton energies of a few hundred MeV or less, but the cooling rate falls off rapidly with energy ([22], [70]).

Another dissipative process, based on active external intervention through an electronic feedback system, was conceived at CERN and has come to be known at Stochastic Cooling. Stochastic Cooling is the damping of transverse betatron oscillations and longitudinal momentum spread or synchrotron oscillations of a particle beam by a feedback system. In its simplest form (Fig. 1 below), a pick-up electrode (sensor) detects the transverse positions or momenta and longitudinal momentum deviation of particles in a storage ring and the signal produced is amplified and applied downstream to a kicker electrode, which produces electromagnetic fields that deflect the particles, in general in all three directions. The time delay of the cable and electronics is designed to match the transit time of particles along the arc of the storage ring between the pick-up and kicker so that an individual particle receives the amplified version of the signal it produced at the pick-up. If there were only a single particle in the ring, it is obvious that betatron oscillation and momentum off-set (or synchrotron oscillation for a bunched beam) could be damped. However, in addition to its own signal, a particle receives signals from other beam particles (Schottky noise), since more than one particle will be in the pick-up at any time. In the limit of an infinite number of particles, no damping could be achieved; we have Liouville's theorem with constant density of the phase-space fluid. For a finite, albeit large number of particles, there remains a residue of the single particle damping which is of practical use in accumulating low phase-space density beams of particles such as antiprotons.



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T = Signal Delay = Mean particle transit time from PU to K

G:
- $A_1, A_2, \ldots, A_N$: Amplifiers
- PU: Pick-up Electrode Array (Current transformer <u>or</u> position electrode)
- K: Kicker Electrode Array (Accelerating cavity or deflecting magnet)
- F: Filter(s) (for Momentum Cooling)

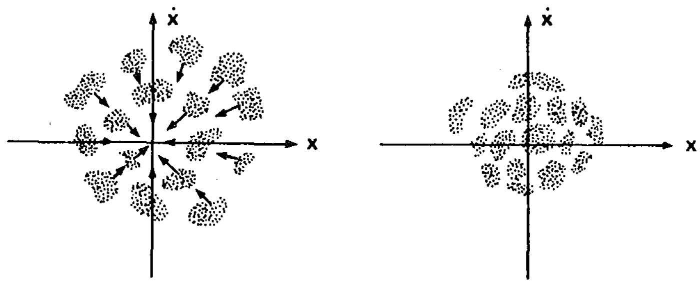
$\psi(\theta_2, \theta_1)$: Betatron Phase-Advance from θ_1 to θ_2

$$\left(= \frac{2n+1}{2}\,\pi \quad \text{for transverse cooling}\right)$$

Typical Stochastic Cooling Feedback Loop in a Storage Ring

Fig. 1

A real beam with a finite number of particles is practically empty almost everywhere in phase-space, with zero mathematical volume (point set of measure zero), as opposed to a continuous fluid (with finite measure). The finite number of particles gives rise to small but non-negligible statistical fluctuations in the phase-space density, especially for small samples of the rather grainy phase-space. These fluctuations can be used to obtain information about the average phase-space coordinates of the small sample. This information can then be used to exchange empty phase volume with volume containing particles in such a way that the latter are concentrated into a smaller phase-volume. This is in principle what is done by stochastic cooling (Fig. 2).



Phase–Space Cooling in Any One Dimension

Fig. 2

The information about the individual particles is essential and the rate at which it can be acquired determines the cooling rate. The feedback system acts to correct on each revolution the mean deviation in some property (say the transverse position $\langle x \rangle$ or longitudinal momentum $\langle \Delta p \rangle$ of small circumferential sections of beam particles. The sample length is determined by the resolution (rise time T_s) or the bandwidth $W = 1/T_s$ of the system. The quantity $NT_s/T_0 = Nf_0/W$ is a measure of the size or population of the sample of particles treated by the feedback system (T_0 = revolution period of nominal particle and N = total number of particles in the beam). The larger the number of particles in a sample, the less precise is the information about the phase-space co-ordinate of a single particle and hence a lower cooling rate. The bandwidth W and the total number of particles N are thus critical to the cooling rate.

For a single pass, we have in general an insignificant amount of cooling because of the quite small signals induced in nondestructive sensing devices and because of the relatively large sample size obtained with the available bandwidth of practical amplifiers (a few GHz). At present, the method therefore is useful only for circulating beams with repeated interactions, e.g. storage rings.

For effective cooling, however, one needs an all important 'stirring process' in phase-space -- kinematic mixing. For zero spread in the azimuthal velocities of the particles, cooling would stop once the average sample errors are corrected. However, due to the spread in revolution frequencies, particles slip away from each other in phase space and migrate between samples (mixing). The error will reappear, and correction continues until ideally all particles have zero error. It is important however that there be little mixing during the beam's passage between the pick-up and kicker, which will introduce undesirable phase-shifts in the Fourier frequency components of the kicker voltage; i.e., the observed sample will change before being manipulated.

While the particle's own pulse is correlated to the particle's arrival time at the kicker and so can accelerate or decelerate the particle, the pulses of the other particles are uncorrelated with the arrival time of the particle and so their effect only adds up in mean square. This causes the particle to diffuse. Since a particle can, on the average, be accelerated or decelerated only by signals at multiples of its own revolution frequency, the diffusion in beams with small revolution frequency spread is due only to those other particles which have the same revolution frequency as the diffusing particle (i.e. non-overlapping resonances, where $n\omega(p) = m\omega(p')$ only when $n = m$ (integers) and $\omega(p) = \omega(p')$, p being the azimuthal momentum of the particle around the ring). In a beam with sufficient spread in revolution frequency, on the other hand, diffusion of a particle with revolution frequency $\omega(p)$ can be caused by a particle with frequency $\omega(p')$ if the overlapping resonance condition $n\omega(p) = m\omega(p')$ with $n \neq m$ and $p \neq p'$ is satisfied. In a given system of large bandwidth W, there may be many such resonance overlaps at high harmonics ($n\Delta\omega \geqslant \omega_0$ or $n \geqslant (\omega_0/\Delta\omega)$ where $\Delta\omega$ = revolution frequency of a nominal beam particle and $\Delta\omega$ = spread in the revolution frequencies in the beam). Diffusion of a particle then includes contributions from these overlapping resonances as well. We will find later that this incoherent blow-up or diffusion effect varies with the square of the 'gain' of the feedback system, but the coherent cooling effect with its first power. Hence it is always possible to find an optimum value for the gain for over-all cooling.

Since the beam can be bunched by voltages at the particle's revolution frequency, the cooling system can cause the particle's arrival times to become correlated -- a cooperative dielectric type effect. This process is known as the "feedback through the beam" or the "collective signal suppression". The kicker signal will induce modulations in the beam, which will propagate coherently around the beam, determined by the collective response properties of the beam. In general the effect is a collective screening or shielding of incoherent beam signals by a suppression factor, similar to the dynamical screening effect in many-body systems. Accordingly, the pick-up detects only these collectively (or dynamically) screened signals and in general both the cooling and the diffusion effects are diminished. The suppression factor is a function of the local beam phase-space distribution in time and changes as the cooling progresses. The effect can become significant at late stages of cooling with increased phase-space density. With suitable phase relationships, this effect may also cause an instability -- the collective instability induced in the beam by the feedback system.

In summary, we may say that:

Stochastic Cooling = (In-phase single particle signal feedback correction) U (Diffusion due to incoherent Schottky noise of other particles) U (Kinematic mixing) U (Collective screening or dielectric shielding of Schottky signals by the kicker induced correlations.)

This general introduction to stochastic cooling provides us now with the platform on which to pose the very specialized problem studied in this report: stochastic cooling of bunched beams. The average luminosity of p-$\bar{p}$ collisions over a long period of time depends not only on the density of particles in each beam and the frequency of their interaction, but also on the life-time of each beam in the colliding mode. In the colliding beam mode operation of a storage ring at the highest energy, the beams are usually bunched or confined within a small angular extent in the ring by external radio-frequency cavity fields in order to have increased density in configuration space and hence higher luminosity. In this high-energy colliding bunched beam mode operation, storing the bunches in the ring for several hours requires preserving the beam emittance against

(a) intrinsic electronic amplitude and phase noise in the radio-frequency bunching voltage from RF cavities causing diffusion of the beam on a time-scale of 8~10 hours ([14], [18], [41], [43], [74]).

(b) beam-beam interaction (non-nuclear!) in the colliding mode leading to a beam blow-up in phase-space typically in 8-12 hours.

(c) intra-beam and rest gas scattering of the high-density beams with a diffusion time of again 10 hours approximately [81].

[We note here a very special feature of the intra-beam scattering or multiple scattering of particles within a beam in a storage ring. Particles in a storage ring exhibit the phenomenon of 'transition' described by the off-energy function η, describing the dispersion of the angular frequency ω in the ring with respect to azimuthal momentum p:

$$\eta = \frac{p}{\omega}\frac{d\omega}{dp} = \frac{1}{\gamma^2} - \alpha = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \tag{1.1}$$

where γ is the relativistic energy factor for the particle (particle energy divided by its rest energy) and γ_t the 'transition energy' and $\alpha = 1/\gamma_t^2$ the 'momentum compaction factor' are properties characteristic of the particular storage ring. Below transition $\gamma < \gamma_t$ and $\eta > 0$, an increase in azimuthal momentum leads to increase in angular frequency ω. Above transition, $\gamma > \gamma_t$ and $\eta < 0$, an increase in azimuthal momentum leads to a decrease in angular velocity or frequency ω. This is a manifestation of the fact that at higher energies, increments in energy or momentum leads to decreasingly smaller increases in the velocity of the particle, ultimately falling short of the amount necessary to compensate for the extra time required to travel a larger equilibrium orbit path around the ring at this higher energy.

The influence of the dispersion can be neglected far below transition energy and the particles in the beam behave like the particles of a gas in a closed box, with the focusing forces playing the same role as the walls of the box. Since the collisions within a gas cannot increase the temperature, the collisions within the beam cannot, below transition, lead to an increase of the total oscillation energy and the longitudinal energy spread. One can only expect a transfer of oscillation energies between diffeent degrees of freedom. Thus there must exist an equilibrium distribution where the intra-beam scattering does not change the beam dimensions. However, the particles at the high-velocity tail of the distribution will be continuously scraped off by the walls of the chamber and will not be confined within the beam, leading to a degradation and depletion of the beam.

Above transition the situation is changed by the "negative mass" behavior of the particles, described before. The comparison with a gas in a closed box is not valid here, and it has been shown by Piwinski [81] that the total oscillation energy can increase. The behavior of the beam under collisions, can be described by a 'collisional invariant,' derived by Piwinski [81], and given by:

$$\frac{1}{\hat{b}}\left(\frac{1}{\gamma^2} - \alpha\right)\left\langle\frac{(\Delta p)^2}{p^2}\right\rangle + \left\langle x'^2\right\rangle + \left\langle z'^2\right\rangle = \text{constant.} \tag{1.2}$$

where Δp is the momentum deviation (longitudinal) and $x' = dx/ds$, $z' = dz/ds$ are the transverse betatron angles for horizontal and vertical directions respectively at location s of the distance around the ring. The factor $\hat{b}$ is 1 for bunched beams and 2 for unbunched beams. Thus if $(1/\gamma^2 - \alpha)$ is positive, i.e. below transition, the three mean values are limited. But for negative $(1/\gamma^2 - \alpha)$ the three mean values can increase so far as they do not exceed other limitations and an equilibrium distribution does not exist.

It is this mechanism of beam heating and diffusion above transition that we are concerned with in bunched beam cooling, since the high energy p-p collisions under interest will occur at energies far above the transition energy of the ring.]

So we need to cool the beams as they collide in order to simultaneously counter the growth in emittance due to effects (a), (b) and (c) above.

Note that for beam maintenance we only need to preserve the beam emittance, so a cooling system with a typical cooling time of about ten hours should be sufficient. However, one may also be interested in real cooling of bunches leading to increase in phase-space density. Also RF manipulations of beams during various stages of stacking coasting beams at low intensity in the Accumulator Ring are easier if such bunched beam cooling is practical [97].

We also note that if the time for accumulation of high density beams in the accumulator ring prior to injection in the collider is significantly less than the bunched beam life-time in the collider, the collider can be refilled with new batches of freshly prepared bunches before the bunches have degraded significantly and the luminosity is not affected by the lifetime of the bunched beam. However increased life-time in the colliding-beam mode allows for longer accumulation times and hence higher density beams at injection to the collider.

With the above motivations for bunched beam cooling, we then specifically pose the following problem:

Given a distribution $f_0(\underset{\sim}{I})$ of particles of a 'bunched beam' in action $\underset{\sim}{I}$-space and an electronic feedback loop characterized by an overall transfer function $\underset{\sim}{G}$,

(a) What are the features that distinguish bunched beam cooling from a continuous (coasting) beam cooling?

(b) What is the specific form of the time-evolution equation of the bunch distribution $f(\underset{\sim}{I},t)$?

(c) What range of cooling times could be achieved?

(d) Are there ways of improving the cooling rate?

The underlying kinematic mechanisms and the cooling dynamics experienced by the particles in a bunch in the stochastic cooling of bunched beams differ nontrivially from the situation of stochastic cooling of continuous coasting beams. These significant differences arise mainly from the topologically different longitudinal particle orbits (synchrotron oscillations) in a bunched beam (as opposed to coasting beam free-streaming particle orbits) and the spatially confined nature of bunched beams as opposed to continuous ring-filling coasting beams. These differences manifest in a qualitatively distinctive frequency-space structure of the spectrum of single particle and collective signals derived from and experienced by particles in a bunched beam. While the theory of stochastic cooling of continuous beams in circular accelerators has been extensively investigated and developed until now ([5], [6], [7], [8], [9], [25], [33], [70], [71], [86]), as will be evident from a look at the history of the subject discussed in

Chapter 2, bunched beam stochastic cooling has been a subject of less intensive study limited to qualitative preliminary analyses only ([10], [11], [48], [69], [71]).

2. PLAYERS IN THE ORDER OF THEIR APPEARANCE: A HISTORICAL REVIEW OF STOCHASTIC COOLING

Since its conception in 1968 by S. van der Meer, stochastic cooling has been the subject of much theoretical and experimental work. The reader is directed to the references for a thorough review of its development.

For long the idea of stochastic cooling was regarded as too far fetched to be practical. It was already known at the time of van der Meer's first proposal in 1972 [100] that systems of a finite number of particles might not be completely subject to gross phase-space volume invariance (Liouville's theorem for 'smoothed out' phase-space distributions) if information about individual particle orbits could be suitably processed to modify those same orbits. The means for developing such information, so-called Schottky scans, was already in use (Borer et al. 1974 [13]) at the Intersecting Storage Rings (ISR) at CERN, where fluctuation spectra of coasting beams were routinely used to measure beam properties. In addition, feedback systems had been in use for many years (Sacherer 1974 [87]) to damp coherent instabilities of intense beams interacting with their environments. The damping of individual particle motion, however, required an electronic feedback system with bandwidth large enough to resolve a relatively small (compared to the whole beam) sample of particles in the beam in phase-space. The pace of research increased in the early seventies with the knowledge of the availability of wide-band power amplifiers. The first and earliest experimental demonstration of stochastic cooling was tried and succeeded only nine years after the invention (three years after the first publication) in the ISR at CERN (Schnell 1977 [93]). Little power and bandwidth were available for those experiments, whose purpose was only to demonstrate that cooling occurred.

The inventor and the early workers had mainly emittance cooling (i.e., transverse phase-space cooling) of high intensity beams in mind in order to improve the luminosity in the ISR. A new era began in 1975 when Strolin and Thorndahl realized the importance of stochastic cooling, both in emittance and in momentum spread of low intensity antiproton beams for the purpose of stacking. Stochastic stack-cooling at low intensity is different from the original van der Meer cooling and the extension of the theory first done by Hereward and Thorndahl as well as the design of the momentum cooling hardware (Thorndahl, Carron [25,99]) are perhaps as fundamental as the original invention and the earlier feasibility studies (van der Meer, Schnell, 1972 [92,100]).

Following this broadening of the scope, Strolin and Thorndahl worked out in 1975 $\bar{p}$ collection schemes for the ISR using stacking in momentum space and Rubbia et al.

[82,83] made first proposals of the $p\text{-}\bar{p}$ scheme for the Super Proton Synchrotron (SPS) using similar techniques of stochastic cooling and accumulation. This work gave new life to the idea at a time when the ISR was routinely stacking such high proton currents that proton beam cooling became unnecessary or even impossible. Further mile-stones in 1975-78 were the invention of the 'filter method' of momentum cooling and the refinement of the theory and of the stacking schemes.

The successful conclusion of the ISR experiments encouraged the CERN workers to go ahead with much more extensive experiments on the ICE (Initial Cooling Experiment), a storage ring transformed from the muon storage ring used for the earlier (g-2) experiments. Longitudinal and transverse cooling systems with approximately 1 kW power and band-width from 100 to 180 MHz were installed for these experiments. With 7×10^7 protons of 1 GeV energy, a longitudinal mean cooling time of 15 seconds was observed. With a different circulating intensity, 3.9×10^8, horizontal and vertical mean cooling times of approximately 4 minutes were observed. Agreement between theory and experiment was good. It even provided some exciting physics results on the life-time of the anti-proton -- a minimum life-time of 32 hours was established (Carron et al. 1978 [26,27]). The ICE studies firmly established the stochastic cooling technique.

Stochastic cooling tests have been carried out with the Fermilab Cooling Ring of 135-m circumference in collaboration with LBL (Lambertson et al. 1980 [61,62]). A circulating beam of approximately 10^5 protons of 200 MeV energy was cooled by a factor of 3 longitudinally in 3 to 4 seconds and by a factor of 3 vertically in a time of the order of 1 minute. Turning on the radio-frequency bunching voltage, initial indications of positive bunched beam cooling were also obtained (G. Lambertson, private communication).

In both these experiments, observed beam life-times were compatible with beam loss caused only by large-angle single scatterings. Small-angle multiple scattering, usually the major cause of beam loss, is overcome by the cooling.

The Novosibirsk group in the U.S.S.R. has also carried out a stochastic cooling experiment on their storage ring NAP-M and report good agreement between theory and experiment [31,32,78,79].

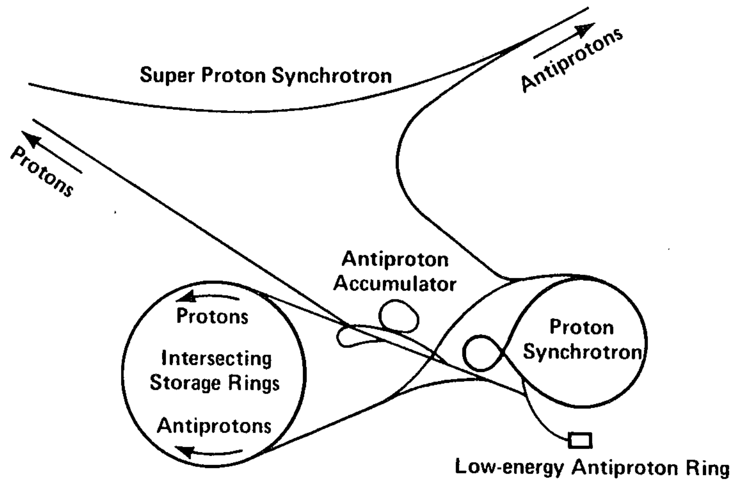
Cooling of a 'bunched' beam has also been observed in ICE at CERN and was applied for stacking of antiprotons [48]. The stored particles were tightly bunched by an RF system working at the first harmonic of the revolution frequency. Injection and RF were synchronized in such a way that the new beam could be injected onto the free part of the circumference without causing losses of the stack. The bucket height was limited to

$\Delta p/p = \pm 8.4 \times 10^{-4}$ (hardware limit) whereas the injected particles had a $\Delta p/p = \pm 2 \times 10^{-3}$. Injection took place while the RF was on. With momentum cooling the particles progressively entered the bucket and accumulated in its center until $\Delta p/p$ had reached about $\pm 3 \times 10^{-4}$ ("stochastic capture"). A similar equilibrium value had been observed with low intensity unbunched beams. With the available RF amplitude then the bunch length was about one half of the circumference. With this simple scheme the number of antiprotons could be increased by a factor of 100 leading to 15,000 stored antiprotons. These initial experiments indicated only that bunched beams can be cooled. On the other hand, the bunching ratio ($\leqslant 3$) was small, the bunch length (~ 25 m) much larger than the sample length (~ 3 m) and the number of particles low.

At present CERN has already completed development of an intense source of antiprotons and has begun initial collision experiments of p-$\bar{p}$ with 270 GeV beams. Fermilab is developing an intense $\bar{p}$ source and is scheduled to do p-$\bar{p}$ experiments with 1 TeV beam by 1985.

In the general scheme of antiproton-proton colliders at CERN and Fermilab, high-energy protons are focused on a target; the antiprotons produced are then transported to a storage ring (cooling ring) called the Accumulator Ring, which provides for cooling for the transverse and longitudinal temperatures of the antiproton beams and also provides for "stacking" of the accumulated antiprotons. Once greater than 10^{11} antiprotons are collected, they are injected into a high-energy storage ring and accelerated along with protons for antiproton-proton collisions.

For the experiments at CERN the particles are directed through a complex sequence of interconnected beam manipulating devices (Fig. 3). First a beam of protons is accelerated to an energy of 26 GeV in the Proton Synchrotron (PS), the original accelerator ring at CERN, completed in 1959. The proton beam is then directed at a copper target, producing a spray of particles, including a small number of antiprotons. Those antiprotons with an energy of 3.5 GeV and momentum spread $\Delta p/p$ of 0.7×10^{-2} are collected and transferred to a wide-aperture storage ring called the Antiproton Accumulator (AA), where they are first precooled repeatedly by the filter method, to reduce their momentum spread by a factor of 9 in two seconds. They are then moved to a slightly smaller orbit, from which they are stochastically accelerated with the accumulating stack of previously injected bunches and subjected to further cooling in all three phase planes. After a few hundred billion (~ 10^{11}) antiprotons have been collected, they are sent back to the PS ring, where they are accelerated to 26 GeV before being injected into the SPS ring in the direction counter to protons. The counter-rotating beams are finally accelerated to 270 GeV each in the SPS ring. The beams collide at intersection sites, at two of which large particle detectors are placed. The nuclear interactions are so rare that the beam life-time of several hours is not affected by them.



XBL 827-7066

CERN p-$\bar{p}$ Collider

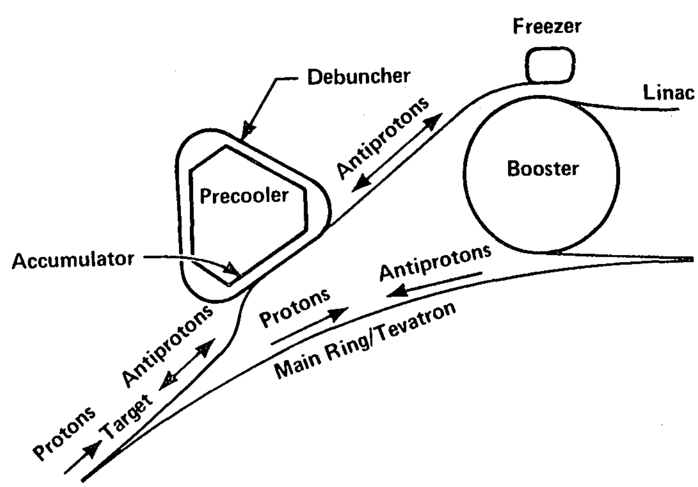
Fig. 3

The reincarnation of the CERN fixed target SPS into a p-$\bar{p}$ collider and the construction of the Antiproton Accumulator ring was completed in July 1981. Many cooling tests have been carried out with good results. The first proton-antiproton collisions at the designed peak energy of 270 GeV per beam were observed in July. By December, more than 250,000 such collisions had been recorded. Because of the comparatively low rate at which intermediate vector bosons are expected to be produced in p-$\bar{p}$ collisions, however, it was not surprising that none were detected in these early runs. The next round of experiments, with an order of magnitude or more improved beam intensity and hence collision rate, may (or may not) reveal the pot of gold!

The single-beam fixed target 1 TeV proton machine at Fermilab, the Tevatron, is scheduled to be operating in 1985, as a p-$\bar{p}$ collider, with a total center-of-mass energy of 2 TeV (2,000 GeV) as opposed to 540 GeV for CERN. The completed Fermilab machine will have the further distinction of being the first large accelerator to employ a ring of superconducting magnets.

The Fermilab Antiproton Source Design Report of February 1982 [97], describes plans for the design of Tevatron I Antiproton Source, which will meet three successive goals. The first goal is to produce p-$\bar{p}$ collisions at a peak luminosity of 10^{30} $cm^{-2}sec^{-1}$ at the maximum energy achievable with the Tevatron. Since the average luminosity of a storage ring ultimately depends on the rate at which the ring can be refilled, the filling time should be much shorter than the luminosity life-time, which is expected to be in excess of thirty hours. Hence, the second goal is to be able to refill the ring with protons and antiprotons in 5 hours or less. The third goal is to design the antiproton source so that the luminosity can be increased to 10^{31} $cm^{-2}sec^{-1}$, when either advances in cooling technology or improvements in beam life-time are made.

Again the process consists of a complex sequence of beam manipulations using the Booster Ring, two fixed energy rings, the Debuncher and the Accumulator and the Main Ring (Fig. 4). The sequence of operations is as follows: One booster-length batch



XBL 827-7063

Fermilab p-$\bar{p}$ Collider

Fig. 4

containing 80 bunches of protons is accelerated in the Main Ring to 125 GeV, followed by a time-compaction by bunch rotation in phase-space. The train of 80 shortened (less than a nano-second wide) bunches is then extracted from the Main Ring. 3×10^{12} protons in 80 bunches then strike a tungsten target producing a train of 80 antiproton bunches, which have the same narrow time-spread as the proton bunches. 1.5×10^8 antiprotons with 8.9 GeV/c are collected and transported to the Debuncher. The momentum spread of the beam is 3% and the transverse beam emittances are 20 π mm-mrad in each plane. Bunch rotation in longitudinal phase-space in the Debuncher leads to momentum compaction and a smeared-out time-structure. The antiprotons spread uniformly around the ring and then precooled transversely in both radial and vertical emittances.

The antiprotons are then extracted from the Debuncher and injected into the Accumulator. Successive batches are accumulated by RF stacking each batch at the edge of the stack.

Between the injections of successive batches onto the tail of the stack, the stack is stochastically cooled using a stack-tail cooling system similar to the type developed by CERN for the AA ring. A new batch of antiprotons with a density of about 7 $\bar{p}$'s per eV is deposited at the edge of the stack tail every 2 sec. The fresh batch is moved by the coherent force of the stochastic cooling system away from the injection channel and toward the center of the stack. The strength of the coherent force diminishes exponentially as the particles move away from the edge of the tail, causing the particle density to increase. Diffusion forces caused by the Schottky noise of the antiproton stack and the thermal noise of the amplifiers cause the anitprotons to move from a region of high density to one of lower density. As long as the coherent force is greater than the diffusion force, the stack will build up in intensity until it reaches the central region where the coherent force is zero. Some antiprotons are lost during transfer and RF stacking and some diffuse away from the stack into the chamber walls. Allowing for losses, 6×10^7 antiprotons are added to the stack with each pulse. If collection is allowed to continue for 4 hours, the core will grow to a density of 1×10^5 $\bar{p}$'s per eV. The total number of $\bar{p}$'s in the core will be 4×10^{11}. After $\bar{p}$ accumulation is complete, bunches of protons, each with at least 8×10^{10} particles, are first prepared in the Main Ring at 150 GeV, then transferred to the Tevatron. Approximately 8×10^{10} antiprotons are then extracted from the stack core, transferred to the Main Ring, accelerated to 150 GeV and transferred to the Tevatron. The p and $\bar{p}$ bunches are then accelerated to full energy and allowed to collide.

Sufficient antiprotons for a luminosity of $10^{30}/\mathrm{cm}^2$-sec can be produced in 4 hours by this scenario, even after allowing for losses in production, cooling and beam transfer. The luminosity is primarily limited by the beam stability, transfer and acceleration schemes. Improved accumulation system and longer collection times can also result in an increased luminosity. The potential luminosity of the proposed Fermilab

$\bar{p}$ source is exhibited in Table I, which shows the relationship between the accumulated $\bar{p}$'s and the luminosity.

In the arena of bunched beam cooling, the possibility of doing a small scale experiment on bunched beam stochastic cooling in the experimental cooling ring at Fermilab seems bright already. Such an experiment will demonstrate the feasibility or otherwise of high-energy bunched beam stochastic cooling in the colliding beam mode and provide much needed insight into the theory of bunched beam stochastic cooling developed in this report.

Table I. LUMINOSITY PROGRESSION

$N_{\bar{p}}$ (10^{11})	N_p (10^{11})	N_B (10^{11})	N_T (10^{11})	β^* (m)	ξ	L (10^{30} cm^{-2}sec^{-1})
0.7	0.7	1	0.7	1	0.002	0.5
0.7	0.7	3	2.1	1	0.002	1.5
1.0	1.0	3	3.0	1	0.003	3.0
1.0	1.0	6	6.0	1	0.003	6.0

$N_{\bar{p}}(N_p)$ is the number of $\bar{p}(p)$ per bunch, N_B is the number of bunches, N_T is the total number of $\bar{p}$, β^* is the value of β at center of the interaction region, ξ is the beam-beam tune shift/crossing, and L is the luminosity.

Along with the experimental work, there has been a rapid increase in the theoretical understanding of the basic process of stochastic cooling of coasting and bunched beams in particle accelerators. Principal theoretical investigators have been Frank Sacherer (until 1978) and Joseph J. Bisognano. Sacherer [86] refined the theory for stochastic cooling based on the frequency domain approach originally developed by Hereward in connection with single particle behavior and collective response of a set of harmonic oscillators perturbed by a packet of frequencies. Sacherer generalized the theory to treat both good and bad mixing situations (overlapping and non-overlapping Schottky bands), advocated the use of the Fokker-Planck transport equation to describe the process of longitudinal cooling, and studied in some detail the collective response of the beam, i.e. the phenomenon of coherent signal suppression. Bisognano [5,6,7,9] has developed an even more rigorous theory of stochastic cooling of coasting beams with no band-overlap based on the kinetic theory of reduced particle phase-space distributions and correlations (up to two-body correlations). The theory includes the coherent signal suppression effect and points to the connection with collective instabilities of the beam in the most natural way. Later, Bisognano [8] generalized the expression for coherent signal suppression to include the case of overlapping Schottky bands, based on Vlasov Plasma theory. This latter generalization has also been done by S. van der Meer [104] in an entirely new way, but with identical results.

This report aims at presenting a theoretical formulation of stochastic cooling of particle beams in a storage ring as a unified whole based on both the kinetic theory in phase-space and the fluctuation theory in the frequency space of a collection of three-dimensional oscillators described most naturally by 'action-angle' variables in phase-space and coupled to each other and to themselves by a retarded, non-hermitian (non-conservative) cooling interaction. The fundamental dichotomy between frequency-domain and time-domain analyses, characteristic of previous approaches is put into perspective clearly. The present formulation has the advantage of providing a natural generalization to a theoretical description of bunched beam cooling, which is a main concern of this report. In addition, the formulation is capable of describing full three-dimensional cooling with coupling between different degrees of freedom, the general tensorial collective response of the beam and includes nonlinear sensing and kicking devices. A detailed, although by no means complete, description of the collective response of a spatially confined (bunched) beam (i.e. coherent signal suppression) is also provided.

Table II summarizes the chronological history of stochastic cooling theory, experiment and practice up to the present.

Table II: HISTORY OF STOCHASTIC COOLING

	PREHISTORY	
Liouville	circa 1850	Invariance of phase-space measure
Schottky	1918	Noise in DC electron beams
- - - - - - - -	- - - - - - - -	- - - - - - - -
van der Meer	1968	Idea of stochastic cooling
ISR Staff (Borer, Bramham, Hereward, Hübner, Schnell, Thorndahl)	1972	Observation of proton beam Schottky noise
van der Meer	1972	Theory of emittance cooling
Schnell	1972	Engineering Studies

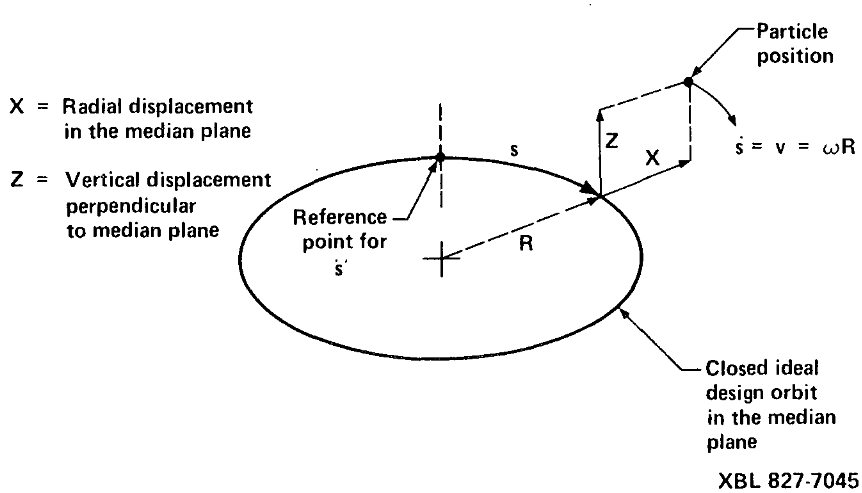
Table II: HISTORY OF STOCHASTIC COOLING (cont'd)

Hereward	1972–74	Refined theory, low intensity cooling
Bramham, Carron, Hereward, Hübner, Schnell, Thorndahl	1975	First experimental demonstration of emittance cooling
Thorndahl, Palmer	1975	Idea of low intensity momentum cooling
Strolin, Thorndahl	1975	$\bar{p}$-accumulation, schemes for ISR using stochastic cooling
Rubbia	1975	$\bar{p}$-accumulation schemes for SPS
Thorndahl	1976	Experimental demonstration of Δp-cooling
Thorndahl	1977	Filter method of Δp-cooling
Sacherer, Thorndahl, van der Meer	1977–78	Refinement of theory; imperfect mixing; Fokker-Planck equations
ICE team	1978	Detailed experimental verification
Herr	1978	Demonstration of bunched beam cooling
Herr, Möhl	1978	Qualitative theory of bunched beam cooling
Fermilab + LBL	1978	Conception of using stochastic cooling for Fermilab p-p scheme
Bisognano	1979	Complete kinetic theory of transverse and longitudinal stochastic cooling of coasting beam with no band overlap
Lambertson et al. LBL	1980	Demonstration experiment of stochastic cooling of 200 MeV protons at Fermilab experimental cooling ring
G. Dôme	1980	Study of bunch diffusion due to RF Noise
Linnecar, Scandale	1980	Development of Schottky noise detectors for bunched beams
S. van der Meer	1980	Theory of signal suppression with band overlap for coasting beams
Bisognano	1981	Independent development of Vlasov theory of signal suppression with band overlap for coasting beams
CERN SPS p-$\bar{p}$ operation	1981	Preliminary p-$\bar{p}$ colliding beam experiment with 270 GeV beams performed
Chattopadhyay, Bisognano	1981	Preliminary Model simulation study of various bunched beams
FNAL + LBL	1982	The Fermilab Antiproton Source Design Report

3. SINGLE PARTICLE DYNAMICS IN A STORAGE RING -- THE UNPERTURBED ORBITS

Single particle orbits in circular accelerators or storage rings in the absence of feedback loops and collective effects, are described in detail in standard texts and classical review papers ([21], [29], [58], [91]). We briefly sketch here some of the relevant properties of these orbits, as needed for a theoretical formulation of stochastic cooling in this report, with a relatively heavier emphasis on the longitudinal (azimuthal) dynamics, since it is the longitudinal orbits that distinguish a bunched beam from a coasting beam.

Particles in a storage ring are confined transversely by magnetic focusing fields and execute betatron oscillations about an equilibrium orbit. Longitudinally the particles either drift in free-streaming orbits with constant angular velocity (coasting beam with no acceleration) or execute synchrotron (energy) oscillations about a synchrotrons particle (bunched beam). The synchronous particle could be either accelerating or moving with constant angular velocity depending on the phase at which it samples the phase-locked radio-frequency cavity voltage at each turn. In most cases, the betatron oscillations in directions transverse to the beam are very weakly coupled to the synchrotron oscillations in energy. This is because the synchrotron oscillation frequencies are usually very small compared to the betatron oscillation frequencies and the betatron oscillations average to zero over a long synchrotron period.



Coordinate System for Single Particle Orbits in a Storage Ring

Fig. 5

In the orthogonal right-handed coordinate system illustrated in Fig. 5, the orbit of a particle having the ideal momentum $p = p_0$ on the design orbit is given by pseudo-harmonic betatron oscillations [91], with both phase and amplitude depending on the instantaneous wavelengths $\beta_{x,z}(s)$ (also called the 'amplitude functions'), which satisfy:

$$\frac{1}{2}\,\beta_{x,z}\,\beta''_{x,z} - \frac{1}{4}\,\beta'^{\,2}_{x,z} + k_{x,z}(s)\,\beta^2_{x,z} = 1 \tag{3.1}$$

$$\beta_{x,z}(s+L) = \beta_{x,z}(s) \tag{3.2}$$

where $\beta' = d\beta(s)/ds$, $L = C/N$ is the circumferential length of one period of the N-fold periodic focusing lattice within the full circumference of length C and $k_{x,z}(s)$ are certain 'field gradients' determined by the magnetic field configuration of the focusing magnets.

When observed at a particular azimuth $s = s_0$, however, the lateral betatron motion is indistinguishable from a sampled simple harmonic oscillation at a frequency $(\omega_\beta)_{x,z} = Q_{x,z}\omega$, called the betatron frequency with betatron displacement, say for the x-motion, being given by

$$x_{s_0}(t_j) = A_x\sqrt{\beta_x(s_0)}\ \text{Sin}\left[Q_x\,\omega\,t_j + \phi^x_{0,s_0}\right] \tag{3.3}$$

where $t_j = j\,T = \frac{2\pi}{\omega}\,j$ are the times for the $j^{\underline{th}}$ passage through the azimuth $s = s_0$ $(j=0,1,2,\ldots)$, ϕ^x_{0,s_0} the phase at zeroth passage $(j=0)$, ω the angular frequency of revolution of the particle, A_x an arbitrary constant depending on initial conditions and

$$Q_{x,z} = \frac{1}{2\pi}\int_0^C \frac{ds}{\beta_{x,z}(s)} \tag{3.4}$$

are known as x,z betatron tunes (number of betatron oscillations in one complete revolution) respectively.

Since for stochastic cooling, it is only the orbit displacements sampled at a particular pick-up location that is relevant, we will use the following amplitude-phase representation of the betatron displacements at the pick-up:

$$x(t) = A_x \sin \phi_x(t); \qquad \dot{x}(t) = A_x(Q_x\omega) \cos \phi_x(t)$$

and (3.5)

$$z(t) = A_z \sin \phi_z(t); \qquad \dot{z}(t) = A_z(Q_z\omega) \cos \phi_z(t)$$

where

$$\phi_{x,z}(t) = Q_{x,z}\, \omega t + \phi^0_{x,z} \tag{3.6}$$

For linear betatron oscillations with betatron tunes $Q_{x,z}$ independent of oscillation amplitudes $A_{x,z}$ particles rotate in circles of radius $A_{x,z}$ with frequencies $Q_{x,z}\omega$ in $(x, \dot{x}/Q_x)$ and $(z, \dot{z}/Q_z\omega)$ phase-planes. The variables $(I_{x,z} = 1/2\, A^2_{x,z}\, , \phi_{x,z})$ represent the familiar canonical action and angle variables for a simple harmonic oscillator obtained by a canonical transformation $(x,\dot{x}) \rightarrow (I_x,\phi_x)$ and similarly for the z motion.

A particle of momentum $p = p_0 + \Delta p$ deviating from the design momentum p_0 will execute its betatron oscillation about a closed orbit $\alpha_p(s)\ (\Delta p/p)$ where $\alpha_p(s)$ is known as the "dispersion" of the machine. The total horizontal displacement is $x = x_\beta + \alpha_p(\Delta p/p)$. The change in β_x and β_z with p has negligible effect on the amplitudes, but the wave numbers or betatron tunes get modified to

$$Q_{x,z}(p) = Q_{x,z}(p_0) \left(1 + \xi_{x,z} \frac{\Delta p}{p}\right) \tag{3.7}$$

where $\xi_{x,z}$ are the horizontal and vertical "chromaticities" usually determined and controlled by the multipole fields (sextupoles etc.) of the focusing lattice.

Longitudinally, the particles in the beam can coast in free-streaming orbits with constant angular velocity and filling the whole ring, if the purpose is just to store the beam in the storage ring for many hours. The beam is usually called a "coasting beam".

However for purposes of acceleration and colliding beam experiments with high luminosity the beam is "bunched" out of necessity; i.e., the beam is confined to a finite angular extent in the ring by external radio-frequency voltages. The radio-frequency voltage at a cavity provides the necessary acceleration to a central synchronous particle each time it passes the cavity in phase with the voltage and confines the other particles in the beam in phase-stable oscillatory orbits around the synchronous particle determined by the potential well created by the rf cavity. Thus the longitudinal dynamics of particles in a bunched beam is that of oscillatory trapped orbits.

The equation of motion for the phase ϕ of a nonsynchronous particle relative to the phase of the rf voltage is given by:

$$\ddot{\phi} + \frac{qV_0 h\eta\omega_0^2}{(2\pi)E_0\beta^2\gamma}\left(\text{Sin}\ \phi - \text{Sin}\ \phi_s\right) = 0 \tag{3.8}$$

where ϕ_s is the phase of the synchronous particle relative to the rf voltage, qV_0 is the peak energy gain per revolution, $h = \omega_{rf}/\omega_0$ is the harmonic number of the rf cavity, ω_{rf} the circular frequency of the rf cavity, ω_0 the revolution frequency of the synchronous particle $(d\phi_s/dt = \dot{\phi}_s = h\omega_0)$, $\gamma = E/E_0 = E/mc^2$ the total energy in units of the rest energy $(\gamma = (1-\beta^2)^{-1/2})$ and η the 'off-energy function' defined in Eq. (1.1) where $\alpha = 1/\gamma_t^2 = \frac{p}{C_p}\frac{dC_p}{dp}$ is the relative change in the orbit length C_p per revolution with respect to momentum p. Equation (3.8) describes the longitudinal synchrotron oscillation in phase generated by the rf cavity, about the phase of the synchronous particle. These phase oscillations are accompaied by oscillations in the angular momentum P_ϕ canonically conjugate to ϕ, about the rising momentum P_s of the synchronous particle. There is also a slow radial oscillation $\alpha_p(\Delta p/p)$ associated with these oscillations.

For a stationary bucket with no acceleration of the synchronous particle $\phi_s = 0$ and Eq. (3.8) becomes that of a simple pendulum

$$\ddot{\phi} + \omega_s^2\ \text{Sin}\ \phi = 0 \tag{3.9}$$

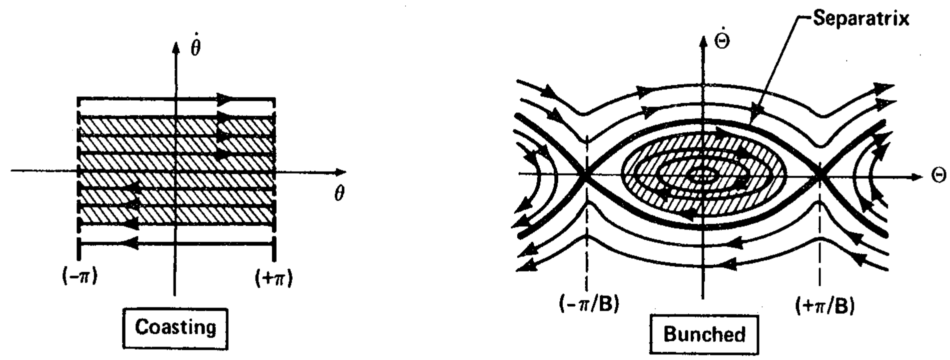
with

$$\omega_s = \sqrt{\frac{q\ V_0\ h\ \eta}{(2\pi)E_0\ \beta^2\ \gamma}}\ \omega_0\ . \tag{3.10}$$

Orbits are thus the familiar simple pendulum trapped oscillatory orbits bounded by a separatrix, beyond which there exist untrapped streaming trajectories. In terms of real angular position in the ring, we have

$$\theta = \text{Ⓗ} + \omega_0\ t \tag{3.11}$$

where Ⓗ is the angular position of the non-synchronous particle with respect to the synchronous particle and is related to ϕ via $\phi = h$Ⓗ. Typical particle orbits in a coasting beam and a bunched beam are illustrated in Fig. 6(a) and 6(b) below.



Longitudinal Particle Orbits in Coasting and Bunched Beams

Fig. 6

Orbits are topologically different in the two cases and both the cooling and the collective dynamics differ nontrivially. For small amplitudes $\phi - \phi_s << 1$, Eq. (3.8) becomes

$$\ddot{\phi} + \frac{q\, V_0\, h\eta\, \omega_0^2 \operatorname{Cos} \phi_s}{(2\pi)\, E_0\, \beta^2\, \gamma} (\phi - \phi_s) = 0 \tag{3.12}$$

This is a simple harmonic motion about ϕ_s with circular frequency

$$\omega_s(\phi_s) = \sqrt{\frac{q\, V_0\, h\eta \operatorname{Cos} \phi_s}{(2\pi)\, E_0\, \beta^2\, \gamma}}\, \omega_0 \tag{3.13}$$

Again for a stationary bucket with no acceleration $(\phi_s = 0)$, we have:

$$\ddot{\phi} + \omega_s^2\, \phi = 0 \tag{3.14}$$

where ω_s is given by Eq. (3.9). Again, in terms of real angular position $\theta = \Theta + \omega_0 t$ in the ring, we have:

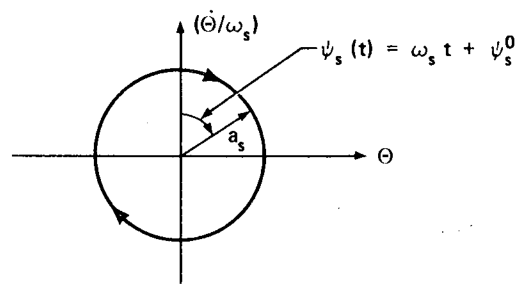
$$\ddot{\Theta} + \omega_s^2 \Theta = 0 \tag{3.15}$$

The orbits are thus

$$\begin{aligned} \theta &= \omega_0 t + a \operatorname{Sin} \psi(t) = \omega_0 t + \Theta \\ \theta &= \omega_0 + a\,\omega_s \operatorname{Cos} \psi(t) = \omega_0 + \dot{\Theta} \end{aligned} \tag{3.16}$$

where $\psi(t) = \omega_s t + \psi^0$ describes the synchrotron phase of the oscillating particle.

The amplitude-phase (a,ψ) representation given in Eq. (3.16) defines circular synchrotron orbits in $(\Theta, \dot{\Theta}/\omega_s)$ phase-space with radius a_s, as shown in Fig. 7 below.



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Amplitude and Phase Representation of Rotation in Phase-Space

Fig. 7

Again a familiar canonical transformation from $(\Theta, \dot{\Theta})$ to the action-angle variables (J,ψ) for a harmonic oscillator yields $J = 1/2\, a^2$ and the orbits

$$\begin{aligned} \Theta &= \sqrt{2J} \operatorname{Sin} \psi(t) \\ \dot{\Theta} &= \sqrt{2J}\ \omega_s \operatorname{Cos} \psi(t) \end{aligned} \tag{3.17}$$

We will see later that for stochastic cooling of bunched beams, it is crucial that different particles in the beam have different synchrotron frequencies, i.e. the beam must have a finite spread in synchrotron oscillation frequencies. This implies that we cannot treat the orbits in a bunch as linear simple harmonic oscillations as given by (3.14), where all the particles oscillate with the same frequency but rather should really treat the orbits as solutions of the full nonlinear equation (3.9), corresponding to a pendulum. It is well known that the pendulum equations of motion can be integrated in terms of elliptic functions. However, a theoretical treatment of stochastic cooling of bunched beams using these real pendulum orbits involving elliptic functions becomes unnecessarily complicated.

We therefore adopt a simple model instead where the nonlinearity of the synchrotron oscillations is given by some functional dependence on amplitude <u>or</u> action of the synchrotron frequency: $\omega_s \equiv \omega_s(J)$, so that different particles oscillating in synchrotron orbits with different amplitudes have different frequencies. However, the shape of the trajectories are still taken to be sinusoidal as given by Eq. (3.16) but with $\psi(t)$ now given by:

$$\psi(t) = \omega_s(J)\ t + \psi^0 \ . \tag{3.18}$$

In using this model we have appealed to the well-known fact that for most physical systems, the eigenvalues or eigen-frequencies are more sensitive to perturbations than the eigenfunctions themselves. Typically up to first order in perturbations, the eigenfunctions distort insignificantly while the eigenvalues shift by finite amounts.

In particular, an asymptotic perturbation series solution of the pendulum equation (3.9) in the first order approximation (limiting ourselves to the first two terms in the expansion for $\text{Sin}\ x = x + x^3/3! + \ldots$) gives [12]:

$$\phi_1 = \phi_0\ \text{Sin}\ \psi(t)$$

where

$$\psi(t) = \omega_s^1(\phi_0)\ t + \psi^0$$

and

$$\frac{\left[\omega_s^1(\phi_0)\right]^2}{\omega_s^2} = 1 - \frac{\phi_0^2}{8}\ ; \quad \frac{d\phi_0}{dt} = 0\ . \tag{3.19}$$

In terms of the actual azimuthal position $Ⓗ = \phi/h$ of the particle in the ring, we thus get:

$$Ⓗ_1 = a \text{ Sin } \psi(t) = \sqrt{2J} \text{ Sin } \psi(t)$$

where

$$\psi(t) = \omega_s^1(J)\, t + \psi^0$$

and

$$\frac{\left[\omega_s^1(J)\right]^2}{\omega_s^2} = \left[1 - \frac{(ha)^2}{8}\right] = \left[1 - \frac{h^2}{4} J\right] \tag{3.20}$$

and

$$\frac{dJ}{dt} = 0 \text{ .}$$

For reasonably small amplitude ϕ_o far from the separatrix, we then get:

$$\frac{\omega_s^1(\phi_o)}{\omega_s} = \left[1 - \frac{\phi_o^2}{8}\right]^{1/2} \approx 1 - \frac{\phi_o^2}{16} \tag{3.21}$$

and

$$\frac{\omega_s^1(J)}{\omega_s} \simeq 1 - \frac{h^2}{8} J \tag{3.22}$$

For an arbitrary rf potential $V(\phi)$, the synchrotron oscillations are governed by the Hamiltonian

$$H = \frac{1}{2} P_\phi^2 + V(\phi) \tag{3.23}$$

where $\phi(t)$ represent the deviation at time t of the particle's rf phase from the synchronous value and $P_\phi(t) = \dot{\phi}(t)$ the conjugate momentum. The equation of motion corresponding to (3.23) is

$$\ddot{\phi} + V'(\phi) = 0 \tag{3.24}$$

We can perform a canonical Hamilton-Jacobi transformation $(\phi, P_\phi) \to (\tau, E)$ by introducing a generating function $W(\phi,E)$ determined by [40]:

$$\frac{1}{2}\left[\frac{\partial W(\phi,E)}{\partial \phi}\right]^2 + V(\phi) = E \tag{3.25}$$

The new canonical variables (τ,E) are related to (ϕ,P_ϕ) via

$$\tau = \frac{\partial W(\phi,E)}{\partial E} = \int_0^\phi \frac{d\phi'}{\sqrt{2[E-V(\phi')]}} \tag{3.26a}$$

$$P_\phi = \frac{\partial W(\phi,E)}{\partial \phi} = \sqrt{2[E-V(\phi)} \tag{3.26b}$$

The solution of (3.26a) is given by $\phi(E,\tau)$ after inversion and the transformed Hamiltonian is

$$\tilde{H} = E \tag{3.27}$$

The new equations of motion are

$$\dot{\tau} = 1 \tag{3.28a}$$

$$\dot{E} = 0 \tag{3.28b}$$

which can be trivially integrated to give:

$$\begin{aligned} \tau &= \tau_0 + t \\ E &= E_0 = \text{constant.} \end{aligned} \tag{3.29}$$

Thus E and τ are really the total energy and the conjugate time along the particle orbit respectively.

We can now define the action variable $J(E)$ by:

$$J(E) = \oint P_\phi \, d\phi = \oint \sqrt{2[E-V(\phi)]} \, d\phi \tag{3.30}$$

and

$$\frac{\partial J(E)}{\partial E} = \oint \frac{d\phi}{\sqrt{2[E-V(\phi)]}} = T(E) \tag{3.31}$$

where $T(E)$ is the period of the synchrotron oscillation. Corresponding angle variable ψ is defined by

$$\psi = \frac{2\pi}{T(E)} \tau = \omega(E)\tau \quad = \omega_s(J)\ t + \psi^0 \tag{3.32}$$

where

$$\omega_s(J) = \omega\Big(E(J)\Big) \ .$$

Similar analysis can be performed for nonlinear betatron oscillations as well. Particles in a beam in a storage ring can thus be described as three-dimensional oscillators with canonical action-angle variables $(\underset{\sim}{I}, \underset{\sim}{\psi}) = (I_x, I_z, J;\ \phi_x, \phi_z, \psi)$ satisfying equations of motion:

$$\begin{aligned} \dot{\underset{\sim}{I}} &= 0 \Rightarrow \left\{I_x, I_z, J\right\} \text{ constants of motion} \\ \dot{\underset{\sim}{\psi}} &= \underset{\sim}{\omega} \Rightarrow \underset{\sim}{\psi} = \underset{\sim}{\omega} t + \underset{\sim}{\psi}^0 \end{aligned} \tag{3.33}$$

where $\underset{\sim}{\omega} = \left\{\omega_x(I_x), \omega_z(I_z), \omega_s(J)\right\}$ describes x and z transverse betatron oscillation frequencies and the longitudinal synchrotron oscillation frequency as functions of corresponding action variables. For linear oscillations $\underset{\sim}{\omega} = \left\{Q_x\omega, Q_z\omega, \omega_s\right\}$ are constants independent of action and $I_x = 1/2\ A_x^2$, $I_z = 1/2\ A_z^2$, $J = 1/2\ a^2$ where (A_x, A_z, a) are the amplitudes of two transverse betatron and longitudinal synchrotron oscillations respectively. Oscillation displacements, even in the general case of nonlinear oscillations, can be looked at as functions of action and angle and what is more, they are periodic in the angle variables $\underset{\sim}{\psi} = (\phi_x, \phi_z, \psi)$ $(x = x(I_x, \phi_x),\ \ z = z(I_z, \phi_z),\ \ H = H(J, \psi))$ with period 2π.

For coasting beams, there are no synchrotron oscillations; however, we can still represent the free-streaming motion of particles by action and angle variables as:

$$J = 2\pi R(\Delta p_\theta) \propto E - E_0 \tag{3.34}$$

and

$$\psi = \theta(t) - \theta_0(t) = (\Delta\omega)\ t + \psi^0 = (\omega - \omega_0)\ t + \psi^0 ,$$

where E_0, ω_0 are the energy and revolution frequency of a nominal reference particle in the beam.

4. GENERAL DISCUSSION OF COOLING DYNAMICS

In this chapter we consider some general aspects of the dynamics and the various processes involved in stochastic cooling. Although we are primarily interested in bunched beam cooling, this general discussion considers both coasting and bunched beams in parallel whenever possible. This chapter then serves both as a review of the basic concepts of the well-developed theory of coasting beam stochastic cooling and as an introduction to some of the essential differences and distinguishing features of bunched beams as opposed to coasting beams. These differences will become even sharper in Chapter 5.

4.1 Stochastic Cooling and Liouville's Theorem

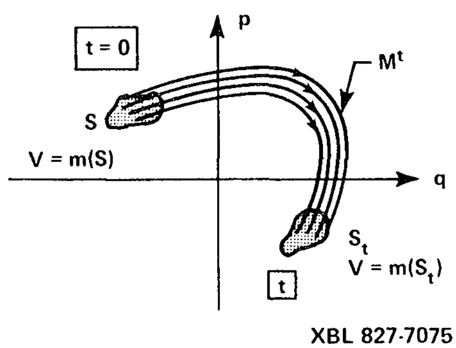
It is usual to think of Liouville's theorem as implying conservation of phase-space density of particle beams in the presence of electromagnetic fields, whether constant or variable in space-time. It occupies a central place in beam optics in the beam manipulating devices such as accelerators and storage rings. It is thus appropriate to elucidate the compression of the phase-space induced by stochastic cooling in the context of Liouville's theorem.

In a nutshell, we can summarize the process of stochastic cooling as follows: the stochastic cooling feedback loop, by virtue of introducing a nonconservative (non-Hamiltonian) and dissipative self-interaction force in the dynamics of a single particle, induces a genuinely non-Liouvillian compressible flow in the projected phase-space of the physical dynamical variables related to the particles only. The possibility of introducing such a nonconservative force by a feedback loop alone, however, is dependent crucially on the finiteness of the number of particles in the beam. For a hypothetical beam containing an infinite number of particles, it would be impossible to introduce such nonconservative self-forces by means of a feedback loop alone.

In a system of N particles, the motion of the system is defined by the motion of a point in the 6N-dimensional space of the canonical coordinates and momenta of the particles (Γ-space). An ensemble of S systems is represented by a set of S points in Γ-space. A precise mathematical statement of Liouville's theorem reads (Khinchin 1949 [56]):

"Let S be any measurable (in the sense of Lebesgue) set of points of the phase-space Γ of the given mechanical system. In the natural motion of this space, described by Hamiltonian dynamics (which maps the phase-space onto itself under a one-parameter group of diffeomorphisms $M^t: \Gamma^{6N} \to \Gamma^{6N}$), the set S gets mapped into another set S_t during an interval of time t: $S_t = M^t S$. The measure $m(S_t)$ of the set S_t for any t coincides with the measure $m(S)$ of the set S. In other words, the measure of measurable point sets is an invariant of the natural Hamiltonian motion of the space Γ:

$$\frac{d}{dt}\left[m(S_t)\right] = 0 \qquad "$$



Hamiltonian Mapping Generating Incompressible Liouvillian Flow in Phase-Space

Fig. 8

Physically, one can imagine preparing an ensemble of systems with all possible initial coordinates and momenta that occupy a finite non-zero volume V in the phase-space Γ. Obviously the volume V then contains an infinite number of system points. Under Hamiltonian dynamics, according to Liouville, the image points at a later time t still occupy the same amount of volume V. The Hamiltonian phase-flow thus resembles that of an incompressible fluid of volume V in phase space (Fig. 8).

In the absence of mutual interactions among the N identical particles of the system, we can consider each particle to be an independent system with a given initial condition and can look at the motion of N-particles as the motion of an ensemble of N discrete points in the 6-dimensional phase-space of a single particle (μ-space). In the limit $N \to \infty$, the motion of the actual volume in μ-space also becomes that of an incompressible Liouvillian flow. However, this motion has little to do with the actual motion of the N discrete points of the real system, which has zero mathematical volume and is not really a fluid in phase-space.

A real beam, even when N is very large but finite, is empty almost everywhere in μ-space. A useful definition of "physical volume" emerges however [84] when we divide the μ-space in six-dimensional cells, each large enough to contain a very large number of particles and yet small enough so that the coordinates do not change appreciably across their volume and consider only those cells that at a given time are occupied by particles. The sum of the volumes of all these cells can be defined as the "volume of the beam or beam emittance" in the case of finite N. Similarly a phase-function measuring density $\rho(p,q)$ in phase-space can also be introduced by taking the ratio of the number of particles in a given cell to the volume of the same cell. So defined ρ is a discontinuous function that can be approximated by a smooth one. This is a local averaging process and is very sensitive to fluctuations from cell to cell, which is precisely what a stochastic cooling system takes advantage of in sensing information about particle co-ordinates. However, if one considers a large number of particles uniformly spread in each cell, all the particles occupying a particular cell at an initial time t_0 are expected to occupy at a later time t another one with the same volume, apart from statistical fluctuations. This expectation is based on the continuity of Hamiltonian flow and the fact that no two flow lines can intersect, since Hamiltonian flow is uniquely deterministic. This approximate conservation of the "physical phase-space volume" of the beam, whose definition is based on local averaging and neglecting statistical fluctuations, plays a useful and dominant role in considerations of beam optics and is often referred to as Liouville's theorem also. Note that for a set of non-interacting particles, the actual Liouville's theorem in the 6-dimensional μ-space, referring to the mathematical phase-space volume (i.e. the measure of measurable point sets in μ-space) remains strictly valid. The fact that the conservation of the "physical phase-space volume" is only an approximation is where the stochastic cooling concept begins to be potentially useful, for the possibility of detecting the graininess of the "physical phase-space" allows one to introduce suitable force-fields to affect the same.

In using Liouville's theorem, it is important to remember that there should not be any mutual interaction between the ensemble points and that the phase-space should include all degrees of freedom of the system, i.e. the phase-space should describe a closed system. The phase-space volume of some smaller subset of the system is not necessarily conserved. External force fields do not constitute additional degrees of freedom. They appear in the system Hamiltonian as given functions -- the potential functions, not as dependent variables. On the other hand other particles that interact with the original system <u>or</u> radiation emitted by particles of the original system constitute additional degrees of freedom. Phase-space volume can be interchanged between the original system and these new degrees of freedom. Thus for example synchrotron radiation can decrease the phase-space volume occupied by a particle beam. For a set of interacting particles then, Liouville's theorem is strictly not valid in the 6-dimensional space of a single particle. For conservative interactions, however, e.g. hard-sphere elastic collisions between neutral particles or electromagnetic interaction between charged particles (velocity and time-dependent potentials), Liouville's theorem remains valid in the larger 6N-dimensional Γ-space of the set of particles, where each ensemble point represents a closed system of conservatively interacting particles (provided one neglects retardation effects of signal propagation and thus ignores the dynamical space of the infinite number of field variables at each point in space-time needed for a Lagrangian description of a closed system of charged particles). Moreover for investigation of certain processes, the time-scales of interest are considerably shorter than the time-scales over which significant number of interparticle interactions or collisions takes place. For such cases, one can visualize each particle as moving under the influence of a self-consistent conservative time-dependent average or mean field of all the other particles (Vlasov averaged or Hatree mean field) and Liouville's theorem remains approximately valid even in the single particle smoothed-out 6-dimensional phase-space for such time-scales. Such is the case when one uses the collisionless self-consistent Vlasov analysis to study the collective processes in a plasma, which occur with frequencies much higher than the collision frequency. For longer time-scales inter-particle correlation effects become non-negligible and leads to systematic transport in single-particle phase-space in general. However the flow in 6N-dimensional Γ-space still remains Liouvillian, apart from the radiation effects related to the dynamical degrees of freedom of the electromagnetic field variables.

A stochastic cooling system introduces inter-particle interactions through the external electronic feedback loop and so the flow in 6-space is not Liouvillian. What is more, even the flow in 6N-space is not Liouvillian because the feedback loop introduces nonconservative and dissipative self-correction forces in the single particle dynamics, depending on the feedback system and the cooled particle alone and independent of the phase-space coordinates of the other particles. This is so because we do not even hope to be able to use all the dynamical degrees of freedom of the system (i.e. the phase-space coordinates of the beam particles and the infinite number of dynamical variables of the electromagnetic fields involved in the electronic feedback loop including power supplies etc.) necessary for a closed system description. Instead we are interested in the evolution dynamics of the set of particles in the beam only. We thus really care about the properties of the dynamics projected onto the subspace of the particle variables only, in terms of the response functions and transfer characteristics of the feedback loop. And in this space of course the cooling interaction takes the form of a nonconservative non-Hamiltonian dynamics describing a dissipative process. Liouville's theorem simply does not apply. This notion of stochastic cooling as a nonconservative dissipative process is not surprising. Most macroscopic dissipative phenomena such as kinematic friction can be traced back to a microscopic feedback effect from a conservative Lagrangian or Hamiltonian dynamics operating in an underlying deeper and larger space of dynamical variables that includes the environmental heat bath. The process of projection into a smaller subspace necessitates the introduction of nonconservative dynamics (e.g. drag forces on a charged particle due to radiation reaction can only be included as velocity dependent nonconservative forces if one does not wish to include the degrees of freedom of the radiation fields in the dynamics of the charged particle).

The conservative or nonconservative nature of the relevant forces is sensitive to the level of hierarchial description of the system. As we go deeper into the hierarchy, however, becoming increasingly systematic and mechanistic, we lose simplicity in describing aspects of the particulate. Aside from the broad question as to whether the universe is closed or open in principle, it is only natural to develop laws of evolution or flow of finite systems in the general context of nonconservative dynamics, with conservative dynamics as a special case dictated by the particular physical situation. And indeed, Liouville himself, having proved his theorem on conservation of phase-space flow under conservative Hamiltonian dynamics in 1837, proceeded in 1838 to study the effect of nonconservative dynamics on the transformation of volumes in phase-space and presented the law of evolution of phase-space volumes and phase-functions for such cases [68], which we present below.

Under nonconservative, non-Hamiltonian dynamics, one can write equations of motions which can again be interpreted as a continuous point transformation in a proper phase-space, as illustrated in Fig. 8, but now the Jacobian of the transformation is not unity and the phase-space density is not preserved under this transformation. Thus if $(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_N)$ are the phase-space coordinates of the particles in the beam $(\underset{\sim}{x}_i \equiv \{p_x^i,p_y^i,p_z^i,q_x^i,q_y^i,q_z^i\})$, we can write the cooling equations of motion as:

$$\dot{\underset{\sim}{x}}_i = \underset{\sim}{G}_i\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_i,\ldots,\underset{\sim}{x}_N\right) \qquad (i = 1,2,\ldots,N) \tag{4.1.1}$$

This determines a velocity at each point $[x] = (\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_N)$ of the 6N-dimensional phase-space. From each initial point $[a] = (\underset{\sim}{a}_1,\underset{\sim}{a}_2,\ldots,\underset{\sim}{a}_N)$ a trajectory starts out which describes the corresponding solution of (4.1.1). We can now consider an ensemble of such points in the 6N-dimensional phase-space and define a phase-function $\rho(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_N;t) = \rho([x];t)$ describing the probability density of the fluid element in phase-space. Conservation of the number of ensemble systems then implies the following continuity equation:

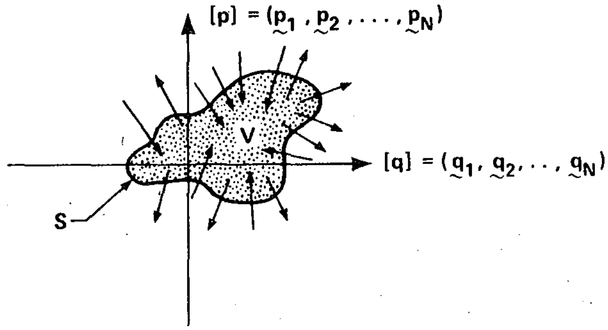
$$\frac{\partial\rho([x];t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial \underset{\sim}{x}_i} \cdot \left[\dot{\underset{\sim}{x}}_i \; \rho([x];t)\right] = 0$$

With dynamics given by (4.1.1), we then have

$$\begin{aligned} \frac{\partial\rho([x];t)}{\partial t} &= -\sum_{i=1}^{N} \frac{\partial}{\partial \underset{\sim}{x}_i} \cdot \left[\underset{\sim}{G}_i([x]) \; \rho([x];t)\right] \\ &= -\nabla_{[x]}^{(6N)} \cdot \left[\underset{\sim}{G}^{(6N)}([x]) \; \rho([x];t)\right] \end{aligned} \tag{4.1.2}$$

This is just a statement in differential form of the continuity of flow in phase-space: rate of accumulation inside a volume V is just the difference between the rate of inflow (inward flux) and the rate of outflow (outward flux) across the surface S that bounds V (Fig. 9).

A continuity equation as above is just a general statement of conservation of probability and is true whether the flow is incompressible or not. The flow is not incompressible for nonconservative cooling interaction and so the divergence operator $\partial/\partial\underset{\sim}{x}$ remains outside in front of $\underset{\sim}{G}$. As a consequence the solution of (4.1.2) is not obtained



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Continuity in Phase-Space - Flux Across a Surface S
Leading to Accumulation Inside Volume V

Fig. 9

by just taking ρ constant along each trajectory, but a Jacobian determinant will appear. If we write the solution of (4.1.1) as a mapping:

$$M^t: \ [a] \rightarrow [x]$$

or $$[x] = M\big([a];t\big) \qquad (4.1.3)$$

with inverse $[a] = M^{-1}([x];t)$ which should always exist in order to have solubility, then solution of (4.1.2) may be written as:

$$\rho\big([x];t\big) = \rho\big(M^{-1}([x],t);0\big) \ \frac{d\{M^{-1}([x])\}}{d\{[x]\}}$$

where $\frac{d\{M^{-1}([x])\}}{d\{[x]\}} \equiv J$ is the Jacobian determinant of the mapping (4.1.3). For Hamiltonian flow:

$$\frac{\partial}{\partial \underset{\sim}{x}_i} \cdot \underset{\sim}{G}_i([x]) = 0 \Rightarrow \nabla^{(6N)}_{[x]} \cdot \underset{\sim}{G}^{(6N)}([x]) = 0$$

That is the generalized force is divergenceless and the continuity equation becomes

$$\frac{\partial\rho([x];t)}{\partial t} + \sum_{i=1}^{N} \underset{\sim}{G}_i([x]) \cdot \frac{\partial\rho([x];t)}{\partial \underset{\sim}{x}_i} = \frac{d\rho([x];t)}{dt} = 0 \ .$$

The Jacobian determinant would simply be 'unity' for all time for a Hamiltonian flow. If on the other hand

$$\dot{\underset{\sim}{x}}_j = \underset{\sim}{G}_j([x]) = -\gamma\, \underset{\sim}{x}_j$$

for some j with γ = constant, the Jacobian determinant is $e^{\gamma t}$ and we have:

$$\rho\Big([x];t\Big) = \rho\Big(M^{-1}([x];t);0\Big)\, e^{\gamma t}$$

i.e. a damping force proportional to the particle phase space co-ordinate increases the phase-space density carried along the trajectory, exponentially in time.

We will use the grand continutiy equation (4.1.2) for compressible flow in 6N-dimensional phase-space later when developing a complete kinetic theory of stochastic cooling in Section 9.2.

In the next sections, we gain further insight into the nature and properties of the cooling force $\underset{\sim}{G}_i(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_N)$. In particular we will show that the cooling force can be decomposed as:

$$\underset{\sim}{x}_i = \underset{\sim}{G}_i = \underset{\sim}{G}(i,i) + \sum_{\substack{j\neq i\\=1}}^{N} \underset{\sim}{G}(i,j)$$

where $\underset{\sim}{G}(i,i)$ is a nonconservative force, not derivable from a Hamiltonian, describing the interaction of a particle with itself (self-force <u>or</u> self-action) induced by the transit-time matched feedback loop and $\sum_{j(\neq i)=1}^{N} \underset{\sim}{G}(i,j)$, the total force exerted by all the <u>other</u> particles in the beam, can be looked at as a conservative force, derivable from a pseudo-Lagrangian or -Hamiltonian and satisfying the Hamiltonian flow condition:

$$\frac{\partial}{\partial \underset{\sim}{x}_i} \cdot \left[\dot{\underset{\sim}{x}}_i - \underset{\sim}{G}(i,i)\right] = \frac{\partial}{\partial \underset{\sim}{x}_i} \cdot \left[\sum_{\substack{j\neq i\\1}}^{N} \underset{\sim}{G}(i,j)\right] = 0 \qquad (4.1.4)$$

We thus look at the process of stochastic cooling of a beam of particles as the time-evolution of a many-body system consisting of a collection of three-dimensional oscillators, interacting with each other conservatively (Hamiltonian-wise) and with themselves nonconservatively (non-Hamiltonian self-action). In other words we are going to study the nonequilibrium statistical mechanics of a collection of three-dimensional oscillators coupled to each other via a retarded, nonconservative (non-Hermitian) cooling interaction.

It is important to remember, however, that the possibility of introducing a nonconservative, dissipative force to particles in the beam by an external electronic feedback loop is crucially dependent on the finiteness of the number of particles in the beam. Stochastic cooling is the process of acquiring effective information regarding the phase-space microstructure of the beam (i.e. knoweldge about the preparation at any time of that particular representation of the ensemble of beam systems which actually represents the beam under process) in successive approximations and simultaneous application of the same information to the beam to induce cooling by generating suitable force fields appropriately in time. It is obvious that if the beam is infinitely dense, i.e. $N \to \infty$ (fluid in phase-space) then there is no statistical fluctuations in a sample and hence no signal containing single particle information can be induced in the pick-up by the beam. Only information regarding the coherent motion of the beam as a whole would be obtained. We cannot then effectively use the feedback loop as an information processor of microscopic phase-space and we cannot induce any cooling.

The process of information extraction can never be totally nondestructive. For a beam containing a large number of particles, this information is not available to us a priori unless one uses experimental devices as diagnostic probes to extract this information, which always disturbs the initial state, even if infinitesimally. In the process of information extraction, the sensing or probing device gains information, but always at the cost of heating up the system that is probed, whose entropy thus increases. With minimal coupling and adequate observation time, like the practically non-destructive Schottky signal pick-up electrodes, one may be able to obtain information signals, extremely small in amplitude but precise enough to resolve small clusters of microscopic phase-space structure of the beam. The process will of course heat up the beam by finite but small amount. Once the information is available though, one can apply it back to the beam with large enough gain and right phase in order to compensate the heating induced in the act of information gathering and in addition cool the beam by a small but finite amount. This is what the stochastic cooling technique does.

A large amount of power is required to be fed externally into the feedback loop to amplify the extremely small signals derived at the pickup and to guarantee transmission of signals with sufficient energy density in the transmission lines required to produce electromagnetic fields at the kicker strong enough to affect single particle motion in the beam. This power together with the relatively insignificant amount of heat deposited into the loop from the cooling beam, is dissipated across a resistor (50–100 ohms typically).

Since stochastic cooling is an entropy-reducing process for the beam of particles, it is tempting to formulate an information-theoretic approach to the process of stochastic cooling. We refrain from such an attempt in this report.

4.2 The Cooling Interaction and the Two Fundamental Processes

We seek to study in general form the nature of the interparticle interaction and the self-interaction induced among the beam particles by the stochastic cooling feedback loop. For high energy beams under consideration, the direct electromagnetic interaction between particles in the beam generally becomes considerably weaker than the interaction between these particles and the external elements in the environment (vacuum chamber walls, localized cavities or resonators, probes or pick-up monitors etc.). Similarly the cooperative collective effects arising from the direct interaction is weaker than the collective effects that are coupled through the impedance or gain of the external elements and feedback loops. Moreover, interactions within the beam can only cause coupling between various degrees of freedom and a slow Coulomb diffusion or heating associated with the overall slowing of the beam. It cannot reduce the total phase volume. In our model of cooling then, interactions beween the beam particles are always induced by the external feedback loop only.

The set $j = 1,\ldots,N$ particles in the beam executing betatron oscillations transversely and either free-streaming (coasting beam) or executing synchrotron oscillations (bunched beam) longitudinally, generate a small electromagnetic signal at the outport of the localized pick-up, which is then transferred to the kicker by a linear electronic transfer element with certain propagation or transfer characteristics. The kicker then produces a time-varying electromagnetic field, which is sampled by, say, the $i^{\underline{th}}$ particle in the beam.

Thus an individual particle in the beam, as it passes through the kicker, sees a time-varying electromagnetic field, which can be described by a scalar potential $\phi(\underset{\sim}{r},t)$ and a vector potential $\underset{\sim}{A}(\underset{\sim}{r},t)$ in the general case. Since the vector potential $\underset{\sim}{A}$ at the kicker is derived from currents or velocities of all the particles $j = 1,\ldots,N$ in the beam at the pickup, $\underset{\sim}{A}$ is a superposition of N terms, each being a function of the corresponding velocity $\underset{\sim}{v}^j$. Thus the electromagnetic potentials sampled at the kicker by an individual particle, say the i^{th}, can in general be separated into two parts: (a) a part ϕ^s, $\underset{\sim}{A}^s$ depending on its own velocity $\underset{\sim}{v}^i$ at the pick-up; this is the coherent self-action, i.e. the dissipative part and (b) a fluctuating time-dependent part ϕ^{ns}, $\underset{\sim}{A}^{ns}$, whose time-dependence is governed by the velocities of all the other particles $(j(\neq i) = 1,\ldots,N)$ and the propagation characteristic of the feedback system.

It is important to recognize that one cannot write down an interaction Lagrangian $\mathcal{L}^i_{int}$ for a particle in an electromagnetic field, if the potentials ϕ or $\underset{\sim}{A}$ of the field depend on the velocity $\underset{\sim}{v}^i$ of the particle of interest. A closer look at a conventional derivation of $\mathcal{L}^i_{int}$ of a charged particle in an electromagnetic field ([40], [55]) makes this fact obvious (for velocity dependent $\underset{\sim}{A}$, different gradient operators do not commute). Such intrinsically nonconservative velocity dependent forces, arising not only from conventional velocity-dependent electromagnetic forces but also from velocity-dependent electromagnetic potentials themselves, is a manifestation of the feedback from the dynamics in a space (radiation field space of the combined particle and feedback system) which has been projected out and no 'potential' formulation is available for them. Consequently they enter into the equations of motion directly as nonconservative driving force terms, with no underlying Lagrangian or Hamiltonian.

The influence of the other part of the potentials ϕ^{ns}, $\underset{\sim}{A}^{ns}$ can be described in terms of an interaction Lagrangian of the i^{th} particle of the beam in this time-varying electromagnetic field at the kicker due to all the other particles and is given by:

$$\mathcal{L}^i_{int} = - q\, \phi^{ns}\left(\underset{\sim}{r}^i(t);t\right) + \frac{q}{c}\, \underset{\sim}{v}^i(t) \cdot \underset{\sim}{A}^{ns}\left(\underset{\sim}{r}^i(t);t\right) \qquad (4.2.1)$$

where $[\underset{\sim}{r}^i(t),\underset{\sim}{v}^i(t)]$ characterizes the orbit or trajectory of the $i^{\underline{th}}$ particle. Written in this form, it represents the interaction Lagrangian sampled along the particle trajectory. The interaction Lagrangian $\mathcal{L}^i_{int}$ for the i^{th} particle as written above thus contains only the electromagnetic potentials or fields generated by all the other particles $j(\neq i) = 1,\ldots,N$ in the beam and not the field generated by the particle i itself. It describes the inteaction between two different particles i,j $(i\neq j)$ for i fixed and j summed over $1,\ldots,N$, but not equal to i. In other words, it describes the interaction of a given particle in the beam with all the <u>other</u> particles through the feedback loop. Since it is based on a Lagrangian, the corresponding force can be derived from a suitable derivative of the interaction potential. This is the fluctuation Schotty noise interaction and should obey, according to the previous arguments, Hamiltonian dynamics based on the pseudo-Lagrangian (<u>or</u> Hamiltonian) given by (4.2.1). We wish to express the $\mathcal{L}^i_{int}$ in terms of the particle cooroinates and velocities alone of all the beam particles, without any field variables. We thus need to express $\underset{\sim}{A}^{ns}(\underset{\sim}{r}^i(t),t)$ and $\phi^{ns}(\underset{\sim}{r}^i(t),t)$ in terms of $\underset{\sim}{r}^j(t'), \underset{\sim}{v}^j(t')$ $(j\neq i)$ of the other beam particles. But first let us do away with the scalar potential which brings in considerable simplification.

The electromagnetic potentials ϕ and $\underset{\sim}{A}$ are however not unique and defined only up to the addition of the gradient (for $\underset{\sim}{A}$) or the time-derivative (for ϕ) of an arbitrary function -- the gauge function $\Lambda(\underset{\sim}{r},t)$. The nonuniqueness of the potentials gives us the possibility of chosing them so that they fulfill one auxiliary condition chosen by us. In particular, since we are not concerned with manifest Lorentz covariance of the electromagnetic field, we can always gauge away the scalar potential to zero $\phi(\underset{\sim}{r},t) = 0$ for all space-time, if we chose a gauge function $\Lambda(\underset{\sim}{r},t)$ that satisfies

$$\frac{1}{c}\frac{\partial\Lambda(\underset{\sim}{r},t)}{\partial t} = \phi(\underset{\sim}{r},t)$$

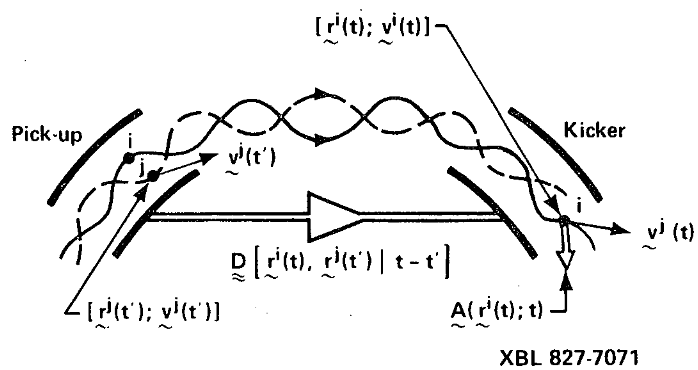
<u>or</u>

$$\Lambda(\underset{\sim}{r},t) = c\int_0^t \phi(\underset{\sim}{r},t')\,dt'$$

This gauge in which $\phi(\underset{\sim}{r},t) = 0$ is the so called 'radiation gauge'. Thus the scalar potential is not really a dynamical degree of freedom. For a source-free region, we can impose the additional constraint $\nabla\cdot\underset{\sim}{A}(\underset{\sim}{r},t) = 0$. This gauge in which $\phi = 0$ and $\nabla\cdot\underset{\sim}{A} = 0$ is known as the 'radiation Coulomb gauge' <u>or</u> 'transverse gauge'. Thus in a source-free region the longitudinal part of $\underset{\sim}{A}$ and the scalar potential are really not dynamical degrees of freedom. For the purpose of this illustration, however, we need only chose the radiation gauge where $\phi = 0$, so that we do not have to talk about scalar potential at all. The interaction Lagrangian then takes the form

$$\mathcal{L}^i_{int} = \frac{q}{c}\,\underset{\sim}{v}^i(t)\cdot\underset{\sim}{A}^{ns}\left[\underset{\sim}{r}^i(t);t\right] = \frac{q}{c}\,v^i_\alpha(t)\,A^\alpha_{ns}\left[\underset{\sim}{r}^i(t);t\right]$$

where the subscripts and superscripts mean various spatial components of the vectors $\underset{\sim}{v}$ and $\underset{\sim}{A}$ and summation over the repeated indices (α) is implied.



