WATER RESOURCES RESEARCH, VOL. 21, NO. 11, PAGES 1575–1584, NOVEMBER 1985

An Approach to Parameter Estimation and Stochastic Control in Water Resources With an Application to Reservoir Operation

HUGO A. LOAICIGA AND MIGUEL A. MARIÑO

Department of Land, Air, and Water Resources and Department of Civil Engineering, University of California, Davis

This paper presents an algorithm for the estimation of parameters in state-space models that represent hydrologic processes in which the state variables are observed with error. The algorithm is based on the maximization of the conditional expectation of the likelihood function of the state equation. The estimation algorithm is numerically stable and guarantees local convergence under mild conditions. It is also shown that the estimation algorithm can be coupled with an optimal control method to yield a combined control estimation technique that can be easily implemented. An application of the theory and methods developed herein is given for flood routing via reservoir operation.

INTRODUCTION

The applicability of state-space models to describe hydrologic processes is well documented (see, for example, *O'Connell* [1977], *Chiu* [1978], and *Kitanidis and Bras* [1980]). In general, state-space models consist of a linear or nonlinear state equation, representing the time evolution of the process in question, and an observation equation, representing the observed subset of state variables through a period of time. The need for estimating unknown parameters in state-space models has led to a variety of schemes to represent the structural relationship between state variables and parameters (see, for example, *Otter* [1978]). The versatility of state-space models to represent the structural relationship between state and observation variables and parameters has led to its popularity in various fields of science where state and parameter estimation is of interest.

The development of adequate algorithms to solve for parameters present in state-space models is an area of continued interest for the following reasons: (1) physical models differ from each other in the structural relationship between state, observations, and parameters, and algorithms that may be satisfactory for some problems may break down in others; (2) the structure of stochastic inputs (e.g., noise terms) in both the state and observation equations may differ from problem to problem, and such structure demands a specialized treatment in each case; and (3) there is continued improvement in numerical techniques and in the software and hardware available.

This paper presents an approach for state and parameter estimation in state-space models of hydrologic processes. The innovative features of the theory and solution method proposed herein are as follows: (1) transition matrices in the state equation as well as the covariances of noise terms in the state and observation equations are simultaneously estimated via maximum likelihood; (2) filtered and smoothed state estimators are obtained as part of the results computed from the estimation algorithm; (3) the initial conditions, i.e., the initial state and state covariance, can also be estimated based on the data set for the period of interest; (4) under very mild statistical conditions, parameter estimates converge to a local optimum of the conditional likelihood function (see below) in a finite number of iterations; (5) the nonlinear estimation problem is converted into the sequential solution of a series of analytical expressions which are suitable for implementation in a digital computer; (6) decision or control variables can be included in the state equation and the estimation technique can be coupled with a management model yielding a combined control-estimation technique; and (7) stochastic inputs in the state equation are modeled as an autocorrelated process, whose parameters are estimated.

In the remainder of this paper the problem statement and the estimation approach are presented first. The extension to a combined control-estimation technique follows afterwards, and an application to an actual water resource system illustrates the implementation of the methodology developed in this study.

Paper number 5W4016.
0043-1397/85/005W-4016$05.00

PROBLEM STATEMENT

The basic state-space model studied in this paper is given by

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t + \mathbf{y}_t + \mathbf{w}_t \tag{1}$$

$$\mathbf{w}_t = \phi\mathbf{w}_{t-1} + \mathbf{e}_t \tag{2}$$

$$\mathbf{z}_t = M\mathbf{x}_t + \mathbf{v}_t \tag{3}$$

in which $\mathbf{x}_t$ is an n-dimensional vector of state variables at time t; $\mathbf{u}_t$ is a p-dimensional vector of decision variables; $\mathbf{y}_t$ is an n-dimensional vector of known quantities (called the exogenous variables henceforth); $\mathbf{w}_t$ is an n-dimensional vector of stochastic inputs; A is an unknown $n \times n$ transition matrix to be estimated; and B is a known $n \times p$ matrix determined by the physical process being represented by the state equation (1), as is explained below. Equation (2) specifies the time structure of the $(n \times 1)$ stochastic, uncontrollable, inputs $\mathbf{w}_t$ as being a first-order autoregressive process (AR(1)). The AR(1) representation is more general than it would appear at first glance, because, by state augmentation, higher-order processes can be reduced to the AR(1) model (see, for example, *Anderson* [1978]). Notice that the AR formulation of the stochastic input $\mathbf{w}_t$ nests as a subcase the simpler white-noise specification which is usually adopted in state-space models. The $n \times n$ matrix ϕ in (2) must be estimated; the error term $\mathbf{e}_t$ in (2) is an unmodeled $n \times 1$ white-noise sequence, with unknown covariance Q to be estimated. In (3), $\mathbf{z}_t$ is an $m \times 1$ $(\leq n)$ vector of observations of $\mathbf{x}_t$ defined by the known $m \times n$ matrix M, which defines the linear combination of state variables being observed, and $\mathbf{v}_t$ is a $n \times 1$ white-noise sequence with unknown covariance R to be estimated, and accounts for

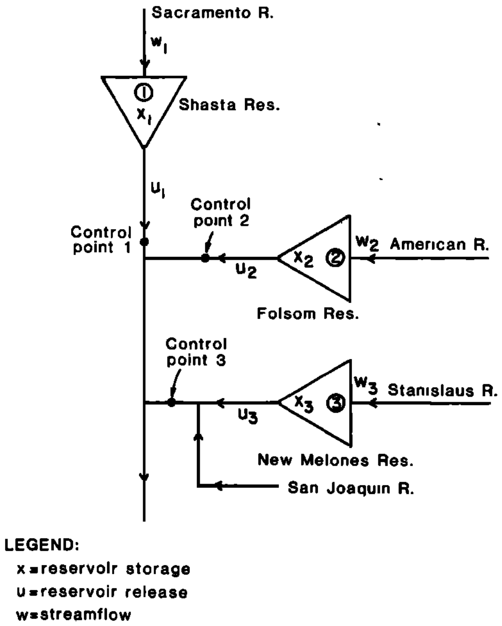

Fig. 1. Flood control reservoir system.

errors in measurement of the pertinent subset of observed state variables (i.e., $M\mathbf{x}_t$). It is assumed the $\mathbf{e}_t$ and $\mathbf{v}_t$ are independent of each other and of the processes $\mathbf{w}_t$ and $\mathbf{x}_t$ also. The full specification of the state-space model requires to specify the initial conditions, i.e.,

$$E(\mathbf{x}_1) = \boldsymbol{\mu}_1 \tag{4}$$

$$\mathrm{Cov}\,(\mathbf{x}_1) = \Sigma_1 \tag{5}$$

where the expected value $\boldsymbol{\mu}_1$ and covariance Σ_1 of the initial state $\mathbf{x}_1$ are unknown quantities to be estimated (*Shumway et al.* [1981] and *Restrepo-Posada and Bras* [1982] presented different initial condition estimators).

In total, the unknown parameter set θ is given by

$$\theta = (\boldsymbol{\mu}_1, \Sigma_1, Q, R, A, \phi) \tag{6}$$

The data set available up to time t to estimate θ consists of the observations $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$. The decision and exogenous sequences, $\{\mathbf{u}_l\}_{l=1}^{l=t-1}$ and $\{\mathbf{y}_l\}_{l=1}^{l=t-1}$, respectively, are known. It is necessary to differentiate between $\mathbf{u}_t$ and $\mathbf{y}_t$ even though both are deterministic exogenous inputs, because when estimation is combined with control, $\mathbf{u}_t$ must be optimally and independently computed.

The applicability of (1)–(5) in reservoir operation management is illustrated next with a three-reservoir system (Figure 1). Clearly, the state equation is given by

$$\mathbf{x}_{t+1} = I\mathbf{x}_t - I\mathbf{u}_t + \mathbf{0} + \mathbf{w}_t \tag{7}$$

Notice that $A = I$ and $B = -I$, where I is a 3×3 identity matrix. Vector $\mathbf{u}_t$ contains the decision variables which are in this case reservoir releases, and $\mathbf{x}_t$ denotes reservoir storages. The stochastic inputs $\mathbf{w}_t$ are the river streamflows into the reservoirs, which follow the model given by (2). Clearly, $\mathbf{y}_t = \mathbf{0}$ in this case. The observation equation becomes

$$\mathbf{z}_t = I\mathbf{x}_t + \mathbf{v}_t \tag{8}$$

i.e., $M = I$. One can justify the presence of (8) because measurements of reservoir levels may have some degree of error; if the measurements are for all practical purposes exact, then the estimation method yields as an estimate for the covariances R and Σ_1 null or approximately null $n \times n$ matrices. Thus error-free or close to error-free observations are automatically handled or nested as a subcase of the general specification (1)–(5). The parameter set to be estimated becomes $\theta = (\boldsymbol{\mu}_1, \Sigma_1, Q, R, \phi)$. The system of Figure 1 is used as an application example subsequently.

