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SPECTRAL GAPS OF RANDOM HECKE OPERATORS

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Abstract

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by

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In this dissertation, we address a number of issues dealing with the edge spectrum of random Hecke operators. In particular, we focus on the spectral gap property of random $2d$-regular graphs, which can be thought of as random Hecke operators of form $z = \pi_1 + \pi_1^{-1} + \cdots + \pi_d + \pi_d^{-1}$ with $\pi_i \in S_n$ under the permutation representation of symmetric group S_n. This dissertation is organized in essentially two parts.

In the first portion of the dissertation, we deal with the spectral gap property of random $2d$-regular graphs. Puder and Parzanchevski ([Puder11], [PP12]) developed a crucial theorem of cancellations between certain collections of closed walks in random n lifts, which simplifies the work of counting the expected number of closed walks in random lifts. Our observation of the connection between generalized forms and core graphs enables us to adopt the cancellation theorem to estimate the expected number of closed walks under the permutation model. The resulting contribution from the cancellation theorem can be expanded to any order by using Friedmans expansion method [Fri91]. The freedom of choosing expansion order leads to an optimal estimation of the spectral gap. However, it is challenging to control high order terms from the expansion. We solve this key issue by separating the summation of irreducible walks into “a good part” and “a bad part” , and showing the probability of the bad part occurring is small. With a lemma of complex random variables [Fri08], and Bartholdi identity [OS09], we

provide an alternative proof of Friedman's strong Alon's conjecture $\lambda(G) \leq 2\sqrt{2d-1}+\varepsilon$ for any $\varepsilon > 0$ with probability $1 - \frac{c}{n^{\lceil \frac{\sqrt{d-1}-1}{2} \rceil}}$ in a simpler way.

There is a strong connection between random Hecke operators/random regular graphs and random matrices. It is conjectured that the edge spectra of random regular graphs can be modeled by certain Tracy-Widom distributions from random matrices. Due to the lack of a proper normalization factor, only indirect evidence [MNS08] is known. In Chapter 3, we consider a normalization factor obtained by matching the first four moments of random Hecke operators/random regular graphs with the corresponding moments of general β ensembles. The validity of this normalization factor is supported by numerical analysis, where we are able to demonstrate that the edge spectra of both random Cayley graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ and the Fourier transform of random Hecke operators over $SU(2)$ at irreducible representations can be modeled by certain Tracy-Widom distributions.

To my parents and my wife Bei

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Chapter 1

Introduction

For any finite integer $N \geq 2$, independently and randomly picking $g_1, \cdots, g_N$ from a group G, one can associate a random Hecke type of operator z in the form

$$zf(x) = \sum_{i=1}^{N} (f(g_i x) + f(g_i^{-1} x)) \quad \text{for} \quad f \in L^2(G). \tag{1.1}$$

To understand the spectrum of z, one can study the spectrum of $\hat{\delta}_{z\rho}$, the Fourier transform of $\delta_z = \sum_{i=1}^{N} \delta_{g_i} + \delta_{g_i^{-1}}$ at a representation $\rho \in \hat{G}$ on a vector space W. Here the Fourier transform of a complex valued function f on G at ρ is defined as

$$\hat{f} = \sum_{g \in G} f(g)\rho(g). \tag{1.2}$$

By assuming z is self-adjoint, $\hat{z}$ is also self-adjoint with respect to a suitable inner product of W. Therefore, the spectrum $spec(\hat{z})$ is real and is contained in the interval $[-2N, 2N]$ as follows,

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{\dim(\rho)}, \tag{1.3}$$

where $dim(\rho)$ is the dimension of the representation ρ.

Let $\|\hat{z}\|$ be the norm of matrix $\hat{z}$. If ρ is an irreducible representation, then $\|\hat{z}\| = \max_{i=1,\cdots \dim(\rho)}\{|\lambda_i|\}$. And if ρ is the permutation/regular representation (which contains the trivial representation), $\|\hat{z}\| = \max\{|\lambda_2|, |\lambda_{\dim(\rho)}|\}$. There is a spectral gap, if $\lim_{\dim(\rho)\to\infty} \|\hat{z}\| < 2N$. For a given z, does spectral gap exist? if so, how large would it be?

The above two questions are especially important for d-regular graphs (i.e. each vertex has the same degree d). A d-regular graph (for even d) on n vertices can also be considered as the Fourier transform of a Hecke operator z at the permutation representation ρ of the symmetric group S_n, where the Hecke operator z is constructed by picking $d/2$ permutations $\pi_1, \cdots, \pi_{d/2}$ independently and randomly from S_n. If there is a spectral gap, the spectral gap property of a regular graph shows that the graph is an expander, a highly-connected sparse graph. Here "sparse" is in terms of a linear relation between the number of edges and the number of vertices. A thorough survey of expanders can be found in [HLW06]. Expanders can be described by the expansion coefficient:

$$h(G) = \min_{\{S\subset|V|\,|\,|S|\leq\frac{|V|}{2}\}} \frac{|E(S,V/S)|}{|S|}, \tag{1.4}$$

where V is the vertex set of a graph G, and $|E(S,V/S)|$ is the number of edges between a subset S of the vertex set and its complement set V/S. G is an expander, if and only if, $h(G) \geq \varepsilon > 0$ for any $\varepsilon > 0$. In addition, there is a relationship by Alon and Milman ([Alon86], [AM85]) between the spectral gap property and the expansion coefficient, $\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}$ for a d-regular graph with eigenvalues $d = \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$.