Interaction through the Pick-Up-Transfer Element-Kicker Feedback Loop

Fig. 10

But the vector potential field $\underset{\sim}{A}^{ns}(\underset{\sim}{r},t)$ at the kicker is generated by the current of all the beam particles $j = 1,\ldots,N$ $(j \neq i)$ at the pick-up (Fig. 10) and is given by:

$$\underset{\sim}{A}^{ns}(\underset{\sim}{r},t) = \int_{-\infty}^{t} dt' \int d^3r' \; \underset{\approx}{D}\left(\underset{\sim}{r},\underset{\sim}{r}'|t-t'\right) \cdot \underset{\sim}{J}(\underset{\sim}{r}',t')$$

or

$$A^{\alpha}_{ns}(\underset{\sim}{r},t) = \int_{-\infty}^{t} dt' \int d^3r' \; D^{\alpha\beta}\left(\underset{\sim}{r},\underset{\sim}{r}'|t-t'\right) \; J_{\beta}(\underset{\sim}{r}',t')$$

where

$$J_{\beta}(\underset{\sim}{r}',t') = \sum_{\substack{j=1 \\ j\neq i}}^{N} j^j_{\beta}(\underset{\sim}{r}',t')$$

and summation over the repeated indices (β) is implied again. $A^{\alpha}_{ns}(\underset{\sim}{r},t)$ is governed by the propagator or Green's function $D^{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}'|t-t')$ which describes the propagation of signals from $(\underset{\sim}{r}',t')$ to $(\underset{\sim}{r},t)$ through the feedback loop. The explicit form of $D^{\alpha\beta}$ depends on the particular structure of the external feedback loop.

If $\underset{\sim}{r}^i(t) \equiv \left\{\theta^i(t),x^i(t),z^i(t)\right\}$ are the longitudinal angle and transverse betatron co-ordinates of $i^{\underline{th}}$ particle, then the particle sees the interaction fields only when $\theta^i(t) = \theta_k + 2\pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$ since the kicker is localized at $\theta = \theta_k$. Also particle currents are generated at the pick-up only when $\theta^i(t') = \theta_p + 2\pi m$ $(m = 0, \pm 1, \pm 2, \ldots)$, since the pick-up is localized at $\theta = \theta_p$. We thus have:

$$A^{\alpha}_{ns}\left[\underset{\sim}{r}^i(t);t\right] = A^{\alpha}_{ns}\left[\theta^i(t), x^i(t), z^i(t)\right]$$

$$= \sum_{n=-\infty}^{+\infty} \delta\left[\theta^i(t) - \theta_k - 2\pi n\right] \int_{-\infty}^{t} dt' \int d^3\underset{\sim}{r}' \; D^{\alpha\beta}\left(\underset{\sim}{r}^i(t), \underset{\sim}{r}' | t-t'\right) J_\beta(\underset{\sim}{r}', t')$$

and

$$J_\beta(\underset{\sim}{r}', t') = q \sum_{\substack{j=1 \\ j \neq i}}^{N} v^j_\beta(t') \, \delta\left[\underset{\sim}{r}' - \underset{\sim}{r}^j(t')\right] \sum_{m=-\infty}^{+\infty} \delta\left[\theta^j(t') - \theta_p - 2\pi m\right]$$

and the interaction Lagrangian or potential due to all the other particles $j = 1, \ldots, N$ $(j \neq i)$ can be written as:

$$\mathcal{L}^i_{int} = V\left(i(t)|t\right) = \sum_{\substack{j=1 \\ j \neq i}}^{N} V\left(i(t), j(t)\right)$$

where

$$V\left(i(t), j(t)\right) = \frac{q^2}{c} \int_{-\infty}^{+\infty} dt' \left[\sum_{n=-\infty}^{+\infty} \delta\left[\theta^i(t) - \theta_k - 2\pi n\right] \left\{ v^i_\alpha(t) \; D^{\alpha\beta}\left(\underset{\sim}{r}^i(t), \underset{\sim}{r}^j(t') | t-t'\right) v^j_\beta(t') \right\} \otimes \sum_{m=-\infty}^{+\infty} \delta\left[\theta^j(t') - \theta_p - 2\pi m\right]\right]$$

We have extended the range of integration to $+\infty$ at the upper limit by assuming the Green's function to be causal, i.e.

$$\underset{\approx}{D}\left(\underset{\sim}{r}^i(t), \underset{\sim}{r}^j(t') | \tau\right) = 0 \quad \text{for} \quad \tau = (t-t') < 0$$

Noting that $qv_\alpha^i(t) = j_\alpha^i(t)$ is just the $\alpha^{\underline{th}}$ component of the current due to the $i^{\underline{th}}$ particle at time t, we can write the effect of j on i, $V(i,j)$ as follows:

$$V\Big(i(t),j(t)\Big) = \frac{1}{c}\int_{-\infty}^{+\infty} dt' \left[\left\{\sum_{n=-\infty}^{+\infty} \delta\left[\theta^i(t)-\theta_k-2\pi n\right]\right\} \cdot M_{ij}(t|t') \cdot \left\{\sum_{m=-\infty}^{+\infty} \delta\left[\theta^j(t')-\theta_p-2\pi m\right]\right\}\right] \tag{4.2.2}$$

where

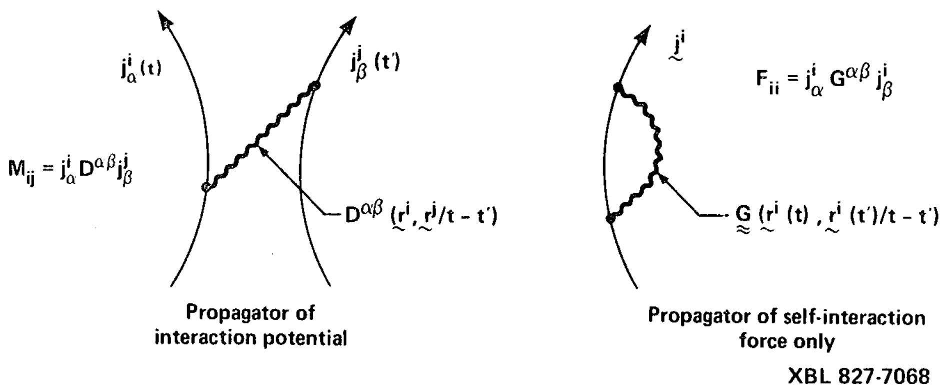
$$M_{ij}(t|t') = \left[j_\alpha^i(t)\ D^{\alpha\beta}\Big(\underset{\sim}{r}^i(t),\underset{\sim}{r}^j(t')|t-t'\Big)\ j_\beta^j(t')\right] \tag{4.2.3}$$

The delta function to the far right in Eq. (4.2.2) corresponds to the signal derived at a pick-up localized at $\theta = \theta_p$ from particles $j(\neq i) = 1,\ldots,N$ at each revolution $(m = 0,1,2,\ldots)$ and the delta function to the left in (4.2.2) corresponds to the 'sampling' of the signal at the kicker localized at $\theta = \theta_k$ by particle i at each revolution $(n = 0,1,2,\ldots)$. $\mathcal{L}^i_{int}$ given above involves products like $j^i_\alpha j^j_\beta = q^2 v^i_\alpha v^j_\beta$ where $\alpha = 1,2,3$ represent the horizontal betatron, vertical betatron and longitudinal coasting or synchrotron degrees of freedom respectively and generally represents the full three-dimensional cooling with coupling between all degrees of freedom. For cooling in any one dimension with no coupling to other degrees of freedom, we have $M_{ij} = j^i_\alpha D^{\alpha\alpha} j^j_\alpha = q^2 v^i_\alpha D^{\alpha\alpha} v^j_\alpha$ with no summation over α, where v_α could represent any one of $\dot{x}$, $\dot{z}$ and $\dot{\theta}$ corresponding to transverse betatron or longitudinal (coasting or synchrotron oscillation) velocities of the particles only.

We can visualize the term described by (4.2.2) and (4.2.3) diagrammatically as corresponding to the scattering of the two particles i and j $(i\neq j)$ with currents j^i_α and j^j_β, where the interaction is mediated by the propagator $D^{\alpha\beta}$ determined by the Green's function of the feedback loop [39] (see Fig. 11(a) below).

As we have mentioned before, the above does not include the nonconservative coherent self-force, which is the component of the interaction that induces real cooling. There is no underlying Larangian for this self-action, because the corresponding vector potential $\underset{\sim}{A}$ is a function of the particle's own velocity $\underset{\sim}{v}$. This self-action can only be included in the equation of motion as a dissipative force. The self-action can be visualized diagrammatically as in Fig. 11(b), where the particle interacts with itself through the transit-time matched propagator $\underset{\approx}{G}(\underset{\sim}{r}^i(t),\underset{\sim}{r}^i(t')|t-t')$ (giving rise to an instantaneous interaction of the particle with itself at the kicker).

We have thus outlined two fundamental single-particle processes involved in a stochastic cooling system -- the direct self-action effect and the incoherent two-particle scattering effect. They are known as the coherent cooling proess and Schottky heating process respectively. Accordingly the electromagnetic signal at the kicker is usually



(a) (b)

The Two Fundamental Processes in Stochastic Cooling - (a) Incoherent Scattering of Two Different Particles and (b) the Self-Interaction Force

Fig. 11

decomposed into a coherent cooling signal depending only on the phase of the cooling particle and an incoherent Schottky fluctuation or noise signal depending on the randomly distributed phases of all the other particles in the beam. We have given pictorial representation of the two processes in Fig. 11(b) and (a) for visualization and outlined the important distinction of non-Hamiltonian vs.Hamiltonian nature of the processes (b) and (a) respectively.

In the next section we look at the explicit form of the equations of motion arising from the dynamics described above.

4.3 Harmonic Representation of the Cooling Equations of Motion in Action-Angle Variables

We are interested in describing the ordinary classical mechanics of a system of interacting charged particles in the beam with the aid of a Lagrangian (or Hamiltonian) and a nonconservative self-force which depend only on the coordinates and velocities of these particles at one and the same time. Due to finite velocity of propagation of electromagnetic interactions (retardation effects), however, one must consider the Lagrangian density associated with the dynamical degrees of freedom of the particles and the fields together for a rigorous description of interacting charged particle systems. In principle then it is impossible to describe such a system rigorously with the aid of only the physical phase-space variables of the particles at the same time and no quantities related to the infinite number of the degrees of freedom of the fields, except in the limit of infinite propagation speed of interactions (classical mechanics with instantaneous interaction i.e. no retardation) or under low-order relativistic effects only (classical Darwin Lagrangian up to order $(v/c)^2$ ([55], [63])).

Retardation of the propagation of interaction is essential for stochastic cooling; however, the necessary retardation is a very special one, namely the one that is matched to the transit time of the particles between the pick-up and the kicker. This special retardation (phase-shift in the feedback loop), together with the spatially localized nature of the interaction (particles interact effectively only when they pass through the localized kicker) allows us to write down, in the adiabatic approximation of slow cooling, a Lagrangian for the two-particle scattering interaction and an instantaneous non-conservative self-force, discussed in Section 4.2, in terms of particle coordinates and momenta at one and the same time. In this picture all the particles interact with each other conservatively and nonconservatively instantaneously at a localized region in space (at the kicker) and discretely in time.

Following the single particle orbits introduced in Chapter 3, we can represent the unperturbed coordinates and velocities of the $i^{\underline{th}}$ particle in the frame of the beam in terms of action-angle variables as

$$\left.\begin{aligned} r_\alpha^i &= r_\alpha^i\left(I_\alpha^i,\ \psi_\alpha^i\right) \\ v_\alpha^i &= v_\alpha^i\left(I_\alpha^i,\ \psi_\alpha^i\right) \end{aligned}\right. \qquad (\alpha = 1,2,3) \tag{4.3.1}$$

where

$$\left\{r_\alpha^i\right\}_{\alpha=1,2,3} = \left(x^i, z^i, \text{Ⓗ}^i\right)$$

$$\left\{v_\alpha^i\right\}_{\alpha=1,2,3} = \left(\dot{x}^i, \dot{z}^i, \dot{\text{Ⓗ}}^i\right)$$

$$\left\{\psi_\alpha^i\right\}_{\alpha=1,2,3} = \left(\phi_x^i,\ \phi_z^i,\ \psi^i\right)$$

$$\psi_\alpha^i(t) = \omega_\alpha^i\left(I_\alpha^i\right) t + \psi_\alpha^{io}$$

and

$$\left\{\omega_\alpha^i\left(I_\alpha^i\right)\right\}_{\alpha=1,2,3} = \left(\omega_x^i\left(I_x^i\right),\ \omega_z^i\left(I_z^i\right),\ \omega_s^i(J^i)\right)$$

In this action-angle formulation, the v_α^i and r_α^i are periodic functions of the angle variable ψ_α^i with period 2π. Hence $\mathcal{L}_{int}^i$ and $V(i,j)$, introduced in Section 4.2 (Eq. 4.2.2) are periodic functions of ψ_α^i and ψ_β^j separately and we can write a Fourier-series decomposition of $\mathcal{L}_{int}^i$ and $V(i,j)$ in terms of harmonics of ψ_α^i and ψ_β^j.

Referred back to the laboratory frame, the action-angle tranformations for the two transverse betatron oscillations remain the same for both coasting and bunched beams. Longitudinally, however, the synchrotron oscillations in a bunched beam in the laboratory frame are described by equations, transformed from Eq. (4.3.1), as follows:

$$\begin{aligned} \theta^{i,r}(t) &= \left[\theta_r^0 + \omega_0 t\right] + Ⓗ^{i,r}\left(J^{i,r}, \psi^{i,r}\right) \\ \dot{\theta}^{i,r} &= \omega_0 + \dot{Ⓗ}^{i,r}\left(J^{i,r}, \psi^{i,r}\right) \end{aligned} \tag{4.3.2}$$

where the first terms on the right-hand sides represent the free-streaming parts of the orbits in the laboratory frame. Here ω_0 is the angular velocity of the reference synchronous particle in the $r^{\underline{th}}$ bunch $(r = 1,2,3,\ldots,h$ where h = harmonic number of rf cavity), characterized by its initial phase or azimuth θ_r^0 in the storage ring. In Eq. (4.3.2), i can be any particle in any one of the h bunches $(r = 1,\ldots,h)$ that the storage ring can ideally store within its circumference. We are thus considering h identical stationary bunches, each containing $j = 1,\ldots,N$ particles within the storage ring. The total number of particles is then (hN).

We can write a general Fourier series representation of the periodic δ-functions in Eq. (4.2.2) as follows:

$$\sum_{\ell=-\infty}^{+\infty} \delta\left[\theta^{i,r}(t) - \theta_k - 2\pi\ell\right] = \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} e^{i\ell[\theta^{i,r}(t)-\theta_k]}$$

$$= \frac{1}{2\pi} \sum_{\ell=-\infty}^{+\infty} e^{i\ell\theta_r^0+i\ell\omega_0 t-i\ell\theta_k} \cdot e^{i\ell \Theta^{i,r}(J^{i,r},\psi^{i,r})} \tag{4.3.3}$$

where we have used the orbits in Eq. (4.3.2). Using (4.3.1), (4.3.2) and (4.3.3), we can write for $\mathcal{L}_{int}^{i,r}$ defined in Section 4.2 the following:

$$\mathcal{L}_{int}^{i,r} = \sum_{\substack{j(\neq i)=1,\ldots,N \\ q=1,\ldots,h}} V\left(i^r(t),j^q(t)\right)$$

$$= \sum_{\substack{j(\neq i)=1,\ldots,N \\ q=1,\ldots,h}} \int_{-\infty}^{+\infty} dt' \; V\left(\underset{\sim}{I}^{i,r}(t),\underset{\sim}{\psi}^{i,r}(t);\; \underset{\sim}{I}^{j,q}(t'),\underset{\sim}{\psi}^{j,q}(t')\,|t-t'\right)$$

$$= \sum_{\substack{j(\neq i)=1,\ldots,N \\ q=1,\ldots,h}} \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} \int_{-\infty}^{+\infty} dt' \; V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^{i,r}(t);\; \underset{\sim}{I}^{j,q}(t')|t-t'\right) e^{i[\underset{\sim}{n}\cdot\underset{\sim}{\psi}^{i,r}(t)-\underset{\sim}{n}'\cdot\underset{\sim}{\psi}^{j,q}(t')]} \tag{4.3.4}$$

where

$$V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^{i,r}(t);\; \underset{\sim}{I}^{j,q}(t')|t-t'\right) =$$

$$= \sum_{\ell}\sum_{m}{}_{(-\infty)}^{(+\infty)} e^{i(\ell\omega_0 t+m\omega_0 t')} \; e^{-i(\ell\theta_k+m\theta_p)} \; e^{i(\ell\theta_r^0+m\theta_q^0)} \cdot V_{\underset{\sim}{n}\underset{\sim}{n}'}^{\ell m}\left(\underset{\sim}{I}^{i,r}(t);\; \underset{\sim}{I}^{j,q}(t')|t-t'\right) \tag{4.3.5}$$

In Eq. (4.3.5), $V_{\underset{\sim}{n}\underset{\sim}{n}'}^{\ell m}(\underset{\sim}{I}^{i,r}(t);\; \underset{\sim}{I}^{j,q}(t')|t-t')$ is defined as the components of

$$\frac{q^2}{c}\left(\frac{1}{2\pi}\right)^2 \Bigg[e^{i\ell \textcircled{H}^{i,r}\left(J^{i,r}(t),\psi^{i,r}(t)\right)} \Bigg\{ v_\alpha^{i,r}\left(I_\alpha^{i,r}(t),\psi_\alpha^{i,r}(t)\right)$$

$$D^{\alpha\beta}\left(I_\alpha^{i,r}(t),\ \psi_\alpha^{i,r}(t);\ I_\beta^{j,q}(t'),\ \psi_\beta^{j,q}(t')|t-t'\right)\ v_\beta^{j,q}\left(I_\beta^{j,q}(t'),\ \psi_\beta^{j,q}(t')\right)\Bigg\}$$

$$\cdot\ e^{im\textcircled{H}^{j,q}\left(J^{j,q}(t'),\ \psi^{j,q}(t')\right)}\Bigg]$$

in a harmonic Fourier series decomposition in the periodic angle variables $\underset{\sim}{\psi}^{i,r}(t) = \left\{\psi_\alpha^{i,r}(t)\right\}_{\alpha=1,2,3}$ and $\underset{\sim}{\psi}^{j,q}(t') = \left\{\psi_\beta^{j,q}(t')\right\}_{\beta=1,2,3}$ with period 2π. Here each of $\underset{\sim}{n}$ and $\underset{\sim}{n}'$ denote a triplet set, $\underset{\sim}{n} \equiv (n_x, n_z, \mu)$ and $\underset{\sim}{n}' \equiv (n_x', n_z', \mu')$, of horizontal betatron harmonics (n_x, n_x'), vertical betatron harmonics (n_z, n_z') and longitudinal synchrotron harmonics (μ, μ') of the particles i and j respectively. The phase $\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i(t)$ in $e^{i\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i}$ for harmonic $\underset{\sim}{n}$ is given by

$$\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i(t) = n_x\ \phi_x^i(t) + n_z\ \phi_z^i(t) + \mu\psi^i(t)$$

$$= \sum_{\alpha=1,2,3} n_\alpha\ \psi_\alpha^i(t)$$

$$= \sum_{\alpha=1,2,3} n_\alpha\left[\omega_\alpha^i\left(I_\alpha^i\right)t + \psi_\alpha^{i,o}\right]$$

where

$$\left\{\omega_\alpha^i\left(I_\alpha^i\right)\right\}_{\alpha=1,2,3} = \left\{\omega_x^i\left(I_x^i\right),\ \omega_z^i\left(I_z^i\right),\ \omega_s^i\left(J^i\right)\right\}$$

is the set of horizontal and vertical betatron and longitudinal synchrotron oscillation frequencies of the $i^{\underline{th}}$ particle. For linear transverse betatron oscillations described by horizontal and vertical betatron tunes Q_x and Q_z, this phase becomes

$$\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i(t) = \left[\left(n_xQ_x + n_zQ_z\right)\omega_o + \mu\omega_s^i\left(J^i\right)\right]t + n_x\ \phi_x^{i,0} + n_z\ \phi_z^{i,0} + \mu\psi^{i,0}$$

neglecting chromatic corrections to the tunes Q_x, Q_z, given by Eqs. (3.7), due to longitudinal energy variations arising from synchrotron oscillations in the bunch.

$\mathcal{L}^i_{int}$ as given by Eqs. (4.3.4) and (4.3.5) involves products like $v^i_\alpha v^j_\beta$ and will contain terms $(\omega_0 \cdot \omega_0)$, $(\omega_0 \cdot \dot{Ⓗ}^j)$, $(\dot{Ⓗ}^i \cdot \omega_0)$ and $(\dot{Ⓗ}^i \cdot \dot{Ⓗ}^j)$ for the longitudinal part involving $v^i_\theta\, v^j_\theta$ only. We are often interested in terms which are only first order in $\dot{Ⓗ}/\omega_0$, in which case we neglect the $\dot{Ⓗ}^i \cdot \dot{Ⓗ}^j$ term. We note that $\dot{Ⓗ} \propto \omega_s(J)$ and in real storage rings $\omega_s << \omega_0$ ($\omega_s \sim .001\, \omega_0$ typically) and $(\omega_s/\omega_0)^2$ is a very small quantity indeed compared to (ω_s/ω_0). Also the term involving the product $(\omega_0 \cdot \omega_0)$ can only affect the gross macroscopic motion of the bunch as a coherent whole and is ineffective in influencing the microscopic motion of smaller samples of particles in phase-space individually. In real cooling systems effort is made to suppress this term as much as possible by a careful design of the feedback system (notch filter for example) so that it exhibits zero gain <u>or</u> minimum transfer of signal corresponding to frequencies $\Omega = n\omega_0$ (n an integer) which are harmonics of the revolution frequency of the central reference particle, but still allows other particles with revolution frequencies distributed around ω_0 to experience finite gain corrections and to cool towards the velocity center ω_0 of the beam.

We observe that

$$\underset{\sim}{n} \cdot \underset{\sim}{\psi}^{i,r}(t) - \underset{\sim}{n}' \cdot \underset{\sim}{\psi}^{j,q}(t') = \underset{\sim}{n} \cdot \underset{\sim}{\psi}^{i,r}(t) - \underset{\sim}{n}' \cdot \underset{\sim}{\psi}^{j,q}(t) + \underset{\sim}{n}' \cdot \left[\underset{\sim}{\psi}^{j,q}(t) - \underset{\sim}{\psi}^{j,q}(t')\right]$$

$$= \underset{\sim}{n} \cdot \underset{\sim}{\psi}^{i,r}(t) - \underset{\sim}{n}' \cdot \underset{\sim}{\psi}^{j,q}(t) + \underset{\sim}{n}' \cdot \underset{\sim}{\omega}^{j,q}(t-t') \qquad (4.3.6)$$

where $\underset{\sim}{\psi}^{j,q}(t) = \underset{\sim}{\omega}^{j,q}\, t + \underset{\sim}{\psi}^{j,q}_0$ according to (3.33). With the aid of Eqs. (4.3.5) and (4.3.6), we can then write (4.3.4) as

$$\mathcal{L}^{i,r}_{int} = \sum_{\substack{j(\neq i)=1,\ldots,N \\ q=1,\ldots,h}} \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} e^{i[\underset{\sim}{n}\cdot\underset{\sim}{\psi}^{i,r}(t)-\underset{\sim}{n}'\cdot\underset{\sim}{\psi}^{j,q}(t)]}$$

$$\sum_{\substack{\ell \\ =(-\infty)}}^{+\infty} \sum_{m} e^{i(\ell+m)[\omega_0 t-\theta_k+\theta^0_r]-im(\theta_p-\theta_k)+im(\theta^0_q-\theta^0_r)}$$

$$\int_{-\infty}^{+\infty} dt' \, V^{\ell m}_{\underset{\sim}{n}\underset{\sim}{n}'} \left(\underset{\sim}{I}^{i,r}(t);\ \underset{\sim}{I}^{j,q}(t')|t-t'\right) e^{i(\underset{\sim}{n}'\cdot\underset{\sim}{\omega}^{j,q}-m\omega_0)(t-t')} \tag{4.3.7}$$

We now assume an adiabatic slow cooling process so that in the interaction we can use the approximation $\underset{\sim}{I}(t) = \underset{\sim}{I}(t')$ in the time-scale in which the frequency spectrum of the interaction is established. Usually the frequency spectrum gets established on a fast time-scale of typically a few hundreds to a thousand turns while actual cooling becomes noticeable much more slowly in several thousands of turns. With this slow cooling approximation and a change of variables to $\tau = (t'-t)$ the integral in Eq. (4.3.7) becomes just the Fourier transform of $V^{\ell m}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I}^{i,r},\ \underset{\sim}{I}^{j,q}|\tau)$ in frequency defined by:

$$\tilde{V}^{\ell m}_{\underset{\sim}{n}\underset{\sim}{n}'} \left(\underset{\sim}{I}^{i,r};\ \underset{\sim}{I}^{j,q}|\Omega\right) = \int_{-\infty}^{+\infty} d\tau \, e^{-i\Omega\tau} \, V^{\ell m}_{\underset{\sim}{n}\underset{\sim}{n}'} \left(\underset{\sim}{I}^{i,r};\ \underset{\sim}{I}^{j,q}|\tau\right) \tag{4.3.8}$$

We can then write the interaction Lagrangian in Eq. (4.3.7) as

$$\mathcal{L}^{i,r}_{int} = \sum_{\substack{j(\neq i)=1,\dots,N\\ q=1,\dots,h}} \sum_{\tilde{n}} \sum_{\tilde{n}'} e^{i[\tilde{n}\cdot\tilde{\psi}^{i,r}-\tilde{n}'\cdot\tilde{\psi}^{j,q}]} \sum_{\substack{\ell\\=(-\infty)}}^{(+\infty)} \sum_{m} e^{i(\ell+m)[\omega_0 t-\theta_k+\theta^0_r]}$$

$$e^{-im(\theta_p-\theta_k)}\; e^{im(\theta^0_q-\theta^0_r)}\quad \tilde{V}^{\ell m}_{\tilde{n}\tilde{n}'}\left(\tilde{I}^{i,r};\, \tilde{I}^{j,q}\middle|\tilde{n}'\cdot\tilde{\omega}^{j,q}-m\omega_0\right)$$

$$= \sum_{\substack{j(\neq i)=1,\dots,N\\ q=1,\dots,h}} \sum_{\tilde{n}} \sum_{\tilde{n}'} e^{i[\tilde{n}\cdot\tilde{\psi}^{i,r}-\tilde{n}'\cdot\tilde{\psi}^{j,q}]} \sum_{\substack{k\\=(-\infty)}}^{(+\infty)} \sum_{\ell} e^{ik[\omega_0 t-\theta_k+\theta^0_r]}$$

$$e^{-i(k-\ell)(\theta_p-\theta_k)}\; e^{i(k-\ell)(\theta^0_q-\theta^0_r)}\cdot \tilde{V}^{\ell,k-\ell}_{\tilde{n}\tilde{n}'}\left(\tilde{I}^{i,r};\, \tilde{I}^{j,q}\middle|\tilde{n}'\cdot\tilde{\omega}^{j,q}+(\ell-k)\omega_0\right) \tag{4.3.9}$$

We note that in addition to the translation-invariant part of $V(i^r(t), j^q(t'))$ depending on $\tau = (t'-t)$, which allowed us to use the Fourier transform in (4.3.7), we also have a rapidly oscillating part $e^{ik\omega_0 t}$ which depends periodically on the time t at which the interaction is considered. In order to obtain an interaction describing the slow time-evolution in terms of the coordinates $[\tilde{I}(t), \tilde{\psi}(t)]$ alone with no explicit time-dependence we now average over the fast periodic time-dependence $e^{ik\omega_0 t}$ arising from the periodically discrete interaction with the pickup and kicker of the feedback loop. This long time averaged interaction Lagrangian is obtained by setting $k = 0$ in (4.3.9) and we get

$$\left\langle \mathcal{L}^{i,r}_{int}\right\rangle = \sum_{\substack{j(\neq i)=1,\dots,N\\ q=1,\dots,h}} \sum_{\tilde{n}} \sum_{\tilde{n}'} e^{i[\tilde{n}\cdot\tilde{\psi}^{i,r}-\tilde{n}'\cdot\tilde{\psi}^{j,q}]}\; V_{\tilde{n}\tilde{n}'}\left(\tilde{I}^{i,r};\, \tilde{I}^{j,q}\right) \tag{4.3.10}$$

where

$$V_{\tilde{n}\tilde{n}'}\left(\tilde{I}^{i,r};\, \tilde{I}^{j,q}\right) = \sum_{\ell=-\infty}^{+\infty} e^{i\ell(\theta_p-\theta_k)}\; e^{-i\ell(\theta^0_q-\theta^0_r)}\; \tilde{V}^{\ell,-\ell}_{\tilde{n}\tilde{n}'}\left(\tilde{I}^{i,r};\, \tilde{I}^{j,q}\middle|\tilde{n}'\cdot\tilde{\omega}^{j,q}+\ell\omega_0\right) \tag{4.3.11}$$

Equation (4.3.11) describes the interaction potential between the particle i in the $r^{\underline{th}}$ bunch and particle j in the $q^{\underline{th}}$ bunch in the storage ring. For cooling of a single bunch, $\theta^0_q = \theta^0_r = \theta^0$ and the extra indices q,r and the sum over $q = 1,\dots,h$

can be dropped and Eqs. (4.3.10) and (4.3.11) for a single bunch with $j = 1,\ldots,i,\ldots,N$ particles become:

$$\left\langle \mathcal{L}^i_{int} \right\rangle = \sum_{j(\neq i)=1}^{N} \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} e^{i[\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i - \underset{\sim}{n}'\cdot\underset{\sim}{\psi}^j]} V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i;\, \underset{\sim}{I}^j\right) \tag{4.3.12}$$

where

$$V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i;\, \underset{\sim}{I}^j\right) = \sum_{\ell=-\infty}^{+\infty} e^{i\ell(\theta_p-\theta_k)}\, \tilde{V}^{\ell,-\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i,\underset{\sim}{I}^j \middle| \underset{\sim}{n}'\cdot\underset{\sim}{\omega}^j+\ell\omega_o\right) \tag{4.3.13}$$

We note that only the particle coordinates $[\underset{\sim}{I}^i(t),\underset{\sim}{\psi}^i(t);\, \underset{\sim}{I}^j(t),\underset{\sim}{\psi}^j(t)]$ at one and the same time t enter in the interaction potential or Lagrangian given by Eqs. (4.3.10) or (4.3.12). According to classical Lagrangian or Hamiltonian theory in canonical variables (action $\underset{\sim}{I}$ and angle $\underset{\sim}{\psi}$ in this case), we can write the contribution to the force on a particle i due to another particle $j \neq i$ in the Hamiltonian form

$$\left[\dot{\underset{\sim}{I}}^i\right]_j = \underset{\sim}{G}(i,j) = -\frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}^i}$$

and (4.3.14)

$$\left[\dot{\underset{\sim}{\psi}}^i\right]_j = \underset{\sim}{H}(i,j) = +\frac{\partial V(i,j)}{\partial \underset{\sim}{I}^i}$$

where

$$V(i,j) = V\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\, \underset{\sim}{I}^j,\underset{\sim}{\psi}^j\right) = \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i;\, \underset{\sim}{I}^j\right) e^{i[\underset{\sim}{n}\cdot\underset{\sim}{\psi}^i - \underset{\sim}{n}'\cdot\underset{\sim}{\psi}^j]} \tag{4.3.15}$$

No such potential exists for the self-action force, which induces real cooling. We have to enter this self-action in the equations of motion directly as a nonconservative force and not as the derivative of a potential. However, we can put the descriptions of both the conservative Hamiltonian interaction with other particles and the non-conservative self-interaction on equal footing by using the generalized action and phase force $\underset{\sim}{G}(i,j)$ and $\underset{\sim}{H}(i,j)$ on particle i due to particle j as the basic physically defined quantity, which formally goes over to the nonconservative self-action force under the substitution $j = i$. We thus write:

$$\left[\dot{\underset{\sim}{I}}^i\right]_i = \underset{\sim}{G}(i,i) = \underset{\sim}{G}(i,j)\Big|_{j=i} = \left[-\frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}_i}\right]_{j=i}$$

$$\left[\dot{\psi}^i\right]_i = \underset{\sim}{H}(i,i) = \underset{\sim}{H}(i,j)\Big|_{j=i} = \left[\frac{\partial V(i,j)}{\partial \underset{\sim}{I}_i}\right]_{j=i} \tag{4.3.16}$$

The complete system of equations of motion for the set of $i = 1,\ldots,N$ particles including self-forces and mutual interactions is thus given by

$$\begin{aligned}
\dot{\underset{\sim}{I}}^i &= \left[-\frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}^i}\right]_{j=i} - \sum_{\substack{j\neq i\\=1}}^{N} \frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}^i} = \underset{\sim}{G}(i,i) + \sum_{j(\neq i)=1}^{N} \underset{\sim}{G}(i,j) \\
\dot{\underset{\sim}{\psi}}^i &= \underset{\sim}{\omega}^i + \left[\frac{\partial V(i,j)}{\partial \underset{\sim}{I}^i}\right]_{j=i} + \sum_{\substack{j\neq i\\=1}}^{N} \frac{\partial V(i,j)}{\partial \underset{\sim}{I}^i} = \underset{\sim}{\omega}^i + \underset{\sim}{H}(i,i) + \sum_{j(\neq i)=1}^{N} \underset{\sim}{H}(i,j)
\end{aligned} \tag{4.3.17}$$

The motion under the influence of all the <u>other</u> particles is thus described by a time-dependent Lagrangian or Hamiltonian, where the time-dependence is governed by the motion of all the other particles $j(t)$, $j = 1,\ldots,N$ $(j\neq i)$. The corresponding time-dependent potential is thus given by

$$V(i|t) = \sum_{j(\neq i)=1}^{N} V\Big(i\,|\,j(t)\Big) \tag{4.3.18}$$

We will see later (Chapter 6 and Chapter 9) that the single particle damping rate for action due to the nonconservative self-force in the absence of interaction with other particles is given by

$$\begin{aligned}
\frac{d\underset{\sim}{I}^i(\tau)}{d\tau} &= \left\langle \dot{\underset{\sim}{I}}^i \right\rangle = \left\langle \underset{\sim}{G}(i,i) \right\rangle \\
&= \frac{1}{(2\pi)^3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} d\underset{\sim}{\psi}^i\, \underset{\sim}{G}\Big(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\, \underset{\sim}{I}^i,\underset{\sim}{\psi}^i\Big) \\
&= -\left\langle \left[\frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}^i}\right]_{j=i} \right\rangle
\end{aligned} \tag{4.3.19}$$

for damping on the slow time-scale τ determined by an average $\langle\ldots\rangle$ over the fast oscillation phase $\underset{\sim}{\psi}^i$ of the particle i (characterized by frequency $\underset{\sim}{\omega}^i$). Both the direct interaction (scattering or Schottky noise diffusion) with other particles and the cooperative collective effect between particles tend to reduce this damping.

We see from Eq. (4.3.13) that the spectrum of interaction harmonics determining the cooling process generally consists of frequencies

$$\Omega_{\ell,\underset{\sim}{n}} = \ell\omega_o + \underset{\sim}{n} \cdot \underset{\sim}{\omega}(\underset{\sim}{I})$$

or

$$\Omega_{\ell,n_x,n_z,\mu} = \ell\omega_o + n_x\,\omega_x(I_x) + n_z\,\omega_z(I_z) + \mu\omega_s(J) \qquad (4.3.20)$$

where ℓ, n_x, n_z and μ are integers $-\infty < \ell, n_x, n_z, \mu < +\infty$. For linear transverse betatron oscillations, we have

$$\Omega_{\ell,n_x,n_z,\mu} = \left(\ell + n_xQ_x + n_zQ_z\right)\omega_o + \mu\omega_s(J) \qquad (4.3.21)$$

where Q_x and Q_z are the two transverse betatron tunes. For longitudinal cooling it is only the $\Omega_{\ell,0,0,\mu}$ and for transverse cooling with linear dipole pick-ups and kickers it is the $\Omega_{\ell,\pm1,0,\mu}$ and $\Omega_{\ell,0,\pm1,\mu}$ that are useful. Equation (4.3.21) describes the frequency of the general revolution-synchro-betatron line (ℓ,n_x,n_z,μ) in the single particle Schottky spectrum generated at the pick-up and analyzed by a spectrum analyzer.

The dependence of $V^{\ell,-\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I}^i,\underset{\sim}{I}^j|\Omega)$ in (4.3.13) on particles j and i i.e. on the 'kicker' and the 'kicked' particle variables separate in most physical cases, since the pick-up and kicker are macroscopically separated in configuration space. Then the functional dependence of $\tilde{V}^{\ell,-\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}$ on the actions $\underset{\sim}{I}^i$ and $\underset{\sim}{I}^j$ can be factored with good accuracy into the separated variable form as follows:

$$\tilde{V}^{\ell,\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^j|\Omega\right)\Big|_{\Omega=(\ell\omega_o+\underset{\sim}{n}'\cdot\underset{\sim}{\omega}^j)} = K^{\ell}_{\underset{\sim}{n}}(\underset{\sim}{I}^i)\ P^{-\ell}_{\underset{\sim}{n}'}(\underset{\sim}{I}^j)\ \tilde{D}(\Omega)\Big|_{\Omega=(\ell\omega_o+\underset{\sim}{n}'\cdot\underset{\sim}{\omega}^j)}$$

and

$$V_{\underset{\sim}{n}\underset{\sim}{n}'}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^j\right) = \sum_{\ell=-\infty}^{+\infty} e^{i\ell(\theta_k-\theta_p)}\ K^{\ell}_{\underset{\sim}{n}}(\underset{\sim}{I}^i)\ P^{-\ell}_{\underset{\sim}{n}'}(\underset{\sim}{I}^j)\ \tilde{D}\left(\ell\omega_o + \underset{\sim}{n}'\cdot\underset{\sim}{\omega}^j\right) \qquad (4.3.22)$$

We note that written in the form of a harmonic decomposition in action and angle variables, the interaction is naturally expressed in terms of oscillation action and angle of particles within a bunch in its own frame of reference. In this frame the bunch is macroscopically stationary (at rest) and periodic revolutions through the feedback system manifest as rapidly oscillating periodic time dependence of the interaction as experienced by a particle in the bunch. The slow long time averaged interaction between two particles with actions $\underset{\sim}{I}$ and $\underset{\sim}{I}'$ in the bunch frame in the angle harmonics $\underset{\sim}{n}$ and $\underset{\sim}{n}'$ in phase-space, will have an effective infinite sum over all the revolution harmonics ℓ, describing a continuous smooth interaction in time between the particles at pick-up and kicker, with no wild fluctuations due to discrete passages through pick-up and kicker. The "effective interaction" seen by a particle in a bunched beam, as given by (4.3.22), thus contains the correlated Schottky signal strengths $P^{-\ell}_{\underset{\sim}{n}'}(\underset{\sim}{I}')$ and the sampling oscillator strengths $K^{\ell}_{\underset{\sim}{n}}(\underset{\sim}{I})$ and the transfer function $\tilde{D}(\Omega)$ at the harmonics $\Omega = \ell\omega_0 + \underset{\sim}{n}'\cdot\underset{\sim}{\omega}'$ summed over all the revolution harmonics ℓ. We will explicitly derive this correlated structure of the effective interaction for longitudinal and transverse bunched beam cooling in Chapter 6. This enhanced effective gain experienced by a particle in a bunch has significant effect on its cooling rate, as we will see later.

The situation is quite different for continuous coasting beams, where the revolution harmonics become part of the oscillation harmonics described by frequencies $\Omega = \ell\omega_0 + n_x\omega_x(I_x) + n_z\omega_z(I_z) = \underset{\sim}{n}\cdot\underset{\sim}{\omega}$ where $\underset{\sim}{n} = (n_x, n_z, \ell)$. The laboratory frame is a natural frame for a description of coasting beams in action-angle variables and the full potential separates in $\underset{\sim}{I}$, $\underset{\sim}{I}'$ as

$$V_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') = K_{\underset{\sim}{n}}(\underset{\sim}{I})\, P_{\underset{\sim}{n}'}(\underset{\sim}{I}')\, \tilde{D}(\Omega)\Big|_{\Omega=\underset{\sim}{n}'\cdot\underset{\sim}{\omega}'} \tag{4.3.23}$$

where $\underset{\sim}{n}\cdot\underset{\sim}{\omega} = \ell\omega_0 + n_x\omega_x(I_x) + n_z\omega_z(I_z)$, with no summation over ℓ, which becomes part of the harmonic $\underset{\sim}{n}$. This is an essential difference between the cooling interactions experienced by particles in coasting and bunched beams.

In practical feedback systems, the electrodynamic interactions between particles is most conveniently described in terms of impedences, admittances and transfer functions of the pick-up-amplifier-kicker loop. These electrodynamic quantities relate the particle currents or charge densities at the pick-up directly to the voltages or electromagnetic fields at the kicker which determine the forces on the particles. Hence we need not talk about the potential $V(i,j)$ at all. Instead we use the equations of motion (4.3.17) in terms of the generalized forces $\underset{\sim}{G}$ and $\underset{\sim}{H}$ which are easily computed in terms of the lumped parameters (impedances, gain, etc.) of the electronic feedback loop. Such a description also has the desirable property of putting both the conservative forces from other particles and the nonconservative self-force on equal footing based on generalized forces. From now on, we will then employ the following equations of motion in general form to describe the cooling dynamics:

$$\begin{aligned} \dot{\underset{\sim}{I}}^i &= \sum_{j=1}^{N} \underset{\sim}{G}\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j\right) \\ \dot{\underset{\sim}{\psi}}^i &= \underset{\sim}{\omega}^i + \sum_{j=1}^{N} \underset{\sim}{H}\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j\right) \end{aligned} \tag{4.3.24}$$

where the summation over j now includes the $j = i$ term. Note that $\underset{\sim}{G}$ and $\underset{\sim}{H}$ are vector quantities now, with components in all three directions x, z and θ in general and are functions of the full three-dimensional action and angle variables $(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j)$ of the kicker and kicked particles. Equation (4.3.24) then describes a cooling interaction for action and phase that couples all three degrees of freedom of the system, and in its general form thus applies to the case where the pick-up sensor derives signals involving all three degrees of freedom of the particle inducing the signal and the voltage or electromagnetic field at the kicker, which in turn affects all three degrees of freedom of the cooled particle. The nature of the coupling between various degrees of freedom will determine the specific functional dependences of the various components $G_{x,z,\theta}$ and $H_{x,z,\theta}$ on their arguments $(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j)$.

We will also use the general Fourier series representation of $\underset{\sim}{G}$ and $\underset{\sim}{H}$ in harmonics of the periodic angle variables $\underset{\sim}{\psi}^i,\underset{\sim}{\psi}^j$ as follows:

$$\begin{aligned} \underset{\sim}{G}\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j\right) &= \sum_{\underset{\sim}{n}_i}\sum_{\underset{\sim}{n}_j} \underset{\sim}{G}_{\underset{\sim}{n}_i\underset{\sim}{n}_j}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^j\right) e^{i[\underset{\sim}{n}_i\cdot\underset{\sim}{\psi}^i+\underset{\sim}{n}_j\cdot\underset{\sim}{\psi}^j]} \\ \underset{\sim}{H}\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^j,\underset{\sim}{\psi}^j\right) &= \sum_{\underset{\sim}{n}_i}\sum_{\underset{\sim}{n}_j} \underset{\sim}{H}_{\underset{\sim}{n}_i\underset{\sim}{n}_j}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^j\right) e^{i[\underset{\sim}{n}_i\cdot\underset{\sim}{\psi}^i+\underset{\sim}{n}_j\cdot\underset{\sim}{\psi}^j]} \end{aligned} \tag{4.3.25}$$

We note that written in the form of Eqs. (4.3.24) and (4.3.25) with harmonic decomposition into a general harmonic set $\{\underset{\sim}{n}_i, \underset{\sim}{n}_j\}$, the above describes not only the coupling between the degrees of freedom induced by cooling but also general nonlinear pick-ups and kickers which detect and affect higher harmonics $(|n_x|, |n_z| > 1)$ of betatron motion in addition to the first harmonic $n_x, n_z = \pm 1$ (corresponding to linear pick-ups and kickers detecting the dipole betaton signal and affecting the dipole moment of the betatron oscillations only).

We incorporate in this generalized force formulation the conservative Hamiltonian nature of the interaction of a particle i with all the <u>other</u> particles $j(\neq i) = 1, \ldots, N$, derived from a time-dependent Hamiltonian or Lagrangian involving a time-dependent potential $V(i|t)$, by demanding that the following Hamiltonian flow condition be satisfied for each particle:

$$\frac{\partial}{\partial \underset{\sim}{I}^i} \cdot \left[\dot{\underset{\sim}{I}}^i - \underset{\sim}{G}(i,i)\right] = - \frac{\partial}{\partial \underset{\sim}{\psi}^i} \cdot \left[\dot{\underset{\sim}{\psi}}^i - \underset{\sim}{H}(i,i)\right]$$

or (4.3.26)

$$\frac{\partial}{\partial \underset{\sim}{I}^i} \cdot \left[\sum_{j(\neq i)=1}^{N} \underset{\sim}{G}(i,j)\right] + \frac{\partial}{\partial \underset{\sim}{\psi}^i} \cdot \left[\sum_{j(\neq i)=1}^{N} \underset{\sim}{H}(i,j)\right] = 0$$

In general again, we will also have the separated variable representation, analogous to (4.3.22) and (4.2.23), of $\underset{\sim}{G}$ and $\underset{\sim}{H}$ as follows:

$$G^{\alpha}_{\underset{\sim}{n}_i \underset{\sim}{n}_j}\left(\underset{\sim}{I}_i; \underset{\sim}{I}_j\right) = \sum_{\ell=-\infty}^{+\infty} e^{-i\ell(\theta_p - \theta_k)} \, K^{\alpha,\ell}_{\underset{\sim}{n}_i}(\underset{\sim}{I}^i) \cdot P^{\alpha,-\ell}_{\underset{\sim}{n}_j}(\underset{\sim}{I}^j) \, \tilde{G}\left(\ell\omega_0 + \underset{\sim}{n}_j \cdot \underset{\sim}{\omega}^j\right)$$

$$\left[\underset{\sim}{n}_j = \left(n^j_x, n^j_z, \mu^j\right)\right] \qquad (4.3.27)$$

$$\alpha = (x, z, \theta)$$

defining the effective interaction or gain for the action of particles in a bunched beam, with the inherent sum over the correlated revolution harmonics ℓ and

$$G^{\alpha}_{\underset{\sim}{n}_i \underset{\sim}{n}_j}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^j\right) = e^{-i\ell(\theta_p-\theta_k)}\ K^{\alpha}_{\underset{\sim}{n}_i}(\underset{\sim}{I}^i)\ \cdot P^{\alpha}_{\underset{\sim}{n}_j}(\underset{\sim}{I}^j)\ \tilde{G}(\underset{\sim}{n}_j \cdot \underset{\sim}{\omega}_j)$$

$$\left[\underset{\sim}{n}_j = (n^j_x,\ n^j_z,\ \ell)\right] \qquad (4.3.28)$$

$$\alpha = (x,z,\theta)$$

for the action interaction of particles in a coasting beam. Note that the above decomposition holds for each component $\alpha = x,z,\theta$ separately. Similar decompositions hold for the phase-interaction $H^{\alpha}_{\underset{\sim}{n}_i \underset{\sim}{n}_j}(\underset{\sim}{I}^i;\underset{\sim}{I}^j)$ also.

We have lumped in the function $P^{\alpha,-\ell}_{\underset{\sim}{n}_j}(\underset{\sim}{I}^j)$ the oscillation amplitude or action $(\underset{\sim}{I}^j)$-dependence of the single particle signal at the pick-up and also the oscillation amplitude or action-sensitivity of the pick-up itself. Similarly the function $K^{\alpha,\ell}_{\underset{\sim}{n}_i}(\underset{\sim}{I}^i)$ contain the amplitude-sensitivity of the kicker as well as the dependence on amplitude arising from the sampling of signals at the kicker by the oscillator with its own amplitude and phase. We will see in Chapter 5, 6 and 7 that for linear transverse dipole pick-ups and kickers $(n_x,n_z = \pm 1)$ with no coupling between degrees of freedom, the pick-up and kicker functions $P^{(x,z),-\ell}_{\pm 1,\mu}(I^j_{x,z},J^j)$ and $K^{(x,z),\ell}_{\pm 1,\mu}(I^i_{x,z},J^i)$ are simply proportional to the amplitudes $A_{x,z} = [2I_{x,z}]^{1/2}$ of transverse betatron oscillations and are given by:

$$P^{(x,z),-\ell}_{\pm 1,\mu}\left(I^j_{x,z},\ J^j\right) \propto \left[2I^j_{x,z}\right]^{1/2}\ J_{\mu}\left((-\ell\pm Q_{x,z})\sqrt{2J^j}\right) = A^j_{x,z}\ J_{\mu}\left((-\ell\pm Q_{x,z})a^j\right)$$

$$K^{(x,z),\ell}_{\pm 1,\mu}\left(I^i_{x,z},\ J^i\right) \propto \left[2I^i_{x,z}\right]^{1/2}\ J_{\mu}\left((\ell\pm Q_{x,z})\sqrt{2J^i}\right) = A^i_{x,z}\ J_{\mu}\left((\ell\pm Q_{x,z})a^i\right) \qquad (4.3.29)$$

for bunched beams, where J_{μ} is an ordinary Bessel function of order μ. The corresponding functions for pure longitudinal cooling of synchrotron oscillations with action $J = 1/2\ a^2$ are

$$P^{\theta,-\ell}_{\mu}(J^j) \propto J_{\mu}\left(-\ell\sqrt{2J^j}\right) = J_{\mu}\left(-\ell a^j\right)$$

$$(4.3.30)$$

$$K^{\theta,\ell}_{\mu}(J^i) \propto J_{\mu}\left(\ell\sqrt{2J^i}\right) = J_{\mu}\left(\ell a^i\right)$$

For coasting beams, the dependence on the transverse amplitudes or actions $I_{x,z}$ for linear dipole cooling remains the same. The factors giving dependence on the longitudinal action J or momentum (energy) deviation are either constants (e.g. in notch filter cooling where dependence on longitudinal momentum deviation $\Delta p = p - p_0$ arises solely from the filter transfer function $G(\Omega = \ell\omega(p))$, see [25]) or linear functions of the longitudinal action or momentum (energy) deviation (e.g. energy sensitive pick-ups and kickers) as follows:

$$P_{\underline{n}}(J^j) = 1 + \alpha_{\underline{n}}\left(E^j - E^0\right) = 1 + \beta_{\underline{n}}\left(p^j - p^0\right) = 1 + \delta_{\underline{n}}\, J^j$$
$$K_{\underline{n}'}(J^i) = 1 + \alpha_{\underline{n}'}\left(E^i - E^0\right) = 1 + \beta_{\underline{n}'}\left(p^i - p^0\right) = 1 + \delta_{\underline{n}'}\, J^i \tag{4.3.31}$$

The most general dependence of $P^{x,z}_{n_x,n_z}(I^j_{x,z})$ and $K^{x,z}_{n_x,n_z}(I^i_{x,z})$ on I_x and I_z for coasting beams for arbitrary harmonics (n_x,n_z) for spatially finite pick-ups or kickers is given by complicated integrals over Bessel functions whose arguments depend linearly on $[2I_{x,z}]^{1/2}$ and is discussed in detail in [9].

The Fourier series expansion of the interaction in phase-angle harmonics of the three-dimensional oscillations is exact and superior to the Taylor-series expansion in amplitudes $A^\alpha_i = [2I^\alpha_i]^{1/2}$ and $A^\alpha_j = [2I^\alpha_j]^{1/2}$ often used in the literature. Thus for example, we will see later in Chapters 5, 6, and 7 that the interaction harmonics for longitudinal cooling of synchrotron oscillations are given by

$$G_{\mu\mu'}\left(a_i,a_j\right) \sim \sum_{m=-\infty}^{+\infty} \frac{\mu}{(-m)}\, J_\mu(-ma_i)\, J_{\mu'}(ma_j)\, \tilde{G}\left[m\omega_0 + \mu'\omega_s(a_j)\right]\, e^{im(\theta_p-\theta_k)} \tag{4.3.32}$$

and goes over to the Taylor expanded form $a_i^\mu\, a_j^{\mu'}$ only in the limit of $ma_i \to 0$, $ma_j \to 0$ when one uses the small argument limit of Bessel functions. However since the amplitudes of the transverse betatron oscillations in storage rings $A_{x,z} = A_\perp$ are small in comparison with the effective transverse aperture $R_\perp$ of the machine, the two expansions are the same to within terms $O(A_\perp^2/R_\perp^2)$. The harmonics $G^\perp_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I}^i,\underset{\sim}{I}^j)$ is thus proportional to the corresponding power of the amplitudes of the transverse oscillations $(I^i_\perp)^{n_\perp/2}\,(I^j_\perp)^{n'_\perp/2}$.

The function $\tilde{G}(\Omega)$ in Eq. (4.3.27) unambiguously characterizes the complex gain (amplitude and phase) as a function of frequency of the signal transfer line connecting the pick-up and kicker, including amplifiers, etc. It includes transfer functions of filters and cables and the amplifier gain itself, all characterized by a net electronic gain $g(\Omega)$. In addition $\tilde{G}(\Omega)$, the total electronic gain, also contains a factor associated with the delay τ in the transfer line given by

$$T(\Omega) = e^{-i\Omega \frac{L}{c}} = e^{-i\Omega\tau} \tag{4.3.33}$$

so that

$$\tilde{G}(\Omega) = g(\Omega)\; T(\Omega) = g(\Omega)\; e^{-i\Omega \frac{L}{c}} = g(\Omega)\; e^{-i\Omega\tau}$$

where L represents the total electrical length of the system. We then have for G appearing in (4.3.27) the following:

$$\begin{aligned} \tilde{G}\left(\Omega = \ell\omega_o + \mathbf{n}_j \cdot \boldsymbol{\omega}^j\right) &= g\left(\ell\omega_o + \mathbf{n}_j \cdot \boldsymbol{\omega}^j\right)\, e^{-i(\ell\omega_o + \mathbf{n}_j\cdot\boldsymbol{\omega}^j)\tau} \\ &= g\left(\ell\omega_o + \mathbf{n}_j \cdot \boldsymbol{\omega}^j\right)\, e^{-i[\ell\omega_o + n_x^j\omega_x^j(I_x^j) + n_z^j\omega_z^j(I_z^j) + \mu^j\omega_s^j(J^j)]\tau} \end{aligned} \tag{4.3.34}$$

Typically the delay is set to be the same as the transit time between pick-up and kicker of a reference particle in the beam with angular revolution frequency ω_o, i.e.

$$\tau = \frac{\theta_k - \theta_p}{\omega_o} = \frac{\Delta\theta}{\omega_o} \tag{4.3.35}$$

Thus

$$\tilde{G}\left(\ell\omega_o + \mathbf{n}_j \cdot \boldsymbol{\omega}^j\right) = e^{-i\ell(\theta_k - \theta_p)}\; e^{-i[n_x^j\omega_x^j(I_x^j) + n_z^j\omega_z^j(I_z^j) + \mu^j\omega_s^j(J^j)]\tau}\; g\left(\ell\omega_o + \mathbf{n}_j \cdot \boldsymbol{\omega}^j\right) \tag{4.3.36}$$

The factor $\exp[-i\ell(\theta_k-\theta_p)]$ is cancelled by the factor $\exp[i\ell(\theta_k-\theta_p)]$ appearing in Eq. (4.3.27). For linear transverse dipole cooling in one dimension, z say, $n_x^j = 0$ and $n_z^j = \pm 1$, so that we are left with a phase factor of

$$e^{\mp i\omega_z^j(I_z^j)\tau}\, e^{-i\mu^j\omega_s^j(J^j)\tau} = e^{\mp i\phi_z^{jpk} - i\mu\psi_s^{jpk}} \tag{4.3.37}$$

corresponding to the phase-advance ϕ_z^{jpk} and ψ_s^{jpk} of the betatron and synchrotron oscillations of particle j from pick-up to kicker. There is an optimum choice of these phase advances for effective cooling.

Single particle cooling for action, $\dot{\underset{\sim}{I}}^i$ is obtained from (4.3.24) and (4.3.25) by setting $j = i$ and taking the long time average yielding $\underset{\sim}{n}_j = -\underset{\sim}{n}_i$ (see Eq. (4.3.19)). We thus obtain

$$\frac{dI_z^i(\tau)}{d\tau} = \left\langle \dot{I}_z^i \right\rangle = \left\langle G^z\left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ \underset{\sim}{I}^i,\underset{\sim}{\psi}^i\right)\right\rangle_{\underset{\sim}{\psi}^i}$$

$$= \sum_{\underset{\sim}{n}_i} G^z_{\underset{\sim}{n}_i,-\underset{\sim}{n}_i}\left(\underset{\sim}{I}^i;\ \underset{\sim}{I}^i\right)$$

$$= \sum_{n_z}\sum_{\mu} G^z_{(n_z,\mu)(-n_z,-\mu)}\left(I_z^i,\ J^i;\ I_z^i,\ J^i\right) \tag{4.3.38}$$

for cooling of betatron oscillation in z-direction only. From the prescription

$$\left\langle \dot{\underset{\sim}{I}}^i \right\rangle = \left\langle \underset{\sim}{G}(i,i) \right\rangle = \left\langle \left[-\frac{\partial V(i,j)}{\partial \underset{\sim}{\psi}^i}\right]_{j=i} \right\rangle \tag{4.3.39}$$

we observe that we obtain a multiplying factor of $(-in_z)$ in $\langle \underset{\sim}{G}(i,i)\rangle$ when we take the derivative of the potential. Thus we can write:

$$\left\langle \dot{I}_z^i \right\rangle = \sum_{n_z} \sum_{\mu} (-in_z) \sum_{\ell} \left\{ K^{z,\ell}_{n_z,\mu}\left(I_z^i, J^i\right)\ g\left(\ell\omega_0 + n_z\omega_z^i\left(I_z^i\right) + \mu\omega_s^i\left(J^i\right)\right)\ P^{z,-\ell}_{-n_z,-\mu}\left(I_z^i, J^i\right) \right\}$$

$$e^{i[\mp\phi_z^{ipk} - \mu\psi_s^{ipk}]}$$

$$= \sum_{(\mp)} \sum_{\mu} (\mp i) \sum_{\ell} \left\{ K^{z,\ell}_{\pm 1,\mu}(i) \cdot g_\ell^{(\pm)}(i) \cdot P^{z,-\ell}_{\mp 1,-\mu}(i) \right\} e^{i[\mp\phi_z^{pk} - \mu\psi_s^{pk}]}$$

$$= \sum_{\ell} \sum_{(\mp)} \sum_{\mu} D_\mu^\ell(\pm) \tag{4.3.40}$$

Let us set $\chi^i = \arg[K^{(\pm)}(i) \cdot g^{(\pm)}(i) \cdot P^{(\pm)}(i)]$ in Eq. (4.3.40). Then

$$\mathrm{Re}\left\{D_\mu^\ell(\pm)\right\} \propto \pm \operatorname{Sin}\left[\chi^i \mp \phi_z^{ipk} - \mu\psi_s^{ipk}\right] \tag{4.3.41}$$

We now assume that the synchrotron oscillations are much slower than the revolution times and the betatron oscillations, so that the synchrotron phase-advance ψ_s^{ipk} of particle i between pick-up and kicker is negligible: $\psi_s^{ipk} \approx 0$. Moreover we assume that $\mu\omega_s^i << \omega_0$ for the highest synchrotron harmonic contributing within the bandpass of the feedback system so that $\mu\psi_s^{ipk} \approx 0$ also. We will see later in Chapter 14 that this is indeed the case for realistic cooling systems. The optimum compensation of phase in (4.3.41) give the solutions

$$\phi_z^{ipk} = \frac{2n+1}{2}\pi, \qquad n = 0,2,4,\ldots \qquad \text{for} \quad \chi^i = 0$$

$$\phi_z^{ipk} = \frac{2n-1}{2}\pi, \qquad n = 2,4,\ldots \qquad \text{for} \quad \chi^i = \pi\,. \tag{4.3.42}$$

Thus by adjusting the electrical phase-shift of the feedback system to correspond to a betatron phase-advance of an odd multiple of $\pi/2$ between the pick-up and kicker, we can compensate for the betatron phase advance ϕ_z^{ipk} optimally. Note that this optimal compensation can only be achieved ideally for a single frequency $\underset{\sim}{\omega}^j = \underset{\sim}{\omega}(I_z^j, J^j)$ satisfying

$$\omega_z^j\left(I_z^j\right) \cdot \tau = \frac{2n\pm1}{2}\,\pi, \quad n = 0,2,4,\ldots \tag{4.3.43}$$

There is a residual of uncompensated phase for other particles with different frequencies around $\underset{\sim}{\omega}^j$. It is not possible to compensate simultaneously the phase-advances between the pick-up and kicker of all the particles unless they all have the same frequencies. This sets an upper limit to the frequency spread in the beam tolerable for cooling purposes without introducing large phase mismatches between pick-up and kicker for all the particles, which degrades cooling.

Finally, in the situation of cooling several bunches, we note that Eq. (4.3.11) describes the interaction potential between the particle i in the $r^{\underline{th}}$ bunch and particle j in the $q^{\underline{th}}$ bunch in the storage ring. We assume that each bunch in the ring can be cooled separately independent of the other bunches by using suitable gating techniques. Indeed if the fields at the kicker last only for a length of time comparable to but no more than a single bunch duration, separate bunches will not feel each other through the feedback loop, but will only feel themselves. Such would be the case if the interaction harmonic $G^{\ell,-\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}(\Omega = \underset{\sim}{n}' \cdot \underset{\sim}{\omega}^{j,q} + \ell\omega_0)$ (i.e. the gain or transfer function $g(\Omega = \underset{\sim}{n}' \cdot \underset{\sim}{\omega}^{j,q} + \ell\omega_0)$ of the feedback loop, which is embedded in $G^{\ell,-\ell}_{\underset{\sim}{n}\underset{\sim}{n}'}$) is fairly flat <u>or</u> has almost constant value at the sampled frequencies $\Omega = \ell\omega_0 + \underset{\sim}{n}' \cdot \underset{\sim}{\omega}^{j,q}$ for a region $|\Delta\ell|$ in ℓ such that $|\Delta\ell| \geqslant 2\pi/\Delta\theta = h$ where $\Delta\theta = 2\pi/h$ is the minimum separation in azimuth between two bunches. (This is easily seen by noting that Eq. (4.3.11) contains the phase-factor $\exp[-i\ell(\theta_q^0-\theta_r^0)] = \exp[-i\ell(q-r)\Delta\theta]$ and that for constant $V^{\ell,-\ell}_{nn'}$ within $(-\ell_m)$ and $(+\ell_m)$, $\sum_\ell e^{-i\ell(\theta_q^0-\theta_r^0)}\, V^{\ell,-\ell}_{nn'} \to V \int_{-\ell_m}^{+\ell_m} d\ell\, e^{-i\ell(q-r)\Delta\theta} =$ $2V\left[\frac{\mathrm{Sin}[\ell_m(q-r)\Delta\theta]}{(q-r)\;\Delta\theta}\right]$ which has significant values only when $\ell_m(q-r)\Delta\theta < 2\pi$ i.e. $\ell_m < \frac{2\pi}{\Delta\theta}\frac{1}{(q-r)} \leqslant h$ and decreases rapidly to zero for $\ell_m > h$ with oscillating phase). We therefore consider single bunch cooling only in this report.

We will derive the action and phase equations of motion explicitly for the transverse (dipole) and longitudinal cooling of bunched beams in the following chapters. To gain more physical insight into the cooling interaction for bunches we consider now a simple example of a model cooling system, first studied by Derbenev and Kheifets [33] in the context of coasting beam cooling.

An example

We consider transverse cooling of a longitudinally bunched beam under a model dipole cooling interaction [33] described by the potential

$$V(i,j) = \sqrt{I^i}\,\sqrt{I^j}\,\mathrm{Sin}\left(\phi^i - \phi^j\right)\, g\left(\theta^i - \theta^j\right) \tag{4.3.44}$$

between particles i and j ($j \neq i$) where $(I^i = 1/2(A^i)^2,\ \phi^i)$ and $(J^i = 1/2(a^i)^2,\ \psi^i)$ are the action and angle variables for the transverse betatron and longitudinal synchrotron oscillations of the $i^{\underline{th}}$ particle. The longitudinal synchrotron orbits are given by

$$\theta^i(t) = \omega_0\, t + a^i \sin\left[\omega_s^i\, t + \psi^i(0)\right] = \omega_0\, t + \sqrt{2J^i}\,\mathrm{Sin}\,\psi^i(t)$$

$$\dot{\theta}^i(t) = \omega_0 + \omega_s^i\, a^i \cos\left[\omega_s^i\, t + \psi^i(0)\right] = \omega_0 + \omega_s^i\,\sqrt{2J^i}\,\mathrm{Cos}\,\psi^i(t) \tag{4.3.45}$$

In (4.3.44) $g(\theta)$ determines the azimuthal distance of effective interaction between the particles and depends on the feedback loop. Periodicity in θ implies a Fourier series representation

$$g(\theta) = \sum_{\ell=-\infty}^{+\infty} g_\ell\, e^{i\ell\theta} \tag{4.3.46}$$

and reality of $g(\theta)$ implies $g_\ell = g^*_{-\ell}$. With $g(\theta)$ non-antisymmetric, the interaction given by (4.3.44) is then explicitly nonhermitian $(V(i,j) \neq V(j,i))$. Note that we are considering only the dipole betatron interaction $n_\perp = \pm 1$, which is sufficient for accuracies of the order of $O(A^2/R_\perp^2)$ when the amplitude A of the displacement from the equilibrium orbit is small compared to the dimension $R_\perp$ (transverse) of characteristic apertures. Also since we are interested in slow cooling with large damping times, the interaction (4.3.44) is taken to be a function of only the phase-difference between particles $(\phi^i - \phi^j)$, with the fast phases $(\phi^i + \phi^j)$ averaged about. This is a good approximation when the relative frequency spread $\Delta\omega/\omega$ in the beam is small, so that the only slow phase in $\underset{\sim}{n}_i \cdot \underset{\sim}{\psi}^i - \underset{\sim}{n}_j \cdot \underset{\sim}{\psi}^j = [\underset{\sim}{n}_i \cdot \underset{\sim}{\omega}^i(\underset{\sim}{I}^i) - \underset{\sim}{n}_j \cdot \underset{\sim}{\omega}^j(\underset{\sim}{I}^j)]t + \underset{\sim}{n}_i \cdot \underset{\sim}{\psi}^i(0) - \underset{\sim}{n}_j \cdot \underset{\sim}{\psi}^j(0)$ corresponds to $\underset{\sim}{n}_j = \underset{\sim}{n}_i = \underset{\sim}{n}$ and is given by $\underset{\sim}{n} \cdot [\underset{\sim}{\psi}^i - \underset{\sim}{\psi}^j]$. The equations of motion in the transverse phase-space in the presence of cooling are written as

$$\dot{I}^{i} = \sum_{j=1}^{N} G(i,j)$$

$$\dot{\phi}^{i} = \omega_{\perp}(i) + \sum_{j=1}^{N} H(i,j) \tag{4.3.47}$$

where

$$G(i,j) = -\frac{\partial V(i,j)}{\partial \phi^{i}} = -\sqrt{I^{i}}\,\sqrt{I^{j}}\,\mathrm{Cos}\left(\phi^{i} - \phi^{j}\right) g\left(\theta^{i} - \theta^{j}\right)$$

$$H(i,j) = \frac{\partial V(i,j)}{\partial I^{i}} = \frac{1}{2}\sqrt{\frac{I^{j}}{I^{i}}}\,\mathrm{Sin}\left(\phi^{i} - \phi^{j}\right) g\left(\theta^{i} - \theta^{j}\right) \tag{4.3.48}$$

The self-action terms $(j=i)$ are thus

$$G(i,i) = -\,g(0)\,I^{i} = -\left(\sum_{\ell=-\infty}^{+\infty} g_{\ell}\right) I^{i} \tag{4.3.49}$$

$$H(i,i) = 0 \tag{4.3.50}$$

Thus the quantity $g(0) = \sum_{\ell=-\infty}^{+\infty} g_{\ell}$ determines the single particle damping rate due to the self-action alone as follows:

$$\dot{I}^{i} = -\,g(0)\,I^{i}$$

$$I^{i}(t) = I^{i}(t=0)\,e^{-g(0)t} = I^{i}(0)\,e^{-\left(\sum_{\ell=-\infty}^{+\infty} g_{\ell}\right)t} \tag{4.3.51}$$

Interactions $G(i,j)$ and $H(i,j)$ with other particles j $(j \neq i)$ will reduce this ideal damping rate.

Using the identity [1],

$$e^{ix\,\mathrm{Sin}\,y} = \sum_{\mu=-\infty}^{+\infty} J_{\mu}(x)\,e^{i\mu y} \tag{4.3.52}$$

where $J_{\mu}(x)$ is an ordinary Bessel function of order μ and Eqs. (4.3.45) and (4.3.46), we find

$$g\left(\theta^i - \theta^j\right) = \sum_{\substack{\mu \\ =(-\infty)}}^{(+\infty)} \sum_{\mu'} G_{\mu\mu'}(i,j)\, e^{i\mu\psi^i}\, e^{i\mu'\psi^j} \tag{4.3.53}$$

where

$$G_{\mu\mu'}(i,j) = \sum_{\ell=-\infty}^{+\infty} g_\ell\, J_\mu\left(\ell\sqrt{2J^i}\right) J_{\mu'}\left(-\ell\sqrt{2J^j}\right) \tag{4.3.54}$$

describes the effective interaction between the $i^{\underline{th}}$ particle in synchrotron mode μ and $j^{\underline{th}}$ particle in synchrotron mode μ'. We call this the "effective gain".

For general particle orbits in arbitrary potential wells, we use the general action-angle representation of the orbit $\theta(t) = \omega_0 t + Ⓗ(J,\psi)$ in synchrotron phase-space (J,ψ), by defining a suitable canonical transformation $(Ⓗ, \dot{Ⓗ}) \rightarrow (J,\psi)$ as discussed at the end of Chapter 3. Since ψ is a periodic angle variable, we can define an "orbit integral" $O_\mu(\ell,J)$ as the coefficients in a Fourier series expansion of $e^{i\ell Ⓗ}$ in the variable ψ as follows:

$$e^{i\ell Ⓗ(J,\psi)} = \sum_{\mu=-\infty}^{+\infty} O_\mu(\ell,J)\, e^{i\mu\psi} \tag{4.3.55}$$

where

$$O_\mu(\ell,J) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\ell Ⓗ(J,\psi) - i\mu\psi}\, d\psi \tag{4.3.56}$$

In particular,

$$O_\mu(\ell,J) = \begin{cases} J_\mu\left(\ell\sqrt{2J}\right) & \text{for harmonic sinusoidal orbits as in (4.3.45)} \\[2ex] \dfrac{\mathrm{Sin}\left[\frac{\pi}{2}(\alpha-\mu)\right]}{[\alpha^2-\mu^2]} \quad \dfrac{2\alpha}{\pi} & \text{for a square-well potential well or bucket of angular extent } \theta_0 \text{ in the ring and } \alpha = \ell\left(\dfrac{2\theta_0}{\pi}\right) \end{cases} \tag{4.3.57}$$

The second orbit integral is derived in Appendix A. The "effective interaction" $G_{\mu\mu'}(i,j)$, defined by (4.3.53), takes the following form for general oscillatory orbits in arbitrary potential well within the bunch:

$$G_{\mu\mu'}(i,j) = \sum_{\ell=-\infty}^{+\infty} g_\ell \, O_\mu\left(\ell,J^i\right) \, O_{\mu'}\left(-\ell,J^j\right) \tag{4.3.58}$$

From (4.3.55) and (4.3.56) follows the following important properties of $O_\mu(\ell,J)$:

$$O_\mu(\ell,J) = O^*_{-\mu}(-\ell,J) \quad \text{and} \quad \sum_{\mu}\sum_{\mu'}{}_{=(-\infty)}^{(+\infty)} O_\mu(\ell,J) \, O_{\mu'}(-\ell,J) \, e^{i(\mu+\mu')\psi} = 1 \tag{4.3.59}$$

Note that for sinusoidal orbits given by (4.3.45), (4.3.59) reduces to the special case

$$J_\mu\left(\ell\sqrt{2J}\right) = J_{-\mu}\left(-\ell\sqrt{2J}\right); \quad \sum_{\mu}\sum_{\mu'}{}_{=(-\infty)}^{(+\infty)} J_\mu\left(\ell\sqrt{2J}\right) J_{\mu'}\left(-\ell\sqrt{2J}\right) e^{i(\mu+\mu')\psi} = 1 \tag{4.3.60a}$$

and

$$\sum_{\mu=-\infty}^{+\infty} J_\mu^2\left(\ell\sqrt{2J}\right) = 1 \ . \tag{4.3.60b}$$

We can write the cooling interaction (4.3.44) and (4.3.48) for general oscillatory orbits (J,ψ) in synchrotron phase-space as

$$V(i,j) = \sum_{(\pm)} \sum_{\mu}\sum_{\mu'}{}_{(-\infty)}^{(+\infty)} V_{(\mu,\pm1)(\mu',\mp1)}\left(I^i,J^i;\ I^j,J^j\right) e^{i\mu\psi^i+i\mu'\psi^j} \ e^{\pm i\phi^i} \ e^{\mp i\phi^j}$$

$$G(i,j) = \sum_{(\pm)} \sum_{\mu}\sum_{\mu'}{}_{(-\infty)}^{(+\infty)} G_{(\mu,\pm1)(\mu',\mp1)}\left(I^i,J^j;\ I^i,J^j\right) e^{i\mu\psi^i+i\mu'\psi^j} \ e^{\pm i\phi^i} \ e^{\mp i\phi^j} \tag{4.3.61}$$

where

$$V_{(\mu,\pm1)(\mu',\mp1)} = \frac{(\pm1)}{2i} \sum_{\ell=-\infty}^{+\infty} g_\ell \left[(I^i)^{1/2} \, O_\mu(\ell,J^i)\right] \left[(I^j)^{1/2} \, O_{\mu'}(-\ell,J^j)\right]$$

$$G_{(\mu,\pm1)(\mu',\mp1)} = -\frac{1}{2} \sum_{\ell=-\infty}^{+\infty} g_\ell \left[(I^i)^{1/2} \, O_\mu(\ell,J^i)\right] \left[(I^j)^{1/2} \, O_{\mu'}(-\ell,J^j)\right] \tag{4.3.62}$$

which is of the form advocated in Eqs. (4.3.22), (4.3.27) and (4.3.29). We observe that in the bunch frame, all the revolution harmonics ℓ of single particle Schottky signals contribute in a correlated fashion to determine an enhanced "effective strength or gain" of interaction experienced by particles with different synchrotron modes within the bunch.

4.4 Mixing and Correlations in Phase-Space

In the simple model of transverse cooling of a bunched beam discussed in the example at the end of Section 4.3, we can rewrite the equations of motion for action and angle given by (4.3.47) and (4.3.48) as equations of motion for the transverse betatron amplitude $A^i(I^i = 1/2(A^i)^2)$ and phase ϕ^i as follows:

$$\frac{dA^i}{dt} = \dot{A}^i = - \sum_{j=1}^{N} \left\{ \frac{1}{2} \mathrm{Cos}(\phi^i - \phi^j)\ g(\theta^i - \theta^j) \right\} A^j \tag{4.4.1a}$$

$$\frac{d\phi^i}{dt} = \dot{\phi}^i = \omega_\perp(I^i) + \sum_{j=1}^{N} \left\{ \frac{1}{2} \mathrm{Sin}(\phi^i - \phi^j)\ g(\theta^i - \theta^j) \right\} \frac{A^j}{A^i} \tag{4.4.1b}$$

In terms of the complex variables

$$\begin{aligned} x_i &= A^i\, e^{-i\phi^i} \\ x_i^* &= A^i\, e^{i\phi^i} \end{aligned} \tag{4.4.2}$$

the equations of motion, as follows from (4.4.1), are:

$$\begin{aligned} \dot{x}_i + i\omega_\perp(i)\, x_i &= - \sum_{j=1}^{N} g\left(\theta^i - \theta^j\right) x_j = - \sum_{j=1}^{N} g(i,j)\, x_j \\ \dot{x}_i^* - i\omega_\perp(i)\, x_i^* &= - \sum_{j=1}^{N} g^*\left(\theta^i - \theta^j\right) x_j^* = - \sum_{j=1}^{N} g(i,j)\, x_j^* \end{aligned} \tag{4.4.3}$$

where we have used the reality of $g(i,j) = g(\theta^i - \theta^j) = g^*(i,j)$. From (4.4.3) we obtain the time evolution of $|x_i|^2 = (x_i x_i^*)$ and $(x_i x_k^*)$ as follows:

$$\frac{d}{dt}\,|x_i|^2 = \frac{d}{dt}\left(x_i x_i^*\right) = -\,2g(i,i)\ |x_i|^2 - \sum_{\substack{j=1\\ \neq i}}^{N} g(i,j)\ \left(x_i x_j^* + x_i^* x_j\right) \tag{4.4.4}$$

$$\frac{d}{dt}\left(x_i x_k^*\right) = i\left[\omega_\perp(k) - \omega_\perp(i)\right]\left(x_i x_k^*\right) - g(k,i)\ |x_i|^2 - g(i,k)\ |x_k|^2$$
$$-\left[g(i,i) + g(k,k)\right]\left(x_i x_k^*\right) - \sum_{j\neq i,k}\left[g(k,j)(x_i x_j^*) + g(i,j)(x_j x_k^*)\right] \tag{4.4.5}$$

We note that, according to (4.3.53)

$$g(i,j) = \sum_{\substack{\mu\\(-\infty)}}^{(+\infty)}\sum_{\mu'} G_{\mu\mu'}\left(J^i,J^j\right)\ e^{i\mu\psi^i}\ e^{i\mu'\psi^j} \tag{4.4.6}$$

Let us define

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}\ |x_i|^2 \tag{4.4.7}$$

and a correlation function of particles i and j in their (μ,ν) Fourier harmonic modes in synchrotron phase angles as:

$$C_{\mu\nu}\left(\omega_s(i),\omega_s(j);\ t\right) = \frac{\left\langle (x_i x_j^*)\ e^{i\mu\psi^i - i\nu\psi^j}\right\rangle_{\psi_i,\psi_j,\phi_i,\phi_j}}{f\left[\omega_s(J^i)\right]\ f\left[\omega_s(J^j)\right]} \tag{4.4.8}$$

One can similarly define a three-body correlation function $T_{\mu\nu\lambda}(\omega_s(i),\omega_s(j),\omega_s(k);t)$ and so on.

Initially $\langle x_i x_j^*\rangle = 0$ for $i \neq j$ and we see from (4.4.4) that $|x_i|^2$ damps at the ideal rate $2g(i,i)$. However, we see from (4.4.5) that $\mathrm{Re}(x_i x_k^*)$ becomes negative as cooling progresses and degrades the overall cooling rate. Thus although we start with totally uncorrelated particles in the beam with transverse and longitudinal oscillation phases distributed randomly between 0 and 2π, the feedback loop introduces negative correlations between particles, which grows in magnitude as a function of time, since feedback is a systematic non-random process. However, synchrotron frequency spread between particles tend to mix and separate them and destroy these correlations. Two particles with different synchrotron frequencies $\omega_s(i)$ and $\omega_s(j)$ slip away from each other in their synchrotron phase (and hence in azimuthal relative coordinate) according to

$$\Delta\psi^{ij} = \psi^j(t) - \psi^i(t) = \left[\omega_s(j) - \omega_s(i)\right]t + \psi_0^j - \psi_0^i$$

$$= \left(\Delta\omega_s^{ij}\right) t + \Delta\psi_0^{ij}$$

and

$$\frac{d}{dt}\left(\Delta\psi^{ij}\right) = \left(\Delta\dot{\psi}^{ij}\right) = \Delta\omega_s^{ij} \ . \tag{4.4.9}$$

The competition between these two processes, feedback correlation and kinematic mixing determine the overall cooling rate of the beam.

Taking appropriate averages on both sides of (4.4.4) and (4.4.5) one obtains the time-evolution of $\langle x_1^2\rangle$, $\langle x_1 x_2\rangle$, etc. From (4.4.4), we see that the two-particle correlations become important when

$$(N-1)\, g_{\mu\nu}\, C_{\mu\nu} \sim g_{\mu\nu}\, \sigma^2 \tag{4.4.10}$$

i.e. when $C_{\mu\nu} \sim O(\sigma^2/N-1)$. From (4.4.5) the characteristic time (Δt) associated with this value of the correlation is given by

$$\frac{C_{\mu\nu}}{(\Delta t)} \cong \frac{\sigma^2}{N(\Delta t)} \sim g_{\mu\nu}\, \sigma^2$$

<u>or</u>

$$(\Delta t) \sim O\left(1/(g_{\mu\nu}\, N)\right) \tag{4.4.11}$$

By the definition (4.4.8), synchrotron frequency variations will destroy correlations on a time scale given by

$$\left[\mu\omega_s(i) - \nu\omega_s(j)\right] t \sim 1$$

i.e.

$$\frac{1}{2}\,(\mu+\nu)\,\Delta\omega_s^{ij}\, t + (\mu-\nu)\,\overline{\omega_s}\, t \sim 1 \tag{4.4.12}$$

where $\bar{\omega}_s = \frac{1}{2}[\omega_s(i) + \omega_s(j)]$ is the average synchrotron frequency of particles i and j and $\Delta\omega_s^{ij} = \omega_s(i)-\omega_s(j)$ is the spread in their synchrotron frequencies. For small spread $\Delta\omega_s^{ij}$ in synchrotron frequencies $(\Delta\omega_s/\omega_s << 1)$, the fast phase variation in (4.4.12) is provided by the $\bar{\omega}_s$ term which averages to yield $\delta_{\mu\nu}$. The slow phase-variation then is given by

$$\mu \cdot \Delta\omega_s \cdot t \sim 1$$

i.e.

$$t \sim 0\left(1/(\mu\Delta\omega_s)\right) \tag{4.4.13}$$

Therefore two-particle correlations will be significant when

$$\frac{1}{(g_{\mu\mu} N)} \lesssim \frac{1}{(\mu\Delta\omega_s)} \tag{4.4.14}$$

We can thus associate a small parameter ϵ with the relative strengths of correlations where

$$\epsilon = 0\left(\frac{g_{\mu\mu} N}{\mu\Delta\omega_s}\right) \tag{4.4.15}$$

for small frequency spreads $\Delta\omega_s$ in the beam. With single-particle self-correlation normalized to unity (totally correlated) i.e. $0(\epsilon^0)$, two-body correlation is of order $0(\epsilon)$, three-body correlations $0(\epsilon^2)$ and so on. For sufficiently small $g_{\mu\mu}$ or large $(\Delta\omega_s)$, $\epsilon << 1$ and we can ignore correlations higher than the two-body correlations $(T_{\mu\nu\lambda} \sim 0)$. This allows us to use a small ϵ expansion in the kinetic treatment of stochastic cooling in Chapter 9 (Section 9.2).