Another example of a linear state-space model is the governing equation for groundwater flow in a confined aquifer. From standard work in groundwater modeling (see, for example, *Pinder and Frind* [1972] and *Narasimhan et al.* [1978]) it can be shown that the discrete form of the governing flow equation is given by (1), to which an observation equation (3) is appended. In this case, the state variables $\mathbf{x}_t$ customarily represent piezometric heads; the decision variables $\mathbf{u}_t$ are pumping and/or recharging rates; $\mathbf{y}_t$ represents across-boundary fluxes and other sinks and sources; $\mathbf{w}_t$, the stochastic term, is added a posteriori to account for errors in model specification. The observation equation (3) is justified by the errors present in measuring a subset of the nodal piezometric heads. The matrices A and B are functions of storativities and transmissivities as well as other parameters of the numerical scheme chosen to discretize the differential equation of flow.

It is useful in later developments to write (1)–(3) in an equivalent augmented form as follows:

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{w}_{t+1} \end{bmatrix} = \begin{bmatrix} A & I \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{w}_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{u}_t + \begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{y}_t + \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_{t+1} \end{bmatrix} \tag{9}$$

$$\mathbf{z}_{t+1} = (M \quad 0) \begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{w}_{t+1} \end{bmatrix} + \mathbf{v}_{t+1} \tag{10}$$

in which 0 is an $n \times n$ null matrix and $\mathbf{0}$ is an $n \times 1$ null vector. In compact notation, in terms of the augmented state $\mathbf{x}_t^{*\prime} = (\mathbf{x}_t', \mathbf{w}_t')$, (9) and (10) become

$$\mathbf{x}_{t+1}^* = A^*\mathbf{x}_t^* + B^*\mathbf{u}_t + \mathbf{y}_t^* + \mathbf{e}_{t+1}^* \tag{11}$$

$$\mathbf{z}_{t+1} = M^*\mathbf{x}_{t+1}^* + \mathbf{v}_{t+1} \tag{12}$$

The advantage of using (11) and (12) stems from the fact that (11) is in first difference form and standard filtering [e.g., *Jazwinski*, 1970] and smoothing [e.g., *Rauch et al.*, 1965] estimators can be applied to (11) subject to (12), as is explained later. Also, (11) and (12) are very suitable when combining estimation and control.

Parameter Estimation in the Linear State-Space Model

The first step in developing parameter estimates for the state-space model defined by (1)–(5) is to write the joint log-likelihood function of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_t$ and $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$ as follows:

$$\begin{aligned} \log L = &-\tfrac{1}{2}\log|\Sigma_1| - \tfrac{1}{2}(\mathbf{x}_1 - \boldsymbol{\mu}_1)'\Sigma_1^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1) \\ &-\frac{(t-1)}{2}\log|Q| \\ &-\frac{1}{2}\sum_{l=2}^{t}[(\mathbf{x}_l - A\mathbf{x}_{l-1} - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1} - \phi\mathbf{w}_{l-2})'Q^{-1} \\ &\cdot(\mathbf{x}_l - A\mathbf{x}_{l-1} - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1} - \phi\mathbf{w}_{l-2}) \\ &-\left(\frac{t-1}{2}\right)\log|R| - \frac{1}{2}\sum_{l=2}^{t}[(\mathbf{z}_l - M\mathbf{x}_l)'R^{-1}(\mathbf{z}_l - M\mathbf{x}_l)] \end{aligned} \tag{13}$$

The joint log-likelihood function in (13) implies that the noises $\mathbf{e}_t$ and $\mathbf{v}_t$ in (2) and (3) are normally distributed. The normality assumption on $\mathbf{v}_t$ is quite appropriate because errors in measurements have instrumental, human, and other sources which can be considered to be independent of each other. By virtue of the central limit theorem (see, for example, *Rao* [1965]), the combination of such different sources results in an approximately normal random variable. The central limit theorem can also be invoked to justify the normality of the error term $\mathbf{e}_t$.

Since the log-likelihood function in (13) depends on the unobserved state variables $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_t$, one must take the expectation of (13) with respect to the observed series $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$ and maximize the resulting expression with respect to the parameters $\boldsymbol{\mu}_1$, Σ_1, Q, R, A, and ϕ. This approach of maximizing the conditional expectation of the log-likelihood function to estimate the parameters in a linear state-space model constitutes a generalization of the so-called expectation maximization (EM) algorithm [*Hartley*, 1958; *Dempster et al.*, 1978; *Shumway and Stoffer*, 1982]. *Wu* [1983] showed that estimation techniques based on the EM algorithm converge to a local maximum of the log-likelihood function in a finite number of iterations, provided that local convexity of the (negative) log-likelihood function holds. An application of the EM algorithm to parameter estimation of mixture distributions, as well as a discussion of its features can be found in the work by *Leytham* [1984].

In order to implement the estimation algorithm, assume that at the (r + 1)st iteration one has available $\boldsymbol{\mu}_1(r)$, $\Sigma_1(r)$, $Q(r)$, $R(r)$, $A(r)$, and $\phi(r)$, which denote the rth iteration value of the parameter set. Let

$$G(\boldsymbol{\mu}_1, \Sigma_1, Q, R, A, \phi) = E_r\,(\log L|\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t) \qquad (14)$$

where E_r denotes the conditional expectation of (13) with respect to the observations $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$ of a distribution containing the rth iterate values $\boldsymbol{\mu}_1(r)$, $\Sigma_1(r)$, $Q(r)$, $R(r)$, $A(r)$, and $\phi(r)$. The idea is to compute the conditional expectation on the right-hand side of (14), maximize the resulting expression with respect to $\boldsymbol{\mu}_1$, Σ_1, Q, R, A, and ϕ, to obtain their $(r + 1)$st iterate values. This two-step, expectation first-maximization second is repeated until convergence is achieved. The conditional expectation of the log-likelihood function with respect to the observations $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$ yields

$$\begin{aligned} G(\boldsymbol{\mu}_1, \Sigma_1, Q, R, A, \phi) &= -\tfrac{1}{2}\log|\Sigma_1| - \tfrac{1}{2}\operatorname{tr}\{\Sigma_1^{-1}[P_1{}^t + (\mathbf{x}_1{}^t - \boldsymbol{\mu}_1)(\mathbf{x}_1{}^t - \boldsymbol{\mu}_1)']\} \\ &\quad - \frac{(t-1)}{2}\log|Q| - \tfrac{1}{2}\operatorname{tr}\{Q^{-1}(A_1 - A_2A' - AA_2' + AA_3A')\} \\ &\quad - \frac{(t-1)}{2}\log|R| - \tfrac{1}{2}\operatorname{tr}\left\{R^{-1}\sum_{l=2}^{t}[(\mathbf{z}_l - M\mathbf{x}_l{}^t)\right. \\ &\quad \left.\cdot(\mathbf{z}_l - M\mathbf{x}_l{}^t)' + MP_l{}^tM']\right\} \end{aligned} \qquad (15)$$

in which tr denotes the trace of a matrix and

$$A_1 = \sum_{l=2}^{t}[P_l{}^t + (\mathbf{x}_l{}^t - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1} - \phi\mathbf{w}_{l-2})(\mathbf{x}_l{}^t - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1} - \phi\mathbf{w}_{l-2})'] \qquad (16)$$

$$A_2 = \sum_{l=2}^{t}[P_{l,l-1}{}^t + (\mathbf{x}_l{}^t - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1} - \phi\mathbf{w}_{l-2})\mathbf{x}_{l-1}{}^{t\prime}] \qquad (17)$$

$$A_3 = \sum_{l=2}^{t}[P_{l-1}{}^t + \mathbf{x}_{l-1}{}^t\mathbf{x}_{l-1}{}^{t\prime}] \qquad (18)$$

In (15)–(18), the following notation was introduced:

$$\mathbf{x}_l{}^k = E(\mathbf{x}_l|\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_k) \qquad (19)$$

$$P_l{}^k = \operatorname{Cov}(\mathbf{x}_l|\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_k) \qquad (20)$$

$$P_{l,l-1}{}^k = \operatorname{Cov}(\mathbf{x}_l, \mathbf{x}_{l-1}|\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_k) \qquad (21)$$

When $k = t$ in (19), $\mathbf{x}_l{}^k = \mathbf{x}_l{}^t$ is the smoothed estimator of $\mathbf{x}_l$, $l = 1, 2, \ldots, t-1$, which has covariance $P_l{}^t$; for $k = l$, $\mathbf{x}_l{}^k = \mathbf{x}_l{}^l$ is the Kalman filter estimator of $\mathbf{x}_l$, which has covariance $P_l{}^l$. When $k = t$, (20) represents the smoothed state estimate covariance matrix. When $k = l$, (20) represents the filtered state estimate covariance matrix. Equation (21) denotes the lagged covariances, of which, of interest for use in the estimation algorithm is the case when $k = t$ (and similarly for equations (19) and (20)). Due to the structure of (1) and (2), filter and smoothed state estimators and covariances must be obtained by using the augmented state and observation equations (11) and (12). Clearly, one is only interested in the subvectors and submatrices that correspond to the true state vector $\mathbf{x}_t$ (see the appendix). The recursions for the Kalman filter and smoothed state estimators as well as for the lagged covariances are presented in the appendix. It should be noted that the conditioning of the log-likelihood function on the observation data $\mathbf{z}_2, \mathbf{z}_3, \cdots, \mathbf{z}_t$ results in parameter estimators that are a function of the smoothed state estimators.

