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IN TWO DIMENSIONS

Berkeley, California

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ABSTRACT

The numerical solution of the second-order, elliptic, quasi-linear, partial-differential equation arising in a two-dimensional magnetostatic-field problem, where the magnetic permeability varies with the field, is considered. A set of nonlinear difference equations approximating the original differential equation is derived, and in solving a test problem the method of nonlinear successive overrelaxation is shown to be superior both to Newton's method and to a commonly used method based on a small-magnetic-field approximation.

This method, as here presented, could also be used to numerically solve similar equations, such as those for Plateau's problem or for irrotational compressible fluid flow.

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1. Introduction.   In this paper, the numerical solution of the
second-order, elliptic, quasi-linear, partial differential equation arising
in two-dimensional magnetostatic field problems is discussed. The type
of problems considered are those arising, for example, in the design of
particle accelerators where the desired magnetic field strength is so large
as to be principally in the domain of nonlinear behavior of the magnetic
material. For such a problem, the usual successive-approximation methods
[1] based on the technique of linearizing about small magnetic fields may be
inadequate, and a technique involving the more essential nonlinear features
should be used.

A numerical method of the latter type is presented, which is based
on the iterative solution, by nonlinear successive overrelaxation, of a set of
nonlinear difference equations approximating the differential equation. Non-
linear successive overrelaxation was recently investigated by Ortega and
Rockoff (or, as they more specifically describe it, extrapolated-Gauss-
Seidel-Newton iteration), and they found that the method compared favorably
to other methods in solving a mildly nonlinear elliptic equation [2]. The
method presented here for a quasi-linear equation is similar to those pro-
posed by Lieberstein [3], Schechter [4], and Greenspan [5], but the approx-
imating difference equations are set up differently. The performance of the
method in numerically solving a sample problem is compared to those of the

usual small-magnetic-field approach and of Newton's method. The results show that the method described here is superior in solving the sample problem. The method could be used to numerically solve other problems governed by a similar quasi-linear differential equation, such as Plateau's problem, or that of irrotational compressible fluid flow.

2. Formulation. Consider a two-dimensional simply connected region R in the x-y plane with boundary Γ. Let a current density in the z direction $\underset{\sim}{J}(x, y) = J(x, y)\underset{\sim}{k}$ be given in R; then the magnetic vector potential $\underset{\sim}{A}(x, y) = A(x, y)\underset{\sim}{k}$ satisfies

(1) $$\underset{\sim}{\nabla} \cdot (\gamma \underset{\sim}{\nabla} A) = -4\pi J \quad \text{in} \quad R,$$

and the magnetic field $\underset{\sim}{B}$ is given by $\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = (\partial A/\partial y)\underset{\sim}{i} - (\partial A/\partial x)\underset{\sim}{j}$. The quantity γ is the magnetic reluctivity (reciprocal of the magnetic permeability μ) of the material occupying R, and is a given function of $|\underset{\sim}{B}|^2$. Since in two dimensions, $|\underset{\sim}{B}|^2 = |\underset{\sim}{\nabla} \times \underset{\sim}{A}|^2 = A_x^2 + A_y^2 = |\underset{\sim}{\nabla} A|^2$, γ is a function of $|\underset{\sim}{\nabla} A|^2$, so that when the differentiations in Eq. (1) are performed, the equation becomes

(1a) $$[\gamma + 2\gamma' A_x^2] A_{xx} + 4\gamma' A_x A_y A_{xy} + [\gamma + 2\gamma' A_y^2] A_{yy} = -4\pi J,$$

where the prime denotes differentiation with respect to $|\underset{\sim}{\nabla} A|^2$, and the subscripts denote partial differentiation. The reluctivity for ideal materials satisfies

$$M \geqslant \gamma \geqslant m > 0 \quad \text{and} \quad M' \geqslant \gamma + 2\gamma'|\underset{\sim}{\nabla} A|^2 \geqslant m' > 0$$

and hence Eq. (1) is quasi-linear and/elliptic. uniformly The boundary conditions for A are normally that A equals A_0, a constant, (no flux leakage) along a

portion or all of Γ, and that $\partial A/\partial n$, the normal derivative of A, is zero (symmetry condition) along the remainder of Γ.