The parameter corresponding to ϵ for a coasting beam is [6]

$$\delta = 0\left(\frac{g_\ell N}{\ell(\Delta\omega)}\right) \tag{4.4.16}$$

For a bunch, $g_{\mu\mu}$, the effective gain of the interaction, has an inherent sum over all the revolution harmonics (Eqs. (4.3.54) and (4.3.58)) and hence enhanced over the gain g of the feedback loop at a single harmonic only. Also the mixing factor

$(\mu\Delta\omega_s) \sim \ell(a\Delta\omega_s) < \ell_{max}(a_{max}\Delta\omega_s) << \ell_{max}(\Delta\omega)$ unless $\Delta\omega_s \sim (\Delta\omega/a_{max}) \sim \omega_s$ i.e. $\Delta\omega_s/\omega_s \approx 1$ corresponding to a coasting beam with revolution frequency spread $\Delta\omega = a_{max} \cdot \omega_s$ where a_{max} is the maximum amplitude of synchrotron oscillations in the bunch. (Note that $J_\mu(\ell\sqrt{2J}) = J_\mu(\ell a)$ in Eq. (4.3.54) has significant values only up to $\mu \sim \ell a$ for fixed ℓa.) Thus the parameter ϵ tends to be large compared to the paràmeter δ for a coasting beam, leading to strong two-body correlations in synchrotron phase-space of a bunch.

For large spreads in synchrotron frequencies or for values of μ such that $\mu\Delta\omega_s/\omega_s \gtrsim 1$, the fast phase variation in Eq., (4.4.12) is given by the first term involving $\Delta\omega_s^{ij}$, which averages to yield $\delta_{\mu,-\nu}$. The slow phase variation then is given by

$$2\mu\ \overline{\omega}_s\ t \sim 1$$

i.e.

$$t \sim 0\left(\frac{1}{\mu\ \omega_s^m}\right)$$

where ω_s^m is the maximum synchrotron frequency in the beam. The small parameter then is given by

$$\epsilon = 0\left(\frac{g_{\mu\mu}\ N}{\mu\omega_s^m}\right) \tag{4.4.17}$$

$$\sim 0\left(\frac{g_{\mu\mu}\ N}{\ell_m\Delta\omega}\right) \tag{4.4.18}$$

where $\mu\omega_s^m \sim \ell_m\Delta\omega$ for an equivalent coasting beam with revolution frequency spread $\Delta\omega$ and ℓ_m the highest harmonic in the bandpass of the feedback system.

Thus for bunches with sufficiently large synchrotron frequency spread, the kinematic mixing factor is comparable to that of a coasting beam. However ϵ still remains determined by the enhanced effective gain $g_{\mu\mu}$ in the numerator in (4.4.18), which is a manifestation of the fact that particles in a bunch are forever correlated in such a way so as to be confined within the finite length of the bunch only.

4.5 Schottky Spectrum, Sampling, Differential Equation for Oscillator Response and All That

In order to obtain the equations of motion for a single particle undergoing stochastic cooling directly from single particle unperturbed orbits and the transfer characteristics of the feedback loop, it is convenient to visualize the basic process in the following stages:

(a) Particles in the beam set up a Schottky noise signal at the pick-up

$$\underset{\sim}{n}^P(t) = q \sum_{j=1}^{N} \underset{\sim}{\xi}_j(t) \tag{4.5.1}$$

where $\underset{\sim}{\xi}_j(t)$ is the signal due to particle j given by

$$\underset{\sim}{\xi}_j(t) = \begin{cases} \dfrac{\omega_j(t)}{2\pi} \displaystyle\sum_{\ell=-\infty}^{+\infty} \delta\left(t - t_\ell^{j,P}\right) & \text{for longitudinal current signal} \\ \dfrac{\omega_j(t)}{2\pi} \displaystyle\sum_{\ell=-\infty}^{+\infty} x_j(t)\ \delta\left(t - t_\ell^{j,P}\right) & \text{for transverse dipole moment signal} \end{cases} \tag{4.5.2}$$

where $t_\ell^{j,P}$ are the times the particle j with angular velocity ω_j is in the pick-up and q the charge of the particles. A Fourier representation of $\underset{\sim}{n}^P(t)$ gives the Schottky noise spectrum $\tilde{\underset{\sim}{n}}^P(\Omega) = q \sum_{j=1}^{N} \tilde{\underset{\sim}{\xi}}_j(\Omega)$ of the beam signal at the pick-up, as seen by a spectrum analyzer in the laboratory frame in the frequency (Ω) domain.

(b) The signal $\underset{\sim}{n}^P(t)$ is then processed by the transfer line characterized by a linear transfer function $\underset{\approx}{G}(t-t')$ and applied as the feedback signal

$$\underset{\sim}{d}^K(t) = \int_{-\infty}^{t} dt'\ \underset{\approx}{G}(t-t') \cdot \underset{\sim}{n}^P(t) = q \sum_{j=1}^{N} \int_{-\infty}^{t} dt'\ \underset{\approx}{G}(t-t') \cdot \underset{\sim}{\xi}_j(t')$$

$$= \sum_{j=1}^{N} \underset{\sim}{d}_j^K(t) \tag{4.5.3}$$

at the kicker. A spectrum analyzer at the kicker will generate noise voltage or electromagnetic field spectrum $\tilde{\underset{\sim}{d}}^k(\Omega) = \tilde{\underset{\approx}{G}}(\Omega) \cdot \tilde{\underset{\sim}{n}}^p(\Omega)$. We have already seen in Section (4.3) that the Schottky spectrum $\underset{\sim}{n}^p(\Omega)$ will contain in general the frequencies

$$\Omega = \ell\omega_o + n_x^j \, \omega_x^j\left(I_x^j\right) + n_z^j \, \omega_z^j\left(I_z^j\right) + \mu^j \, \omega_s^j\left(J^j\right)$$

for all particles $j = 1,\ldots,N$ and it is the gain function $\underset{\approx}{G}(\Omega)$ evaluated at these frequencies that enters into the kicker signal $\tilde{\underset{\sim}{d}}^k(\Omega)$.

(c) An individual particle i in the beam samples the signal at the kicker only when it passes through the kicker periodically at times $t = t_n^{i,K}$, $n = 0,\pm1,\pm2,\ldots$. The signal sampled by the $i^{\underline{th}}$ particle as a function of time is given by

$$\underset{\sim}{\mathscr{S}}^i(t) = \sum_{n=-\infty}^{+\infty} \underset{\sim}{d}^K(t) \; \delta\left(t - t_n^{i,K}\right)$$

Thus

$$\underset{\sim}{\mathscr{S}}^i(t) = \sum_{n=-\infty}^{+\infty} \delta\left(t - t_n^{i,K}\right) q \cdot \sum_{j=1}^{N} \int_{-\infty}^{t} dt' \; \underset{\approx}{G}(t-t') \cdot f_j \sum_{\ell=-\infty}^{+\infty} \delta\left(t' - t_\ell^{j,P}\right) \begin{bmatrix} 1 \\ x_j(t') \end{bmatrix} \tag{4.5.4}$$

Note the similarity between this expression for the signal seen by $i^{\underline{th}}$ particle from all the particles $j = 1,\ldots,N$ in the beam and the interaction potential $V(i,j)$ between particles i and j as given by (4.2.2). Note especially the two periodic δ-functions, one corresponding to particle j setting up a signal at pick-up periodically and the other corresponding to particle i sampling this signal due to particle j at the kicker periodically in time. Note also the transfer function sandwiched between them.

(d) The sampled signal $\underset{\sim}{\mathscr{S}}^i(t)$ seen by the $i^{\underline{th}}$ particle is effective in changing its oscillation amplitude. $\underset{\sim}{\mathscr{S}}^i(t)$ has the frequency and phase information of all the particles in the beam including the information about the particle i that is sampling the signal and acts as the driving term in the differential equation describing the three-dimensional oscillations of the $i^{\underline{th}}$ particle. The oscillator response is thus given by the nonautonomous, nonconservative differential equation:

$$\ddot{x}_i^\alpha + \left(\omega_i^\alpha\right)^2 x_i^\alpha = \mathcal{S}^{i,\alpha}\left(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i;\ t\right) \tag{4.5.5}$$

where $\left\{\omega^\alpha\right\} = (\omega_x, \omega_z, \omega_s)$ and the explicit time-dependence on the right-hand side is given by the phase-space coordinates $\underset{\sim}{x}_j(t)$, $\dot{\underset{\sim}{x}}_j(t)$ of all the other beam particles $j(\neq i) = 1,\ldots,N$.

The autonomous self-interaction part of $\mathcal{S}^{i,\alpha}$ (the coherent term), denoted $C^{i,\alpha}(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i)$, determines the real damping <u>or</u> cooling of the $i^{\underline{th}}$ particle.

The nonautonomous <u>or</u> incoherent part of $\mathcal{S}^{i,\alpha}$, denoted by $S^{i,\alpha}(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i;\ t)$, describes the Schottky noise from other ($j\neq i$, $j=1,\ldots,N$) particles and causes heating or diffusion. We thus have the decomposition into a coherent cooling term and a Schottky noise term as follows:

$$\underset{\sim}{\mathcal{S}}^i\left(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i;\ t\right) = \underset{\sim}{C}^i\left(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i\right) + \underset{\sim}{S}^i\left(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i;\ t\right) \tag{4.5.6}$$

The differential equation (4.5.5) involving $(x, \dot{x})$ can be transformed into equations for action and angle (I^α, ψ^α) (<u>or</u> equivalently amplitude and phase (A^α, ψ^α)) in the general form of Eq. (4.3.24) by various methods, e.g. method of averaging, method of multiple time-scales, etc. We will use the multiple time-scales method in Chapter 6 to derive the action and angle cooling equations for bunched beam stochastic cooling.

Detailed analysis of sampling and amplitude-phase representation in the context of coasting beam cooling is discussed in [6], [9]. In general, the sampled noise $\mathcal{S}^{i,\alpha}(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i; t) = \sum_{j=1}^{N} \mathcal{S}^{i,\alpha}(\underset{\sim}{x}_i, \dot{\underset{\sim}{x}}_i; \underset{\sim}{x}_j, \dot{\underset{\sim}{x}}_j)$ and the corresponding action noise $G^{i,\alpha}(\underset{\sim}{I}^i, \underset{\sim}{\psi}^i; t) = \sum_{j=1}^{N} G^{i,\alpha}(\underset{\sim}{I}^i, \underset{\sim}{\psi}^i; \underset{\sim}{I}^j, \underset{\sim}{\psi}^j)$ depend nonlinearly on $\left(x_i^\alpha, x_j^\alpha\right)$ and $\left(\sqrt{I_\alpha^i}, \sqrt{I_\alpha^j}\right)$ respectively and it is not possible to obtain a differential equation for the time-evolution of $\langle I_\alpha\rangle = 1/2\,\langle A_\alpha^2\rangle = 1/2\,\langle |x_\alpha|^2\rangle$ alone, without involving higher moments of I_α like $\langle I_\alpha \cdot I_\alpha\rangle$, $\langle I_\alpha \cdot I_\alpha \cdot I_\alpha\rangle$, etc. Instead, one usually obtains a partial differential equation for transport in phase-space in the form

$$\frac{\partial f(\underset{\sim}{I},t)}{\partial t} = \left[\hat{D}(\underset{\sim}{I},f)\; f(\underset{\sim}{I},t)\right] \tag{4.5.7}$$

where $f(\underset{\sim}{I},t)$ is the distribution function in action $\underset{\sim}{I}$ of the beam and $\hat{D}(\underset{\sim}{I},f)$ is a partial differential operator involving derivatives with respect to $\underset{\sim}{I}$ ($\partial/\partial \underset{\sim}{I}$, $\partial^2/\partial \underset{\sim}{I}^2$, etc.) and depending on $f(\underset{\sim}{I},t)$ as well. Such is the case in general for longitudinal stochastic cooling of coasting beams [9]. In general Eq. (4.5.7) has the general form of a Fokker-Planck equation up to second order in the strength of the cooling interaction. For the special case where $G^{i,\alpha}$ depends linearly on $(I^i_\alpha)^{1/2}$ and $(I^j_\alpha)^{1/2}$, however, e.g. for linear transverse dipole betatron cooling (Eqs. (4.3.29), (4.3.48), (4.4.1) and (4.4.3), an equation can be obtained for the time-evolution of the first moment $\langle I\rangle = 1/2 \langle A^2\rangle$ alone. For coasting beams with nonoverlapping Schottky bands, this is given by [9]:

$$\frac{d\langle I_\omega\rangle}{dt} = \sum_{n,(\pm)} \left\{ -\, g\left[(n\pm Q)\cdot\omega\right] + \frac{\pi N f(\omega)}{|n \pm Q|}\; g\left[(n\pm Q)\cdot\omega\right]^2 \right\} \left\langle I_\omega \right\rangle \tag{4.5.8}$$

or

$$\gamma_\omega = \frac{1}{\langle I_\omega\rangle}\frac{d\langle I_\omega\rangle}{dt} = \frac{1}{\left\langle A^2_\omega\right\rangle}\frac{d\left\langle A^2_\omega\right\rangle}{dt} = \sum_{n,(\pm)} \left\{ -\, g\left[(n\pm Q)\cdot\omega\right] + \frac{\pi N f(\omega)}{|n \pm Q|} \cdot g\left[(n\pm Q)\cdot\omega\right]^2 \right\} \tag{4.5.9}$$

where $f(\omega) = \int_0^\infty F(\omega,I)\, dI$, $\int_{-\infty}^{+\infty} f(\omega)\, d\omega = 1$, $F(\omega,I)$ being the distribution function of particles with revolution frequency ω and betatron action I, Q the betatron tune and $g[(n\pm Q)\omega]$ the gain of the feedback loop at the betatron harmonics $\Omega^n_\pm = (n\pm Q)\omega$.

The first term on the right-hand side in Eq. (4.5.9) arises from the coherent cooling term $C^i(x_i,\dot{x}_i)$ and the second term on the right-hand side of (4.5.9) arises from the incoherent noise term $S^i(x_i,\dot{x}_i;t)$, as given by Eq. (4.5.6).

Equations (4.5.8) and (4.5.9) for coasting beams are obtained by assuming the noise $\mathcal{S}^{i,\alpha}$ in Eq. (4.5.5) to be determined from uncorrelated particle orbits or beam parameters. In a real cooling system, correlations among the beam particles introduced by the kicker, act to deform the noise spectrum or signal $\mathcal{S}^{i,\alpha}$. This in turn leads to modifications of the cooling rate Eq. (4.5.9), as discussed in the next section.

4.6 Collective Signal Suppression - Cooperative Particle Effects

A beam of particles undergoing stochastic cooling is characterized by collective properties, generated by the inter-particle interactions induced by the feedback loop. The kicker fields introduce correlations between particle's arrival times and their phase-space coordinates. Such correlations are then propagated within the beam by single particle orbits.

Quite generally, the collective dynamics is describable in terms of response functions or propagators $D(\Gamma,\Gamma';t,t')$ which describe how a disturbance $f(\Gamma';t')$ centered around Γ' at time t' in the six-dimensional phase-space of the beam propagate through the beam to the neighbourhood of the point Γ in phase-space at a later time t. The response is generally causal for physical systems like beams and its structure depends both on the single particle orbits in the absence of interparticle interactions and on the nature of the interaction between particles. The specific form of this response function is sensitive to the boundary conditions of the system and causality generally imposes certain analytic structure on it in the frequency space conjugate to time.

It is easy to see from Fig. 12, describing the self-consistency between the stochastic feedback loop and the 'feedback through the beam' loop, that the collective propagation through the beam leads in general to a shielding or suppression of uncorrelated single particle signals (i.e. kicker fields in the absence of kicker-induced modulations) by a factor $\varepsilon(\Omega)$, similar to the dielectric function of a medium. In presence of kicker-induced modulations, the total current at the pick-up is modified to

$$I_p^T(\Omega) = I_o(\Omega) + \lambda(\Omega) \tag{4.6.1}$$

where $I_o(\Omega)$ is the unperturbed Schottky current and $\lambda(\Omega)$ is the collective modulation to the current due to the kicker.

If the beam is described by a "system response function" $D(\Omega)$, then

$$\lambda(\Omega) = D(\Omega) \cdot V_k^T(\Omega)$$

and

$$V_k^T(\Omega) = G(\Omega)\ I_p^T(\Omega) = G(\Omega)\left[I_o(\Omega) + \lambda(\Omega)\right]$$

$$= G(\Omega) \cdot I_o(\Omega) + G(\Omega) \cdot D(\Omega) \cdot V_k^T(\Omega)$$

Thus

$$V_k^T(\Omega) = \frac{G(\Omega)\ I_0(\Omega)}{\left[1 - G(\Omega) \cdot D(\Omega)\right]} = \frac{V_0^k(\Omega)}{\varepsilon(\Omega)} \qquad (4.6.2)$$

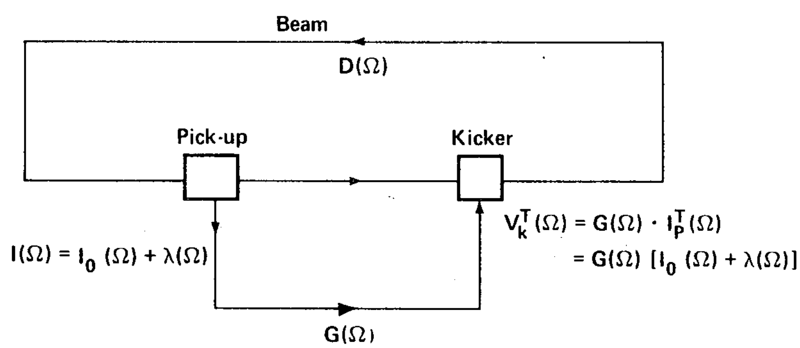
where

$$\varepsilon(\Omega) = 1 - G(\Omega) \cdot D(\Omega) \qquad (4.6.3)$$

and

$$V_0^k(\Omega) = G(\Omega) \cdot I_0(\Omega) \qquad (4.6.4)$$

The Beam Response Function $D(\Omega)$ has to be evaluated from the appropriate equation describing collective propagation of signals or perturbations through the beam, usually the Vlasov equation for single particle distribution in phase-space.



Collective Signal Suppression by Feedback through the Beam Response

Fig. 12

Thus in presence of collective correlations, the Schottky signal spectrum at the kicker is distorted from the spectrum with uncorrelated dynamics and this leads to a modification of the cooling rate. For example the coasting beam cooling rate, given by Eq. (4.5.9) for linear betatron cooling, is modified to [9]

$$\gamma_\omega = \frac{1}{\langle I_\omega \rangle} \frac{d\langle I_\omega \rangle}{dt} = \sum_{n,(\pm)} \left\{ - \frac{g[(n \pm Q)\omega]}{\varepsilon[(-n \mp Q)\omega]} + \frac{\pi N f(\omega)}{|n \pm Q|} \frac{|g[(n \pm Q)\omega]|^2}{|\varepsilon[(n \pm Q)\omega]|^2} \right\} \qquad (4.6.5)$$

in the case of non-overlapping Schottky bands, where the suppression factor $\varepsilon[(n\pm Q)\omega]$, as derived from a Vlasov analysis, is given by ([5], [6]):

$$\varepsilon\left[(n\pm Q)\cdot\omega\right] = 1 + \frac{N\ g[(-n\pm Q)\omega]}{|n\pm Q|} \int_{\eta\to 0^+} d\omega' \frac{f(\omega')}{\eta \pm i(\omega-\omega')} \tag{4.6.6}$$

A more general expression for the signal suppression $\varepsilon(\Omega)$ at frequency Ω including the situation of revolution Schottky-band overlap and localized interactions, has been derived by Bisognano ([8], [9]) and van der Meer [104] for cooling of coasting beams.

Calculation of the collective response or the signal suppression factor for coasting continuous beams is simplified by the fact that the response of such a beam at an aximuth θ' and time t' due to a perturbation at an azimuth θ and time t, is invariant with respect to arbitrary rotations in azimuth and stationary with respect to arbitrary shifts in the origin of time, i.e. the response is a function of $(\theta-\theta')$ and $(t-t')$ alone. Hence eigen-states or normal modes are plane or circular waves of the form

$$|\ell,\Omega\rangle^{(\pm)} \sim e^{i(\ell\theta-\Omega t)}\, f^{(\pm)}(x,z) \tag{4.6.7}$$

as long as the set (ℓ,Ω) satisfy a certain Dispersion Relation:

$$\varepsilon_\ell^{(\pm)}(\Omega) = 0\ . \tag{4.6.8}$$

determined by the particular collective interaction under consideration, e.g. space-charge, external impedances, feedback loops, etc. For spatially localized interactions (e.g. at a cavity or the kicker) one obtains a single scalar function $\varepsilon(\Omega)$, involving sums over the revolution harmonics ℓ, that determines the normal modes through the condition

$$\varepsilon(\Omega) = 0\ . \tag{4.6.9}$$

Collective response to an arbitrary density excitation $\tilde{\rho}^0(\ell,\Omega) \sim e^{i(\ell\theta-\Omega t)}$ where (ℓ,Ω) do not satisfy (4.6.8) or to $\tilde{\rho}^0_{\theta_k}(\Omega)$ at a localized region $\theta = \theta_k$ where Ω does not satisfy (4.6.9), is a shielding effect described by

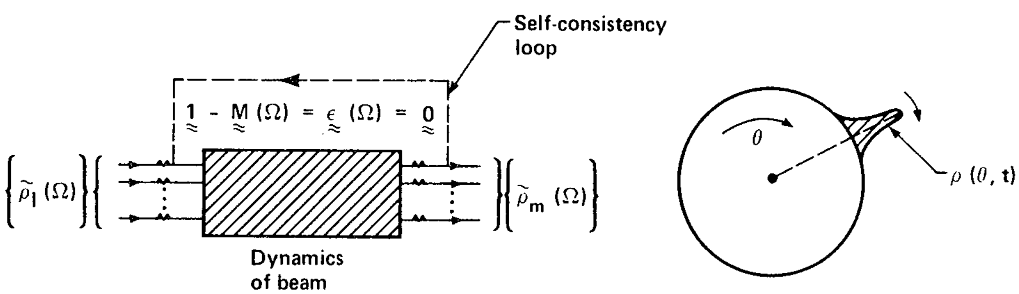
$$\tilde{\rho}(\ell,\Omega) = \frac{\tilde{\rho}^0(\ell,\Omega)}{\epsilon_\ell(\Omega)} \quad \underline{\text{or}} \quad \tilde{\rho}_{\theta_k}(\Omega) = \frac{\tilde{\rho}^0_{\theta_k}(\Omega)}{\epsilon(\Omega)} \tag{4.6.10}$$

The collective response of a bunched beam differ significantly from a coasting beam, owing to the spatial confinement property of a bunched beam and the topologically different particle orbits in a bunch. A bunch has a finite length and a nonuniform azimuthal density distribution in general. Hence a circular wave $e^{i(\ell\theta-\Omega t)}$ is not an eigenstate, except under very special periodic boundary conditions. So we expect all the angular Fourier components to be coupled to each other, as depicted in Fig. 13 below, and the self-consistent propagation of perturbations $\tilde{\rho}_\ell(\Omega)$, to be desdribed in general by a matrix relation:

$$\tilde{\rho}_\ell(\Omega) = \sum_{k=-\infty}^{+\infty} \tilde{M}_{\ell k}(\Omega)\, \tilde{\rho}_k(\Omega) \tag{4.6.11}$$

or

$$\underset{\approx}{\epsilon}(\Omega) \cdot \underset{\sim}{\tilde{\rho}}(\Omega) = 0 \tag{4.6.12}$$



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Bunched Beam Response

Fig. 13

where $\underset{\sim}{\tilde{\rho}}(\Omega)$ is a column vector involving $\{\tilde{\rho}_\ell(\Omega)\}_{\ell=-\infty,\ldots,+\infty}$ and $\underset{\approx}{\epsilon}(\Omega)$ is a matrix $\{\epsilon_{\ell k}(\Omega)\}_{\ell,k=-\infty,\ldots,+\infty}$ given by:

$$\epsilon_{\ell k}(\Omega) = \delta_{\ell k} - \tilde{M}_{\ell k}(\Omega)$$

<u>or</u> (4.6.13)

$$\underset{\approx}{\epsilon}(\Omega) = \underset{\approx}{I} - \underset{\approx}{\tilde{M}}(\Omega)$$

Eigenstates or normal modes must satisfy the Dispersion Relation

$$\det\left[\underset{\approx}{\varepsilon}(\Omega)\right] = |\underset{\approx}{I} - \underset{\approx}{\tilde{M}}(\Omega)| = 0 \tag{4.6.14}$$

which is a condition for solution of (4.6.11) or (4.6.12) with non-zero $\underset{\sim}{\tilde{\rho}}(\Omega)$. Eigenfrequencies are thus given by the roots of the infinite-order determinantal equation (4.6.14). For localized interactions at a fixed azimuth θ_k, the matrix equation (4.6.11) translates into a coupling of the localized perturbation $\tilde{\rho}_{\theta_k}(\Omega)$ to all the revolution frequency translates $\tilde{\rho}_{\theta_k}(\Omega + k\omega_0)$ in the following way:

$$\tilde{\rho}_{\theta_k}(\Omega) = \sum_{k=-\infty}^{+\infty} M_k(\Omega)\, \tilde{\rho}_{\theta_k}\left(\Omega + k\omega_0\right) \tag{4.6.15}$$

This will be seen more clearly in Chapter 10.

For stochastic cooling, one has an arbitrary initial excitation (incoherent Schottky signal of particles) $\underset{\sim}{\tilde{\rho}}^0(\Omega) \equiv \left\{\tilde{\rho}_\ell^0(\Omega)\right\}_{\ell=-\infty,\ldots,+\infty}$ where Ω is in general not a root of (4.6.14). Thus the collective shielding or suppression of the original signal $\underset{\sim}{\tilde{\rho}}^0(\Omega)$ is given by:

$$\underset{\approx}{\varepsilon}(\Omega) \cdot \underset{\sim}{\tilde{\rho}}(\Omega) = \underset{\sim}{\tilde{\rho}}^0(\Omega) \quad \underline{\text{or}} \quad \underset{\sim}{\tilde{\rho}}(\Omega) = \left[\underset{\approx}{\varepsilon}(\Omega)\right]^{-1} \cdot \underset{\sim}{\tilde{\rho}}^0(\Omega) \tag{4.6.16}$$

Calculation of the inverse of the infinite matrix $\underset{\approx}{\varepsilon}(\Omega)$ poses considerable mathematical difficulty.

A second distinguishing feature of a bunch which complicates the collective dynamics even further is the following: a bunch is most conveniently described in terms of action-angle variables (J, ψ) in longitudinal phase-space, as outlined in Chapter 3. The natural "complete set" to describe disturbances in bunch phase-space is thus

$$\left\{\mu, \Omega\right\}: \quad \rho(J, \psi;\, t) = \int_{-\infty}^{+\infty} \sum_{\mu=-\infty}^{+\infty} \rho_\mu(J;\, \Omega)\, e^{i(\mu\psi + \Omega t)}\, d\Omega \tag{4.6.17}$$

whereas the natural "complete set" to describe perturbations in configuration space of a storage ring is:

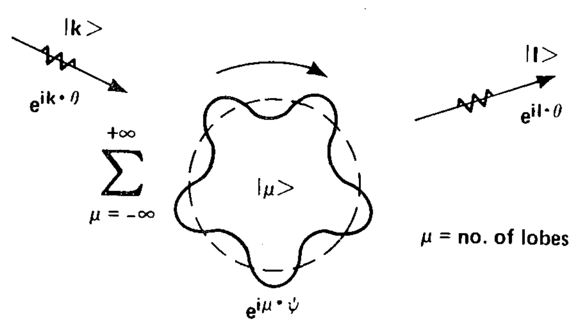
$$\{\ell,\Omega\}:\quad \rho(\theta;\,t) = \int_{-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} \rho_\ell(\Omega)\, e^{i(\ell\theta+\Omega t)}\, d\Omega \tag{4.6.18}$$

since the electrodynamics or the response of the external impedances, feedback transfer line, etc. to current perturbations is conveniently described in terms of response to single frequency periodic circular waves $|\ell,\Omega\rangle \sim e^{i(\ell\theta-\Omega t)}$. This basic incompatibility is one of the major sources of the difficulty in solving for the collective dynamics of a bunch.

Externally imposed disturbances, characterized by plane-wave states $|\ell,\Omega\rangle$, will be carried along by the intrinsically circular dynamics of particle orbits in a bunch and projected onto the plane-wave states again, as demanded by self-consistency. Hence each matrix element $\tilde{M}_{k\ell}(\Omega)$ has an effective sum over the internal bunch harmonics μ:

$$\langle\ell|k\rangle = \sum_{\mu=-\infty}^{+\infty} \langle\ell|\mu\rangle\,\langle\mu|k\rangle \tag{4.6.19}$$

The situation is pictorially represented in Fig. 14 below.



Coupling of Internal Bunch Modes (Synchrotron Modes) to External Electromagnetic Disturbances

Fig. 14

Hence we expect

$$\tilde{M}_{k\ell}(\Omega) = \sum_{\mu=-\infty}^{+\infty} \tilde{M}^{\mu}_{k\ell}(\Omega) \tag{4.6.20}$$

Rotation with frequency ω_s in phase-space implies that $\psi(t) = \omega_s t + \psi(0)$ and $\exp(i\mu\psi) = \exp[i\mu\omega_s t + i\mu\psi(0)]$. Fourier transformed in frequency, we thus expect $\tilde{M}^{\mu}_{k\ell}(\Omega)$ to have simple poles at $\Omega = \mu\omega_s$, so that

$$\tilde{M}_{k\ell}(\Omega) = \sum_{\mu=-\infty}^{+\infty} \tilde{M}^{\mu}_{k\ell}(\Omega) = \sum_{\mu=-\infty}^{+\infty} \frac{F^{\mu}_{k\ell}(\Omega)}{[\Omega - \mu\omega_s]} \tag{4.6.21}$$

where $F^{\mu}_{k\ell}(\Omega)$ describes some kind of form-factor of the bunch in the coupling of waves $|k>$ and $|\ell>$ through an internal phase state of harmonic μ. For nonlinear oscillation orbits with action dependent frequency $\omega_s \equiv \omega_s(J)$ we would have

$$\tilde{M}_{k\ell}(\Omega) = \sum_{\mu=-\infty}^{+\infty} \int_0^{\infty} dJ \, \frac{S^{\mu}_{k\ell}(\Omega; J)\, f_0(J)}{[\Omega - \mu\omega_s(J)]} \tag{4.6.22}$$

For localized interactions at a location $\theta = \theta_k$, an extra summation $\sum\limits_{m=-\infty}^{+\infty}$ would appear within $\tilde{M}_{k\ell}(\Omega)$ to reproduce the delta function nature of the interaction.

We will see in Chapter 10 that the exact form of $\tilde{M}_{k\ell}(\Omega)$ derived from a Vlasov analysis does indeed have the same structure as Eq. (4.6.22).

It is important to recognize that the quantity $\underset{\approx}{\varepsilon}(\Omega)$ determines the frequencies and growth rates of collective modes, excited by the feedback loop; through the condition:

$$\det\left[\underset{\approx}{\varepsilon}\left(\Omega^n_{coh}\right)\right] = 0 \tag{4.6.23}$$

The damping <u>or</u> growth rates of these modes are determined by:

$$\delta^n_{coh} = \mathrm{Im}\left(\Omega^n_{coh}\right) \tag{4.6.24}$$

Hence associated with any feedback system, there is a characteristic time-scale:

$$\tau^n_{coh} \sim \left(\delta^n_{coh}\right)^{-1} \tag{4.6.25}$$

over which collective oscillations excited by the feedback loop would grow, if it had the appropriate phase. The situation is similar to beam instabilities induced by external elements.

In the context of cooling, τ_{coh}^{n} could describe how fast the process of collective signal suppression is established in a beam with no frequency spread.

For a beam with non-zero frequency spread, however, one needs to take into account 3-body correlations in order to have a correct evaluation of the coherent signal suppression time scale. Such an analysis requires a kinetic theory based on a hierarchy of correlations for the particles in the beam. (See Chapter 9, Section 9.2.)

4.7 Various Time Scales

The fastest frequency present in the system is usually the betatron oscillation frequency $\omega_{x,z} = Q_{x,z}\omega_0$ where $Q_{x,z}$ is the tune and ω_0 the revolution frequency in the storage ring. Nothing significant happens to the beam in a single turn except for a few betatron oscillations. At the other end of the frequency scale is the cooling rate of the beam, γ. We are considering a stochastic cooling feedback system that gives rise to cooling on a slow time-scale $\tau_{cool} \sim 1/\gamma$. Much before any cooling has occurred, there has been several betatron oscillations, the beam has made several turns, there has been several longitudinal synchrotron oscillations and the Schottky spectrum of the beam noise signal at the pick-up has been established.

The synchrotron oscillation frequency ω_s is usually much slower than the revolution frequency ω_0 and yet we need quite a few of them in order to establish the synchrotron harmonic structure in the Schottky noise spectrum, before any cooling occurs. Typically synchrotron periods T_s correspond to thousands of revolutions.

As discussed in Section 4.4, there is also a time-scale of mixing in phase-space corresponding to the $\mu^{\underline{th}}$ harmonic:

$$\left[\tau_{mix}^{\mu}\right]^{-1} \sim \mu\,\Delta\omega_s \sim \mu \left|\frac{d\omega_s}{da}\right| \cdot a_m$$

for maximum synchrotron amplitude a_m in the beam.

There is also the coherent damping time as discussed in Section 4.6, corresponding to the time-scale in which cooperative collective particle effects screen <u>or</u> suppress single particle Schottky signals. Typically this time-scale is given by [9]

$$\tau_{coh} \sim \left(\delta_{coh}\right)^{-1} \approx \frac{1}{n \cdot G^{T}_{eff.}}$$

where n is the number of particles that interact at a given time, i.e. the number of particles within the system pass-band <u>or</u> in each sample handled at a given time by the feedback loop and G^{T}_{eff} is the total effective gain <u>or</u> interaction strength experienced by a particle.

In our formulation, we are going to work in the following regime of hierarchial time or frequency scales:

$$\left(\omega_0, Q\omega_0\right) >> \mu(\Delta\omega_S) \gtrsim \omega_S \approx \delta_{coh} >> \gamma_{cool}$$

<u>or</u>

$$\left(T_0, T_\beta\right) << \left[\tau^{\mu}_{mix}\right] < T_S \approx \tau_{coh} << \tau_{cool}$$

Fast cooling schemes involving a time-scale of cooling faster than the synchrotron period so that the synchrotron band structure of noise signal does not get time to be established, is not a subject of this report.

5. SCHOTTKY SPECTRUM OF A BUNCH IN THE ABSENCE OF COHERENT MODULATIONS

We describe the longitudinal synchrotron and transverse betatron oscillations of particles in a bunch by the action-angle variables $(J = 1/2\ a^2, \psi)$ and $(I = 1/2\ A^2, \phi)$ respectively. The longitudinal orbit of a particle i in the beam undergoing synchrotron oscillations is given by (Chapter 3):

$$\theta_i(t) = \omega_0\, t + a_i \operatorname{Sin} \psi_i(t)$$

$$\dot{\theta}_i(t) = \omega_i = \omega_0 + \Delta\omega_i \operatorname{Cos} \psi_i(t) = \omega_0 + \omega_s(a_i) \cdot a_i \operatorname{Cos} \psi_i(t)$$

where

$$a_i = \left[2J_i\right]^{1/2}$$

and

$$\psi_i(t) = \omega_s(a_i)\, t + \psi_i^0$$

The current at a pick-up located azimuthally at $\theta = \theta_p$ due to a particle j is:

$$I_j^p(t) = q\, \omega_j \sum_{m=-\infty}^{+\infty} \delta\left[\theta_j(t) - \theta_p - 2\pi m\right]$$

$$= q\left[f_0 + \left(\frac{\Delta\omega_j}{2\pi}\right) \operatorname{Cos}\left(\omega_s(a_j)t + \psi_j^0\right)\right] \sum_{m=-\infty}^{+\infty} e^{im[\omega_0 t + a_j \operatorname{Sin} \psi_j(t) - \theta_p]} \tag{5.1}$$

We now use the identity (4.3.52). The current then is given by:

$$I_j^p(t) = qf_0 \sum_{\substack{m \\ (-\infty)}}^{+\infty} \sum_{\substack{\mu \\ (-\infty)}}^{+\infty} J_\mu(ma_j)\, e^{i[m\omega_0 + \mu\omega_s(a_j)]t - im\theta_p + i\mu\psi_j^0}$$

$$+ q\left(\frac{\Delta f_j}{2}\right) \sum_{\substack{m \\ (-\infty)}}^{(+\infty)} \sum_{\substack{\mu \\ (-\infty)}}^{(+\infty)} J_\mu(ma_j) \left[e^{i[m\omega_0 + (\mu+1)\omega_s(a_j)]t + i(\mu+1)\psi_j^0 - im\theta_p}\right.$$

$$\left. + e^{i[m\omega_0 + (\mu-1)\omega_s(a_j)]t + i(\mu-1)\psi_j^0 - im\theta_p}\right]$$

The first term gives a spectrum of lines at the revolution harmonics $m\omega_0$, each one of which is accompanied by synchrotron satellite bands (an infinite number of them in principle) $[m\omega_0 \pm \mu\omega_s(a_j)]$ whose strengths are given by $J_\mu(ma_j)$. The second term gives a set of second order synchrotron satellite bands at $[m\omega_0 \pm (\mu \pm 1)\omega_s(a_j)]$ displaced by $\pm\,\omega_s(a_j)$ from the first order bands and whose relative strengths are given by

$$\frac{(\Delta f_j/2)}{f_0} = \frac{1}{2}\,\frac{a_j\ \omega_s(a_j)}{\omega_0} << 1$$

since synchrotron oscillations are usually much slower than the revolution time ($a\omega_s << \omega_0$). We thus neglect these second order bands from our analysis. The first-order longitudinal Schottky signal due to particle j at the pick-up is then

$$I_j^p(t) = qf_0 \sum_{\substack{m \\ (-\infty)}}^{(+\infty)} \sum_{\substack{\mu \\ (-\infty)}}^{(+\infty)} J_\mu(ma_j)\ e^{i[m\omega_0 + \mu\omega_s(a_j)]t - im\theta_p + i\mu\psi_j^0}$$

The total Schottky current signal at $\theta = \theta_p$ due to all the particles $j=1,\ldots,N$ in the beam is given by

$$I^p(t) = \sum_{j=1}^{N} I_j^p(t) = qf_0 \sum_{j=1}^{N} \sum_{\substack{m \\ (-\infty)}}^{(+\infty)} \sum_{\substack{\mu \\ (-\infty)}}^{(+\infty)} J_\mu(ma_j)\ e^{i[m\omega_0 + \mu\omega_s(a_j)]t - im\theta_p + i\mu\psi_j^0} \tag{5.2}$$

Equation (5.2) gives us the time-domain representation of the spectral form of $I^p(t)$, the longitudinal Schottky noise signal of a bunched beam at a pick-up. The usual form for unbunched (coasting beam) particles can be derived from (5.2) in the limit $a_j \to 0$

$$I^p(t) = qf_0 \sum_{j=1}^{N} \sum_{m=-\infty}^{+\infty} e^{im(\omega_j t - \theta_p) + im\theta_j^0} \tag{5.3}$$

where ω_j is the revolution frequency of the $j^{\underline{th}}$ particle.

The first order transverse signal of the $j^{\underline{th}}$ particle is given by the dipole moment

$$d_j^p(t) = x_j(t) \cdot I_j^p(t)$$

where

$$x_j(t) = A_j(t)\ \mathrm{Cos}\left(Q\omega_o t \quad \phi_j^0\right)$$

for a particle executing betatron oscillations with tune Q and amplitude $A_j(t)$. The total transverse dipole signal at the pick-up is given by:

$$d^p(t) = \sum_{j=1}^{N} d_j^p(t)$$

$$= \sum_{j=1}^{N} q \left(\frac{A_j}{2}\right) \cdot f_o \cdot \sum_{m}\sum_{\mu}{}_{=(-\infty)}^{(+\infty)} \left\{ J_\mu\left[(m+Q)a_j\right] e^{i[(m+Q)\omega_o + \mu\omega_s(a_j)]t + i\mu\psi_j^0 + i\phi_j^0 - im\theta_p} \right.$$

$$\left. + J_\mu\left[(m-Q)a_j\right] e^{i[(m-Q)\omega_o + \mu\omega_s(a_j)]t + i\mu\psi_j^0 - i\phi_j^0 - im\theta_p} \right\} \tag{5.4}$$

If one wishes to include the effect of the machine chromaticity, the arguments of the Bessel functions are modified as [67]

$$J_\mu\left[(m\pm Q)a_j\right] \longrightarrow J_\mu\left[(m\pm Q)a_j - Q\,\frac{\xi}{\eta}\,a_j\right] \tag{5.5}$$

where ξ is the 'chromaticity' defined by:

$$\frac{\Delta Q}{Q_o} = \xi\,\frac{\Delta p}{p_o} = \frac{\xi}{\eta}\,\frac{\Delta\omega}{\omega_o} = \frac{\xi}{\eta}\,\frac{\Delta f}{f_o}$$

and $\eta = \gamma_{tr}^{-2} - \gamma^{-2}$ is the "off-energy function".

Again, the usual form for unbunched particles can be derived from (5.4) in the limit $a_j \to 0$

$$d^P(t) = \sum_{j=1}^{N} q \left(\frac{A_j}{2}\right) f_0 \sum_{m=-\infty}^{+\infty} \left\{ e^{i\left[(m+Q)\omega_j t - m\theta_p + \phi_j^0 + m\theta_j^0\right]} + e^{i\left[(m-Q)\omega_j t - m\theta_p - \phi_j^0 + m\theta_j^0\right]} \right\} \tag{5.6}$$

where ω_j is the revolution frequency of the $j^{\underline{th}}$ particle.

As expected, Eq. (5.6) tells us that the Schottky spectrum of an unbunched beam contains an infinity of betatron lines each having a central frequency $\omega_m = (m \pm Q_0)\omega_0$ and a spread, due to momentum and transverse tuning dispersion, given by:

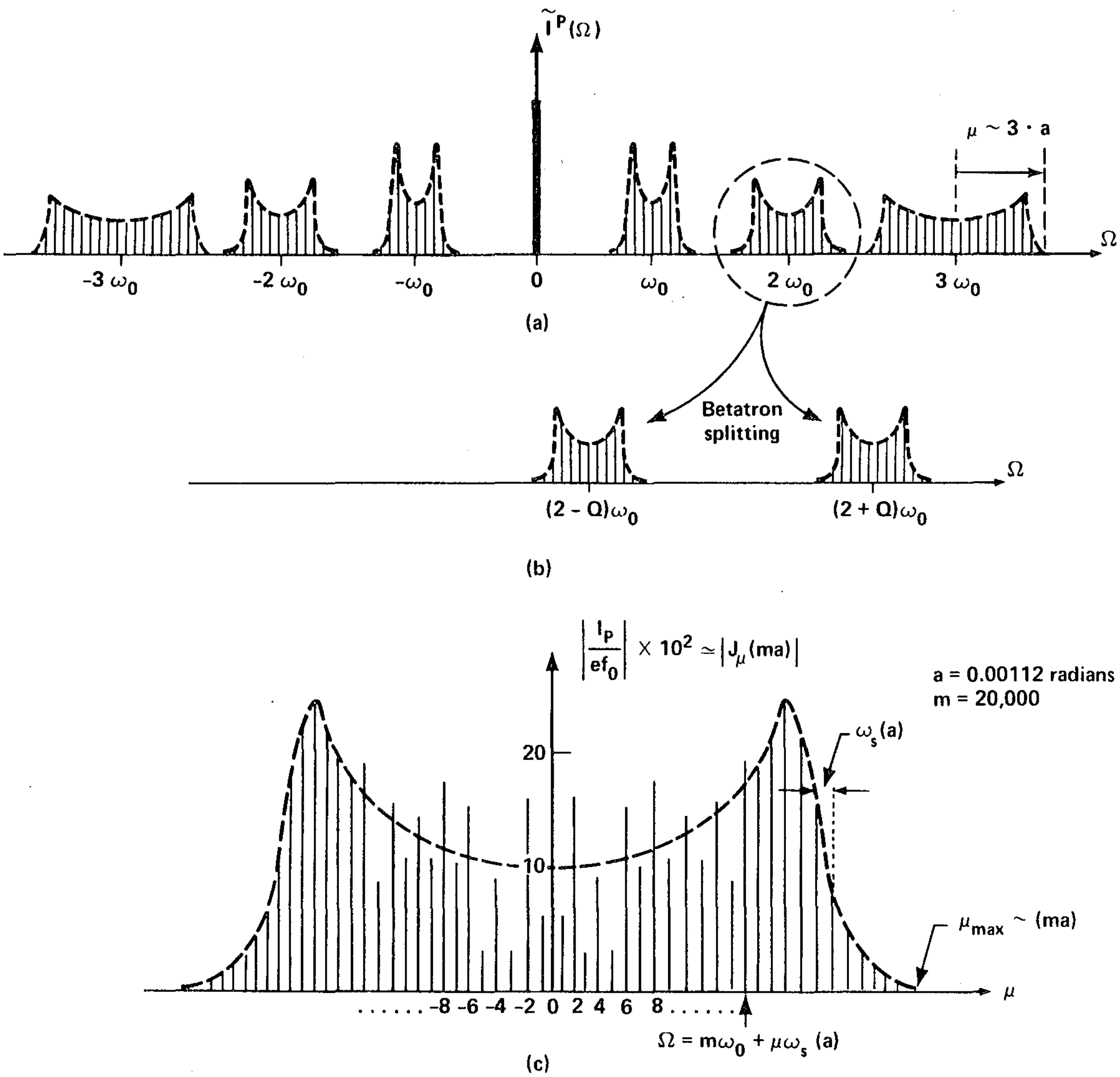
$$\Delta\omega_m = (\eta m \pm Q_0 \xi)\, \omega_0 \frac{\Delta p}{p_0} \tag{5.7}$$

For a bunched beam, the additional synchrotron oscillation transforms each of these bands into a central betatron line associated with an infinite set of synchrotron satellites, spaced at the synchrotron frequency and intensity modulated by Bessel functions of increasing order. In a linear machine without ripple, the central betatron lines are sharp while the synchrotron satellites reproduce the momentum distribution of the beam. The relative height of the satellites compared to the central line contains the information about the machine chromaticity now.

The single-particle longitudinal Schottky spectrum, as seen by a spectrum analyzer attached to the pick-up, is shown on the global frequency scale in Fig. 15(a). The splitting of each revolution band $m\omega_0$ into two betatron side-bands $(m+Q)\omega_0$ and $(m-Q)\omega_0$ for the transverse dipole signal is illustrated in Fig. 15(b). The detailed satellite band structure of each revolution band due to synchrotron oscillations is shown on a magnified scale in Fig. 15(c), for a revolution harmonic $m = 20{,}000$ and synchrotron oscillation amplitude $a = .00112$ radians in a bucket of maximum length $a_{max} =$ .0014 radians corresponding to an example of a $h = 2226$ rf system for the Fermilab main ring <u>or</u> the Tevatron.

The shape of the current spectrum due to a beam of particles is obtained by adding up the single particle contributions by taking the proper distribution of synchrotron amplitudes

$$0 \leqslant a \leqslant a_{max}$$



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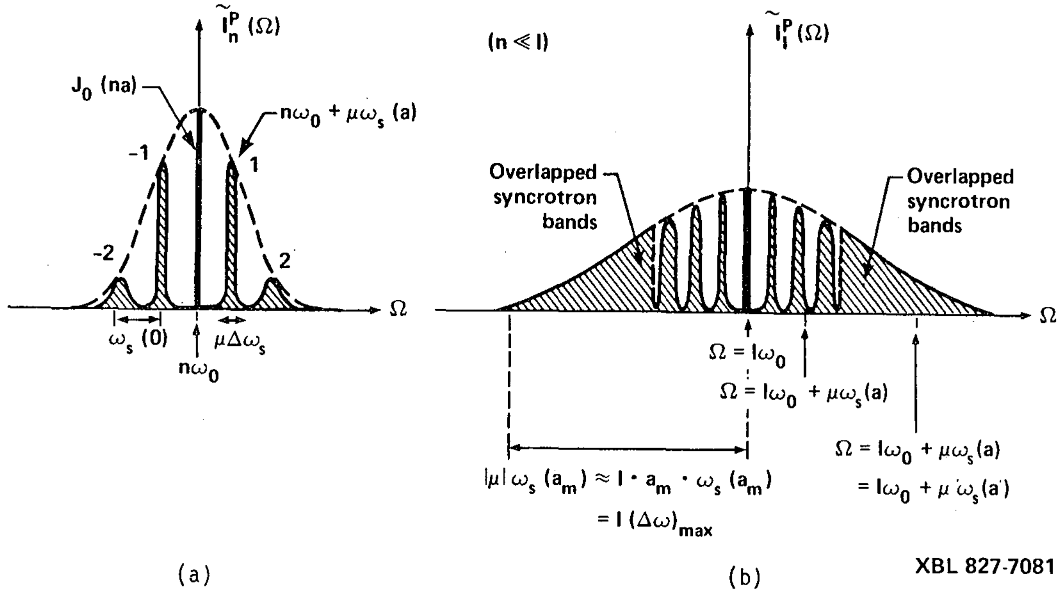
Schottky Spectrum of a Particle in a Bunch

Fig. 15

and phases $$0 \leqslant \psi \leqslant 2\pi$$

into account. The profile of the Schottky band at a given revolution harmonic $m\omega_0$ should resemble the projection of the phase-space distribution of the bunch along the velocity-axis and thus duplicates the longitudinal velocity distribution of the bunch.

The Schottky spectrum (longitudinal) for a distribution of particles in a bunch at a given revolution harmonic is shown in Fig. 16(a) for a low harmonic $n\omega_0$ and in Fig. 16(b) for a high harmonic $\ell\omega_0$ ($\ell >> n$).



Schottky Spectrum for a Bunch Distribution

Fig. 16

Since $J_\mu(\ell a)$ has significant magnitudes only up to $\mu \sim \ell a$ and falls off to zero rapidly for $\mu \gtrsim \ell a$ $\left(J_\mu(x) \underset{\mu\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\mu}}\left(\frac{ex}{2\mu}\right)^\mu\right)$, the side-band spectrum for a single particle of amplitude a extends up to $\mu \approx \ell a = \ell \cdot \frac{\Delta\omega(a)}{\omega_s(a)}$. For a distribution of particles as in Fig. 16(a) and (b), the side-band structure extends up to $\mu_m \approx \ell a_m = \ell \cdot \frac{\Delta\omega(a_m)}{\omega_s(a_m)}$ where a_m is the maximum synchrotron amplitude present in the bunch. With line spacing $\omega_s(a_m)$, the width $\mu_m\omega_s(a_m) = \ell\Delta\omega(a_m)$ approaches that of the coasting beam case with $\Delta\omega$ equal to the frequency modulation.

For low revolution harmonics $n\omega_0$, we have around each n a line spectrum essentially (where the $\mu^{\underline{th}}$ side-band has a width $\mu\Delta\omega_s$, Fig. 16(a)) rather than a continuous band. Thus as compared to a coasting beam, the noise density at the side-bands tends to be increased by

$$\Gamma_\mu = \frac{\omega_s}{(\mu\Delta\omega_s)}$$

until the side-bands overlap, i.e. $\Gamma_\mu \lesssim 1$.

This noise concentration at the side-bands is however limited to small revolution harmonic numbers n and a few synchrotron side-bands μ around each n. We observe that $|J_\mu(\ell a)|$ falls off to zero rapidly only for $\mu \gtrsim \ell a \equiv \hat{\mu}$. Thus for large argument (ℓa), i.e. large harmonic numbers ℓ, many high order side-bands contribute for whom the overlapping condition $\Gamma_\mu \lesssim 1$ is satisfied. This situation is illustrated in Fig. 16(b), where we observe that except for a few synchrotron bands near the center, most of them overlap partially <u>or</u> completely throughout the rest of the revolution harmonic band.

In the band-overlapped region of the spectrum, it is impossible to assign a definite amplitude a and synchrotron harmonic μ to a given frequency Ω. Or in other words given a certain frequency Ω that falls within the revolution band, several different particles with different synchrotron amplitudes and different synchrotron harmonic numbers will generate the same frequency Ω:

$$\Omega = m\omega_o + \mu\omega_s(a) = m\omega_o + \mu'\omega_s(a') = m\omega_o + \mu''\omega_s(a'') = \ldots$$

It is interesting to ask: given a certain maximum spread $(\Delta\omega_s)^b$ in the synchrotron frequencies of the particles with maximum synchrotron oscillation amplitude a_m in the bunch what is the range in μ over which there is significant band overlap in a given resolution harmonic $n\omega_o$? And secondly, given a certain μ for a particle of amplitude a lying within this overlapped region, how many other synchrotron bands μ' overlap with it?

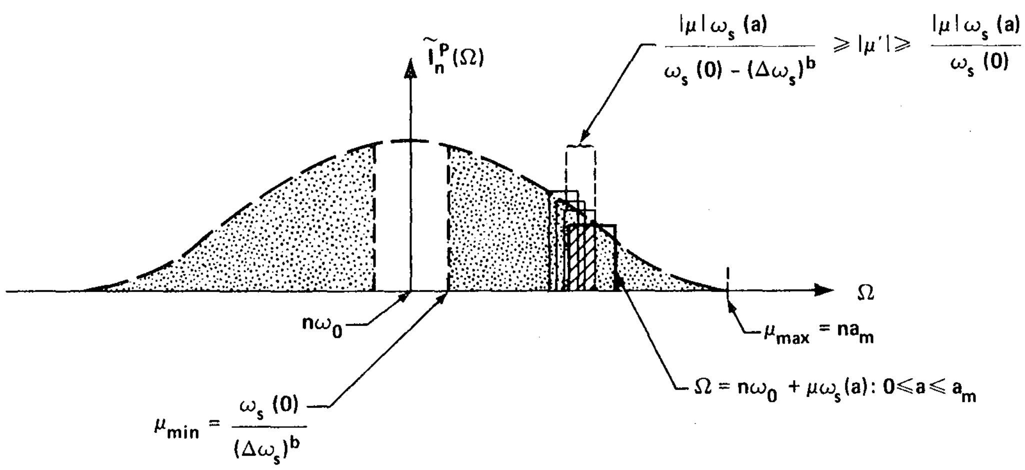
It is easy to see that if $\omega_s(0)$ is the synchrotron oscillation frequency at the center of the bucket for small amplitude synchrotron oscillations, then the range in μ over which bands overlap in revolution harmonic $n\omega_o$ is given by

$$\frac{\omega_s(0)}{(\Delta\omega_x)^b} = \mu_{min} \leqslant \mu \leqslant \mu_{max} = na_m \tag{5.8}$$

and the range of μ' overlapping with the frequency $\Omega = n\omega_0 + \mu\omega_s(a)$ for a particle of amplitude a such that $0 \leqslant a \leqslant a_m$ and in a synchrotron oscillation mode μ lying within the range given by (5.8), is given by

$$\frac{|\mu|\omega_s(a)}{\omega_s(0) - (\Delta\omega_s)^b} \geqslant |\mu'| \geqslant \frac{\mu\ \omega_s(a)}{\omega_s(0)} \tag{5.9}$$

We illustrate these ranges of band overlap in Fig. 17 below.



Synchrotron Band Overlap Structure of Bunched Beam Schottky Signal

Fig. 17

The general picture of overlapped and non-overlapped bunched beam Schottky signal is as follows: Schottky bands corresponding to low revolution harmonic numbers $n < \omega_s(0)/[a_m(\Delta\omega_s)^b]$ are separate and non-overlapping. Moreover within each such revolution harmonic band, we have separate and distinct non-overlapping synchrotorn bands where a given frequency $\Omega = n\omega_0 + \mu\omega_s(a)$ corresponds to one amplitude a and one synchrotron mode μ only (Fig. 16(a)).

For higher revolution harmonic numbers satisfying

$$\frac{\omega_s(0)}{a_m(\Delta\omega_s)^b} < n < \frac{\omega_0}{2a_m}$$

different revolution harmonic Schottky bands $n\omega_o$ are still separate and non-overlapping. However within each such band, n, synchrotron bands μ overlap throughout the band except for a narrow strip containing a few low μ's ($\mu=0,\pm1,\pm2,\ldots$) near the center, where the synchrotron bands are distinct and separate (Fig. 16(b)). This is the case most likely to be encountered in practical cooling systems. A given frequency Ω within the band in this case, corresponds to a definite revolution harmonic $n\omega_o$ but no definite synchrotron harmonic μ or amplitude a: $\Omega = n\omega_o + \mu\omega_s(a) = n\omega_o + \mu'\omega_s(a') = \ldots$.

At still higher revolution harmonics $n > \frac{\omega_o}{2a_m}$ even the revolution bands begin to touch and overlap. In this case a given frequency Ω corresponds to no fixed revolution harmonc n or synchrotron mode and amplitude (μ, a): $\Omega = n\omega_o + \mu\omega_s(a) = n'\omega_o + \mu'\omega_s(a') = \ldots$. This will be the situation for cooling systems with bandwidths in the extremely high-frequency region.

The Schottky noise spectrum of a coasting beam, given by Eq. (5.6) for the transverse signal for example, is distinguished by the particular feature that the total power per band is proportional to N, the number of circulating particles, due to incoherencies of particle motion (random initial betatron and longitudinal phases). More explicitly we find:

$$\left\langle d_{n+Q}^{p^2} \right\rangle_{\text{coasting}} = \frac{N}{2}\, q^2\, \bar{f}^2 \, \langle A^2 \rangle \tag{5.10}$$

per betatron band of an unbunched beam, where $\bar{f}$ is the average frequency of revolution of the particles in the beam. Similarly for longitudinal signal the average current squared in the $n^{\underline{th}}$ revolution band is given by:

$$\left\langle I_n^{p^2} \right\rangle_{\text{coasting}} = q^2\, \bar{f}^2 \, (2N)$$

or

$$\left(I_n^p \right)_{\text{rms}} = q\, \bar{f} \sqrt{2N}$$

If this current were analyzed by a spectrum analyzer with resolution $\Delta\Omega$, about a frequency $\Omega = n\omega_o$, then only particles such that $|n(\omega_o - \omega_j)| < (\Delta\Omega/2)$ would enter into the sum over particles and we have [9]:

$$\left[I_n^p(\Omega)\right]_{rms} = q\,\bar{f}\sqrt{2N}\,\sqrt{f\left(\frac{\Omega}{n}\right)\left(\frac{\Delta\Omega}{n}\right)}$$

where $f(\omega)$ is the distribution of particles in their angular velocities. Thus, for random initial phases of particles i.e. incoherent (uncorrelated) motions of particles, the power (proportional to $|[I_n(\Omega)]|^2_{rms}$) in the $n^{\underline{th}}$ Schottky band mirrors the angular velocity distribution with its width and height proportional to n and $1/n$ respectively.

Similarly, for the transverse Schottky signal of a bunched beam, we easily verify from Eqs. (5.4) and (5.5) that

$$\langle d^2\rangle_{\text{bunched}} = \sum_{(+,-)} \sum_n \sum_\mu \left\langle d^2_{n,\mu}\right\rangle^{(\pm)}_{\text{bunched}}$$

where

$$\left\langle d^2_{n,\mu}\right\rangle^{(\pm)}_{\text{bunched}} = \frac{N}{2}\,q^2\,f_0^2\,S^{(\pm)}_{n,\mu} = \frac{N}{2}\,q^2\,f_0^2\,\langle A^2\rangle\,F^{(\pm)}_{n,\mu} \tag{5.11}$$

per betatron (μ=0) <u>or</u> synchro-betatron ($\mu\neq0$) band of a bunched beam. The form factor $S^{(\pm)}_{n,\mu}$ is the integral along the bunch of the Bessel function squared $J^2_\mu\left[(m\pm Q)a - Q\frac{\xi}{\eta}a\right]$ times the betatron amplitude squared A^2 weighted by the normalized momentum <u>or</u> amplitude distribution $f_0(a,A)/N$ i.e.

$$S^{(\pm)}_{n,\mu} = \frac{1}{N}\int_0^\infty\!\!\int_0^\infty da\cdot dA\cdot f_0(a,A)\cdot A^2\,J^2_\mu\left[(n\pm Q)a - Q\,\frac{\xi}{\eta}\,a\right]$$

with

$$\int_0^\infty\!\!\int_0^\infty dA\cdot da\cdot f_0(a,A) = N$$

If the distribution is a separate function of a and A, we have

$$f_0(a,A) = g_0(a)\cdot h_0(A)$$

and

$$<A^2> = \int_0^\infty dA \cdot A^2 \, h_o(A)$$

then

$$S_{n,\mu}^{(\pm)} = <A^2> \, F_{n,\mu}^{(\pm)}$$

and

$$F_{n,\mu}^{(\pm)} = \frac{1}{N} \int_0^\infty da \cdot g_o(a) \, J_\mu^2 \left[(n \pm Q) a - Q \, \frac{\xi}{n} \, a \right]$$

Since

$$\sum_{\mu=-\infty}^{+\infty} J_\mu^2(x) = 1$$

we have the following 'sum rule' for the form factors:

$$\sum_{\mu=-\infty}^{+\infty} F_{n,\mu}^{(+)} = \sum_{\mu=-\infty}^{+\infty} F_{n,\mu}^{(-)} = 1$$

and the total power per betatron band $(n+Q)\omega_o$ of a bunched beam, summed over all the synchrotron bands, is given by:

$$\left\langle d_{n+Q}^2 \right\rangle_{\text{bunched}} = \frac{N}{2} \, q^2 \, f_o^2 \, <A^2> \sum_{\mu=-\infty}^{+\infty} F_{n,\mu}^{(+)}$$

$$= \frac{N}{2} \, q^2 \, f_o^2 \, <A^2> \tag{5.12}$$

in full agreement with the result for coasting beams (Eq. (5.10)).

The agreement between the total power per betatron band for the transverse Schottky signal of a coasting beam and a bunched beam is a manifestation of the fact that all the bands $(n \pm Q)\omega_o + \mu\omega_s^i$ including the center bands $(\mu=0)$ are randomized due to the betatron phase $\pm\phi_i^o$ appearing in the total phase factor for the $(n \pm Q, \mu)^{\underline{th}}$ line: $i\left[\mu\psi_i^0 \pm \phi_i^0 - n\theta_p\right] \equiv \psi_{n,\mu}^{i,o^{(\pm)}}$. An essential difference appears for the longitudinal signal, where the sidebands $(\mu \neq 0)$ are still randomized by the synchrotron phases ψ_i^o, however there is no synchrotron or betatron phases left to randomize the $\mu = 0$ central bands since the $(n,\mu)\underline{\text{th}}$ line in the longitudinal signal has phase given by $i\left[\mu\psi_i^o - n\theta_p\right]$.

It is easily seen from Eq. (5.2) that

$$\left\langle |I_m^p|^2 \right\rangle = q^2\ f_o^2 \sum_{\substack{j\ i \\ (=1)}}^{(N)} \sum_{\substack{\mu\ \mu' \\ (-\infty)}}^{(+\infty)} J_\mu(ma_j)\ J_{\mu'}(ma_i)$$

$$\left\langle \begin{array}{l} e^{i[m\omega_o + \mu\omega_s(a_j)]t - im\theta_p + i\mu\psi_j^0} \\ \cdot\ e^{-i[m\omega_o + \mu'\omega_s(a_i)]t + im\theta_p - i\mu'\psi_i^0} \end{array} \right\rangle$$

$$= q^2\ f_o^2 \left[\sum_{\substack{j\ i \\ (=1)}}^{(N)} J_o(ma_j)\ J_o(ma_i) + \sum_{j=1}^{N} \sum_{\mu=-\infty}^{+\infty} J_\mu^2(ma_j) \right]$$

$$= q^2\ f_o^2 \left\{ \left[\sum_{j=1}^{N} J_o(ma_j) \right]^2 + \sum_{j=1}^{N} (1) \right\}$$

$$= q^2\ f_o^2 \left\{ \left[\int_0^\infty da\ g_o(a)\ J_o(ma) \right]^2 + N \right\}$$

$$\sim q^2\ f_o^2 \left\{ O(N^2) + N \right\} \tag{5.13}$$

Thus for the longitudinal signal, the central bands $(\mu=0)$ add up linearly $(J_o(ma) \approx 1)$ with intensity. Instead of Schottky noise $\left\langle I_n^p \right\rangle_{RMS} = \sqrt{\sum_{j=1}^{N} I_n^2} \propto \sqrt{N}$, we have the bunch current $\left\langle I_n^p \right\rangle_{RMS} \propto N$ at the revolution harmonic. This systematic coherent signal at the $\mu = 0$ central bands corresponds to the gross macroscopic total current at the revolution band m due to the bunch as a whole and tends to blind the cooling system. Fortunately, this is also the signal that is suppressed the most by collective feedback in the case of transverse cooling, as we will see later. One can also design in the case of longitudinal cooling, a cooling system with a proper notch filter that removes the coherent central $\mu = 0$ bands by virtue of a zero in the gain function at frequencies $\Omega = n\omega_0$ corresponding to $\mu = 0$.

Another essential difference between the Schottky signals (in this case both the transverse and the longitudinal) of bunched beams and coasting beams, is the coherency of adjacent revolution bands for the same synchrotron harmonic. From (5.2) or (5.4) we see that the sidebands at a given harmonic $m\omega_0$ add up rms wise due to the synchrotron phase factor $\exp[i\mu\psi_i^0]$. However sidebands with the same μ but belonging to neighbouring revolution harmonics m have the same phase-factor and a similar weighting factor, $J_\mu(ma)$. Summing the noise power from n_ℓ coherent harmonics we have:

$$\left[\sum_m I_{m\mu}\right]^2 \approx n_\ell^2 \left\langle I_{m\mu}^2 \right\rangle \tag{5.14}$$

whereas in the case of a coasting beam (Eq. (5.3)):

$$\left[\sum_m I_m\right]^2 = n_\ell \left\langle I_m^2 \right\rangle$$

In other words, the noise of a particle and hence its disturbing influence on other particles adds, according to (5.14), in a coherent manner [48].

This coherency of adjacent revolution bands in the same synchrotron mode μ is seen more explicitly in the notion of effective gain developed in Section 4.3 and in Appendix B. The summation over the revolution harmonics ℓ in Eq. (4.3.54) for the same synchrotron mode $\mu = \mu'$ suggests a coherent enhancement of the effective gain of the interaction as felt by a particle of amplitude a in the Fourier harmonic μ of its synchrotron oscillations.

The analysis of transverse Schottky noise is a powerful technique for continuous and nondestructive beam diagnosis. Two particular features of the $p\bar{p}$ colliders (both the Fermilab Tevatron and the CERN SPS) are relevant to the considerations of Schottky noise -- the low D.C. current and the tight bunching. The low D.C. current, due to the low production rate of antiprotons and to the big dimensions of colliders, implies a low total power per spectrum line as given by Eqs. (5.10) and (5.12). The tight bunching required to increase the luminosity, produces a coherent enhancement of the parasitic longitudinal component as described by Eq. (5.13) in the transverse pick-up as well as a decrease in the width of the betatron lines. The latter effect increases the spectral power density of the Schottky signal compared to an unbunched beam and reduces the total power of the superimposed thermal noise. Therefore the most critical parameters in a Schottky detector system are the sensitivity of the pick-up, the noise factor of the electronics and the rejection level of the parasitic common mode signal. In particular the order of magnitude of this last parameter, required to avoid the risk of amplifier saturation, is fixed by the ratio of the power in the coherent longitudinal line to that in the incoherent transverse Schottky line and corresponds roughly to N, the number of circulating particles.

We mentioned earlier that the $\mu = 0$ coherent longitudinal signal can be removed by a notch filter designed to have zeros at frequencies $\Omega = n\omega_0$. However, for $N \sim 10^8 - 10^{12}$, the depth of the notches, as given by the ratio of powers in $\mu = 0$ coherent line and $\mu \neq 0$ incoherent Schottky line, is required to be 80-120 db (decibels).

With parabolic density profile for the bunches, the macroscopic bunch current falls off as $1/n$ with the revolution harmonic number n. For large bandwidth feedback system (4-8 GHz), $n \sim 10^5$ and the macroscopic bunch current is down by 10^5 requiring only a 60 db notch depth. Best available filters are characterized by 15-20 db notches, in the 1-2 GHz bandwidth region. When cascaded, notches of 50db-60db can be achieved again in the 1-2 Ghz range but at the cost of adding the phases of the filters and thus losing desirable phase-characteristics over the entire 1-2 Ghz range. With the new technology of superconducting cables (losses in cables minimized), notches can be made deeper and narrower with 50db-60db in a single filter again at 1-2 Ghz. These db figures for filters deteriorate as we go to larger bandwidth systems, e.g. 4-8 Ghz. Sensing of the central $\mu = 0$ line can also be suppressed by using a difference pick-up in a dispersive region of the storage ring. However, one then suffers from the very low sensitivity of the pick-up, since the electronic noise is not filtered out as in filter cooling and so the signal-to-noise ratio is very low. This scheme is thus typically noise dominated.

To evaluate explicitly the change in the signal to noise ratio in going from a tightly bunched to an unbunched beam of the same intensity, note that in both cases the total power per betatron line of the Schottky signal is practically the same as illustrated by Eqs. (5.10) and (5.12). This is because for short bunches and for a large range of the harmonic numbers n, including normally the revolution frequency harmonic tuned by the Schottky receiver, the form factor $F_{n,\mu=0}$ in (5.11) becomes approximately equal to 1. Therefore the change in signal to noise ratio depends only on the superimposed noise power, determined by the transverse line width ratio. As already mentioned the transverse frequency spread of a bunched beam is strongly reduced compared to that given by Eq. (5.7) and reflects only the non-linear tune spread coming from the multipole components of the machine (second order sextupoles, octupoles, etc.) and the space-charge effect. For the CERN SPS collider, the line width ratio has a value between 10 and 100 depending on the operating conditions, in good agreement with the measurements [67].

We mention now two other important points relevant to Schottky noise. We have studied the transverse signal for the dipole moment only, which splits each revolution harmonic $m\omega_0$ into two betatron side-bands $(m+Q)\omega_0$ and $(m-Q)\omega_0$. In reality, we have nonlinear sensing or pick-up devices which detect not only the transverse dipole moment but higher moments as well. Each harmonic $m\omega_0$ then gets split into a series of betatron harmoncs $(m\pm pQ)\omega_0$ with $p=1,2,\ldots$. Harmonics higher than the dipole $p=1$ however fall off rapidly in strength for real pick-ups. The cooling theory developed in this report includes such nonlinear sensing and kicking devices in terms of a general cooling interaction that involves all possible and relevant harmonics.

The other important point is the fact that all the above analysis of Schottky noise is based on uncorrelated beam parameters, i.e. random phase of the betatron and synchrotron oscillations of all the particles in the beam. Correlations develop among the phases of the oscillating particles as cooling progresses since the kicker electromagnetic fields tend to correlate the arrival times and betatron phases of the particles at the pick-up. The correlations created by the stochastic cooling feedback system act to deform the Schottky spectrum and in fact rapid "Schottky signal suppression" is commonly observed when cooling systems are turned on.

To illustrate this point, we introduce a collective variable

$$z_\ell(t) = \sum_{j=1}^{N} x_j(t)\, e^{-i\ell\theta_j(t)}$$

where $x_j(t)$ is the transverse betatron position as a function of time of the $j^{\underline{th}}$ particle and $\theta_j(t)$ is the aximuthal position and ℓ denotes a revolution harmonic. A quantity equivalent to the Schottky band intensity for harmonic ℓ is given by:

$$\left\langle |z_\ell(t)|^2 \right\rangle_{\theta,x} = \left\langle \left\{ z_\ell(t) \cdot z_\ell^*(t) \right\} \right\rangle_{\theta,x}$$

$$= \left\langle \left\{ \sum_{i=1}^{N} x_i \, e^{-i\ell\theta_i} \right\} \left\{ \sum_{j=1}^{N} x_j^* \, e^{i\ell\theta_j} \right\} \right\rangle_{\theta,x}$$

$$= \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j^* \, e^{-i\ell(\theta_i - \theta_j)} \right\rangle_{\theta,x}$$

$$= \left\langle \left\{ \sum_{i=1}^{N} |x_i|^2 + \sum_{\substack{i \neq j \\ 1}}^{N} \left(x_i x_j^* \right) e^{-i\ell(\theta_i - \theta_j)} \right\} \right\rangle_{\theta,x}$$

$$= N \langle |x|^2 \rangle + \left\langle \sum\sum_{\substack{i \neq j \\ =1}}^{N} \left(x_i x_j^* \right) e^{-i\ell(\theta_i - \theta_j)} \right\rangle_{\theta,x}$$

Using the particle orbits in a bunch defined in Chapter 3 and the identity (4.3.52) we write

$$\left\langle |z_\ell(t)|^2 \right\rangle_{\theta,x} = N \left\langle |x^2| \right\rangle + \sum\sum_{\substack{i \neq j \\ (=1)}}^{N} \sum\sum_{\substack{\mu \;\mu' \\ (-\infty)}}^{(+\infty)} J_\mu(-\ell a_i) \, J_{\mu'}(\ell a_j) \left\langle \left(x_i x_j^* \right) e^{i\mu\psi_i(t) + i\mu'\psi_j(t)} \right\rangle_{\theta,x}$$

If we define a 2-body correlation function, analogous to Eq. (4.4.8), in the $\mu^{\underline{th}}$ and $\mu'^{\underline{th}}$ Fourier coefficients in the synchrotron phase-angles of particles i and j at time t by:

$$C_{\mu,\mu'}(i,j;t) = \left\langle x_i(t) \, x_j^*(t) \, e^{i\mu\psi_i(t)} \, e^{-i\mu'\psi_j(t)} \right\rangle_{(x_i, x_j; \psi_i, \psi_j)}$$

then

$$\left\langle |z_\ell(t)|^2 \right\rangle = N \left\langle |x^2| \right\rangle + \sum\sum_{\substack{i \neq j \\ (=1)}}^{(N)} \sum\sum_{\substack{\mu \;\mu' \\ (-\infty)}}^{(+\infty)} J_\mu(-\ell a_i) \, J_{-\mu'}(\ell a_j) \, C_{\mu\mu'}(i,j;t)$$

Our previous analysis assumed $C^{\ell}_{\mu,\mu'}(i,j;t) = 0$ for all time. Even with incoherent random initial phases at $t = 0$ when $C^{\ell}_{\mu,\mu'}(i,j;t{=}0) = 0$, finite and nonzero

$C^{\ell}_{\mu,\mu'}(i,j;t)$ develops for $t > 0$ as cooling progresses (Section 4.4) and distorts the shape of the Schottky spectrum $<|z_{\ell}(t)|^2>$.

This phenomenon has been called "collective distortion of fluctuation spectra" in Chapter 4 and is common to other types of collective particle interactions as well, e.g. interaction through beam space-charge and wall impedances, which can also induce correlations and modify the Schottky signals. This topic is the subject of much quantitative study in Chapter 10. We will only mention here that this effect is analogous to the polarization and Debye shielding effect of plasma physics, where a "dielectric function or permittivity" is used to describe the details. We will derive a similar "collective signal suppression factor" later.

A similar Scottky signal analysis can be performed for particles confined by any general potential well and circulating in the ring with revolution frequency ω_0, as long as the particle orbits in the general potential well created by the bunching rf cavity (cavities) are known explicitly. In Appendix A, we derive the single particle longitudinal Schottky signal for a particle confined by a rectangular potential well or a square bucket, where the particles stream freely within the bucket except at the walls of the potential well where they reflect specularly like hard spheres. Such a bucket can be constructed by adding a cavity with a voltage of proper amount operating at a third harmonic (or any small amount of an odd harmonic as necessary to make the bucket square to the desired degree) relative to the main rf cavity operating at a fundamental harmonic. Such a bucket has maximal nonlinearity in some sense and provides spreads in the synchrotron oscillation frequencies comparable to coasting beams.

As is indicated in Section 4.3 the whole formulation of this report can be generalized to very general oscillatory synchrotron orbits (not necessarily sinusoidal) by defining certain orbit integrals $O_{\mu}(\ell,a)$ which for the simple case of harmonic sinusoidal orbits reduce to the Bessel functions $J_{\mu}(\ell a)$ used in this section.

6. SAMPLED SIGNAL AND AMPLITUDE AND PHASE EQUATIONS OF MOTION

6.1 Sampled Signal Seen by an Individual Particle

The Schottky signal generated at the pick-up is transferred by a linear transfer element characterized by a transfer function $G(\tau)$ to the kicker where a corresponding voltage or electric field signal is produced. The voltage $V_k(t)$ across the kicker gap due to the longitudinal signal or the electric field $E^k(t)$ at the kicker due to the transverse signal, is then given by

$$\begin{Bmatrix} V^k(t) \\ E^k(t) \end{Bmatrix} = \int_{-\infty}^{+\infty} dt' \; G(t-t') \begin{Bmatrix} I^p(t') \\ d^p(t') \end{Bmatrix}$$

where we have assumed a 'causal' transfer function $G(\tau)$ such that

$$G(\tau) = 0 \quad \text{for} \quad \tau < 0$$

The transfer function G includes the pick-up and kicker impedances (<u>or</u> admittances) describing their sensitivities and efficiencies as well as the linear 'gain' functions of amplifiers and filters in the feedback loop. With the convention for Fourier transforms

$$\tilde{G}(\Omega) = \int_{-\infty}^{+\infty} dt \; G(t) \, e^{-i\Omega t}$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\Omega \; \tilde{G}(\Omega) \, e^{i\Omega t}$$

we get for the longitudinal voltage $V^k(t)$ at the kicker $\theta = \theta_k$, by using the Schottky current formula (5.2) given in Chapter 5, the following:

$$V^k(t) = qf_0 \sum_{j=1}^{N} \sum_{m \, (-\infty)}^{(+\infty)} \sum_{\mu} \tilde{G}\left[m\omega_0 + \mu\omega_s(a_j)\right] \; J_\mu(ma_j) \; e^{i[m\omega_0 + \mu\omega_s(a_j)]t + i\mu\psi_j^0 - im\theta_p} \tag{6.1.1}$$

Thus each frequency component gets multiplied by the gain of the feedback loop at that frequency, as expected. A similar formula can be derived for the electric field signal $E^k(t)$ at the kicker derived from the transverse signal.