Differentiation of (15) with respect to $\boldsymbol{\mu}_1$, Σ_1, Q, R, A, and ϕ yields the following analytical expressions for their respective estimators, in the $(r + 1)$st iteration,

$$\boldsymbol{\mu}_1(r+1) = \mathbf{x}_1{}^t \qquad (22)$$

$$\Sigma_1(r+1) = P_1{}^t \qquad (23)$$

$$Q(r+1) = \left(\frac{1}{t-1}\right)(A_1 - A_2A_3{}^{-1}A_2') \qquad (24)$$

$$R(r+1) = \left(\frac{1}{t-1}\right)\sum_{l=2}^{t}[(\mathbf{z}_l - M\mathbf{x}_l{}^t)(\mathbf{z}_l - M\mathbf{x}_l{}^t)' + MP_l{}^tM'] \qquad (25)$$

$$A(r+1) = A_2A_3{}^{-1} \qquad (26)$$

$$\phi(r+1) = B_2B_3{}^{-1} \qquad (27)$$

in which

$$B_2 = \sum_{l=2}^{t}(\mathbf{x}_l{}^t - A\mathbf{x}_{l-1}{}^t - B\mathbf{u}_{l-1} - \mathbf{y}_{l-1})\mathbf{w}_{l-2}' \qquad (28)$$

$$B_3 = \sum_{l=2}^{t}\mathbf{w}_{l-2}\mathbf{w}_{l-2}' \qquad (29)$$

Notice that (22)–(29) require only basic vector-matrix algebraic operations (addition and multiplication) as well as a numerically stable subroutine for matrix inversion [*Stewart*, 1973]. The simple structure of the basic equations needed to estimate the state-space parameters (i.e., equations (22)–(29)) is one of the attractive features of the EM algorithm relative to other methods based on nonlinear equation solvers [e.g., *Mehra*, 1970; *Gupta and Mehra*, 1974].

Other maximum likelihood approaches [e.g., *Kashyap*, 1970; *Gupta and Mehra*, 1974; *Anderson*, 1977] utilize nonlinear solution techniques (e.g., method of scoring, which requires the inverse of the matrix of second partial derivatives, Newton-Raphson or constrained optimization algorithms). The computation of the expected value of the matrix of second derivatives (the so-called information matrix) and its inverse usually leads to computational difficulties. The convergence of the Newton-Raphson is highly dependent on the choice of initial estimators. On the other hand, constrained optimization methods require substantial knowledge of mathematical programming and their implementation may require a good deal of expertise in numerical analysis and computer programming, which is not necessary in the implementation of the EM algorithm. For the case of the normal distribution, *Wu* [1983] has shown that the log-likelihood function increases at every iteration monotonically, and such convergence to a local optimum is not affected by the choice of initial estimator. Although the computation of the information matrix in the method of scoring provides an estimate of the standard error of estimators (see, for example, *Rao* [1965, pp. 302–309]), this can be also done numerically by perturbing the log-likelihood function about the computed solution.

The estimation algorithm can be summarized as follows.

1. For the first iteration, guess values $\mathbf{\mu}_1(1)$, $\Sigma_1(1)$, $Q(1)$, $R(1)$, $A(1)$, and $\phi(1)$ (see application below).
2. Compute $\mathbf{x}_l^t$, P_l^t, and $P_{l,l-1}{}^t$ (see the appendix).
3. At the rth iteration, based on the estimates $\mathbf{\mu}_1(r)$, $\Sigma_1(r)$, $Q(r)$, $R(r)$, $A(r)$, and $\phi(r)$, compute the estimators (22)–(27) for the $(r+1)$st iteration. Go to step 2.
4. Repeat steps 2 and 3 until a user-specified convergence criterion is satisfied (the convergence test suggested by *Gill et al.* [1981, p. 306] is adopted in this study).

Notice that the estimation algorithm yields, in addition to the set of estimators presented above, the relevant estimator of the unobservable past state variables, namely, the smoothed state estimator. One of the attractive features of the EM algorithm is that under convexity of the (negative) log-likelihood function, the initial estimates to be provided in step 1 need not be close to the convergence values. In fact, initial estimates affect the rate of convergence but not the convergence itself to a local maximum [*Wu*, 1983].

COMBINING ESTIMATION AND CONTROL

In many cases, water resources managers, in addition to estimating unknown parameters, must make decisions to optimally control the values of the state variables. Examples of this can be the operation of a reservoir to minimize flood damages by regulating reservoir releases during flooding events, or the regulation of pumping rates in an irrigation district to keep piezometric heads in an aquifer close to some target levels. The following quadratic objective function is considered (the augmented state-space model equations (11) and (12) are used henceforth)

$$\min_{\mathbf{u}_l, \forall l} E\left\{ \sum_{l=1}^{t} (\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*)' S_l^* (\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*) + \sum_{l=1}^{t-1} (\mathbf{u}_l - \bar{\mathbf{u}}_l)' Z_l (\mathbf{u}_l - \bar{\mathbf{u}}_l) \right\} \tag{30}$$

subject to

$$\mathbf{x}_{l+1}^* = A^* \mathbf{x}_l^* + B^* \mathbf{u}_l + \mathbf{y}_l^* + \mathbf{e}_{l+1}^* \qquad l = 1, 2, \ldots, t-1 \tag{31}$$

$$\mathbf{z}_{l+1} = M^* \mathbf{x}_{l+1}^* + \mathbf{v}_{l+1} \qquad l = 1, 2, \ldots, t-1 \tag{32}$$

in which

$$S_l^* = \begin{bmatrix} S_l & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{\mathbf{x}}_l^{*\prime} = (\bar{\mathbf{x}}_l', \mathbf{0}') \tag{33}$$

where S_l is a penalty matrix; $\bar{\mathbf{x}}_l$ is a desired target path for the state $\mathbf{x}_l$, $\forall l$; and $\bar{\mathbf{u}}_l$ is a reference value for the decision $\mathbf{u}_l$, $\forall l$. The structure of S_l^* implies that no penalties are imposed on $\mathbf{w}_l$, $\forall l$. The matrix Z_l is a penalty matrix assumed positive definite and without loss of generality, symmetric. The problem specified by (30)–(32) minimizes deviations of the state and decision vectors $\mathbf{x}_l^*$ and $\mathbf{u}_l$ about the values $\bar{\mathbf{x}}_l^*$ and $\bar{\mathbf{u}}_l$, respectively. *Noel and Howitt* [1982] and *Wasimi and Kitanidis* [1983] presented additional examples on the use of models with quadratic objective functions in water resources modeling. *Yakowitz* [1982] discussed solutions of quadratic problems via dynamic programming.

The solution to (30)–(32) is developed herein. Additional discussion on how to select $\bar{\mathbf{x}}_l$, $\bar{\mathbf{u}}_l$, S_l, and Z_l, $\forall l$, is provided in the application example. Problem (30)–(32) is solved via dynamic programming. The solution to problem (30)–(32) follows partly the steps outlined by *Bertsekas* [1976, pp. 129–133] for the case in which $\bar{\mathbf{x}}_l^*$, $\mathbf{y}_l^*$, and $\bar{\mathbf{u}}_l$ are zero.

Define the "cost-to-go" function for period l as

$$J_l = \min_{\mathbf{u}_l, \forall l} \{E[(\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*)' S_l^* (\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*) + (\mathbf{u}_l - \bar{\mathbf{u}}_l)' Z_l (\mathbf{u}_l - \bar{\mathbf{u}}_l) + J_{l+1}(A\mathbf{x}_l^* + B^* \mathbf{u}_l + \mathbf{y}_l^* + \mathbf{e}_{l+1}^*)] | \mathbf{z}_2, \ldots, \mathbf{z}_l\} \tag{34}$$

for $l = t-1, t-2, \ldots, 1$. The terminal cost to go is given by

$$J_t = E[(\mathbf{x}_t^* - \bar{\mathbf{x}}_t^*)' S_t^* (\mathbf{x}_t^* - \bar{\mathbf{x}}_t^*)] \tag{35}$$

By letting $l = t-1$ in (34), differentiating with respect to $\mathbf{u}_{t-1}$, setting the resulting expression equal to zero, and solving for $\mathbf{u}_{t-1}$, one obtains (noticing that $E(\mathbf{e}_t^*) = \mathbf{0}$ and independent of $\mathbf{x}_l^*$, $\forall l$),

$$\mathbf{u}_{t-1}^* = -(Z_{t-1} + B^{*\prime} G_t B^*)^{-1} [B^{*\prime} G_t (A^* \mathbf{x}_{t-1}{}^{*t-1} + \mathbf{y}_{t-1}^*) - B^{*\prime} \mathbf{a}_t - Z_{t-1} \bar{\mathbf{u}}_{t-1}] \tag{36}$$

in which

$$G_t = S_t^* \tag{37}$$

$$\mathbf{a}_t' = \bar{\mathbf{x}}_t^{*\prime} S_t^* \tag{38}$$

and $\mathbf{x}_{t-1}{}^{*t-1}$ is the Kalman filter estimate of the state $\mathbf{x}_{t-1}^*$ (see the appendix).