The problem may also be formulated in variational terms. Find a function $A(x, y)$, twice differentiable in R satisfying the boundary conditions on Γ, that minimizes the integral,

$$(2) \qquad I = \iint_R [g(|\nabla A|^2) - 8\pi J A]\ dx\ dy.$$

The given function $g(|\nabla A|^2)$ is proportional to the magnetostatic energy and is related to the reluctivity by

$$(3) \qquad \gamma = \frac{dg}{d(|\nabla A|^2)}.$$

Equation (1) is the Euler equation corresponding to Eq. (2).

In accelerator-design problems, R is usually divided into two regions, R_1 and R_2, by a curve Γ_1, and Eq. (1) [or Eq. (2)] holds separately for each region. The regions are characterized by different permeability functions. For region R_1, which is the region occupied by the ferromagnetic material, γ varies with $|\nabla A|^2$, whereas for region R_2, which is not occupied by ferromagnetic materials, γ is identically 1. In region R_2, Eq. (1) simplifies to the Poisson equation, and correspondingly, $g(|\nabla A|^2)$ in Eq. (2) simplifies to $|\nabla A|^2$. The appropriate matching condition along Γ_1 is that $\gamma(\partial A/\partial n)$ and A be continuous. In the usual case, one has $J \equiv 0$ in region R_1 and $J \neq 0$ in region R_2.

The numerical solution of Eq. (1) inside of region R_2 presents little problem, since standard finite-difference methods for the Laplace operator can be used. The main difficulty arises in region R_1, where Eq. (1) is not linear. Methods commonly in use today for solving Eq. (1)

are based upon obtaining the succession of linear approximating equations for A^{n+1}, the (n+1)th approximation to A,

(4) $$\nabla \cdot (\gamma^n \nabla A^{n+1}) = -4\pi J,$$

where γ^n denotes γ as calculated from the nth approximation to A. Such methods, however, can be slowly converging or unstable when the range of A is such that γ differs significantly from a constant over R_1; this is the case when the current density J in region R_2 is large enough to partly saturate the magnetic material in R_1 [6].

In this paper another method of solving Eq. (1) is investigated which takes into account variations of γ with $|\nabla A|^2$. The method essentially corresponds to Newton's method, which obtains A^{n+1}, the (n+1)th approximation to A, by solving the equation

(5) $$\nabla \cdot [\gamma^n \nabla(A^n + \epsilon) + 2(\nabla A^n \cdot \nabla \epsilon)(\gamma^n)' \nabla A^n] = -4\pi J$$

for the quantity ϵ, subject to the appropriate boundary conditions, and adding it to A^n,

$$A^{n+1} = A^n + \epsilon.$$

Equation (5) is derived from Eq. (1) by neglecting all terms $O(\epsilon^2)$. Notice that Eq. (4) lacks the term containing $(\gamma^n)'$ in Eq. (5). Thus, it is a special case of Eq. (5) when the term containing $(\gamma^n)'$ is negligible in comparison to the other retained terms, which is the case when the magnetic field is small and γ is nearly constant.

In the following sections, a set of nonlinear difference equations is derived to approximate Eq. (1), and methods of solution using successive approximations analogous to Eqs. (4) and (5) are compared for a sample problem.

3. Finite-difference equations. The finite-difference equations corresponding to Eq. (1) for a rectangular mesh are given in this section, and these equations are the only ones explicitly discussed in the remainder of the paper. The same method of solution may be applied to other mesh configurations as well [6].

Let the region R be covered with a rectangular mesh (not necessarily uniformly spaced) parallel to the x and y directions, and, to avoid the additional complications of boundary interpolation, let the lines intersect Γ and Γ_1 only at mesh points. Replace any curved portion of Γ and Γ_1 with a polygonal one consisting of the chords joining adjacent mesh points. The region R is thus divided into rectangular mesh cells in its interior and either rectangular or right triangular mesh cells at the boundary and interfaces, with each cell lying entirely in either R_1 or R_2 [7, Sec. 6.3]. Let the x and y mesh spacings be denoted by

$$(6) \qquad h_{\overline{i}} = x_i - x_{i-1} \quad \text{and} \quad k_{\overline{j}} = y_j - y_{j-1} \; ,$$

respectively. The difference equations satisfied by $A_{i,j}$, the discrete approximation to $A(x, y)$, can be derived by first considering a discrete analog to the variational form of the problem, Eq. (2). From this (Ritz method), a set of nonlinear difference equations can then be obtained corresponding to Eq. (1).