An individual particle i in the beam does not see the total voltage $V^k(t)$ at the kicker for all time, but samples it whenever it passes through the kicker, i.e. whenever

$$\theta_i(t) = \theta_k + 2\pi n \qquad (n = -\infty,\ldots,-1,0,+1,\ldots,+\infty)$$

We define

$$\begin{aligned}\theta_i(t) &= \omega_0 + \Theta_i(t)\\ &= \omega_0\Big[t - \tau_i(t)\Big]\end{aligned}$$

where $\Theta_i(t)$ is the azimuthal coordinate of particle i at time t in a frame moving with the bunch at angular velocity ω_0 and

$$\tau_i(t) = -\frac{\Theta_i(t)}{\omega_0} = -\frac{a_i}{\omega_0}\sin\Big[\omega_s(a_i)t + \psi_i^0\Big]$$

is the time-lag of the $i^{\underline{th}}$ particle with respect to the synchronous particle moving at constant ω_0. The sampling times are then given by

$$\omega_0\Big[t - \tau_i(t)\Big] = \theta_k + 2\pi n$$

or

$$t = \tau_i(t) + \frac{\theta_k}{\omega_0} + nT_0 \qquad \left(T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}\right)$$

We do not solve this transcendental equation for the sampling times t_n but instead convert the δ-function sampling in time into a periodic δ-function sampling in angle in the following. The sampled voltage signal seen by particle i is then

$$\mathscr{S}^i(t) = \sum_{n=-\infty}^{+\infty} V^k(t)\ \delta\left[t - \tau_i(t) - \frac{\theta_k}{\omega_0} - nT_0\right]$$

$$= \omega_0 \sum_{n=-\infty}^{+\infty} V^k(t)\ \delta\left[\theta_i(t) - \theta_k - 2\pi n\right]$$

Using Eq. (6.1.1) for $V^k(t)$ and expanding the periodic δ-functions in Fourier series, we get the total longitudinal voltage signal $\mathscr{S}^i[a_i, \psi_i, t]$ sampled by the $i^{\underline{th}}$ particle on its orbit $a_i(t)$, $\psi_i(t)$ at the kicker as a function of time, in the absence of coherent modulations by the kicker, as

$$\mathscr{S}^i\left[a_i, \psi_i;\ t\right] = q(f_0)^2 \sum_{j=1}^{N} \sum_{n} \sum_{m\,(-\infty)}^{(+\infty)} \sum_{\mu} \sum_{\nu} \tilde{G}\left[m\omega_0 + \mu\omega_s(a_j)\right]\ J_\mu(ma_j)\ J_\nu(na_i)$$

$$e^{i(m+n)\omega_0 t}\ e^{i\mu\omega_s(a_j)t}\ e^{i\mu\psi_j^0}\ e^{i\nu\psi_i(t)}\ e^{-i(m\theta_p + n\theta_k)} \qquad (6.1.2)$$

In the slow cooling approximation, $a_i(t) = a_i = \text{constant}$ and $\psi_i(t) = \omega_s(a_i)t + \psi_i^0$ and we get the zero-order sampled signal (neglecting the adiabatic time-dependence due to slow cooling) as follows:

$$\mathscr{S}_0^i(t) = q(f_0)^2 \sum_{j=1}^{N} \sum_{n} \sum_{m\,(-\infty)}^{(+\infty)} \sum_{\mu} \sum_{\nu} \tilde{G}\left[m\omega_0 + \mu\omega_s(a_j)\right]\ J_\mu(ma_j)\ J_\nu(na_i)$$

$$e^{i(m+n)\omega_0 t}\ e^{i[\mu\omega_s(a_j) + \nu\omega_s(a_i)]t}\ e^{i(\mu\psi_j^0 + \nu\psi_i^0)}\ e^{-i(m\theta_p + n\theta_k)} \qquad (6.1.3)$$

Fourier transformed in frequency, one gets

$$\tilde{\mathscr{S}}_0^i(\Omega) = (2\pi)\ q\ (f_0)^2 \sum_{j=1}^{N} \sum_{n} \sum_{m} \sum_{\mu\,(-\infty)}^{(+\infty)} \sum_{\nu} \tilde{G}\left[m\omega_0 + \mu\omega_s(a_j)\right]\ J_\mu(ma_j)\ J_\nu(na_i)$$

$$e^{-i(m\theta_p + n\theta_k)}\ e^{i(\mu\psi_j^0 + \nu\psi_i^0)}\ \delta\left[\Omega - m\omega_0 - n\omega_0 - \mu\omega_s(a_j) - \nu\omega_s(a_i)\right] \qquad (6.1.4)$$

In the situation of non-overlapping revolution bands (i.e. $m\omega_o + \mu\omega_s(a_j) = n\omega_o + \nu\omega_s(a_i)$ possible only with $m = n$ within the band-pass of the system), the term $\exp[i(m+n)\omega_o t]$ in the above expression (6.1.3) describes the rapid fluctuations in the sampled signal due to periodic traversals through the kicker. We can average over these rapidly oscillating terms when only the $n = -m$ contributes, giving a smoothed-out signal as follows:

$$\mathcal{S}^i\left[a_i,\psi_i;\, t\right] = q(f_o)^2 \sum_{j=1}^{N} \sum_m \sum_{\mu\,(-\infty)}^{(+\infty)} \sum_\nu \tilde{G}\left[m\omega_o + \mu\omega_s(a_j)\right] J_\mu(ma_j)\, J_\nu(-ma_i)$$

$$e^{i[\mu\psi_j t)+\nu\psi_i(t)]}\, e^{-im(\theta_p-\theta_k)} \qquad (6.1.5)$$

A similar analysis gives the transverse smoothed-out sampled betatron signal as follows:

$$\mathcal{S}_T^i\left[a_i,\psi_i;\, A_i,\phi_i;\, t\right] = \sum_{j=1}^{N} \left(\frac{A_j}{2}\right) (f_o)^2 \sum_m \sum_{\mu\,(-\infty)}^{(+\infty)} \sum_\nu e^{i[\mu\psi_j(t)+\nu\psi_i(t)]}\, e^{-im(\theta_p-\theta_k)}$$

$$\Big[\tilde{G}\left[(m+Q)\omega_o + \mu\omega_s(a_j)\right] J_\mu\left[(m+Q)a_j\right] J_\nu(-ma_i)\, e^{i\phi_j(t)}$$

$$+ \tilde{G}\left[(m-Q)\omega_o + \mu\omega_s(a_j)\right] J_\mu\left[(m-Q)a_j\right] J_\nu(-ma_i)\, e^{-i\phi_j(t)}\Big] \qquad (6.1.6)$$

6.2 Sampled Signal for Amplitude and Phase by the Method of Multiple Time-Scales Perturbation

The sampled longitudinal signal given by Eqs. (6.1.2) or (6.1.3) in the previous section is a "voltage/second" signal sampled by the $i^{\underline{th}}$ particle which causes a change in its 'longitudinal energy'. A change in 'energy' translates into a change in its synchrotron oscillation 'amplitude' and 'phase', due to rotation in phase-space. Similarly the transverse sampled signal is an "impulse/second" signal and changes the betatron velocity and momentum of the particle and translates into a change in the betatron oscillation amplitude and phase. We thus need to know the sampled amplitude and phase noise as experienced by a particle and as suitable for use in the amplitude and phase equations of motion

$$\frac{da_i}{dt} = n^i\left[a^i, \psi^i(t);\ t\right] \qquad \frac{dA_i}{dt} = n_T^i\left[a^i, \psi^i(t);\ A^i, \phi^i(t);\ t\right]$$

$$\frac{d\psi_i}{dt} = \omega_s^i + \xi^i\left[a^i, \psi^i(t);\ t\right] \qquad \frac{d\phi_i}{dt} = \omega_\perp^i + \xi_T^i\left[a^i, \psi^i(t);\ A^i, \phi^i(t);\ t\right] \tag{6.2.1}$$

for the synchrotron and betatron oscillations respectively.

In Eq. (6.2.1) the explicit time-dependence of the amplitude and phase noise signals is through the time-dependence of the orbits of all the other beam particles that set up the signal at the kicker while the implicit time-dependence is through the orbit of the particle i $(a^i(t), \psi^i(t))$ that is sampling the signal at the kicker. In the adiabatic slow cooling approximation, orbits of all the particles, including the sampling particle, can be replaced by zero-order unperturbed orbits, which are explicitly known as functions of time. The zero-order sampled amplitude and phase noise signals $n_0^i(t)$ and $\xi_0^i(t)$ then become explicitly known functions of time only.

We derive the longitudinal amplitude and phase noise signals now. The longitudinal energy satisfies

$$\dot{E}^i(t) = \frac{dE^i(t)}{dt} = q\,\mathcal{V}^i\left[a^i(t), \psi^i(t);\ t\right]$$

But

$$\dot{\theta}(t) = \omega(E) = \kappa E$$

where

$$\kappa = \frac{d\omega(E)}{dE}$$

is the machine parameter. So

$$\ddot{\theta}_i(t) = \frac{d\dot{\theta}_i(t)}{dt} = \frac{d\omega^i(E)}{dt} = \kappa\,\dot{E}^i(t) = q\,\kappa\mathcal{V}^i\left[a^i, \psi^i;\ t\right]$$

Thus in presence of synchrotron oscillations and sampled voltage noise signal, we have

$$\ddot{\Theta}_i + \omega_s^2(a_i)\,\Theta_i = q\,\kappa\mathcal{V}^i\left[a_i, \psi_i;\ t\right] \tag{6.2.2}$$

where $\textcircled{H}_i(t) = \theta_i(t) - \omega_0 t$, $a_i(t)$ and $\psi_i(t)$ are the synchrotron amplitude and phase of $i^{\underline{th}}$ particle at time t, and $\omega_s(a_i)$ the synchrotron oscillation frequency of particle i.

Note that we have used the rapid time-averaged (over periodic traversals through the kicker) form $\bar{\mathcal{S}}^i(a_i,\psi_i;t)$ of the sampled voltage as given by Eq. (6.1.5) in Eq. (6.2.2). This is because synchrotron oscillations are usually much slower than the revolution time $(\omega_s << \omega_0)$ and hence the use of a differential equation to describe synchrotron oscillations implies that such averages over rapidly fluctuating revolution-periodic dependences be taken in all the forces entering the synchrotron equation of motion (6.2.2). Such need not be the case for the betatron motion which is usually comparable or slightly faster than the revolution time $(\omega_\perp = Q\omega_0 > \omega_0)$. For betatron motion then, we use

$$\ddot{x}_i + \omega_{\perp i}^2 \, x_i = q \, \mathcal{S}_T^i\left[a_i,\psi_i;\, A_i,\phi_i;\, t\right] \tag{6.2.3}$$

where $\mathcal{S}_T^i$ is the full transverse sampled signal with no time-averaging and thus containing all the rapidly fluctuating terms in it.

Since the sampled signals on the right-hand sides of Eqs. (6.2.2) and (6.2.3) are small compared to the strong synchrotron <u>or</u> betatron oscillation restoring forces, a number of perturbation methods are available for the determination of approximate solutions of these equations. We use the method of "multiple time-scales perturbation" to determine first order expansions, which are valid for large t ([72],[73]). The essence of the multiple time-scales perturbation method is to consider the expansion representing the solution $\textcircled{H}_i(t)$ <u>or</u> $x_i(t)$ to be a function of multiple independent and dispar- ate time variables or scales, instead of a single variable.

We introduce a small dimensionless parameter ε that represents the strength of the cooling signal (the sampled signal on the right-hand side of Eqs. (6.2.2) and (6.2.3)) relative to the restoring spring constants $\omega_\perp = Q\omega_0$ <u>or</u> ω_s for betatron and synchrotron oscillations. Thus, for example we write Eq. (6.2.3) as

$$\ddot{x}_i + \omega_{\perp i}^2 \, x_i = \varepsilon q \, \mathcal{S}_T^i\left[a_i,\psi_i;\, A_i,\phi_i;\, t\right] \tag{6.2.4}$$

and Eq. (6.2.2) as

$$\ddot{H}_i + \omega_s^2(a_i) H_i = \epsilon q \kappa \mathcal{J}^i \left[a_i, \psi_i; t \right] \tag{6.2.5}$$

After having done the perturbation analysis and obtaining the solution to the desired order in ϵ, we eventually let $\epsilon \equiv 1$ since the gain of the feedback loop per particle per frequency line embedded in $\mathcal{J}^i$ already provides us with such a small parameter.

It is also convenient to visualize the right hand sides (6.2.5) and (6.2.4) as functions $f(H_i, \dot{H}_i, t)$ so that for example

$$\ddot{H}_i + \omega_s^2(a_i) \, H_i = \epsilon \, f\left[H_i, \dot{H}_i; t \right] \tag{6.2.6}$$

since there is always a transformation relating $[J_i = 1/2 \, a_i^2, \psi_i]$ to $[H_i, \dot{H}_i]$.

Some comments are in order regarding the sampled voltage <u>or</u> electric field noise function $f[H_i, \dot{H}_i; t]$ appearing on the right-hand sides of these equations. The explicit time-dependence of these functions comes from the signals of all the particles $j = 1$ to N derived at the pick-up and applied at the kicker. The dependence on H_i and $\dot{H}_i$ comes from the fact the $i^{\underline{th}}$ particle samples this signal on its orbit $[H_i(t), \dot{H}_i(t)]$ only. Moreover, the sum $\sum_{j=1}^{N}$ contains the term $j = i$, the coherent part describing the $i^{\underline{th}}$ particle sampling its own signal and the $\sum_{j(\neq i)=1}^{N}$, the incoherent Schottky noise part describing the signals of all the other particles. We thus have the decomposition (for longitudinal cooling, say):

$$\begin{aligned} f\left(H_i, \dot{H}_i; t \right) &= \sum_{j=1}^{N} s\left(H_i, \dot{H}_i; H_j, \dot{H}_j \right) \\ &= C\left(H_i, \dot{H}_i \right) + S\left(H_i, \dot{H}_i; t \right) \end{aligned}$$

where C depends on $(H_i, \dot{H}_i)$ alone corresponding to the $i = j$ coherent part. This is the nonconservative but 'autonomous' part of the dynamics for particle i. The second term describes the force due to all the <u>other</u> particles in the beam and gives the manifest time-dependence. This is the 'nonautonomous' part of the dynamics of the $i^{\underline{th}}$ particle.

If we represent zero order particle orbits by

$$\textcircled{H}_i = a_i \operatorname{Sin}\left(\omega_s(a_i)t + \psi_i^0\right)$$

then both C and S have oscillating time dependences of the form $\sum_n \sum_m e^{i\Omega_{nm}t}$ where $\Omega_{nm} = n\omega_s(a_i)\delta_{nm}$ for C and $\Omega_{nm} = n\omega_s(a_i) + m\omega_s(a_j)$ for S.

The linear slow cooling or damping rate γ is given by

$$\gamma \sim \varepsilon\omega_\perp \quad \text{for transverse cooling}$$

$$\sim \varepsilon\omega_s \quad \text{for longitudinal cooling}$$

Thus we make use of the fact that the characteristic time-scale for cooling γ^{-1} is much longer than the time-scale for synchrotron or betatron oscillations (ω_s^{-1} or $\omega_\perp^{-1}$) i.e.

$$\frac{\gamma^{-1}}{\omega_\perp^{-1}} = \frac{1}{\varepsilon} >> 1 \quad \text{and} \quad \frac{\gamma^{-1}}{\omega_s^{-1}} = \frac{1}{\varepsilon} >> 1 \tag{6.2.7}$$

Relation (6.2.7) can be used as a definition of the small parameter ε if one wishes.

To incorporate this disparity between oscillation and damping time scales in the expansion procedure, we arbitrarily extend the number of time variables from one variable t to many variables by introducing new independent time variables τ_n according to

$$\tau_n = \varepsilon^n t \quad \text{for} \quad n = 0,1,2,\ldots \tag{6.2.8}$$

Thus

$$\left(\tau_0,\tau_1,\tau_2,\ldots\right) \equiv \left(t,\varepsilon t,\varepsilon^2 t,\ldots\right)$$

and

$$\frac{d\tau_0}{dt} = 1, \quad \frac{d\tau_1}{dt} = \varepsilon, \quad \frac{d\tau_2}{dt} = \varepsilon^2,\ldots \tag{6.2.9}$$

The derivatives with respect to t thus become expansions in terms of the partial derivatives with respect to the operationally independent τ_n's according to

$$\frac{d}{dt} = \frac{d\tau_o}{dt}\frac{\partial}{\partial\tau_o} + \frac{d\tau_1}{dt}\frac{\partial}{\partial\tau_1} + \ldots = D_o + \epsilon D_1 + \ldots$$

$$\frac{d^2}{dt^2} = D_o^2 + 2\epsilon\, D_o\, D_1 + \epsilon^2\left(D_1^2 + 2D_oD_2\right) + \ldots \tag{6.2.10}$$

where

$$D_n \equiv \frac{\partial}{\partial\tau_n}$$

We visualize Ⓗ <u>or</u> x to be a function of the various new time-scales and assume that they can be represented by an expansion having the form

$$Ⓗ = Ⓗ^{(0)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon\, Ⓗ^{(1)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon^2 Ⓗ^{(2)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \ldots \tag{6.2.11}$$

The number of independent time-scales needed depends on the order to which the expansion is carried out. If the expansion is carried out to $O(\epsilon)$, then only τ_o and τ_1 are needed.

For longitudinal synchrotron oscillations that are linear, $\omega_s(a_i)$ is a constant independent of a_i and we need only the expansion (6.2.11). For nonlinear synchrotron oscillations however, $\omega_s(a_i)$ is a function of amplitude of the particle and hence a slow function of time since the expansion (6.2.11) implies a similar expansion for the amplitude:

$$a = a^{(0)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon\, a^{(1)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon^2\, a^{(2)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \ldots \tag{6.2.12}$$

Hence we also need an expansion for ω_s as follows:

$$\omega_s = \omega_s^{(0)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon\,\alpha\,\omega_s^{(1)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \epsilon^2\,\alpha^2\,\omega_s^{(2)}\left(\tau_o,\tau_1,\ldots,\tau_n,\ldots\right) + \ldots \tag{6.2.13}$$

where $\alpha = \frac{a}{\omega_s(a)} \left[\frac{d\omega_s(a)}{da}\right]$ is the nonlinearity parameter for sychrotron oscillations, assumed small $(\alpha < 1)$.

We note that $\omega_s^{(0)}(\tau_0, \tau_1, \ldots)$ is not the small amplitude synchrotron oscillation frequency but the frequency as a function of the synchrotron amplitude evaluated to zero order in ε i.e.

$$\omega_s^{(0)}\left(\tau_0, \tau_1, \ldots\right) = \omega_s \left\{a^{(0)}\left(\tau_0, \tau_1, \ldots\right)\right\} \tag{6.2.14}$$

Substituting (6.2.8), (6.2.9), (6.2.10), (6.2.11) and (6.2.13) into (6.2.5) and (6.2.6) and equating coefficients of successive like powers of ε, we obtain

Order $\varepsilon^0 \alpha^0$: $$\left[D_0^2 + \left(\omega_s^{i(0)}\right)^2\right] Ⓗ_i^{(0)} \left(\tau_0, \tau_1, \tau_2, \ldots\right) = 0 \tag{6.2.15}$$

Order $\varepsilon^0 \alpha^1$: No term

Order $\varepsilon^1 \alpha^0$: $$\left[D_0^2 + \left(\omega_s^{i(0)}\right)^2\right] Ⓗ_i^{(1)} \left(\tau_0, \tau_1, \tau_2, \ldots\right) = -2D_0 D_1 Ⓗ_i^{(0)} + f\left[Ⓗ_i^{(0)}, D_0 Ⓗ^{(0)};\ \tau_0\right]$$

$$\vdots \tag{6.2.16}$$

Some comments are in order. Extending the number of time variables provides us with considerable freedom to remove order by order, any time secularities which may occur in the solutions for $Ⓗ_i^{(0)}(\tau_0, \tau_1, \ldots)$, $Ⓗ_i^{(1)}(\tau_0, \tau_1, \ldots)$, etc. This assures that the perturbation solution represented by (6.2.11) is uniformly valid, order by order. The reason for removing time secularities is simply that for the physical situation under consideration $|Ⓗ(t)| < \infty$ i.e. $|Ⓗ(t)|$ cannot grow without bound, except in cases where the cooling loop induces collective instabilities, in which case we cannot cool the beam anyway. The advantage of a multiple time-scale perturbation analysis is that the condition that the first-order solution $Ⓗ_i^{(1)}(\tau_0, \tau_1, \ldots)$ be nonsecular as $\tau_0 \to \infty$ determines the slow evolution (and saturation, if nonlinear) of the zero-order oscillation amplitude and phase on the τ_1 time-scale. Once the multiple time scale perturbation solution is obtained in this way to the desired order of accuracy and secularities removed, we return to the physical time-variable t in the final expressions for $Ⓗ_i^{(0)}$, $Ⓗ_i^{(1)}$, ... etc.

We concern ourselves here with a two time-scale approach -- a fast time-scale τ_0 of the synchrotron or betatron oscillations and a slow time-scale τ_1 of overall cooling over a long period of time. We thus have Eqs. (6.2.15) and (6.2.16) only to consider.

We write the general solution of (6.2.15) in the complex form

$$\Theta_i^{(0)} = R_i\left(\tau_1,\tau_2,\ldots\right) e^{i\omega_s^{i(0)}\tau_0} + R_i^*\left(\tau_1,\tau_2,\ldots\right) e^{-i\omega_s^{i(0)}\tau_0} \tag{6.2.17}$$

The function $R_i(\tau_1,\tau_2,\ldots)$ is still arbitrary at this level of approximation. It is determined by eliminating the secular terms (invoking the so called "solvability conditions") at the higher levels of approximation.

Substituting for $\Theta_i^{(0)}$ into (6.2.16), we have

$$\begin{aligned}\left[D_0^2 + \left(\omega_s^{i(0)}\right)^2\right] \Theta_i^{(1)} &= - 2i\omega_s^{i(0)}\, D_1 R_i\, e^{i\omega_s^{i(0)}\tau_0} + 2i\omega_s^{i(0)}\, D_1 R_i^*\, e^{-i\omega_s^{i(0)}\tau_0} \\ &\quad + f\left[R_i\, e^{i\omega_s^{i(0)}\tau_0} + R_i^*\, e^{-i\omega_s^{i(0)}\tau_0},\ i\omega_s^{i(0)}\, R_i\, e^{i\omega_s^{i(0)}\tau_0}\right. \\ &\qquad \left. - i\omega_s^{i(0)}\, R_i^*\, e^{-i\omega_s^{i(0)}\tau_0};\ \tau_0\right]\end{aligned} \tag{6.2.18}$$

Depending on the function R_i, all particular solutions of (6.2.18) contain terms proportional to $\tau_0 \exp(\pm i\omega_s^{i(0)}\tau_0)$ -- these are the so called "secular terms". Thus $\epsilon\Theta_i^{(1)}$ can dominate $\Theta_i^{(0)}$ for large t, resulting in a nonuniform expansion. We choose the function R_i so that secular terms are eliminated from $\Theta_i^{(0)}$ and thereby obtain a uniformly valid expansion. To this end we now use the decomposition

$$f\left[\Theta_i^{(0)}, D_0\Theta_i^{(0)};\ \tau_0\right] = C\left[\Theta_i^{(0)}, D_0\Theta_i^{(0)}\right] + S\left[\Theta_i^{(0)}, D_0\Theta_i^{(0)};\ \tau_0\right]$$

and expand C and S in a Fourier series as follows:

$$C = \sum_{n=-\infty}^{+\infty} c_n\left(R_i,R_i^*\right) e^{in\omega_s^{i(0)}\tau_o} \tag{6.2.19}$$

$$S = \sum_{\substack{j(\neq i)\\=1}}^{N} \sum_{n}^{(+\infty)}\sum_{m} S_{nm}\left(R_i,R_i^*;\ R_j,R_j^*\right) e^{i[n\omega_s^{i(0)}+m\omega_s^{j(0)}]\tau_o} \tag{6.2.20}$$

where

$$c_n\left(R_i,R_i^*\right) = \frac{\omega_s^{i(0)}}{2\pi} \int_0^{2\pi/\omega_s^{i(0)}} d\tau_o\ e^{-in\omega_s^{i(0)}\tau_o}\ c\left[Ⓗ_i^{(0)},D_oⒽ_i^{(0)}\right] \tag{6.2.21}$$

and so on.

Then Eq. (6.2.18) becomes:

$$\left[D_o^2 + \left(\omega_s^{i(0)}\right)^2\right] Ⓗ_i^{(1)} = -\,2i\omega_s^{i(0)}\ D_1\ R_i\ e^{i\omega_s^{i(0)}\tau_o} + 2i\omega_s^{i(0)}\ D_1\ R_i^*\ e^{-i\omega_s^{i(0)}\tau_o}$$

$$+ \sum_{n=-\infty}^{+\infty} c_n\left(R_i,R_i^*\right) e^{in\omega_s^{i(0)}\tau_o} + \sum_{j(\neq i)=1}^{N} \sum_{n}^{(+\infty)}\sum_{m} S_{nm}\left(R_i,R_i^*;\ R_j,R_j^*\right)$$

$$e^{i[n\omega_s^{i(0)}+m\omega_s^{j(0)}]\tau_o} \tag{6.2.22}$$

Equation (6.2.22) has the form of a forced harmonic oscillator equation for $Ⓗ_i^{(1)}(\tau_o,\tau_1,\ldots)$. The terms on the right-hand side which are proportional to $\exp[\pm\, i\omega_s^{i(0)}\tau_o]$ drive the oscillator at its natural frequency $\omega_s^{i(0)}$ and give rise to secular contributions to $Ⓗ_i^{(1)}(\tau_o,\tau_1,\ldots)$. The other terms on the right-hand side of (6.2.22) produce oscillatory modulations of $Ⓗ_i^{(1)}$ at harmonics of $\omega_s^{i(0)}$ e.g. $2\omega_s^{i(0)}$, $3\omega_s^{i(0)}$ and so on.

One such secular term is $c_1(R_i,R_i^*)e^{i\omega_s^{i}(0)\tau_o}$. In the last term on the right hand side, there are driving terms of the form $\exp[\pm i\omega_s^{i(0)}\tau_o]$ whenever the resonance condition $n\omega_s^{i(0)} + m\omega_s^{j(0)} = \pm\ \omega_s^{i}(0)$ is satisfied. In particular the terms $n = \pm 1$, $m = 0$ drive the oscillator with its natural frequency with right phase $\pm\ \psi_s^{i(0)}$. However, such terms correspond to the $i^{\underline{th}}$ particle sampling with its first harmonic

$(n = \pm 1)$ the signal produced by the $j^{\underline{th}}$ particle in its zeroth harmonic $(m = 0)$. For transverse pick-up, no signal is produced when there is no dipole moment $(m = 0)$ i.e. no betatron oscillations, if the pick-up is properly aligned and hence this term is zero. For longitudinal pick-up, the signal is nonzero but falls exactly at one of the revolutions harmonics $p\omega_0$ of the center of the bunch $(\tilde{G}(p\omega_0 + m\omega_s^{j(0)}) = \tilde{G}(p\omega_0)$ for $m = 0)$. Since we do not wish to affect the center of the bunch but cool other particles towards it this gain $\tilde{G}(p\omega_0)$ is usually set to zero by using a notch filter. Thus contribution from this term again is negligible. There are other values of n,m and i,j which however satisfy the resonance condition above and drive the oscillator at its natural frequency $\omega_s^{(0)}$; however they do so with random phases $e^{i[n\psi_s^{i(0)} + m\psi_s^{j(0)}]}$, $j = 1,\ldots,N(\neq i)$ and their average effect on $Ⓗ_i^{(1)}$ vanishes. However their average effect on the amplitude R_i does not vanish and cause a slow diffusion in the mean square $<|R|^2>$. We wish to retain this average effect on the mean square.

Let us write

$$Ⓗ_i^{(1)}\left(\tau_0,\tau_1,\ldots\right) = \left[Ⓗ_i^{(1)}\left(\tau_0,\tau_1,\ldots\right)\right]_{sec} + \left[Ⓗ_i^{(1)}\left(\tau_0,\tau_1,\ldots\right)\right]_{non-sec}$$

where $[Ⓗ_i^1]_{sec}$ diverages as $\tau_0 \to \infty$ and $[Ⓗ_i^1]_{non-sec}$ remains bounded as $\tau_0 \to \infty$. This could be obtained by integrating (6.2.22) on the τ_0-scale since τ_0 and τ_1 are independent.

In order that the solution for $Ⓗ_i^{(1)}(\tau_0,\tau_1,\ldots)$ be uniformly valid for all τ_0, we remove the secular behavior by setting:

$$\left[Ⓗ_i^{(1)}\left(\tau_0,\tau_1,\ldots\right)\right]_{sec} = 0$$

It is evident from (6.2.22) and the above discussion of resonances, that we can eliminate the secular terms for all values of τ_0 and still preserve the average effect on the mean square, if we set:

$$2i\omega_s^{i(0)}\; D_1\, R_i = C_1\left(R_i, R_i^*\right) + S\left[Ⓗ_i^{(0)}, D_0 Ⓗ_i^{(0)}; \tau_0\right] e^{-i\omega_s^{i(0)}\tau_0}$$

$$= \frac{\omega_s^{i(0)}}{2\pi} \int_0^{2\pi/\omega_s^{i(0)}} d\tau_0\; e^{-i\omega_s^{i(0)}\tau_0}\, C\left[Ⓗ_i^{(0)}, D_0 Ⓗ_i^{(0)}\right]$$

$$+ S\left[Ⓗ_i^{(0)}, D_0 Ⓗ_i^{(0)}; \tau_0\right] e^{-i\omega_s^{i(0)}\tau_0} \qquad (6.2.23)$$

Since we are considering only two-time scales τ_0 and τ_1, we conider R_i to be a function of τ_1 only and end the solution here. To solve (6.2.23) it is convenient to express $R_i(\tau_1)$ in the polar form as

$$R_i(\tau_1) = \frac{1}{2}\, a_i(\tau_1)\, e^{i\beta_i(\tau_1)} \qquad (6.2.24)$$

so that we rewrite (6.2.17) as:

$$Ⓗ_i^{(0)} = a_i(\tau_1)\ \mathrm{Cos}\ \psi_i; \qquad \psi_i = \omega_s^{i(0)}\tau_0 + \beta_i(\tau_1) \qquad (6.2.25)$$

Substituting (6.2.24) into (6.2.23), we get:

$$i\left(a_i' + i a_i \beta_i'\right) = \frac{1}{2\pi\omega_s^{i(0)}} \int_0^{2\pi} d\psi_i\; e^{-i\psi_i}\, C\left(a_i, \psi_i\right) + \frac{1}{\omega_s^{i(0)}}\, S\left(a_i, \psi_i; \tau_0\right) e^{-i\psi_i}$$

$$(6.2.26)$$

where the primes denote differentiation with respect to τ_1 e.g. $a' = da(\tau_1)/d\tau_1$ etc.

Separating the real and imaginary parts in (6.2.26) and going back to the original t variables now, we see that we may write the solution of (6.2.5) as

$$Ⓗ_i = a_i(t)\ \mathrm{Cos}\ \psi_i(t)$$

$$\psi_i(t) = \omega_s(a_i)\, t + \beta_i(t) \qquad (6.2.27)$$

where

$$\dot{a}_i = \frac{da_i}{dt} = \frac{-1}{2\pi\omega_s(a_i)} \int_0^{2\pi} d\psi_i \; \mathrm{Sin}\,\psi_i \; C\left(a_i,\psi_i\right) - \frac{1}{\omega_s(a_i)} \mathrm{Sin}\,\psi_i \; S\left(a_i,\psi_i;\, t\right)$$

$$\dot{\psi}_i = \omega_s(a_i) + \dot{\beta}_i = \omega_s(a_i) - \frac{1}{2\pi\omega_s(a_i)a_i} \int_0^{2\pi} d\psi_i \; \mathrm{Cos}\,\psi_i \; C\left(a_i,\psi_i\right)$$

$$- \frac{1}{\omega_s(a_i)a_i} \mathrm{Cos}\,\psi_i \; S\left(a_i,\psi_i;\, t\right) \tag{6.2.28}$$

By removing the time-secularities on a fast time-scale, we have thus obtained the differential equation which determines the time-development of the amplitude and phase on a slower time-scale. Equation (6.2.28) thus gives us the amplitude and phase equations of motion corresponding to (6.2.2) in terms of the coherent part C and the Schottky noise part S of the sampled signal $\mathcal{S}^i$ where

$$q\kappa\mathcal{S}^i\left(a_i,\psi_i;\, t\right) = C\left(a_i,\psi_i\right) + S\left(a_i,\psi_i;\, t\right) \tag{6.2.29}$$

We note that if the synchrotron oscillations were purely linear so that $\omega_s(a_i)$ is a constant independent of a_i, then Eq. (6.2.27) and (6.2.28) would be obtained without assuming the small nonlinearity parameter α. As outlined in Section 4.4 of Chapter 4 (and as we will see later) the presence of nonlinearity i.e. amplitude-dependent synchrotron oscillation frequency is crucial to effective stochastic cooling of bunches. We can then use Eqs. (6.2.27) and (6.2.28) for nonlinear synchrotron oscillations, provided we restrict ourselves to modestly small nonlinearities $\alpha = \frac{a}{\omega_s(a)} \frac{d\omega_s}{da} < 1$. In using Eqs. (6.2.27) and (6.2.28) to bunches captured in buckets with a higher amount of nonlinearity of the oscillation orbits, one has to exercise considerable amount of care. The same remark also applies to fast cooling schemes where cooling time γ^{-1} is smaller than the synchrotron oscillation time T_s since then $(\varepsilon\alpha\omega_s t)$ is not necessarily small compared to $(\omega_s t)$. Our formulation for stochastic cooling of <u>longitudinal synchrotron oscillations</u> is thus restricted to cooling times slow compared to synchrotron oscillation times.

The situation is considerably less involved for transverse betatron cooling which does not interfere with the longitudinal nonlinearity of synchrotron oscillations. Transverse cooling rate is not crucially sensitive to whether betatron oscillations are linear <u>or</u> not as long as synchrotron oscillations provide enough nonlinearity and hence mixing in the longitudinal orbits. Steps leading to (6.2.27) and (6.2.28) for purely linear betatron oscillations do not involve any assumption of a small nonlinearity parameter α and Eqs. (6.2.27) and (6.2.28) for betatron amplitude A_i and phase $\phi_i = \omega_\perp^i t + \delta_i(t)$ are then <u>exact</u> for arbitrary nonlinearity.

Noting that $I_i = 1/2\, A_i^2$ and $J_i = 1/2\, a_i^2$ are the 'action' variables and $\phi_i(t) = \omega_\perp^i t + \delta_i(t)$ and $\psi_i(t) = \omega_s(a_i)t + \beta_i(t)$ the corresponding phase or angle variables for the betatron and synchrotron oscillations respectively, we thus have the following cooling dynamics equations for the action and angle variables for transverse and longitudinal cooling:

<u>LONGITUDINAL</u>

$$\begin{aligned}
\dot{J}_i &= G_i^0\left(J_i, \psi_i\right) + G_i^1\left(J_i, \psi_i;\, t\right) \\
&= G(i,i) + \sum_{j(\neq i)=1}^{N} G(i,j) \\
&= -\frac{[2J_i]^{1/2}}{2\pi\omega_s(J_i)} \int_0^{2\pi} d\psi_i \,\mathrm{Sin}\, \psi_i \; C\left(J_i, \psi_i\right) - \frac{[2J_i]^{1/2}}{\omega_s(J_i)} \,\mathrm{Sin}\, \psi_i \; S\left(J_i, \psi_i;\, t\right)
\end{aligned} \tag{6.2.30}$$

$$\begin{aligned}
\dot{\psi}_i &= \omega_s(J_i) + H_i^0\left(J_i, \psi_i\right) + H_i^1\left(J_i, \psi_i;\, t\right) \\
&= \omega_s(J_i) + H(i,i) + \sum_{j(\neq i)=1}^{N} H(i,j) \\
&= \omega_s(J_i) - \frac{1}{2\pi\omega_s(J_i)[2J_i]^{1/2}} \int_0^{2\pi} d\psi_i \,\mathrm{Cos}\, \psi_i \; C\left(J_i, \psi_i\right) \\
&\quad - \frac{1}{\omega_s(J_i)[2J_i]^{1/2}} \,\mathrm{Cos}\, \psi_i \; S\left(J_i, \psi_i;\, t\right)
\end{aligned} \tag{6.2.31}$$

TRANSVERSE

$$\dot{I}_i = G_i^{T0}\left(J_i,\psi_i;\ I_i,\phi_i\right) + G_i^{T1}\left(J_i,\psi_i;\ I_i,\phi_i;\ t\right)$$

$$= G^T(i,i) + \sum_{j(\neq i)=1}^{N} G^T(i,j)$$

$$= -\frac{[2I_i]^{1/2}}{2\pi\ \omega_\perp^i}\int_0^{2\pi} d\phi_i \ \mathrm{Sin}\ \phi_i\ C\left(J_i,\psi_i;\ I_i,\phi_i\right)$$

$$-\frac{[2I_i]^{1/2}}{\omega_\perp^i}\ \mathrm{Sin}\ \phi_i\ S\left(J_i,\psi_i;\ I_i,\phi_i;\ t\right) \qquad (6.2.32)$$

$$\dot{\phi}_i = \omega_\perp^i + H_i^{T0}\left(J_i,\psi_i;\ I_i,\phi_i\right) + H_i^{T1}\left(J_i,\psi_i;\ I_i,\phi_i;\ t\right)$$

$$= \omega_\perp^i + H^T(i,i) + \sum_{j(\neq i)=1}^{N} H^T(i,j)$$

$$= \omega_\perp^i - \frac{1}{2\pi\omega_\perp^i[2I_i]^{1/2}}\int_0^{2\pi} d\phi_i\ \mathrm{Cos}\ \phi_i\ C\left(J_i,\psi_i;\ I_i,\phi_i\right)$$

$$-\frac{[2I_i]^{1/2}}{\omega_\perp^i}\ \mathrm{Cos}\ \phi_i\ S\left(J_i,\psi_i;\ I_i,\phi_i;\ t\right) \qquad (6.2.33)$$

$$\left.\begin{aligned}\dot{J}_i &= 0\\ \dot{\psi}_i &= \omega_s(J_i)\end{aligned}\right\}\ \begin{array}{l}\text{No}\\ \text{Longitudinal}\\ \text{Cooling}\end{array} \qquad (6.2.34)$$

7. HARMONIC REPRESENTATION AND THE HAMILTONIAN FLOW CONDITION FOR ACTION AND PHASE SIGNAL

Using the explicit form of the longitudinal sampled signal $\mathcal{S}^i(J^i,\psi^i;t)$ given by Eq. (6.5) into Eq. (7.2.39) and using the identity [111]:

$$e^{ix \, \mathrm{Sin}\, y} \, \mathrm{Cos}\, y = \sum_{\mu=-\infty}^{+\infty}{}' \frac{\mu}{x} J_\mu(x) \, e^{i\mu y} \tag{7.1}$$

where $\sum'$ means $\mu = 0$ is not included in the sum and $J_\mu(x)$ is an ordinary Bessel function of order μ, we obtain a Fourier series representation of $\dot{J}^i$ in harmonics of ψ^i and ψ^j as

$$\dot{J}^i = \sum_{j=1}^{N} G\left(J^i,\psi^i;\, J^j,\psi^j\right) = \sum_{j=1}^{N} \sum_{\mu}^{(+\infty)}\sum_{\nu}{}' G_{\mu\nu}\left(J^i,J^j\right) e^{i[\mu\psi^j(t)+\nu\psi^i(t)]} \tag{7.2}$$

where

$$G_{\mu\nu}\left(J^i,J^j\right) = \frac{(qf_0)^2\kappa}{\omega_s(J^i)} \left[1 + \delta_{ij}(\delta_{\mu,-\nu}-1)\right] \left[1 - \delta_{\nu,0}\right] \sum_{m=-\infty}^{+\infty} \left(\frac{\nu}{-m}\right) \tilde{G}\left[m\omega_0 + \mu\omega_s(J^j)\right]$$

$$\cdot\, J_\mu\left(m\sqrt{2J^i}\right) J_\nu\left(-m\sqrt{2J^i}\right) e^{im(\theta_p-\theta_k)} \tag{7.3}$$

Writing

$$\dot{J}^i = G(i,i) + \sum_{j(\neq i)=1}^{N} G(i,j) = G_i^0 + G_i^1 \tag{7.4}$$

we identify.

$$G_i^0 = G(i,i) = \sum_{\mu=-\infty}^{+\infty}{}' G_{\mu,-\mu}\left(J^i,J^i\right) \tag{7.5}$$

$$G_i^1 = \sum_{j(\neq i)=1}^{N} G(i,j) = \sum_{j(\neq i)=1}^{N} \sum_{\mu}^{(+\infty)}\sum_{\nu}{}' G_{\mu\nu}\left(J^i,J^j\right) e^{i[\mu\psi^j+\nu\psi^i]} \tag{7.6}$$

where

$$G_{\mu\nu}\left(J^i,J^j\right) = \frac{(qf_o)2\kappa}{\omega_s(J^i)} \sum_{m=-\infty}^{+\infty} \left(\frac{\nu}{-m}\right) \tilde{G}\left[m\omega_o + \mu\omega_s(J^j)\right] J_\mu\left(m\sqrt{2J^i}\right) J_\nu\left(-m\sqrt{2J^j}\right) e^{im(\theta_p-\theta_k)} \tag{7.7}$$

Similarly using the identity [111]:

$$e^{ix \operatorname{Sin} y} \operatorname{Sin} y = (-i) \sum_{\nu=-\infty}^{+\infty} J_\nu'(x)\, e^{i\nu y} \tag{7.8}$$

we get from (6.2.40) for the phase equation:

$$\dot{\psi}^i = \omega_s(J^i) - \sum_{j=1}^{N} \sum_{\mu}^{(+\infty)}\sum_{\nu}_{(-\infty)} H_{\mu\nu}\left(J^i,J^j\right) e^{i[\mu\psi^j+\nu\psi^i]} \tag{7.9}$$

where

$$H_{\mu\nu}\left(J^i,J^j\right) = \frac{(qf_o)^2\kappa}{i\omega_s(J^i)\ 2J^i} \left[1+\delta_{ij}(\delta_{\mu,-\nu}-1)\right] \sum_{m=-\infty}^{+\infty} \tilde{G}\left[m\omega_o + \mu\omega_s(J^j)\right] J_\mu\left(m\sqrt{2J^j}\right)$$

$$\cdot\ J_\nu'\left(-m\sqrt{2J^i}\right) e^{-im(\theta_p-\theta_k)} \tag{7.10}$$

and equations similar to (7.5), (7.6) and (7.7) for H_i^0 and H_i^1 where

$$\dot{\psi}^i = \omega_s(J^i) - H(i,i) - \sum_{j(\neq i)=1}^{N} H(i,j) = \omega_s(J^i) - H_i^0 - H_i^1 \tag{7.11}$$

Note that $J_\nu'(x)$ above means a derivative of $J_\nu(x)$ with respect to its argument.

Using all the equations above, we verify the following approximate relation for Hamiltonian Flow:

$$\frac{\partial}{\partial J^i}\left[\dot{J}^i - G(i,i)\right] \simeq -\frac{\partial}{\partial\psi^i}\left[\dot{\psi}^i - H(i,i)\right]$$

i.e.

$$\frac{\partial}{\partial J^i} G_i^1\left(J^i,\psi^i;\ t\right) \simeq -\frac{\partial}{\partial\psi^i} H_i^1\left(J^i,\psi^i;\ t\right) \tag{7.12}$$

This relationship is only approximate because there is a term $-\frac{1}{J^i}\left[\frac{J^i}{\omega_s(J^i)}\frac{d\omega_s(J^i)}{dJ^i}\right]$ $\dot{J}^i$ to the left hand side of Eq. (7.12). Since $\dot{J}^i \sim O(\epsilon)$, this term is of order $O(\epsilon\alpha)$ where $\alpha = \frac{a^i}{\omega_s(a^i)}\frac{d\omega_s(a^i)}{da^i}$ is the nonlinearity parameter. Consistent with our assumption of small nonlinearity $\alpha < 1$ used in deriving the action and phase equations of motion, we neglect this term in order to obtain the approximate condition (7.12). Thus except for the self-interaction part, the longitudinal cooling dynamics in our model is approximately Hamiltonian.

For the linear transverse dipole cooling, we use the expression for the sampled transverse signal $\tilde{\mathcal{J}}^i_T$ given in Eq. (6.16) into Eq. (6.2.41) to obtain

$$\dot{I}^i = \sum_{j=1}^{N} G(i,j) = \sum_{j=1}^{N} G\left(I^i,\phi^i;\ J^i,\psi^i;\ I^j,\phi^j;\ J^j,\psi^j\right)$$

$$= \sum_{j=1}^{N}\sum_{\substack{\mu \\ (-\infty)}}^{+\infty}\sum_{\nu}\ \sum_{\substack{\beta \\ =(\pm 1)}}\ \sum_{\substack{\beta' \\ =(\pm 1)}} G_{(\mu\beta),(\nu,\beta')}\left(I^i,J^i;\ I^j,J^j\right) e^{i(\mu\psi^j+\nu\psi^i)+i[\beta\phi^j+\beta'\phi^i]} \tag{7.13}$$

where

$$G_{(\mu\beta),(\nu,\beta')}\left(I^i,J^i;\ I^j,J^j\right) = \frac{\sqrt{2I^i}\sqrt{2I^j}}{4}\frac{(qf_o)^2}{\omega_\perp^i}\sum_{m=-\infty}^{+\infty} e^{im(\theta_p-\theta_k)}\left[1+\delta_{ij}(\delta_{\beta,-\beta'}-1)\right] .$$

$$\left\{\tilde{G}\left[(m+Q)\omega_o+\mu\omega_s(J^j)\right] J_\mu\left[(m+Q)\sqrt{2J^j}\right] J_\nu\left(-m\sqrt{2J^i}\right)\left[\delta_{\beta 1}+\delta_{\beta,-1}\right]\delta_{\beta',1}\right.$$

$$\left.+\ \tilde{G}\left[(m-Q)\omega_o+\mu\omega_s(J^j)\right] J_\mu\left[(m-Q)\sqrt{2J^j}\right] J_\nu\left(-m\sqrt{2J^i}\right)\left[\delta_{\beta 1}+\delta_{\beta,-1}\right]\delta_{\beta',-1}\right\} \tag{7.14}$$

It is straightforward to find the corresponding expression for the harmonic coefficient $H_{(\mu,\beta),(\nu,\beta')}(I^i,J^i;\ I^j,J^j)$ in the betatron phase equation of motion derived from (6.2.42) and to verify the Hamiltonian flow condition:

$$\frac{\partial}{\partial I^i}\left[\dot{I}^i - G(i,i)\right] = -\frac{\partial}{\partial \phi^i}\left[\dot{\phi}^i - H(i,i)\right] \tag{7.15}$$

In contrast to the longitudinal case (7.12), Eq. (7.15) is exact, since we are considering linear betatron oscillations (i.e. betatron frequencies independent of betatron amplitudes) in our model and remains valid for nonlinear transverse pick-ups and kickers (i.e. $|\beta|,|\beta'| > 1$) as well.

Note that the approximate nature of the explicit demonstration of the Hamiltonian flow condition for nonlinear oscillations (e.g. Eq. 7.12) is only a reflection of the approximate orbits ((3.20a) and (3.20b)), derived from an asymptotic perturbation series and used in our model. In principle, the flow condition is exact, provided one uses the exact canonical action-angle variables for the full nonlinear problem as in (3.30) and (3.32) and can be explicitly demonstrated [110].

8. STATISTICAL (SPECTRAL) PROPERTIES OF BUNCHED BEAM SAMPLED NOISE

8.1 Nonstationarity of Bunched Beam Sampled Noise

Longitudinal and transverse incoherent Schottky signals $S^i(t)$ sampled by a particle i in a bunched beam are not 'stationary', i.e. their statistical properties are dependent on time. In particular, the autocorrelation function of the sampled incoherent noise signal at two different times is not a function of the time-difference alone:

$$\left\langle S^i(t)\, S^i(t')\right\rangle = \mathbb{R}^i(t,t') \neq R^i(t-t')$$

where $<\ >$ denotes an ensemble average over the phases of all the particles in the beam. There are three sources of nonstationarity in the problem: (a) adiabatic nonstationarity due to slow cooling imposed by the feedback system; the situation is similar to the adiabatic nonstationarity of the electromagnetic fluctuation spectrum of an infinite homogenous plasma due to the slow damping induced by electromagnetic radiation (b) periodic discrete kicks from the kicker which render even continuous and homogenous coasting beam sampled noise nonstationary (c) finite extent of the bunch which is a manifestation of the oscillatory synchrotron orbits.

Since the fluctuation spectrum gets established in a time much shorter than any significant cooling time, the nonstationarity due to slow cooling only makes the fluctuation spectrum a slowly varying function of time determined by the instantaneous local distribution of the particles. The time-evolution of the beam is thus determined by a locally time-stationary fluctuation spectrum. The beam distribution stays almost a constant during the establishment of this spectrum. The discreteness of the kicks introduces an essential nonstationarity viz. the noise signal is stationary only with respect to certain fixed translations in time(translations by multiples of T_0, the revolution period) and not with respect to arbitrary translations. For correlation times much longer than a revolution period, one bypasses this nonstationarity by averaging over the fast revolutions. For coasting beams, this rapid time-averaging is enough to render the sampled incoherent noise stationary

$$\overline{\left\langle S^i(t)\, S^i(t')\right\rangle}_{\text{coasting}} = R(t-t')$$

However, even with this rapid revolution time average, the bunched beam sampled signal is not rendered stationary owing to synchrotron oscillations within the bunch. One needs to average over the synchrotron oscillations to obtain a smoothed out stationary sampled noise for bunched beams. The stationarity properties of the signal at the kicker and the incoherent signal as sampled by a particle in the beam, for the continuous coasting beams and bunched beams and various averaging procedures that render them stationary are listed in Table III below.

TABLE III

NOISE CONSIDERED	COASTING BEAM		BUNCHED BEAM	
	PROPERTY	SOURCE	PROPERTY	SOURCE
Noise signal at the kicker	stationary	Continuous and uniform beam filling the ring	non-stationary	Finite length of bunch + synchrotron oscillations.
Noise signal sampled by a particle at the kicker	non-stationary	Periodic dis crete sampling of kicks	non-stationary	Finite bunch length + synchrotron oscillations + periodic discrete sampling
Rapid revolution time averaged correlation of sampled signal	stationary	Smoothing out of discrete sampling	non-stationary	Modulations due to synchrotron oscillations still present
Correlation of sampled signal by averaging over both rapid revolution times and synchrotron oscillation times	Not applicable		stationary	Modulations due to fast oscillations up to synchrotron oscillations smoothed out.

Stationary noise signals can be described by single-frequency spectral functions, e.g. power spectral function $\tilde{R}(\Omega)$ defined as the Fourier transform of the autocorrelation function $R(\tau) = \langle S(t)\, S(t-\tau)\rangle$, which is independent of t. No such single-frequency spectral functions exist for nonstationary noise, which has to be described by spectral functions of the form $\tilde{R}(\Omega, \Omega')$ etc. We avoid mathematical complications arising from nonstationary noise by considering the smoothed out stationary correlations only. To this end we illustrate the various smoothing procedures in the next section. A rigorous analysis including the nonstationarity due to synchrotron oscillations and discrete kicks would require solving a difference equation implicitly.

8.2 Auto-Correlation and Spectral Function Obtained by Smoothing

Let us consider the auto-correlation function of the zero-order longitudinal sampled incoherent noise seen by the i^{th} particle as given by (6.1.3) (without the $j = i$ term):

$$\left\langle S_o^i(t)\; S_o^i(t')\right\rangle = \left(qf_o^2\right)^2 \sum_{\substack{j,k=1\\(\neq i)}}^{N} \sum_{\substack{n,n'\\(-\infty)}}^{(+\infty)} \sum_{\substack{m,m'\\(-\infty)}}^{(+\infty)} \sum_{\substack{\mu,\mu'\\(-\infty)}}^{(+\infty)} \sum_{\substack{\nu,\nu'\\(-\infty)}}^{(+\infty)} J_\mu(ma_j)\; J_{\mu'}(m'a_k)\; J_\nu(na_i)\; J_{\nu'}(n'a_i)$$

$$\cdot\; \tilde{G}\left[m\omega_o + \mu\omega_s(a_j)\right]\; \tilde{G}\left[m'\omega_o + \mu'\omega_s(a_k)\right]\; e^{-i(m+m')\theta_p}\; e^{-i(n+n')\theta_k}$$

$$\cdot\; e^{i(m+n)\omega_o t}\; e^{i(m'+n')\omega_o t'}\; e^{i[\mu\omega_s(a_j)t+\mu'\omega_s(a_k)t']}\; e^{i(\nu t+\nu' t')\omega_s(a_i)}$$

$$e^{i(\nu+\nu')\psi_i^o}\; \left\langle e^{i\left[\mu\psi_j^o+\mu'\psi_k^o\right]}\right\rangle \tag{8.2.1}$$

This is explicitly nonstationary since the averaging over ψ_j^o and ψ_k^o cannot convert the exponential $\exp[i(m+n)\omega_o t + i(m'+n')\omega_o t']$ into a function of $(t-t')$ alone. We thus need to average over the fast revolution periods. Changing variables to $\ell = m + n$ and $\ell' = m' + n'$ and averaging over the fast phase given by the first factor in

$$e^{i\omega_o(\ell t+\ell' t')} = e^{i\frac{1}{2}\omega_o(\ell+\ell')(t+t')}\; e^{i\frac{1}{2}\omega_o(\ell-\ell')(t-t')}$$

one reduces the sum over ℓ,ℓ' by a sum over ℓ only by virtue of $\delta_{\ell,-\ell'}$ and we get

$$\overline{\left\langle S_o^i(t)\; S_o^i(t')\right\rangle} = \left(qf_o^2\right)^2 \sum_{\substack{j,k\\=1\\(\neq i)}}^{N} \sum_{\substack{\ell\\(-\infty)}}^{(+\infty)} \sum_{\substack{\mu,\mu'\\(-\infty)}}^{(+\infty)} \sum_{\substack{\nu,\nu'\\(-\infty)}}^{(+\infty)} \left[X_{\mu\nu}^{\ell}(a_j,a_i)\cdot X_{\mu'\nu'}^{-\ell}(a_k,a_i)\right]$$

$$e^{i\ell\omega_o(t-t')}\; e^{i[\mu\omega_s(a_j)t+\mu'\omega_s(a_k)t']}\; e^{i(\nu t+\nu' t')\omega_s(a_i)}$$

$$e^{i(\nu+\nu')\psi_i^o}\; \left\langle e^{i\left[\mu\psi_j^o+\mu'\psi_k^o\right]}\right\rangle \tag{8.2.2}$$

where

$$X^{\ell}_{\mu\nu}(a_j,a_i) = \sum_{m=-\infty}^{+\infty} \left\{ J_\mu(ma_j)\; J_\nu\Big((\ell-m)a_i\Big)\; \tilde{G}\Big[m\omega_o + \mu\omega_s(a_j)\Big]\; e^{im(\theta_k-\theta_p)} \right\}. \quad (8.2.3)$$

In the averaging over ψ_j^o and ψ_k^o for $j \neq k$, we obtain $\delta_{\mu,o}\,\delta_{\mu',o}$ and the time-dependence is given by $\{\exp[i\ell\omega_o(t-t')] \cdot \exp[i(\nu t+\nu' t')\omega_s(a_i)]\}$. For $j = k$, the average yields $\delta_{\mu',-\mu}$ and a time dependence $\{\exp[i\ell\omega_o(t-t')] \cdot \exp[i\mu\omega_s(a_j)(t-t')] \cdot \exp[i(\nu t+\nu' t')\omega_s(a_i)]\}$. Neither of these are functions of $(t-t')$ alone. We then average over the phases given by the first factor in the synchrotron phase term

$$e^{i(\nu t+\nu' t')\omega_s(a_i)} = e^{i\frac{1}{2}(\nu+\nu')(t+t')\omega_s(a_i)}\; e^{i\frac{1}{2}(\nu-\nu')(t-t')\omega_s(a_i)}$$

which yields $\delta_{\nu,-\nu'}$ and a smoothed-out stationary auto-correlation function:

$$\overline{\overline{\left\langle S_o^i(t)\; S_o^i(t')\right\rangle}} = \left(qf_o^2\right)^2 \sum_{\substack{j(\neq i)\\=1}}^{N} \sum_{\ell}^{(+\infty)}\sum_{\nu}^{(-\infty)} \Bigg[\sum_\mu X^{\ell}_{\mu\nu}(a_j,a_i)\; X^{-\ell}_{-\mu,-\nu}(a_j,a_i)\; e^{i\mu\omega_s(a_j)(t-t')}$$

$$+ \sum_{k\neq j} X^{\ell}_{o,\nu}(a_j,a_i)\; X^{-\ell}_{o,-\nu}(a_k,a_i)\Bigg]$$

$$e^{i\ell\omega_o(t-t')+i\nu\omega_s(a_i)(t-t')} \quad (8.2.4)$$

which is a function of $(t-t')$ alone. Fourier transformed in the time-difference $\tau = t - t'$, the spectral function $\tilde{R}(\Omega)$ is thus:

$$\tilde{R}^i(\Omega) = \left(qf_o^2\right)^2 (2\pi) \sum_{\substack{j(\neq i)\\=1}}^{N} \sum_{\ell}^{(+\infty)}\sum_{\nu}^{(-\infty)} \Bigg\{\sum_\mu X^{\ell}_{\mu\nu}(a_j,a_i)\; X^{-\ell}_{-\mu,-\nu}(a_j,a_i)\; \delta\Big[\Omega - \ell\omega_o - \mu\omega_s(a_j) - \nu\omega_s(a_i)\Big]$$

$$+ \sum_{k\neq j}^{N} X^{\ell}_{o\nu}(a_j,a_i)\; X^{-\ell}_{o,-\nu}(a_k,a_i)\; \delta\Big[\Omega - \ell\omega_o - \nu\omega_s(a_i)\Big]\Bigg\}. \quad (8.2.5)$$

Moreover, for non-overlapping revolution Schottky bands, $|\mu\omega_s(a_j)+\nu\omega_s(a_j)| < \omega_0$ and only the $\ell = 0$ term contributes in (8.2.5). One can easily verify that the expression thus obtained for non-overlapping revolution bands is the same as one would get by considering the synchrotron fast phase average of the sampling particle alone i.e., if one started out by considering the auto-correlation of the revolution time-averaged forms $\bar{\mathcal{S}}^i$ and $\bar{\mathcal{S}}^i_T$ of the signals, as given in (6.1.5) and (6.1.6), from the very beginning. One also notes that the term involving $X^{\ell}_{0\nu}$ and $X^{-\ell}_{0,-\nu}$ corresponds to the macroscopic gross signal derived from the coherent motion of the bunch as a whole $(\mu=0)$ and affects the coherent motion of the bunch only, since it contais no single particle information. Accordingly we ignore this term from our analysis . Alternatively we can imagine this coherent signal to be filtered out by a notch filter (with zeros at $\Omega = n\omega_0$, $n = 0,\pm1,\pm2,\ldots$) before being applied at the kicker, as discussed in Chapter 5 before. With this in mind, we can then write the power spectral function of the sampled longitudinal Schottky voltage for a particle in a bunch, in the region of no revolution band overlap, as

$$\tilde{R}^i(\Omega) = \left(qf_0^2\right)^2 (2\pi) \sum_{\substack{j\neq i\\=1}}^{N} \sum_{\nu}^{(+\infty)}\sum_{\mu}^{(-\infty)} \left[X^0_{\mu\nu}(a_j,a_i)\; X^0_{-\mu,-\nu}(a_j,a_i)\right] \delta\left[\Omega - \mu\omega_s(a_j) - \nu\omega_s(a_i)\right] \tag{8.2.6}$$

The auto-correlation function of the longitudinal action noise (equivalently, amplitude noise), in the region of no revolution band overlap, can be calculated by using

$$\dot{J}^i = n^i\left[J^i,\psi^i(t);t\right] = \sum_{j=1(\neq i)}^{N} G\left[i(t),j(t)\right] \tag{8.2.7}$$

where $G[i,j]$ is obtained from the revolution time-averaged sampled voltage Schottky signal $\bar{S}^i$ as in Chapter 6 and 7. Thus

$$R^i(t,t') = \left\langle n^i\left[J^i,\psi^i(t);t\right] n^i\left[J^i,\psi^i(t');t'\right]\right\rangle$$

$$= \left\langle \left\{ \sum_{j(\neq i)=1}^{N} G\left[i(t),j(t)\right]\right\} \left\{ \sum_{k(\neq i)=1}^{N} G\left[i(t'),k(t')\right]\right\} \right\rangle_{j,k}$$

$$= \sum_{\substack{j\\(\neq i)\\=1}}^{N} \sum_{k} \sum_{\mu\mu'} \sum_{\substack{\nu\nu'\\(-\infty)}}^{(+\infty)} \left[G_{\mu\nu}(J^i,J^j)\; G_{\mu'\nu'}(J^i,J^k)\right]$$

$$\left\langle e^{i\left[\mu\psi^i(t)+\nu\psi^j(t)+\mu'\psi^i(t')+\nu'\psi^k(t')\right]}\right\rangle_{j,k} \quad . \tag{8.2.8}$$

Using the orbits $\psi^i(t) = \omega_s(J^i)t + \psi_0^i$, averaging over the phase ψ_0^i of the test particle i that is sampling the noise, in addition to the ensemble average over the phases ψ_0^j, ψ_0^k and again neglecting the $\nu = \nu' = 0$ terms corresponding to the coherent macroscopic signal as before, one obtains

$$\bar{R}^i(t,t') = \sum_{j(\neq i)}^{N} \sum_{\mu} \sum_{\substack{\nu\\(-\infty)}}^{(+\infty)} \left[G_{\mu\nu}(J^i,J^j)\; G_{-\mu,-\nu}(J^i,J^j)\right]$$

$$\cdot\; e^{i[\mu\omega_s(a_i)+\nu\omega_s(a_j)](t-t')} \quad . \tag{8.2.9}$$

Using the reality condition for $G(i,j)$

$$G^*_{\mu\nu}(J^i;J^j) = G_{(-\mu)(-\nu)}(J^i,J^j) \tag{8.2.10}$$

and Fourier transforming in $\tau = (t-t')$ in (8.2.9), the power spectral function of sampled action noise is obtained as

$$\tilde{\bar{R}}^i(\Omega) = (2\pi) \sum_{\substack{j(\neq i)\\=1}}^{N} \sum_{\mu} \sum_{\substack{\nu\\(-\infty)}}^{(+\infty)} \left[G_{\mu\nu}(J^i,J^j)\; G^*_{\mu\nu}(J^i,J^j)\right] \delta\left[\Omega - \mu\omega_s(J^i) - \nu\omega_s(J^j)\right] \tag{8.2.11}$$

Employing a distribution function $f(J)$ describing the angle-independent distribution of particles in action space, we can replace the sum $\sum_{j=1}^{N}$ by $[N \int dJ' f(J')]$ ($f(J)$ normalized to unity) and the spectral function (8.2.11) can be written as

$$\tilde{R}\,(\Omega;J) = (2\pi)\cdot N \cdot \sum_{\mu}\sum_{\nu}{}^{(+\infty)}_{(-\infty)} \int dJ'\; f(J') \left[G_{\mu\nu}(J,J')\cdot G^{*}_{\mu\nu}(J,J')\right]\cdot\delta\left[\Omega - \mu\omega_s(J) - \nu\omega_s(J')\right] \tag{8.2.12}$$

Substituting the explicit expression for $G_{\mu\nu}(J,J')$ from (7.7), one gets

$$\tilde{R}(\Omega;J) = (2\pi)\cdot N\cdot\left[\frac{(qf_0)^2\kappa}{\omega_s(J)}\right]^2 \sum_{\mu}\sum_{\nu}{}^{(+\infty)}_{(-\infty)} \int dJ'\; f(J')\;\delta\left[\Omega - \mu\omega_s(J) - \nu\omega_s(J')\right]$$

$$\left|\sum_{m=-\infty}^{+\infty} e^{im(\theta_p-\theta_k)}\left(\frac{\nu}{-m}\right)\tilde{G}\left[m\omega_0 + \mu\omega_s(J')\right]\; J_\mu(m\sqrt{2J'})\; J_\nu(-m\sqrt{2J})\right|^2. \tag{8.2.13}$$

Note that the above procedure of averaging over the phase ψ_0^i of the sampling particle in the auto-correlation function, is not equivalent to replacing the interaction

$$G\left[i(t),j(t)\right] = \sum_{\mu}\sum_{\nu}{}^{(+\infty)}_{(-\infty)} G_{\mu\nu}(J^i,J^j)\; e^{i[\mu\psi^i(t)+\nu\psi^j(t)]}$$

by the simplified form of $G[i(t),j(t)]$ which is a function of the phase-difference alone

$$\overline{G[i(t),j(t)]} = \sum_{\mu=-\infty}^{+\infty} G_{\mu,-\mu}(J^i,J^j)\; e^{i\mu[\psi^i(t)-\psi^j(t)]} \tag{8.2.14}$$

obtained after eliminating the rapidly oscillating terms by averaging over times of the order of the synchrotron oscillation period. This would be a good approximation for cases where the relative frequency spread in the motion of the particles, $\Delta\omega_s(J)/\omega_s(0)$, is small so that the harmonics of the longitudinal synchrotron motion, $\mu \gtrsim \frac{\omega_s(0)}{\Delta\omega_s(J)}$, are unimportant. As already discussed in Chapter 5, for large bandwidth, high frequency systems there is considerable amount of synchrotron band overlap within each revolution harmonic and so large synchrotron harmonics μ are indeed important.

The auto-correlation in time domain and the corresponding spectral function in the conjugate frequency space for the general three dimensional cooling interaction that couples all three degrees of freedom and includes nonlinear pick-ups and kickers can be derived by using similar averaging procedures to smooth out the nonstationarity due to rapid oscillations thus leading to a stationary sampled noise.

With coupled degrees of freedom, however, the full auto-correlation or spectral function for both the action noise and phase noise is a tensor of rank 'three' with typical elements like $\left\langle n^i_\alpha(t)\ n^i_\beta(t')\right\rangle$ for action noise and $\left\langle \xi^i_\alpha(t)\ \xi^i_\beta(t')\right\rangle$ for phase noise where $\alpha = \left\{x,z,\theta\right\}$, $\beta \equiv \left\{x,z,\theta\right\}$ and $\underset{\sim}{n}$ and $\underset{\sim}{\xi}$ are the noise or fluctuation functions entering in the equations of motion

$$\begin{aligned} \dot{\underset{\sim}{I}}{}^i &= \underset{\sim}{n}^i\left[\underset{\sim}{I}^i,\underset{\sim}{\psi}^i(t);t\right] \\ \underset{\sim}{\psi}^i &= \underset{\sim}{\omega}^i + \underset{\sim}{\xi}^i\left[\underset{\sim}{I}^i,\underset{\sim}{\psi}^i(t);t\right] . \end{aligned} \tag{8.2.15}$$

Thus for example the spectral function of the action noise will have the form

$$\underset{\approx}{\tilde{R}}{}^i(\Omega;\underset{\sim}{I}^i) = \left\{\tilde{R}^i_{\alpha\beta}(\Omega;\underset{\sim}{I}^i)\right\}_{\alpha,\beta=x,z,\theta} \equiv \begin{bmatrix} \left\langle n^i_x\ n^i_x\right\rangle & \left\langle n^i_x\ n^i_z\right\rangle & \left\langle n^i_x\ n^i_\theta\right\rangle \\ \left\langle n^i_z\ n^i_x\right\rangle & \left\langle n^i_z\ n^i_z\right\rangle & \left\langle n^i_z\ n^i_\theta\right\rangle \\ \left\langle n^i_\theta\ n^i_x\right\rangle & \left\langle n^i_\theta\ n^i_z\right\rangle & \left\langle n^i_\theta\ n^i_\theta\right\rangle \end{bmatrix} \tag{8.2.16}$$

and similarly for the spectral function of phase-noise. Using dyadic notation for the tensor $\underset{\approx}{R}(t,t')$ obtained from the product $\underset{\sim}{n}\,\underset{\sim}{n}$ and using the general expression for $\dot{\underset{\sim}{I}}$ given in Chapter 4, we obtain, following the by now familiar procedure, the following (with properly aligned pick-ups and a filter in the feedback loop to eliminate $\underset{\sim}{G}_{\underset{\sim}{n},0}$ term):

$$\underset{\approx}{R}(t-t') =$$

$$= N \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}' \, f(\underset{\sim}{I}') \left[G_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \, G^{*}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \right] \cdot e^{i[\underset{\sim}{n}\cdot\underset{\sim}{\omega}(\underset{\sim}{I})+\underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}')](t-t')} \tag{8.2.17}$$

and

$$\tilde{\underset{\approx}{R}}(\Omega;\underset{\sim}{I}) = (2\pi) \cdot N \cdot \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}' \, f(\underset{\sim}{I}') \left[G_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \, G^{*}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \right] \cdot \delta\left[\Omega - \underset{\sim}{n}\cdot\underset{\sim}{\omega}(I) - \underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}')\right] \tag{8.2.18}$$

where

$$\int d\underset{\sim}{I}' \, f(\underset{\sim}{I}') = 1 \quad \text{with} \quad d\underset{\sim}{I}' \equiv dI_x' \, dI_z' \, dJ'$$

A similar expression can be obtained for the auto-correlation and spectral tensor of the phase-noise. Equation (8.2.17) describes the correlation in the sampled noise developing in a time-scale of $\tau \sim (1/\Omega)$ from all possible scattering events within the beam (through the feedback loop) between the primed and unprimed particles satisfying the resonance condition $\Omega = \underset{\sim}{n} \cdot \underset{\sim}{\omega}(\underset{\sim}{I}) + \underset{\sim}{n}' \cdot \underset{\sim}{\omega}(\underset{\sim}{I}')$. We will see in Section 9.1 that long time slow diffusion due to these scatterings is determined by the $\Omega \to 0$ limit of $\tilde{\underset{\approx}{R}}(\Omega;I)$ corresponding to two particles falling exactly on top of each other in frequency space: $\underset{\sim}{n} \cdot \underset{\sim}{\omega}(\underset{\sim}{I}) = -\underset{\sim}{n}' \cdot \underset{\sim}{\omega}(\underset{\sim}{I}')$.

9. THE TIME-EVOLUTION OF THE ANGLE-INDEPENDENT (PHASE-AVERAGED) DISTRIBUTION FUNCTION -- THE FOKKER-PLANCK EQUATION AND THE TRANSPORT COEFFICIENTS

In this chapter, we employ two formalisms for arriving at the equation for the time-evolution of the uniform angle-independent distribution function (zeroth Fourier-component in angle, conjugate to action, of the distribution function) of the beam undergoing stochastic cooling. The first method is based on the classical fluctuation theory used in the Langevin equation to describe Brownian motion and also various nonequilibrium (stationary and nonstationary) processes in statistical mechanics of many-body systems. This method has a very broad scope and is applicable to any system evolving under noise or fluctuations with finite correlation times. The theory of fluctuations is an active and growing field of research in itself, and we will only demonstrate a simple and clear procedure for the application of well-established fluctuation-theoretic techniques to the process of stochastic cooling employing a suitable model. The second method is based on the canonical kinetic theory in phase-space for many-body systems, employing the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy of reduced distributions and correlations in phase-space. This kinetic method is of wide-spread use in plasma physics and also in nonequilibrium statistical mechanics.

Up to second order in a small parameter λ measuring the strength of the fluctuations or the cooling interaction and first order in particle correlations, both methods yield identical transport equations in phase-space when the collective screening effects are negligible and in particular demonstrate the Fokker-Planck nature of the transport. The fluctuation theory becomes cumbersome, however, if one wishes to include the collective many-body aspects of the beam as a whole. The collective signal suppression or dynamic screening effect is thus ignored and not integrated in this fluctuation-theoretic formulation. This latter effect has to be evaluated independently by using the appropriate collective dynamics (Vlasov equation) and later put in by hand in our fluctuation-theoretic results. However this method lends itself to a quick and easy evaluation of the transport coefficients (Friction and Diffusion coefficients), even when the spectrum of the fluctuations is not obtainable from theory as long as one has adequate knowledge of the spectrum (e.g. power spectral function) experimentally.

The kinetic theory method provides the most satisfactory description of transport, that integrates within itself the collective effects of dynamic screening (or dielectric signal suppression) in a holistic way. However, this method becomes complicated and poses considerable mathematical difficulty in solution for the most general situation of coupled degrees of freedom and general particle orbits (i.e. other than free-streaming or simple oscillatory orbits), unless one ignores collective correlations, in which case one recovers fluctuation-theoretic results. A solution including collective signal suppression can be obtained however for the simple cases of no coupling between degrees of freedom and either free-streaming (coasting) or anharmonic oscillator (bunched) orbits, as demonstrated in Section 9.2.

We discuss the effect of extraneous electronics noise (due to amplifiers) on the diffusion term in Section 9.3. In Section 9.4 we reduce the Fokker-Planck transport equation for transverse cooling to an equation describing the evolution of the second moment of the distribution function, in the special case of linear transverse dipole interaction. For this special case we also demonstrate an explicit solution and the Green's function of the Fokker-Planck equation. In Section 9.5 we write down explicitly the components of the friction vector and diffusion tensor in terms of generalized interaction harmonics for the general case with coupling between degrees of freedom.

The fluctuation theory discussed in Section 9.1, when supplemented with the expression for the collective response or signal suppression factors derived from general Vlasov theory in phase-space in Chapters 10 and 12 and with a prescription as to how to modify incoherent spectral functions by these factors (Chapter 11), becomes equivalent to the kinetic theory discussed in Section 9.2

9.1 Fluctuation Theoretic Model of Stochastic Cooling

The classical fluctuation theoretic formulation used in this section is an already well-established and growing field and has been discussed, exposed and used in various contexts involving fluctuations or stochastic processes ([28], [30], [51], [53], [54], [77], [95], [98], [107]). In our formulation, we follow the treatment by Van Kampen [107] closely since it is particularly well suited for the two-storage ring model of stochatic cooling adopted in this section.

If $\underset{\sim}{x} = (\underset{\sim}{I}_i, \underset{\sim}{\psi}_i)$, $i = 1,\ldots,N$ denote the canonical action-angle phase-space coordinate of the $i^{\underline{th}}$ particle in the beam, then the cooling dynamics is generally of the form

$$\dot{\mathcal{X}}_i = \mathcal{G}_i\left(\mathcal{X}_1,\mathcal{X}_2,\ldots,\mathcal{X}_N\right) = \mathcal{G}_i\left[\{\mathcal{X}_j\}_{j=1,\ldots,N}\right] \tag{9.1.1}$$

where $\underset{\sim}{G}$ is a general nonlinear vector function of $\{\underset{\sim}{x}_i\}_{i=1,\ldots,N}$ determined by the cooling interaction imposed by the feedback loop (Eqs. (7.2) through (7.14)).

As we have discussed before, a single particle experiences a coherent corrective force that depends on the 'state' or 'phase' of its own position in phase-space alone and also experiences an incoherent fluctuating force or Schottky noise due to all the other particles in the beam, which depends on the phase-space coordinates of all these other particles.

$\underset{\sim}{G}$ thus has a general decomposition

$$G_i\left[\{X_j\}\right] = G_i\left[X_i,t;\sigma\right] = G_i^0(\underset{\sim}{X}_i) + G_i^1\left[X_i,t;\sigma\right] \tag{9.1.2}$$

where $\underset{\sim}{G}_i^0$ is a "sure function," determined by the coherent cooling term and depends on the phase of the cooled particle alone and $\underset{\sim}{G}_i^1$ is a "stochastic function" or random function depending on a stochastic or random parameter σ describing the initial phases of all the particles in the beam. The time dependence of $\underset{\sim}{G}_i^1$ enters through the orbits of all the other particles $j(\neq i) = 1,\ldots,N$. As the phases range between 0 and 2π, σ is a particular realization of an ensemble of random phases each belonging to a set Σ whose range is thus mod $(0,2\pi)$. A probability distribution defined on Σ may be specified by its density $P(\sigma)$ obeying

$$P(\sigma) \geqslant 0 \quad \text{and} \quad \int_\Sigma P(\sigma)\, d\sigma = 1 \tag{9.1.3}$$

It is quite generally true for our case that

$$\left\langle G_i^1[X_i,t] \right\rangle_{\sigma\in\Sigma} = 0 \tag{9.1.4}$$

where the average $<\ldots>$ is taken over an ensemble of the random set Σ of σ's. Thus the noise from the other particles adds only in the mean square (and higher moments) which is non-zero e.g.

$$\left\langle \underset{\approx}{R}(t,t') \right\rangle = \left\langle G^1[X;t]\, G^1[X';t'] \right\rangle_{\sigma\in\Sigma} \neq 0 \tag{9.1.5}$$

In the derivations to follow, it is not necessary to use the decomposition (9.1.2) or use the assumption (9.1.4) until later. At this stage, it only provides a useful insight into the process of cooling.

With the introduction of the stochastic variable $\sigma \in \Sigma$, the differential equation (9.1.1) becomes a stochastic differential equation for $\underset{\sim}{x}$. In our simple model then, cooling is described by the general stochastic nonlinear differential equation given by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{G}(\underset{\sim}{x},t;\sigma) \tag{9.1.6}$$

When we prescribe given initial values $\underset{\sim}{x}(t=0) = \underset{\sim}{a}$, this equation determines a stochastic process $\underset{\sim}{x}(t;\sigma)$, provided that for each individual $\sigma \in \Sigma$, Eq. (9.1.6) has a unique solution. However, practically one is more interested in the statistical properties of $\underset{\sim}{x}$ such as $\langle \underset{\sim}{x} \rangle$, $\langle \underset{\sim}{x}\,\underset{\sim}{x} \rangle$, etc. that arises from the stochastic nature of the force $\underset{\sim}{G}(\underset{\sim}{x},t;\sigma)$ in Eq. (9.1.6).

For linear equations, where the right hand side of (9.1.6) depends linearly on the $\underset{\sim}{x}$, techniques exist for solving (9.1.6) for the statistical properties of $\underset{\sim}{x}$ directly. However, a solution is extremely difficult to obtain for the general nonlinear $\underset{\sim}{x}$-dependence of $\underset{\sim}{G}$. If we define $\langle \underset{\sim}{x}(t) \rangle$ to be the ensemble average of $\underset{\sim}{x}(t;\sigma)$ over the distribution $P(\sigma)$ of σ defined in (9.1.3), i.e. if

$$\left\langle \underset{\sim}{x}(t) \right\rangle = \int_{\Sigma} d\sigma\; P(\sigma)\; \underset{\sim}{x}(t;\sigma) \tag{9.1.7}$$

then it is obvious that we do not expect to find a differential equation for $\langle \underset{\sim}{x}(t) \rangle$ by itself, because in general the nonlinearity of $\underset{\sim}{G}(\underset{\sim}{x},t;\sigma)$ necessarily brings in the higher moments. We can however consider a second probability distribution or density function of $\underset{\sim}{x}$, $f(\underset{\sim}{x},t)$ by defining a fluid in phase-space $\underset{\sim}{x}$ and a phase-function $f(\underset{\sim}{x})$ which measures the density of the fluid at $\underset{\sim}{x}$ (<u>or</u> the probability of being within $\underset{\sim}{x}$ and $\underset{\sim}{x} + d\underset{\sim}{x}$ being $f(\underset{\sim}{x})\, d\underset{\sim}{x}$). This reduces the nonlinear ordinary differential equation (9.1.6) for $\underset{\sim}{x}$ into a linear partial differential equation for $f(\underset{\sim}{x},t)$ as follows.

We look at $\underset{\sim}{x}$ as a point in the 6-dimensional phase-space of the particle. Equation (9.1.6) determines a velocity at each point of this phase-space. Each initial value point $\underset{\sim}{x}(t=0) = \underset{\sim}{a}$ has a trajectory passing through it describing the corresponding solution of (9.1.6). We hold σ fixed for the moment. Let us consider a cloud of initial points and a phase function $f(\underset{\sim}{x};0)$ describing the density of these points in phase-space. All the points in phase-space move according to (9.1.6) and the conservation of probability $\left[\int f(\underset{\sim}{x};t)\, d\underset{\sim}{x} = 1\right]$ or the conservation of the number of system points in the ensemble means that the flow must satisfy the continuity equation

$$\frac{\partial f(\underset{\sim}{x};t)}{\partial t} = - \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[\underset{\sim}{G}(\underset{\sim}{x},t;\sigma)\; f(\underset{\sim}{x};t)\right] \tag{9.1.8}$$

As explained under Section 4.1, the flow is not incompressible for the stochastic cooling interaction and $\partial/\partial\underset{\sim}{x}$ remains outside of $\underset{\sim}{G}$ in general. Consequently, a solution of (9.1.8) is not obtained by taking $f(\underset{\sim}{x};t)$ constant along each trajectory, but a Jacobian determinant will appear later. Note that Eq. (9.1.8) is an equation for the flow in 6-dimensional $\underset{\sim}{x}$-space for the one-particle distribution $f(\underset{\sim}{x};t)$ of a set of test-particles and is different from Eq. (4.1.2) defined in 6N-dimensional space for the N-particle distribution $\rho[\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N;t]$.

Now we consider all values of $\sigma \in \Sigma$ with their probability distribution (9.1.3). Then (9.1.8) is a stochastic partial differential equation for $f(\underset{\sim}{x};t)$ which is linear in $f(\underset{\sim}{x};t)$.

Again, given the initial condition for the distribution $f(\underset{\sim}{x};\, t=0) = g(\underset{\sim}{x})$, Eq. (9.1.8) determines a stochastic process $f(\underset{\sim}{x};t;\sigma)$, provided that for each $\sigma \in \Sigma$, (9.1.8) has a unique solution. We can then define

$$\left\langle f(\underset{\sim}{x};t)\right\rangle = \int_{\Sigma} f(\underset{\sim}{x};t;\sigma)\; P(\sigma)\; d\sigma \tag{9.1.9}$$

We wish to connect $\langle f(\underset{\sim}{x};t)\rangle$ with the probability density $p(\underset{\sim}{x};t)$ of $\underset{\sim}{x}(t)$ arising from the random variable σ as determined by Eq. (9.1.6). It is known, in fact, in the theory of fluctuations that

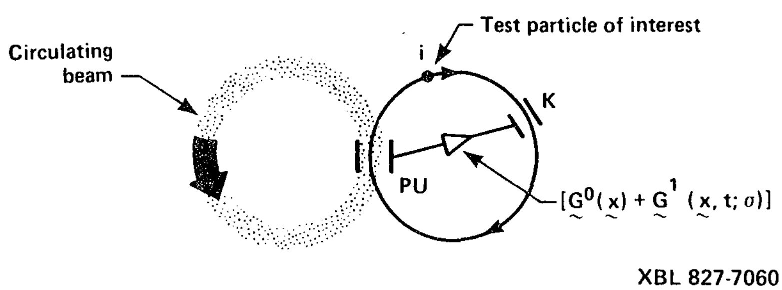
$$\left\langle f(\underset{\sim}{x},t)\right\rangle = p(\underset{\sim}{x},t)$$

so that the solution $\langle f(\underset{\sim}{x},t)\rangle$ of (9.1.8) leads to a solution $p(\underset{\sim}{x},t)$ for Eq. (9.1.6) automatically. We reproduce a formal proof of this lemma in Appendix C after Van Kampen [107].

The proof is based on assuming 'sure' initial values in the form of delta functions as follows:

$$f(\underset{\sim}{x};\, t=0) = g(\underset{\sim}{x}) = \delta^{(n)}\,(\underset{\sim}{x} - \underset{\sim}{a})$$

We can however consider other initial values corresponding to solutions of (9.1.6) in which the initial value $\underset{\sim}{a}$ is also random with probability distribution $f(\underset{\sim}{a};0) = P(\underset{\sim}{a})$ with $\underset{\sim}{a} \in X$, the set of all possible initial conditions. In this case $\langle f(\underset{\sim}{x},t)\rangle$ is identical with the probability density $p(\underset{\sim}{x},t)$ of $\underset{\sim}{x}$ arising from the randomness of the Eq. (9.1.6) <u>and</u> of the initial value. It is required however that the distribution of initial values $P(\underset{\sim}{a})$ be statistically independent of that of σ, i.e. $P(\sigma)$. In other words, it is required that the random variables a and σ be independent and uncorrelated. This formulation so far then describes exactly the situation in the model of a hypothetiocal two-storage-ring cooling as depicted in Fig. 18 below.



Two Storage-Ring Model of Stochastic Cooling

Fig. 18

A beam of particles circulates in one ring, generates a Schottky noise signal at an azimuthal pick-up, which is then transferred to a kicker located azimuthally at a separate hypothetical storage ring where a test-particle, circulating in the second ring, sees this amplified noise signal together with the coherent signal generated by itself at the same pick-up. We thus imagine the cooling particle, whose dynamics is of interest, as a test particle cooling under the influence of the 'sure' function $\underset{\sim}{G}^0$ and diffusing under the influence of the noise or fluctuation function $\underset{\sim}{G}^1(t;\sigma)$, determined by all the other particles in the other storage ring.

The time-evolution of the probability density $p(\underset{\sim}{x},t)$ of such a test particle would be of purely academic interest unless we identify $p(\underset{\sim}{x},t)$ with the actual coarse-grained 6-dimensional phase-space density of the actual beam circulating in a single storage ring and interacting with the PU-Amplifier-kicker feedback loop simultaneouly, i.e., we identify the test particle as one of the actual particles in the beam. Complications of an essential nature arise, however, in this process of identification which we are forced to do in order for our idealized two-storage-ring model to correspond to a real stochastic cooling situation. The probability spaces Σ and X or the densities $P(\sigma)$ and $P(X)$ no longer remain independent. The parameter σ which measures the randomness of Schottky noise $\underset{\sim}{G}^1(\underset{\sim}{x},t,\sigma)$ is the same parameter as X which defines the single particle distribution of the whole beam. Both parameters arise from the same random initial phases of the beam particles and are totally correlated.