The substitution of (36) into (34) yields J_{t-1}. By letting $l = t-2$ in (34) and using the known J_{t-1} in the cost-to-go function J_{t-2}, one can compute $\mathbf{u}_{t-2}^*$ similarly, as done for $\mathbf{u}_{t-1}^*$. By repeating this substitution-minimization procedure for $\mathbf{u}_{t-3}^*$, $\mathbf{u}_{t-4}^*$, etc., the optimal decision for any period l is

$$\mathbf{u}_l^* = -(Z_l + B^{*\prime} G_{l+1} B^*)^{-1} [B^{*\prime} G_{l+1} (A^* \mathbf{x}_l^{*l} + \mathbf{y}_l^*) - B^{*\prime} \mathbf{a}_{l+1} - Z_l \bar{\mathbf{u}}_l] \tag{39}$$

for $l = t-1, t-2, \ldots, 1$. In (39), $\mathbf{x}_l^{*l}$ is the Kalman filter estimate of the state $\mathbf{x}_l^*$ (see the appendix). The backward recursions G_{l+1} and $\mathbf{a}_{l+1}$ are computed by the following expressions:

$$G_l = A^{*\prime} [G_{l+1} - G_{l+1} B^* (Z_l + B^{*\prime} G_{l+1} B^*)^{-1} B^{*\prime} G_{l+1}] A^* + S_l^* \tag{40}$$

and

$$\mathbf{a}_l' = \bar{\mathbf{x}}_l^{*\prime} S_l^* - \mathbf{y}_l^{*\prime} G_{l+1} [A^* - B^* (Z_l + B^{*\prime} G_{l+1} B^*)^{-1} B^{*\prime} G_{l+1} A^*] + \mathbf{a}_{l+1}' [A^* - B^* (Z_l + B^{*\prime} G_{l+1} B^*)^{-1} B^{*\prime} G_{l+1} A^*] - \bar{\mathbf{u}}_l' Z_l (Z_l + B^{*\prime} G_{l+1} B^*)^{-1} B^{*\prime} G_{l+1} A^* \tag{41}$$

In (40) and (41), $l = t-1, t-2, \ldots, 2$, with G_t and $\mathbf{a}_t'$ defined in (37) and (38) respectively. Equations (39)–(41) provide the

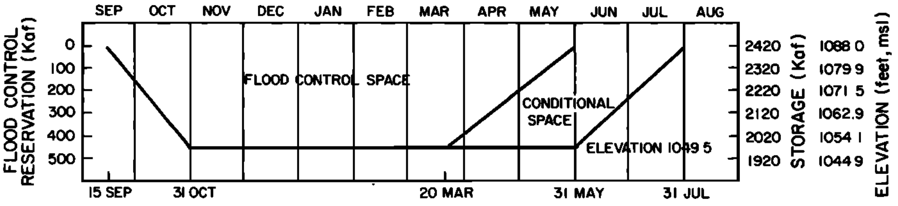

Fig. 2. Flood control diagram for New Melones Reservoir.

optimal solution to the management problem (30)–(32). Given (1) the quadratic nature of (30); (2) the structure of the augmented state-space model (31)–(32), which is equivalent to the model specified by (1)–(3); and (3) the statistical assumptions on the error terms $\mathbf{e}_l^*$ and $\mathbf{v}_l$, the solution provided by (39)–(41) is a strict local optimum solution.

The control solution given by (39)–(41) implies that both states ($\mathbf{x}_l^*$) and decision or control variables ($\mathbf{u}_l^*$) are unconstrained. The weighting matrices play to some extent the role of constraining $\mathbf{x}_l^*$ and $\mathbf{u}_l^*$ about the desired target values $\bar{\mathbf{x}}_l^*$ and $\bar{\mathbf{u}}_l$, since the objective function can be interpreted as quadratic penalties on deviations about $\bar{\mathbf{x}}_l^*$ and $\bar{\mathbf{u}}_l$. It is not possible to derive analytical solutions for the problem (30)–(32) in the presence of constraints. If infeasibilities are detected when applying the control given by (39), one could "clip" the control at its maximum or minimum permissible value. This would lead to suboptimal and only approximately correct control sequences since the actual process of deriving (39) did not (and cannot) consider the effect of constraints. The reader is referred to *Deyst and Price* [1973], *Bryson and Ho* [1975], and *Murray and Yakowitz* [1979] for a discussion on dynamic deterministic control with linear constraints.

In order to compute the solution (39)–(41), it is required to have an estimate of A^*, which is given by the application of the parameter estimation algorithm, i.e.,

$$A^* = \begin{bmatrix} \hat{A} & I \\ 0 & \hat{\phi} \end{bmatrix} \qquad (42)$$

in which $\hat{A}$ and $\hat{\phi}$ are the convergence values in the parameter estimation algorithm. The value of A^* (and the rest of the parameters) can be computed by running the estimation algorithm on a data set prior to the beginning of the control horizon (i.e., $l < 1$). Such estimates are then used in (39)–(41). Notice that the other unknown parameters ($\boldsymbol{\mu}_1$, Σ_1, Q, and R) enter in (39)–(41) through the sufficient statistic $\mathbf{x}_l^{*l}$, the Kalman filter estimator of the state $\mathbf{x}_l^*$. Cascading the combined parameter estimation-control approach into (1) parameter estimation and (2) control proper (taking a certainty equivalence point of view) has been termed the separation property since its introduction by *Joseph and Tou* [1961] and has been widely used to solve optimal control problems ever since. It can be verified in the appendix the dependence of $\mathbf{x}_l^{*l}$ on the parameter set θ. $\mathbf{x}_l^{*l}$ is a sufficient statistic, because, to the controller, all the necessary information about the state-space model is summarized into the state estimator $\mathbf{x}_l^{*l}$. The estimate of the parameter set θ can be updated within the control horizon (i.e., $1 < l < t$), if desired, by running the estimation algorithm based on data up to the time in which the update of parameters is performed. The extent to which periodic revision of parameter estimates is important to the control or management problem depends on the nature of each specific problem. An example is provided in the next section.

APPLICATION AND ANALYSIS OF RESULTS

The implementation of the previous theoretical developments is illustrated with an application that combines control and estimation to solve a multiperiod management problem. Figure 1 shows a three-reservoir system from the northern portion of the California Central Valley Project (NCVP). One of the main functions of the NCVP is to provide flood control protection to the highly developed Sacramento and San Joaquin valleys. During periods of large river streamflows, the NCVP managers must operate the reservoirs jointly, so as to satisfy flood control regulations imposed by the U.S. Corps of Engineers, and minimize flood control damages. Figure 2 shows the flood control diagram for New Melones reservoir, which specifies flood control target storages through the year. In addition to maintaining reservoir storages close to the targets levels, the NCVP system operators have the goal of minimizing losses caused by large reservoir releases at control points 1, 2, and 3 (Figure 1). The authors conducted a study to relate water-surface profiles downstream of the reservoirs to economic losses. The following equations expressing damages as functions of releases were developed:

$$D_l^{(1)} = -7.79 + 13.5u_l^{(1)} - 1.866(u_l^{(1)})^2 \qquad (43)$$

$$710 \le u_l^{(1)} \le 3700$$

$$D_l^{(2)} = -0.187 + 6.74u_l^{(2)} - 0.961(u_l^{(2)})^2 \qquad (44)$$

$$230 \le u_l^{(2)} \le 3400$$

$$D_l^{(3)} = 0.205 + 18.1u_l^{(3)} - 0.374(u_l^{(3)})^2 \qquad (45)$$

$$60 \le u_l^{(3)} \le 850$$

in which the damage $D_l^{(i)}$, at control point $i = 1, 2, 3$, is in million of dollars, and the releases $u_l^{(i)}$ are in m^3/s. Equations (43)–(45) are used to set up the penalty matrices Z_l, $\forall l$. Matrices Z_l contain diagonal elements that are equal to the relative value of release damages at control point i, $i = 1, 2, 3$ (see Figure 1). Notice that of interest is the relative (as opposed to the absolute) level of economic loss. Thus the diagonal elements of Z_l are given by $z_l^{(i)} = D_l^{(i)}/\min(D_l^{(1)}, D_l^{(2)}, D_l^{(3)})$, provided that $\min(D_l^{(1)}, D_l^{(2)}, D_l^{(3)})$ is strictly larger than zero. If all $D_l^{(i)}$ ($i = 1, 2, 3$) are equal to zero, then Z_l is set equal to the identity matrix. If $\min(D_l^{(1)}, D_l^{(2)}, D_l^{(3)})$ equals zero, but not all the $D_l^{(i)}$ are zero, then $z_l^{(i)} = D_l^{(i)}/\min D_l^{(k)}$, for all i and k such that $D_l^{(i)}$ and $D_l^{(k)}$ are nonzero, and $z_l^{(i)} = \varepsilon$, for all i such that $D_l^{(i)} = 0$, in which ε is a small positive number. In view of the NCVP system operation goals, the objective function of the flood control operation problem is stated as

$$\min_{\mathbf{u}_l, \forall l} E\left\{ \sum_{l=1}^{t} (\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*)' S_l^* (\mathbf{x}_l^* - \bar{\mathbf{x}}_l^*) + \sum_{l=1}^{t-1} (\mathbf{u}_l - \bar{\mathbf{u}}_l)' Z_l (\mathbf{u}_l - \bar{\mathbf{u}}_l) \right\} \qquad (46)$$

which is subject to the state equation

$$\mathbf{x}_{l+1}^* = A^*\mathbf{x}_l^* + B^*\mathbf{u}_l + \mathbf{e}_{l+1}^* \tag{47}$$

in which

$$A^* = \begin{bmatrix} I & I \\ 0 & \phi \end{bmatrix} \qquad B^* = \begin{bmatrix} -I \\ 0 \end{bmatrix} \tag{48}$$

where I and 0 are 3×3 identity and null matrices, respectively, and $\mathbf{x}_l^*$ and $\mathbf{e}_{l+1}^*$ are defined in (9). The upper 3×1 subvector of $\mathbf{x}_l^*$ represents the storage (state) variables, and $\mathbf{u}_l$ is the 3×1 vector of reservoir releases. From (9), (47), and (48) it follows that the streamflow vector $\mathbf{w}_l$ is modeled by a first-order autoregressive model. An observation equation is added,

$$\mathbf{z}_{l+1} = \mathbf{x}_{l+1}^* + \mathbf{v}_{l+1} \tag{49}$$

which is similar to (12) with M^* defined as

$$M^* = (I \quad 0) \tag{50}$$

The solution to (46)–(49) is given by (39)–(41), after setting $\mathbf{y}_l^* = \mathbf{0}$. The filtering, smoothing, and lagged covariances given in the appendix should be modified by setting $\mathbf{y}_l^* = \mathbf{0}$ and $M^* = (I \quad 0)$ where appropriate. No other modifications are required. Estimation is needed to compute the unknown parameter set $\theta = \{\boldsymbol{\mu}_1, \Sigma_1, Q, R, \phi\}$, and this is accomplished by the parameter estimation technique developed earlier. The optimal control solution given by (39) requires the filtered estimate $\mathbf{x}_l^{*l}$ that can be obtained by the filtering equations given in the appendix. The matrices S_l^* in (46) are obtained by first calculating the economic damage stemming from potential overtopping at each dam site. Subsequently, the first three diagonal elements of S_l^* (see equation (33)) are made equal to the ratio between the potential damage from dam overtopping at site i and the minimum potential overtopping damage, for $i = 1, 2, 3$ (notice that relative figures are used, as was done with Z_l). Clearly, S_l^* is then a time-invariant matrix, i.e., $S_l^* = S^*$, $\forall l$. The matrix Z_l, $\forall l$, in (46), as is explained above, was obtained by setting its diagonal elements z_l^i equal to the ratio of the damage for period $l-1$ (as obtained from equations (43)–(45)) to the minimum damage caused during period $l-1$, for $i = 1, 2, 3$. It must be pointed out that the last known damage information corresponds to period $l-1$, for $\mathbf{u}_l$ is unknown at the beginning of the lth decision period. Releases vary at most $\pm 20\%$ in the short decision intervals used in this application. Since the most recent information on downstream damage is that associated with $\mathbf{u}_{l-1}$, the weights $z_l^{(i)}$ are based on damage information for period $l-1$. Such weighting matrices Z_l only affect the computation of the backward recursions for period l. For period $l+1$, the backward recursions are recomputed based on damage information for period $l+1$, and so on. Additional details for computing $S_l^* = S^*$ and Z_l are given in the work by H. A. Loaiciga and M. A. Mariño (unpublished manuscript, 1984) where a simpler management problem is considered that does not combine control and estimation. The diagonal elements of S_l^* are $s_l^{*i} = 4.5$, 1.0, and 2.4 for $i = 1$, 2, and 3, respectively. All other diagonal and off-diagonal elements of S_l^* are zeroes. The off-diagonal elements of Z_l are also zero. For the first period, $l = 1$, the matrix Z_1 is set equal to the identity matrix.

The combined control-estimation method is used to develop an optimal release strategy to route large runoff volumes generated by heavy rainfall occurring during February 25 to March 5, 1983. Seventy-two decision periods, each of a three-hour duration, make up the entire control horizon. As a by-product of applying the combined control-estimation technique, one obtains the parameter estimates as well as the smoothed state estimators. The steps to implement the proposed control-estimation technique to the reservoir operation problem are as follows.

1. For the first period, provide values for $\boldsymbol{\mu}_1$, Σ_1, Q, R, and ϕ. In this study, initial estimates were obtained from moment estimators for $\boldsymbol{\mu}_1$, Σ_1, Q, and R [see *Todini et al.*, 1977]. A standard least squares estimator [*Anderson*, 1978] provided the initial estimate for ϕ. These values are used to compute filtered estimates of the storage variables. Based on these estimates, the releases, as given by (39), are obtained and implemented.

2. After six (3-hour) decision periods, the estimation algorithm is run as explained above, to update the guessed parameter estimates made at the beginning of the first operation problem.

3. With the updated parameter estimates, compute filtered storages and releases up to the twelfth decision period, where parameter estimates are reupdated again by running the estimation algorithm.

4. Continue to obtain and implement the controls for the remaining periods within the control horizon, reupdating the parameter estimates every 12 periods, i.e., at the beginning of the decision periods 24, 36, ..., 72.

Table 1 contains initial data to start the control-estimation algorithm. Figures 3–5 display the optimal (smoothed) storage trajectories and optimal release policies for Shasta, Folsom, and New Melones reservoirs. It can be observed in Figure 3 that the storage at Shasta reservoir is driven towards the

TABLE 1. Initial Data to Begin the Control-Estimation Algorithm

Initial Value	Shasta	Folsom	New Melones
Initial storage, $\boldsymbol{\mu}_1^{(1)}$, (m^3)	4.3045×10^9	7.8795×10^8	2.2784×10^9
Initial streamflows, $\boldsymbol{\mu}_1^{(2)}$, (m^3/s)	0.5851×10^3	0.3034×10^3	0.0833×10^3
Target storages, $\bar{\mathbf{x}}_l$, $\forall l$, (m^3)	4.0114×10^9	0.7542×10^9	2.4293×10^9
Reference releases, $\bar{\mathbf{u}}_l$, $\forall l$, (m^3)	0.7100×10^3	0.2300×10^3	0.0600×10^3
Diagonal weight submatrix, S_l, $\forall l$ (dimensionless)	4.5	1.0	2.4
Diagonal weight submatrix, Z_1, (dimensionless)	1.0	1.0	1.0
Diagonal covariance matrix, Σ_1, (m^3)^2	1.3693×10^{15}	7.6074×10^{13}	6.0859×10^{14}
Diagonal covariance matrix, Q, (m^3)^2	1.0650×10^{13}	1.5215×10^{12}	1.5215×10^{12}
Diagonal covariance matrix, R, (m^3)^2	1.3693×10^{15}	7.6074×10^{13}	6.0859×10^{14}

In the augmented state model the initial condition vector is a 6×1 array whose upper and lower 3×1 subvectors contain initial storages and streamflows, respectively.

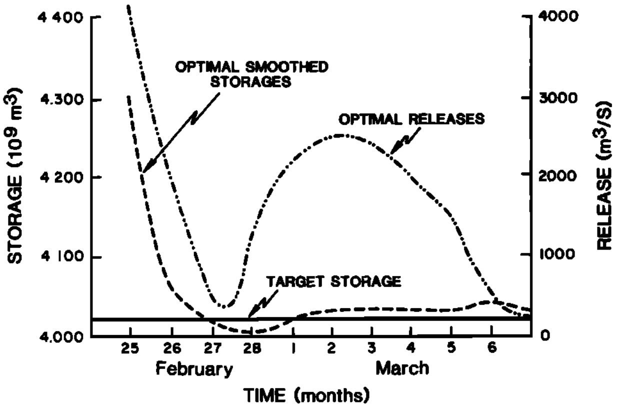

Fig. 3. Optimal smoothed storage and releases for Shasta reservoir.