Approximate the integral in Eq. (2) by taking the integrand to be constant over each mesh cell. Then the integral is replaced by the sum

$$(7) \qquad I \approx \sum_r [g(|\nabla A|^2) - 8\pi J A]_r \times \text{area}_r \; ,$$

where the sum is taken over all cells into which R has been divided. The

specific form of each term in the sum will depend upon whether the corresponding cell is rectangular or triangular.

If the $\underline{r}$th cell is rectangular, the appropriate value of the integrand to use is its value at the center of the cell (midpoint rule). The explicit expression for the term corresponding to cell $\overline{i},\overline{j}$ [the one with center at $(x_{i-1} + h_{\overline{i}}/2,\ y_{i-1} + k_{\overline{i}}/2)$, see Fig. 1] that is used here is

$$(8) \qquad [g(|\underset{\sim}{\nabla}A|^2) - 8\pi J A]_{\overline{i},\overline{j}} = g(|\underset{\sim}{\nabla}A|^2_{\overline{i},\overline{j}}) - 8\pi J_{\overline{i},\overline{j}} A_{\overline{i},\overline{j}},$$

where

$$(9) \qquad |\underset{\sim}{\nabla}A|^2_{\overline{i},\overline{j}} = \frac{1}{2}\left\{ \left(\frac{A_{i,j} - A_{i-1,j}}{h_{\overline{i}}} \right)^2 + \left(\frac{A_{i,j-1} - A_{i-1,j-1}}{h_{\overline{i}}} \right)^2 \right.$$
$$\left. + \left(\frac{A_{i,j} - A_{i,j-1}}{k_{\overline{j}}} \right)^2 + \left(\frac{A_{i-1,j} - A_{i-1,j-1}}{k_{\overline{j}}} \right)^2 \right\},$$

an approximation giving $|\underset{\sim}{\nabla}A|^2$ at the center of the cell to $O(h_{\overline{i}}^2 + k_{\overline{j}}^2)$,

$$A_{\overline{i},\overline{j}} = \frac{1}{4}(A_{i,j} + A_{i-1,j} + A_{i,j-1} + A_{i-1,j-1}),$$

and $J_{\overline{i},\overline{j}}$ is the given value for the average current density J crossing cell $_{\overline{i},\overline{j}}$. The area of the cell is

$$\text{area}_{\overline{i},\overline{j}} = h_{\overline{i}}\, k_{\overline{j}}.$$

If the x and y differences of $A_{i,j}$ are denoted by

$$\delta_{\overline{i},j} = \frac{A_{i,j} - A_{i-1,j}}{h_{\overline{i}}} \quad \text{and} \quad \eta_{i,\overline{j}} = \frac{A_{i,j} - A_{i,j-1}}{k_{\overline{j}}},$$

then Eq. (9) can be written more simply as

(10)
$$|\nabla A|^2_{\bar{i},\bar{j}} = \frac{1}{2}\left\{ \delta^2_{\bar{i},j} + \delta^2_{\bar{i},j-1} + \eta^2_{i,\bar{j}} + \eta^2_{i-1,\bar{j}} \right\}.$$

If the _rth_ cell is right triangular, the corresponding explicit expression in Eq. (7) depends upon the orientation of the triangle. For example, for cell$_{\bar{i},\bar{j},III}$ (the one with its right angle in the third quadrant of rectangular cell$_{\bar{i},\bar{j}}$, see Fig. 2) the expression is

(11)
$$[g(|\nabla A|^2) - 8\pi J A]_{\bar{i},\bar{j},III} = g(|\nabla A|^2_{\bar{i},\bar{j},III}) - 8\pi J_{\bar{i},\bar{j},III} A_{\bar{i},\bar{j},III},$$

where

(12)
$$|\nabla A|^2_{\bar{i},\bar{j},III} = \delta^2_{\bar{i},j-1} + \eta^2_{i-1,\bar{j}},$$

(13)
$$A_{\bar{i},\bar{j},III} = \frac{1}{2} A_{i-1,j-1} + \frac{1}{4}(A_{i-1,j} + A_{i,j-1}),$$

and $J_{\bar{i},\bar{j},III}$ is the given value for the average current density crossing cell$_{\bar{i},\bar{j},III}$. The area of the cell is

$$\text{area}_{\bar{i},\bar{j},III} = 1/2\ h_{\bar{i}}\ k_{\bar{j}}.$$

The choice for $A_{\bar{i},\bar{j},III}$ is made so that the formulas for the triangular and rectangular regions are consistent. The formula for $|\nabla A|^2$ for the triangle is in general only first order, however, since it is determined by three, rather than four, values of A.