Thus formally Eq. (9.1.8) is linear in $f(\underset{\sim}{x},t)$ but deceptively so. σ is determined by a particular realization of the ensemble of initial phases of the beam particles. As the cooling process continues, $f(\underset{\sim}{x},t)$ changes and σ becomes a function of time through the changing distribution function, i.e. $\sigma = \sigma\{f(\underset{\sim}{x},t)\}$. Thus $\left\langle [G(\underset{\sim}{x},t;\sigma)\ f(\underset{\sim}{x};t)] \right\rangle_\sigma \sim [G(\underset{\sim}{x},p(\underset{\sim}{x};t))\ p(\underset{\sim}{x};t)]$. Hence for a real cooling system, Eq. (9.1.8) is an inherently nonlinear partial differential equation for $f(\underset{\sim}{x},t)$. Such nonlinearity is always inherent whenever there is a correlation between the two probability spaces Σ and X, one of which measures the randomness of the coefficients in the differential equation while the other the randomness of the initial values. The application of our model then to a real cooling system becomes moot.

However, a careful multiple time-scale analysis based on an inspection of the various disparate time-scales involved provides us with a physical argument to bypass this difficulty.

Equation (9.1.8) determines three distinct time-scales. The first one is the scale on which $f(\underset{\sim}{x};t)$ varies. This is measured by λ^{-1} (taking $\underset{\sim}{G}$ to be of order unity) where λ is a measure of the strength of $\underset{\sim}{G}$ in some sense. This is the relaxation time of slow cooling. The second one is the scale on which $\underset{\sim}{G}(t)$ varies and is measured by T_0, the revolution time-period or the periodicity with which the kicks are applied -- this time-scale is irrelevant for the following arguments. The third one is given by the correlation time τ_c of $\underset{\sim}{G}(t)$ -- this is the time-scale on which the random nature of the function $\underset{\sim}{G}(t)$ becomes appreciable.

Now if $(\lambda\tau_c)$ is small it is possible to subdivide the time-axis in intervals (Δt) such that $\Delta t >> \tau_c$ and yet $(\lambda\,\Delta t) << 1$. That is $f(\underset{\sim}{x},t)$ does not vary much during a time Δt in which $\underset{\sim}{G}(t)$ has forgotten its past. Then we can use Eq. (9.1.8) during the time-interval Δt between 't' and 't+Δt' and solve it to express $<f(\underset{\sim}{x};\, t+\Delta t)>$ in terms of $<f(\underset{\sim}{x};t)>$ assuming (9.1.8) to be linear in $f(\underset{\sim}{x};t)$, but the coefficients in the solution determined by the instantaneous averages of $\underset{\sim}{G}(t)$ and its moments over $f(\underset{\sim}{x};t)$. For the next interval one may use the same method of solution to express $<f(\underset{\sim}{x};\, t+2\Delta t)>$ in terms of $<f(\underset{\sim}{x};\, t+\Delta t)>$. The crucial point is that the values of $\underset{\sim}{G}(t')$ during the second interval are practically uncorrelated with those during the previous Δt. This makes it possible to use in the second interval the same unbiased averages of $\underset{\sim}{G}(t')$ rather than the averages conditioned by the knowledge of how $f(\underset{\sim}{x};t)$ behaved in the previous interval. The coefficients in the solution become averages over the instantaneous distribution function of the particles and thus we have finally resurrected the basic nonlinear nature of the transport, without having to solve the nonlinear equation.

All the above arguments amount to saying that on the coarse-grained level determined by Δt the process is (approximately) Markovian. This means in general that the probability distribution $p(\underset{\sim}{x},t) = <f(\underset{\sim}{x};t)>$ obeys a differential equation, rather than an integral equation with long-time memory. At this point it is prudent to state then what our goal is. We wish to derive a deterministic differential equation for $p(\underset{\sim}{x},t)$ of the form:

$$\frac{\partial p(\underset{\sim}{x},t)}{\partial t} = \hat{D}_{\underset{\sim}{x}}\, p(\underset{\sim}{x};t)$$

where $\hat{D}_{\underset{\sim}{x}} = \hat{D}_{\underset{\sim}{x}}\,\{p(\underset{\sim}{x},t)\}$ is an operator acting on the $\underset{\sim}{x}$-dependence of $p(\underset{\sim}{x},t)$, but not on its t-dependence and may in addition be a function of $p(\underset{\sim}{x};t)$ itself. Later on we will identify $p(\underset{\sim}{x},t)$ with the actual phase-space distribution $f(\underset{\sim}{x};t)$ of the beam of particles.

Another complication arises from the fact that the kicker electromagnetic fields in a stochastic feedback loop introduces correlations in the trajectories (and hence arrival times) of the particles which propagate collectively through the beam and distorts the fluctuation spectrum. This collective correlation between the probability spaces Σ and X is not included in our model, which thus precludes us from obtaining the collective signal suppression factors. Solving a nonlinear stochastic differential equation where the probability spaces of the initial value distribution and the original stochastic variable in the differential equation are correlated dynamically as a function of time is extremely complicated and is the leading edge of research today in such diverse fields as non-stationary nonlinear stochastic processes, irreversible statistical mechanics of many-body systems and quantum field theory. The only satisfactory method known to the author is a full-blown kinetic theory of microscopic dynamics in phase-space, which we will discuss in the next section. In this section we will solve the problem by assuming that this correlation is small and can be neglected. Indeed, if the modulations induced by the kicker are small enough, the dynamic correlation between the two ensembles: the noise ensembles and the test particle ensemble is weak and time-evolution of an ensemble of such test particles or a distribution of them will coincide with the time evolution of the actual beam.

Later on we will cure this desease of not including the collective distortion of the flucutation spectrum (owing to the additional feedback loop through the beam) by slightly modifying our model. In this modified model, we let the test particle sample not the incoherent fluctuation spectrum generated by the single particle motions, but the total collectively distorted or screened signal generated at the kicker. This latter spectrum is obtained simply by suppressing the incoherent fluctuation signal $\xi^0(\Omega)$ at the kicker by a shielding factor $\varepsilon(\Omega)$, i.e. setting $\xi(\Omega) = \frac{\xi^0(\Omega)}{\varepsilon(\Omega)}$. However, $\varepsilon(\Omega)$ then has to be evaluated separately and independently from the Vlasov equation in phase-space.

This approach is analogous to the "Test particle" approach used in Plasma Theory to get the Fokker-Planck coefficients for transport in a plasma, where the fluctuations arise from the Coulomb interaction of particles with each other, which gets dynamically screened by the Debye shielding effect ([30], [51], [53], [54], [98]).

Let us write Eq. (9.1.8) formally as:

$$\frac{\partial f(\underset{\sim}{x};t)}{\partial t} = \hat{A}_{\underset{\sim}{x},\sigma}(t)\; f(\underset{\sim}{x};t) \tag{9.1.10}$$

where the operator $\hat{A}_{\underset{\sim}{x},\sigma}(t)$ is defined as

$$\hat{A}_{\underset{\sim}{x},\sigma}(t) = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}(\underset{\sim}{x},t;\sigma) = -\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{G}(\underset{\sim}{x},t;\sigma) \tag{9.1.11}$$

It is easily seen that Eq. (9.1.10) is solved by:

$$f(\underset{\sim}{x},t) = \left[1 + \int_0^t dt_1\, \hat{A}_{\underset{\sim}{x},\sigma}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2\, \hat{A}_{\underset{\sim}{x},\sigma}(t_1)\, \hat{A}_{\underset{\sim}{x},\sigma}(t_2) + \ldots\right] f(\underset{\sim}{x};0)$$

$$= \left[\sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, \hat{A}(t_1)\, \hat{A}(t_2)\, \hat{A}(t_3)\ldots\hat{A}(t_n)\right] f(\underset{\sim}{x};0)$$

where we have suppressed the subscripts $\underset{\sim}{x},\sigma$ for convenience. If the operators $\hat{A}$ commute, all integrations can be extended from $t_i = 0$ to $t_i = t$ $(i=1,2,\ldots,n)$ provided that a factor $(1/n!)$ is supplied to compensate for the larger integration domain. But in our case $\hat{A}$'s are operators involving gradients or derivatives in phase-space variables of the functions $\underset{\sim}{G}(\underset{\sim}{x};t)$ for different times and do not necessarily commute. However, even though $\hat{A}(t_i)$ do not commute we may still write

$$f(\underset{\sim}{x};t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n\, \mathbf{T}\left[\hat{A}(t_1)\, \hat{A}(t_2)\ldots\hat{A}(t_n)\right] f(\underset{\sim}{x},0)$$

where the "time-ordering" symbol $\mathbb{T}[\ldots]$ indicate that the operators have to be shuffled so as to appear in the order of decreasing values of their time arguments.

We can write the result in a more compact form as

$$f(\underset{\sim}{x};t) = \mathbb{T}\left[\exp\left\{\int_0^t \hat{A}_{\underset{\sim}{x},\sigma}(t')\, dt'\right\}\right] f(\underset{\sim}{x},0) \tag{9.1.12}$$

where the time-ordering symbol mean that one should first expand the exponential and in each term order the operators chronologically. Equation (9.1.12) gives us the formal solution of (9.1.10) for each individual value of σ as a time-ordered product. This gives for the average

$$p(\underset{\sim}{x};t) = \left\langle f(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[\left\langle \exp\left\{\int_0^t \hat{A}_{\underset{\sim}{x}}(t')\,dt'\right\}\right\rangle\right] f(\underset{\sim}{x},0) \tag{9.1.13}$$

since averaging over the ensemble of $\sigma \in \Sigma$ and time-ordering operations commute. Furthermore, inside the time-ordering symbol one may freely commute operators because they have to be put ultimately in chronological order anyway.

The integral over $\hat{A}_{\underset{\sim}{x},\sigma}(t')$ is itself a stochastic quantity Z. On expanding the exponential one obtains a series in successive moments $\langle Z^n\rangle$ which may be written as multiple integrals over moments of $\hat{A}$

$$\left\langle f(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[1 + \int_0^t \left\langle \hat{A}(t_1)\right\rangle dt_1 + \frac{1}{2}\int_0^t\int_0^t dt_1\, dt_2 \left\langle \hat{A}(t_1)\, \hat{A}(t_2)\right\rangle + \ldots\right] f(\underset{\sim}{x},0) \tag{9.1.14}$$

This expression, however, does not provide good successive approximations, because any finite number of terms constitute a bad representation of the function defined by the whole series -- much the same way as the behavior of e^{-t} for large t is badly represented by any finite number of terms of its expansion. In other words, if the magnitude of the fluctuations characteristic of $\hat{A}(t)$ is measured by some parameter λ, then Eq. (9.1.14) is not a suitable expansion because the successive terms are not only of increasing order in λ, but also in t. That is, it is actually an expansion in powers of (λt) and is therefore only valid for limited time. [Equation (9.1.14) has worked successfully in the past in the theory of scattering, where it is known as the Schwinger-Dyson formula. The reason it worked is because in a scattering process, the interaction Hamiltonian acts during a short collisional time τ_s and practically vanishes at all other times t. Equation (9.1.14) then becomes an expansion in powers of $(\lambda\tau_s)$ and our objections are empty in that case.]

We bypass this difficulty by defining "irreducible connected parts" or "cumulants", which are certain combinations of the moments $\langle\hat{A}(t_1)\ldots\hat{A}(t_n)\rangle$ and we will denote them by a bar on top connecting the elements, e.g. $\overline{\langle\hat{A}(t_1)\ldots\hat{A}(t_n)\rangle}$. For the single random variable Z, we define the cumulants by means of the generating function

$$\left\langle e^{tZ}\right\rangle = \exp\left\{\sum_{m=1}^{\infty}\frac{(t)^m}{m!}\,\overline{\langle\underbrace{Z\cdot Z\cdots Z}_{\text{m-times}}\rangle}\right\}$$

With

$$Z = \int_0^t \hat{A}(t')\,dt'$$

we get

$$\left\langle \exp\left\{t\int_0^t \hat{A}(t')\,dt'\right\}\right\rangle = \exp\left\{\sum_{m=1}^{\infty}\frac{(t)^m}{m!}\int_0^t\cdots\int_0^t \overline{\left\langle \hat{A}(t_1)\ldots\hat{A}(t_m)\right\rangle}\,dt_1\ldots dt_m\right\} \tag{9.1.15}$$

The connection with the moments is given by the following hierarchy of equations (we write 1,2,... etc. for the operators $\hat{A}(t_1),\hat{A}(t_2),\ldots$etc.):

$$<1> = <\overline{1}>$$

$$<12> = <\overline{1}><\overline{2}> + <\overline{12}>$$

$$<123> = <\overline{1}><\overline{2}><\overline{3}> + <\overline{1}><\overline{23}> + <\overline{2}><\overline{13}> + <\overline{3}><\overline{12}> + <\overline{123}>$$

$$\ldots \text{ and so on.} \tag{9.1.16}$$

The cumulant expansion (9.1.15) is far better behaved than the moment expansion (9.1.14) for processes $\hat{A}(t)$ having a short correlation time τ_c. To illustrate this, let us suppose that $\hat{A}(t_1)$ and $\hat{A}(t_2)$ are statistically independent quantities when $|t_1-t_2| >> \tau_c$. Then the moment $<\hat{A}(t_1)\ \hat{A}(t_2)>$ factorizes into $<\hat{A}(t_1)>\ <\hat{A}(t_2)>$ and it is easily seen that the cumulant $\overline{<\hat{A}(t_1)\ \hat{A}(t_2)>}$ vanishes. Thus the moments have the "product" property while the cumulants have the "cluster" property. Accordingly (9.1.16) is known as the cluster decomposition. In the language of scattering theory, the cumulants express the totally "connected" <u>or</u> correlated part of the moment $<\hat{A}(t_1)\ldots\hat{A}(t_m)>$, which expresses m successive scatterings from points $t_1,\ldots,t_m$. Thus the cumulant expansion is an expansion in a hierarchy of "disconnected" and "connected" parts of the moments.

More generally the $m^{\underline{th}}$ cumulant $\overline{\langle\hat{A}(t_1)\ldots\hat{A}(t_m)\rangle}$ vanishes as soon as the sequence of times $t_1,\ldots,t_m$ contains a gap large compared to τ_c. This is what we mean intuitively by a "rapidly fluctuating random process".

As a result, each integrand in (9.1.15) vanishes unless $t_1,t_2,\ldots,t_m$ are close together. The only contribution to the integral comes from a m-dimensional tube of diameter of order τ_c along the diagonal $t_1 \sim t_2 \sim\ldots\sim t_m$ in the m-dimensional integration space; m integrations with one relation between the $(t_1,\ldots,t_m)$ above leave one time-integration free. Hence for large t, the contribution of each term is proportional to t so that

$$\left\langle e^Z \right\rangle \sim e^{ct}$$

where c can be found from the cumulants of the operator $\hat{A}(t,\sigma)$.

Let us illustrate this for the term involving the second-order cumulant

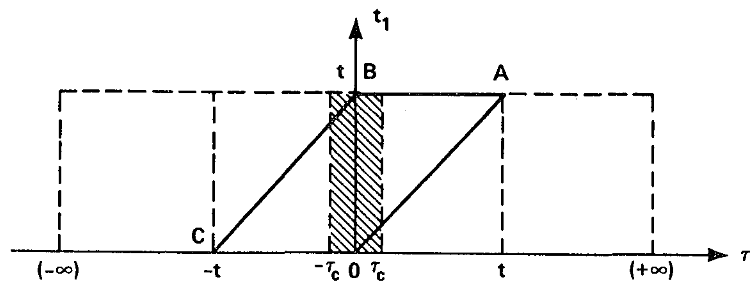
$$\hat{\mathbb{R}} = \int_0^t dt_1 \int_0^t dt_2 \quad \overline{\left\langle \hat{A}(t_1)\ \hat{A}(t_2) \right\rangle}$$

Let us change one of the integration variables to τ via $\tau = t_1 - t_2$. Then

$$\hat{\mathbb{R}} = \int_0^t dt_1 \int_{t_1-t}^{t_1} d\tau \quad \overline{\left\langle \hat{A}(t_1)\ \hat{A}(t_1-\tau) \right\rangle}$$

$$= \int_0^t dt_1 \int_{t_1-t}^{t_1} d\tau\ R\left(t_1,\ t_1-\tau\right)$$

The domain of the double integration in the above equation is illustrated in the Fig. 19 below, as the parallelogram ABCO. By the definition of the correlation time τ_c, the integrand $R(t_1,t_1-\tau)$ takes on significant values only in the shaded domain in Fig. 19 (tube in 2-dimensions), where $|\tau| \leqslant \tau_c$ <u>or</u> $-\tau_c \leqslant \tau \leqslant \tau_c$. Hence, errors coming from the contributions of the small triangular domains outside the parallelogram in the vicinity of 0 and B, we may extend the upper and lower limits of τ-integration to



Domain of Integration for the Second Order Cumulant

Fig. 19

$+\infty$ and $-\infty$, respectively; the errors involved are of the order of (τ_c/t) and for large t are negligible. Thus we can write

$$\hat{R} = \int_0^t dt_1 \int_{-\infty}^{+\infty} d\tau \, R\left(t_1, t_1-\tau\right)$$

For a "stationary process", i.e. a process which is invariant under time-translations $R(t_1,t_1-\tau)$ is a function of the time-difference $t_1-(t_1-\tau) = \tau$ only i.e.

$$\left\langle \overline{\hat{A}(t_1)\ \hat{A}(t_1-\tau)} \right\rangle \equiv R(\tau)$$

In general we do not have a stationary process. Nonstationarity enters into the process either due to slow relaxation processes (slow cooling in our case) <u>or</u> due to rapid (compared to τ_c) oscillations (synchro-betatron oscillations and discrete periodic kicks in our case). In the former case, stationarity holds adiabatically in time and in the latter case we can define a smoothed-out time-stationary cumulant by averaging over the rapidly oscillating terms, as discussed in Ch. 8. For the purposes of illustration, let us assume a time-stationary cumulant $R(t_1,t_1-\tau) \equiv \bar{R}(\tau)$ then, although the general statements remain valid independent of this assumption. We then have

$$\hat{\mathbb{R}} = \int_0^t dt_1 \int_{-\infty}^{+\infty} d\tau\, \bar{R}(\tau)$$

If we define the Fourier transform $\tilde{\bar{R}}(\Omega)$ of $\bar{R}(\tau)$ by the definition

$$\tilde{\bar{R}}(\Omega) = \int_{-\infty}^{+\infty} d\tau\, \bar{R}(\tau)\, \bar{e}^{i\Omega\tau}$$

then

$$\hat{\mathbb{R}} = t \cdot \left[\tilde{\bar{R}}(\Omega)\right]_{\Omega=0} = t \cdot \left[\overline{\langle \hat{A}\,\hat{A}\rangle}\,(\Omega)\right]_{\Omega=0}$$

The term in Eq. (9.1.15) for the expansion into cumulants which involves the second-order cumulant is thus proportional to t with the coefficient determined by the power spectrum of the second-order cumulant. Similar arugments show that all the terms in the exponent in (9.1.15) involving higher-order cumulants are proportional to t, with coefficients determined by the various spectral properties of the cumulants.

Using the cumulant expansion (9.1.15), (9.1.13) gives us:

$$\left\langle f(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[\exp\left\{\int_0^t \overline{\left\langle \hat{A}(t_1)\right\rangle}\, dt_1 + \frac{1}{2}\int_0^t\int_0^t dt_1 dt_2\, \overline{\left\langle \hat{A}(t_1)\,\hat{A}(t_2)\right\rangle} + \dots\right\}\right] f(\underset{\sim}{x},0) \tag{9.1.17}$$

If $\hat{A}$ is measured by some parameter λ, then apart from the first term, the successive terms in the exponent are of order $(\lambda^2\tau_c),(\lambda^3\,\tau_c^2),\dots$ etc. Moreover, they all grow linearly with t when $t \gg \tau_c$. We have assumed that $\hat{A}(t)$ has a finite correlation time τ_c in the sense that all cumulants of $\hat{A}(t)$ vanish, i.e.

$$\overline{\left\langle \hat{A}(t_1)\,\hat{A}(t_2)\dots\hat{A}(t_m)\right\rangle} \approx 0$$

as soon as the time arguments in them have a gap $|t_i - t_j|$ large compared to τ_c. The successive terms are then of order $(\lambda^m\,\tau_c^{m-1}\,t)$ for $t \gg \tau_c$. We note that although successive term in (9.1.17) do not have any meaning yet since the time-ordering operator $\mathbb{T}$ mixes them up eventually, the above comments on orders of magnitude remain correct.

We now solve (9.1.17) step by step. First, we omit all terms of $O(\lambda^2\tau_c)$. Then

$$\left\langle f(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[\exp\left\{\int_0^t \left\langle \hat{A}(t_1)\right\rangle dt_1\right\}\right] f(\underset{\sim}{x};0)$$

By the definition of $\mathbb{T}$, this is the solution of

$$\frac{\partial}{\partial t}\left\langle f(\underset{\sim}{x};t)\right\rangle = \left\langle \hat{A}(t)\right\rangle f(\underset{\sim}{x};t); \qquad \left\langle f(\underset{\sim}{x},0)\right\rangle = f(\underset{\sim}{x},0) \tag{9.1.18}$$

We now use the decomposition (9.1.2) and the definition (9.1.11) to write

$$\hat{A}(t) = \hat{A}_0 + \hat{A}_1(t)$$

where

$$\hat{A}_0 = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0(\underset{\sim}{x}) \quad \text{and} \quad \hat{A}_1(t) = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^1(\underset{\sim}{x};t;\sigma)$$

Then

$$\left\langle \hat{A}(t)\right\rangle = \left\langle \hat{A}_0\right\rangle + \left\langle \hat{A}_1(t)\right\rangle = \hat{A}_0$$

since $\underset{\sim}{G}^0(x)$ is a sure function (hence $\langle\hat{A}_0\rangle = \hat{A}_0$) and $\langle\underset{\sim}{G}^1(\underset{\sim}{x};t)\rangle = 0$ by Eq. (9.1.4).

We now use this result to define an 'interaction representation'. Let $\hat{U}(t|t')$ be the evolution operator belonging to (9.1.18). We set

$$f(\underset{\sim}{x};t) = \hat{U}(t|0)\, g(\underset{\sim}{x};t)$$

and accordingly transform $\hat{A}_1(t)$ into $\hat{B}(t)$:

$$\hat{A}_1(t) = \hat{U}(t|0)\, \hat{B}(t)\, \hat{U}(0|t)$$

The evolution operator satisfies the equation

$$\frac{\partial}{\partial t}\hat{U}(t|t') = \hat{A}_0(t)\,\hat{U}(t|t') \quad \text{with} \quad \hat{U}(t'|t') = \hat{\mathbb{1}}\,.$$

In this representation, we have

$$\frac{\partial g(\underset{\sim}{x};t)}{\partial t} = \hat{B}(t)\,g(\underset{\sim}{x};t)$$

and the solution is written analogous to (9.1.17)

$$\left\langle g(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[\exp\left\{\frac{1}{2}\int_0^t\int_0^t dt_1 dt_2\,\overline{\left\langle \hat{B}(t_1)\,\hat{B}(t_2)\right\rangle} + \ldots\right\}\right] g(\underset{\sim}{x};0)$$

The second step consists in omitting the terms of order $(\lambda^3\tau_c^2)$ and higher in the exponent. We should now expand the exponential and in each term of the expansion rearrange the operators $\hat{B}$ chronologically. We partially fulfill this requirement by writing

$$\left\langle g(\underset{\sim}{x};t)\right\rangle = \mathbb{T}\left[\exp\left\{\int_0^t dt_1\int_0^{t_1} dt_2\,\overline{\left\langle \hat{B}(t_1)\,\hat{B}(t_2)\right\rangle}\right\}\right] g(\underset{\sim}{x};0) \qquad (9.1.19)$$

Let us denote

$$\hat{K}(t_1) = \int_0^{t_1} dt_2\,\overline{\left\langle \hat{B}(t_1)\,\hat{B}(t_2)\right\rangle}$$

and consider the differential equation

$$\frac{\partial}{\partial t}\left\langle g(\underset{\sim}{x};t)\right\rangle = \hat{K}(t)\left\langle g(\underset{\sim}{x};t)\right\rangle \qquad (9.1.20)$$

Its solution is

$$\left\langle g(\underset{\sim}{x};t)\right\rangle = \mathbf{T}\left[\exp\left\{\int_0^t \hat{K}(t_1)\, dt_1\right\}\right]\left\langle g(\underset{\sim}{x};0)\right\rangle \tag{9.1.21}$$

with $<g(\underset{\sim}{x};0)> = g(\underset{\sim}{x};0)$. This is almost identical with (9.1.19) but not exactly. We will show later that the difference is of order $(\lambda\tau_c)$.

Equation (9.1.20) in the original representation is given by

$$\frac{\partial}{\partial t}\left\langle f(\underset{\sim}{x};t)\right\rangle = \left[\hat{A}_0 + \hat{L}(t)\right]\left\langle f(\underset{\sim}{x};t)\right\rangle \tag{9.1.22}$$

where $\hat{L}(t)$ is the operator $\hat{K}(t)$ transformed back to the original representation:

$$\begin{aligned}\hat{L}(t) &= \hat{U}(t|0)\ \hat{K}(t)\ \hat{U}(0|t) \\ &= \int_0^t dt'\ \hat{U}(t|0)\left\langle \overline{\hat{B}(t)\ \hat{B}(t')}\right\rangle \hat{U}(0|t) \\ &= \int_0^t dt'\left\langle \overline{\hat{A}_1(t)\ \hat{U}(t|t')\ \hat{A}_1(t')}\right\rangle \hat{U}(t'|t)\end{aligned}$$

With

$$\hat{A}(t) = \hat{A}_0 + \hat{A}_1(t) \quad \text{and} \quad \left\langle \hat{A}_1(t)\right\rangle = 0,$$

we have

$$\hat{U}(t|t') = \exp\left\{\hat{A}_0(t-t')\right\}$$

Then we can write (9.1.22) as

$$\frac{\partial}{\partial t}\left\langle f(\underset{\sim}{x};t)\right\rangle = \left[\hat{A}_0 + \int_0^t d\tau \left\langle \hat{A}_1(t)\ e^{\hat{A}_0\tau}\ \hat{A}_1(t-\tau)\right\rangle e^{-\hat{A}_0\tau}\right]\left\langle f(\underset{\sim}{x};t)\right\rangle$$

Recalling our definitions

$$\hat{A}_0(\underset{\sim}{x}) = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0(\underset{\sim}{x})$$

$$\hat{A}_1(\underset{\sim}{x},t;\sigma) = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^1(\underset{\sim}{x},t;\sigma)$$

$$p(\underset{\sim}{x};t) = \langle f(\underset{\sim}{x};t)\rangle$$

we write

$$\frac{\partial}{\partial t} p(\underset{\sim}{x},t) = \nabla_{\underset{\sim}{x}} \cdot \left[-\underset{\sim}{G}^0(\underset{\sim}{x}) + \int_0^t d\tau \left\langle \underset{\sim}{G}^1(\underset{\sim}{x};t)\, e^{(-\tau\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{G}^0)}\, \nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^1(\underset{\sim}{x},t-\tau)\right\rangle \, e^{(\tau\nabla_{\underset{\sim}{x}}\cdot\underset{\sim}{G}^0)} \right] p(\underset{\sim}{x};t)$$

$$= \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[-\underset{\sim}{G}^0(\underset{\sim}{x}) + \int_0^t d\tau \left\langle \underset{\sim}{G}^1(\underset{\sim}{x};t)\, e^{-\tau\frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\sim}{G}^0(\underset{\sim}{x})}\, \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{G}^1(\underset{\sim}{x},t-\tau)\right\rangle \, e^{\tau\frac{\partial}{\partial x}\cdot\underset{\sim}{G}^0(\underset{\sim}{x})} \right]$$

$$p(\underset{\sim}{x};t) \qquad (9.1.23)$$

When $t >> \tau_c$, we an take the upper limit of integration to be $(+\infty)$. We have thus obtained Eq. (9.1.23) to second-order $O(\lambda^2)$ in the strength λ of $\underset{\sim}{G}^1$, without assuming absence of correlations or cutting off secular terms.

Now we justify the use of (9.1.21) instead of (9.1.19). When we expand the exponential in (9.1.19) and apply the time-ordering, it may happen that the two operators $\hat{B}(t_1)$ and $\hat{B}(t_2)$ in the cumulant become separated by one or more operators from the other factors. In contrast in (9.1.21) the operators $\hat{B}$ in a cumulant stay together to make $\hat{K}$ and only the operators $\hat{K}$ are time-ordered among themselves. This is the difference between (9.1.21) and (9.1.19). For an estimate of this difference, let us look at a typical term of the expansion of (9.1.19)

$$\frac{\lambda^{2n}}{n!} \mathbb{T} \left[\left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \, \overline{\langle \hat{B}(t_1)\, \hat{B}(t_2)\rangle} \right\}^n \right] \qquad (9.1.24)$$

where λ measures the strength of $\hat{B}$ (i.e. $\hat{B}$ is of the order of unity in this expression.) There are n pairs of time-arguments (t_1,t_2) and one has to integrate over a 2n-dimensional domain. There is a subdomain where no two pairs overlap and which therefore is correctly represented in (9.1.21). The volume of that subdomain is of order $t^n\tau_c^n$, because each pair ranges over an interval of order t, while the pair t_1, t_2 must be within τ_c from one another. The contribution of this subdomain is therefore of order $\lambda^{2n}t^n\tau_c^n$.

The order of the operators $\hat{B}$ in the remaining subdomain is quite different in (9.1.21) and (9.1.19). However, if two pairs have to overlap, all four times involved must be within a distance of order τ_c from one another, so that the range of integration for those two pairs is only of order $(t\tau_c^3)$. Hence the difference between (9.1.21) and (9.1.19) as far as the term (9.1.24) is concerned, is at most of order $\lambda^{2n}t^{n-1}\tau_c^{n+1}$. Since this is proportional to t^{n-1}, we have to compare this with the principal contribution of the previous term in the expansion of (9.1.19), which is $\lambda^{2n-2}t^{n-1}\tau_c^{n-1}$. Hence the difference between (9.1.21) and (9.1.19) is or order $\lambda^2\tau_c^2$. A general expression involving these higher-order terms in the expansion of $\hat{L}$ can be found but our present report does not concern itself with orders beyond $(\lambda^2\tau_c)$.

Equation (9.1.23) has the structure of a Fokker-Planck transport equation. We have thus demonstrated that such an equation arises most naturally from the differential equation for stochastic cooling up to order $(\lambda^2\tau_c)$, when the process is characterized by a finite correlation time τ_c, which is smaller than the time-scales of interest for the relaxation process of cooling.

The operator $\exp(-\tau\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0)$ in (9.1.23) provides a solution to the equation

$$\frac{\partial f(\underset{\sim}{x},t)}{\partial t} = -\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0 f(\underset{\sim}{x},t)$$

Thus

$$f(\underset{\sim}{x},t) = \left[\exp(-t\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0)\right] f(\underset{\sim}{x},0) \tag{9.1.25}$$

However, the single particle equation

$$\dot{\underset{\sim}{x}} = \underset{\sim}{G}^0(\underset{\sim}{x})$$

determines a mapping

$$g^t: \underset{\sim}{x} \longrightarrow \underset{\sim}{x}^t$$

with inverse

$$(\underset{\sim}{x}^t)^{-t} = \underset{\sim}{x}$$

for fixed t. Then we also have

$$f(\underset{\sim}{x},t) = f(\underset{\sim}{x}^{-t},0)\,\frac{d[\underset{\sim}{x}^{-t}]}{d[\underset{\sim}{x}]}$$

The effect of the operator $\exp(-t\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0)$ is thus given by

$$\exp\left(-t\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{G}^0\right) f(\underset{\sim}{x}) = f(\underset{\sim}{x}^{-t})\,\frac{d[\underset{\sim}{x}^{-t}]}{d[\underset{\sim}{x}]} \tag{9.1.26}$$

where the last factor is the Jacobian determinant of the mapping. Using this, we can write Eq. (9.1.23) in the following two alternative forms:

$$\frac{\partial}{\partial t} p(\underset{\sim}{x},t) = \nabla_{\underset{\sim}{x}} \cdot \left[-\underset{\sim}{G}^0(\underset{\sim}{x}) + \int_0^\infty d\tau\, \frac{d[\underset{\sim}{x}^{-\tau}]}{d[\underset{\sim}{x}]} \left\langle \underset{\sim}{G}^1(\underset{\sim}{x},t)\underset{\sim}{\nabla}_{-\tau} \cdot \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau},t-\tau)\right\rangle \frac{d[\underset{\sim}{x}]}{d[\underset{\sim}{x}^{-\tau}]} \right] p(\underset{\sim}{x},t) \tag{9.1.27}$$

or

$$\frac{\partial}{\partial t} p(\underset{\sim}{x},t) = \nabla_{\underset{\sim}{x}} \cdot \left[-\underset{\sim}{G}^0(\underset{\sim}{x}) + \int_0^\infty d\tau \left\langle \underset{\sim}{G}^1(\underset{\sim}{x},t)\; \underset{\sim}{\nabla}_{-\tau} \cdot \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau},t-\tau)\right\rangle \right.$$

$$\left. + \int_0^\infty \left\langle \underset{\sim}{G}^1(\underset{\sim}{x},t)\; \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau},t-\tau)\right\rangle \cdot \underset{\sim}{Y}(\underset{\sim}{x}^{-\tau}) \right] p(\underset{\sim}{x},t) \tag{9.1.28}$$

where $\underset{\sim}{\nabla}_{-\tau}$ means differentiation with respect to $\underset{\sim}{x}_{-\tau}$ and $\underset{\sim}{Y}$ is a vector function

$$\underset{\sim}{Y}(\underset{\sim}{x}) = \nabla_{\underset{\sim}{x}} \left[\log \frac{d[\underset{\sim}{x}^{\tau}]}{d[\underset{\sim}{x}]} \right] \tag{9.1.29}$$

So far we have assumed the fluctuation term $\underset{\sim}{G}^1$ to be of order λ. For a cooling system both $\underset{\sim}{G}^0$ and $\underset{\sim}{G}^1$ are of the same order λ and hence the Jacobian of the transformation of orbits under $\underset{\sim}{G}^0$ given by (9.1.26) is

$$\frac{d[\underset{\sim}{x}^{\tau}]}{d[\underset{\sim}{x}]} = 1 + O(\lambda) \tag{9.1.30}$$

Since the second and third terms on the right-ahdn side of (9.1.28) are already $O(\lambda^2)$, being of the form $(\underset{\sim}{G}^1 \cdot \underset{\sim}{G}^1)$, the added $O(\lambda)$ contribution from (9.1.30) adds $O(\lambda^3)$ terms and higher and we neglect them. Then $\underset{\sim}{Y}$ in (9.1.29) vanishes (logarithm of unity being zero). Moreover, the fluctuation term $\underset{\sim}{G}^1$ in the equation of motion (9.1.2) satisfies the Hamiltonian flow condition for stochastic cooling as explained before and so

$$\underset{\sim}{\nabla}_{-\tau} \cdot \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau}, t-\tau) = 0$$

Thus $\underset{\sim}{\nabla}_{-\tau}$ and $\underset{\sim}{G}^1$ commute and we can write:

$$\begin{aligned}\underset{\sim}{\nabla}_{-\tau} \cdot \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau}, t-\tau)\, p(\underset{\sim}{x},t) &= \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau}, t-\tau) \cdot \underset{\sim}{\nabla}_{-\tau}\, p(\underset{\sim}{x},t) \\ &= \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau}, t-\tau) \cdot \underset{\sim}{\nabla}\, p(\underset{\sim}{x},t)\end{aligned}$$

where in the last step we have used (9.1.30) again.

Thus for fluctuations satisfying the condition for Hamiltonian flow and strength of cooling and fluctuations given by a small dimensionless parameter $\lambda << 1$ (we incorporate λ within $\underset{\sim}{G}^0$ and $\underset{\sim}{G}^1$ so they do not appear explicitly), the Fokker-Planck equation for transport has the form:

$$\frac{\partial p(\underset{\sim}{x},t)}{\partial t} = - \nabla_{\underset{\sim}{x}} \cdot \left[\underset{\sim}{G}^0(\underset{\sim}{x})\, p(\underset{\sim}{x},t) \right] + \nabla_{\underset{\sim}{x}} \cdot \int_0^{\infty} d\tau \left\langle \underset{\sim}{G}^1(\underset{\sim}{x},t)\, \underset{\sim}{G}^1(\underset{\sim}{x}^{-\tau}, t-\tau) \right\rangle \nabla_{\underset{\sim}{x}}\, p(\underset{\sim}{x},t) \tag{9.1.31}$$

We write (9.1.31) as follows:

$$\frac{\partial p(\underset{\sim}{x},t)}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[\underset{\sim}{E}(\underset{\sim}{x})\; p(\underset{\sim}{x},t)\right] + \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[\underset{\approx}{D}(\underset{\sim}{x}) \cdot \frac{\partial p(\underset{\sim}{x};t)}{\partial \underset{\approx}{x}}\right] \tag{9.1.32}$$

where

$$E(\underset{\sim}{x}) = \underset{\sim}{G}^0(\underset{\sim}{x})$$

$$\underset{\approx}{D}(\underset{\sim}{x}) = 2\int_0^{\infty} \left\langle \underset{\sim}{G}^1(\underset{\sim}{x},t) \quad \underset{\sim}{G}^1\Big(\underset{\sim}{x}(t-\tau),t-\tau\Big)\right\rangle d\tau \tag{9.1.33}$$

are the Friction and Diffusion coefficients respectively.

We note that in action-angle variables $(\underset{\sim}{x} = \underset{\sim}{I}, \underset{\sim}{\psi})$ the distribution function $p(\underset{\sim}{x}) = p(\underset{\sim}{I},\underset{\sim}{\psi})$ is a function of $\underset{\sim}{I},\underset{\sim}{\psi}$. If we are interested in the time evolution of a homogenous distribution $f(\underset{\sim}{I})$ independent in angle $\underset{\sim}{\psi}$, the terms in (9.1.32) involving gradients in angle (e.g. $\partial^2/\partial\underset{\sim}{\psi}\,\partial\underset{\sim}{\psi}$, $\partial^2/\partial\underset{\sim}{I}\,\partial\underset{\sim}{\psi}$ etc.) drop out and we only have to consider transport in action $\underset{\sim}{I}$-space alone. We then have

$$\frac{\partial p(\underset{\sim}{I};t)}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[F(\underset{\sim}{I})\; p(\underset{\sim}{I};t)\right] + \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[\underset{\approx}{D}(\underset{\sim}{I}) \cdot \frac{\partial p(\underset{\sim}{I},t)}{\partial \underset{\sim}{I}}\right] \tag{9.1.34}$$

where

$$E(\underset{\sim}{I}) = \underset{\sim}{G}^0(\underset{\sim}{I})$$

$$\underset{\approx}{D}(\underset{\sim}{I}) = 2\int_0^{\infty} d\tau \left\langle \underset{\sim}{G}^1\Big[\underset{\sim}{I}(t),\underset{\sim}{\psi}(t);t\Big]\; \underset{\sim}{G}^1\Big[\underset{\sim}{I}(t-\tau),\underset{\sim}{\psi}(t-\tau);t-\tau\Big]\right\rangle \tag{9.1.35}$$

To the order of the approximation, the co-ordinates $\underset{\sim}{I}(t)$, $\underset{\sim}{\psi}(t)$, appearing in (9.1.35) are to be evaluated just as zero-order orbits $\underset{\sim}{I}^0(t) = \underset{\sim}{I} = \text{const.}$ and $\underset{\sim}{\psi}^0(t) = \underset{\sim}{\omega} \cdot t + \underset{\sim}{\psi}(0)$ and then the expectation value over the ensemble $<\underset{\sim}{G}^1\underset{\sim}{G}^1>$ in (9.1.35) becomes just the auto-correlation function $\underset{\approx}{R}(t,t-\tau)$ of the sampled noise without the self-interaction as defined in (8.2.17):

$$\underset{\approx}{R}(t,t-\tau) = \left\langle \underset{\sim}{G}^1(t)\; \underset{\sim}{G}^1(t-\tau)\right\rangle$$

Averaging over rapid oscillations as in Section (8.2), we get a smoothed out auto-correlation which is a function of τ alone:

$$\underset{\approx}{\bar{R}}(\tau) = \overline{\left\langle \mathcal{G}^1(t)\ \mathcal{G}^1(t-\tau)\right\rangle}$$

For time-stationary auto-correlation, we have then the property $\underset{\approx}{\bar{R}}(\tau) = \underset{\approx}{\bar{R}}(-\tau)$ and the Diffusion coefficient becomes:

$$\underset{\approx}{D}(\underset{\sim}{I}) = 2\int_0^{\infty} d\tau\ \underset{\approx}{\bar{R}}(\tau)$$

$$= \int_{-\infty}^{+\infty} d\tau\ \underset{\approx}{\bar{R}}(\tau) = \underset{\approx}{\tilde{\bar{R}}}(\Omega)\Big|_{\Omega=0}$$

where $\underset{\approx}{\tilde{\bar{R}}}(\Omega)$ is the Fourier transform of $\underset{\approx}{\tilde{\bar{R}}}(\tau)$ in τ:

$$\underset{\approx}{\tilde{\bar{R}}}(\Omega) = \int_{-\infty}^{+\infty} d\tau\ e^{-i\Omega\tau}\ \underset{\approx}{\bar{R}}(\tau)$$

Using the explicit expressions for $\underset{\sim}{\mathcal{G}}^0(\underset{\sim}{I})$ and $\underset{\approx}{\tilde{\bar{R}}}(\Omega)$ given by (7.1.16) and (8.2.17) we obtain for the Friction and Diffusion coefficients in the Fokker-Planck equation (9.1.34) the following:

$$\underset{\sim}{F}(\underset{\sim}{I}) = \underset{\sim}{\mathcal{G}}^0(\underset{\sim}{I}) = \sum_{\underset{\sim}{n}} \mathcal{G}_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}) \tag{9.1.36}$$

and

$$\underset{\approx}{D}(\underset{\sim}{I}) = (2\pi)\cdot N \sum_{\underset{\sim}{n}}\sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}'\ f(\underset{\sim}{I}') \left[\mathcal{G}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\ \mathcal{G}^*_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\right] \times \delta\left(\underset{\sim}{n}\cdot\underset{\sim}{\omega}(\underset{\sim}{I}) + \underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}')\right) \tag{9.1.37}$$

Substituting for $\underset{\sim}{G}_{\underset{\sim}{nn'}}(\underset{\sim}{I},\underset{\sim}{I}')$ for the relevant cooling interactions, e.g. Eq. (7.4) through (7.11) for longitudinal cooling or Eq. (7.14) for transverse cooling, one gets the explicit Fokker-Planck equation for stochastic cooling.

Had one anticipated the Markovian nature of the process beforehand, one could have arrived at the same conclusion with considerable economy of time. Recognizing the stochastic nature of the phenomenon, we seek a description in terms of the transition probability in phase-space starting from a given initial distribution. It is important to recognize, as of the essence of the cooling process, that there exist time intervals (Δt) during which the phase-space coordinates of the cooling particle change by infinitesimal amounts while there occur a very large number of kicks <u>or</u> fluctuations characteristic of the motion and arising from the interaction with the other particles in the beam. Thus we assume

$$\tau_d \gg \Delta t \gg \tau_c$$

where τ_c is the correlation time of interaction and τ_d is the relaxation time of slow cooling and diffusion.

The time-evolution of the distribution function $p(\underset{\sim}{x},t)$ for the cooled particle may then be described by the integral equation

$$p(\underset{\sim}{x};\, t+\Delta t) = \int p(\underset{\sim}{x}-\Delta\underset{\sim}{x};\, t)\, W_{\Delta t}(\underset{\sim}{x}-\Delta\underset{\sim}{x};\, \Delta\underset{\sim}{x})\, d(\Delta\underset{\sim}{x}) \qquad (9.1.38)$$

Here $W_{\Delta t}(\underset{\sim}{x};\Delta\underset{\sim}{x})$ represents the transition probability that $\underset{\sim}{x}$ changes by $\Delta\underset{\sim}{x}$ during Δt, with normalization

$$\int W_{\Delta t}(\underset{\sim}{x};\Delta\underset{\sim}{x})\, d(\Delta\underset{\sim}{x}) = 1\ . \qquad (9.1.39)$$

In expecting the integral equaton to be true, we are actually supposing that the course which a cooling particle will take depends only on the instantaneous values of its physical parameters and is entirely independent of its whole previous history, i.e. we are assuming it to be a Markov process. It is far from obvious that we can idealize the cooling process by a Markov one. However from the arguments given before and experimental observations of the actual evolution dynamics in the stochastic cooling experiments at CERN, we expect the assumption to be physically plausible.

Expanding Eq. (9.1.38) in a Taylor series with respect to Δt and $\Delta \underset{\sim}{x}$ we get [53]

$$p(\underset{\sim}{x},t) + \frac{\partial p}{\partial t}\Delta t + 0\left(\frac{(\Delta t)^2}{\tau_d^2}\right) = \int d(\Delta \underset{\sim}{x}) \left[p(\underset{\sim}{x};t) - \Delta \underset{\sim}{x} \cdot \frac{\partial p}{\partial \underset{\sim}{x}} + \frac{1}{2}(\Delta \underset{\sim}{x} \Delta \underset{\sim}{x}): \frac{\partial^2 p}{\partial \underset{\sim}{x} \partial \underset{\sim}{x}} - \dots\right]$$

$$\otimes \left[W_{\Delta t}(\underset{\sim}{x};\Delta \underset{\sim}{x}) - \Delta \underset{\sim}{x} \cdot \frac{\partial W_{\Delta t}}{\partial \underset{\sim}{x}} + \frac{1}{2}(\Delta \underset{\sim}{x} \Delta \underset{\sim}{x}): \frac{\partial^2 W_{\Delta t}}{\partial \underset{\sim}{x} \partial \underset{\sim}{x}} - \dots\right]$$

With the normalization (9.1.29), we may define and calculate the average values of the increments according to

$$\langle \Delta \underset{\sim}{x} \rangle = \int d(\Delta \underset{\sim}{x})(\Delta \underset{\sim}{x})\, W_{\Delta t}(\underset{\sim}{x};\Delta \underset{\sim}{x})$$

$$\langle \Delta \underset{\sim}{x}\, \Delta \underset{\sim}{x} \rangle = \int d(\Delta \underset{\sim}{x})(\Delta \underset{\sim}{x}\, \Delta \underset{\sim}{x})\, W_{\Delta t}\, (\underset{\sim}{x};\Delta \underset{\sim}{x})$$

Neglecting the third order terms on the left hand side, which is smaller by a factor $0(\Delta t/\tau_d)$ than the second-order term, we obtain

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[\frac{\langle \Delta \underset{\sim}{x} \rangle}{\Delta t} p\right] + \frac{1}{2}\frac{\partial^2}{\partial \underset{\sim}{x}\, \partial \underset{\sim}{x}} : \left[\frac{\langle \Delta \underset{\sim}{x}\, \Delta \underset{\sim}{x} \rangle}{\Delta t} p\right] + 0\left(\frac{\langle \Delta \underset{\sim}{x}\, \Delta \underset{\sim}{x}\, \Delta \underset{\sim}{x} \rangle}{\Delta t}\right)$$

We thus recover the basic Fokker-Planck transport equation for the time-evolution of the one-particle distribution $p(\underset{\sim}{x};t)$ for a particle experiencing fluctuating fields at the kicker valid up to two-body correlation effects, in the form:

$$\frac{\partial p(\underset{\sim}{x};t)}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[E(\underset{\sim}{x})\, p\right] + \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{x}}\frac{\partial}{\partial \underset{\sim}{x}} : \left[\underset{\approx}{D}(\underset{\sim}{x})\, p\right] \tag{9.1.40}$$

where

$$\underset{\sim}{F}(\underset{\sim}{x}) = \frac{1}{\tau} \langle \Delta \underset{\sim}{x} \rangle \tag{9.1.41}$$

$$\underset{\approx}{D}(\underset{\sim}{x}) = \frac{1}{\tau} \langle \Delta \underset{\sim}{x} \, \Delta \underset{\sim}{x} \rangle \tag{9.1.42}$$

are the usual Friction and Diffusion coefficients and $\langle(\ldots)\rangle$ denotes an average over ensemble of particles (which in practical computation would mean an average over initial conditions or phases of the particles) and τ is a time which is much shorter than the time-scale over which the distribution function changes significantly but still long enough so that a large number of fluctuation kicks characteristic of the motion (arising from the kicker fields) occur.

To evaluate $\underset{\sim}{F}$ and $\underset{\approx}{D}$ we write the fluctuation equations of motion in presence of kicker fields as:

$$\dot{\underset{\sim}{x}} = \lambda \, \underset{\sim}{G}\left[\underset{\sim}{x}(t);\, t\right] \qquad (\lambda \ll 1)$$

where λ is a "strength of fluctuations" parameter and $\underset{\sim}{G}[\underset{\sim}{x}(t);t]$ is the signal sampled by the particle on its orbit $\underset{\sim}{x}(t)$ at time t. In terms of action and angle variable $\underset{\sim}{x} = (\underset{\sim}{I},\underset{\sim}{\psi})$, the fluctuation equation becomes:

$$\begin{aligned} \dot{\underset{\sim}{I}} &= \lambda \, \underset{\sim}{G}^1\left[\underset{\sim}{I}(t),\underset{\sim}{\psi}(t);\, t\right] + \lambda \, \underset{\sim}{G}^0\left[\underset{\sim}{I}(t)\right] \\ \dot{\underset{\sim}{\psi}} &= \underset{\sim}{\omega}(\underset{\sim}{I}^0) + \lambda \, \underset{\sim}{H}^1\left[\underset{\sim}{I}(t),\underset{\sim}{\psi}(t);\, t\right] + \lambda \, \underset{\sim}{H}^0\left[\underset{\sim}{I}(t)\right] \end{aligned} \tag{9.1.43}$$

where the superscripts 0 and 1 denote the coherent self-interaction signal and the Schottky noise part respectively. Then:

$$\underset{\sim}{I}(t) = \underset{\sim}{I}(0) + \lambda \int_0^t dt' \, \underset{\sim}{G}^0\left[\underset{\sim}{I}(t')\right] + \lambda \int_0^t dt' \, \underset{\sim}{G}^1\left[\underset{\sim}{I}(t'),\underset{\sim}{\psi}(t');\, t'\right] \tag{9.1.44}$$

and

$$\underset{\sim}{\psi}(t) = \underset{\sim}{\psi}(0) + \underset{\sim}{\omega}(\underset{\sim}{I}^0)\, t + \lambda \int_0^t dt' \, \underset{\sim}{H}^0\left[\underset{\sim}{I}(t')\right] + \lambda \int_0^t dt' \, \underset{\sim}{H}^1\left[\underset{\sim}{I}(t'),\underset{\sim}{\psi}(t');\, t'\right] \tag{9.1.45}$$

Since we are interested in the time-evolution of the angle-averaged one particle distribution $p(I;t)$, we need to calculate $\underset{\sim}{F}(\underset{\sim}{I})$ and $\underset{\approx}{D}(\underset{\sim}{I})$ only since terms of the form $\frac{\partial}{\partial\underset{\sim}{\psi}}$ and $\frac{\partial}{\partial\underset{\sim}{\psi}}\frac{\partial}{\partial\underset{\sim}{I}}$ and $\frac{\partial^2}{\partial\underset{\sim}{\psi}\partial\underset{\sim}{\psi}}$ vanish on averaging over $\underset{\sim}{\psi}$ over 0 to 2π. We are considering particle distributions which are homogeneous in the phase-angle.

One can now evaluate the coefficients $\frac{1}{\tau}\langle\Delta\underset{\sim}{I}(\tau)\rangle = \underset{\sim}{F}(\underset{\sim}{I})$ and $\frac{1}{\tau}\langle\Delta\underset{\sim}{I}(\tau)\Delta\underset{\sim}{I}(\tau)\rangle = \underset{\approx}{D}(\underset{\sim}{I})$ to the order λ^2, neglecting $O(\lambda^3)$ terms. For $\underset{\approx}{D}(\underset{\sim}{I})$ one needs only to keep the first order term in λ in the expansion for $\Delta\underset{\sim}{I}(\tau) = \underset{\sim}{I}(\tau)-\underset{\sim}{I}(0)$ as given by (9.1.44) since the product $\Delta\underset{\sim}{I}\,\Delta\underset{\sim}{I}$ already introduces λ^2. One then obtains a result identical to (9.1.35) for $\underset{\approx}{D}(\underset{\sim}{I})$ as is easily verified. There is no $O(\lambda)$ term in the $D(I)$. The $O(\lambda)$ term appears in $\underset{\sim}{F}(\underset{\sim}{I})$ and is given by the expression for $\underset{\sim}{F}(\underset{\sim}{I})$ as given in Eq. (9.1.35). The $O(\lambda^2)$ term in $\underset{\sim}{F}(\underset{\sim}{I})$ transforms into a term of the form $-\frac{1}{2}\frac{\partial}{\partial\underset{\sim}{I}}\cdot\underset{\approx}{D}(\underset{\sim}{I})$ when one expands by iterating the orbit (9.1.44) up to second order in 'λ' ($O(\lambda^2)$) by using the first order orbit expansion for $\Delta\psi(\tau)$ from (9.1.45) into (9.2.44) and one uses the Hamiltonian property of $\underset{\sim}{G}^1$, namely

$$\frac{\partial\, \underset{\sim}{G}^1(\underset{\sim}{I},\underset{\sim}{\psi};t)}{\partial\underset{\sim}{\psi}} = -\frac{\partial\, \underset{\sim}{G}^1(\underset{\sim}{I},\underset{\sim}{\psi};t)}{\partial\underset{\sim}{I}}$$

This terms thus simply modifies the diffusion term to $\frac{1}{2}\frac{\partial}{\partial\underset{\sim}{I}}\cdot\left[\underset{\approx}{D}(\underset{\sim}{I})\cdot\frac{\partial f^0}{\partial\underset{\sim}{I}}\right]$.

We note that the general form of the Fokker-Planck equation as given by (9.1.40) and (9.1.28) has two partial derivatives $\partial^2/\partial\underset{\sim}{x}\,\partial\underset{\sim}{x}$ to the left acting on $[\underset{\approx}{D}(\underset{\sim}{x})\,p]$ to the right. For fluctuation equations governed by Hamiltonian dynamics, aside from the self-interaction part, however, the diffusion term in the Fokker-Planck equation takes the form of $\frac{\partial}{\partial\underset{\sim}{x}}\cdot\left[\underset{\approx}{D}(\underset{\sim}{x})\cdot\frac{\partial p}{\partial\underset{\sim}{x}}\right]$ with one derivative acting only on $p(\underset{\sim}{x},t)$. This is indeed the case for stochastic cooling as also for Coulomb interactions in a plasma.

9.2 Kinetic Theory in Phase Space

We now illustrate how a transport equation in phase-space of the form of a Fokker-Planck equation arises naturally from a consistent kinetic theory of the N-particle system. In general this description includes both the time-coherence of Schottky signals (Schottky noise diffusion) and the shielding induced by the feedback system. Both of these effects are manifestations of correlations developing between particles in phase-space. In the usual derivation of the Vlasov equation which describes instabilities in particle beams in accelerators two-body and higher correlation effects are assumed negligible. In addition, there is no dissipative self-interaction; the equations of motion are Hamiltonian. In stochastic cooling systems, it is such a self-interaction term which increases the phase-space density. In the following, equations are derived for the one-particle distribution function and the totally correlated part of the two-particle distribution function. The one-particle distribution is just the usual distribution function used in the Vlasov equation for collective modes. In the context of stochastic cooling, however, the equation for the one-particle distribution will describe the transport or time-evolution of such a function in the presence of cooling. The two-particle distribution will describe the effects of correlations. The equation for the two-particle distribution thus describes the propagation of correlations and gives us the noise diffusion and collective shielding effects. Accordingly, this second equation has the nature of a Vlasov equation for the two-particle correlations. To describe the time-dependence of the two-particle correlations, one has to consider the equation for the three-particle correlation distribution function and so on. The hierarchy thus obtained for higher and higher order correlations is known as the BBGKY (Bogolubov-Born-Green-Kirkwood-Yvon) hierarchy and closes on itself only at the N^{th} level which describes the totally correlated part of the full N-body distribution. We will truncate our hierarchy beyond the two-particle correlations thus preserving information about the slow damping due to cooling, diffusion due to Schottky noise with long-time coherence and collective shielding due to two-body correlations generated by the feedback loop but throwing away information about the fast time-development of such two-body correlations leading to collective shielding. In order to learn about how fast the process of shielding establishes itself we have to go one step further by taking into account the three-body effects as well. We will not do this last step.

One further remark is in order. The system of particles in a beam is characterized by a set of frequencies having a discrete spectrum in principle (N finite). The average frequency spacing between particles is of the order of $\Delta\omega/N$ where $\Delta\omega$ is the full frequency spread in the beam. One would be able to resolve such small frequency spacing if one waits long enough until the characteristic return time $\tau_r \sim N/\Delta\omega$. Hence a kinetic theory description is valid only for characteristic relaxation times γ^{-1} (cooling and diffusion times) short compared to τ_r: $\gamma^{-1} << \tau_r$. In actual machines there are always enough nonlinearities and intrinsic noise to cause an intrinisic frequency jitter of the order of or more than $\Delta\omega/N$. We then have a smooth smeared out frequency spectrum with arbitrarily small frequency spacing between particles as a function of time and a kinetic description becomes valid for arbitrarily large times. Moreover, overlapping resonances arising from nonlinear dependence of frequencies on amplitudes guarantees an almost stochastic single particle motion, according to the Chirikov criterion, in higher order islands corresponding to high resonances involved in a large bandwidth system. Thus under the assumption that the feedback loop does not induce any collective instabilities in the beam or that the conditions are such that coherent oscillations are damped or suppressed by a rather large frequency spread, a kinetic description is expected to be valid for times long enough so that $t \gg (\delta_{coh})^{-1}$ and $t \gg \tau_{mix}$ where $(\delta_{coh})^{-1}$ is the damping time-scale of coherent oscillations and τ_{mix} is the characteristic mixing time as introduced in Section 4.7.

We set the stage by considering a 6N-dimensional ensemble space whose elements are vectors $\left(\{\underset{\sim}{I}^i\}_{i=1,2,\ldots,N};\{\underset{\sim}{\psi}^i\}_{i=1,\ldots,N}\right) \equiv \left(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i;\ldots\underset{\sim}{I}^N,\underset{\sim}{\psi}^N\right)$ as defined in Section 4.1. Each vector represents one whole system of N particles each with 6-dimensional coordinates $(\underset{\sim}{I}^i,\underset{\sim}{\psi}^i)$ in action and angle variables for the two transverse betatron degrees of freedom and the longitudinal coasting or synchrotron oscillation degree of freedom. We consider the ensemble distribution $D\left(\underset{\sim}{I}^1,\underset{\sim}{\psi}^1;\ldots\underset{\sim}{I}^N,\underset{\sim}{\psi}^N\right) \equiv D\left[\{\underset{\sim}{I}^i\}_{i=1,\ldots,N};\{\underset{\sim}{\psi}^i\}_{i=1,\ldots,N}\right]$ describing a collection of these N-particle systems each with different initial conditions, normalized so that

$$\int d\Gamma_N\, D\left[\{\underset{\sim}{I}^i\}_{i=1,\ldots,N};\ \{\underset{\sim}{\psi}^i\}_{i=1,\ldots,N}\right] = 1$$

where $d\Gamma_N$ is the 6N-dimensional phase-space volume element

$$d\Gamma_N = \left(d\underset{\sim}{I}^1 \, d\underset{\sim}{\psi}^1\right) \ldots \left(d\underset{\sim}{I}^N \, d\underset{\sim}{\psi}^N\right)$$

$$= \prod_{i=1}^{N} \left(d\underset{\sim}{I}^i \, d\underset{\sim}{\psi}^i\right)$$

$$= \prod_{i=1}^{N} \left(dI_x^i \, d\phi_x^i\right) \left(dI_z^i \, d\phi_z^i\right) \left(dJ^i \, d\psi^i\right)$$

$$= \prod_{i=1}^{N} \left(A_x^i \, dA_x^i \, d\phi_x^i\right) \left(A_z^i \, dA_z^i \, d\phi_z^i\right) \left(a^i \, da^i \, d\psi^i\right)$$

where we have used $I_{x,z} = 1/2 \, A^2_{x,z}$ and $J = 1/2 \, a_s^2$ as the betatron and synchrotron oscillation actions.

As introduced in Section 4.1, we have the following continuity equation expressing the conservation of probability or the number of ensemble systems:

$$\frac{\partial D}{\partial t} + \underset{\sim}{\nabla}^N \cdot \left[\underset{\sim}{u}^N D\right] = 0 \qquad (9.2.1)$$

where

$$\underset{\sim}{u}^N = \left(\dot{\underset{\sim}{I}}^1, \dot{\underset{\sim}{\psi}}^1, \ldots ; \dot{\underset{\sim}{I}}^N, \dot{\underset{\sim}{\psi}}^N\right)$$

and

$$\underset{\sim}{\nabla}^N \equiv \left(\underset{\sim}{\nabla}_{\underset{\sim}{I}^1}, \underset{\sim}{\nabla}_{\underset{\sim}{\psi}^1}, \ldots, \underset{\sim}{\nabla}_{\underset{\sim}{I}^N}, \underset{\sim}{\nabla}_{\underset{\sim}{\psi}^N}\right)$$

with notation:

$$\underset{\sim}{\nabla}_{\underset{\sim}{I}^i} \equiv \frac{\partial}{\partial \underset{\sim}{I}^i}$$

$$\underset{\sim}{\nabla}_{\underset{\sim}{\psi}^i} \equiv \frac{\partial}{\partial \underset{\sim}{\psi}^i}$$

Explicitly we can write (9.2.1) as:

$$\frac{\partial D}{\partial t} + \sum_{i=1}^{N} \left\{ \frac{\partial}{\partial \underset{\sim}{I}^i} \cdot \left[\dot{\underset{\sim}{I}}^i D\right] + \frac{\partial}{\partial \underset{\sim}{\psi}^i} \cdot \left[\dot{\underset{\sim}{\psi}}^i D\right] \right\} = 0 \qquad (9.2.2)$$

We have already discussed the distinction between the continuity Eq. (9.2.1) describing compressible flow in 6N-space for the dissipative, non-Hamiltonian stochastic cooling systems and the continuity equation in the form of Liouville's theorem describing incompressible flow in 6N-space for conservative Hamiltonian systems in Chapter 4.

We obtain reduced one-particle, two-particle,...,etc. distributions by integrating over the variables we do not wish to care about. Thus we define a one-particle distribution by

$$f_1(1;t) \equiv f_1\left(\underset{\sim}{I}^1,\underset{\sim}{\psi}^1;t\right) = \int d\Gamma_{N-1}\, D\left[\{\underset{\sim}{I}^i\}_{i=1,\ldots,N},\ \{\underset{\sim}{\psi}^i\}_{i=1,\ldots,N};t\right]$$

$$= \underbrace{\int\cdots\int}_{2(N-1)} \left(d\underset{\sim}{I}^2\, d\underset{\sim}{\psi}^2\right)\cdots\left(d\underset{\sim}{I}^N\, d\underset{\sim}{\psi}^N\right)\, D\left(\underset{\sim}{I}^1,\underset{\sim}{\psi}^1;\ldots;\underset{\sim}{I}^N,\underset{\sim}{\psi}^N;t\right)$$

and similarly a two-particle distribution by

$$f_2(1,2;t) \equiv f_2\left(\underset{\sim}{I}^1,\underset{\sim}{\psi}^1;\underset{\sim}{I}^2,\underset{\sim}{\psi}^2;t\right) = \int d\Gamma_{N-2}\, D\left[\{\underset{\sim}{I}^i\}_{i=1,\ldots,N},\{\underset{\sim}{\psi}^i\}_{i=1,\ldots,N};t\right]$$

$$= \underbrace{\int\cdots\int}_{2(N-2)} \left(d\underset{\sim}{I}^3\, d\underset{\sim}{\psi}^3\right)\cdots\left(d\underset{\sim}{I}^N\, d\underset{\sim}{\psi}^N\right)\, D\left(\underset{\sim}{I}^1,\underset{\sim}{\psi}^1;\ldots\underset{\sim}{I}^N\underset{\sim}{\psi}^N;t\right)$$

and so on.

We can now start integrating Eq. (9.2.2) over $2(N-1)$ or $2(N-2)$ particle variables to yield equations for one and two particle distribution functions.

Thus integrating (9.2.2) over particles $(2,\ldots,N)$ we get

$$\frac{\partial f_1}{\partial t} = -\int \left(d\underset{\sim}{\psi}_2\, d\underset{\sim}{I}_2\right)\cdots\left(d\underset{\sim}{\psi}_N\, d\underset{\sim}{I}_N\right)\left[\left\{\frac{\partial}{\partial\underset{\sim}{\psi}_1}\cdot\left[\dot{\underset{\sim}{\psi}}_1\, D\right] + \frac{\partial}{\partial\underset{\sim}{I}_1}\cdot\left[\dot{\underset{\sim}{I}}_1\, D\right]\right\}\right.$$

$$\left. + \sum_{j=2}^{N}\left\{\frac{\partial}{\partial\underset{\sim}{\psi}_j}\cdot\left[\dot{\underset{\sim}{\psi}}_j\, D\right] + \frac{\partial}{\partial\underset{\sim}{I}_j}\cdot\left[\dot{\underset{\sim}{I}}_j\, D\right]\right\}\right]$$

The second term on the right-hand side may be represented as $\int \underset{\sim}{\nabla}^{N-1} \cdot [\underset{\sim}{u}^{N-1} D]$ $d\Gamma_{N-1}$ and converts to a surface integral of $(\underset{\sim}{u}^{N-1} D)$ on the $[6(N-1)-1]$ dimensional surface of the $6(N-1)$ dimensional space of Γ^{N-1}, by Stoke's theorem. Consequently this term vanishes because of boundary conditions on $(\underset{\sim}{u}^{N-1} D)$ which must vanish at infinity. So we have

$$\frac{\partial f_1}{\partial t} = -\int \left(d\psi_2\, dI_2\right)\ldots\left(d\psi_N\, dI_N\right)\left[\frac{\partial}{\partial \psi_1}\cdot\left(\dot{\psi}_1 D\right) + \frac{\partial}{\partial I_1}\cdot\left(\dot{I}_1 D\right)\right]$$

$$= -\frac{\partial}{\partial \psi_1}\cdot\left[\int \left(d\psi_2\, dI_2\right)\ldots\left(d\psi_N\, dI_N\right)\left[\dot{\psi}_1 D\right]\right]$$

$$-\frac{\partial}{\partial I_1}\cdot\left[\int \left(d\psi_2\, dI_2\right)\ldots\left(d\psi_N\, dI_N\right)\left[\dot{I}_1 D\right]\right] \tag{9.2.3}$$

With cooling dynamics of the form given most generally by (4.3.17) as

$$\dot{I}^i = \sum_{j=1}^{N} G(i,j) = G(i,i) + \sum_{j\neq i}^{N} G(i,j) \tag{9.2.4}$$

and

$$\dot{\psi}^i = \omega^i + \sum_{j=1}^{N} H(i,j) = \omega^i + H(i,i) + \sum_{\substack{j\neq i\\ =1}}^{N} H(i,j)$$

and the Hamiltonian flow condition

$$\frac{\partial}{\partial I^i}\cdot\left[\dot{I}^i - G(i,i)\right] = -\frac{\partial}{\partial \psi^i}\cdot\left[\dot{\psi}^i - H(i,i)\right] \tag{9.2.5}$$

given by (4.3.26) and under the usual symmetry assumption for D under the interchange of particle indices we obtain:

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{\omega}_1 \cdot \frac{\partial f_1}{\partial \underset{\sim}{\psi}_1} + (N-1) \int \left(d\underset{\sim}{\psi}_2 \, d\underset{\sim}{I}_2\right) \underset{\sim}{G}(1,2) \cdot \frac{\partial f_2(1,2;t)}{\partial \underset{\sim}{I}_1}$$

$$+ (N-1) \int \left(d\underset{\sim}{\psi}_2 \, d\underset{\sim}{I}_2\right) \underset{\sim}{H}(1,2) \cdot \frac{\partial f_2(1,2;t)}{\partial \underset{\sim}{\psi}_1}$$

$$+ \left\{ \frac{\partial}{\partial \underset{\sim}{I}_1} \cdot \left[\underset{\sim}{G}(1,1)\, f_1\right] + \frac{\partial}{\partial \underset{\sim}{\psi}_1} \cdot \left[\underset{\sim}{H}(1,1)\, f_1\right] \right\} = 0 \qquad (9.2.6)$$

The last term within the curly brackets on the left-hand side is the addition to the usual kinetic equation without self-interaction and expresses the dissipative non-conservative flow aside from the Liouvillian part of the flow. It is already of the form of the frictional part of a Fokker-Planck equation for transport and induces compression of phase-space. The integrals on the left-hand side describes interaction with other beam particles and includes the "usual" Vlasov average field and correlation effects that describe shielding or signal suppression through beam feedback and diffusion due to noise from other particles. Both these effects tend to decrease the phase-space density and suppress overall cooling. Both terms together may be interpreted as the divergence of a 'particle flux'. Note that the partial derivatives under the integrals act on f_2 alone -- the partial derivatives on $\underset{\sim}{G}$ and $\underset{\sim}{H}$ combine, canceling each other by the Hamiltonian flow condition (9.2.5).

Similarly, integrating Eq. (9.2.2) over particles $(3,\ldots,N)$ and using the same cooling dynamics as (9.2.4) and (9.2.5) and the symmetry assumption for D, we obtain:

$$\frac{\partial f_2}{\partial t} + \underset{\sim}{\omega}_1 \cdot \frac{\partial f_2}{\partial \underset{\sim}{\psi}_1} + \underset{\sim}{\omega}_2 \cdot \frac{\partial f_2}{\partial \underset{\sim}{\psi}_2} + (N-2) \int \left(d\underset{\sim}{\psi}_3 \, d\underset{\sim}{I}_3\right) \left[\underset{\sim}{G}(1,3) \cdot \frac{\partial f_3}{\partial \underset{\sim}{I}_1} + \underset{\sim}{G}(2,3) \cdot \frac{\partial f_3}{\partial \underset{\sim}{I}_2}\right.$$

$$\left. + \underset{\sim}{H}(1,3) \cdot \frac{\partial f_3}{\partial \underset{\sim}{\psi}_1} + \underset{\sim}{H}(2,3) \cdot \frac{\partial f_3}{\partial \underset{\sim}{\psi}_2}\right]$$

$$+ \underset{\sim}{G}(1,2) \cdot \frac{\partial f_2}{\partial \underset{\sim}{I}_1} + \underset{\sim}{G}(2,1) \cdot \frac{\partial f_2}{\partial \underset{\sim}{I}_2} + \underset{\sim}{H}(1,2) \cdot \frac{\partial f_2}{\partial \underset{\sim}{\psi}_1} + \underset{\sim}{H}(2,1) \cdot \frac{\partial f_2}{\partial \underset{\sim}{\psi}_2}$$

$$+ \left\{ \frac{\partial}{\partial \underset{\sim}{I}_1} \cdot \left[\underset{\sim}{G}(1,1)\, f_2\right] + \frac{\partial}{\partial \underset{\sim}{I}_2} \cdot \left[\underset{\sim}{G}(2,2)\, f_2\right] + \frac{\partial}{\partial \underset{\sim}{\psi}_1} \cdot \left[\underset{\sim}{H}(1,1)\, f_2\right] + \frac{\partial}{\partial \underset{\sim}{\psi}_2} \cdot \left[\underset{\sim}{H}(2,2)\, f_2\right] \right\}$$

$$= 0 \qquad (9.2.7)$$

where $f_3 = f_3(1,2,3;t)$ is the three particle distribution. Again the terms within the curly brackets represent contributions from the dissipative, nonconservative self-interaction part.

As expected, an infinite hierarchy of relations between the reduced distributions $f_n(1,2,\ldots,n;t)$ is developing. At this point some approximation is needed to terminate the sequence.

First we disentangle the totally and irreducibly correlated or connected parts of the distributions by writing the following "cluster decomposition"

$$
\begin{aligned}
f_1(1;t) &= f(1;t) \\
f_2(1,2;t) &= f(1;t)\, f(2;t) + g(1,2;t) \\
f_3(1,2,3;t) &= f(1;t)\, f(2;t)\, f(3;t) + f(1;t)\, g(2,3;t) \\
&\quad + f(2;t)\, g(3,1;t) + f(3;t)\, g(1,2;t) \\
&\quad + h(1,2,3;t) \qquad (9.2.8) \\
&\vdots \\
&\text{etc.}
\end{aligned}
$$

We note the similarity of this decomposition to the decomposition of moments into cumulants given by (9.1.16) in Section 9.1.

To truncate the hierarchy of equations beyond f_2, we now assume that correlation effects are small and in particular $h(1,2,3;t) \approx 0$ but non-negligible $g(1,2;t) \neq 0$, i.e. three-body correlations are small compared to two-body correlations. We also assume $N \approx (N-1) \approx (N-2)$ for large N. With these assumptions Eqs. (9.2.6) and (9.2.7) yield

$$
\begin{aligned}
&\frac{\partial f(1;t)}{\partial t} + \boldsymbol{\omega}_1 \cdot \frac{\partial f(1;t)}{\partial \boldsymbol{\psi}_1} + N \int d\Gamma_2\, \mathbf{G}(1,2) \cdot \frac{\partial f(1;t)}{\partial \mathbf{I}_1} f(2;t) + N \int d\Gamma_2\, \mathbf{H}(1,2) \cdot \frac{\partial f(1;t)}{\partial \boldsymbol{\psi}_1} f(2;t) \\
&\quad + N \int d\Gamma_2\, \mathbf{G}(1,2) \cdot \frac{\partial g(1,2;t)}{\partial \mathbf{I}_1} + N \int d\Gamma_2\, \mathbf{H}(1,2) \cdot \frac{\partial g(1,2;t)}{\partial \boldsymbol{\psi}_1} \\
&\quad + \left\{ \frac{\partial}{\partial \mathbf{I}_1} \cdot \Big[\mathbf{G}(1,1)\, f(1;t) \Big] + \frac{\partial}{\partial \boldsymbol{\psi}_1} \cdot \Big[\mathbf{H}(1,1)\, f(1;t) \Big] \right\} = 0 \qquad (9.2.9)
\end{aligned}
$$

where $d\Gamma_2 = (d\underline{\psi}_2 \, d\underline{I}_2)$ and

$$\frac{\partial g(1,2;t)}{\partial t} + \omega_1 \cdot \frac{\partial g(1,2;t)}{\partial \underline{\psi}_1} + \omega_2 \cdot \frac{\partial g(1,2;t)}{\partial \underline{\psi}_2}$$

$$+ N \frac{\partial g(1,2;t)}{\partial \underline{I}_1} \cdot \int d\Gamma_3 \, \underline{G}(1,3) \, f(3;t) + N \frac{\partial g(1,2;t)}{\partial \underline{I}_2} \cdot \int d\Gamma_3 \, \underline{G}(2,3) \, f(3;t)$$

$$+ N \frac{\partial g(1,2;t)}{\partial \underline{\psi}_1} \cdot \int d\Gamma_3 \, \underline{H}(1,3) \, f(3;t) + N \frac{\partial g(1,2;t)}{\partial \underline{\psi}_2} \cdot \int d\Gamma_3 \, \underline{H}(2,3) \, f(3;t)$$

$$+ N \frac{\partial f(1;t)}{\partial \underline{I}_1} \cdot \int d\Gamma_3 \, \underline{G}(1,3) \, g(2,3;t) + N \frac{\partial f(2;t)}{\partial \underline{I}_2} \cdot \int d\Gamma_3 \, \underline{G}(2,3) \, g(3;1;t)$$

$$+ N \frac{\partial f(1;t)}{\partial \underline{\psi}_1} \cdot \int d\Gamma_3 \, \underline{H}(1,3) \, g(2,3;t) + N \frac{\partial f(2;t)}{\partial \underline{\psi}_2} \cdot \int d\Gamma_3 \, \underline{H}(2,3) \, g(3,1;t)$$

$$+ \left[\!\left[\underline{G}(1,2) \cdot \frac{\partial f(1;t)}{\partial \underline{I}_1} f(2;t) + \underline{G}(2,1) \cdot \frac{\partial f(2;t)}{\partial \underline{I}_2} f(1;t) + \underline{H}(1,2) \cdot \frac{\partial f(1;t)}{\partial \underline{\psi}_1} f(2;t) \right.\right.$$

$$\left.\left. + \underline{H}(2,1) \cdot \frac{\partial f(2;t)}{\partial \underline{\psi}_2} f(1,t) \right]\!\right] \text{①}$$

$$+ \left[\!\left[\underline{G}(1,2) \cdot \frac{\partial g(1,2;t)}{\partial \underline{I}_1} + \underline{G}(2,1) \cdot \frac{\partial g(1,2;t)}{\partial \underline{I}_2} + \underline{H}(1,2) \cdot \frac{\partial g(1,2;t)}{\partial \underline{\psi}_1} + \underline{H}(2,1) \cdot \frac{\partial g(1,2;t)}{\partial \underline{\psi}_2} \right]\!\right] \text{②}$$

$$+ \left[\!\left[\frac{\partial}{\partial \underline{I}_1} \cdot \left[\underline{G}(1,1) \, g(1,2;t) \right] + \frac{\partial}{\partial \underline{I}_2} \cdot \left[\underline{G}(2,2) \, g(1,2;t) \right] + \frac{\partial}{\partial \underline{\psi}_1} \cdot \left[\underline{H}(1,1) \, g(1,2;t) \right] \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial \underline{\psi}_2} \left[\underline{H}(2,2) \, g(1,2;t) \right] \right]\!\right] \text{③}$$

$$= 0 \qquad (9.2.10)$$

where $d\Gamma_3 = (d\psi_3\ dI_3)$

Again the cooling or damping of phase-space due to dissipative interaction appears in the last term within curly brackets on the left-hand side of Eq. (9.2.9). The terms labelled (3) in Eq. (9.2.10) are an addition from the dissipative, nonconservative non-Liouvillian flow and are of the same order as terms labelled (2).

All the terms labelled under (2) and 3 are of the form $\frac{\partial}{\partial \underset{\sim}{I}} \cdot (\underset{\sim}{G} g)$ or $\frac{\partial}{\partial \psi} \cdot (\underset{\sim}{H} g)$. In general we assume the following hierarchy of strengths of correlations

$$\cdots \quad \frac{h(1,2,3)}{g(1,2)} \sim \frac{g(1,2)}{f(1)\ f(2)} \sim \frac{\partial}{\partial \underset{\sim}{I}} \cdot \underset{\sim}{G}(1,2) \sim \varepsilon \ll 1 \ .$$

Then

$$g \sim \varepsilon f \cdot f$$

and

$$h \sim \varepsilon g \sim \varepsilon^2 f \cdot f$$

At this level of approximation then, we drop terms labelled (2) since they are second order relative to terms labelled (1) which are of the form $\frac{\partial}{\partial \underset{\sim}{I}} (\underset{\sim}{G} ff) \sim \varepsilon ff \sim g$. Terms labelled (2) are of order (εg) and so of same order as h. We likewise drop terms labelled (3) from our analysis. The integrals in Eq. (9.2.10) are of order $(N\ \varepsilon g) = N \cdot h$. Since N is large, we cannot in general consider them as negligible. With these approximations Eq. (9.2.10) is formally identical, except for the explicit form of interaction $\underset{\sim}{G}(i,j)$, to the usual kinetic equatons for two-body correlations obtained in the Lenard-Balescu analysis of plasma physics ([53], [57]). Non-Liouvillian damping appears only in the last term in Eq. (9.2.9).

In Eq. (9.2.10) the terms labelled (1) are the direct effect of other beam particles, i.e. the Schottky noise. The last four integrals on the left-hand side of (9.2.10) describe the suppression of both the coherent cooling rate and Schottky noise diffusion through collective correlations introduced by the feedback system. The $\underset{\sim}{\omega}_1 \cdot \frac{\partial g}{\partial \underset{\sim}{\psi}_1}$ and $\underset{\sim}{\omega}_2 \cdot \frac{\partial g}{\partial \underset{\sim}{\psi}_2}$ terms describe the effect of mixing or relative phase slippage between particles through the frequency spread in the beam.

In Eq. (9.2.9) the last two integrals on the left-hand side describe Schottky noise and correlative shielding through feedback effects. The first two integrals on the left-hand side of (9.2.9) are Vlasov-like expressions which vanish for stochastic cooling systems. Amplifier noise may be added with the appearance of the usual noise term in (9.2.9) and an additional term in (9.2.10) describing shielding of the noise signal through the beam feedback.

One final comment is in order before proceeding further. For plasmas with long range forces, the assumption is usually made that the n-particle correlation effects vanish as the $(n-1)^{\underline{th}}$ power of the ratio of interaction energy to thermal energy. In the context of stochastic cooling, the corresponding ratio is the strength of the correction relative to the frequency spread in the beam. Frequency spread is a measure of the temperature or thermal energy of the beam in a frame where the beam is macroscopically stationary.