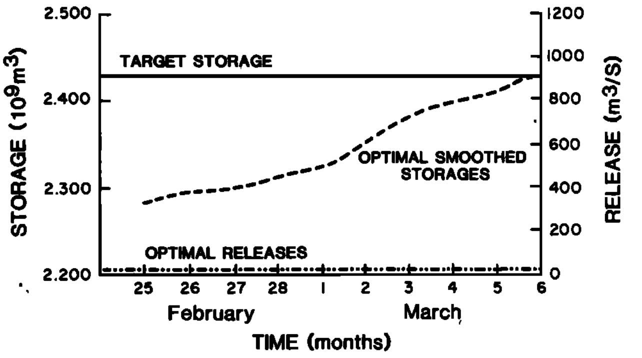

Fig. 5. Optimal smoothed storage and releases for New Melones reservoir.

target storage from above and is subsequently kept close to it for the rest of the control horizon. The increase of the optimal releases after February 28 is due to the large streamflow volumes occurring during that period. A similar behavior is observed for Folsom reservoir (Figure 4). For New Melones reservoir (Figure 5) it can be observed that the storage starts below the target storage. The optimal strategy in this case is to maintain the releases at a zero level for the entire horizon, while the storage is driven toward its target value from below, entirely by the streamflow volumes. Actually, the computed releases from the analytical expression shown in (39) yielded negative values for New Melones. This implies that the reservoir is to be filled so that the storage could be driven promptly toward its target value. Since the reservoirs are not in tandem, such an increase in reservoir storage can be achieved at a maximum rate established by the streamflow volumes into the reservoir, which is obtained by setting $\mathbf{u}_l^* = 0$, $\forall l$, for New Melones reservoir. It is emphasized that despite the fact that the weighting matrices S^* and Z_l, $\forall l$, are diagonal, the optimal control solution to the flood control reservoir operation problem is not decoupled, i.e., the releases from one reservoir affect the optimal strategy of the remaining reservoirs. This is due to (1) the nondiagonal structure of matrices A^* and B^* in (47) and (2) the dependence of the control $\mathbf{u}_l^*$ (see equation (39)) on the augmented state filtered estimator $\mathbf{x}_l^{*l}$, which in turn depends on the values of the parameter set $\theta = (\boldsymbol{\mu}_1, \Sigma_1, Q, R, \phi)$. Furthermore, given the penalties on releases and storage deviations, and the state-space model structure (see equations (47)–(49)), the computed release and smoothed storage policies constitute an optimal solution to

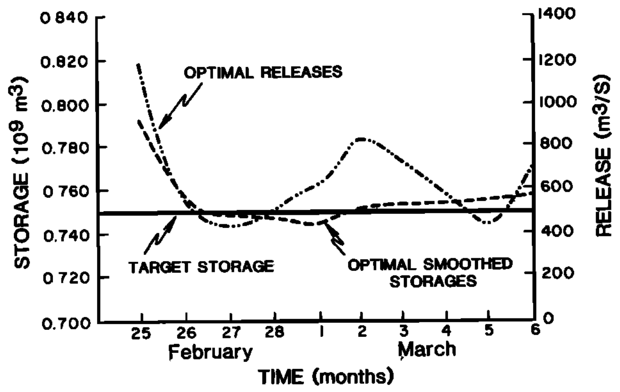

Fig. 4. Optimal smoothed storage and releases for Folsom reservoir.

the flood control management problem. The release sequences shown in Figures 3–5 did not lead to infeasibilities in the reservoir storages (i.e., reservoir levels were kept below their maximum permissible values of 5.61×10^9, 1.25×10^9, and 3.36×10^9 m^3 for Shasta, Folsom, and New Melones, respectively). The restrictions on maximum permissible levels were nonbinding and did not affect the control sequences in Figures 3–5. To compare the results shown in Figures 3–5 with those policies that actually took place for the three reservoir system, it is useful to explain how the reservoirs are managed. Figure 6 shows the flood operation criteria enforced by the U.S. Army Corps of Engineers for Shasta reservoir. It can be observed in Figure 6 that the actual operation criteria consider storage targets (i.e., $\mathbf{x}_l$, the upper 3×1 subvector of $\bar{\mathbf{x}}_l^*$) and release targets ($\bar{\mathbf{u}}_l$) that are contingent on forecasted inflows and desired flood control levels. For the estimation-control application of this study, we have used constant values for the reservoir storage target (corresponds to the horizontal lower line in Figure 6*a*) and for the release target (see Table 1). By following the flood control operation criteria for Shasta reservoir, and a similar one for Folsom reservoir, the actual storage sequences were kept within $\pm 5\%$ of the specified target values shown in Table 1 for the entire control horizon at Shasta and Folsom reservoirs (except for the first day of operation, when the storages were decreased from the initial storage value toward the targets). For New Melones, the storage was increased to about 2.350×10^9 m^3 by March 5 and kept relatively constant afterwards.

It was found that the term on the objective function composed of penalties on the square of the releases was approximately 5% smaller by following an actual (heuristic) operation criterion implemented by the NCVP managers than that yielded by the optimization model. The term in the objective function penalizing the square deviations on storages was 20% larger using the actual policies implemented by the NCVP managers. Overall, the value of the objective function obtained by the actual policies implemented by the NCVP staff was 15% larger than that obtained from the optimal control approach (recall that the objective function implies a minimization problem). The estimated release damage downstream of Shasta and Folsom reservoirs, following the computed release sequences (and shown in Figures 3–5), was approximately $750,000 (1985 U.S. dollars), whereas the actually implemented releases caused damages of nearly $\$10^6$ (a 25% difference between both schedules). These figures can be contrasted with estimated averted damages of nearly $\$70 \times 10^6$

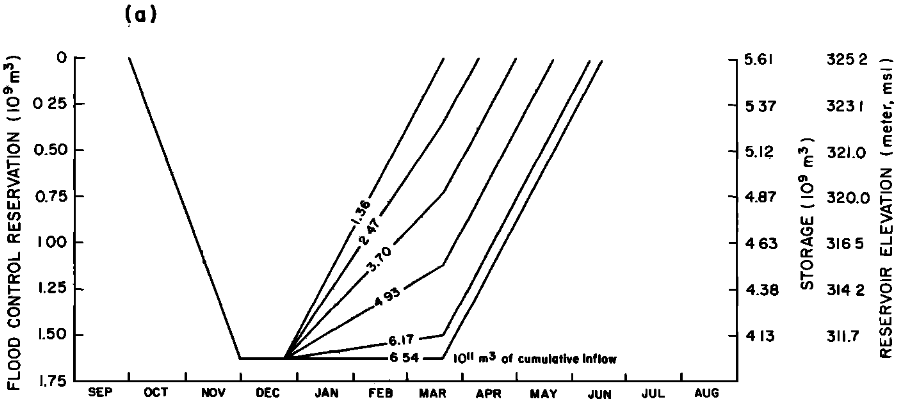

Fig. 6*a*. Shasta flood control diagram. (source: Central Valley Operations Office, U.S. Bureau of Reclamation, Sacramento, California).

(1985 U.S. dollars) during rainy seasons in northern California [*Madsen and Coleman*, 1974].

The estimated values of the parameter set θ at the end of the control horizon, when the estimation algorithm was based on the largest data set (i.e., up to the 72th period), are shown in Table 2. The initial storage estimators were within $\pm 4\%$ of the estimated values shown in Table 1. The covariance matrices Σ_1, Q, and R had their first diagonal element decreased about two orders of magnitude from their originally estimated values in Table 1. Their second and third diagonal elements were about one order of magnitude smaller than the guessed values (see Table 1). In particular, the estimated matrix R was diagonal, indicating the lack of cross-correlation between errors in measurements. The substantial decrease in the magnitude of the diagonal elements in the covariance matrix R indicates that the storage observations are relatively accurate. The relative small values of the off-diagonal elements in the covariances Σ_1 and Q indicate some degree of statistical decoupling between the storage and streamflow values at the different reservoir sites. The log-likelihood function was decreased from its original to its final value at convergence within 10 iterations. Fifteen additional iterations were needed to satisfy the convergence test used in this study [see *Gill et al.*, 1981]. The estimation algorithm showed slower convergence rate on the final iterations every time it was run, and this is in agreement with the discussion on the EM algorithm performance features provided by *Leytham* [1984]. *Shumway and Stoffer* [1982] reported the same slow convergence and suggested switching to a more rapidly convergent algorithm (e.g., Newton-Raphson) once in the vicinity of the local optimum, although this was not done in the application presented herein. However, after having obtained the parameter estimates, the control equations (39)–(41) were easily implemented. Notice also that (26) need not be implemented in this application, alleviating somewhat the computational burden. As was shown by *Wu* [1983], the EM estimators have all the desirable (asymptotic) properties of maximum likelihood estimators. The CPU processing time for the control-estimation algorithm was approximately 240 s in a VAX-11/780 computer.