The difference equations corresponding to Eq. (1) are then obtained by requiring that the partial derivative of I with respect to each of the unknown values of A_{ij} in Eq. (7) be zero. The resulting equation obtained for a general interior mesh point surrounded by four rectangles is

$$f_{ij} = \gamma_{\bar{i}\,\bar{j}} \left(\delta_{\bar{i}\,j}\, k_{\bar{j}} + \eta_{i\bar{j}}\, h_{\bar{i}} \right) + \gamma_{\bar{i}+1,\,\bar{j}} \left(-\delta_{\bar{i}+1,\,j}\, k_{\bar{j}} + \eta_{i,\,\bar{j}}\, h_{\bar{i}+1} \right)$$

$$+ \gamma_{\bar{i},\,\bar{j}+1} \left(\delta_{\bar{i},\,j}\, k_{\bar{j}+1} - \eta_{i,\,\bar{j}+1}\, h_{\bar{i}} \right) + \gamma_{\bar{i}+1,\,\bar{j}+1} \left(-\delta_{\bar{i}+1,\,j}\, k_{\bar{j}+1} - \eta_{i,\,\bar{j}+1}\, h_{\bar{i}+1} \right)$$

$$- 2\pi \left(J_{\bar{i},\,\bar{j}}\, h_{\bar{i}}\, k_{\bar{j}} + J_{\bar{i}+1,\,\bar{j}}\, h_{\bar{i}+1}\, k_{\bar{j}} + J_{\bar{i},\,\bar{j}+1}\, h_{\bar{i}}\, k_{\bar{j}+1} + J_{\bar{i}+1,\,\bar{j}+1}\, h_{\bar{i}+1}\, k_{\bar{j}+1} \right) = 0,$$

(14)

where Eq. (3) was used to substitute for $dg/d(|\nabla A|^2)$. Here $\gamma_{\bar{i}\,\bar{j}}$ denotes the reluctivity evaluated for cell $\bar{i}\,\bar{j}$ with the use of Eq. (10). Equation (14) is, in general, a nonlinear one relating each A_{ij} to its eight neighbors, $A_{i\pm 1,\,j\pm 1}$, $A_{i,\,j\pm 1}$, and $A_{i\pm 1,\,j}$. When γ is a constant, the equation reduces to that derived from the usual five-point difference approximation to the Poisson equation.

For points along an interface or boundary bordered by triangular regions, the resulting equations are slightly more complex. For example, the equation for A_{ij} in Fig. 3 is

$$f_{ij} = \gamma_{\bar{i}\,\bar{j}} \left(\delta_{\bar{i}\,j}\, k_{\bar{j}} + \eta_{i\bar{j}}\, h_{\bar{i}} \right) + \gamma_{\bar{i}+1,\,\bar{j},\,\text{III}}\, \eta_{i\bar{j}}\, h_{\bar{i}+1} - \gamma_{\bar{i}+1,\,\bar{j},\,\text{I}}\, \delta_{\bar{i}+1,\,j}\, k_{\bar{j}}$$

$$+ \gamma_{\bar{i},\,\bar{j}+1,\,\text{III}}\, \delta_{\bar{i},\,j}\, k_{\bar{j}+1} - \gamma_{\bar{i},\,\bar{j}+1,\,\text{I}}\, \eta_{i,\,\bar{j}+1}\, h_{\bar{i}} + \gamma_{\bar{i}+1,\,\bar{j}+1} \left(-\delta_{\bar{i}+1,\,j}\, k_{\bar{j}+1} - \eta_{i,\,\bar{j}+1}\, h_{\bar{i}+1} \right)$$

$$- 2\pi \left(J_{\bar{i},\,\bar{j}}\, h_{\bar{i}}\, k_{\bar{j}} + 1/2 \left[J_{\bar{i}+1,\,\bar{j},\,\text{III}} + J_{\bar{i}+1,\,\bar{j},\,\text{I}} \right] h_{\bar{i}+1}\, k_{\bar{j}} + 1/2 \left[J_{\bar{i},\,\bar{j}+1,\,\text{III}} + J_{\bar{i},\,\bar{j}+1,\,\text{I}} \right] \right.$$

(15) $$\left. \times h_{\bar{i}}\, k_{\bar{j}+1} + J_{\bar{i}+1,\,\bar{j}+1}\, h_{\bar{i}+1}\, k_{\bar{j}+1} \right) = 0.$$

This equation also relates A_{ij} to its eight neighbors, and reduces to the usual five-point formula when γ is a constant.