We now use the following Fourier series representation of f, g and G in the periodic angle variables $\underset{\sim}{\psi}$ (period 2π) as follows:

$$f(1;t) = f\left(\underset{\sim}{I}_1,\underset{\sim}{\psi}_1;t\right) = \sum_{\underset{\sim}{n}_1} f_{\underset{\sim}{n}_1}\left(\underset{\sim}{I}_1;t\right) e^{i\underset{\sim}{n}_1\cdot\underset{\sim}{\psi}_1} \tag{9.11a}$$

$$g(1,2;t) = g\left(\underset{\sim}{I}_1,\underset{\sim}{\psi}_i;\underset{\sim}{I}_2,\underset{\sim}{\psi}_2;t\right) = \sum_{\underset{\sim}{n}_1}\sum_{\underset{\sim}{n}_2} g_{\underset{\sim}{n}_1\underset{\sim}{n}_2}\left(\underset{\sim}{I}_1,\underset{\sim}{I}_2;t\right) e^{i(\underset{\sim}{n}_1\cdot\underset{\sim}{\psi}_1+\underset{\sim}{n}_2\cdot\underset{\sim}{\psi}_2)} \tag{9.11b}$$

and

$$\underset{\sim}{G}(1,2) = \underset{\sim}{G}\left(\underset{\sim}{I}_1,\underset{\sim}{\psi}_1;\underset{\sim}{I}_2,\underset{\sim}{\psi}_2\right) = \sum_{\underset{\sim}{n}_1}\sum_{\underset{\sim}{n}_2} \underset{\sim}{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}\left(\underset{\sim}{I}_1,\underset{\sim}{I}_2\right) e^{i(\underset{\sim}{n}_1\cdot\underset{\sim}{\psi}_1+\underset{\sim}{n}_2\cdot\underset{\sim}{\psi}_2)} \tag{9.2.11c}$$

We are looking for the time-evolution of the angle-independent distribution $f_0(\underset{\sim}{I};t)$. Using the Fourier series representations given by (9.2.11) in Eq. (9.2.9) and solving for the time evolution of $f_0(\underset{\sim}{I};t)$ (setting $\underset{\sim}{n}_1 = 0$) by harmonic balance, we find

$$\frac{\partial f_0(\underset{\sim}{I};t)}{\partial t} + \frac{\partial}{\partial \underset{\sim}{I}}\cdot\sum_{\underset{\sim}{n}}\left[G_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I})\ f_0(\underset{\sim}{I};t)\right] = -\frac{\partial}{\partial\underset{\sim}{I}}\cdot\left[\sum_{\underset{\sim}{n}} \underset{\sim}{R}^*_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I})\right] \tag{9.2.12}$$

where $\underset{\sim}{R}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(\underset{\sim}{I}_1,\underset{\sim}{I}_2)$ is defined by

$$\underset{\sim}{R}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(\underset{\sim}{I}_1,\underset{\sim}{I}_2) = N\sum_{\underset{\sim}{n}_3}\int(d\underset{\sim}{I}_3)\ \underset{\sim}{G}^*_{\underset{\sim}{n}_1\underset{\sim}{n}_3}(\underset{\sim}{I}_1,\underset{\sim}{I}_3)\ g_{\underset{\sim}{n}_2\underset{\sim}{n}_3}(\underset{\sim}{I}_2,\underset{\sim}{I}_3;t) \tag{9.2.13}$$

Similarly using the Fourier series representation (9.2.11) in Eq. (9.2.10) for the 2-body correlation $g(1,2;t)$ and Laplace transforming in time and assuming that the one-particle distribution remains a constant during the rapid time-development of $g(1,2;t)$ it is shown in Appendix D that the quantity $\mathbf{R}_{\mathbf{n}_1\mathbf{n}_2}$ appearing in (9.2.12) and defined by (9.2.13) satisfies the following integral equation:

$$
\begin{aligned}
R_{\mathbf{n}_2\mathbf{n}_1}(\mathbf{I}_2,\mathbf{I}_1) = -\pi N \sum_{\mathbf{n}_3} \int d\mathbf{I}_3\, \delta_+\left[\mathbf{n}_3\cdot\boldsymbol{\omega}_3-\mathbf{n}_1\cdot\boldsymbol{\omega}_1\right] \mathbf{G}^*_{\mathbf{n}_2\mathbf{n}_3}(\mathbf{I}_2,\mathbf{I}_3) \\
\otimes \left[\mathbf{G}_{\mathbf{n}_1\mathbf{n}_3}(\mathbf{I}_1\mathbf{I}_3) \cdot \frac{\partial f_0(1,t)}{\partial \mathbf{I}_1} f_0(3;t) - \mathbf{G}^*_{\mathbf{n}_3\mathbf{n}_1}(\mathbf{I}_3,\mathbf{I}_1) \cdot \frac{\partial f_0(3;t)}{\partial \mathbf{I}_3} f_0(1;t) \right. \\
\left. + \frac{\partial f_0(1;t)}{\partial \mathbf{I}_1} \cdot \mathbf{R}^*_{\mathbf{n}_1\mathbf{n}_3}(\mathbf{I}_1,\mathbf{I}_3) - \frac{\partial f_0(3;t)}{\partial \mathbf{I}_3} \cdot \mathbf{R}_{\mathbf{n}_3\mathbf{n}_1}(\mathbf{I}_3,\mathbf{I}_1) \right]
\end{aligned}
\tag{9.2.14}
$$

where one has again used phase-averaging to get relations involving $f_0(\mathbf{I};t)$ only. Here we have used the notation

$$
\pi\, \delta_+(x) = \pi\, \delta(x) + P\left(\frac{1}{x}\right) = \lim_{\eta\to 0^+} \frac{1}{[\eta + i x]}
$$

where P denotes the principal value part.

The first two terms on the right hand side of Eq. (9.2.14) contains the contributions from two-body scattering events due to the pair-interactions induced by the feedback system. Of these two terms, the first one represents the direct diffusion of particle 1, due to noise from <u>or</u> interaction with all the other particles. The second one represents an 'induced polarization' effect, i.e. the polarization of all the other particles 3 in the beam due to particle 1. This 'induced polarization' acts back on particle 1 and causes an 'induced friction' <u>or</u> drag force on particle 1. In the context of plasma physics, this term usually determines the 'stopping power' of a test particle traversing a plasma, which gets polarized by the test particle and drags it back. Accordingly this term redistributes itself and modifies the coherent cooling term in (9.2.12) (the second term on the left-hand side) by polarization factors. We note that this term, when substituted in (9.2.12) does indeed have the form of a friction term as follows:

$$-\frac{\partial}{\partial \underset{\sim}{I}_1} \cdot \left\{ \left[\pi N \sum_{\underset{\sim}{n}_3} \int d\underset{\sim}{I}_3 \; \delta_+\left[\underset{\sim}{n}_3\cdot\underset{\sim}{\omega}_3 - \underset{\sim}{n}_1\cdot\underset{\sim}{\omega}_1\right] \underset{\sim}{G}^*_{\underset{\sim}{n}_1\underset{\sim}{n}_3}(\underset{\sim}{I}_1,\underset{\sim}{I}_3)\; \underset{\sim}{G}^*_{\underset{\sim}{n}_3\underset{\sim}{n}_1}(\underset{\sim}{I}_3,\underset{\sim}{I}_1) \cdot \frac{\partial f_o}{\partial \underset{\sim}{I}_3} \right] \otimes f_o(\underset{\sim}{I}_1;t) \right\}$$

$$= -\frac{\partial}{\partial \underset{\sim}{I}_1} \cdot \left[\underset{\sim}{K}(\underset{\sim}{I}_1)\; f_o(\underset{\sim}{I}_1;t)\right]$$

The last two terms in Eq. (9.2.14) include the effect of signal suppression or shielding due to collective correlations propagating in the beam. In the plasma physics context this is known as the dynamic screening effect. These terms distort the fluctuation spectrum of the beam from their uncorrelated Schottky noise spectrum values.

In principle then if one can solve the integral equation (9.2.14) for $\underset{\sim}{R}_{\underset{\sim}{n}_2\underset{\sim}{n}_1}$, then substitution into Eq. (9.2.12) leads to the general transport equation for the cooling beam, including the collective signal suppression effect, the noise diffusion effect and induced polarization feedback from all the other particles.

To get a quick look at the resulting transport equation when the collective signal suppression and induced polarization effects are small, we ignore the last three terms in Eq. (9.2.14) and substitute in (9.2.12). The resulting equation is as follows:

$$\frac{\partial f_o(\underset{\sim}{I};t)}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[\sum_{\underset{\sim}{n}} \underset{\sim}{G}_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I})\; f_o(\underset{\sim}{I};t)\right]$$

$$+ \pi N \frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[\sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}' \; \delta[\underset{\sim}{n}'\cdot\underset{\sim}{\omega}' - \underset{\sim}{n}\cdot\underset{\sim}{\omega}]\; \underset{\sim}{G}_{-\underset{\sim}{n},\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\; \underset{\sim}{G}^*_{-\underset{\sim}{n},\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\; f_o(\underset{\sim}{I}')\right]$$

$$\cdot \frac{\partial f_o(\underset{\sim}{I};t)}{\partial \underset{\sim}{I}} \tag{9.2.15}$$

This equation thus has the form of a Fokker-Planck equation

$$\frac{\partial f_o(\underset{\sim}{I};t)}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[\underset{\sim}{F}(\underset{\sim}{I})\; f_o(\underset{\sim}{I};t)\right] + \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{I}} \cdot \left[\underset{\approx}{D}(\underset{\sim}{I}) \cdot \frac{\partial f_o(\underset{\sim}{I},t)}{\partial \underset{\sim}{I}}\right]$$

where

$$\underset{\sim}{E}(\underset{\sim}{I}) = \sum_{\underset{\sim}{n}} \underset{\sim}{G}_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}) \tag{9.2.16}$$

and

$$\underset{\approx}{D}(\underset{\sim}{I}) = (2\pi) \cdot N \cdot \sum_{\underset{\sim}{n}} \sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}' \; \delta[\underset{\sim}{n}'\cdot\underset{\sim}{\omega}'-\underset{\sim}{n}\cdot\underset{\sim}{\omega}] \left[\underset{\sim}{G}_{-\underset{\sim}{n},\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \; \underset{\sim}{G}^{*}_{-\underset{\sim}{n},\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\right] f_o(\underset{\sim}{I}') \tag{9.2.17}$$

in agreement with the results (9.1.36) and (9.1.37) derived from Fluctuation theory Note that (9.2.16) and (9.2.17) are general and valid for overlapping synchrotron bands but does not include signal suppression.

We now include the effects of correlations and signal suppression by keeping all the terms on the right-hand side of (9.2.14) and solving it.

The situation is complicated for general vector interactions $\underset{\sim}{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}$ which gives rise to tensorial correlation properties. The disentanglement of collective propagation of $\underset{\sim}{R}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}$, governed by a tensor and given by (9.2.14) and its ultimate inversion in order to be a useful input on the right-hand side of (9.2.12), is extremely difficult in general. We therefore consider a one dimensional model now where $\underset{\sim}{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}$ is a scalar function describing cooling in any one of the three phase-planes. The important physics of suppression of both cooling and noise diffusion terms by collective screening effect is retained.

The quantity $R_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I})$, on the right-hand side of (9.2.12) is given by (9.2.14) as:

$$\begin{aligned} R_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}) = -\,\pi N \sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}' \; \delta_{+}[\underset{\sim}{n}'\cdot\underset{\sim}{\omega}'-\underset{\sim}{n}\cdot\underset{\sim}{\omega}] \, G^{*}_{-\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \\ \otimes \Bigg[G_{-\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') \frac{\partial f_o(\underset{\sim}{I})}{\partial I} f_o(\underset{\sim}{I}') - G^{*}_{-\underset{\sim}{n}'\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I}) \frac{\partial f_o(\underset{\sim}{I}')}{\partial I'} f_o(\underset{\sim}{I};t) \\ + \frac{\partial f_o(\underset{\sim}{I})}{\partial I} R^{*}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}') - \frac{\partial f_o(\underset{\sim}{I}')}{\partial I'} R_{\underset{\sim}{n}'\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I}) \Bigg] \end{aligned} \tag{9.2.18}$$

where $\underset{\sim}{I}$ is the general action in three degrees of freedom of the particles while I is the action in the degree of freedom in which cooling is taking place.

To simplify the problem further, we consider the region of non-overlapping bands, i.e. we ignore overlapping resonances where $\underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}') = \underset{\sim}{n}\cdot\underset{\sim}{\omega}(\underset{\sim}{I})$ is satisfied with $\underset{\sim}{n} \neq \underset{\sim}{n}'$. In this region then two particles can resonate with each other only if their frequencies $\underset{\sim}{\omega}'(\underset{\sim}{I}')$ and $\underset{\sim}{\omega}(\underset{\sim}{I})$ are the same i.e. $\underset{\sim}{I}' \sim \underset{\sim}{I}$. For such non-overlapping resonances, only the "diagonal" harmonics of the interaction $G_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}')$ with $\underset{\sim}{n}' \equiv \underset{\sim}{n}$ contribute in (9.2.18). The sum $\sum_{\underset{\sim}{n}'}$ thus drops off from the right-hand side. Equation (9.2.18) then becomes:

$$R_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}) = -\pi N \int d\underset{\sim}{I}' \ \delta_+\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right] G^*_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}')$$

$$\otimes \left[G_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}') \frac{\partial f_o(\underset{\sim}{I})}{\partial I} f_o(\underset{\sim}{I}') - G^*_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I}) \frac{\partial f_o(\underset{\sim}{I}')}{\partial I'} f_o(\underset{\sim}{I})\right.$$

$$\left. + \frac{\partial f_o(\underset{\sim}{I})}{\partial I} R^*_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}') - \frac{\partial f_o(\underset{\sim}{I}')}{\partial I'} R_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I})\right] \tag{9.2.19}$$

We now introduce the following quantities:

$$H_{\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}') = R^*_{\underset{\sim}{n}\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}') + G_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}') f_o(\underset{\sim}{I}') \tag{9.2.20}$$

and

$$\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I}) = 1 + \pi N \int d\underset{\sim}{I}' \ \delta_+\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right] G^*_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I}') \frac{\partial f(\underset{\sim}{I}')}{\partial I'} \tag{9.2.21}$$

We will see later that the quantity $\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I})$ so defined determines the signal suppression and plays a role analogous to that of the dielectric permittivity in a plasma.

For cooling of coasting beams, we can assume the separated variable form for the interaction, as introduced before:

$$G_{\underline{n}\underline{n}'}(\underline{I},\underline{I}') = K_{\underline{n}}(\underline{I})\; P_{\underline{n}'}(\underline{I}') \tag{9.2.22a}$$

With this form for the interaction, we note the following important property:

$$G^{*}_{\underline{n}\underline{n}'}(\underline{I},\underline{I}')\; R_{\underline{n}'\underline{n}}(\underline{I}',\underline{I}) =$$

$$= \left[K^{*}_{\underline{n}}(\underline{I})\; P^{*}_{\underline{n}'}(\underline{I}')\right] \cdot N \sum_{\underline{n}''} \int d\underline{I}''\; G^{*}_{\underline{n}'\underline{n}''}(\underline{I}',\underline{I}'')\; g_{\underline{n}\underline{n}''}(\underline{I},\underline{I}'')$$

$$= \left[K^{*}_{\underline{n}}(\underline{I})\; P^{*}_{\underline{n}'}(\underline{I}')\right] \cdot N \sum_{\underline{n}''} \int d\underline{I}'' \left[K^{*}_{\underline{n}'}(\underline{I}')\; P^{*}_{\underline{n}''}(\underline{I}'')\right] g_{\underline{n}\underline{n}''}(\underline{I},\underline{I}'')$$

$$= \left[K^{*}_{\underline{n}'}(\underline{I}')\; P^{*}_{\underline{n}'}(\underline{I}')\right] \cdot \left\{N \sum_{\underline{n}''} \int d\underline{I}'' \left[K^{*}_{\underline{n}}(\underline{I})\; P^{*}_{\underline{n}''}(\underline{I}'')\right] g_{\underline{n}\underline{n}''}(\underline{I},\underline{I}'')\right\}$$

The quantity within the curly brackets is just $R_{\underline{n}\underline{n}}(\underline{I},\underline{I})$. We thus have the important identity:

$$G^{*}_{\underline{n}\underline{n}'}(\underline{I},\underline{I}')\; R_{\underline{n}'\underline{n}}(\underline{I}',\underline{I}) = G^{*}_{\underline{n}'\underline{n}'}(\underline{I}',\underline{I}')\; R_{\underline{n}\underline{n}}(\underline{I},\underline{I}) \tag{9.2.22b}$$

Using (9.2.20), (9.2.21) and (9.2.22b), we find that (9.2.19) is equivalent to the following:

$$\epsilon_{\underline{n}}(\underline{I})\; H^{*}_{\underline{n}}(\underline{I},\underline{I}) = G_{-\underline{n},\underline{n}}(\underline{I},\underline{I})\; f_o(\underline{I}) - \pi N \frac{\partial f_o(\underline{I})}{\partial I} \int d\underline{I}'\; \delta_+\left[\underline{n}\cdot(\underline{\omega}'-\underline{\omega})\right] H_{\underline{n}}(\underline{I},\underline{I}')\; G^{*}_{-\underline{n},\underline{n}}(\underline{I},\underline{I}') \tag{9.2.23}$$

It is readily seen that an iterative solution for H with the second term on the right hand side assumed small is consistent with

$$H^{*}_{\underline{n}}(\underline{I},\underline{I}) = \frac{G_{-\underline{n},\underline{n}}(\underline{I},\underline{I})}{\epsilon_{\underline{n}}(\underline{I})}\, f_o(\underline{I}) - \pi N \frac{\partial f_o(\underline{I})}{\partial I} \int d\underline{I}'\; \delta_+\left[\underline{n}\cdot(\underline{\omega}'-\underline{\omega})\right] \frac{\left|G_{-\underline{n},\underline{n}}(\underline{I},\underline{I}')\right|^2}{\left|\epsilon_{\underline{n}}(\underline{I})\right|^2}\, f_o(\underline{I}') + \ldots \tag{9.2.24}$$

Substituting (9.2.24) in (9.2.12) and using (9.2.20), we get:

$$\frac{\partial f_o(\underset{\sim}{I},t)}{\partial t} = -\frac{\partial}{\partial I}\left[F(\underset{\sim}{I})\ f_o(\underset{\sim}{I};t)\right] + \frac{1}{2}\frac{\partial}{\partial I}\left[D(\underset{\sim}{I})\ \frac{\partial f_o(\underset{\sim}{I};t)}{\partial I}\right]$$

where:

$$F(\underset{\sim}{I}) = \sum_{\underset{\sim}{n}} \frac{G_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I})}{\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I})} \tag{9.2.25}$$

and

$$D(\underset{\sim}{I}) = (2\pi)\cdot N\sum_{\underset{\sim}{n}} \int d\underset{\sim}{I}'\ \delta\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right]\frac{\left|G_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}')\right|^2}{\left|\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I})\right|^2} f_o(\underset{\sim}{I}') \tag{9.2.26}$$

where $\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I})$ is given by (9.2.21).

The above results have been derived from an iterative solution and we have assumed approximate cancellation between different $\pm\underset{\sim}{n}$ of the principal value integral coming from

$$\pi\ \delta_+\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right] = \pi\ \delta\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right] + P\left\{\frac{1}{\left[\underset{\sim}{n}\cdot(\underset{\sim}{\omega}'-\underset{\sim}{\omega})\right]}\right\}$$

when the summation $\sum_{\underset{\sim}{n}}$ is performed in Eq. (9.2.12). The final solution obtained is however exact, as is known in plasma physics. We do not reiterate the rigorous proof here. Such a proof requires the application of Wiener-Hopf techniques for analytic functions in different half-planes in the complex plane to match along a common boundary by analytic continuation ([6],[57]).

Note that for cooling a system describable by one degree of freedom only, $\underset{\sim}{n}$, $\underset{\sim}{\omega}$ and $\underset{\sim}{I}$ become scalar quantities n, ω and I and one can then perform the δ-function integration in (9.2.26) explicitly giving:

$$D(I) = (2\pi)\cdot N\cdot\sum_n \frac{1}{|n|\ \left|\frac{d\omega(I)}{dI}\right|}\ \frac{\left|G_{-n,n}(I,I)\right|^2}{\left|\epsilon_n(I)\right|^2} f_o(I) \tag{9.2.27}$$

Such is the case for longitudinal cooling, for example, when one does not need to care about the transverse betatron oscillations of the beam.

In obtaining (9.2.25), (9.2.26), we have used the definitions (9.2.20) and (9.2.21) and the property (9.2.22b) based on the separated variable representation of $G_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')$ given by (9.2.22a). For bunched beams, such a representation as (9.2.22a) is not possible, instead we have (4.3.27) (e.g. (4.3.62)) where all the revolution harmonics couple together. Equation (9.2.22b) is then no longer valid. However, a solution can be obtained for bunched beams also if we throw away the principal value integral and retain contributions only from the pole term $\underset{\sim}{I}' = \underset{\sim}{I}$ $(\underset{\sim}{\omega}(\underset{\sim}{I}') = \underset{\sim}{\omega}(\underset{\sim}{I}))$ in (9.2.19). The δ_+ - function then simply becomes an ordinary δ-function. Replacing the definition (9.2.21) by

$$\epsilon_{\underset{\sim}{n}}(\underset{\sim}{I}) = 1 + \pi N \int d\underset{\sim}{I}' \; \delta\Big[\underset{\sim}{n} \cdot \big(\underset{\sim}{\omega}(\underset{\sim}{I}')-\underset{\sim}{\omega}(\underset{\sim}{I})\big)\Big] \; G^{*}_{-\underset{\sim}{n},\underset{\sim}{n}}(\underset{\sim}{I}',\underset{\sim}{I}') \frac{\partial f(\underset{\sim}{I}')}{\partial I'} \tag{9.2.28}$$

where δ_+-function in (9.2.21) is replaced by a δ-function, one can verify that Eqs. (9.2.25) and (9.2.26) still remain valid but with this new definition (9.2.28) of $\epsilon_{\underset{\sim}{n}}(I')$. We note that $\delta\big[\underset{\sim}{n} \cdot (\underset{\sim}{\omega}(I')-\underset{\sim}{\omega}(I))\big]$ is just $\dfrac{1}{|\mu| \left|\dfrac{d\omega_s(J)}{dJ}\right|}\,\delta(J'-J)$ for bunched beam cooling.

For transverse cooling $\underset{\sim}{n} \equiv (\mu,\pm 1)$ and for longitudinal cooling $\underset{\sim}{n} \equiv (\mu,0)$ and so we have the following signal suppression factors

$$\epsilon_{\mu,\pm 1}(J) = \epsilon_{\mu}^{(\pm)}(J), \qquad \epsilon_{\mu}(J)$$

to consider for transverse and longitudinal cooling. The kinetic theory without band-overlap thus automatically includes the signal suppression effect as manifested in the suppressed Friction and Diffusion coefficients in (9.2.25) and (9.2.26) and provides the explicit expression (9.2.28) for the suppression factor in the case of no band-overlap. This is a new result, not obtained from the Fluctuation theory in Section 9.1 and is an indication of the power of self-consistency inherent in a kinetic theoretic formulation.

It is mathematically difficult to obtain kinetic theoretic expressions for $F(\underset{\sim}{I})$ and $D(\underset{\sim}{I})$ for bunched beams in the situation of band-overlap including the suppression effect. We develop a general theory of signal suppression independently from Vlasov theory in Ch. 10. In Ch. 11 then we study how the coefficients $F(\underset{\sim}{I})$, $D(\underset{\sim}{I})$ get modified due to these suppression effects when there is synchrotron band overlap, by using a slightly modified fluctuation theory, as discussed in Section 9.1. We will also see that the general expression for signal suppression for overlapping bands derived in Ch. 10 reduces to the expression (9.2.28) for the special case of no band-overlap. An independent derivation of $\varepsilon_\mu^{(\pm)}(J)$ in Appendix E also provides an expression consistent with (9.2.28).

9.3 Beam Heating (Diffusion) Due to Amplifier Noise

Electronic components in the feedback loop generate intrinsic noise. For stochastic cooling systems, the preamplifier noise gets further amplified by the power amplifier and the kicker fields always carry these amplified noise components. Usually the noise can be regarded as a Gaussian thermal random noise, characterized by a temperature (kT). Such random noise causes diffusion of particles in phase-space and heats up the beam.

Let e(t) be the amplified transverse electric field noise at the kicker. Betatron motion subject to this noise satisfies

$$\ddot{x} + Q^2\omega^2 x = q \sum_{n=-\infty}^{+\infty} e(t)\, \delta\left(t - \tau(t) - \frac{\theta_k}{\omega_0} - nT_0\right)$$

$$= \mathcal{E}(t) \tag{9.3.1}$$

According to Section (5.2) the action-noise for the action $I = 1/2\, A^2$ of betatron oscillations $x = \sqrt{2I}\, \mathrm{Cos}[\phi(t-\tau(t))]$ is given by:

$$\dot{I} = -\frac{\sqrt{2I}}{Q\omega_0} \mathrm{Sin}\, \phi\Big(t-\tau(t)\Big)\, \mathcal{E}(t) = \eta(t) \tag{9.3.2}$$

Using betatron phase $\phi(t) = Q\omega_0 t + \phi(0)$ and expanding the periodic δ-function in (9.3.1) and using the identity (4.3.52), we get as in Chapter 5

$$\eta(t) = \frac{q\sqrt{2I}}{2Q}(i) \sum_n \sum_\mu \sum_{(\pm)} J_\mu\Big[(n\pm Q)a\Big]\, e^{i(n\pm Q)\omega_0 t}\, e^{i\mu\omega_s(a)t} \otimes e^{-in\theta_k + i\mu\psi(0) \pm i\phi(0)}\, e(t) \tag{9.3.3}$$

Auto-correlation of this action noise is given by

$$R(\tau) = \left\langle n(t)\ n(t-\tau)\right\rangle$$

where $<\ldots>$ denotes average over the noise ensemble. In addition we also have to average over the phase $\psi(t)$, $\phi(t)$ of the particle sampling the noise, in order to render the autocorrelation stationary. Performing these averages, one gets

$$R(\tau) = \frac{q^2 I}{2Q^2} \sum_n \sum_\mu \sum_{(\pm)} J_\mu^2\left[(n\pm Q)a\right] e^{-i[(n\pm Q)\omega_o + \mu\omega_s(a)]\tau} \left\langle e(t)\ e(t-\tau)\right\rangle \tag{9.3.4}$$

If $P^T(\Omega)$ is the power spectrum of noise $e(t)$ at the kicker defined by

$$P^T(\Omega) = \int_{-\infty}^{+\infty} d\tau \left\langle e(t)\ e(t-\tau)\right\rangle e^{-i\Omega\tau} \tag{9.3.5}$$

then

$$\tilde{R}(\Omega) = \int_{-\infty}^{+\infty} d\tau\ R(\tau)\ e^{-i\Omega\tau}$$

$$= \frac{q^2 I}{2Q^2} \sum_n \sum_\mu \sum_{(\pm)} J_\mu^2\left[(n\pm Q)a\right] P^T\left[\Omega + (n\pm Q)\omega_o + \mu\omega_s(a)\right] \tag{9.3.6}$$

According to Section (9.1) on fluctuation theory, the diffusion due to amplifier noise is thus given by a Diffusion Coefficient $D^T_{noise}(I)$ appearing in the Fokker-Planck equation (9.1.34) and given by:

$$D^T_{noise}(I) = \tilde{R}(\Omega)\Big|_{\Omega=0} = \frac{q^2 I}{2Q^2} \sum_n \sum_\mu \sum_{(\pm)} J_\mu^2\left[(n\pm Q)a\right] P^T\left[(n\pm Q)\omega_o + \mu\omega_s(a)\right] \tag{9.3.7}$$

for particles with fixed synchrotron action $J = 1/2\ a^2$ diffusing in betatron action I-space.

Thus diffusion due to amplifier noise for a particle of amplitude a is determined by the strength of the noise power spectrum at all the synchro-betatron bands $\Omega = (n \pm Q)\omega_0 + \mu\omega_s(a)$ of the diffusing particle multiplied by the square of the corresponding strength of the orbit integrals $J_\mu[(n \pm Q)a]$ at those bands.

Similarly if $v(t)$ is the longitudinal voltage noise at the kicker and if $P^L(\Omega)$ is the power spectrum of $v(t)$ defined by

$$P^L(\Omega) = \int_{-\infty}^{+\infty} d\tau \left\langle v(t)\ v(t-\tau) \right\rangle e^{-i\Omega\tau} \tag{9.3.8}$$

then the diffusion in synchrotron phase-space for a particle with action J of synchrotron motion, is described by a Diffusion coefficient $D^L_{noise}(J)$ given by

$$D^L_{noise}(J) = \frac{\left(q \frac{\omega_0}{2\pi}\right)^2}{[\omega_s(J)]^2} \sum_\mu \sum_n \left[\frac{\mu\ J_\mu(na)}{n}\right]^2 P^L\left[n\omega_0 + \mu\omega_s(a)\right] \tag{9.3.9}$$

where

$$J = \frac{1}{2} a^2 .$$

If collective signal suppression is important, it follows from the kinetic theory developed in Section (9.2) that the signal suppression factors $\epsilon_\mu(a)$ appear in the denominators of $D^{T,L}_{noise}$ (I or J) similar to the Schottky noise diffusion terms and the coefficients are then given by:

$$D^T_{noise}(I,J) = \frac{q^2 I}{2Q^2} \sum_\mu \sum_{(\pm)} \frac{\left\{\sum_{(n)} J^2_\mu\left[(n \pm Q)a\right] P^T[(n \pm Q)\omega_0 + \mu\omega_s(a)]\right\}}{\left|\epsilon^{T(\pm)}_\mu(a)\right|^2} \tag{9.3.10}$$

and

$$D^L_{noise}(J) = \frac{\left(q \frac{\omega_0}{2\pi}\right)^2}{[\omega_s(J)]^2} \sum_\mu \frac{\left\{\mu^2 \sum_n \left[\frac{J_\mu(na)}{n}\right]^2 P^L[n\omega_0 + \mu\omega_s(a)]\right\}}{\left|\epsilon^L_\mu(a)\right|^2} \tag{9.3.11}$$

where $\epsilon_\mu^{T(\pm)}(a)$ and $\epsilon_\mu^{L}(a)$ are given by expressions similar to (9.2.28) and written explicitly for the transverse and longitudinal cooling in Chs. 10, 13 and Appendix E. The amplifier noise acts as additional beam particles and gets shielded by the collective dynamics as experienced by a sampling particle, but does not enter into the $\epsilon(\Omega)$ factors, because noise does not introduce interparticle interactions that are responsible for collective screening.

Typically the noise $e(t)$ or $v(t)$ is the amplified version of the noise $r(t)$ from an equivalent resistor before the amplifier stage. If $K(t-t')$ is the transfer function of the amplifier then

$$e(t) = \int_{-\infty}^{+\infty} dt'\ K(t-t')\ r(t')$$

so that

$$\tilde{e}(\Omega) = \tilde{K}(\Omega)\ \tilde{r}(\Omega)\ .$$

and

$$\left\langle e(t)\ e(t-\tau)\right\rangle = \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt''\ K(t-t')\ K(t-\tau-t'')\ \left\langle r(t')\ r(t'')\right\rangle$$

and

$$P^{T}(\Omega) = \int_{-\infty} d\tau\ \left\langle e(t)\ e(t-\tau)\right\rangle\ e^{-i\Omega\tau}$$

$$= \left|K(\Omega)\right|^{2} S(\Omega)$$

where

$$S(\Omega) = \int_{-\infty}^{+\infty} d\tau\ \left\langle r(t)\ r(t-\tau)\right\rangle\ e^{-i\Omega\tau}$$

Typically for a resistor R one uses

$$S(\Omega) = 2kTR$$

i.e. a flat power spectrum depending on temperature and the resistance corresponding to white thermal noise. For such noise, we also have the property

$$\left\langle r(\Omega)\, r^*(\Omega')\right\rangle = S(\Omega)\ \delta(\Omega-\Omega')\ .$$

We will see later that for transverse cooling with dipole interaction, the noise diffusion determines the ultimate asymptotic level of squared amplitude or action for betatron motion.

9.4 The Time-Evolution of Mean Squared Betatron Amplitude for Linear Transverse Dipole Cooling

For transverse dipole cooling in any one transverse phase-plane of a bunch, we can write the friction and diffusion coefficients neglecting signal suppressions, as

$$F(I,J) = \sum_{(\pm)} \sum_{\substack{\mu \\ (-\infty)}}^{+\infty} G_{(\mu,\pm1),(-\mu,\mp1)}(I,J;\ I,J) \tag{9.4.1}$$

$$D(I,J) = (2\pi)\cdot N \cdot \sum_{(\pm)} \sum_{\mu}\sum_{\mu'}{}_{(-\infty)}^{(+\infty)} \iint dI'dJ'\ f(I',J')\ \left|G_{(\mu,\pm1)(\mu',\pm1)}(I,I';\ J,J')\right|^2$$

$$\delta\left[\mu\omega_s(J) + \mu'\omega_s(J')\right] \tag{9.4.2}$$

with μ,μ' being the synchrotron harmonics and (±1) the only betatron harmonics for dipole transverse cooling. I and J are the betatron and synchrotron action variables respectively. From (7.14), we have:

$$\left|G_{(\mu,\pm1),(\mu',\pm1)}(I,I';\ J,J')\right|^2 = \frac{I\ I'\,(q\omega_0)^4}{4(2\pi)^4\ Q^2\ \omega_0^2}\left|\tilde{g}^{(\pm)}_{\mu\mu'}(J,J')\right|^2 \tag{9.4.3}$$

where

$$\tilde{g}^{(\pm)}_{\mu\mu'}(J,J') = \sum_{m=-\infty}^{+\infty} \tilde{G}\left[(m\pm Q)\omega_0 + \mu\omega_s(J')\right]\ J_\mu\left[(m\pm Q)\sqrt{2J'}\right]\ J_{\mu'}\left(-m\sqrt{2J}\right) \tag{9.4.4}$$

and we have absorbed the phase factor $e^{im(\theta_p-\theta_k)}$ into the gain function $\tilde{G}$. We define a mean betatron action $\langle I\rangle(J)$ for particles with synchrotron action J as follows:

$$\int dI \cdot I \cdot f(I,J) = \langle I\rangle(J) \cdot f_0(J) \tag{9.4.5}$$

Then

$$D(I,J) = \frac{N(q^2\omega_0)^2}{4(2\pi)^3 Q^2} I \cdot \sum_{(\pm)} \sum_{\mu} \sum_{\mu'} \int dJ' \langle I\rangle(J') \left|\tilde{g}^{(\pm)}_{\mu\mu'}(J,J')\right|^2 f_0(J')$$

$$\delta\left[\mu\omega_s(J) + \mu'\omega_s(J')\right] \tag{9.4.6}$$

and

$$F(I,J) = \alpha(J) \cdot I$$

where

$$\alpha(J) = \frac{(q^2\omega_0)}{2(2\pi)^2 Q} \sum_{\mu=-\infty}^{+\infty} \sum_{(\pm)} \tilde{g}^{(\pm)}_{\mu,-\mu}(J,J) \tag{9.4.7}$$

Considerable simplification occurs when there is no synchrotron band overlap. We then get

$$D(I,J) = (2\pi) \cdot N \cdot \left[\frac{q^2\omega_0}{2(2\pi)^2 Q}\right]^2 I \cdot \langle I\rangle(J) \sum_{(\pm)} \sum_{\mu} \frac{f_0(J)}{|\mu| \left|\frac{d\omega_s(J)}{dJ}\right|} \left|\tilde{g}^{(\pm)}_{\mu,-\mu}(J,J)\right|^2$$

$$= D(J) \cdot I \tag{9.4.8}$$

where

$$D(J) = \beta(J) \cdot \langle I\rangle(J) \tag{9.4.9}$$

and

$$\beta(J) = (2\pi) N \left[\frac{q^2\omega_0}{2(2\pi)^2 Q}\right]^2 \sum_{(\pm)} \sum_{\mu=-\infty}^{\infty} \frac{f_0(J)}{|\mu| \left|\frac{d\omega_s}{dJ}\right|} \left|\tilde{g}^{(\pm)}_{\mu,-\mu}(J,J)\right|^2 \tag{9.4.10}$$

The Fokker-Planck transport equation for linear dipole transverse cooling without signal suppression, then reads:

$$\frac{\partial}{\partial t} f(I,J;t) = -\frac{\partial}{\partial I}\left[\alpha(J) \cdot I \cdot f(I,J;t) - \frac{1}{2} D(J) \cdot I \cdot \frac{\partial}{\partial I} f(I,J;t)\right] \tag{9.4.11}$$

Taking first moment by operating with $\int dI \cdot I$ on both sides we get:

$$\frac{d}{dt} \langle I \rangle (J) = - \alpha(J) \langle I \rangle (J) + \frac{D(J)}{2} . \tag{9.4.12}$$

Noting that $D(J) = \beta(J) \langle I \rangle (J)$, we get the equation for the evolution of means betatron action <u>or</u> mean squared betatron amplitude for particles with synchrotron action J as follows:

$$\frac{1}{\langle I \rangle (J)} \frac{d}{dt} \langle I \rangle (J) = \frac{1}{\langle A^2 \rangle_J} \frac{d}{dt} \langle A^2 \rangle_J = - \gamma(J) = - \left[\alpha(J) - \frac{\beta(J)}{2} \right] \tag{9.4.13}$$

and

$$\langle I \rangle (J;t) = \langle I \rangle (J;0)\, e^{-\gamma(J)t} \tag{9.4.14}$$

where

$$\gamma(J) = + \left[\alpha(J) - \frac{\beta(J)}{2} \right]$$

$$= + \frac{(q^2 \omega_0)}{2(2\pi)^2 Q} \sum_{(\pm)} \sum_{\mu} \tilde{g}^{(\pm)}_{-\mu,\mu}(J,J) - (\pi N) \left[\frac{q^2 \omega_0}{2(2\pi)^2 Q} \right]^2 \sum_{(\pm)} \sum_{\mu} \frac{f_0(J)}{|\mu| \left| \frac{d\omega_s}{dJ} \right|} \otimes \left| \tilde{g}^{(\pm)}_{-\mu,\mu}(J,J) \right|^2 \tag{9.4.15}$$

When signal suppression is important we can include this in the cooling rate without much complication as follows:

$$\gamma(J) = + \frac{(q^2 \omega_0)}{2(2\pi)^2 Q} \sum_{(\pm)} \sum_{\mu} \left\{ \frac{\tilde{g}^{(\pm)}_{-\mu,\mu}(J,J)}{\epsilon^{\mu^{(\pm)}}_T(J)} - \pi N \left(\frac{q^2 \omega_0}{2(2\pi)^2 Q} \right) \frac{f_0(J)}{|\mu| \left| \frac{d\omega_s}{dJ} \right|} \frac{\left| \tilde{g}^{(\pm)}_{-\mu,\mu}(J,J) \right|^2}{\left| \epsilon^{\mu^{(\pm)}}_T(J) \right|^2} \right\} \tag{9.4.16}$$

where

$$\epsilon^{\mu^{(\pm)}}_T(J) = 1 + \frac{\pi N f_0(J)}{|\mu|} \tilde{g}^{(\pm)}_{-\mu,\mu}(J,J) \cdot \left(\frac{q^2 \omega_0}{2(2\pi)^2 Q} \right) \tag{9.4.17}$$

as derived in Appendix E.

From (9.4.16) and (9.4.17), we can rewrite $\gamma(J)$ as:

$$\gamma(J) = \frac{q^2\omega_0}{2(2\pi)^2 Q} \sum_{(\pm)} \sum_{\mu} \frac{\tilde{g}^{(\pm)}_{-\mu,\mu}(J,J)}{\left|\epsilon_T^{\mu(\pm)}(J)\right|^2} \tag{9.4.18}$$

$$= \frac{q^2\omega_0}{2(2\pi)^2 Q} \sum_{(\pm)} \sum_{\mu} \frac{\tilde{g}^{(\pm)}_{-\mu,\mu}(J,J)}{\left|1 + \frac{q^2\omega_0}{2(2\pi)^2 Q} \cdot \frac{\pi N f_0(J)}{|\mu|} \tilde{g}^{(\pm)}_{-\mu,\mu}(J,J)\right|^2} \tag{9.4.19}$$

where $\tilde{g}^{(\pm)}_{-\mu,\mu}(J,J)$ is given by (9.4.4).

When there is amplifier noise present, we have

$$D^T_{noise}(I,J) = I \cdot \lambda(J)$$

where

$$\lambda(J) = \frac{q^2}{2Q^2} \sum_{\mu} \sum_{(\pm)} \frac{\left\{\sum_n J_\mu^2\left[(n\pm Q)\sqrt{2J}\right] P^T\left[(n\pm Q)\omega_0 + \mu\omega_s(J)\right]\right\}}{\left|\epsilon_T^{\mu(\pm)}(J)\right|^2} \tag{9.4.20}$$

as discussed in Section 9.3.

The Fokker-Planck equation becomes:

$$\frac{\partial f(I,J;t)}{\partial t} = -\frac{\partial}{\partial I}\left[\alpha(J) \cdot I \cdot f(I,J,t) - \frac{\beta(J)}{2} \langle I\rangle(J) \cdot I \cdot \frac{\partial}{\partial I} f(I,J;t) - \frac{\lambda(J)}{2} \cdot I \frac{\partial}{\partial I} f(I,J;t)\right] \tag{9.4.21}$$

The equation for the first moment becomes:

$$\frac{d}{dt}\langle I\rangle(J) = -\left[\alpha(J) - \frac{\beta(J)}{2}\right]\langle I\rangle(J) + \frac{\lambda(J)}{2}$$

$$= -\gamma(J)\,\langle I\rangle(J) + \frac{\lambda(J)}{2}\,. \tag{9.4.22}$$

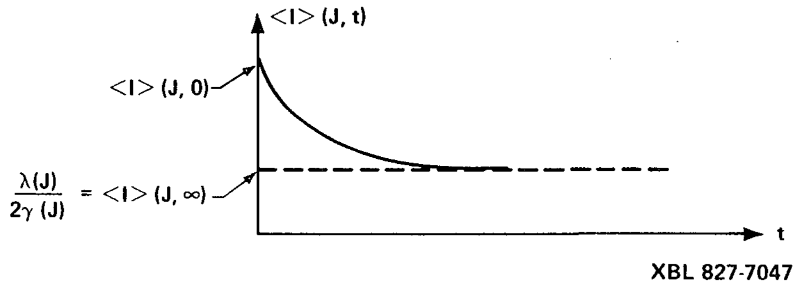
as must be the case.

Solution of (9.4.22) is given by:

$$\langle I\rangle(J;t) = \Big[\langle I\rangle(J,0) - \langle I\rangle(J,\infty)\Big]\, e^{-\gamma(J)t} + \langle I\rangle(J,\infty) \tag{9.4.23}$$

where

$$\langle I\rangle(J,\infty) = \lambda(J)/2\gamma(J) \tag{9.4.24}$$



Time-Evolution of Mean Squared Betatron Amplitude for Dipole Cooling

Fig. 20

Thus the asymptotic level of the distribution is determined by a combination of amplifier noise contribution $\lambda(J)$ and cooling rate $\gamma(J)$ in the absence of such noise (Fig. 20).

We can also find an equilibrium distribution $f_{eq}(I,J)$ such that $\partial f_{eq}/\partial t = 0$ in the case when the amplifier noise is dominant over the Schottky noise. The Fokker-Planck then reads:

$$\frac{\partial f(I,J;t)}{\partial t} = -\frac{\partial}{\partial I}\left[\alpha(J)\cdot I\cdot f(I,J;t) - \frac{\lambda(J)}{2}\, I\cdot\frac{\partial}{\partial I} f(I,J;t)\right] \tag{9.4.25}$$

and equilibrium distribution is given by:

$$f_{eq}(I;J) = \exp\left[-\frac{I}{I_0(J)}\right] \tag{9.4.26}$$

where

$$I_o(J) = \frac{\lambda(J)}{2\alpha(J)} . \tag{9.4.27}$$

In this special case it is also possible to find eigen-functions of (9.4.25). We assume

$$f(I,J;t) = f(I,J)\, e^{-\delta t} \tag{9.4.28}$$

Then (11.5.25) becomes:

$$\delta f - \frac{d}{dI}\left[\alpha I f - \frac{\lambda}{2}\, I\, \frac{df}{dI}\right] = 0 . \tag{9.4.29}$$

We change variables to $I = -\frac{\lambda}{2\alpha} x$ and $f = he^{x}$. Equation (9.4.29) then reduces to the equation for Laguerre polynomials:

$$xh'' + (1-x)h' + kh = 0 \tag{9.4.30}$$

where $k = \frac{\delta}{(-\alpha)} = n = 0,1,2,\ldots$ defines the eigensolutions, given by:

$$f_n = L_n(x)\, e^{-x-n\alpha t} \tag{9.4.31}$$

<u>or</u>

$$f_n(I,J;t) = L_n\left(-\frac{2\alpha(J)}{\lambda(J)}\, I\right) e^{\frac{2\alpha(J)}{\lambda(J)} I - n\alpha(J)t} . \tag{9.4.32}$$

The general solution is given by

$$f(I,J;t) = \int dI'\; G(I,I',J|t)\; f(I',J;0) \tag{9.4.33}$$

where $G(I,I';J|t)$ is the Green's function given by

$$G(I,I';\,J|t) = H(x,x'|t)$$

$$= e^{-x} \sum_{n=0}^{\infty} L_n(x)\, L_n(x')\, e^{-n\alpha t} \tag{9.4.34}$$

$$= \frac{\exp\left[-\dfrac{x-x'e^{-\alpha t}}{1-e^{-\alpha t}}\right]}{\left[1-e^{-\alpha t}\right]}\, I_o \left[\frac{\sqrt{xx'}}{\sinh\left(\frac{\alpha}{2}\, t\right)}\right] \tag{9.4.35}$$

where I_o is the Bessel function of the second kind with imaginary argument. We verify the following:

$$H(x,x'|0) = \delta(x-x') \tag{9.4.36}$$

$$H(x,x'|\infty) = e^{-x} \tag{9.4.37}$$

and

$$\int H(x,x'|t)\, dx' = 1\ . \tag{9.4.38}$$

9.5 Fokker-Planck with Coupled Degrees of Freedom

The coupling between the degrees of freedom induced by the feedback loop will produce particle fluxes in all three directions in the three-dimensional action $\underset{\sim}{I}$-space (or velocity $\underset{\sim}{v}$-space) and the resulting transport equation for the time-evolution of the distribution function will involve at third rank "Diffusion Tensor" and a "cooling flux vector" of dimension three. In particular, the coupling of degrees of freedom will produce cross-correlations and generate cross-moments like $\langle I_x I_z \rangle$, $\langle I_x J \rangle$,... etc. on top of the mean squared moments like $\langle I_x^2 \rangle$, $\langle I_z^2 \rangle$ and $\langle J^2 \rangle$, even when such cross-moments were intially set to zero before cooling started (I_x, I_z and J are the action variables in the horizontal betatron, vertical betatron and longitudinal degrees of freedom respectively).

Complications arise also in the presence of strong collective signal suppression effects. In presence of coupling between various degrees of freedom, the collective response of the beam to the modulations induced by the kicker is again described by a dielectric tensor of rank three in general and various components of the diffusion tensor will be modified by various combinations of the components of the inverse dielectric tensor. We will discuss signal suppression for coupled degrees of freedom for coasting beams in Chapter 12. In this section we write the relevant damping and diffusion coefficients ignoring signal suppression.

In general, for a particle experiencing fluctuating fields (kicker voltage or electric field) in all three dimensions, we have the basic Fokker-Planck equation for the time-evolution of the one-particle distribution function $f(\underset{\sim}{I};t)$ valid up to two-body correlations given by Eq. (9.1.34) with transport coefficients given in terms of general interaction harmonics as

$$F^{\alpha}(\underset{\sim}{I}) = \sum_{\underset{\sim}{n}} G^{\alpha}_{\underset{\sim}{n},-\underset{\sim}{n}}(\underset{\sim}{I},\underset{\sim}{I}) \tag{9.5.1}$$

$$D_{\alpha\beta}(\underset{\sim}{I},f) = (2\pi)\cdot N \sum_{\underset{\sim}{n}}\sum_{\underset{\sim}{n}'} \int d\underset{\sim}{I}'\, f(\underset{\sim}{I}') \left[G^{\alpha}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\, G^{\beta^*}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I},\underset{\sim}{I}')\right]$$

$$\delta\left[\underset{\sim}{n}\cdot\underset{\sim}{\omega}(\underset{\sim}{I}) + \underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}')\right] \tag{9.5.2}$$

as follows from (9.1.36) and (9.1.37).

With the separated variable representation given by Eq. (4.3.28) for the gain function $G^{\alpha}_{\underset{\sim}{n}^i\underset{\sim}{n}^j}(\underset{\sim}{I}_i,\underset{\sim}{I}_j)$ for coasting beams, one can write the general elements of the Diffusion tensor and the Friction vector given by (9.5.1) and (9.5.2) as follows:

$$D^{\alpha\beta}(\underset{\sim}{I},f) = (2\pi)\cdot N \sum_{\underset{\sim}{n}}\sum_{\underset{\sim}{n}'} \left[K^{\alpha}_{\underset{\sim}{n}}(\underset{\sim}{I})\, K^{\beta^*}_{\underset{\sim}{n}'}(\underset{\sim}{I})\right] \int d\underset{\sim}{I}'\, f(\underset{\sim}{I}') \left[P^{\alpha}_{\underset{\sim}{n}'}(\underset{\sim}{I}')\, P^{\beta^*}_{\underset{\sim}{n}'}(\underset{\sim}{I}')\right]$$

$$\otimes\, \delta\left[\underset{\sim}{n}\cdot\underset{\sim}{\omega}(\underset{\sim}{I}) + \underset{\sim}{n}'\cdot\underset{\sim}{\omega}(\underset{\sim}{I}')\right] \tag{9.5.3}$$

and

$$F^{\alpha}(\underset{\sim}{I}) = \sum_{\underset{\sim}{n}} K^{\alpha}_{\underset{\sim}{n}}(\underset{\sim}{I}) \cdot P^{\alpha}_{-\underset{\sim}{n}}(\underset{\sim}{I}) \tag{9.5.4}$$

with $\alpha,\beta = x,z,\theta$, for coupled degrees of freedom in coasting beam cooling. For bunched beams, there will be additional sums over the revolution harmonics according to (4.3.27).

10. VLASOV THEORY OF SIGNAL SUPPRESSION

10.1 General Coupled-Mode Matrix for Signal Suppression of Bunched Beams

We have already obtained an expression for collective signal suppression of bunched beams in the special case when there are no overlapping resonances (no band overlap), while deriving the Fokker-Planck equation for the time-evolution of single particle distribution from kinetic theory in Section 9.2. While kinetic theory up to two-body correlations provides the most satisfactory self-consistent derivation of single particle transport in phase-space including collective distortion (suppression) of fluctuations, such a derivation is unavoidably complicated for general situations where there is band overlap or coupling between degrees of freedom or unusual unperturbed particle orbits. A tour de force solution can be obtained only in very special cases (e.g. homogenous plasma etc.) and requires considerable amount of mathematical gymnastics and physical insight into the structure of collective dynamics.

Fortunately there exists a considerably simpler approach leading to identical results up to the order of two-body correlations, when there are two disparate time-scales, as discussed in Section 9.1 under fluctuation theory. More specifically this approach is applicable when the collective distortion of the fluctuation spectrum seen by a single particle occurs in a time much shorter than the slow relaxation time of the distribution of particles leading to transport in phase-space. Such is the case for stochastic cooling as observed experimentally in practical schemes to date. Novel cooling schemes involving cooling times comparable to the time-scale of collective distortion of signals, if feasible, will require special considerations, not studied in this report.

Our approach in this simpler model is that we calculate the transport coefficients (friction and diffusion coefficients) from the simple prescriptions of Eqs. (9.1.33)-(9.1.37). However, we incorporate the collective effects by asserting that the spectral function and the coherent cooling force, in presence of collective dynamics, are screened or shielded from their incoherent values determined by an appropriate operator $\hat{\varepsilon}(\Omega)$ in frequency space, to be derived independently using collective dynamics only. The modifications in the transport coefficients stem from the total collectively distorted signal $\tilde{y}(\Omega)$ at the kicker sampled by a test particle and related to the incoherent signal $\tilde{y}^0(\Omega)$ at the kicker by the relation

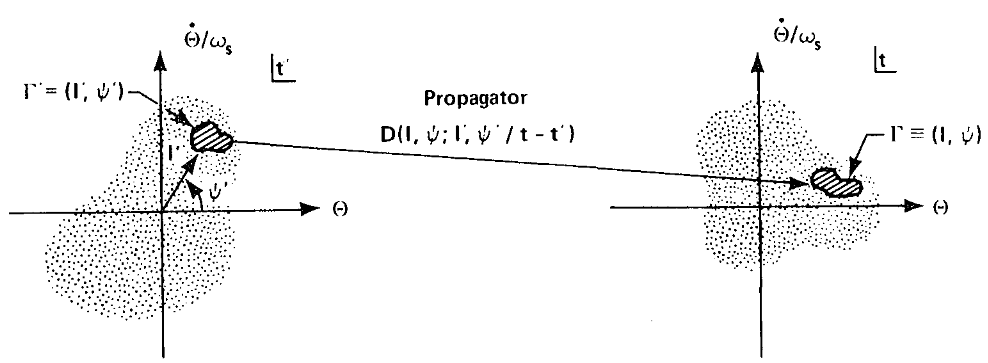
$$\hat{\varepsilon}(\Omega)\ \tilde{y}(\Omega) = \tilde{y}^0(\Omega) \qquad (10.1.1)$$

and

$$\tilde{y}(\Omega) = \left[\hat{\varepsilon}(\Omega)\right]^{-1} \tilde{y}^{0}(\Omega) \qquad (10.1.2)$$

where $\hat{\varepsilon}(\Omega)$ describes the dielectric response operator of the beam when excited with frequency Ω and is to be evaluated from Vlasov theory of dielectric response in a charged particle system. The resulting modifications in the transport coefficients are discussed in Ch. 11. The picture is similar to the familiar test-particle approach in plasma physics where transport coefficients are calculated by assuming that the diffusing or damping particle samples a fluctuation signal which is already dynamically screened by $\hat{\varepsilon}(\Omega)$ in the time-scale in which the particle has not changed its phase-space coordinate signficantly and then performing appropriate ensemble averages over such screened signals.

The proper framework for evaluating $\hat{\varepsilon}(\Omega)$ is the well-known Vlasov equation for the single particle distribution function in phase-space. One assumes the existence of a zero-order stationary (time-independent) distribution function for the beam. One then studies how a small perturbation in phase-space on top of this stationary distribution propagates in time. More specifically one studies how a perturbation $f(I',\psi';t')$ centered around $\Gamma' \equiv (I',\psi')$ in phase-space at time t' propagate collectively to the neighbourhood of the phase point $\Gamma \equiv (I,\psi)$ at time t, where (I,ψ) are the canonical action-angle variables for single degree of freedom longitudinal synchrotron oscillations for example (Fig. 21 below).



Propagation of Perturbations in Bunch Phase-Space

Fig. 21

For small perturbations we expect the dynamics to be linear and describable by a propagator $D(I,\psi;I',\psi'|t-t')$ so that

$$f(I,\psi;t) = \int d\psi' dI' dt' \; D(I,\psi;I',\psi'|t-t') \cdot f(I',\psi';t') + f^0(I,\psi;t) \qquad (10.1.3)$$

where f^0 is an arbitrary excitation at (I,ψ) at time t. Fourier series expanding in the angle variable ψ and Fourier transforming in time, we get

$$\tilde{f}^{\mu}(I;\Omega) = \sum_{\mu'} \int dI' \; \tilde{D}^{\mu\mu'}(I,I'|\Omega) \; \tilde{f}^{\mu'}(I';\Omega) + \tilde{f}^{0\mu}(I;\Omega) \qquad (10.1.4)$$

This is the form of the basic integral equation for propagation of perturbations in phase-space. One needs to find $\tilde{D}^{\mu\mu'}(I,I'|\Omega)$ from Vlasov theory and then solve the integral equation with $\tilde{D}^{\mu\mu'}$ thus obtained.

An essential complication arises however for bunched beams due to the fact that the response Green's function or propagator $D(I,\psi;I',\psi'|t,t')$ is not invariant under translations in time and hence is not a function of $(t-t')$ alone, but in addition depends on one of the time-arguments t <u>or</u> t'. This is a manifestation of the non-stationarity of bunched beam collective response. We note however that the response function $D(I,\psi;I',\psi'|t,t')$ as appearing in an integral equation written in bunch frame, can only be periodically nonstationary, i.e. it will depend on the second time argument t' periodically, corresponding to experiencing interactions at every revolution period through the feedback loop. We may thus write

$$D(I,\psi;I',\psi'|t,t') = D(I,\psi;I',\psi'|t-t',t') = \sum_{k=-\infty}^{+\infty} D_k(I,\psi;I',\psi'|\tau) \; e^{ik\omega_0 t'} \qquad (10.1.5)$$

where $\tau = (t-t')$.

The basic integral equation (10.1.4) for bunched beam collective response then transforms into:

$$\tilde{f}^{\mu}(I;\Omega) = \sum_{\mu'} \sum_{k} \int dI' \, \tilde{D}_k^{\mu\mu'}(I,I'|\Omega) \, \tilde{f}^{\mu'}(I';\Omega+k\omega_0) + \tilde{f}^{0\mu}(I;\Omega) \tag{10.1.6}$$

We thus see that the periodic nonstationarity of the bunched beam response relates any angular harmonic $\tilde{f}^{\mu}(I;\Omega)$ of perturbation at frequency Ω to all its discrete frequency translates $\tilde{f}^{\mu'}(I';\Omega+k\omega_0)$ through the response kernel $\tilde{D}_k^{\mu\mu'}(I,I'|\Omega)$ as in Eq. (10.1.6).

We have already obtained some insight into the structure of $\tilde{D}_k^{\mu\mu'}$ in Section 4.6. We derive the specific form of $\tilde{D}_k^{\mu\mu'}$ from Vlasov theory now.

In absence of cooling, the local beam density at the pick-up at azimuth θ_p is given by

$$\rho^0(\theta_p;t) = q \sum_{i=1}^{N} \sum_{n=-\infty}^{+\infty} \delta\left[\theta_p - \theta_i^0(t) - 2\pi n\right]$$

$$= \sum_{n=-\infty}^{+\infty} \frac{\rho_n^0(t)}{2\pi} e^{in\theta_p} \tag{10.1.7}$$

where

$$\rho_n^0(t) = q \sum_{i=1}^{N} e^{-in\theta_i^0(t)} \tag{10.1.8}$$

and

$$\theta_i^0(t) = \text{unperturbed orbit of } i^{\underline{th}} \text{ particle.}$$

Adding the cooling loop introduces correlations between particles and modifies the particle orbits. The density fluctuation at the pick-up will thus have modified Fourier amplitudes

$$\rho_n(t) = \rho_n^0(t) + \lambda_n(t)$$

where $\lambda_n(t)$ is the coherent first order modulation of density fluctuation at the pick-up due to the introduction of the cooling loop. Fourier transforming to the frequency domain, we get

$$\tilde{\rho}_n(\Omega) = \tilde{\rho}_n^0(\Omega) + \tilde{\lambda}_n(\Omega) \; . \tag{10.1.9}$$

Note that the above relation is in the laboratory frame, fixed at the pick-up. $\lambda_n(\Omega)$ defined in the laboratory frame is related to $\tilde{\lambda}_n^B(\Omega)$ derived from Vlasov equation in the beam frame (frame moving with a central reference particle in the bunch with revolution frequency ω_o) by the following transformation to Doppler-shifted frequency

$$\tilde{\lambda}_n(\Omega) = \tilde{\lambda}_n^B(\Omega - n\omega_o) \tag{10.1.10}$$

Vlasov equation for a distribution $f(a,\psi;t)$ in terms of the amplitude and phase variables (a,ψ) introduced in Chapter 3 reads ([65], [86], [87], [89])

$$\frac{\partial f(a,\psi;t)}{\partial t} + \dot{\psi}\,\frac{\partial f(a,\psi;t)}{\partial \psi} + \dot{a}\,\frac{\partial f(a,\psi;t)}{\partial a} = 0 \tag{10.1.11}$$

Decomposing the distribution into a stationary part and a small perturbation as $f = f_o(a) + f_1(a,\psi;t)$ and linearizing Eq. (10.1.11) in f_1, assuming $df_o/dt = 0$, one gets

$$\frac{\partial f_1(a,\psi;t)}{\partial t} + \omega_s(a)\,\frac{\partial f_1(a,\psi;t)}{\partial \psi} + \dot{a}\,\frac{df^0}{da} = 0\ . \tag{10.1.12}$$

From Section 6.2, the amplitude equation is given by

$$\dot{a} = \frac{(q\kappa)}{\omega_s(a)}\,\mathcal{S}(t)\ \mathrm{Cos}\ \psi(t) \tag{10.1.13}$$

where

$$\mathcal{S}(t) = \omega_o \sum_{n=-\infty}^{+\infty} V^K(t)\ \delta\left[\theta(t) - \theta_k - 2\pi n\right] \tag{10.1.14}$$

where $\mathcal{S}(t)$ is the voltage sampled by the particle as a function of time as it passes through the kicker (with voltage $V^K(t)$) periodically. Fourier expanding the periodic δ-function and using the identity (7.1), one obtains

$$\dot{a} = \frac{(q\ t_o\ \kappa)}{\omega_s(a)\cdot a}\sum_{\nu}\sum_{n} \frac{\nu}{n}\ J_\nu(na)\ e^{-in\theta_k}\left[V^K(t)\ e^{in\omega_o t}\right] e^{i\nu\psi(t)} \tag{10.1.15}$$

Substituting (10.1.15) into (10.1.12) and Fourier expanding in the angle ψ, one obtains

$$\frac{\partial f_1^{\mu}(a,t)}{\partial t} + i\mu\omega_s(a)\, f_1^{\mu}(a,t) + \frac{(q\, f_o\, \kappa)}{\omega_s(a)} \sum_n \left(\frac{\mu}{na}\right) J_\mu(na)\, e^{-in\theta_k} \otimes \left[V^K(t)\, e^{in\omega_o t}\right] \frac{df^0}{da} = 0 \tag{10.1.16}$$

Fourier transforming in time in Eq. (10.1.16) yields:

$$-i\left[\Omega-\mu\omega_s(a)\right] \tilde{f}_1^{\mu}(a;\Omega) = -\frac{(q\, f_o\, \kappa)}{\omega_s(a)} \left[\frac{df^0}{da}\right] \sum_{n=-\infty}^{+\infty} \left(\frac{\mu}{na}\right) J_\mu(na)\, e^{-in\theta_k} \otimes V^K(\Omega + n\omega_o) \tag{10.1.17}$$

Voltage at the kicker is given by

$$V^K(t) = \int_{-\infty}^{+\infty} dt'\, G(t-t')\, I(\theta_p;t') \tag{10.1.18}$$

where $I(\theta_p;t')$ is the total current at the pick-up at time t' and $G(t-t')$ is the linear and causal transfer function of the feedback loop. Fourier transformed in frequency (10.1.18) reads

$$\tilde{V}^K(\Omega) = \tilde{G}(\Omega)\, \tilde{I}(\theta_p;\Omega) \tag{10.1.19}$$

Substitution in (10.1.17) then yields

$$\tilde{f}_1^{\mu}(a;\Omega) = (-i) \left[\frac{(q\, f_o\, \kappa)}{\omega_s(a)}\right] \left[\frac{df^0}{da}\right] \frac{\sum_{n=-\infty}^{+\infty} \left(\frac{\mu}{na}\right) J_\mu(na)\, e^{-in\theta_k}\, \tilde{G}(\Omega+n\omega_o)\, \tilde{I}(\theta_p;\Omega+n\omega_o)}{\left[\Omega - \mu\omega_s(a)\right]} \tag{10.1.20}$$

We now write the distribution as

$$f_1(a,\psi;t) = g(Ⓗ,\dot{Ⓗ};t) = g\left(\theta-\omega_0 t;\ \dot{\theta}-\omega_0;\ t\right) = \Psi(\theta,\dot{\theta};t)$$

Then the beam current at (θ,t) is given by

$$I(\theta;t) = q\omega_0 \int d\dot{\theta}\ \Psi(\theta,\dot{\theta};t) + I_0(\theta,t)$$

Fourier decomposition in θ yields

$$I_m(t) = \frac{q\omega_0}{2\pi} \iint d\theta d\dot{\theta}\ \Psi(\theta,\dot{\theta};t)\ e^{-im\theta} + I_{om}(t)$$

Using the invariance of the volume element

$$d\theta d\dot{\theta} = d Ⓗ d\dot{Ⓗ} = \omega_s\ d\left(\frac{1}{2}\ a^2\right)\ d\psi$$

for the transformation $(Ⓗ,\dot{Ⓗ}/\omega_s) \to (J = \frac{1}{2}\ a^2, \psi)$, we obtain

$$I_m(t) = \frac{q\omega_0}{2}\ \omega_s \int d(a^2) \sum_{\mu=-\infty}^{+\infty} f_1^\mu(a;t)\ J_\mu(ma)\ e^{-im\omega_0 t} + I_{om}(t)$$

so that for $\lambda(\theta_p;t) = I(\theta_p;t) - I_0(\theta_p;t)$ we find:

$$\lambda(\theta_p;t) = \sum_m \lambda_m(t)\ e^{im\theta_p} = \frac{q\omega_0}{2}\ \omega_s \int d(a^2) \sum_m\sum_\mu f_1^\mu(a;t)\ J_\mu(ma)\ e^{-im\omega_0 t} \otimes e^{im\theta_p}$$

and

$$\tilde{\lambda}(\theta_p;\Omega) = \frac{q\omega_0}{2}\ \omega_s \int d(a^2) \sum_m\sum_\mu J_\mu(ma)\ e^{im\theta_p}\ \tilde{f}_1^\mu(a,\Omega-m\omega_0) \qquad (10.1.21)$$

We now split the total current $\tilde{I}(\theta_p;\ \Omega+n\omega_0)$ appearing in Eq. (10.1.20), according to (10.1.9), as

$$\tilde{I}\left(\theta_p;\ \Omega+n\omega_0\right) = \tilde{I}_0\left(\theta_p;\ \Omega+n\omega_0\right) + \tilde{\lambda}\left(\theta_p;\ \Omega+n\omega_0\right) \tag{10.1.22}$$

where $\tilde{I}_0(\theta_p;\Omega)$ is the sum of single particle unperturbed Schottky currents and $\tilde{\lambda}$ corresponds to the coherent modulation of the current due to feedback, as given by (10.1.21).

Substituting (10.1.21) into (10.1.20) and using (10.1.22) and absorbing the factor $(q\omega_0)^2\ \kappa/2\pi$ into the gain function by a redefinition

$$\tilde{G}(\Omega) \longrightarrow \frac{(q\omega_0)^2}{(2\pi)}\ \kappa\ \tilde{G}(\Omega)$$

we find

$$\tilde{f}_1^{\mu}(J;\Omega) = \sum_{\mu'} \sum_{k} \int dJ'\ \tilde{D}_k^{\mu\mu'}(J,J'\,|\Omega)\ \tilde{f}_1^{\mu'}\left(J';\ \Omega+k\omega_0\right) + \tilde{f}^{0\mu}(J,\Omega)$$

where

$$\tilde{D}_k^{\mu\mu'}(J,J'\,|\Omega) = -i\left[\frac{df^0}{dJ}\right] \frac{\sum_n \left(\frac{\mu}{n}\right) J_\mu(n\sqrt{2J})\ J_{\mu'}[(n-k)\sqrt{2J'}]\ \tilde{G}(\Omega+n\omega_0)\ e^{in(\theta_p-\theta_k)}}{\left[\Omega-\mu\omega_s(J)\right]}\ e^{-ik\theta_p} \tag{10.1.23}$$

We have thus obtained the specific form of the kernal $\tilde{D}_K^{\mu\mu'}(J,J'|\Omega)$ given by (10.1.23) above that appears in the integral Eq. (10.1.6) for bunched beam collective response.

The above describes the collective response in the frame of the bunch. We can obtain a corresponding equation relating various frequency components (and their translates) of the current at the pick-up or the voltage at kicker by substituting (10.1.20) in (10.2.21) and using (10.1.22) again. One obtains

$$V_K(\Omega) = V_K^0(\Omega) + \sum_{k=-\infty}^{+\infty} D_k(\Omega)\ V_K(\Omega+k\omega_0) \tag{10.1.24}$$

where

$$V_K(\Omega) = \tilde{G}(\Omega)\ \tilde{I}(\theta_p,\Omega) = \text{Total voltage at kicker at frequency } \Omega.$$

$$V_K^0(\Omega) = \tilde{G}(\Omega)\cdot\tilde{I}_0(\theta_p,\Omega) = \text{Voltage at kicker due to unperturbed Schottky signals only at frequency } \Omega.$$

$$D_k(\Omega) = \int_0^\infty da \left[\frac{df^0}{da}\right] \sum_{\mu=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} A_{n-k}^{\mu}(\Omega;a)\ B_n^{\mu}(a)\right] \tag{10.1.25}$$

$$B_n^{\mu}(a) = \frac{J_\mu(na)}{n}\ e^{-in\theta_k} \tag{10.1.26}$$

$$A_m^{\mu}(a) = \frac{(-i)\ \mu\ J_\mu(ma)\ e^{im\theta_p}}{\left[\Omega - m\omega_0 - \mu\omega_s(a)\right]}\ \tilde{G}(\Omega) \tag{10.1.27}$$

If we define a "translation operator $\hat{T}^k$" by

$$\hat{T}^k\ V_K(\Omega) = V_K(\Omega + k\omega_0) \tag{10.1.28}$$

and a kernel operator by

$$\hat{D}(\Omega) = \sum_{k=-\infty}^{+\infty} D_k(\Omega)\ \hat{T}^k \tag{10.1.29}$$

then we can rewrite Eq. (10.1.24) as

$$\left[1 - \sum_k D_k(\Omega)\ \hat{T}^k\right] V_K(\Omega) = V_K^0(\Omega)$$

or

$$\left[1 - \hat{D}(\Omega)\right] V_K(\Omega) = V_K^0(\Omega) \tag{10.1.30}$$

or

$$\hat{\epsilon}(\Omega)\cdot V_K(\Omega) = V_K^0(\Omega) \tag{10.1.31}$$

Equation (10.1.30) has unique solution if the operator

$$\hat{\varepsilon}(\Omega) = \left[1 - \hat{D}(\Omega)\right] = \left[1 - \sum_k D_k(\Omega)\ \hat{T}^k\right] \tag{10.1.32}$$

has an inverse, i.e. if '1' does not belong to the spectrum of $\hat{D}(\Omega)$ $[1 \notin \sigma(\hat{D})]$. Then formally, the collective distortion of current or voltage spectrum from the unperturbed Schottky value is given by

$$V_K(\Omega) = \left[\hat{\varepsilon}(\Omega)\right]^{-1} V_K^0(\Omega) \tag{10.1.33}$$

Collective signal suppression effects are obtained by an effective inversion of the operator $\hat{\varepsilon}(\Omega) = 1 - \hat{D}(\Omega)$. We will see below that in a matrix representation of $\hat{\varepsilon}(\Omega)$, this amounts to inverting an infinite matrix, a nontrivial task in general.

We have outlined some of the interesting properties of the various quantities entering into the coupled mode response Eq. (10.1.24) in Appendix F.

Let us now turn to a matrix representation of $\hat{\varepsilon}(\Omega)$. We define vectors in an infinite-dimensional space by

$$\underline{V}_K = \left\{V_K^\ell(\Omega)\right\}: \quad V_K^\ell(\Omega) = T^\ell\ V_K(\Omega) = V_K\left(\Omega + \ell\omega_o\right) \quad (\ell = 0, \pm 1, \pm 2, \dots)$$

From (10.1.24), we then obtain

$$V_K^\ell(\Omega) = V_K^{0,\ell}(\Omega) + \sum_k T^\ell\ D_k(\Omega)\ V_K\left(\Omega + k\omega_o\right)$$

$$= V_K^{0,\ell}(\Omega) + \sum_k D_k\left(\Omega + \ell\omega_o\right) V_K\left(\Omega + (\ell + k)\omega_o\right)$$

Redefining indices as $m = \ell + k$ gives

$$V_K^{\ell}(\Omega) = V_K^{0,\ell}(\Omega) + \sum_m D_{m-\ell}\left(\Omega + \ell\omega_o\right) V_K^m(\Omega)$$

$$= V_K^{0,\ell}(\Omega) + \sum_m M_{\ell m}(\Omega)\; V_K^m(\Omega) \qquad (10.1.34)$$

We then find that we have an equivalent matrix representation of the operator Eq. (10.1.31) in the following form:

$$\underset{\approx}{\varepsilon}(\Omega) \cdot \underset{\sim}{V}_K(\Omega) = \underset{\sim}{V}_K^0(\Omega) \qquad (10.1.35)$$

where

$$\underset{\approx}{\varepsilon}(\Omega) = \underset{\approx}{\mathbb{1}} - \underset{\approx}{M}(\Omega) \qquad (10.1.36)$$

is an infinite matrix with elements

$$\varepsilon_{\ell m}(\Omega) = \delta_{\ell m} - M_{\ell m}(\Omega) \qquad (10.1.37)$$

and

$$M_{\ell m}(\Omega) = D_{m-\ell}\left(\Omega + \ell\omega_o\right)$$

$$= \int_0^{\infty} da \left[\frac{df^0}{da}\right] \sum_{\mu}^{(+\infty)}\sum_{n}^{(-\infty)} \frac{(-i)\;\mu\; J_\mu[(n-m+\ell)a]\; e^{i(n-m+\ell)\theta_p}}{\left[\Omega - (n-m)\omega_o - \mu\omega_s(a)\right]}\; \tilde{G}\left(\Omega + \ell\omega_o\right) \frac{J_\mu(na)}{n} \otimes e^{-in\theta_k} \qquad (10.1.38)$$

The infinite matrix $\underset{\approx}{\varepsilon}(\Omega)$ represents a generalized dielectric function or signal suppression and $\underset{\approx}{M}(\Omega)$ represents a generalized susceptibility of the bunch in presence of collective interactions determined by the feedback loop. When we are interested in the instability <u>or</u> stability of collective modes induced by the feedback system we set the right-hand side of (10.1.35) to zero $(V_K^0(\Omega)=0)$ and the real frequencies ω_n and the corresponding growth <u>or</u> decay rates γ_n of the collective modes are determined by the complex (in general) frequencies $\Omega = \omega_n + i\gamma_n$ satisfying the dispersion relation

$$\det\left[\underset{\approx}{\varepsilon}(\Omega)\right] = 0 \tag{10.1.39}$$

One thus solves for the roots of the infinite order determinantal equation (10.1.39). For collective modes having frequencies close to the harmonics of the natural oscillation frequencies of the particles in the beam (i.e. when the system is close to a resonance), one can usually truncate the matrix safely beyond a certain low but finite order (3x3, 5x5, etc.) and roots of such a low order determinant then gives the collective mode frequencies fairly accurately. The corresponding eignefunctions give the mode pattern in phase-space ([87], [88], [89], [90]).

When excited by a finite $V_K^0(\Omega)$ at a frequency Ω away from the resonances <u>or</u> collective mode frequencies of the system, the general effect is a distortion of the perturbation spectrum $V_K(\Omega)$ away from $V_K^0(\Omega)$ as determined by (10.1.35) through (10.1.38) and given by:

$$\underset{\sim}{V}_K(\Omega) = \left[\underset{\approx}{\varepsilon}(\Omega)\right]^{-1} \underset{\sim}{V}_K^0(\Omega) = \left[\underset{\approx}{\varepsilon}(\Omega)\right]^C \left[\frac{\underset{\sim}{V}_K^0(\Omega)}{\det\left[\underset{\approx}{\varepsilon}(\Omega)\right]}\right]$$

where $[\underset{\approx}{\varepsilon}(\Omega)]^C$ is the co-factor matrix of $\underset{\approx}{\varepsilon}(\Omega)$. Note that since we are off-resonance, i.e. away from collective instabilities, the determinant is not zero but has a finite non-zero value, measuring the suppression of signals, aside from the complication of mode-coupling expressed by the matrix $[\underset{\approx}{\varepsilon}(\Omega)]^C$. This is the case for stochastic cooling, since we certainly do not want to excite collective modes in the beam by the feedback loop. We thus have to effectively invert the infinite dielectric matrix $\underset{\approx}{\varepsilon}(\Omega)$ rather than finding approximate eigenvalues and eigenfunctions of $\underset{\approx}{\varepsilon}(\Omega)$. It is this former task that makes a solution of bunched beam collective response uniquely difficult and interesting.

Equations (10.1.35) through (10.1.38) defines the basic coupled mode matrix for the collective response of a bunched beam. Note the structural similarity between these equations and Eq. (4.6.22) obtained in Chapter 4 from a general discussion of bunched beam collective dynamics.

A similar analysis can be perfomed for transverse collective effects leading to structurally similar results with slightly different matrix elements $M_{\ell m}$. Rather than reiterate the procedure from Vlasov theory, we provide an alternative derivation of the coupled mode matrix for transverse perturbations based on an integral equation satisfied by certain "collective coordinates" in Appendix E.

One may look at Eq. (10.1.35) as a matrix relation for $\underset{\approx}{\varepsilon}^{-1}(\Omega)$ itself <u>or</u> as an integral equation for $\underset{\approx}{\varepsilon}^{-1}$ in the time-domain as follows.

By definition, we have

$$\sum_{m'} \varepsilon_{\ell m'}(\Omega)\ \varepsilon^{-1}_{m'm}(\Omega) = \delta_{\ell m} \tag{10.1.40}$$

From (10.1.37), it follows

$$\sum_{m'} \left[\delta_{\ell m'} - M_{\ell m'}(\Omega)\right] \varepsilon^{-1}_{m'm}(\Omega) = \delta_{\ell m}$$

or

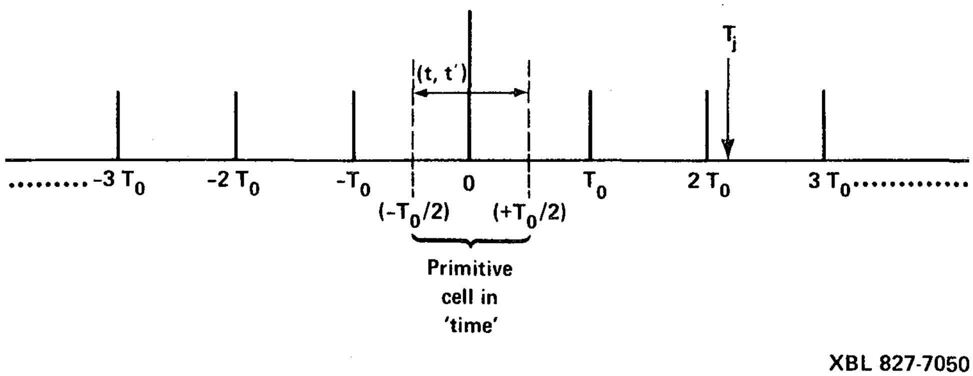
$$\varepsilon^{-1}_{\ell m}(\Omega) = \delta_{\ell m} + \sum_{m'} M_{\ell m'}(\Omega)\ \varepsilon^{-1}_{m'm}(\Omega) \tag{10.1.41}$$

This integral relation for $\varepsilon^{-1}_{\ell m}(\Omega)$ in the discrete (ℓ,m) space can be transformed to an equivalent integral equation for a suitably defined $\varepsilon^{-1}(t,t';\Omega)$ in (t,t') space by defining the transforms (Bloch Representation):

$$\begin{aligned} \varepsilon^{-1}(t,t';\Omega) &= \sum_{\ell}\sum_{m}{}_{(-\infty)}^{(+\infty)} e^{i\ell\omega_0 t}\ e^{-im\omega_0 t'}\ \varepsilon^{-1}_{\ell m}(\Omega) \\ M(t,t';\Omega) &= \sum_{\ell}\sum_{m}{}_{(-\infty)}^{(+\infty)} e^{i\ell\omega_0 t}\ e^{-im\omega_0 t'}\ M_{\ell m}(\Omega) \end{aligned} \tag{10.1.42}$$

where (t,t') are confined to within a periodic primitive cell in time: $-T_0/2 < (t,t') < +T_0/2$ with origin at the center as in Fig. 22 below. $(T_0 = 2\pi/\omega_0)$. Using (10.1.42), we can rewrite (10.1.41) as:

$$\varepsilon^{-1}(t,t';\Omega) = \sum_{\ell} e^{i\ell\omega_0(t-t')} + \sum_{\ell}\sum_{m}\sum_{m'} e^{i\ell\omega_0 t - im\omega_0 t'}\ M_{\ell m'}(\Omega)\ \varepsilon^{-1}_{m'm}(\Omega) \tag{10.1.43}$$



Time Cells for a Bloch Representation of Inverse Response Kernel

Fig. 22

Using the identity

$$\delta_{m'm''} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} dt'' \, e^{-i(m'-m'')\omega_0 t''}$$

we can write (10.1.43) as:

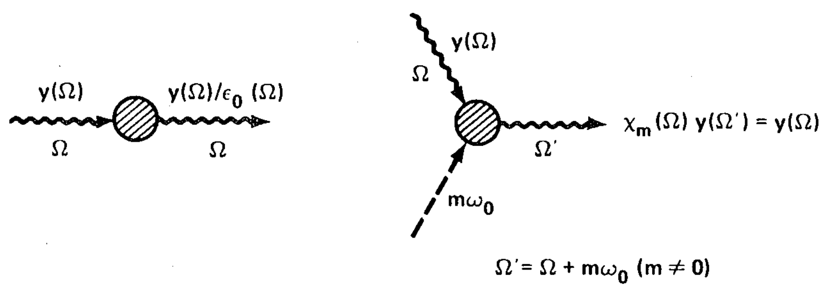
$$\epsilon^{-1}(t,t';\,\Omega) = \sum_{\ell} e^{i\ell\omega_0(t-t')} + \left(\frac{\omega_0}{2\pi}\right) \int_{-T_0/2}^{T_0/2} dt''\, M(t,t'';\,\Omega)\; \epsilon^{-1}(t'',t';\,\Omega) \tag{10.1.44}$$

with $-T_0/2 < (t,t') < T_0/2$. Equation (10.1.44) is the integral equation in the time-domain satisfied by the inverse collective response kernel $\epsilon^{-1}(t,t';\Omega)$.

From this viewpoint, the coupling to frequencies translated by $m\omega_0$ where $m = \pm 1, \pm 2, \pm 3, \ldots$ in (10.1.24) corresponds to rapid fluctuations within a cell in time and gives the actual field locally in time whereas the $m = 0$ process gives the non-interfering coarse field over many turns of the bunch. The two types of processes ($m = 0$ and $m \neq 0$) can be pictorially represented as in Fig. 23(a) and (b) below.

m = 0
Non-Interfering Schottky
Noise Signal Screening

m ≠ 0
Interfering Schottky Noise
Signal Screening or "Local
Field Shielding"



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(a) (b)

Non-Interfering and Interfering Schottky Signal Screening

Fig. 23

We do not have a complete solution to the inversion problem (i.e. finding $\underset{\approx}{\varepsilon}^{-1}(\Omega)$ in closed form). However solutions under special cases of no synchrotron band overlap and for particular distributions (e.g. water-bag distribution) can be obtained and are discussed in Section 10.2 and 10.3 below. Solution to the infinite determinant problem for collective instabilities has been obtained under various approximations in the past ([4], [60], [65], [80], [87], [88], [89], [90], [108], [109], [113]).