CONCLUSIONS

An innovative estimation approach to estimate the parameters (i.e., initial conditions, transition and covariance matrices) has been developed and combined with a stochastic control method. The combined control-estimation algorithm was used to solve a multistage reservoir management problem. In addition to yielding a set of optimal decisions for the management problem, the state-space parameters as well as the smoothed estimators of the storages were also derived.

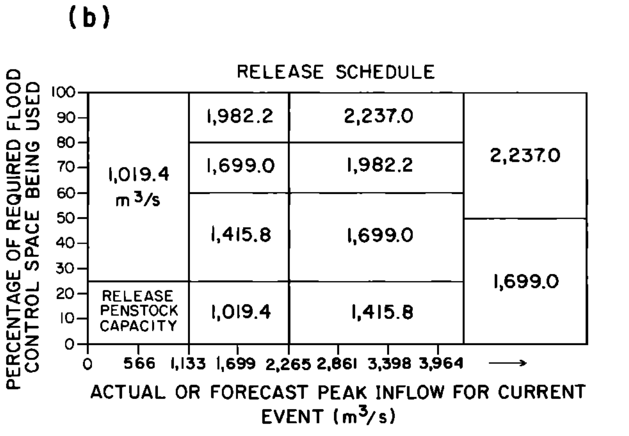

Fig. 6*b*. Shasta release schedule (source: Central Valley Operations Office, U.S. Bureau of Reclamation, Sacramento, California).

TABLE 2. Estimated Parameter Values

Estimated Value	Shasta	Folsom	New Melones
μ_1, (m^3)	4.100×10^9	0.760×10^9	2.300×10^9
Σ_1, $(m^3)^2$			
Shasta	0.2283×10^{14}	0.0001×10^{14}	-0.0002×10^{14}
Folsom	0.0001×10^{14}	0.1249×10^{13}	-0.0045×10^{14}
New Melones	-0.0002×10^{14}	-0.0045×10^{14}	0.8878×10^{13}
Q, $(m^3)^2$			
Shasta	0.2922×10^{14}	0.0003×10^{14}	-0.0002×10^{14}
Folsom	0.0003×10^{14}	0.1755×10^{13}	-0.0015×10^{14}
New Melones	-0.0002×10^{14}	-0.0015×10^{14}	0.1168×10^{14}
R, $(m^3)^2$			
Shasta	0.2036×10^{14}	0.0000	0.0000
Folsom	0.0000	0.1219×10^{13}	0.0000
New Melones	0.0000	0.0000	0.8147×10^{13}
ϕ (dimensionless)			
Shasta	0.8785	0.5900	−0.4605
Folsom	−0.0140	0.0304	0.1732
New Melones	−0.0097	0.0668	0.8904

The initial condition estimator μ_1 estimates the initial storages.

The parameter estimation technique is based on very mild statistical assumptions; thus it does not require the introduction of any strong limiting statistical condition. The technique leads to analytical expressions for updating the parameter estimators, which is ideal for implementation in digital computers. The control method uses the filtered storage estimator as a basis for decision, and the analytical control law is also based on suitably computable recursions, of easy coding and implementation as software. The application example shows that the control-estimation technique provides realistic and credible decision policies as well as parameter estimators within reasonable computational time.

APPENDIX: FILTERING, SMOOTHING, AND LAGGED COVARIANCES

Filtering

The filtering forward recursions for the augmented state-space model [*Jazwinski*, 1970] (11)–(12) are ($l = 2, 3, \ldots, t$).

State estimate extrapolation:

$$\mathbf{x}_l{}^{*l-1} = A^*\mathbf{x}_{l-1}{}^{*l-1} + B^*\mathbf{u}_{l-1} + \mathbf{y}_{l-1}{}^* \qquad \text{(A1)}$$

Error covariance extrapolation:

$$P_l{}^{*l-1} = A^*P_{l-1}{}^{*l-1}A^{*\prime} + Q^* \qquad \text{(A2)}$$

where

$$Q^* = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \qquad \text{(A3)}$$

State estimate update:

$$\mathbf{x}_l{}^{*l} = \mathbf{x}_l{}^{*l-1} + K_l(\mathbf{z}_l - M^*\mathbf{x}_l{}^{*l-1}) \qquad \text{(A4)}$$

Error covariance update:

$$P_l{}^{*l} = P_l{}^{*l-1} - K_lM^*P_l{}^{*l-1} \qquad \text{(A5)}$$

Kalman gain matrix:

$$K_l = P_l{}^{*l-1}M^{*\prime}(M^*P_l{}^{*l-1}M^{*\prime} + R)^{-1} \qquad \text{(A6)}$$

The initial conditions are $E(\mathbf{x}_1{}^*) = \boldsymbol{\mu}_1{}^*$ and cov $(\mathbf{x}_1{}^*) = \Sigma_1{}^*$. The estimator $\mathbf{x}_l{}^l$ is given by the upper $n \times 1$ subvector of (A4). The $P_l{}^{l-1}$ and $P_l{}^l$ covariances are given by the upper-left $n \times n$ submatrices of (A2) and (A5), respectively, For parameter estimation via (22)–(27), one is interested in the smoothed estimators given below.

Smoothing

The smoothing backward recursions for the augmented state-space model [*Rauch et al.*, 1965] (11)–(12) are ($l = t - 1, t - 2, \ldots, 1$).

Smoothed state estimate:

$$\mathbf{x}_l{}^{*t} = \mathbf{x}_l{}^{*l} + H_l(\mathbf{x}_{l+1}{}^{*t} - \mathbf{x}_{l+1}{}^{*l}) \qquad \text{(A7)}$$

Smoothed covariance matrix:

$$P_l{}^{*t} = P_l{}^{*l} + H_l(P_{l+1}{}^{*t} - P_{l+1}{}^{*l})H_l' \qquad \text{(A8)}$$

$$H_l = P_l{}^{*l}A^{*\prime}(P_{l+1}{}^{*l})^{-1} \qquad \text{(A9)}$$

The estimator $\mathbf{x}_l{}^t$ is obtained from the upper $n \times 1$ subvector of (A7). The $P_l{}^t$ matrix is obtained from the upper-left $n \times n$ submatrix of (A8).

Lagged Covariances

The lagged covariance matrix backward recursions for the augmented state-space model (11)–(12) are ($l = t, t - 1, \ldots, 3$)

$$P_{l-1,l-2}{}^{*t} = P_{l-1}{}^{*l-1}H_{l-2}' + H_{l-1}(P_{l,l-1}{}^{*t} - A^*P_{l-1}{}^{*l-1})H_{l-2}' \qquad \text{(A10)}$$

where

$$P_{t,t-1}{}^{*t} = (I - K_tM^*)A^*P_{t-1}{}^{*t-1} \qquad \text{(A11)}$$

The lagged covariances $P_{l,l-1}{}^t$ can be obtained from (A10) and (A11) from their respective upper-left $n \times n$ submatrices.

NOTATION

The following notation is used in this paper.

- A — $n \times n$ parameter matrix in the state-space model.
- A_1, A_2, A_3 — $n \times n$ matrices appearing in the expressions for parameter estimates A and Q.
- A^* — $2n \times 2n$ transition matrix in the augmented state-space model.
- $\mathbf{a}_l$ — $2n \times 1$ recursion in the control algorithm.
- B — $n \times p$ known matrix in the state-space model.
- B_2, B_3 — $n \times n$ matrices appearing in the expressions to compute ϕ.
- B^* — $2n \times p$ matrix in the augmented state-space model.
- $D_l{}^i$ — damage at control point i, during period l, due to release from reservoir i.
- $\mathbf{e}_l$ — $n \times 1$ noise vector in the AR(1) process.
- $\mathbf{e}_l{}^*$ — $2n \times 1$ noise vector in the augmented state-space model.
- $G(\ )$ — conditional expectation of the state-space process log-likelihood function.
- G_l — $2n \times 2n$ matrix recursion in the control algorithm.
- H_l — $2n \times 2n$ matrix recursion appearing in the smoothed state and covariance estimators.
- J_l — cost-to-go function at time l in the dynamic programming formulation of the control problem.
- K_l — $2n \times 2n$ Kalman gain matrix in the state filtering recursions of the augmented state-space model.
- l — time index, $l = 1, 2, \ldots, t - 1$.
- M — $m \times n$ observation matrix in the state-space model.
- M^* — $m \times n$ observation matrix in the augmented state-space model.
- $P_l{}^k$ — $n \times n$ state estimate covariance in the augmented state-space model. If $k > l$, it represents the smoothed covariance. If $k = l$, it represents the filtered covariance. In either case, it is given by the upper-left $n \times n$ submatrix of $P_l{}^{*k}$.
- $P_l{}^{*k}$ — $2n \times 2n$ state estimate covariance in the augmented state-space model. If $k > l$, it represents the smoothed covariance. If $k = l$, it represents the filtered covariance.
- $P_{l,l-1}{}^k$ — lagged covariance obtained from the upper left $n \times n$ submatrix of $P_{l,l-1}{}^{*k}$.
- $P_{l,l-1}{}^{*k}$ — lagged covariance in the augmented state-space model.
- Q — $n \times n$ covariance of the noise vector $\mathbf{e}_l$.