For boundary points along which $\partial A/\partial n = 0$, the obtained finite difference equations automatically correspond to this boundary condition, because it is the natural one for the variational problem. For boundary points along which $A = A_0$, a constant, an additional nonlinear finite

difference equation may be obtained for A_0 by considering it to be unknown. This equation would correspond to the application of Ampere's law to the entire region R, and it may be used during the iterative solution of the problem to improve convergence by adjusting A_0 to correspond to the current approximate solution for A [8].

Because Eq. (10) or Eq. (12) was used to approximate $\left| \underset{\sim}{\nabla} A \right|^2$, the same finite difference equations as those derived above could have been derived by using the line-integral equivalent to Eq. (1) obtained by the application of Green's theorem (in this case, Ampere's law) to each auxiliary mesh region, and approximating the normal derivatives by central differences [7, Sec. 6. 4]. This method would be equivalent to the variational one used above, and in some cases may be algebraically more convenient. The two main features to note here are that γ is a function of the unknown A values, making Eq. (14) and alterations such as Eq. (15) nonlinear in general, and that the use of Eq. (10) or (12) yields difference equations that have a symmetric Jacobian. Although in general, the Jacobian is not diagonally dominant, its positive-definiteness follows from Schechter's arguments [4, Sec. 9] when they are applied to the differencing scheme used here.

4. **Solution of difference equations.** The task of solving the simultaneous nonlinear difference equations--Eq. (14) for general interior points and possible alterations such as Eq. (15) for points near interfaces and boundaries--is approached by the commonly used small-magnetic-field method by taking γ to be a known function at each iteration, as calculated from A at previous iterations,

$$(16) \qquad \gamma_{i,j}^{n} = \gamma_{i,j}^{n-1} - \omega_1 \left[\gamma_{i,j}^{n-1} - \gamma \left(\left| \underset{\sim}{\nabla} A^n \right|_{i,j}^2 \right) \right].$$

This method, which is the discrete analogue to Eq. (4), then solves (or approximately solves) the resulting set of linear equations to obtain the next approximation for A. The comparison methods of interest here are based essentially on Newton's method, the discrete analogue to Eq. (5), which linearizes the equations taking into account the dependence of γ on the unknown A values.

Newton's method for solving $f_{ij} = 0$ gives A_{ij}^{k+1}, the $\underline{(k+1)}$th approximation to A_{ij}, as

$$A_{ij}^{k+1} = A_{ij}^{k} + \epsilon_{ij}^{k} ,$$

where the ϵ_{ij}^{k} satisfy the set of linear equations

(17) $$\sum_{\ell, m} (\partial f_{ij}/\partial A_{\ell m})^{k} \epsilon_{\ell m}^{k} = -f_{ij}^{k} .$$

Equation (17) is a nine-point difference approximation to Eq. (5), having a coefficient matrix that is positive-definite, symmetric, and block tridiagonal, each block of which is itself tridiagonal. It need be solved only approximately at each step before computing the next Newton's iterate.

The computational scheme of special interest is that of nonlinear successive overrelaxation, which is [2, 3, 4, 5]

(18) $$A_{ij}^{k+1} = A_{ij}^{k} - \omega \frac{f_{ij}[A_{11}^{k+1}, \cdots, A_{i-1,j}^{k+1}, A_{ij}^{k}, \cdots, A_{n,n-1}^{k}]}{\partial f_{ij}/\partial A_{ij} [A_{11}^{k+1}, \cdots, A_{i-1,j}^{k+1}, A_{ij}^{k}, \cdots, A_{n,n-1}^{k}]} ,$$

and is equivalent to performing one sweep with successive point overrelaxation on Eq. (17) when the second derivatives of f_{ij} do not vary much. Notice that only the diagonal coefficients of Eq. (17) need be computed with this scheme, but that these and f must be updated each time a new A value

is calculated. The method is equivalent to performing one Newton's iteration on each equation $f_{ij} = 0$, successively, considering A_{ij} to be the only unknown and using the latest available values for the other values of A. Kronrod has suggested a variation of the method, in which one Steffensen's iteration rather than one Newton's iteration is performed on each equation [9]; his method may be a useful substitute when $\partial f_{ij}/\partial A_{ij}$ cannot be easily calculated.