10.2 Solution in the Dominant Pole Approximation Neglecting Revolution and Synchrotron Band Overlap

We are interested in evaluating the signal suppression at a certain frequency, say Ω. In general several revolution and synchrotron harmonics for particles with different amplitudes will correspond to the same Ω, i.e. the resonant denominator in Eq. (10.1.38) will contain resonances like

$$\ell\omega_0 + \mu\omega_s(a) = \Omega = p\omega_0 + \nu\omega_s(a') \qquad (10.2.1)$$

with $p \neq \ell$, $\nu \neq \mu$ and $a' \neq a$ i.e. different revolution and synchrotron bands for the distribution of particles will overlap. However, as discussed in Chapter 5, for bunched beams the $\ell \neq p$ resonances do not occur until very high revolution harmonics. Let us assume then that the different revolution bands do not overlap within the band-pass of the feedback system. This amounts to saying that given a frequency Ω, we can always associate with it a certain revolution harmonic, k, say i.e.

$$\Omega = k\omega_0 + \Omega'$$

As we sum over n in the matrix element $M_{\ell m}(k\omega_0 + \Omega')$ given by (10.1.38), we only pick out the resonant denominator that corresponds to

$$(n-m)\ \omega_0 = k\omega_0$$

i.e.

$$n = k + m$$

We are thus only considering resonances (10.2.1) with $\ell = p$. The matrix element (10.1.38) does not involve a sum over n in this case and becomes:

$$M_{\ell m}\left(k\omega_0 + \Omega'\right) = \int_0^\infty da \left[\frac{df^0}{da}\right] \sum_{\mu=-\infty}^{+\infty} \frac{(-i)\ \mu\ J_\mu[(k+\ell)a]\ e^{i(k+\ell)\theta_p}}{\left[\Omega' - \mu\omega_s(a)\right]} \tilde{G}\left(\Omega' + (k+\ell)\omega_0\right)$$

$$\cdot \frac{J_\mu[(k+m)a]}{(k+m)} e^{-i(k+m)\theta_k} \tag{10.2.2}$$

The resonant denominator in (10.2.2) still contain resonances like

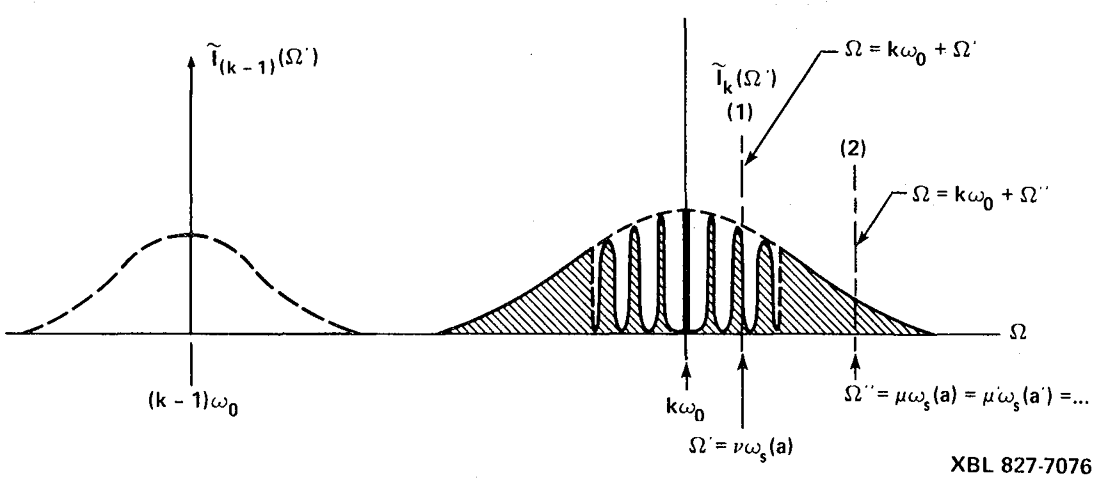
$$\mu\omega_s(a) = \Omega' = \nu\omega_s(a') \tag{10.2.3}$$

with $\mu \neq \nu$ and $a \neq a'$. Contributions from these resonances are nonnegligible since in a large band-width feedback system, there is considerable amount of synchrotron-band overlap as discussed in Chapter 5. Thus one really has to sum over the synchrotron harmonics μ in (10.2.2) for a large number of resonant μ's in order to get an $\epsilon(\Omega)$ that includes synchrotron band overlap. Since synchrotron band overlap is an intrinsically nonlocal phenomenon in amplitude space (different a and a' will contribute to the same frequency with different synchrotron harmonics μ and ν, see Eq. (10.2.3)), we expect an integral equation in amplitude <u>or</u> action space (a <u>or</u> J) for some properly defined collective signal $X(a|\Omega)$ for certain amplitude a at a given frequency Ω, of the following form:

$$X(a|\Omega) = \int_0^\infty da' \, \mathbb{K}(a,a'|\Omega) \; X(a'|\Omega) + X^0(a|\Omega) \qquad (10.2.4)$$

where $\mathbb{K}(a,a'|\Omega)$ is an appropriate kernel.

The above complication arises when the frequency Ω' falls in the synchrotron band-overlap region of a given revolution harmonic band $(k\omega_0)$ e.g. at line marked (2) in Fig. 24 below. We do not yet have a solution to the integral equation (10.2.4) for the region of synchrotron band overlap.



Collective Response Frequencies Inside and Outside the Region of Synchrotron Band Overlap

Fig. 24

For frequencies Ω' falling in the non-overlap region of a given revolution band '$k\omega_0$' e.g. line marked (1) in Fig. 24 above, we need only consider resonances (10.2.3) with $\mu = \nu$ only. In this section, then, we derive an approximate expression for the signal suppression by keeping only a dominant pole in expression (10.2.2) corresponding to the non-overlapping resonance

$$\mu\omega_s(a) = \Omega' = \nu\omega_s(a')$$

with $\mu = \nu$ only. In expression (10.1.38) this corresponds to keeping only the non-overlapping revolution and synchrotron resonances

$$k\omega_o + \nu\omega_s(a) = \Omega = (n-m)\,\omega_o + \mu\omega_s(a')$$

with $k = (n-m)$ and $\nu = \mu$. The sum over μ then drops out of the expression for $M_{\ell,m}$ in (10.2.2) and we write the resonant integral as

$$\lim_{\varepsilon \to 0^{\pm}} \int dx \, \frac{g(x)}{x + i\varepsilon} = PV \int \frac{g(x)}{x}\, dx \mp i\pi \int dx \; g(x)\; \delta(x)$$

where 'PV' stands for the principal value integral. To simplify further, we neglect the principal value term and keep only the dominant δ-function pole term. With these approximations of non-overlapping revolution and synchrotron bands and dominant pole contribution only (no principal value integral contribution), the matrix element $M_{\ell m}(\Omega = k\omega_o + \nu\omega_s(a))$ corresponding to signal suppression at $\Omega = \Omega^{k,\nu}(a) = k\omega_o + \nu\omega_s(a)$ for particles with synchrotron amplitude in the neighbourhood of a in the $(k,\nu)\underline{^{th}}$ revolution-synchrotron band, reduces to

$$M_{\ell m}\left[\Omega^{k,\nu}(a)\right] = -\pi \int da' \; \delta\left(\nu\left[\omega_s(a) - \omega_s(a')\right]\right)\left[\frac{df^0}{da'}\right] \cdot \nu \cdot$$

$$\left\{J_\nu\left[(k+\ell)a'\right] e^{i(k+\ell)\theta_p}\, \tilde{G}\left[\Omega^{k,\nu}(a) + \ell\omega_o\right]\right\} \otimes \left\{\frac{J_\nu[(k+m)a']}{(k+m)}\, e^{-i(k+m)\theta_k}\right\}$$

$$= -\pi \frac{\left[\frac{df^0}{da'}\right]_{a'=a}}{\left|\frac{d\omega_s(a')}{da'}\right|_{a'=a}} \left\{J_\nu\left[(k+\ell)a\right] e^{i(k+\ell)\theta_p}\, \tilde{G}\left[\Omega^{k,\nu}(a) + \ell\omega_o\right]\right\}$$

$$\left\{\frac{J_\nu[(k+m)a]}{(k+m)}\, e^{-i(k+m)\theta_p}\right\} \qquad (10.2.4)$$

which is thus decoupled into factors depending on ℓ and m separately and we write:

$$M_{\ell m}\left[\Omega^{k,\nu}(a)\right] = -\pi \frac{\left[\frac{df^0}{da'}\right]_{a'=a}}{\left|\frac{d\omega_s(a')}{da'}\right|_{a'=a}} P_\ell^{k,\nu}(a) \cdot Q_m^{k,\nu}(a) \qquad (10.2.5)$$

where

$$P_\ell^{k,\nu}(a) = \left\{J_\nu\left[(k+\ell)a\right] e^{i(k+\ell)\theta_p} \tilde{G}\left[(k+\ell)\omega_o + \nu\omega_s(a)\right]\right\}$$

and (10.2.6)

$$Q_m^{k,\nu}(a) = \left\{\frac{J_\nu[(k+m)a]}{(k+m)} e^{-i(k+m)\theta_p}\right\}$$

From (10.1.34), we then get

$$V^\ell\left[\Omega^{k,\nu}(a)\right] = V^{0,\ell}\left[\Omega^{k,\nu}(a)\right] + \frac{1}{2} \frac{\left[\frac{df^0}{da}\right]}{\left|\frac{d\omega_s(a)}{da}\right|} P_\ell^{k,\nu}(a) \sum_m Q_m^{k,\nu}(a)\ V^m\left(\Omega^{k,\nu}(a)\right) \qquad (10.2.7)$$

Multiplying (10.2.7) by $Q_\ell^{k,\nu}(a)$ and summing over ℓ, we get

$$X\left[\Omega^{k,\nu}(a)\right] = \frac{X_o\left[\Omega^{k,\nu}(a)\right]}{\varepsilon\left[\Omega^{k,\nu}(a)\right]} \qquad (10.2.8)$$

where

$$X\left[\Omega^{k,\nu}(a)\right] = \sum_\ell Q_\ell^{k,\nu}(a)\ V^\ell\left[\Omega^{k,\nu}(a)\right] = \sum_\ell \frac{J_\nu[(k+\ell)a]}{(k+\ell)} e^{-i(k+\ell)\theta_k} V\left[(k+\ell)\omega_o + \nu\omega_s(a)\right]$$

$$X_o\left[\Omega^{k,\nu}(a)\right] = \sum_\ell Q_\ell^{k,\nu}(a)\ V^{0,\ell}\left[\Omega^{k,\nu}(a)\right] = \sum_\ell \frac{J_\nu[(k+\ell)a]}{(k+\ell)} e^{-i(k+\ell)\theta_k} V^0\left[(k+\ell)\omega_o + \nu\omega_s(a)\right] \qquad (10.2.9)$$

and

$$\varepsilon\left[\Omega^{k,\nu}(a)\right] = 1 + \pi \frac{\left[\frac{df^0}{da}\right]}{\left|\frac{d\omega_s(a)}{da}\right|} \sum_{\ell} Q_{\ell}^{k,\nu}(a)\ P_{\ell}^{k,\nu}(a)$$

$$= 1 + \pi \frac{\left[\frac{df^0}{da}\right]}{\left|\frac{d\omega_s(a)}{da}\right|} \sum_{\nu} \frac{J_\nu^2[(k+\ell)a]}{(k+\ell)}\ e^{i(k+\ell)(\theta_p-\theta_k)}\ \tilde{G}\left[(k+\ell)\omega_0 + \nu\omega_s(a)\right] \tag{10.2.10}$$

We thus see that the total signal $X[\Omega^{k,\nu}(a)]$ (incoherent signal plus collective modulation), the incoherent signal $X_0[\Omega^{k,\nu}(a)]$ and the signal suppression factor $\varepsilon[\Omega^{k,\nu}(a)]$ at $\Omega = k\omega_0 + \nu\omega_s(a)$ are all independent of k and depend only on ν and a. Hence the signals and their suppression at the $\nu^{\underline{th}}$ synchrotron band for particles with amplitude a are all the same at all revolution harmonics. Defining a normalized distribution

$$f_N^0(a) = \frac{1}{N} f^0(a)$$

so that

$$\int_0^\infty f_N^0(a)\ da = \frac{1}{N}\int_0^\infty f^0(a)\ da = 1$$

where N is the total number of particles in the beam, we can thus express local signal suppression effect by the equation

$$X^\mu(a) = \frac{X_0^\mu(a)}{\varepsilon^\mu(a)} \tag{10.2.11}$$

where

$$X^\mu(a) = \sum_n \frac{J_\mu(na)}{n}\ e^{-in\theta_k}\ V\left[n\omega_0 + \mu\omega_s(a)\right]$$

$$X_0^\mu(a) = \sum_n \frac{J_\mu(na)}{n}\ e^{-in\theta_k}\ V^0\left[n\omega_0 + \mu\omega_s(a)\right] \tag{10.2.12}$$

and

$$\varepsilon^\mu(a) = 1 + \pi N \frac{\left[\frac{df_N^0}{da}\right]}{\left|\frac{d\omega_s(a)}{da}\right|} \sum_n \frac{J_\mu^2(na)}{n}\ \tilde{G}\left[n\omega_0 + \mu\omega_s(a)\right] \otimes e^{in(\theta_p-\theta_k)} \tag{10.2.13}$$

Note that this expression is in complete agreement with Eq. (9.2.28) obtained from kinetic theory when we use $J = 1/2\ a^2$ and $\omega_s(a)$ for $\underset{\sim}{\omega}(\underset{\sim}{I})$ and μ for $\underset{\sim}{n}$.

We also note that the proper definition of a signal, as given by (10.2.12), which allows the total signal (coherent + incoherent) to be expressed simply as a suppressed or dynamically screened incoherent signal (Eq. 10.2.11), is compatible with the 'sampled amplitude signal' defined in Eqs. (10.1.13) through (10.1.15).

For hard-edge distributions, the 'steepness parameter' $[df_N^0(a)/da]$ is large at the edge leading to a correspondingly large signal suppression $\varepsilon^\mu(a)$, if we have finite mixing in phase-space, i.e. $|d\omega_s(a)/da| \neq 0$. Hence signals from particles at the edge of the distribution in amplitude, get screened by a large factor. For an infinitely steep slope at the edge, no signals can be obtained from the edge particles, which will thus be harder to cool.

The same is true if the synchotron oscillations do not have any frequency spread, i.e. $d\omega_s(a)/da = 0$. Particles do not slip away from each other in phase-space but stay together for arbitrarily large number of synchrotron oscillation. There is no mixing in phase-space, and signals get totally suppressed by collective feedback $(\varepsilon^\mu(a) \to \infty)$. Thus no effective signal could be derived from the bunch beyond the coherent damping time $(\delta_{coh})^{-1}$ and hence no cooling could be achieved. Such strong suppression of Schottky signals for bunches with very small synchrotron frequency spread have been observed experimentally at CERN [41].

However, for moderate slope <u>or</u> steepness $[df_N^0/da]$ and moderate mixing $|d\omega_s(a)/da|$ in the bulk of phase-space, $\varepsilon^\mu(a)$ may remain a modest number and cooling, no matter how slow, is possible even for relatively small synchrotron frequency spread. The interplay between the effects of kinematic mixing and phase-space distribution is thus contained in the factor $[df_n^0/da]/|d\omega_s(a)/da|$ in $\varepsilon^\mu(a)$.

The effect of the transit time between the pick-up and the kicker is contained in the factor $e^{in(\theta_p-\theta_k)}$ and get cancelled by the factor $e^{-in(\theta_p-\theta_k)}$ implicit in $\tilde{G}$.

The sum $\sum_n \left[J_\mu^2(na)/n\right]\tilde{G}[n\omega_0+\mu\omega_s(a)]$ converges and we get a finite limiting value for the sum even for constant gain system ($\tilde{G}$ = constant), due to the factor n in the denominator.

We also see that the summation in (10.2.13) involves both positive and negative n and hence $\varepsilon^\mu(a)$ contains real and imaginary parts of $\left\{e^{in(\theta_p-\theta_k)}\ \tilde{G}[n\omega_0+\mu\omega_s(a)]\right\}$.

Finally the expressions derived in this section are valid locally at amplitude in the neighbourhood of a and synchrotron harmonic μ and so are ideally applicable in a Fokker-Planck description for no band-overlap situations, where the relevant transport equation is a partial differential equation describing cooling and diffusion locally at amplitude a.

10.3 Solution for the Water-Bag Distribution Including Principal Value Integral but Neglecting Band Overlap

We now consider a special case where an expression for signal suppression in the approximation of no revolution and synchrotron band overlap can be obtained without neglecting the principal value integral, as was done in Section 10.2.

We consider a very special distribution for the particles in the bunch, namely the water-bag distribution where phase-space density is uniform and a constant up to a certain ammplitude $a = \Delta$, beyond which the density is strictly zero (no particles beyond a certain amplitude Δ). Such a distribution, normalized to the total number N of particles, may be written as

$$f^0(a) = -\frac{N}{\Delta}\,\Theta\left(\frac{1}{a} - \frac{1}{\Delta}\right) \tag{10.3.1}$$

and

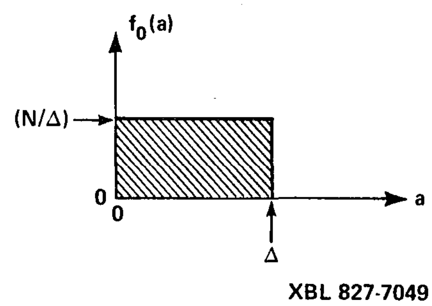
$$\frac{df^0(a)}{da} = \frac{N}{\Delta a^2}\,\delta\left(\frac{1}{a} - \frac{1}{\Delta}\right) \tag{10.3.2}$$

where $\Theta(x)$ is the heavy-side step function. (See Fig. 25 below.)

The slope of $f^0(a)$ is zero everywhere except at $a = \Delta$ where it is a δ-function. The a-integration in (10.2.2) can then be performed exactly with contribution coming from $a = \Delta$ only. We get:

$$M_{\ell m}\Big[k\omega_0 + \nu\omega_s(a)\Big] = \frac{N}{\Delta}\,\frac{(-i)\;J_\nu[(k+\ell)\Delta]}{[\omega_s(a) - \omega_s(\Delta)]}\,e^{i(k+\ell)\Theta_p}\,\tilde{G}\Big[(k+\ell)\omega_0 + \nu\omega_s(a)\Big]$$

$$\left\{\frac{J_\nu[(k+m)\Delta]}{[k+m]}\,e^{-i(k+m)\Theta_k}\right\} \tag{10.3.2}$$



Water-Bag Distribution in Amplitude for a Bunch

Fig. 25

Analogous to Section 10.2, one then obtains signal suppression in the form

$$X^{\nu}(a) = \frac{X_0^{\nu}(a)}{\left[\varepsilon^{\nu}(a)\right]}$$

where

$$X^{\nu}(a) = \sum_n \frac{J_{\nu}(n\Delta)}{n} e^{-in\theta_k} V\left[n\omega_0 + \nu\omega_s(a)\right]$$

$$X_0^{\nu}(a) = \sum_n \frac{J_{\nu}(n\Delta)}{n} e^{-in\theta_k} V^0\left[n\omega_0 + \nu\omega_s(a)\right] \tag{10.3.3}$$

and

$$\varepsilon^{\nu}(a) = 1 + \frac{N}{\Delta} \frac{(i)}{\left[\omega_s(a) - \omega_s(\Delta)\right]} \sum_n \frac{J_{\nu}^2(n\Delta)}{n} \tilde{G}\left[n\omega_0 + \nu\omega_s(a)\right] \otimes e^{+in(\theta_p - \theta_k)} \tag{10.3.4}$$

Again one notes that the signal suppression is singular at the edge $a = \Delta$. Particles at the edge in a water bag distribution will have their incoherent Schottky signal totally suppressed by the feedback system and no signal can be obtained from them beyond the coherent damping time $(\delta_{coh})^{-1}$. The same is true if there is no spread in synchrotron frequencies so that $\omega_s(a) = \omega_s(\Delta) = \omega_s(0)$.

11. COLLECTIVELY SCREENED SPECTRAL FUNCTION AND TRANSPORT COEFFICIENTS IN PRESENCE OF SIGNAL SUPPRESSION

The spectral function $\tilde{R}(\Omega)$ of the sampled signal, calculated in Chapter 8 from uncorrelated particle motion, gets collectively distorted or screened by the feedback loop induced modulations. This in turn modifies the transport coeficients in the time-evolution equation, since such coefficients are determined by the spectral functions as seen in 9.1. We have obtained the transport equation in absence of such screening in 9.1 and the suppression of the total collective signals separately in Chapter 10. In this chapter we study the modifications of the Friction and Diffusion coefficients arising from collectively screened Schottky signals. Note that in Section 9.2 we have already obtained a time-evolution equation including such signal suppression effects from kinetic theory. However, the derivation was limited to non-overlapping synchrotron bands only. We derive formulas for the modified spectral function and transport coefficients in terms of the matrix elements of the inverse signal suppression matrix, $[\underset{\approx}{\epsilon}^{-1}(\Omega)]_{\ell m}$. Usefulness of these formulas however depends on one's ability to invert the matrix $\underset{\approx}{\epsilon}(\Omega)$.

From Chapter 6, the sampled action noise signal is given by

$$\dot{J} = n(t) = \frac{q\kappa\sqrt{2J}}{\omega_s(J)} \mathscr{S}(t) \operatorname{Cos} \psi(t)$$

where

$$\mathscr{S}(t) = \omega_0 \sum_{n=-\infty}^{+\infty} V^K(t)\, \delta\left[\theta(t) - \theta_K - 2\pi n\right]$$

is the sampled voltage signal. Using the identity (7.1), we can write:

$$n(t) = \frac{(qf_0\kappa)}{\omega_s(J)} \sum_{\nu}^{(+\infty)}\sum_{n}^{} \left(\frac{\nu}{n}\right) J_\nu\left(n\sqrt{2J}\right) e^{-in\theta_K} \left[V^K(t)\, e^{in\omega_0 t}\right] e^{i\nu\psi(t)} \qquad (11.1)$$
$$(-\infty)$$

where $\psi(t) = \omega_s(a)t + \psi(0)$ and $a = \sqrt{2J}$ is the synchrotron oscillation amplitude. The auto-correlation can be written as:

$$R(t,t') = \left\langle n(t)\ n(t')\right\rangle$$

$$= \frac{(qf_o\kappa)^2}{\omega_s^2(J)} \sum_\nu \sum_\mu \sum_n \sum_m \frac{(\nu\cdot\mu)}{(n\cdot m)} J_\nu(n\sqrt{2J})\ J_\mu(m\sqrt{2J})\ e^{-i(n+m)\ \theta_K}$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} d\Omega'\ e^{i(\Omega t+\Omega' t')}$$

$$\left\langle e^{i(\nu+\mu)\psi(0)}\ V^K\!\left(\Omega + n\omega_o + \nu\omega_s(J)\right)\ V^K\!\left(\Omega' + m\omega_o + \mu\omega_s(J)\right)\right\rangle \qquad (11.2)$$

Here $V_K(\Omega)$ is the total voltage at the kicker in presence of collective signal suppression and is related to $V_{oK}(\Omega)$, the voltage due to incoherent particle motion only, by the signal suppression factor $\underset{\approx}{\varepsilon}(\Omega)$ as in Chapter 10:

$$\underset{\approx}{\varepsilon}(\Omega)\cdot \underset{\sim}{V}^K(\Omega) = \underset{\sim}{V}^{oK}(\Omega)$$

where

$$\underset{\sim}{V}^K(\Omega) = \left\{V_n^K\ (\Omega)\right\}_{n=0,\pm1,\pm2,\ldots} = \left\{V^K\!\left(\Omega + n\omega_o\right)\right\}_{n=0,\pm1,\pm2,\ldots}$$

Thus:

$$\underset{\sim}{V}^K(\Omega) = \left[\underset{\approx}{\varepsilon}(\Omega)\right]^{-1}\cdot \underset{\sim}{V}^{oK}(\Omega)$$

i.e.

$$V_\ell^K(\Omega) = \sum_m \left[\varepsilon^{-1}(\Omega)\right]_{\ell m} V_m^{oK}(\Omega) \qquad (11.3)$$

where

$$V_\ell^K(\Omega) = V^K\!\left(\Omega + \ell\omega_o\right)$$

Using (11.3), we obtain the auto-correlation function for sampled longitudinal action noise given by (11.2), in the presence of the signal suppression effect described by matrix $\underset{\approx}{\varepsilon}(\Omega)$ as follows:

$$R(t,t') = \frac{(qf_o\kappa)^2}{\omega_s^2(J)} \sum_\nu \sum_\mu \sum_n \sum_m \left(\frac{\nu}{n}\frac{\mu}{m}\right) J_\nu(n\sqrt{2J})\; J_\mu(m\sqrt{2J})\; e^{-i(n+m)\theta_K}\; e^{i(\nu+\mu)\psi(0)}$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} d\Omega'\; e^{i(\Omega t+\Omega' t')} \sum_{n'} \sum_{m'} \left[\varepsilon^{-1}\left(\Omega + \nu\omega_s(J)\right)\right]_{nn'} \left[\varepsilon^{-1}\left(\Omega' + \mu\omega_s(J)\right)\right]_{mm'}$$

$$\left\langle V_{n'}^{oK}\left(\Omega + \nu\omega_s(J)\right)\; V_{m'}^{oK}\left(\Omega' + \mu\omega_s(J)\right)\right\rangle \tag{11.4}$$

The incoherent Schottky noise voltage at the kicker is given by (Chapter 6, Eq. (6.1.4)):

$$V^{oK}(\Omega) = (2\pi)(qf_o) \sum_{j=1}^{N} \sum_{\ell\,(-\infty)}^{(+\infty)} \sum_\lambda G\left[\ell\omega_o + \lambda\omega_s(J_j)\right]\; J_\lambda(\ell\sqrt{2J_j})\; e^{i\lambda\psi_j(0)-i\ell\theta_p}\; \delta\left[\Omega - \ell\omega_o - \lambda\omega_s(J_j)\right] \tag{11.5}$$

The time-stationary auto-correlation function can now be obtained by performing the ensemble average $<\,>$ and averaging over the phase $\psi(0)$ of the sampling test particle, as in Chapter 8. Defining the quantities

$$X_{\nu\lambda}^{n\ell}(J,J') = \left(\frac{\nu}{n}\right) \tilde{G}\left[\ell\omega_o + \lambda\omega_s(J')\right]\; J_\nu(n\sqrt{2J})\; J_\lambda(\ell\sqrt{2J'})\; e^{-in\theta_K-i\ell\theta_p} \tag{11.6}$$

and

$$Z_{\nu,\lambda}^{k}(J,J') = \sum_n \sum_{n'} X_{\nu,\lambda}^{n,(k-n')}(J,J') \left[\varepsilon^{-1}\left(k\omega_o + \lambda\omega_s(J')\right)\right]_{nn'} \tag{11.7}$$

and the reality condition

$$Z_{-\lambda,-\nu}^{-k}(J,J') = \left[Z_{\lambda,\nu}^{k}(J,J')\right]^* \tag{11.8}$$

one obtains, after performing the averaging similar to Chapter 8, for the spectral function

$$\tilde{R}(\Omega) = \frac{(q^2 f_0^2 \kappa)^2}{\omega_s^2(J)} (2\pi) \cdot N \int dJ' \; f(J') \sum_k \sum_\nu \sum_\lambda \left| Z^k_{\nu,\lambda}(J,J') \right|^2$$

$$\delta\left[\Omega - k\omega_0 - \nu\omega_s(J) - \lambda\omega_s(J')\right] \tag{11.9}$$

and for the Diffusion coefficient:

$$D(J) = \tilde{R}(\Omega)\Big|_{\Omega=0}$$

$$= \frac{\left(q^2 f_0^2 \kappa\right)^2}{\omega_s^2(J)} (2\pi) \cdot N \int dJ' \; f(J') \sum_k \sum_\nu \sum_\lambda \left| Z^k_{\nu,\lambda}(J,J') \right|^2 \delta\left[k\omega_0 + \nu\omega_s(J) + \lambda\omega_s(J')\right] \tag{11.10}$$

$$= \frac{\left(q^2 f_0^2 \kappa\right)^2}{\omega_s^2(J)} (2\pi) \cdot N \int dJ' \; f(J') \sum_k \sum_\nu \sum_\lambda$$

$$\left| \sum_n \sum_{n'} X^{n,(k-n')}_{\nu,\lambda}(J,J') \left[\varepsilon^{-1}\left(k\omega_0 + \nu\omega_s(J')\right)\right]_{nn'} \right|^2 \delta\left[k\omega_0 + \nu\omega_s(J) + \lambda\omega_s(J')\right] \tag{11.11}$$

Equation (11.11) is a general result which we did not obtain from the kinetic theory in Section 9.2. It describes how the spectral function and the Diffusion coefficient get modified in presence of beam feedback described by a generalized signal suppression matrix $\underset{\approx}{\varepsilon}(\Omega)$. Evaluation of $\tilde{R}(\Omega)$ and $D(J)$ explicitly however requires an effective inversion of the infinite matrix $\underset{\approx}{\varepsilon}(\Omega)$, since it is the matrix elements of $[\underset{\approx}{\varepsilon}(\Omega)]^{-1}$ that appears in Eqs. (11.9) and (11.11). Let us now recover from (11.11) the result for $D(J)$ obtained from kinetic theory in Section 9.2 in the limit of no revolution and synchrotron band overlap.

In the limit of no revolution band overlap only the $k = 0$ term contributes in (11.11). In addition, if there is no synchrotron band overlap, we only have to consider the resonance $\nu\omega_s(J) + \lambda\omega_s(J') = 0$ with $\nu = -\lambda$ only. The Diffusion coefficient then is given by:

$$D(J) = \frac{\left(q^2 f_0^2 {}_K\right)^2}{\omega_s^2(J)} (2\pi) \cdot N \sum_{\nu} \frac{f(J)}{|\nu| \left|\frac{d\omega_s(J)}{dJ}\right|} \left| \sum_n \sum_{n'} X^{n,-n'}_{\nu,-\nu}(J,J) \cdot \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{nn'} \right|^2 \tag{11.12}$$

where

$$\sum_n \sum_{n'} X^{n,-n'}_{\nu,-\nu}(J,J) \cdot \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{-nn'} = \sum_n \sum_{n'} \left(\frac{\nu}{n}\right) \tilde{G}\left[-n'\omega_0 - \nu\omega_s(J)\right] e^{-in\theta_K + in\theta_P}$$

$$J_\nu(n\sqrt{2J})\; J_{-\nu}(-n'\sqrt{2J}) \cdot \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{nn'} \tag{11.13}$$

where we have used Eqs. (11.6) and (11.7). According to the definitions in (10.2.6), we can then write

$$\sum_n \sum_{n'} X^{n,-n'}_{\nu,-\nu}(J,J) \cdot \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{-nn'} = \sum_n \sum_{n'} \nu \cdot P^{-\nu}_{-n'}(J)\; Q^{\nu}_n(J) \cdot \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{nn'}$$

$$= \sum_{n'} \nu \cdot P^{-\nu}_{-n'}(J) \left\{ \sum_n Q^{\nu}_n(J) \left[\varepsilon^{-1}\left(\nu\omega_s(J)\right)\right]_{nn'} \right\} \tag{11.14}$$

where

$$P^{-\nu}_{-n'}(J) = \tilde{G}\left[-n'\omega_0 - \nu\omega_s(J)\right] J_{-\nu}(-n'\sqrt{2J})\; e^{in'\theta_P}$$

$$Q^{\nu}_n(J) = \frac{J_\nu(n\sqrt{2J})}{n} e^{-in\theta_K} \tag{11.15}$$

From (11.3),

$$V^K_\ell(\Omega) = \sum_m \left[\varepsilon^{-1}(\Omega)\right]_{\ell m} V^{0K}_m(\Omega)$$

Multiplying by $Q^{\nu}_{\ell}(J)$, and summing over ℓ,

$$\sum_{\ell=-\infty}^{+\infty} Q_\ell^\nu(J)\; V_\ell^K(\Omega) = \sum_{\ell}\sum_{m}{}^{(+\infty)}_{(-\infty)} Q_\ell^\nu(J) \left[\epsilon^{-1}(\Omega)\right]_{\ell m} V_m^{oK}(\Omega)$$

$$= \sum_m \left\{\sum_\ell Q_\ell^\nu(J) \left[\epsilon^{-1}(\Omega)\right]_{\ell m}\right\} V_m^{oK}(\Omega)\;. \tag{11.16}$$

From Chapter 10 however, we know that for non-overlapping synchrotron bands, the response at $\Omega = \nu\omega_s(J)$, is given by:

$$\sum_{\ell=-\infty}^{+\infty} Q_\ell^\nu(J)\; V_\ell^K\Big(\Omega = \nu\omega_s(J)\Big) = \frac{\sum_\ell Q_\ell^\nu(J)\; V_\ell^{oK}(\Omega = \nu\omega_s(J))}{\epsilon^\nu(J)} \tag{11.17}$$

where

$$\epsilon^\nu(J) = 1 + \pi N \frac{(df^0/dJ)}{\left|\frac{d\omega_s(J)}{dJ}\right|} \sum_n \frac{J_\nu^2(n\sqrt{2J})}{n} \tilde{G}\Big[n\omega_o + \nu\omega_s(J)\Big]\; e^{in(\theta_P - \theta_K)}$$

Comparing (11.16) and (11.17) we obtain

$$\sum_\ell Q_\ell^\nu(J) \left[\epsilon^{-1}\Big(\Omega = \nu\omega_s(J)\Big)\right]_{\ell m} = \frac{1}{\epsilon^\nu(J)} \cdot Q_m^\nu(J) \tag{11.18}$$

$$\underset{\sim}{Q}^\nu(J) \cdot \underset{\approx}{\epsilon}^{-1}\Big(\Omega = \nu\omega_s(J)\Big) = \left[\frac{1}{\epsilon^\nu(J)}\right] \cdot \underset{\sim}{Q}^\nu(J) \tag{11.19}$$

Thus $[1/\epsilon^\nu(J)]$ is the left eigenvalue of the matrix $\underset{\approx}{\epsilon}^{-1}(\Omega = \nu\omega_s(J))$ with $\underset{\sim}{Q}^\nu(J)$ the corresponding eigenfunctions. Using (11.19) in (11.14) and (11.12) we obtain

$$D(J) = \frac{\left(q^2 f_o^2{}_K\right)^2}{\omega_s^2(J)} (2\pi) \cdot N \sum_\nu \frac{f(J)}{|\nu| \left|\frac{d\omega_s(J)}{dJ}\right|} \left|\frac{\sum_{n'} \nu \cdot P_{-n'}^{-\nu}(J)\; Q_{n'}^\nu(J)}{\epsilon^\nu(J)}\right|^2 \tag{11.20}$$

Recalling the definition

$$G_{\nu,-\nu}(J,J) = \frac{\left(q^2 f_0^2 \kappa\right)}{\omega_s(J)} \sum_{n'} \nu \cdot P_{-n'}^{-\nu}(J)\ Q_{n'}^{\nu}(J)$$

from Chapter 7, we write

$$D(J) = (2\pi) \cdot N \cdot \sum_{\nu} \frac{f(J)}{|\nu| \left| \frac{d\omega_s(J)}{dJ} \right|} \left| \frac{G_{\nu,-\nu}(J,J)}{\epsilon^{\nu}(J)} \right|^2 \tag{11.21}$$

which is what we obtained in Section 9.2. Similarly one can show, starting from Eqs. (11.1), (11.3), (11.5) and using (11.15), (11.17), (11.19), that the modified friction or coherent cooling term, reduces in the limit of no revolution and synchrotron band overlap to

$$F(J) = \sum_{\nu=-\infty}^{+\infty} \frac{\left[G_{\nu,\nu}(J,J)\right]}{\left[\epsilon^{\nu}(J)\right]} \tag{11.22}$$

12. SIGNAL SUPPRESSION MATRIX FOR TRANSVERSELY COUPLED BETATRON COOLING OF COASTING BEAMS

The collective response arising from a betatron cooling interaction that couples both the transverse degrees of freedom is in general described by a dielectric tensor $\underset{\approx}{\varepsilon}(\Omega)$ that couples the components of the collectively screened vector electromagnetic signal $\underset{\sim}{\chi}(\Omega;\theta_k)$ at the kicker to those of the unscreened signal $\underset{\sim}{\chi}^0(\Omega;\theta_k)$ at any given frequency Ω, and may be written as

$$\underset{\approx}{\varepsilon}(\Omega) \cdot \underset{\sim}{\chi}(\Omega;\theta_k) = \underset{\sim}{\chi}^0(\Omega;\theta_k) \tag{12.1}$$

$$\underset{\sim}{\chi}(\Omega;\theta_k) = \left[\underset{\approx}{\varepsilon}(\Omega)\right]^{-1} \cdot \underset{\sim}{\chi}^0(\Omega;\theta_k) \tag{12.2}$$

We derive here $\underset{\approx}{\varepsilon}(\Omega)$ for coasting beams from first-order linearized Vlasov theory. The inversion of this matrix, $[\underset{\approx}{\varepsilon}(\Omega)]^{-1}$ naturally implies a tensorial character of the response and a single scalar signal suppression factor or dielectric response function does not exist in general, except under special circumstances.

For transverse cooling of a coasting beam, the phase-space coordinates of a particle are written as

$$(\underset{\sim}{I},\underset{\sim}{\psi}) = \left(\underset{\sim}{I}_\perp,\omega;\ \underset{\sim}{\phi}_\perp,\theta\right) \tag{12.3}$$

where $\underset{\sim}{I}_\perp \equiv (I_x,I_z)$, $\underset{\sim}{\phi}_\perp \equiv (\phi_x,\phi_z)$, ω the longitudinal angular velocity and θ the angle around the storage ring. The first-order linearized Vlasov equation for perturbed distribution function $f \equiv f(\underset{\sim}{I}_\perp,\underset{\sim}{\phi}_\perp;\omega,\theta;t)$ is

$$\frac{\partial f}{\partial t} + \underset{\sim}{\omega}_\perp \cdot \frac{\partial f}{\partial \underset{\sim}{\phi}_\perp} + \omega \frac{\partial f}{\partial \theta} + \dot{\underset{\sim}{I}}_\perp \cdot \frac{df^0}{d\underset{\sim}{I}_\perp} = 0 \tag{12.4}$$

where $f^0 \equiv f^0(\underset{\sim}{I}_\perp,\omega)$ is the stationary zero-order distribution and $\dot{\omega} \equiv 0$ for no longitudinal cooling.

Our model for cooling interaction is:

$$\dot{I}_\perp\left(I_\perp,\phi_\perp;\ \theta,\omega;\ t\right) = \delta^P\left(\theta-\theta_k\right)\int_0^\infty\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{t} \mathcal{G}\left(I_\perp,\phi_\perp;\ I_\perp',\phi_\perp';\ \omega,\omega';\ t-t'\right)$$

$$\otimes f\left(I_\perp',\phi_\perp';\ \theta_p,\omega';\ t'\right)\, dI_\perp'\, d\phi_\perp'\, d\omega'\, dt' \quad . \tag{12.5}$$

where $\delta^P(\theta-\theta_k) = \sum_{n=-\infty}^{+\infty} \delta(\theta-\theta_k-2\pi n)$ is the periodic δ-function.

Note that $\underset{\sim}{G}$ is a vector interaction, determining the cooling dynamics in the two directions x and z respectively. Using the fact that $\underset{\sim}{\phi}_\perp$, $\underset{\sim}{\phi}_\perp'$ are periodic angle variables with period 2π, we can rewrite Eq. (12.5) as follows:

$$\dot{I}_\perp\left(I_\perp,\phi_\perp;\ \theta,\omega;\ t\right) = \delta^P\left(\theta-\theta_k\right)\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dI_\perp'\, d\omega'\, d\Omega \sum_{\underset{\sim}{n}}\sum_{\underset{\sim}{n}'} e^{i\underset{\sim}{n}\cdot\phi_\perp}\, e^{i\Omega t}$$

$$\mathcal{G}_{\underset{\sim}{n}\underset{\sim}{n}'}\left(I_\perp,I_\perp';\ \omega,\omega';\ \Omega\right) f_{-\underset{\sim}{n}'}\left(I_\perp',\omega';\ \theta_p;\ \Omega\right). \tag{12.6}$$

In Eq. (12.4), we now perform a Fourier series decomposition in harmonics of the angles variables $\underset{\sim}{\phi}_\perp$ and θ and a Fourier transform in time. Using Eqs. (12.6) and (12.4) then gives:

$$i\left[\Omega+\underset{\sim}{n}\cdot\underset{\sim}{\omega}_\perp+\ell\omega\right]\tilde{f}^{\ell}_{\underset{\sim}{n}}\left(I_\perp,\omega;\ \Omega\right)$$

$$+\left(\frac{1}{2\pi}\right) e^{-i\ell\theta_k}\left(\frac{df^0}{dI_\perp}\right)\cdot\iint dI_\perp'\, d\omega' \sum_{\underset{\sim}{n}'}\sum_{m} e^{im\theta_p}\, \mathcal{G}_{\underset{\sim}{n}\underset{\sim}{n}'}\left(I_\perp,I_\perp';\ \omega,\omega';\ \Omega\right)\ \tilde{f}^m_{-\underset{\sim}{n}'}\left(\underset{\sim}{I}_\perp',\omega';\ \Omega\right)$$

$$= \tilde{f}^{0\ell}_{\underset{\sim}{n}}(I_\perp,\omega) \tag{12.7}$$

where $\tilde{f}^{0\ell}_{\underset{\sim}{n}}(\underset{\sim}{I}_\perp,\omega)$ is an arbitrary excitation (can be identified with the incoherent zero-order Schottky signal excitation in the absence of kicker induced modulations).

We now assume the separated variable representation given by Eq. (4.3.28), i.e. we assume that the dependence of the gain $\underset{\sim}{G}_{\underset{\sim}{n}\underset{\sim}{n}'}(\underset{\sim}{I}_\perp,\underset{\sim}{I}_\perp';\omega,\omega';\Omega)$ on the 'kicker' and the 'kicked' particle variables separate in the following way:

$$G^{\alpha}_{\underline{n}\underline{n}'}\left(I_\perp, I'_\perp;\ \omega,\omega';\Omega\right) = K^{\alpha}_{\underline{n}}(I_\perp,\omega)\ P^{\alpha}_{\underline{n}'}\left(I'_\perp,\omega';\Omega\right) \tag{12.8}$$

where $\alpha = \{x,z\}$. In a more general notation, we can then write:

$$\underline{G}_{\underline{n}\underline{n}'}\left(I_\perp, I'_\perp;\ \omega,\omega';\ \Omega\right) = \underset{\approx}{K}_{\underline{n}}(I_\perp,\omega)\ \cdot\ \underline{P}_{\underline{n}'}\left(I'_\perp,\omega';\ \Omega\right) \tag{12.9}$$

where $\underset{\approx}{K}_{\underline{n}}(\underset{\sim}{I}_\perp,\omega)$ is the matrix:

$$\underset{\approx}{K}_{\underline{n}}\left(I_\perp,\omega\right) = \begin{bmatrix} K^{x}_{\underline{n}}(I_\perp,\omega) & 0 \\ 0 & K^{z}_{\underline{n}}(I_\perp,\omega) \end{bmatrix} \tag{12.10}$$

and $\underset{\sim}{P}_{\underline{n}'}(I'_\perp,\omega';\Omega)$ is the vector:

$$\underline{P}_{\underline{n}'}\left(I'_\perp,\omega';\Omega\right) = \begin{bmatrix} P^{x}_{\underline{n}'}(I'_\perp,\omega';\ \Omega) \\ P^{z}_{\underline{n}'}(I'_\perp,\omega';\ \Omega) \end{bmatrix} \tag{12.11}$$

Using (12.9) in (12.7), we obtain:

$$i\left[\Omega + \underline{n}\cdot\underline{\omega}_\perp + \ell\omega\right]\tilde{f}^{\ell}_{\underline{n}}\left(I_\perp,\omega;\ \Omega\right)$$

$$+\left(\frac{1}{2\pi}\right)e^{-i\ell\theta_k}\left(\frac{df^0}{dI_\perp}\right)\iint dI'_\perp\ \ d\omega'\ \underset{\approx}{K}_{\underline{n}}(I_\perp,\omega)\ \cdot \sum_{\underline{n}'}\sum_{m} e^{im\theta_p}\ \underline{P}_{\underline{n}'}\left(I'_\perp,\omega';\ \Omega\right)\otimes\tilde{f}^{m}_{-\underline{n}'}\left(I'_\perp,\omega';\ \Omega\right)$$

$$= \tilde{f}^{0\ell}_{\underline{n}}(I_\perp,\omega) \tag{12.12}$$

We now define a collectively screened vector signal by

$$\underset{\sim}{X}(\Omega) = \int_{-\infty}^{+\infty} d\omega' \sum_{\underset{\sim}{n}'} \sum_{m} e^{im\theta_p} \underset{\sim}{g}^{m}_{n'}(\omega';\Omega) \tag{12.13}$$

where

$$g^{m}_{n'}(\omega';\Omega) = \int_{0}^{\infty} dI'_{\perp} \; P_{n'}\left(I'_{\perp},\omega';\Omega\right) \; \tilde{f}^{m}_{-n'}\left(I'_{\perp},\omega';\Omega\right) \tag{12.14}$$

Then (12.12) gives

$$X(\Omega) + \left[\frac{1}{2\pi}\sum_{n}\sum_{\ell}\int_{-\infty}^{+\infty}\int_{0}^{\infty} d\omega \cdot d\underset{\sim}{I}_{\perp} \; e^{-i\ell(\theta_k-\theta_p)} \frac{P_{n}(I_{\perp},\omega;\Omega) \; \frac{df^{0}}{dI_{\perp}} \cdot \underset{\approx}{K}_{n}(I_{\perp},\omega)}{i\left[\Omega + n \cdot \omega_{\perp} + \ell\omega\right]}\right] \cdot X(\Omega)$$

$$= X^{0}(\Omega) \tag{12.15}$$

where:

$$X^{o}(\Omega) = \int_{-\infty}^{+\infty} d\omega \sum_{n}\sum_{\ell} \frac{\underset{\sim}{g}^{o\ell}_{n}(\omega) \; e^{i\ell\theta_p}}{i\left[\Omega + n \cdot \omega_{\perp} + \ell\omega\right]} \tag{12.16}$$

is the incoherent Schottky or any arbitrary signal.

We can rewrite Eq. (12.15) as:

$$\underset{\approx}{\varepsilon}(\Omega) \cdot X(\Omega) = X^{o}(\Omega) \tag{12.17}$$

where:

$$\underset{\approx}{\varepsilon}(\Omega) = \underset{\approx}{I} + \left(\frac{1}{2\pi}\right)\sum_{n}\sum_{\ell}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} d\omega \cdot dI_{\perp} \; e^{-i\ell(\theta_k-\theta_p)} \frac{P_{n}(I_{\perp},\omega;\Omega) \; \frac{df^{0}}{dI_{\perp}} \cdot \underset{\approx}{K}(I_{\perp},\omega)}{i\left[\Omega + n \cdot \omega_{\perp} + \ell\omega\right]} \tag{12.18}$$

and $\underset{\approx}{I}$ is the unit matrix or diagonal unit tensor.

We have thus derived an expression for the dielectric tensor $\underset{\approx}{\varepsilon}(\Omega)$ given by Eq. (12.18) in presence of coupling between two transverse degrees of freedom. We write the matrix elements of $\underset{\approx}{\varepsilon}(\Omega)$ given by Eq. (12.18) in explicit form as:

$$\epsilon_{\alpha\beta}(\Omega) = \delta_{\alpha\beta} + \left(\frac{1}{2\pi}\right)\sum_{\underset{\sim}{n}}\sum_{\ell}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\omega \cdot dI_{\perp} \frac{e^{-i\ell(\theta_k-\theta_p)}}{i\left[\Omega+\underset{\sim}{n}\cdot\underset{\sim}{\omega}_{\perp}+\ell\omega\right]} P_{\underset{\sim}{n}}^{\alpha}(I_{\perp},\omega;\ \Omega)\left[\frac{df^0}{dI_{\beta}}\ K_{\underset{\sim}{n}}^{\beta}(I_{\perp};\Omega)\right] \tag{12.19}$$

where $\alpha,\beta = (x,z)$. Determinant of this matrix is given by:

$$\Delta = \det\ \underset{\approx}{\epsilon}(\Omega)$$

$$= 1 + \left(\frac{1}{2\pi}\right)\sum_{\underset{\sim}{n}}\sum_{\ell}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} d\omega dI_{\perp} \frac{e^{-i\ell(\theta_k-\theta_p)}}{i\left[\Omega+\underset{\sim}{n}\cdot\underset{\sim}{\omega}_{\perp}+\ell\omega\right]}\frac{df^0}{dI_{\perp}}\cdot \mathbb{G}_{\underset{\sim}{n}\underset{\sim}{n}}\left(I_{\perp},I_{\perp};\ \omega,\omega;\ \Omega\right)$$

$$+ \left(\frac{1}{2\pi}\right)^2\sum_{\underset{\sim}{n}}\sum_{\underset{\sim}{n}'}\sum_{\ell}\sum_{\ell'}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} d\omega d\omega' dI_{\perp} dI_{\perp}' \frac{e^{-i(\ell+\ell')(\theta_k-\theta_p)}(-1)}{\left[\Omega+\underset{\sim}{n}\cdot\underset{\sim}{\omega}_{\perp}+\ell\omega\right]\left[\Omega+\underset{\sim}{n}'\cdot\underset{\sim}{\omega}_{\perp}'+\ell'\omega'\right]}$$

$$\left[\frac{df^0}{dI_x}\ G_{\underset{\sim}{n}\underset{\sim}{n}}^{x}(I_{\perp},\omega;\ \Omega)\ \frac{df^0}{dI_z}\ G_{\underset{\sim}{n}'\underset{\sim}{n}'}^{z}(I_{\perp}',\omega';\ \Omega) - \frac{df^0}{dI_z}\ G_{\underset{\sim}{n}\underset{\sim}{n}'}^{x}(I_{\perp},I_{\perp}',\omega,\omega';\ \Omega)\ \otimes\right.$$

$$\left.\frac{df^0}{dI_x}\ G_{\underset{\sim}{n}'\underset{\sim}{n}}^{z}(I_{\perp}',I_{\perp},\omega'\omega;\Omega)\right] \tag{12.20}$$

The inverse matrix $\underset{\approx}{\epsilon}^{-1}(\Omega)$ is then given by:

$$\underset{\approx}{\epsilon}^{-1}(\Omega) = \frac{1}{\Delta}\begin{bmatrix}\epsilon_{11} & -\epsilon_{21}\\ -\epsilon_{12} & \epsilon_{22}\end{bmatrix} \tag{12.21}$$

where Δ is given by Eq. (12.20) and the elements ϵ_{11}, ϵ_{12}, ϵ_{21} and ϵ_{22} are given by the matrix elements of $\underset{\approx}{\epsilon}(\Omega)$ given in Eq. (12.19).

Analogous to Ch. 11, one can then show that the auto-correlation of collectively screened sampled signal is given by:

$$\underset{\approx}{R}(t,t') = \frac{q^2 f_o^2}{(2\pi)^2} \sum_n \sum_m \iint d\Omega d\Omega' \, e^{i(\Omega t+\Omega' t')} \, e^{-i(n+m)\theta_k} \, e^{i(n+m)\theta(0)}$$

$$\left\langle \left[\underset{\approx}{\varepsilon}^{-1}(\Omega+n\omega) \cdot \tilde{\underset{\sim}{X}}^0(\Omega+n\omega)\right] \left[\underset{\approx}{\varepsilon}^{-1}(\Omega'+m\omega)\cdot\tilde{\underset{\sim}{X}}^0(\Omega'+m\omega)\right]\right\rangle .$$

<u>or</u>

$$R^{\alpha\beta}(t,t') = \frac{q^2 f_o^2}{(2\pi)^2} \sum_n \sum_m \iint d\Omega d\Omega' \, e^{i(\Omega t+\Omega' t')} \, e^{-i(n+m)\theta_k} \, e^{i(n+m)\theta(0)}$$

$$\sum_\gamma \sum_\delta \varepsilon^{-1}_{\alpha\gamma}(\Omega+n\omega) \, \varepsilon^{-1}_{\beta\delta}(\Omega'+m\omega) \left\langle \tilde{X}^{0\gamma}(\Omega+n\omega) \, \tilde{X}^{0\delta}(\Omega'+m\omega)\right\rangle \tag{12.22}$$

and the diffusion coefficient is obtained by the prescription

$$D^{\alpha\beta} = \tilde{R}^{\alpha\beta}(\Omega)\Big|_{\Omega=0}$$

where $\tilde{R}^{\alpha\beta}(\Omega)$ is the Fourier transform of an averaged time-stationary auto-correlation, given by

$$\tilde{R}^{\alpha\beta}(\Omega) = \int_{-\infty}^{+\infty} d\tau \, \bar{R}^{\alpha\beta}(\tau) \, e^{-i\Omega\tau}$$

where $\tau = (t-t')$.

The components of the suppressed signal are given by

$$X^x(\Omega) = \frac{1}{\Delta}\left[\varepsilon_{11}(\Omega) \, X_0^x(\Omega) - \varepsilon_{21}(\Omega) \, X_0^z(\Omega)\right]$$

$$X^z(\Omega) = \frac{1}{\Delta}\left[-\varepsilon_{21}(\Omega) \, X_0^x(\Omega) + \varepsilon_{22}(\Omega) \, X_0^z(\Omega)\right] \tag{12.23}$$

Under some special cases, we might get an isotropic scalar dielectric response. This happens when the betatron bands do not overlap and the coherent signal in harmonic $\underset{\sim}{n}$ is proportional to $\underset{\sim}{n}$.

If bands do not overlap, then we get:

$$\underset{\approx n}{\epsilon}(\Omega) \cdot \underset{\sim}{X}_n(\Omega) = \underset{\sim}{X}_n^0(\Omega)$$

Betatron bands decouple and no sum over $\underset{\sim}{n}$ is present. If then:

$$\underset{\sim}{X}_n(\Omega) = \underset{\sim}{n}\, X(\Omega)$$

we have

$$\left\{\underset{\approx n}{\epsilon}(\Omega) \cdot \underset{\sim}{n}\right\} X(\Omega) = \underset{\sim}{n}\, X^0(\Omega)$$

or

$$\left[\underset{\sim}{n} \cdot \underset{\approx n}{\epsilon}(\Omega) \cdot \underset{\sim}{n}\right] X(\Omega) = X^0(\Omega)\ |\underset{\sim}{n}|^2$$

or

$$X(\Omega) = \frac{X^0(\Omega)}{\epsilon(\Omega)}$$

where

$$\epsilon(\Omega) = \frac{1}{|\underset{\sim}{n}|^2}\left[\underset{\sim}{n} \cdot \underset{\approx n}{\epsilon}(\Omega) \cdot \underset{\sim}{n}\right]$$

$$= \frac{\left[\epsilon_{11}n_x^2 + (\epsilon_{12}+\epsilon_{21})n_x n_z + \epsilon_{22}n_z^2\right]}{n_x^2 + n_z^2} \tag{12.24}$$

where $\epsilon_{11}, \epsilon_{12}, \ldots$etc. are given by elements of Eq. (12.19) with sum over $\underset{\sim}{n}$ omitted. In this case, the friction coefficients are simply divided by $\epsilon(\Omega)$ while the diffusion tensor elements are divided by $|\epsilon(\Omega)|^2$.

13. STUDIES OF A NUMERICAL SIMULATION

We present results of a numerical simulation study with 90 pseudo-particles for transverse and longitudinal stochastic cooling of bunched particle beams. Radio-frequency buckets of various shapes (e.g. rectangular, parabolic well, single sinusoidal waveform) are used to investigate the enhancement of phase-space cooling by nonlinearities of synchrotron motion. The connection between the notions of Landau damping for instabilities and kinematic mixing for stochastic cooling are discussed. In particular, the need for synchrotron frequency spread for both Landau damping and good mixing is seen to be comparable for bunched beams. Replacement of a real bucket orbit by a simple sinusoidal orbit with amplitude-dependent synchrotron frequency is substantiated and a comparison with analytic results is given.

13.1 Particle Orbits Studied

We have investigated four types of particle orbits in a bunch -- orbits in a rectangular potential well, in a parabolic well with linear synchrotron oscillations of amplitude-independent frequency, in a sinusoidal RF bucket, and sinusoidal orbits as in the parabolic case but with an imposed amplitude dependence of the synchrotron frequency. The motivation to investigate the orbits belonging to the last cateogry has been to study directly the effect of a spread in synchrotron frequencies on the cooling rate.

The potential wells and the corresponding orbits for the first three cases are shown in Fig. 26(a), (b) and (c). The orbit in a real rf bucket (Fig. 26(c)) is represented by the first four odd-harmonics only of the exact solution, as follows:

$$\theta(t) = 4 \sum_{n=1}^{4} \frac{\mathrm{Sin}[(2n-1)\omega\, t]}{(2n-1)\cosh\left[\frac{k'(2n-1)\omega}{\omega_0}\right]}$$

where

$$\omega \equiv \omega(\theta_m) = \frac{\pi\omega_0}{K(k)}\,; \quad k = \mathrm{Sin}(\theta_m/2); \quad k' = \left(1-k^2\right)^{1/2}$$

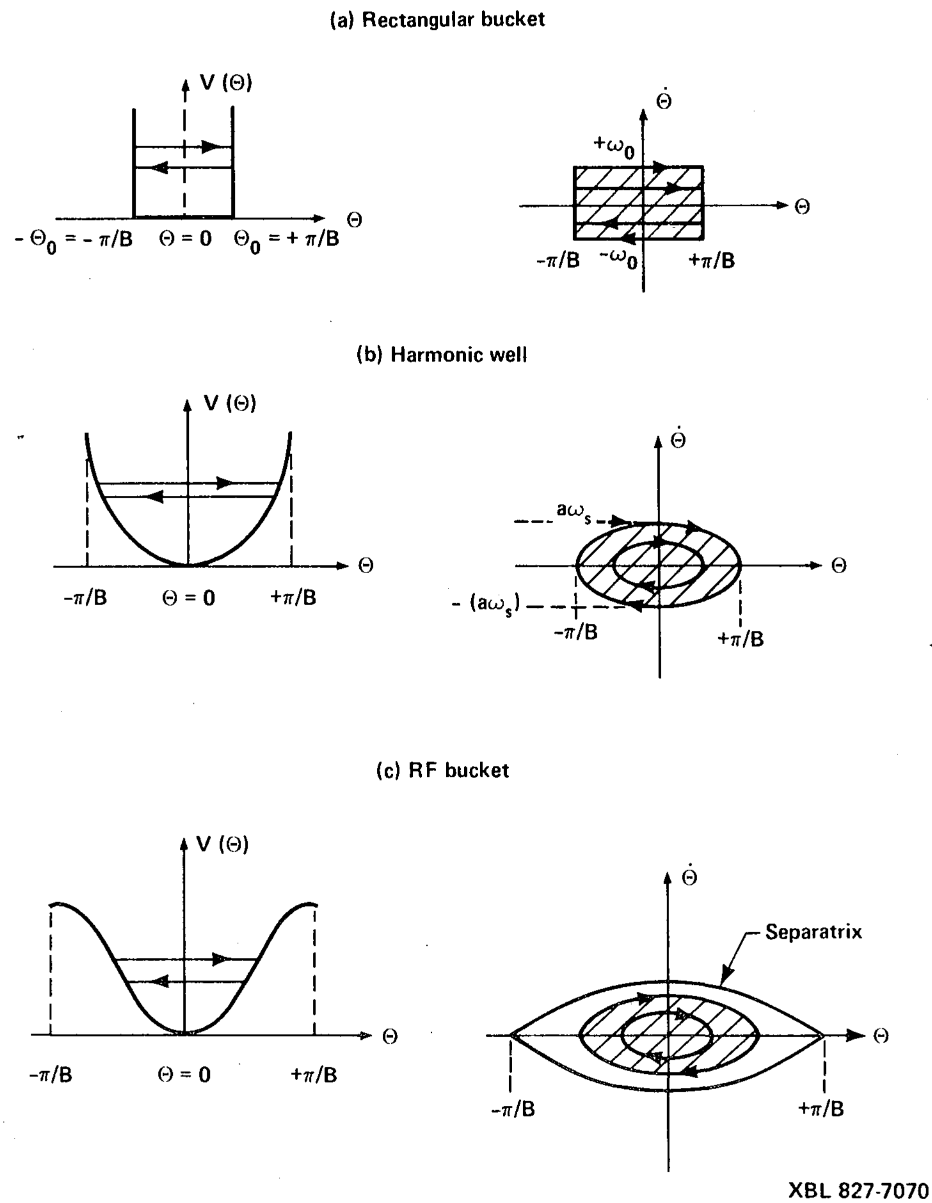
where θ_m is the maximum amplitude of oscillation and $K(k)$ the elliptic integral of the first kind. This approximation is good to about 5% for amplitudes as large as nine-tenths of the separatrix width.

In accordance with discussions at the end of Chapter 3, the orbits for the last case are taken to be

$$\Theta(t) = a \, \mathrm{Sin}\left[\omega_s(a)\ t + \psi(0)\right]$$

where either

$$\omega_s(a) = \omega_s(0)\ \left[1 - \delta a^2\right]$$



Buckets and Longitudinal Particle Orbits in the Simulation Study

Fig. 26

or

$$\omega_s^2(a) = \omega_s^2(0)\left[1 - \beta a^2\right]$$

with $\omega_s(0)$ the small-amplitude synchrotron oscillation frequency and δ or β a variable parameter, designed to model various degrees of nonlinearities.

13.2 Algorithm for Model Cooling System

In the simulation, the correction per step to the cooled variable was taken to be of the form

$$\Delta \underset{\sim}{x}_i = -\sum_{k=1}^{N} g\left(\theta_i - \theta_k\right) \underset{\sim}{x}_k$$

where $\underset{\sim}{x}_k = (x_k, x_k')$ is the cooled phase-space vector, transverse or longitudinal of the $k^{\underline{th}}$ particle and $\theta_k(t)$ describes the longitudinal orbit of $k^{\underline{th}}$ particle (angle around the ring) as a function of time. We report results for cases where the transverse correction was applied to transverse betatron position (x_k) only, and the longitudinal correction was applied to the longitudinal velocity deviation from the synchronous particle, $x' = v_k - v_s$ only. In real cooling systems, corrections are applied impulsively in momentum; however, it is a matter of interpretation in a simulation experiment, since a correction in either position or velocity of the transverse motion will lead to a change in betatron oscillation energy in general. All times are measured in units of the revolution period in the simulation.

The function $g(\theta)$ simulates the response function of the feedback system and determines the azimuthal distance of the effective interaction between particles. The rotational symmetry in the angle θ implies $g(\theta)$ to be periodic in θ. Thus we use a finite number of azimuthal harmonics to simulate a realistic $g(\theta)$:

$$g(\theta) = \sum_{\ell=0}^{m} a_\ell \operatorname{Cos} \ell\theta = \sum_{\ell=-m}^{+m} g_\ell e^{i\ell\theta}$$

We use up to $m = 4$ harmonics, implying a feedback system of effective angular extent $\theta_0 \sim 90^\circ$. Since the bunch length has to be larger than the electrical length of the feedback system in order to have effective cooling, we expect good cooling only for bunches longer than 90° in angular extent in these simulations.

Initial distributions of particles in phase space were constructed out of a random number generator to tailor to a desired amplitude profile. The transverse and longitudinal oscillation amplitudes were chosen according to $a_T = [1-\sqrt{1-R}]^{1/2}$ and $a_L = (1/2)\, F\cdot\sqrt{R}$ where R is a random number between 0 and 1, and F is the fraction of the total length of the ring occupied by the bunch ($F = \theta_B/2\pi$, θ_B = angular extent of the bunch). In the particular case of the rectangular potential well, the revolution frequencies were chosen from a rectangular distribution.

We correct the cooled phase-space variable of a particle at a fixed kicker position at every nominal revolution period of the bunch. Once a central reference particle has arrived at the kicker, all particles are kicked irrespective of their angular position in the ring. The basic code is a modification of a transverse coasting beam cooling simulation developed by Laslett and described in references [5], [6]. In the coasting beam context, the results were well-described by stochastic cooling theory including signal suppression [6]. For this agreement, however, it was found necessary to introduce a small, random frequency variation to destroy very small frequency differences between particle pairs. Physically, this corresponds to energy variations induced by stray fields, noise, etc., which do not change the gross frequency distribution. This feature was retained in the bunch beam code, although it is unnecessary for the longitudinal simulation where Schottky noise effects provide sufficient "wiggle" of frequency.

13.3 Results for Bunched and Coasting Beams and Their Comparison with Theory

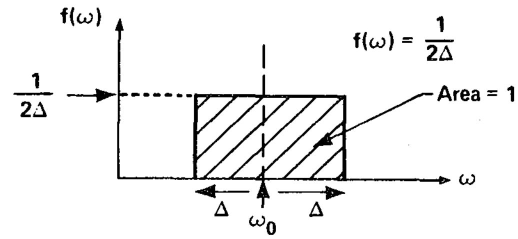
For transverse cooling of coasting beams, the cooling rate γ_ω for the mean squared betatron amplitude for particles of revolution frequency ω, is given by Eqs. (4.6.5) and (4.6.6), in a continuous correction limit without including amplifier noise. Using symmetries in the expression for $\epsilon[(n\pm Q)\omega]$ given by (4.6.6) we can rewrite (4.6.5) as

$$\gamma_\omega = -\sum_{(\pm)} \sum_{n} \frac{g[(n\pm Q)\omega}{\left|\epsilon[(n\pm Q)\omega]\right|^2} \tag{13.3.1}$$

where

$$\epsilon\Big[(n\pm Q)\omega\Big] = 1 + \frac{N\, g[(n\pm Q)\omega]}{|n\pm Q|} \int_{\eta\to 0^+} d\omega' \, \frac{f(\omega')}{\eta \pm i(\omega-\omega')} \tag{13.3.2}$$

In the simulation, it was found that the δ-function part of the singular integral in (13.3.2) describes well the signal suppression effects for a rectangular frequency distribution of half-width Δ, as shown in Fig. 27 below. The suppression factors then are



Rectangular Distribution in Angular Velocity for Coasting Beams

Fig. 27

are given by:

$$\varepsilon\left[(n\pm Q)\omega\right] \approx 1 + \frac{g[(n\pm Q)\omega]\ N}{8\ \Delta\ |n\pm Q|} \tag{13.3.3}$$

The factor $\varepsilon[(n\pm Q)\omega] \geqslant 1$ and from (13.3.1) it is seen that the cooling rate monotonically decreases with decreasing Δ; at no time does the Schottky noise (second term in Eq. (4.6.5)) dominate over the coherent cooling rate (first term in (4.6.5)). Coasting beam simulations were performed and described in [5], [6] with 90 and 180 particles cooled for 1000 correction steps. Averages over 25 cases agreed within 5% to theory, with case to case variations of ±10%. The $\varepsilon[(n\pm Q)\omega]$ factor ranged between 1 to 5. Some growth of oscillation amplitudes was seen for large $g_n \equiv g[(n\pm Q)\omega]$ which is attributable to the discrete nature of corrections and was found to be insensitive to Δ.

For g_n of the "wrong" sign, the condition $\varepsilon^{(\pm)} = \varepsilon[(n\pm Q)\omega] = 0$ gives the condition for coherent instability as discussed in Section 4.6. Thus the condition

$$\left|\frac{g_n \cdot N}{8\ \Delta\ (n\pm Q)}\right| = \left|\frac{g[(n\pm Q)\omega]\ N}{8\ \Delta\ (n\pm Q)}\right| \gtrsim 1 \tag{13.3.4}$$

gives both a condition on the sufficiency for Landau damping of instabilities and a criteria for feedback and/or Schottky noise effects to be important in stochastic cooling. Note that, as discussed in Section 4.7, $|Ng_n|$ is a measure of the magnitude of the coherent growth rate for instabilities or the coherent damping time of Schottky signals in the zero frequency spread limit.

Analogous to the coasting beam situation, approximate criteria for collective stability by Landau damping for bunched beams have been studied by various authors ([3], [4], [15], [52], [60], [65], [80], [87], [88], [90]) and are given by [60]:

$$\delta\omega_s \leq \frac{4}{\sqrt{\mu}} \left|\Delta\Omega_{\parallel\mu\mu}\right| \tag{13.3.5}$$

or

$$\delta\omega_\perp \leq \left|\Delta\Omega_{\perp\mu\mu}\right| \tag{13.3.6}$$

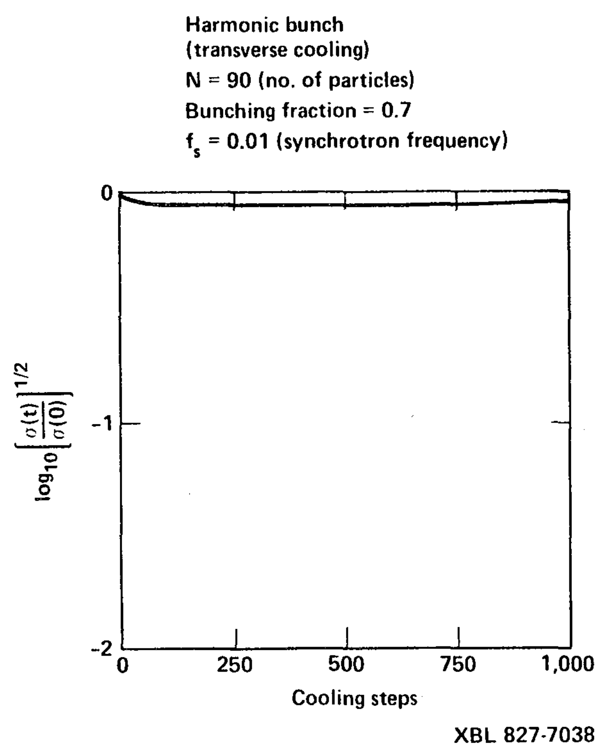
for longitudinal and transverse instabilities respectively where $(\Delta\Omega_{\parallel,\perp})_{\mu\mu}$ are the coherent growth rates of synchrotron mode μ for longitudinal and trasnverse modes and $\delta\omega_s$, $\delta\omega_\perp$ are the full spreads in nonlinear synchrotron oscillation frequencies and within-bunch betatron angular frequencies. If this condition is applied to a stochastic cooling system for bunched beams, we arrive at the condition that the reciprocal of the synchrotron frequency spread must exceed the coherent damping time of the Schottky signals with no mixing.

The two main distingushing features of bunched beam versus coasting beam stochastic cooling are:

1) The frequency variation which provides mixing is now determined (as discussed in the previous paragraph) by the spread of synchrotron frequency rather than revolution frequency or momentum spread. The amplitude dependence of synchrotron frequency, which depends critically on the bucket shape, becomes crucial to cooling.

2) Because of the finite length of the bunch, the Schottky signals at different harmonics become correlated. This effect manifests itself in enhancing beam heating and coupling the signal suppression in different Schottky bands as seen in Section 9.2, 10.2 and Appendix E.

One set of runs was performed with a parabolic potential energy bucket; i.e., the synchrotron oscillation is linear with frequency amplitude-independent. The cooling system used harmonics 1 to 4 with equal weighting, 90 particles, and the gain would provide a perfect cooling rate of .009/turn. The synchrotron frequency f_s = .01 and the bunch length is .7 of the ring circumference. No significant cooling occurs after 1 synchrotron oscillation, as is seen from Fig. 28 below. Similar runs for longitudinal corrections also yields no cooling, and varying synchrotron frequency and gain had no appreciable effect.



Simulation of Transverse Cooling of Linear Harmonic Bunch

Fig. 28

To investigate the effect of synchrotron frequency spread in detail runs were made with a "square" bucket and a sinusoidal "rf" bucket. For the square bucket particles were assigned a range of revolution frrquencies from a rectangular distribution. The particles were advanced in azimuthal angle as in a coasting beam until they reached the end of the bunch. At the ends, particles are elastically reflected with only their angular velocities changing sign. The motivation of this bucket shape was the hope that it would most closely resemble a coasting beam for analysis and offer in some sense a maximal degree of nonlinearity. The sinusoidal bucket corresponded to a harmonic 2 system. The phase orbits were determined by the first 4 terms of an expansion of the orbits in terms of elliptic integrals, as discussed in Section 13.1. In Table IV(a) results for a number of 90 particle runs are tabulated for transverse and longitudinal

TABLE IV(a)

Square Bucket - Transverse - BF = 0.5

	Δ	γ
$g_4 = .0022$	.1	.0036
$g_3 = .0022$	.01	.0012
$g_4 = .0022$	.01	.0015
$g_3 = g_4 = .0022$	.01	.0020
$g_1 = g_2 = g_3 = g_4 = .0022$	.1	.0090
$g_1 = g_2 = g_3 = g_4 = .0022$	.01	.0021

Sinusoidal Bucket - Transverse

	Δf_s	BF	γ
$g_4 = .0022$	.015	.7	.0022
$g_4 = .0022$	.025	.9	.0029
$g_4 = .0022$	.0015	.7	.00051
$g_1 = g_2 = g_3 = g_4 = .0022$	.015	.7	.003
$g_1 = g_2 = g_3 = g_4 = .0022$	.025	.9	.0045
$g_1 = g_2 = g_3 = g_4 = .0022$	.0015	.7	.00065

Sinusoidal-Like Bucket - Longitudinal - BF = .7

	Δf_s	γ_{200}	γ_{1000}
$g_1 = g_2 = g_3 = g_4 = .0022$	.15	.0096	.0065
$g_1 = g_2 = g_3 = g_4 = .0022$	.015	.0054	.0043

TABLE IV(b)

Coasting Beam Theory - Transverse

	Δ	γ
g_3 = .0022	.1	.0038
	.05	.0033
	.01	.0013
	.005	.00062
g_4 = .0022	.1	.004
	.05	.0035
	.025	.0028
	.015	.0022
	.01	.0017
	.005	.00088
$g_1 = g_2 = g_3 = g_4$	.1	.014
	.05	.011
= .0022	.025	.0084
	.015	.0059
	.01	.0042
	.005	.002
$g_3 = g_4$ = .002		
effective	.01	.0022
$g_1 = g_2 = g_3 = g_4$		
effective	.01	.0028

cooling. Table IV(b) gives coasting beam rates from theory. The bunching factor gives the fraction of the ring circumference occupied by the bunch. The results in Table IV(a) are for single cases with 90 particles. Within each category the same seed was used to initialize the random loading to lessen statistical variation as parameters were changed.

For the square bucket, single harmonic (ℓ = 3,4) rates compare remarkably well with coasting beam theory. However, when both harmonics are present, the cooling rate is significantly different from that of coasting beam theory, where rates for each harmonic are simply added. For a square bucket, Schottky signals ℓ and m are coupled with a weighting

$$\left[\frac{\sin(\ell-m)\,\theta_0}{(\ell-m)\,\theta_0}\right]^2$$

where θ_0 is the half length of the bunch (see Appendix A and B). The last entries in Table IV(b) give rates calculated with coasting beam theory, using an effective gain

$$g_{\ell_{eff}} = \sum_m g_m \left[\frac{\sin(\ell-m)\ \theta_0}{(\ell-m)\ \theta_0}\right]^2 \qquad (13.3.7)$$

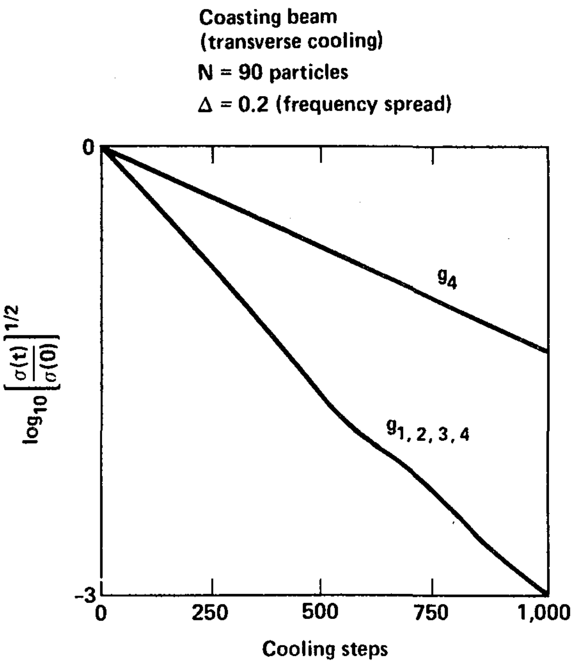
to evaluate the ε_ℓ. Agreement is good. These results clearly demonstrate the interference of neighboring harmonics; but for a long bunch, this interference does not totally cancel the effects of using neighboring harmonics in a cooling system. See Fig. 29 below, which compares single harmonic runs with g_4 only to runs with all four g_1, g_2, g_3, g_4 for coasting and square buckets. Coasting beam rate as in Fig. 29(a) modified by (13.3.7), gives a rate $\gamma_{eff} \sim .00265$ which agrees well with $\gamma \sim .0027$ for the $g_{1,2,3,4}$ in Fig. 29(b).

For transverse cooling in a sinusoidal rf bucket, cooling rates for a synchrotron oscillation spread Δf_s are comparable to coasting beam rates with $\Delta = \Delta f_s$. Again, with several harmonics simultaneously acting, there is a degradation of coasing beam rates by a factor of 2. It should be noted that the longitudinal random load provides a uniform distribution in phase space.

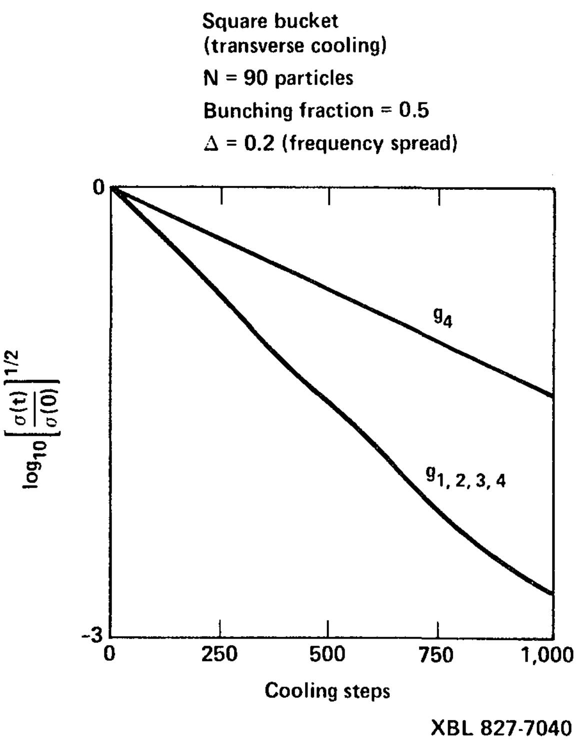
Finally, the last entries in Table IV(a) are for longitudinal runs. Effective cooling rates/step are given after 200 and 1000 correction steps are given. The phase space orbits are elliptical with amplitude variation of synchrotron frequency. Cooling rates degraded as mixing lessens with higher phase space density.

To compare the adequacy of the model orbits in which the orbits are still sinusoidal but with an imposed amplitude dependence of frequencies, we compared cooling rates for a real rf bucket with an equivalent anharmonic well that generates similar nonlinearities. Results are shown in Fig. 30(a) and (b) below and to the degree of accuracy of the simulations, no significant differences are observed.

We observe that synchrotron frequency spread provides the necessary mixing mechanism for bunched beam cooling. In addition, it appears that the natural nonlinearities of a single, long full rf bucket can provide mixing comparable to a coasting beam for harmonics of higher frequency than those associated with the gross bunch structure. However, as the bunch length decreases degradation of cooling occurs as the mixing mechanism couples neighboring Schottky bands.



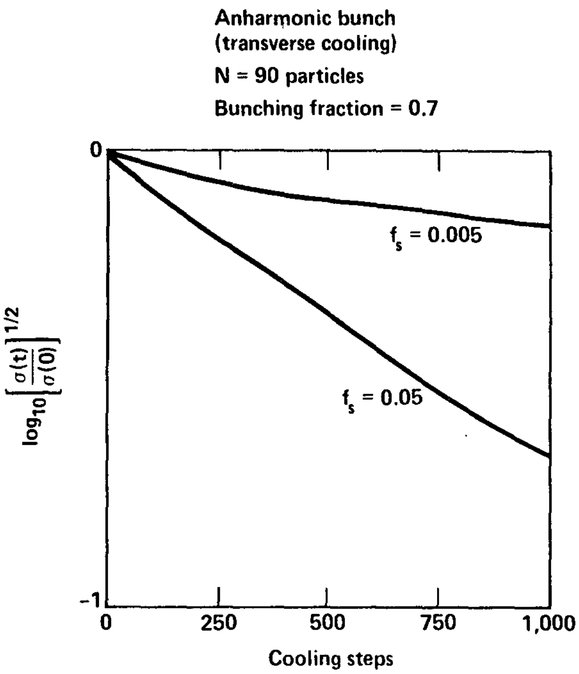
(a)



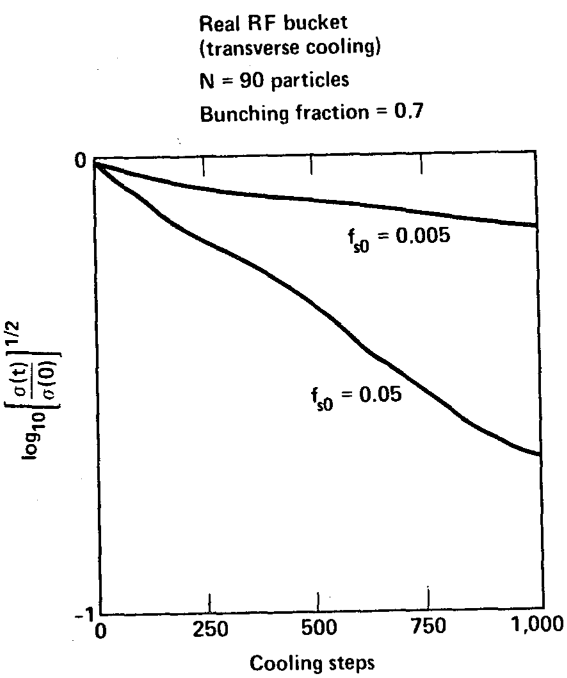
(b)

Manifestation of Effective Gain for Square Bucket

Fig. 29



(a)

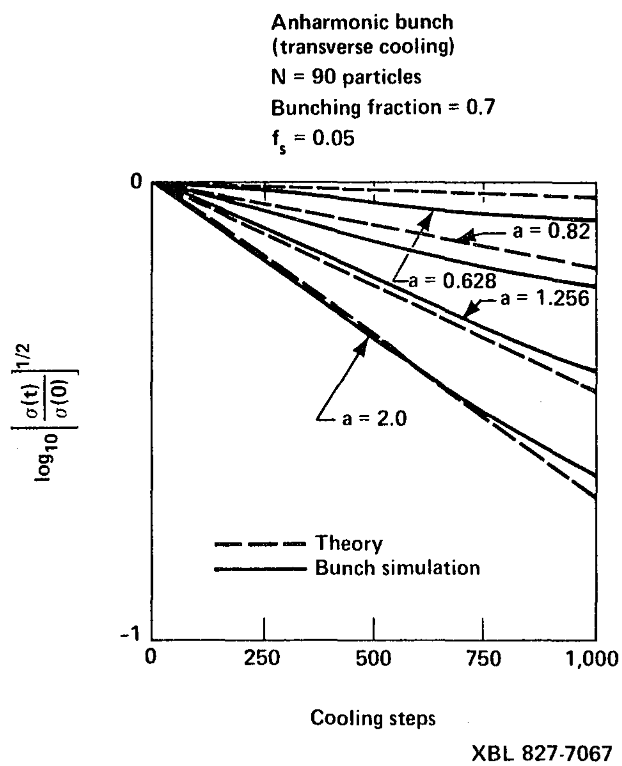


(b) XBL 827-7039

Cooling of Real Orbits vs. First Order Asymptotic Perturbation Orbits of a rf Bucket

Fig. 30

For these model simulations using only low harmonics ($\ell = 1,2,3,4$) of the gain, there is no synchrotron band overlap and our analysis of band non-overlap cooling rate, studied in Chapters 9 and 10 is ideally suited for a comparison with simulation. Figure 31 compares simulation results with cooling rates obtained from our theory with no band overlap (Eq. (9.4.15), (9.4.16)). The dashed curves are the slopes obtained from theory and results agree fairly well for cooling rates at large amplitudes. An equivalent comparison for longitudinal cooling is made difficult by the fact that there is in general no exponential cooling and one really has to compare particle fluxes locally atany synchrotron amplitude, both from theory and simulation. For simulation with small number of particles such as ours (90 or 180), instantaneous local fluxes are very small and a comparison is almost moot.