Q^* $2n \times 2n$ covariance of the augmented noise vector $\mathbf{e}_l^*$.

R $n \times n$ covariance of the noise vector $\mathbf{v}_l$.

S_l^* $2n \times 2n$ penalty matrix in the control problem.

$\mathbf{u}_l$ $p \times 1$ decision vector in the state-space model and control problem.

$\mathbf{u}_l^*$ $p \times 1$ optimal decision vector in the control problem.

$\bar{\mathbf{u}}_l$ $p \times 1$ vector of reference decision values.

$\mathbf{v}_l$ $m \times 1$ noise vector in the state-space model.

$\mathbf{w}_l$ $n \times 1$ stochastic, uncontrollable, input to the state-space model.

$\mathbf{x}_l$ $n \times 1$ state vector.

$\bar{\mathbf{x}}_l$ $n \times 1$ state target vector.

$\mathbf{x}_l^*$ $2n \times 1$ augmented state vector.

$\bar{\mathbf{x}}_l^*$ $2n \times 1$ augmented state vector.

$\mathbf{x}_l^k$ $n \times 1$ state estimator. If $k > l$, it represents the smoothed state estimator. If $k = l$, it represents the filtered state estimator. In either case, it is given by the upper $n \times 1$ subvector of the vector $\mathbf{x}_l^{*k}$.

$\mathbf{x}_l^{*k}$ $n \times 1$ state estimator in the augmented state-space model. If $k > l$, it represents the smoothed state estimator. If $k = l$, it represents the filtered estimator.

$\mathbf{y}_l$ $n \times 1$ vector of exogenous deterministic and known variables in the state-space model.

$\mathbf{y}_l^*$ $2n \times 1$ vector of exogenous deterministic and known variables in the augmented state-space model.

Z_l $p \times p$ positive definite penalty matrix in the control problem.

$\mathbf{z}_l$ $m \times 1$ vector of observable variables in the state-space model.

θ set of parameters $= (\boldsymbol{\mu}_1, \Sigma_1, Q, R, A, \phi)$.

Acknowledgments. The research leading to this report was supported by the University of California, Water Resources Center as part of Water Resources Center Project UCAL-WRC-W-634. We thank R. L. Bras for reviewing an earlier version of this paper and providing valuable suggestions. The comments and suggestions of two anonymous reviewers are also appreciated.

REFERENCES

Anderson, T. W., Estimation for autoregressive moving average models in the time and frequency domain models, *Ann. Stat.*, *5*(5), 842–865, 1977.

Anderson, T. W., Repeated measurements on autoregressive processes, *J. Am. Stat. Assoc.*, *73*(362), 371–378, 1978.

Bertsekas, D. I., *Dynamic Programming and Stochastic Control*, Academic, Orlando, Fla., 1976.

Bryson, A. E., and Y. C. Ho, *Applied Optimal Control*, Hemisphere, Washington, D. C., 1975.

Chiu, C. L. (Ed.), *Applications of Kalman Filter to Hydrology, Hydraulics, and Water Resources*, AGU, Washington, D. C., 1978.

Dempster, A. P., N. M. Laird, and D. B. Rubin, Maximum likelihood from incomplete data via the EM algorithm, *J. R. Stat. Soc., Ser. B*, *39*, 1–38, 1978.

Deyst, J. J., and C. F. Price, Optimal stochastic laws for tactical missiles, *J. Spacecr. Rockets*, *10*(5), 301–308, 1973.

Gill, P. E., W. Murray, and M. H. Wright, *Practical Optimization*, Academic, Orlando, Fla., 1981.

Gupta, N. K., and R. K. Mehra, Computational aspects of maximum likelihood estimation and reduction in sensitivity function calculations, *Inst. Elec. Electr. Eng. Trans. Automat. Contr.*, *AC-19*, 774–783, 1974.

Hartley, H. O., Maximum likelihood estimation from incomplete data, *Biometrics*, *27*, 783–808, 1958.

Jazwinski, A., *Stochastic Processes and Filtering Theory*, Academic, Orlando, Fla., 1970.

Joseph, P. D., and J. T. Tou, On linear control theory, *Trans. Am. Inst. Electr. Eng.*, *80*, 193–196, 1961.

Kashyap, R. L., Maximum likelihood identification of stochastic linear systems, *Inst. Elect. Electr. Eng. Trans. Automat. Contr.*, *AC-15*(1), 25–34, 1970.

Kitanidis, P. K., and R. L. Bras, Real-time forecasting with a conceptual hydrologic model, 2, Applications and results, *Water Resour. Res.*, *16*(6), 1034–1044, 1980.

Leytham, K. M., Maximum likelihood estimates of the parameters of mixture distributions, *Water Resour. Res.*, *20*(7), 896–902, 1984.

Madsen, W. R., and D. G. Coleman, Coordinated operation of Central Valley and State Water Projects, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *100*(HY10), 1399–1411, 1974.

Mehra, R. K., On the identification of variances and adaptive Kalman filtering, *Inst. Elect. Electr. Eng. Trans. Automat. Contr.*, *AC-15*(2), 175–184, 1970.

Murray, D. M., and S. J. Yakowitz, Constrained differential dynamic programming and its applications to multireservoir control, *Water Resour. Res.*, *15*(5), 1017–1027, 1979.

Narasimhan, T. N., S. P. Neuman, and P. A. Witherspoon, Finite element method for subsurface hydrology using a mixed explicit-implicit scheme, *Water Resour. Res.*, *14*(5), 863–877, 1978.

Noel, J. E., and R. E. Howitt, Conjunctive multibasin management: An optimal control approach, *Water Resour. Res.*, *18*(4), 753–763, 1982.

O'Connell, P. E. (Ed.), *Real-Time Hydrological Forecasting and Control*, Institute of Hydrology, Wallingford, Oxon, 1977.

Otter, P. W., The discrete Kalman filter applied to linear regression models: Statistical considerations and an application, *Stat. Neerlandica*, *32*, 41–56, 1978.

Pinder, G. F., and E. O. Frind, Application of Galerkin's procedure to aquifer analysis, *Water Resour. Res.*, *8*(1), 108–120, 1972.

Rao, C. R., *Linear Statistical Inference and Its Applications*, John Wiley, New York, 1965.

Rauch, H. E., F. Tung, and C. T. Striebel, Maximum likelihood estimation of linear dynamic systems, *AIAA J.*, *3*(8), 1445–1450, 1965.

Restrepo-Posada, C., and R. L. Bras, Automatic parameter estimation of a large conceptual rainfall-runoff model: A maximum likelihood approach, *Tech. Rep. 267*, Ralph M. Parsons Lab., Mass. Inst. of Technol., Cambridge, Mass., 1982.

Shumway, R. H., D. E. Olsen, and L. J. Levy, Estimation and tests of hypothesis for the initial mean and covariance in the Kalman filter model, *Commun. Stat. Theory Methods*, *A10*(16), 1625–1641, 1981.

Shumway, R. H., and D. S. Stoffer, An approach to time series smoothing and forecasting using the EM algorithm, *J. Time Series*, *3*(4), 253–264, 1982.

Stewart, G. W., *Introduction to Matrix Computations*, Academic, Orlando, Fla., 1973.

Todini, E., P. E. O'Connell, and D. A. Jones, Basic methodology: Kalman filter estimation problems, in *Real-Time Hydrological Forecasting and Control*, edited by P. E. O'Connell, Wallingford, Oxon, 1977.

Wasimi, S. A., and P. K. Kitanidis, Real-time forecasting and daily operation of a multireservoir system during floods by linear quadratic Gaussian control, *Water Resour. Res.*, *19*(4), 1511–1522, 1983.

Wu, J. C. F., On the convergence properties of the EM algorithm, *Ann. Stat.*, *11*(1), 95–103, 1983.

Yakowitz, S. J., Dynamic programming applications in water resources, *Water Resour. Res.*, *18*(4), 673–696, 1982.

H. A. Loaiciga and M. A. Mariño, Department of Land, Air, and Water Resources and Department of Civil Engineering, University of California, Davis, CA 95616.

(Received January 18, 1985;
revised July 16, 1985;
accepted August 12, 1985.)