The explicit expression for $\partial f_{ij}/\partial A_{ij}$ for an interior mesh point surrounded by four rectangles is obtained by differentiating Eq. (14), and is

$$
\frac{\partial f_{ij}}{\partial A_{ij}} = \left(\frac{k_{\bar{j}}}{h_{\bar{i}}} + \frac{h_{\bar{i}}}{k_{\bar{j}}} \right) \gamma_{\bar{i}\,\bar{j}} + \left(\frac{k_{\bar{j}}}{h_{\overline{i+1}}} + \frac{h_{\overline{i+1}}}{k_{\bar{j}}} \right) \gamma_{\overline{i+1},\,\bar{j}} + \left(\frac{k_{\overline{j+1}}}{h_{\bar{i}}} + \frac{h_{\bar{i}}}{k_{\overline{j+1}}} \right) \gamma_{\bar{i},\,\overline{j+1}}
$$

$$
+ \left(\frac{k_{\overline{j+1}}}{h_{\overline{i+1}}} + \frac{h_{\overline{i+1}}}{k_{\overline{j+1}}} \right) \gamma_{\overline{i+1},\,\overline{j+1}} + \frac{\gamma'_{\bar{i}\,\bar{j}}}{h_{\bar{i}}k_{\bar{j}}} \, (\delta_{\bar{i}j}\, k_{\bar{j}} + \eta_{ij}\, h_{\bar{i}})^2
$$

$$
+ \frac{\gamma'_{\overline{i+1},\,\bar{j}}}{h_{\overline{i+1},}\,k_{\bar{j}}} \, (-\delta_{\overline{i+1},\,j}\, k_{\bar{j}} + \eta_{ij}\, h_{\overline{i+1}})^2 + \frac{\gamma'_{\bar{i},\,\overline{j+1}}}{h_{\bar{i}}\, k_{\overline{j+1}}} \, (\delta_{\bar{i}j}\, k_{\overline{j+1}} - \eta_{i,\,\overline{j+1}}\, h_{\bar{i}})^2
$$

(19) $\qquad + \dfrac{\gamma'_{\overline{i+1},\,\overline{j+1}}}{h_{\overline{i+1}}\, k_{\overline{j+1}}} \, (\delta_{\overline{i+1},\,j}\, k_{\overline{j+1}} + \eta_{i,\,\overline{j+1}}\, h_{\overline{i+1}})^2\ ,$

where $\gamma'_{\bar{i}\,\bar{j}}$ denotes the derivative of γ with respect to $|\nabla A|^2$ evaluated for cell $_{\bar{i}\,\bar{j}}$. Corresponding expressions for points along an interface or boundary bordered by triangular regions are of the same form but, in general, may contain fewer or more terms.

5. Comparison and results. The above methods were compared for the solution of a test problem having some of the essential features encountered in the ferromagnetic region R_1 of an actual accelerator-design problem.

Let the square region $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ be entirely occupied by a ferro-

magnetic substance with reluctivity $\gamma(w) = (10^{-4} + w)/(1 + w)$, where

$w = |\nabla A|^2$ (Fig. 4). Let the current density J be identically zero and the

boundary conditions on A be that $A = 0$ for $x = 0$ and $y = 1$; $A = 0.05 \sin(\pi x/2)$

for $y = 0$; and $(\partial A/\partial x) = 0$ for $x = 1$. The number 0.05 is chosen so that

w is of the order of 10^{-2}, so that γ, in turn, varies significantly over

the rectangle, and the ratio of the second to the first term on the left of

Eq. (5) is maximized. Finally, let the region be covered with a uniform

square mesh so that $h = k = 1/n$.

The results of the various numerical methods for solving this test

problem are shown in Table I for the case where the initial approximation

to A was the solution to the linear problem (constant γ),

$$A = \frac{0.05 \sin(\pi x/2) \sinh[\pi(1-y)/2]}{\sinh(\pi/2)} .$$

The calculations were performed on the IBM 7094 by means of a FORTRAN IV

program. Two meshes were considered: one containing 90 unknown points

($n = 10$) and the other containing 870 unknown points ($n = 30$). The convergence

criterion in the former case was that the sum of the squares of the residuals

be less than 10^{-13}, and in the latter that it be less than 10^{-12}. These corre-

spond to an average residual at each point of the order of 10^{-6} of the maximum

value of A. The iterations were ordered by letting i increase through all

its values for each successively larger value of j. For each method, an

optimal value of the relaxation factor was found for fastest convergence.