Simulation vs. Theory

Fig. 31

14. A NUMERICAL EXAMPLE OF BUNCHED BEAM COOLING IN A HIGH ENERGY STORAGE RING

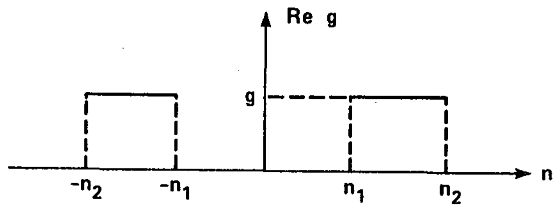
In order to obtain estimates of probable cooling rates for bunches in a real high energy storage ring we have applied our analytical results to bunches in a typical high energy storage ring with parameters as listed in Table IV below as an example. The parameters correspond to a $h = 2226$ harmonic rf system for the Fermilab main ring. The example illustrates the qualitative nature of bunched beam cooling and provides order of magnitude estimates of the cooling rate with a 2-4 Ghz bandwidth cooling system. While the exact cooling rates might differ for actual high-energy storage ring like the proposed Tevatron I at Fermilab with probably a different set of parameters in its final design and for different bandwidth feedback systems, the qualitative picture of synchrotron amplitude dependence of local cooling rate and signal suppression and the nature of band-overlap resnances in a large-bandwidth feedback loop for a high harmonic rf system, remain valid and demonstrate the essential peculiarities of bunched beam cooling.

TABLE IV

Parameter	Value
f_o, Revolution frequency	50 kHz
Q, Betatron tune	19.4
N_T, Total number of antiprotons ($\bar{p}$'s)	6×10^{11}
N, No. of $\bar{p}$/bunch	10^{11}
No. of bunches	6
h, Harmonic of rf cavity	2226
T_s, Synchrotron time-period	5 m-sec.
$\omega_s(0)$, Small amplitude synchrotron oscillation frequency	12×10^2 radians/sec.

14.1 Cooling Rate

We consider a flat gain 2-4 Ghz bandwidth feedback system. With $f_o \sim 50$ kHz, the revolution harmonic within the bandwidth ranges from 40,000 to 80,000 (Fig. 32). The



2-4 GHz Flat Gain System

Fig. 32

number of real positive harmonics is thus $\Delta n = n_2 - n_1 = 40{,}000$ within the pass-band. With harmonic $h = 2226$ for the rf cavity, the maximum angular extent of the bunch is

$$(\Delta\theta)_{max} = 2\ a_{max} = 2\pi/h$$

and

$$a_{max} = \pi/h = 1.411 \times 10^{-3} \text{ radians}$$

where a_{max} is the maximum angular excursion relative to the synchronous particle within the bunch, if the bucket was full.

We take a distribution of particles within the bunch with a sharp edge at $a = a_{max}$ and given by

$$f(a^2) = \frac{3}{2a_m^2}\left[1 - \frac{a^2}{a_m^2}\right]^{1/2} \Theta\left(1 - \frac{a^2}{a_m^2}\right) \qquad (14.1.1)$$

and

$$\int_0^\infty f(a^2)\ da^2 = 1$$

where Θ is a heavy-side step function. Such a distribution produces a parabolic particle density as a function of azimuth along the bunch, as is observed experimentally.

We take as a model of the synchrotron nonlinearity the following amplitude-dependent frequency

$$\omega_s\left(\phi_m^2\right) = \omega_s(0)\left[1 - \frac{\phi_m^2}{16}\right] \tag{14.1.2}$$

where

$$\phi_m = \text{Maximum phase excursion} = ha \ .$$

Then

$$\omega_s(a^2) = \omega_s(0)\left(1 - \frac{h^2 a^2}{16}\right) \tag{14.1.3}$$

The corresponding particle distribution in synchrotron frequency is given by

$$g\left[\omega_s(a^2)\right] = \frac{f(a^2)}{\left|\dfrac{d\omega_s(a^2)}{da^2}\right|} \tag{14.1.4}$$

since

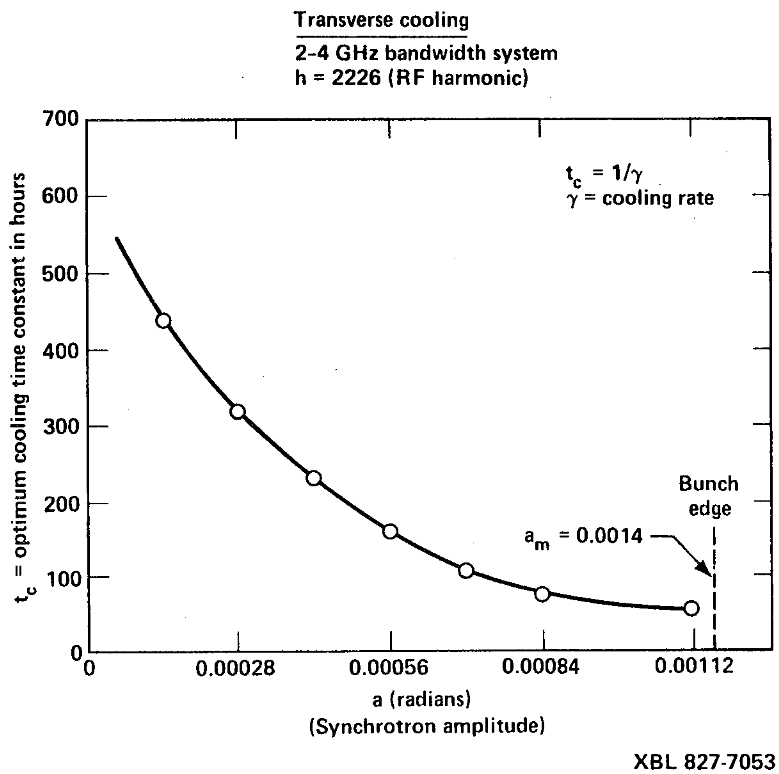
$$f(a^2)\, da^2 = g\ \omega_s(a^2) \cdot d\left[\omega_s(a^2)\right].$$

Since we have a constant flat gain, the transverse cooling rate as given by Eq. (9.4.15) can be written as

$$\gamma(a) = -\ \alpha(a)\ g + \beta(a)\ g^2 \tag{14.1.5}$$

for cooling of particles with synchrotron amplitudes in the neighbourhood of a, where $\alpha(a)$ and $\beta(a)$ are various combinations of Bessel functions, mixing factors and machine parameters. The cooling rate is then maximized by chosing a g that maximizes $\gamma(a)$ in (14.1.5). The maximum cooling times in hours for particles with different synchrotron amplitudes are plotted as a function of the amplitude within the bunch and the results are shown in Fig. 33. Note that these results are obtained from Eq. (9.4.15) for the non-overlapping synchrotron-band case.

Cooling rate decreases (cooling time increases) from the bunch edge to the bunch center from the few tens of hours to a few hundred hours at the core. The reduced cooling at the bunch core is a manifestation of both increased particle density and decreased mixing (less variation of synchrotron frequency with amplitude). This cooling rate is certainly not enough to compensate bunch diffusion due to intra-beam scattering, rf noise or beam-beam interaction with typical diffusion times of 10~12 hours. On the other hand we have used a relatively low band-width system (2-4 Ghz) compared to the bunch duration and a realistic bucket which usually has very little nonlinearity. With a higher band-width system (8-16 Ghz) and a flattened bucket (by adding a small voltage at a third harmonic say), the cooling rate is expected to improve.



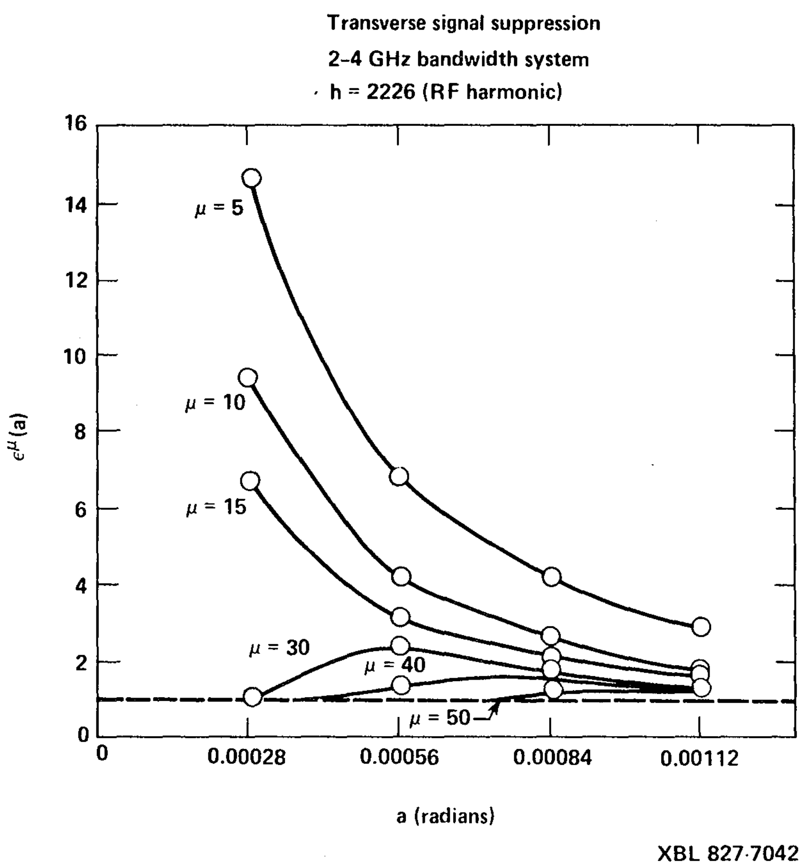
Theoretical Transverse Cooling Rate Neglecting Synchrotron Band Overlap

Fig. 33

14.2 Signal Suppression

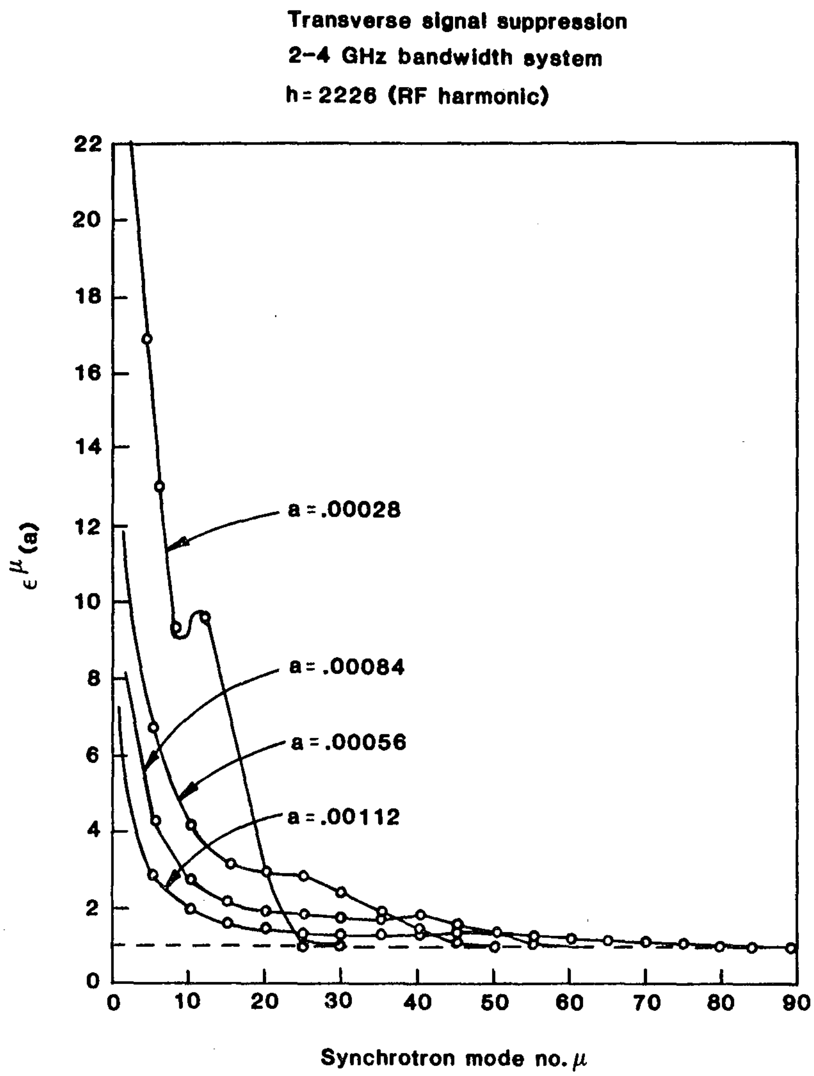
The collective signal suppression factors are evaluated for transverse and longitudinal cooling at different amplitudes within the bunch. The gain used in these calculations is the optimum gain for the highest amplitude particles in the bunch, which is taken to be .00112 radians in this case.

The local suppression factor $\epsilon_T^{\mu}(a)$ for fixed μ as a function of amplitude in the bunch are plotted for different μ's in Fig. 34(a). We see that suppression is enhanced for low synchrotron harmonics in general and increases towards the core for the low synchrotron harmonics. Higher harmonics contribute to larger amplitudes only, but with strength less than the low harmonics. At any given amplitude, only a finite number of μ's contribute. At smaller amplitudes less number of synchrotron modes contribute but with enhanced strengths. The suppression factor at fixed amplitude but with different synchrotron mode numbers are plotted for various amplitudes in Fig. 34(b).



Transverse Signal Suppression

Fig. 34(a)



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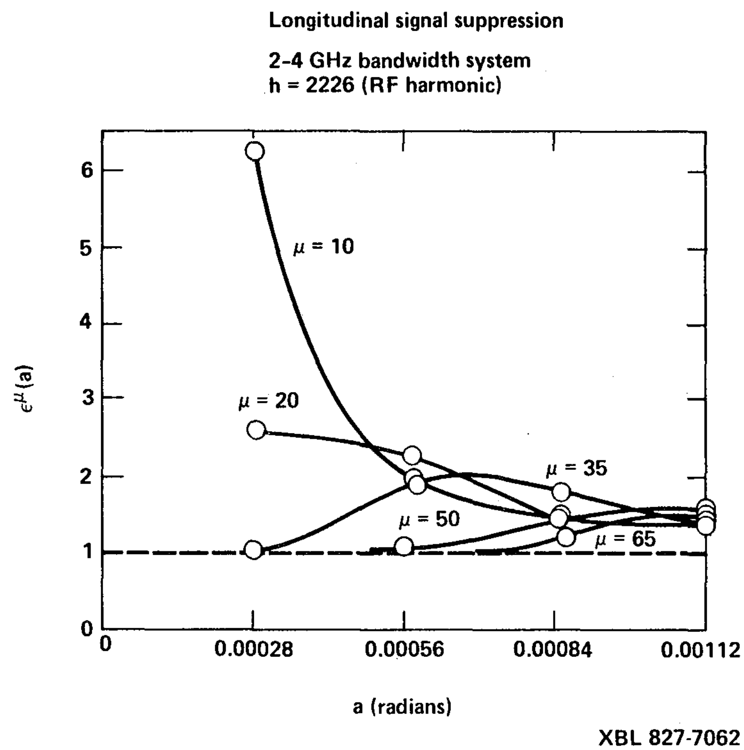
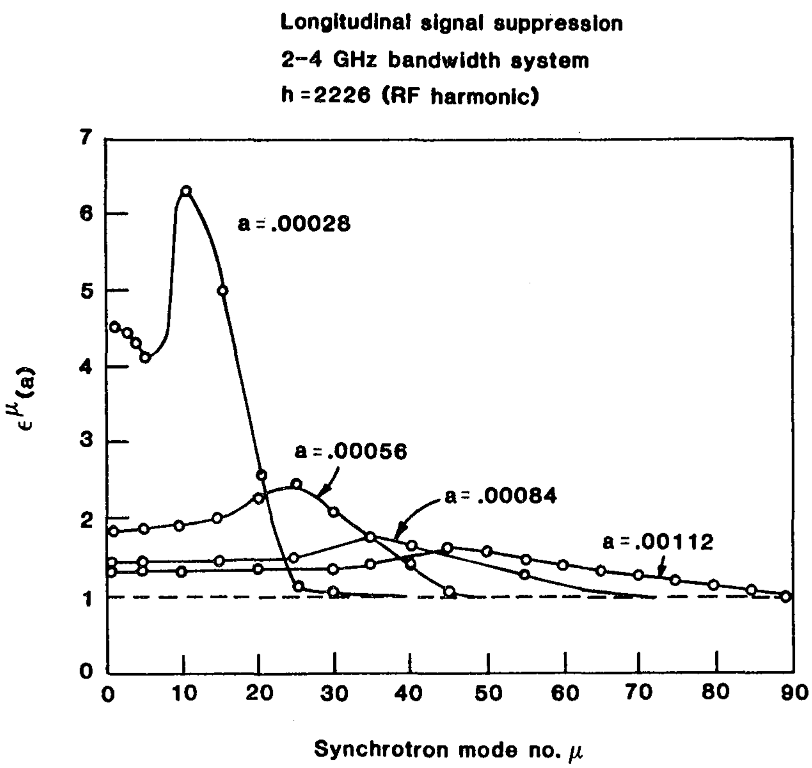
Transverse Signal Suppression

Fig. 34(b)



Longitudinal Signal Suppression

Fig. 35(a)



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Longitudinal Signal Suppression

Fig. 35(b)

Similar curves for the longitudinal signal suppression are plotted in Fig. 35(a) and 35(b). The general trend is similar to the transverse case with two noticeable differences. The suppression at any given amplitude a does not blow up as $1/|\mu|$ as in the transverse case but rather levels to a flat value for almost all amplitudes. Secondly the local maxima for certain synchrotron harmonics at a given amplitude are more pronounced in the longitudinal case.

14.3 Comparison with Coasting Beams

In order to gain more insight into the bunched beam cooling rates, we make various comparisons with well known coasting beam cooling rates. A fair comparison evolves in the course of this investigation.

Particle density in frequency space determines the concentration or density of Schottky noise due to all the particles, as seen by a single particle and limits the cooling rate. Hence for comparable particle distribution in frequency $f(\Omega)$, bunched beam cooling rate should compare with the coasting beam rate reasonably well except for a form-factor describing the gross bunch structure.

For a gain function with harmonics g_ℓ nonzero for $n_1 \leqslant |\ell| \leqslant n_2$ (bandwidth = $(n_2-n_1)\omega_0$) and a rectangular frequency distribution with half-width Δ (Fig. 27) and for total number of particles N, the transverse coasting beam cooling rate is given by

$$\gamma = - \sum_{n_1 \leqslant |\ell| \leqslant n_2} \frac{2g_\ell}{\left|1 + \frac{g_\ell N}{4\Delta\ell}\right|^2} = - \sum_{n_1 \leqslant |\ell| \leqslant n_2} \frac{2g_\ell}{\left|1 + \frac{g_\ell [Nf(\omega)]}{2\ell}\right|^2}$$

We keep $f(\omega)$ the same for coasting and bunched beam. Coasting beam cooling rate corresponding to an $f(\omega)$ for a bunch at $a = .00056$ radians gives an optimized cooling time of 140 hours as opposed to 186 hours for a bunched beam cooling with corresponding local density. Similar comparison corresponding to bunch density at $a = .00112$ radians, leads to an optimized cooling time of 35 hours as opposed to 60 hours for bunched beam local cooling.

A different comparison preserves the total number of particles $N \sim 10^{11}$ in the beam and the frequency spread $\Delta\omega = a_m\omega_s(0)$. However we consider the beam debunched in the whole ring thus decreasing its configuration-space density. For the same optimum gain g, the signal suppression factor is almost 1. Hence there is hardly any collective suppression: $1 + \frac{g_\ell N}{4\Delta\ell} \approx 1$. We get $\gamma \sim 10 \ \sec^{-1}$ or a cooling time of .6 hours <u>or</u> 36 minutes as opposed to 60 hours for a bunch. This is an unfair comparison since we are not keeping the density the same in any space whatsoever (configuration space, frequency space <u>or</u> phase-space).

To preserve density in configuration space, we now consider a coasting beam with total number of particles enhanced by the harmonic number of the rf cavity, i.e. $N = h \times N/\text{Bunch} = 2226 \times 10^{11}$ but with same frequency $\Delta\omega = a_m\omega_s(0)$. Noise density gets enhanced for the coasting beam and $\epsilon_\ell = (1 + g_\ell N/4\Delta\ell)$ is not close to 1 any more but of the order of 2-3.5. Cooling rate is decreased by a factor of 10 approximately. Thus we get a cooling time of 4.8 hours as opposed to 60 hours for a bunch.

And yet a final comparison is in order. This time we preserve the density in phase-space. Thus we keep the total number of particles N and total 'phase-space' area the same. Comparing the area of the rectangular phase-space distribution of a coasting beam in (θ,θ') plane of height Δ and length 2π radians to the elliptical phase-space area of a bunch of semi-axes a_m and $a_m \cdot \omega_s(0)$ we get

$$2\pi\,\Delta = \pi\, a_m \cdot a_m\, \omega_s(0)$$

i.e.

$$\Delta = a_m^2\, \omega_s(0)/2 \simeq \frac{\pi}{h}\, a_m\, \omega_s(0)\ .$$

We obtain a cooling time of 50 hours which is comparable now to 60 hours for a bunch.

14.4 Enhancement of Diffusion Due to Band-Overlapped Noise

So far our estimates of transverse cooling rate did not include contributions from overlapping resonances such as:

$$\ell\omega_0 + \mu\omega_s(a) = \Omega' = m\omega_0 + \nu\omega_s(a')$$

with $\ell \neq m$ and $\mu \neq \nu$.

Typically for a large bandwidth feedback loop, there are high revolution harmonics within which considerable amount of synchrotron band-overlap occur. These extra resonances cause extra diffusion and heating of the beam. We estimate the enhancement of diffusion due to these band-overlapped noise contribution in this section.

Let $\Delta\omega_s(a)$ be the detuning of synchrotron frequency at amplitude a from the zero-amplitude frequency $\omega_s(0)$. Then

$$\frac{\Delta\omega_s(a)}{\omega_s(0)} = \frac{\omega_s(0) - \omega_s(a)}{\omega_s(0)} = 1 - \frac{\omega_s(a)}{\omega_s(0)}$$

Let the maximum synchrotron amplitude a_{max} present in the bunch be 2/3 the full length of the bucket. For a sinusoidal rf bucket corresponding to simple pendulum orbits, we have [12]

$$\left.\frac{\omega_s(a_{max})}{\omega_s(0)}\right|_{a_{max} = \frac{2}{3} \text{ bucket length}} = .7646$$

so that

$$\frac{\Delta\omega_s(a_{max})}{\omega_s(0)} = 1 - .7646 = .2354$$

Synchrotron bands will overlap whenever

$$\mu \, \Delta\omega_s \geqslant \omega_s(0)$$

i.e.

$$\mu \geqslant \frac{\omega_s(0)}{\Delta\omega_s(a_m)} \sim 5 \ .$$

For a 2-4 Ghz system, the range of revolution harmonics is given by:

$$n_1 = 40{,}000$$

$$n_2 = 80{,}000$$

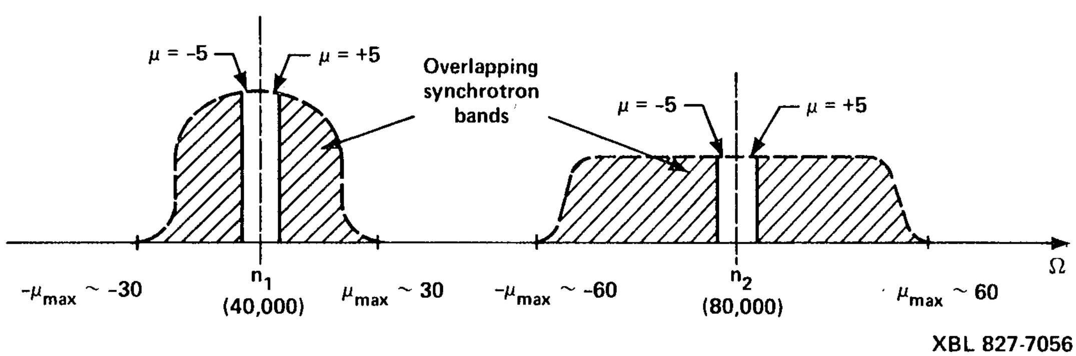
Maximum μ contributing at lower end $n_1 = 40{,}000$: $\quad \mu_{max} \sim n_1 \, a_{max} \sim 30$

Maximum μ contributing at higher end $n_2 = 80{,}000$: $\quad \mu_{max} \sim n_2 \, a_{max} \sim 60$

Therefore except for a few (~10) synchrotron bands at the center, most of the revolution harmonic band, even at the lowest revolution harmonic, contain overlapped synchrotron bands. However, maximum half-width of the revolution bands is given at the highest revolution harmonic $n_2 = 80{,}000$ by:

$$\mu_{max} \cdot \omega_s = \frac{\mu_{max}\ \omega_s}{\omega_o} \cdot \omega_o \cong 0.24\ \omega_o < \omega_o\ .$$

Hence revolution bands do not overlap within the band-pass of the 2-4 GHz system. They will begin to overlap for a 8-16 GHz system. The revolution bands at the lowest n_1 = 40,000 and the highest n_2 = 80,000 harmonics thus look like the ones shown in Fig. 36.



Synchrotron Band Overlap for 2-4 GHz System

Fig. 36

Since we cannot obtain a simple cooling rate corresponding to exponential cooling in time for the situation of band-overlap we compute instead the Diffusion Coefficient including synchrotron band-overlap and compare it to the Diffusion coefficient without band-overlap contribution.

The method of computing the Diffusion coefficient with synchrotron band-overlap is a straightforward computation using the formula (9.4.2) for D(I,J), keeping all the resonances that contribute within the band.

With expressions (9.4.3), (9.4.4) and defining

$$f(I',J') = f(I')\ h(J')$$

and

$$\int dI' \; I' \; f(I') = \langle I \rangle$$

we obtain for $D(I,J)$:

$$D(I,J) = \frac{N\,\omega_o^2}{4(2\pi)^3\,Q^2}\, I \cdot \langle I \rangle \sum_{(\pm)} \sum_{\mu} \sum_{\mu'} dJ' \; h(J') \left| \tilde{g}^{(\pm)}_{\mu\mu'}(J,J') \right|^2 \delta\left[\mu\omega_s(J) + \mu'\omega_s(J')\right]$$

where we have absorbed the q^2 (charged squared) factor within the gain $\tilde{g}_{\mu\mu'}$.

Changing from action to frequency distribution by

$$h(\omega_s)\; d\omega_s = h(J)\; dJ$$

we get:

$$D(I,J) = \frac{N\,\omega_o^2}{4(2\pi)^3\,Q^2}\, I \cdot \langle I \rangle \sum_{(\pm)} \sum_{\mu} \sum_{\mu'} \int d\omega_s' \; h(\omega_s') \frac{\left| \tilde{g}^{(\pm)}_{\mu\mu'}(\omega_s, \omega_s') \right|^2}{|\mu'|} \delta\left[\omega_s' + \frac{\mu}{\mu'}\,\omega_s\right]$$

$$= A \cdot I \cdot \langle I \rangle \sum_{(\pm)} \sum_{\mu} \sum_{\mu'} \frac{\left| \tilde{g}^{(\pm)}_{\mu\mu'}\left(\omega_s, -\frac{\mu}{\mu'}\,\omega_s\right) \right|^2}{|\mu'|} h\left(-\frac{\mu}{\mu'}\,\omega_s\right) \qquad (14.4.1)$$

where the sum over μ and μ' extend within the ranges defined by Eqs. (5.8) and (5.9) in Chapter 5 and $A = N\omega_o^2/4(2\pi)^3 Q^2$.

This has to be compared with the contribution

$$D(I,J) = A \cdot I \cdot \langle I \rangle \sum_{(\pm)} \sum_{\mu} \frac{\left| \tilde{g}^{(\pm)}_{\mu,-\mu}(J,J) \right|^2}{|\mu| \left| \frac{d\omega_s}{dJ} \right|} h(J)$$

to the diffusion coefficient from non-overlapping bands only. The $\tilde{g}^{(\pm)}_{\mu\mu'}(J,J')$ in above expressions is given by (9.4.4) i.e.

$$\tilde{g}^{(\pm)}_{\mu\mu'}(J,J') = \left\{ \sum_{m} \tilde{G}\left[(m\pm Q)\omega_o + \mu\omega_s(J')\right] J_{\mu}\left[(m\pm Q)\sqrt{2J'}\right] J_{\mu'}\left[-m\sqrt{2J}\right] \right\}.$$

As an illustration of the band-overlap structure, let us consider the amplitude $a = .00028$ radians.

Then Eq. (5.8) defines the following range for μ:

$$11 \leqslant \mu \leqslant 21$$

The bands μ' that overlap with some of these values of μ for amplitudes a' within the beam distribution $0 \leqslant a' \leqslant a_{max}$ are shown in Table V below.

TABLE V

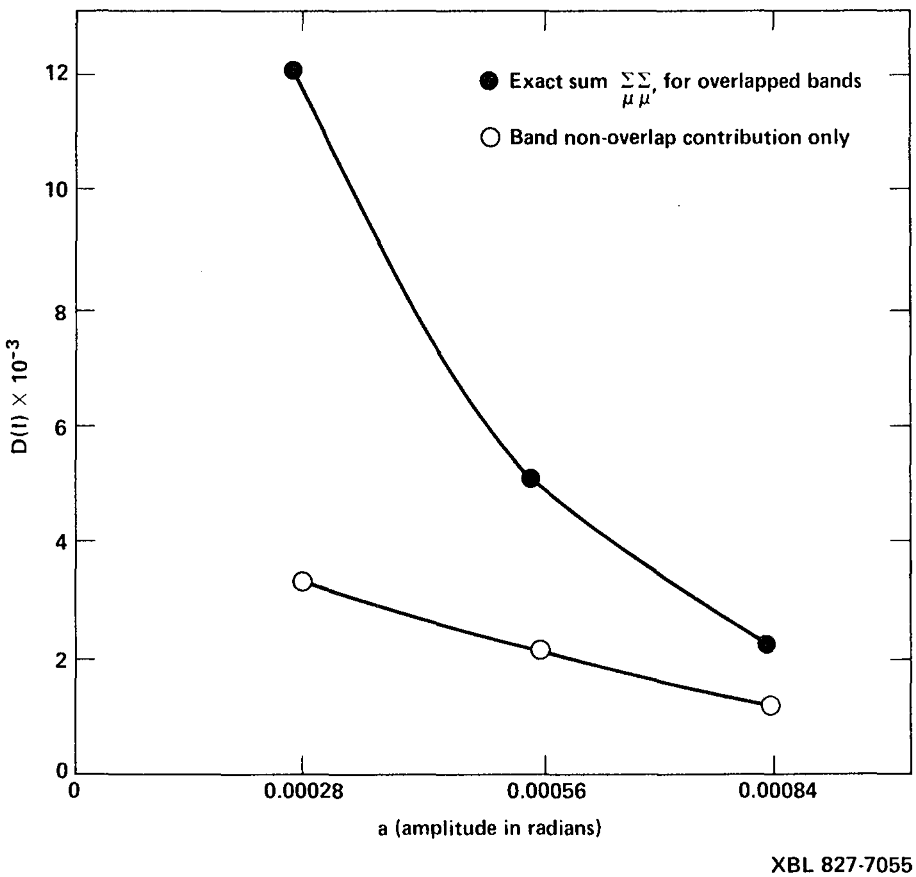
μ	OVERLAP
11	$14 \geqslant \mu' \geqslant 11$
13	$17 \geqslant \mu' \geqslant 13$
15	$19 \geqslant \mu' \geqslant 15$
17	$22 \geqslant \mu' \geqslant 17$
19	$24 \geqslant \mu' \geqslant 19$
21	$27 \geqslant \mu' \geqslant 21$

$a = .00028$

The results for the Diffusion coefficient thus calculated is compared to the band-nonoverlap contribution in Fig. 37. The results of calculation of band-overlapped noise diffusion show a Diffusion coefficient significantly higher than the nonoverlap Diffusion coefficient for small synchrotron amplitudes. Cooling rates for small amplitudes will thus significantly suffer from band-overlapped noise. The enhancement is reasonably modest (factors of 2) at large amplitudes where noise diffusion is relatively small anyway due to improved mixing.

A quick estimate for the best possible cooling rate of a bunched beam follows along these lines: except for the line-structure in the middle, we treat the bunched beam signal as an equivalent coasting beam signal with frequency half-width Δ and an enhanced effective number of particles. The situation then would correspond to a bunch confined by a square bucket. From (13.1.1) and (13.3.3), the cooling rate then is

$$\gamma = \sum_{\pm\ell} \frac{2g_\ell}{\left|1 + \frac{g_\ell N_{eff}}{4\Delta\ell}\right|^2} \tag{14.4.2}$$



Enhancement of Schottky Noise Diffusion Due to Synchrotron Band Overlap

Fig. 37

The optimum gain is then given by

$$g_\ell^{optimum} = \frac{4 \cdot \Delta \cdot \ell}{N_{eff}} \tag{14.4.3}$$

and optimum cooling rate

$$\gamma_{opt} = \sum_{\pm\ell} \frac{2 \cdot \Delta \cdot \ell}{N_{eff}} \tag{14.4.4}$$

For slightly different parameters corresponding to the Fermilab Tevatron I collider [97], with f_o = 48 kHz, rf harmonic h = 1113, transition energy γ_t = 18.75 GeV, $\eta = \gamma^{-2} - \gamma_t^{-2}$ = -0.0028, Synchrotron period T_s = 27 m-sec, each bunch is 1.6 meters long (contains 95% of all the particles with $\sigma \sim$ 40 cm.). Approximately 3925 bunches would have the same length as the ring circumference. Bunches occupy an area of 3 eV-sec within a bucket of area 12.7 eV-sec. Then $\sigma_p/p = 1.2 \times 10^{-4}$ and 95% of the particles are contained within $\pm 2\ \sigma_p/p = \pm 2.4 \times 10^{-4}$ for Gaussian bunches. The frequency spread is

$$\Delta f = f_o\ \eta\ \frac{\Delta p}{p} = .032 \text{ Hz} .$$

For an example of 8-16 Ghz system with $N \sim 10^{11}$ per bunch, we have 160,000 < ℓ < 320,000 harmonics within the band-pass, with $\ell_{average}$ = 240,000, $\Delta \cdot \ell_{average}$ = 7680 Hz and $N_{eff} = 3925 \times N = 3925 \times 10^{11}$. Equation (14.4) gives

$$\gamma_{opt} \simeq 1.25 \times 10^{-5}/\text{sec}$$

and

$$\tau_{opt} \cong 22 \text{ hours.}$$

This is the maximum cooling rate possible for bunched beams with above parameters with full band overlap within each revolution band. Table VI gives cooling rates for different bandwidth systems.

Table VI

Band-width	ℓ	$\bar{\ell}_{average}$	τ_{cool} (transverse)
2-4 GHz	40,000-80,000	60,000	352 hours
4-8 GHz	80,000-160,000	120,000	88 hours
8-16 GHz	160,000-320,000	240,000	22 hours

For the 8–16 GHz feedback system, each revolution band has a half-width of 10 kHz with separation between nearest revolution bands 48 kHz. Thus there is a gap of about (48 - 2x10) kHz = 28 kHz between revolution bands where there is no signal power. In the language of the equivalent coasting beam Schottky spectrum, the cooling rate is thus limited by the situation of bad mixing.

15. CONCLUSION

The theoretical formulation of stochastic cooling of bunched beams presented in this report provides us with the necessary ingredients for a realistic calculation of the time-evolution and rate of cooling of bunched beams for the purposes of prediction, design or comparison with experimental observations. The analysis and preliminary estimates unfold a pattern of behaviour of bunched beams undergoing stochastic cooling. The increased particle density in a bunched beam compared to a coasting beam and relatively small synchrotron frequency spread in a conventional rf bucket makes bunched beam cooling a much slower process than coasting beam cooling in general, for identical feedback systems. The importance of potential well (bucket) nonlinearity in providing sufficient synchrotron frequency spread and hence good mixing in phase space is crucial to bunched beam cooling. The notions of Landau damping for beam instabilities and mixing for stochastic cooling are intimately related. The need for synchrotron frequency spread for both Landau damping to stabilize beam instabilities and good mixing to enhance cooling are seen to be comparable for bunched beams. In particular the reciprocal of the synchrotron frequency spread must exceed the coherent damping time of Schottky signals with no mixing for effective cooling.

Bunched beam cooling is also a strongly local function of amplitudes of particles in the bunch. In general nonlinearities are stronger for larger amplitudes and so the edge particles cool faster than the core particles that do not mix as well in phase space and whose signals get collectively suppressed stronger and faster than those from the edge particles. However, steepness of the bunch distribution at the edge competes against this process and trapped particles at the steep slope region of a hard-edge distribution are harder to cool.

A useful concept in bunched beam cooling is the notion of 'effective gain'. Because of the finite length of the bunch a single particle in the beam sees an enhanced effective gain (relative to a coasting beam) including a sum over correlated Schottky signals at different revolution harmonics. This effect manifests itself in enhanced beam heating and coupling the signal suppression at different Schottky bands.

Typical estimates of transverse cooling rates for bunches containing 10^{11} particles confined in a conventional rf bucket of harmonic number $h = 2226$ for a prototype high-energy storage ring, using a 2-4 Ghz feedback system, indicate a cooling time of hundreds of hours, much too slow to compensate beam blow up on a fast time scale of 10-20 hours due to rf noise, intra-beam scattering and beam-beam interactions. However, with a higher bandwidth system (8-16 Ghz) and rectangular potential well rf bucket containing a relatively long bunch (1/1000 of ring circumference), the cooling rate at the bunch edge is expected to improve significantly (20-50 hours) enough at least to maintain its emittance against countering blow-up effects. Bucket shapes can be altered to the desired degree of nonlinearity by adding extra higher harmonic (odd) rf cavities with proper voltage strengths. Use of a high band-width feedback system is almost implicit in any bunched beam cooling scheme as one desires to resolve sufficiently small sized phase-space samples within a bunch (with number of samples per beam handled by the feedback system approximately comparable to a coasting beam situation) for effective cooling. With growing interest in the use of high energy bunched beam cooling at Fermilab Tevatron and SPS $p\bar{p}$ collider at CERN, a sophisticated numerical calculation and study of novel schemes of bunched beam cooling seems imminent.

The theory developed is consistent and mostly complete except for a closed form expression of the collective signal suppression factor of bunched beams in the region of strong synchrotron band overlap. Note that one can still calculate diffusion coefficients in such a region by properly adding up the contributions from overlapping resonances; however, one has to use the non-overlapping band expression for the signal suppression factor in these calculations. While for small collective effects, this is fairly accurate, for strong suppression effects, the outcome of such a numerical estimate is somewhat suspect. Under a different guise, the problem has plagued the study of bunched beam coherent instabilities with strong synchrotron mode coupling for over a decade and as of today no general solution exists. In the context of instabilities one can however afford to be contented with approximate criteria for thresholds and growth rates of coherent modes and more importantly bounds on stability. These have been obtained in the past ([9], [60], [65], [80], [87], [88], [90], [109]) in various limits. For stochastic cooling the problem is made worse in the sense that one has to solve an "inversion problem" of the infinite coupled mode case in order to obtain finite values of the signal suppression factor <u>or</u> dielectric permittivity. Our musings and efforts in this domain strongly suggest the use of an appropriate space for properly chosen collective signals with a qualitatively different ordering parameter which "concentrates" <u>or</u> "accumulates" all the essential mode-coupling contributions into a single or a few dominant terms of leading order. Future work in this direction will then require an approach of significantly different nature and quality in essence.

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APPENDIX A

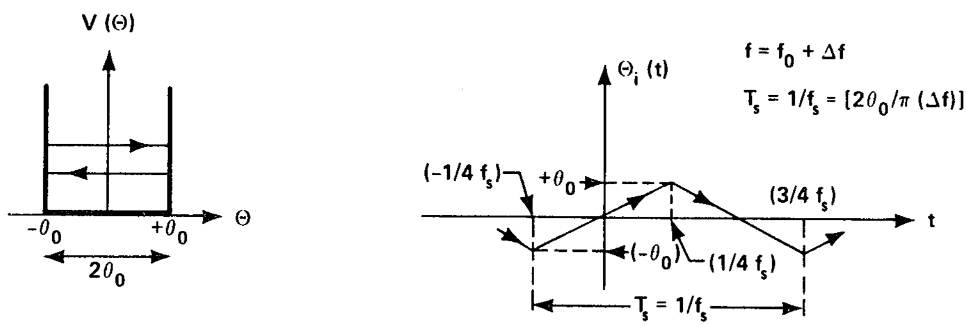
Longitudinal Schottky Spectrum for a Particle in a Square Bucket

The orbit of a particle in a square bucket (Fig. 38(a) and (b) is given by:

$$\theta(t) = 2\pi\, f_0\, t + Ⓗ(t)$$

where

$$Ⓗ(t) = \begin{cases} 2\pi(\Delta f)t & \text{for } \quad |t| < \dfrac{1}{4f_s} \\ \dfrac{\pi(\Delta f)}{fs} - 2\pi(\Delta f)t & \text{for } \quad \left(\dfrac{1}{4f_s}\right) < t < \left(\dfrac{3}{4f_s}\right) \end{cases}$$



(a) (b)

Square-Well Bucket and Particle Orbits

Fig. 38

The current due to the particle at azimuth $\theta = \theta_p$ of the pick-up is:

$$I(t,\theta_p) = 2\pi q\, f_0\, \delta\Big(\theta(t) - \theta_p\Big) = 2\pi q\, f_0 \left[1 + 2\sum_{h=1}^{\infty} \mathrm{Cos}\Big\{h\Big(\theta(t) - \theta_p\Big)\Big\}\right] = \sum_{h=0}^{\infty} I_h$$

where

$$I_h = 2\pi q\ f_o\ \mathrm{Re}\left\{\delta_h\ e^{ih[2\pi f_o t + \text{Ⓗ}(t) - \theta_p]}\right\}$$

$$= 2\pi q\ f_o\ \mathrm{Re}\left\{\delta_h\ v_h(t)\ e^{ih[2\pi f_o t - \theta_p]}\right\}$$

and

$$v_h(t) = e^{ih\text{Ⓗ}(t)}$$

$$\delta_h = \begin{cases} 1 & \text{for} \quad h = 0 \\ 2 & \text{for} \quad h > 0 \end{cases}$$

Fourier expanding $v_h(t)$ in harmonics of the bounce frequency f_s, we write

$$v_h(t) = e^{ih\text{Ⓗ}(t)} = \sum_{\mu=-\infty}^{+\infty} v_\mu\ e^{i\mu 2\pi f_s t}$$

where

$$v_\mu = f_s \int_{-\frac{1}{4f_s}}^{\frac{3}{4f_s}} e^{ih\text{Ⓗ}(t) - i\mu 2\pi f_s t}\ dt$$

In general there will be an initial phase so that $\text{Ⓗ}(t)\Big|_{t=0} = \text{Ⓗ}(t_o) \neq 0$ and

$$v_\mu = f_s \int_{-\frac{1}{4f_s} - t_o}^{\frac{3}{4f_s} - t_o} dt\ e^{ih\text{Ⓗ}(t+t_o) - i\mu 2\pi f_s t}$$

$$= e^{i\mu\psi^0}\ f_s \cdot \int_{-\frac{1}{4f_s}}^{\frac{3}{4f_s}} dt'\ e^{ih\text{Ⓗ}(t') - i\mu 2\pi f_s t'}$$

where $\psi^0 = 2\pi f_s t_o$ is the initial oscillation phase at $t = 0$. We can write:

$$v_\mu = e^{i\mu\psi^0}\left(I_1 + I_2\right)$$

where

$$I_1 = f_s \int_{-\frac{1}{4f_s}}^{+\frac{1}{4f_s}} dt'\, e^{ih2\pi(\Delta f)t' - i\mu 2\pi f_s t'} = \frac{1}{\pi}\,\frac{\operatorname{Sin}\frac{\pi}{2}\left[\frac{h(\Delta f)}{fs} - \mu\right]}{\left[h\,\frac{(\Delta f)}{f_s} - \mu\right]}$$

and

$$I_2 = f_s\, e^{ih\pi\frac{(\Delta f)}{f_s}} \int_{\frac{1}{4f_s}}^{\frac{3}{4f_s}} dt'\, e^{-ih2\pi(\Delta f)t' - i\mu 2\pi f_s t'} = \frac{1}{\pi}\,\frac{\operatorname{Sin}\frac{\pi}{2}\left[\frac{h(\Delta f)}{f_s} - \mu\right]}{\left[h\,\frac{(\Delta f)}{fs} + \mu\right]}$$

Let $\alpha = h\,\frac{(\Delta f)}{f_s} = h\left(\frac{2\Theta_o}{\pi}\right)$ and set $\Theta_p = 0$ without any loss of generality. Focusing attention on one revolution harmonic h, we get

$$I_h = (2\pi)\,(qf_o)\,\delta_h\left(\frac{2\alpha}{\pi}\right)\operatorname{Re}\left\{\sum_{\mu=-\infty}^{+\infty}\frac{\operatorname{Sin}\left[\frac{\pi}{2}\,(\alpha-\mu)\right]}{(\alpha^2-\mu^2)}\, e^{i2\pi(hf_o-\mu f_s)t}\, e^{i\mu\psi^0}\right\}$$

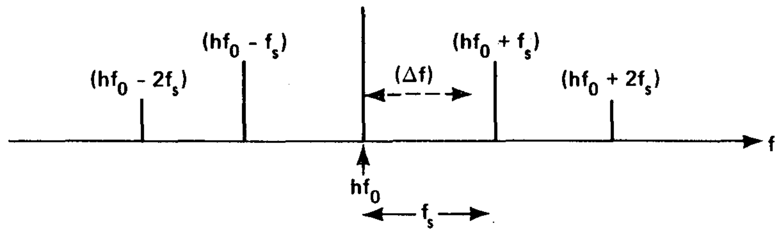
$$= \frac{(2\pi)(2qf_o)}{\pi\alpha}\,\delta_h\,\operatorname{Sin}\left(\frac{\pi\alpha}{2}\right)\operatorname{Cos} 2\pi hf_o t \qquad \underline{\text{CENTRAL HARMONIC}}$$

$$+ \frac{(2\pi)(2qf_o)\cdot\delta_h\cdot\alpha}{\pi\left(\alpha^2-1^2\right)}\operatorname{Cos}\left(\frac{\alpha\pi}{2}\right)\left[\operatorname{Cos}\left\{2\pi(hf_o-f_s)t+\psi^0\right\} - \operatorname{Cos}\left\{2\pi(hf_o+f_s)t-\psi^0\right\}\right] \qquad \underline{\text{FIRST SATELLITE PAIR}}$$

$$- \frac{(2\pi)(2qf_o)\cdot\delta_h\cdot\alpha}{\pi\left(\alpha^2-2^2\right)}\operatorname{Sin}\left(\frac{\alpha\pi}{2}\right)\left[\operatorname{Cos}\left\{2\pi\,(hf_o-2f_s)t+\psi^0\right\} + \operatorname{Cos}\left\{2\pi(hf_o+2f_s)t-\psi^0\right\}\right] \qquad \underline{\text{SECOND SATELLITE PAIR}}$$

+ etc.

See Fig. 39 below.



Single Particle Schottky Spectrum in a Square Bucket

Fig. 39

In the coasting beam limit:

$$\Delta f = \text{constant}$$

$$\left.\begin{array}{l} f_s \longrightarrow 0 \\ \\ T_s \longrightarrow \infty \end{array}\right\} \alpha = h \cdot \frac{\Delta f}{f_s} = h \quad \frac{2\theta_0}{\pi} \longrightarrow \infty$$

i.e. both top and bottom streams wrap around the ring many many times before ever reflecting and exchanging their roles.

The Fourier coefficients have significant magnitude only for

$$\mu \approx \pm\, h \frac{\Delta f}{f_s} \quad \text{i.e.} \quad \mu f_s \approx \pm\, h(\Delta f)$$

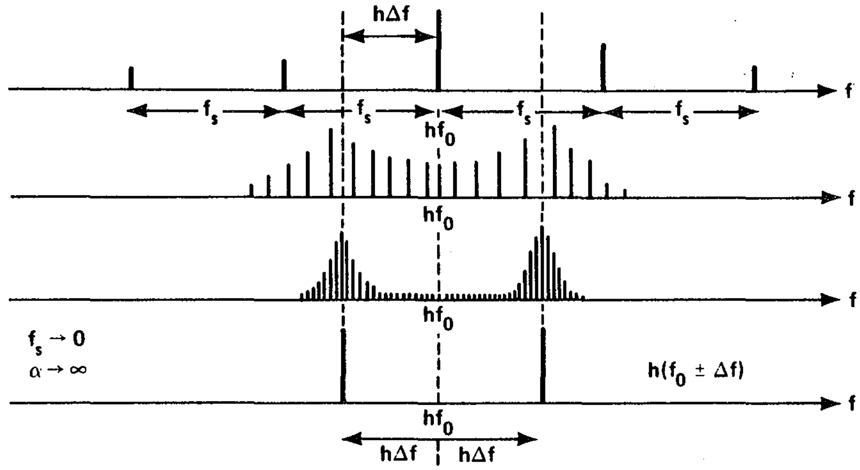
i.e. the spectrum is concentrated around $f = h(f_0 \pm \Delta f)$. See Fig. 40. For small bunch length or bucket:

$$\frac{\pi}{2} \frac{(\Delta f)}{f_s} << 1$$

In this limit, we have:

$$v_0 = 1$$

$$v_\mu = 0 \quad \text{for} \quad \mu = \text{even} \neq 0$$



Square-Well Schottky Spectrum in the Limit of a Coasting Beam

Fig. 40

$$V_\mu = \frac{2}{\pi} h \frac{(\Delta f)}{f_s} \frac{(-1)^{(\mu-1)/2}}{\mu^2} \quad \text{for} \quad \mu = \text{odd.}$$

$$= \frac{2}{\pi} \alpha \frac{(-1)^{(\mu-1)/2}}{\mu^2}$$

The spectrum is then given by:

$$I(t,\theta) = (2\pi)(qf_0) \operatorname{Re} \left\{ \sum_{h=0} \delta_h \left[1 + \frac{2\alpha}{\pi} \sum_{\substack{\mu=-\infty \\ (\text{odd})}}^{+\infty} \frac{(-1)^{(\mu-1)/2}}{\mu^2} e^{i2\pi(hf_0-\mu f_s)t} e^{i\mu\psi^0} \right] \right\}$$

APPENDIX B

Notion of Effective Gain

For a harmonic bunch we introduce the 'effective gain' given by

$$g_{eff.}^{\mu^{(\pm)}} = \sum_{\ell} g_{\ell}\, J_{\mu}(\ell a)\, J_{\mu}\left[(\ell \pm Q) a\right] \tag{A.B.1}$$

The signal suppression factor $\epsilon_{\mu}^{(\pm)}(a)$ as given by (A.E.21) then can be written as

$$\epsilon_{\mu}^{(\pm)}(a) = 1 + \frac{\pi\, N\, f(a)}{|\mu| \left|\frac{d\omega_s(a)}{da}\right|}\, g_{eff}^{\mu^{(\pm)}} \tag{A.B.2}$$

In analogy to above, one can construct, using the orbit integrals in Appendix A, an 'effective gain' for a bunch in a square bucket as

$$g_{eff.}^{n^{(\pm)}} = \sum_{m} g_m \left[\frac{\mathrm{Sin}(n \pm Q - m)\, \theta_o}{(n \pm Q - m)\, \theta_o}\right]^2 \tag{A.B.3}$$

and a signal suppression factor

$$\epsilon_{n}^{(\pm)}(\omega) = 1 + \frac{\pi N f(\omega)}{|n \pm Q|}\, g_{eff.}^{n^{(\pm)}} \tag{A.B.4}$$

where θ_o is the angular extent of the bunch and we have made use of the coupling strengths near the harmonics $\mu = \pm (n \pm Q) \Delta f / f_s$ as given in Appendix A.

This concept of effective gain has been supported reasonably well by the numerical simulation studies of transverse bunched beam cooling as reported in Ch. 13.

APPENDIX C

Proof of $\langle f(\underset{\sim}{x};t)\rangle = p(\underset{\sim}{x};t)$ in Section 9.1

Formally, we write the solution of (9.1.6) as a mapping:

$$M^t: \; \underset{\sim}{a} \longrightarrow \underset{\sim}{x} \qquad \text{for fixed } \sigma$$

or

$$\underset{\sim}{x} = M(\underset{\sim}{a};t;\sigma) \tag{A.C.1}$$

If the differential equation (9.1.6) is 'soluble', then the mapping (A.C.1) has an inverse:

$$\underset{\sim}{a} = M^{-1}(\underset{\sim}{x};t;\sigma)$$

Solution of (9.1.8) may then be written as

$$f(x;t;\sigma) = f\left[M^{-1}(\underset{\sim}{x},t;\sigma),0;\sigma\right] \cdot \left|\frac{d[M^{-1}(\underset{\sim}{x})]}{d[\underset{\sim}{x}]}\right|$$

where $\dfrac{d[M^{-1}(\underset{\sim}{x})]}{d[\underset{\sim}{x}]} = J$ is the Jacobian determinant of the mapping (A.C.1). For incompressible Liouvillian flow, the Jacobian determinant would simply be unity. We can thus write

$$\left\langle f(\underset{\sim}{x},t)\right\rangle = \int_{\Sigma} d\sigma \; f(\underset{\sim}{x};t;\sigma) \; P(\sigma)$$

$$= \int_{\Sigma} d\sigma \; P(\sigma) \; f\left[M^{-1}(\underset{\sim}{x};t;\sigma),0;\sigma\right] \left|\frac{d[M^{-1}(\underset{\sim}{x})]}{d[\underset{\sim}{x}]}\right|$$

Let us take the initial distribution

$$g(\underset{\sim}{x}) = f(\underset{\sim}{x};\ t=0) = \delta^{(n)}(\underset{\sim}{x}-\underset{\sim}{a})$$

Then

$$\left\langle f(\underset{\sim}{x},t)\right\rangle = \int_{\Sigma} d\sigma\ P(\sigma)\ \delta\left[M^{-1}(\underset{\sim}{x},t;\sigma) - \underset{\sim}{a}\right] \left|\frac{d[M^{-1}(\underset{\sim}{x})]}{d[\underset{\sim}{x}]}\right|$$

$$= \int_{\Sigma} d\sigma\ P(\sigma)\ \delta\left[\underset{\sim}{x} - M(\underset{\sim}{a},t;\sigma)\right] \qquad \text{(A.C.2)}$$

The integral in (A.C.2) is just the probability $p(\underset{\sim}{x};t)$ that the solution $M[\underset{\sim}{a},t;\sigma]$ of (9.1.6) takes the value $\underset{\sim}{x}$. Thus

$$\left\langle f(\underset{\sim}{x};t)\right\rangle = p(\underset{\sim}{x},t) \qquad \text{(A.C.3)}$$

This lemma was first demonstrated by Van Kampen [107]. It demonstrates that the solution $<f(\underset{\sim}{x},t)>$ of (9.1.8) leads to a solution $p(\underset{\sim}{x},t)$ for Eq. (9.1.6) automatically.

APPENDIX D

We derive the integral equaton for $R_{\underset{\sim}{n}_2\underset{\sim}{n}_1}(\underset{\sim}{I}_2,\underset{\sim}{I}_1)$ given by Eq. (9.2.14).

Equation (9.2.10) for $g(1,2;t)$ after Fourier analyzing in angles $\underset{\sim}{\psi}_1,\underset{\sim}{\psi}_2$ and phase-averaging over $f(\underset{\sim}{I}_1,\underset{\sim}{\psi}_1;t)$ to retain $f_o(\underset{\sim}{I}_1,t)$ only, gives:

$$\frac{\partial g_{n_1 n_2}(I_1,I_2;t)}{\partial t} + i\left[n_1\cdot\omega_1+n_2\cdot\omega_2\right] g_{n_1 n_2}(I_1,I_2;t) =$$

$$= - N \frac{\partial}{\partial I_1} \cdot \int dI_3 \sum_{n_3} G_{n_1 n_3}(I_1,I_3)\; g_{n_2,-n_3}(2,3;t)\; f_o(1;t)$$

$$- N \frac{\partial}{\partial I_2} \cdot \int dI_3 \sum_{n_3} G_{n_2 n_3}(I_2,I_3)\; g_{-n_3 n_1}(3,1;t)\; f_o(2;t)$$

$$- G_{n_1 n_2}(I_1,I_2) \cdot \frac{\partial f_o(1;t)}{\partial I_1} f_o(2;t) - G_{n_2 n_1}(I_2,I_1) \frac{\partial f_o(2;t)}{\partial \underset{\sim}{I}_2} f_o(1;t) \qquad \text{(A.D.1)}$$

We now perform a Laplace transformation of (A.D.1) assuming $g(1,2;t)$ changes much faster than $f_o(1;t)$ <u>or</u> $f_o(2;t)$; i.e., assuming $f_o(1;t)$ and $f_o(2;t)$ are almost constants in the time-scale of change of $g(1,2;t)$. Then:

$$\tilde{g}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(1,2;s) = \frac{1}{\left[s+i(\underset{\sim}{n}_1\cdot\underset{\sim}{\omega}_1+\underset{\sim}{n}_2\cdot\underset{\sim}{\omega}_2)\right]}$$

$$\left[-\,\mathbb{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(\underset{\sim}{I}_1,\underset{\sim}{I}_2)\cdot\frac{\partial f_o(1)}{\partial \underset{\sim}{I}_1}\,f_o(2)\cdot\frac{1}{s}\right.$$

$$-\,\mathbb{G}_{\underset{\sim}{n}_2\underset{\sim}{n}_1}(\underset{\sim}{I}_2,\underset{\sim}{I}_1)\cdot\frac{\partial f_o(2)}{\partial \underset{\sim}{I}_2}\,f_o(1)\cdot\frac{1}{s}$$

$$-\,N\frac{\partial}{\partial \underset{\sim}{I}_1}\cdot\int d\underset{\sim}{I}_3\sum_{\underset{\sim}{n}_3}\mathbb{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_3}(\underset{\sim}{I}_1,\underset{\sim}{I}_3)\,\tilde{g}_{\underset{\sim}{n}_1,-\underset{\sim}{n}_3}(2,3;s)\,f_o(1)$$

$$\left.-\,N\frac{\partial}{\partial \underset{\sim}{I}_2}\cdot\int d\underset{\sim}{I}_3\sum_{\underset{\sim}{n}_3}\mathbb{G}_{\underset{\sim}{n}_2\underset{\sim}{n}_3}(\underset{\sim}{I}_2,\underset{\sim}{I}_3)\,\tilde{g}_{-\underset{\sim}{n}_3\underset{\sim}{n}_1}(3,1;s)\,f_o(2)\right] \qquad \text{(A.D.2)}$$

The asymptotic behavior for $t \to \infty$ is governed by the $s \to 0^+$ limit. So we pick up the pole $s = 0^+$ and use $\lim\limits_{s\to 0^+}\frac{1}{[s+ix]} = \pi\delta_+(x)$ to get:

$$\tilde{g}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(1,2) = \pi\delta_+\left[\underset{\sim}{n}_1\cdot\underset{\sim}{\omega}_1+\underset{\sim}{n}_2\cdot\underset{\sim}{\omega}_2\right]$$

$$\left[-\,\mathbb{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(\underset{\sim}{I}_1,\underset{\sim}{I}_2)\cdot\frac{\partial f_o(1)}{\partial \underset{\sim}{I}_1}\,f_o(2) - \mathbb{G}_{\underset{\sim}{n}_2\underset{\sim}{n}_1}(\underset{\sim}{I}_2,\underset{\sim}{I}_1)\cdot\frac{\partial f_o(2)}{\partial \underset{\sim}{I}_2}\,f_o(1)\right.$$

$$\left.-\,\frac{\partial}{\partial \underset{\sim}{I}_1}\cdot\left\{\mathbb{B}^*_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(\underset{\sim}{I}_1,\underset{\sim}{I}_2)\,f_o(1)\right\} - \frac{\partial}{\partial \underset{\sim}{I}_2}\cdot\left\{\mathbb{B}_{\underset{\sim}{n}_2\underset{\sim}{n}_1}(\underset{\sim}{I}_2,\underset{\sim}{I}_1)\,f_o(2)\right\}\right] \qquad \text{(A.D.3)}$$

Multiplying $\tilde{g}_{\underset{\sim}{n}_1\underset{\sim}{n}_2}(1,3)$ by $N\sum_{\underset{\sim}{n}_3}\int d\underset{\sim}{I}_3\,\mathbb{G}^*_{\underset{\sim}{n}_2\underset{\sim}{n}_3}(\underset{\sim}{I}_2,\underset{\sim}{I}_3)$ we get:

$$\mathcal{B}_{\underset{\sim}{n}_2\underset{\sim}{n}_1}(\underset{\sim}{I}_2,\underset{\sim}{I}_1) = -\pi N \sum_{\underset{\sim}{n}_3} \int d\underset{\sim}{I}_3\ \delta_+\left[\underset{\sim}{n}_3\cdot\underset{\sim}{\omega}_3-\underset{\sim}{n}_1\cdot\underset{\sim}{\omega}_1\right] \cdot \mathcal{G}^*_{\underset{\sim}{n}_2\underset{\sim}{n}_3}(\underset{\sim}{I}_2,\underset{\sim}{I}_3)$$

$$\left[\mathcal{G}_{\underset{\sim}{n}_1\underset{\sim}{n}_3}(\underset{\sim}{I}_1,\underset{\sim}{I}_3) \cdot \frac{\partial f(1)}{\partial \underset{\sim}{I}_1} f(3) - \mathcal{G}^*_{\underset{\sim}{n}_3\underset{\sim}{n}_1}(\underset{\sim}{I}_3,\underset{\sim}{I}_1) \cdot \frac{\partial f(3)}{\partial \underset{\sim}{I}_3} f(1)\right.$$

$$\left. + \frac{\partial f(1)}{\partial \underset{\sim}{I}_1} \cdot \mathcal{B}^*_{\underset{\sim}{n}_1\underset{\sim}{n}_3}(\underset{\sim}{I}_1,\underset{\sim}{I}_3) - \frac{\partial f(3)}{\partial \underset{\sim}{I}_3} \cdot \mathcal{B}_{\underset{\sim}{n}_3\underset{\sim}{n}_1}(\underset{\sim}{I}_3,\underset{\sim}{I}_1)\right] \qquad \text{(A.D.4)}$$

as advertized in Eq. (9.2.14) before.

We mention here that a careful multiple-time scale perturbation analysis can be performed on Eqs. (9.2.9) and (9.2.10) using two time-scales τ_0, τ_1 where τ_0 is the fast time-scale of variation of $g(1,2;t)$ and τ_1 is the slow relaxation time of $f_0(1;t)$. An analysis similar to Section 6.2 of Chapter 6, then shows that the non-secularity condition (i.e. $g^{(0)}(1,2;\tau_0)$ does not diverge as $(\tau_0 \to \infty)$ on $g(1,2,t)$) on the faster time-scale determines the slow evolution of $f_0(1;\tau_1)$ on the τ_1 scale which is in agreement with Eqs. (9.2.12) and (9.2.13). Thus our assumption of $f_0(1;t)$ being a constant in Eq. (A.D.1) gives results consistent with a more careful analysis of time scales.

We note that the terms $(\underset{\sim}{n}_3\cdot\underset{\sim}{\omega}_3-\underset{\sim}{n}_1\cdot\underset{\sim}{\omega}_1)$ appearing in the formulas in this Appendix effect mixing through frequency spread and enhance the interaction of particles neighboring in frequency.

APPENDIX E

Transverse Signal Suppression Factor for Bunched Beams in a Model Cooling Interaction

We consider transverse cooling of a longitudinally bunched beam in the model dipole cooling interacion described at the end of Section 4.3 and in Section 4.4. We will use Eqs. (4.3.44) through (4.3.50) and (4.4.1) through (4.4.3) in this appendix.

From (4.4.3) and (4.3.46), we write the cooling equation of motion as

$$\dot{x}_i + i\omega_\perp(i)\, x_i = -\sum_{j=1}^{N} g\left(\theta^i - \theta^j\right) x_j = -\sum_{\ell=-\infty}^{+\infty} z_\ell(t)\, g_\ell\, e^{i\ell\theta^i(t)} \tag{A.E.1}$$

where

$$z_\ell(t) = \sum_{j=1}^{N} x_j(t)\, e^{-i\ell\theta^j(t)} \tag{A.E.2}$$

describes a collective variable. Equation (A.E.1) has the formal solution

$$x_i(t) = x_i^0\, e^{-i\omega_\perp(i)t} - \sum_{\ell=-\infty}^{+\infty} g_\ell \int_0^t dt'\, z_\ell(t')\, e^{i\ell\theta^i(t') - i\omega_\perp(i)(t-t')} \tag{A.E.3}$$

Using (A.E.2) and (A.E.3), we find the integral equation satisfied by $z_\ell(t)$ as follows:

$$z_\ell(t) = z_\ell^0(t) - \sum_{i=1}^{N} \sum_{m=-\infty}^{+\infty} g_m \int_0^t dt'\, z_m(t')\, e^{i[m\theta^i(t') - \ell\theta^i(t)] - i\omega_\perp(i)(t-t')} \tag{A.E.4}$$

Using the identity given by (4.3.52) and defining

$$Q_\ell(t) = z_\ell(t)\ {}^{i\ell\omega_0 t} \tag{A.E.5}$$

we get, by use of the orbits (4.3.45), the following:

$$Q_\ell(t) = Q_\ell^0(t) - \sum_{i=1}^{N} \sum_{m=-\infty}^{+\infty} g_m \int_0^t dt' \, Q_m(t') \sum_\mu \sum_{\mu'} J_\mu(ma_i) \, J_{\mu'}(\ell a_i)$$

$$e^{i\omega_s(i)(\mu t-\mu' t')-i\omega_\perp(i)(t-t')} \, e^{i(\mu-\mu')\psi^i(0)} \tag{A.E.6}$$

Note that $\tilde{Q}_\ell(s) = \tilde{z}_\ell(s-i\ell\omega_0)$, as follows from a Laplace transform of (A.E.5), corresponds to the properly Doppler-shifted response in the beam frame.

In the continuous N-body limit, we replace the sum over i by an integration over the distribution function $f(J,\psi) = f\left(\frac{1}{2}\, a^2,\psi\right)$ of the particle in the synchrotron phase-space, normalized as

$$\frac{1}{2\pi}\int_0^\infty dJ \int_0^{2\pi} d\psi \, f(J,\psi) = \frac{1}{2\pi}\int_0^\infty d\left(\frac{1}{2}\, a^2\right) \int_0^{2\pi} d\psi \, f\left(\frac{1}{2}\, a^2,\psi\right) = 1 \tag{A.E.7}$$

Then

$$\sum_{i=1}^{N} F(a_i,\psi_i) \longrightarrow N \cdot \left(\frac{1}{2\pi}\right)\int_0^\infty d\left(\frac{1}{2}\, a^2\right) \int_0^{2\pi} d\psi \, f\left(\frac{1}{2}\, a^2,\psi\right) F(a,\psi) \tag{A.E.8}$$

where N is the total number of particles in the beam. If the particles are distributed randomly and uniformly in phase-angle ψ, $f\left(\frac{1}{2}\, a^2,\psi\right)$ is independent of ψ and we get

$$\sum_{i=1}^{N} F(a_i,\psi_i) \longrightarrow N \cdot \left(\frac{1}{2\pi}\right)\int_0^\infty d\left(\frac{1}{2}\, a^2\right) f\left(\frac{1}{2}\, a^2\right) \int_0^{2\pi} F(a,\psi)\, d\psi \tag{A.E.9}$$

Using

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(\mu-\mu')\psi_i(0)} \, d\psi_i(0) = \delta_{\mu\mu'} \tag{A.E.10}$$

we obtain from (A.E.6)

$$Q_\ell(t) = Q_\ell^0(t) - N \sum_{m=-\infty}^{+\infty} g_m \int_0^t dt' \, Q_m(t') \sum_{\substack{\mu \\ (-\infty)}}^{+\infty} \int_0^\infty d\left(\frac{1}{2}\, a^2\right) f\left(\frac{1}{2}\, a^2\right) J_\mu(ma) \, J_\mu(\ell a)$$

$$e^{-i(\mu\omega_s(a)+\omega_\perp)(t-t')} \tag{A.E.11}$$

or

$$Q_\ell(t) = Q_\ell^0(t) - N \sum_{m=-\infty}^{+\infty} g_m \int_0^t dt'\; Q_m(t')\; f_{m\ell}(t-t') \qquad \text{(A.E.12)}$$

where

$$f_{m\ell}(t-t') = \sum_{\mu=-\infty}^{+\infty} \int_0^\infty d\left(\frac{1}{2}\,a^2\right) f\left(\frac{1}{2}\,a^2\right) J_\mu(ma)\; J_\mu(\ell a)\; e^{-i[\mu\omega_s(a)+\omega_\perp](t-t')} \qquad \text{(A.E.13)}$$

Laplace transforming in time t yields [1]

$$\tilde{Q}_\ell(s) = \tilde{Q}_\ell^0(s) - N\sum_m g_m\, \tilde{f}_{m\ell}(s)\; \tilde{Q}_m(s) \qquad \text{(A.E.14)}$$

or

$$\sum_m \left[\delta_{\ell m} + Ng_m\, \tilde{f}_{m\ell}(s)\right] \tilde{Q}_m(s) = \tilde{Q}_\ell^0(s) \qquad \text{(A.E.15)}$$

where

$$\tilde{f}_{m\ell}(s) = \sum_{\mu=-\infty}^{+\infty} \int_0^\infty d\left(\frac{1}{2}\,a^2\right) \frac{J_\mu(ma)\; J_\mu(\ell a)}{\left[s + i(\mu\omega_s(a) + \omega_\perp)\right]}\; f\left(\frac{1}{2}\,a^2\right) \qquad \text{(A.E.16)}$$

If now we evaluate (A.E.14) at $s = i\Omega$ where

$$\Omega = \mu\omega_s(a) + \omega_\perp = \mu\omega_s(a) + Q\omega_o$$

we get the response $Q_{(\ell+Q),\mu}(a)$ corresponding to the $\mu^{\underline{th}}$ synchrotron satellite band in the $\ell^{\underline{th}}$ revolution harmonic for particles with synchrotron amplitude a and at betatron harmonic $n_x = +1$ as follows:

$$\tilde{Q}_{(\ell+Q),\mu}(a) = \tilde{Q}^0_{(\ell+Q),\mu}(a) - \sum_{\mu'=-\infty}^{\infty} \sum_{m=-\infty}^{+\infty} (Ng_m) \int d\omega_s(a')\; f\left[\omega_s(a')\right]$$

$$\left[\frac{J_{\mu'}(ma')\; J_{\mu'}[(\ell+Q)a']}{\left[(\mu\omega_s(a) + \omega_\perp) - (\mu'\omega_s(a') + \omega_\perp)\right]}\right] \tilde{Q}_{(m+Q),\mu'}(a') \qquad \text{(A.E.17)}$$

where we have used $f\left(\frac{1}{2}a^2\right) d\left(\frac{1}{2}a^2\right) \equiv f(\omega_s(a))\ d[\omega_s(a)]$ with $f(\omega_s(a))$ being the distribution in synchrotron frequency of the particles in the bunch.

For non-overlapping synchrotron bands, we now approximate the integral by keeping only the term corresponding to the non-overlap resonances $[\mu\omega_s(a)+Q\omega_o] \approx [\mu'\omega_s(a')+Q\omega_o]$ with $\mu = \mu'$ and $a = a'$ in the integrand and neglecting the principal value integral. Then

$$\tilde{Q}_{(\ell+Q),\mu}(a) = \tilde{Q}^0_{(\ell+Q),\mu}(a) - N\pi \frac{f(\omega_s(a))}{|\mu|} \sum_m g_m\ J_\mu(ma)\ J_\mu\left[(\ell+Q)a\right] \tilde{Q}_{(m+Q),\mu}(a) \tag{A.E.18}$$

Multiplying both sides by $g_\ell\ J_\mu(\ell a)$ and summing over ℓ yields:

$$X^{(\pm)}_\mu(a) = \frac{X^{0(\pm)}_\mu(a)}{\epsilon^{(\pm)}_\mu(a)} \tag{A.E.19}$$

where

$$X^{(\pm)}_\mu(a) = \sum_\ell g_\ell\ J_\mu(\ell a)\ \tilde{Q}_{(\ell\pm Q),\mu}(a) \tag{A.E.20}$$

$$X^{0(\pm)}_\mu(a) = \sum_\ell g_\ell\ J_\mu(\ell a)\ \tilde{Q}^0_{(\ell\pm Q),\mu}(a)$$

and

$$\epsilon^{(\pm)}_\mu(a) = 1 + \frac{\pi N f(\omega_s(a))}{|\mu|} \sum_\ell g_\ell\ J_\mu(\ell a)\ J_\mu\left[(\ell\pm Q)a\right]$$

$$= 1 + \frac{\pi N f(a)}{|\mu| \left|\frac{d\omega_s(a)}{da}\right|} \sum_\ell g_\ell\ J_\mu(\ell a)\ J_\mu\left[(\ell\pm Q)a\right] \tag{A.E.21}$$

The cooling rate for linear transverse dipole cooling is thus:

$$\gamma = -\sum_{(\pm)} \sum_{\mu} \left[\frac{G^{(\pm)}_{\mu,-\mu}(a,a)}{\epsilon^{(\pm)}_{\mu}(a)} - \frac{\left|G^{(\pm)}_{\mu,-\mu}(a,a)\right|^2}{\left|\epsilon^{(\pm)}_{\mu}(a)\right|^2} \cdot \frac{\pi N f(a)}{|\mu| \left|\frac{d\omega_s(a)}{da}\right|} \right] \tag{A.E.22}$$

$$= -\sum_{(\pm)} \sum_{\mu} \frac{G^{(\pm)}_{\mu,-\mu}(a,a)}{\left|\epsilon^{(\pm)}_{\mu}(a)\right|^2} \tag{A.E.23}$$

where we have used the symmetries of $\epsilon^{(\pm)}_{\mu}(a)$ as given by Eq. (A.E.21).

APPENDIX F

A Few Properties of the Gain Function, The Collectively Modulated Voltage and the Kernel Appearing in the Coupled-Mode Response Equation for a Bunch

We demonstrate various properties of the quantities appearing in the collective response equation

$$V^K(\Omega) = V_0^K(\Omega) + \sum_{k=-\infty}^{+\infty} D_k(\Omega)\, V^K\left(\Omega + k\omega_0\right) \tag{10.1.24}$$

where

$$\begin{aligned} \tilde{V}^K(\Omega) &= \tilde{G}(\Omega)\, \tilde{I}(\theta_p,\Omega) \\ V_0^K(\Omega) &= \tilde{G}(\Omega)\, \tilde{I}_o(\theta_p,\Omega) \end{aligned} \tag{A.F.1}$$

and $D_k(\Omega)$ is defined in Eqs. (10.1.25) through (10.1.27) and Ω is a real frequency.

(1)

$$\begin{aligned} \tilde{I}(\theta_p,\Omega) &= \tilde{I}^*(\theta_p,-\Omega) \\ \tilde{I}_o(\theta_p,\Omega) &= \tilde{I}_o^*(\theta_p,-\Omega) \end{aligned} \tag{A.F.2}$$

Proof. These follow from the reality of $I(\theta_p,t)$ and $I_o(\theta_p,t)$.

(2)

$$\tilde{G}(\Omega) = \tilde{G}^*(-\Omega) \tag{A.F.3}$$

Proof. This again follows from the reality of $G(t-t') = G(\tau)$.

(3)

$$\begin{aligned} V^K(\Omega) &= V^{K*}(-\Omega) \\ V_o^K(\Omega) &= V_o^{K*}(-\Omega) \end{aligned} \tag{A.F.4}$$

Proof. These follow from the definition (A.F.1) and properties (1) and (2) above and also from the reality of $V^K(t)$ and $V_o^K(t)$.

(4) $$D_k^*(\Omega) = D_{-k}(-\Omega) \qquad \text{(A.F.5)}$$

Proof: From (10.1.24) and (A.F.4)

$$V_o^K(-\Omega) + \sum_{p=-\infty}^{+\infty} D_p(-\Omega)\; V^K\!\left(-\Omega + p\omega_o\right) = V_o^{K*}(\Omega) + \sum_{k=-\infty}^{+\infty} D_k^*(\Omega)\; V^{K*}\!\left(\Omega + k\omega_o\right)$$

or

$$V_o^K(-\Omega) + \sum_{p=-\infty}^{+\infty} D_p(-\Omega)\; V^{K*}\!\left(\Omega - p\omega_o\right) = V_o^K(-\Omega) + \sum_{k=-\infty}^{+\infty} D_k^*(\Omega)\; V^{K*}\!\left(\Omega + k\omega_o\right)$$

Equation (A.F.5) follows by comparison of both sides.

(5) The operator $\hat{D}(\Omega) = \sum_{k=-\infty}^{+\infty} D_k(\Omega)\; \hat{T}^k$ is not 'self-adjoint'. (A.F.6)

Proof. In order to be self-adjoint we must have

$$\hat{D}(\Omega) = \left[\hat{D}(\Omega)\right]^+$$

i.e.

$$\sum_k D_k(\Omega)\; \hat{T}^k = \sum_p \left[D_p(\Omega)\; \hat{T}^p\right]^+$$

$$\text{Right hand side} = \sum_p \hat{T}^{-p}\; D_p^*(\Omega)$$

$$= \sum_p D_p^*\!\left(\Omega - p\omega_o\right)\; \hat{T}^{-p}$$

$$= \sum_k D_{-k}^*\!\left(\Omega + k\omega_o\right)\; \hat{T}^k$$

Therefore, to be self-adjoint, we must have $D_p(\Omega) = D_{-p}^*(\Omega + p\omega_o)$ which is not true from property (4) above.

(6) $$\sum_k D_k(\Omega)\, z^{-k} = \int_0^\infty da \left[\frac{df^0}{da}\right] \sum_\mu \left\{ \left[\sum_m A_m^\mu(\Omega;a)\, z^{-m}\right] \left[\sum_n B_n^\mu(a)\, z^{-n}\right] \right\} \qquad \text{(A.F.7)}$$

for any complex number z and in particular for $z = 1$:

$$\sum_k D_k(\Omega) = \int_0^\infty da \left[\frac{df^0}{da}\right] \sum_\mu \left\{ \left[\sum_m A_m^\mu(\Omega;a)\right] \left[\sum_n B_n^\mu(a)\right] \right\} \qquad \text{(A.F.8)}$$

where $A_m^\mu(\Omega;a)$ and $B_n^\mu(a)$ are defined in (10.1.26) and (10.1.27).

Proof: $D_k(\Omega)$ is a discrete convolution of A and B in the form $\sum_p A_{n-p}\, B_p$. Equations (A.F.7) and (A.F.8) follow from the convolution theorem of z-transforms: if

$$C_n = \sum_p A_{n-p}\, B_p$$

then

$$C(z) = A(z) \cdot B(z)$$

where

$$C(z) = \sum_{n=-\infty}^{+\infty} C_n\, z^{-n}, \quad z \text{ a complex number}$$

(7) Let $V^K[\ell] = V^K(\ell\omega_0)$. From (10.1.34)

$$V^K(\ell\omega_0) = V_0^K(\ell\omega_0) + \sum_p D_p(\ell\omega_0)\, V^K\left[(\ell+p)\omega_0\right]$$

Change indices to $(\ell+p) = k$. Then

$$V^K[\ell] = V_0^K[\ell] + \sum_k D\left[k-\ell;\ell\right] V^K[k]$$

This is not a discrete convolution, since $D[k-\ell;\ell]$ depends not only on $(k-\ell)$, but also on ℓ. This is the unique feature of bunched beam response.

(8) (10.1.24) in frequency domain corresponds to an integral equation in time-domain:

$$V^K(t) = \int dt' \, K(t,t') \, V(t') + V_0^K(t) \tag{A.F.9}$$

where $K(t,t') = K(t-t';t')$ is periodic in t' with period $T_0 = 2\pi/\omega_0$. Then

$$K(t-t';t') = \sum_{p=-\infty}^{+\infty} K_p(t-t') \, e^{ip\omega_0 t'} \tag{A.F.10}$$

and using (A.F.10) in (A.F.9) and Fourier transforming in time yields (10.1.24) immediately with the identification

$$\tilde{K}_p(\Omega) = D_p(\Omega) \; .$$

So we can interpret $V^k(\Omega+p\omega_0)$ as the components of a Bloch function $V(t)$ written in the Bloch form

$$V^K(t;\Omega) = e^{i\Omega t} \sum_{p=-\infty}^{+\infty} V^K(\Omega+p\omega_0) \, e^{ip\omega_0 t}$$

$$= e^{i\Omega t} \, F(\Omega;t)$$

where $F(\Omega;t) = \sum_{p=-\infty}^{+\infty} V^K(\Omega+p\omega_0) \, e^{ip\omega_0 t}$ is periodic in t with period $T = 2\pi/\omega_0$.