Two columns of Table I are for Newton's method, Eq. (17). Succes-

sive-point overrelaxation was used in the first, and successive-block over-

relaxation in the second, with the relaxation factor ω. It was found that

taking more than one overrelaxation sweep per iteration did not improve the convergence, signifying it was not worth solving each Newton's iteration too well.

Another column is for the nonlinear successive-overrelaxation method, Eq. (18). The time per iteration is greater than for Newton's method, because one must compute new values of $\left|\nabla A\right|^2$ for each point. The iteration behaved quite stably with respect to changes in ω and in the initial approximation for A. It was found also that the symmetric difference formula for $\left|\nabla A\right|^2_{ij}$, Eq. (10), gave better results than the lower-order, one-sided formulas suggested elsewhere [3,4,5].

The last column is for the small-magnetic-field approximation, Eqs. (4) and (16), for which the difference equations can be obtained by setting $\gamma' = 0$ in $\partial f_{ij}/\partial A_{ij}$ in Eq. (18). It was found that in order for the process to converge, one must underrelax the new values of γ by choosing ω_1, the relaxation parameter in Eq. (16), less than one, and that it is sufficient to perform one Gauss-Seidel iteration (i. e., $\omega = 1$) on the resulting linear equations. The method did not behave as stably as the other methods with respect to changes in the initial approximation or in the relaxation parameter.

Examination of the results shows that the methods based on the non-linear difference equations performed better than the small-magnetic-field approximation, especially for the problem with the larger number of mesh points. Among the former methods, nonlinear successive overrelaxation performed the best. Although it required more calculations per iteration because new values of γ and γ' were calculated each time a new value of A was, the number of iterations required for convergence was significantly smaller than that required by Newton's method, and the total computer time

used was less. From these results, and from the fact that the method is an easy one to program, nonlinear successive overrelaxation shows itself to be a promising tool for solving this type of elliptic equation.

FOOTNOTE AND REFERENCES

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Table I. Comparison of numerical methods.

		Newton		Nonlinear successive overrelaxation	Small-field approximation
		Point	Line		
90 points Σ Res.$^2 < 10^{-13}$	Minutes/iteration	0.0016	0.0022	0.0020	0.0008
	$\omega_{opt.}$	1.50	1.50	1.54	0.45 (ω_1)
	Iterations to converge	25	20	18	52
	Minutes to converge	0.040	0.044	0.036	0.042
870 points Σ Res.$^2 < 10^{-12}$	Minutes/iteration	0.0198	0.0218	0.0229	0.0076
	$\omega_{opt.}$	1.80	1.75	1.82	0.50 (ω_1)
	Iterations to converge	97	101	58	> 400
	Minutes to converge	1.92	2.20	1.33	> 3

FIGURE CAPTIONS

Fig. 1.   General rectangular cell.

Fig. 2.   Triangular cell.

Fig. 3.   Interface bordered with triangular cells.

Fig. 4.   Test problem.
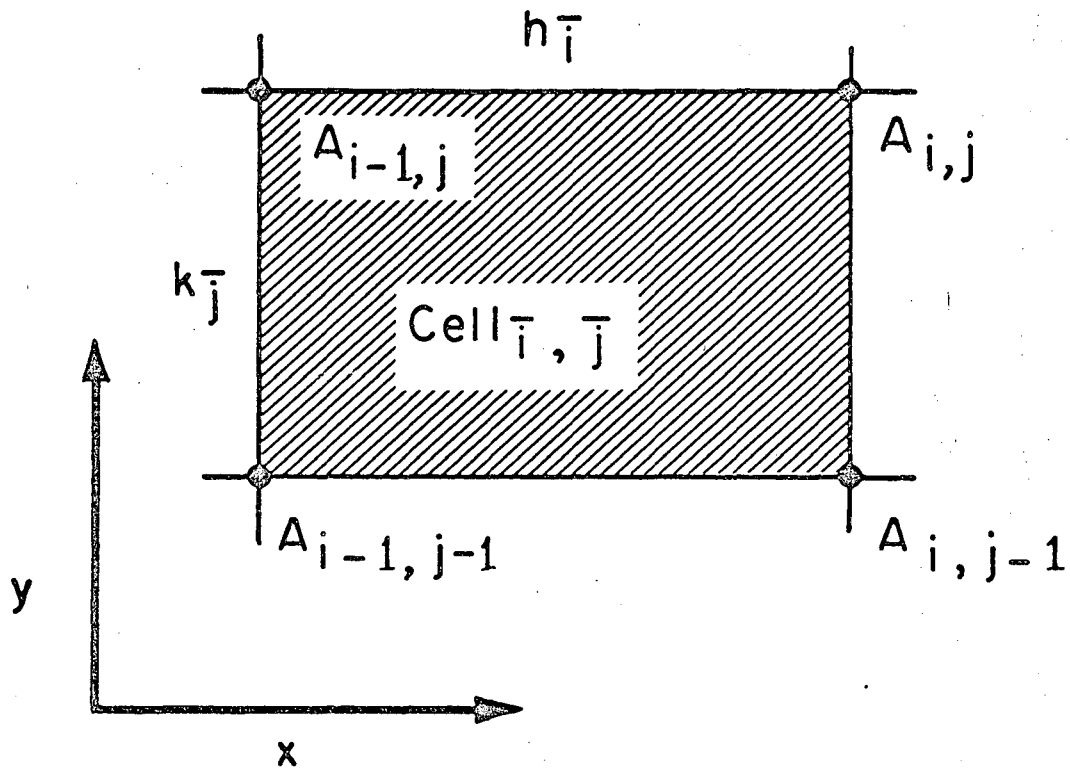
MUB-8847

Fig. 1
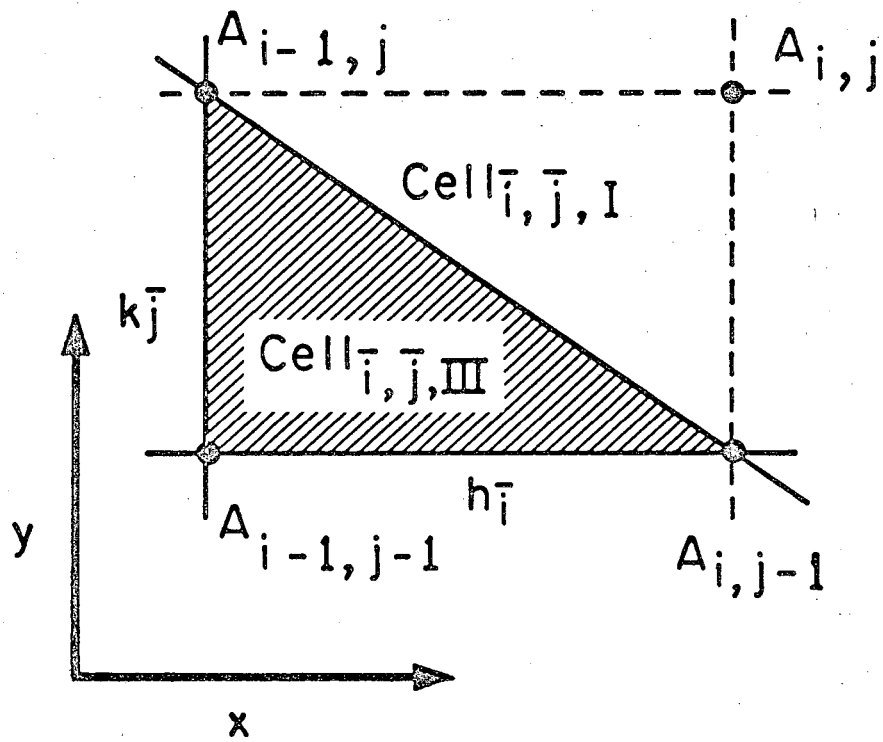
MUB-8848

Fig. 2

Fig. 3

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$A = 0$ $(1,1)$

$A = 0$

$$\mu = \frac{1 + |\nabla A|^2}{10^{-4} + |\nabla A|^2}$$

$$\frac{\partial A}{\partial x} = 0$$

y

$(0,0)$ x

$A = 0.05 \sin(\pi x/2)$

Fig. 4   MUB-7273