Pinsker [Pinsker73] first noted random d-regular graphs are expanders (i.e. positive spectral gaps exist). Quantitatively, how large are those spectral gaps? If denoting $\lambda(G) = \max(\lambda_2, |\lambda_n|)$, for a d-regular graph on n vertices, Alon-Boppana bound [Nil91] tells that $\lambda(G) \geq 2\sqrt{d-1} - o_n(1)$.

Lubotzky, Phillips and Sarnak [LPS88] and Margulis [Margulis88] provided a celebrated construction of a family of Cayley graphs $X^{p,q}$ of $PSL_2(\mathbb{F}_q)$ with respect to a very special choice of a generating set of size $p+1$, which are $2p+2$-regular graphs satisfying $\lambda \leq 2\sqrt{2p+1}$. These particular graphs are called Ramanujan graphs. With respect to Alon-Boppana bound, the spectral gap of a Ramanujan graph is optimal asymptotically.

Alon [Alon86] conjectured that for any $d \geq 3$ and $\varepsilon > 0$, $\lambda(G) \leq 2\sqrt{d-1} + o_n(1)$ for "most" d-regular graphs on a sufficiently large number of vertices. In other words, almost every d-regular graph is near Ramanujan.

Friedman used the trace method (for example, see [BS87] and [FK81]) to attack Alon's conjecture, and obtained a bound $\lambda(G) \leq 2\sqrt{d-1} + 2\log d + c$ in [Fri91] for even d. Later, Friedman noticed that the small spectral gap comes from certain small configurations (so-called "supercritical tangles" in [Fri08]). He constructed a B-selective trace by ruling out all irreducible closed walks which have a length $< B$ subwalk tracing out any supercritical tangle of rank $\lceil \frac{\sqrt{d-1}-1}{2} \rceil$, then additionally he required that subgraphs traced out by the B-selective trace not to contain a minimal collection of the above supercritical tangles. Through this sophisticated construction, Alon's conjecture is finally proved in a monumental paper [Fri08].

Theorem 1.0.1 (Friedman)**.** *For a given* $\varepsilon > 0$ *and even* $d \geq 4$*, there exists a constant* $c = c(d, \varepsilon) > 0$*, such that for a random* d*-regular graph on* n *vertices,*

$$\lambda(G) \leq 2\sqrt{d-1} + \varepsilon \qquad \textit{with probability} \quad 1 - \frac{c}{n^{\lceil \frac{\sqrt{d-1}-1}{2} \rceil}} \tag{1.5}$$

Stemming from Linial and Puder's new machinery [LP10], Puder and Parzanchevski built a crucial theorem ([Puder11], [PP12], recalled in Theorem 2.2.6) to analyze closed walks in random n lifts of a finite connected graph. Namely, using the notation of Stallings' folding core graphs (for example, see [Sta83], [KM02], [MVW07]), they defined a quotient relation between core graphs, and noticed a cancellation among a certain collection of quotient core graphs. The cancellation can be used to simplify the work of counting the expected number of closed walks in random n lifts. Very recently, using random n lifts of a banquet of $d/2$ self loops, Puder [Puder12] provided a simpler but weaker bound of Alon conjecture. This result is followed by a simple computation of an expansion of all closed walks up to order $d/2$ using the cancellation.

Theorem 1.0.2 (Puder)**.** *For a random d-regular graph on n vertices with even d,*

$$\lambda(G) \leq 2\sqrt{d-1} + 0.84 \qquad a.s.. \tag{1.6}$$

There are other similar bounds for $\lambda(G)$ of type $C\sqrt{2d-1}$. The interested reader can refer to [Puder12] where the author provides a list of similar results.

Our work is motivated by the simplicity of counting the expected number of closed walks in random n lifts using Puder and Parzanchevski's cancellation theorem ([Puder11], [PP12]). Relying on our observation of a connection between generalized forms and core graphs, we are able to adopt the cancellation theorem to estimate the expected number of closed walks under the permutation model. The resulting contribution from the cancellation theorem can be expanded to any order by using Friedman's expansion method [Fri91]. However, high order terms from the expansion are challenging to control simultaneously and directly. Through separating a high order term into a good part and a bad part, and showing the probability of the bad part occurring is small, we solve the above key issue. As a result, we are able to arrive at the following Theorem 1.0.3. Compared with Puder's approach which leads to Theorem 1.0.2, our approach has an advantage of choosing any expansion order benefited from using Friedman's expansion method under the permutation model. Later, we will see that the freedom of choosing expansion order leads to an optimal error estimation.

Theorem 1.0.3 (Theorem 2.3.14)**.** *For a $2d$-random regular graph on n vertices with $2d \geq 6$, and any $\varepsilon > 0$, there exists an expansion order r_{exp}, such that $\lambda(G) \leq 2\sqrt{2d-1}+\varepsilon$ with probability $1 - O_{d,r_{exp}}(n^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor})$.*

The result of Theorem 1.0.3 is similar as Friedman's form of Alon's conjecture (see Theorem 1.0.1). Unlike [Fri08], where Friedman constructed an artificial B-selective trace at the beginning, then tediously computed the expansion for the B-selective trace and the expansion for the B-selective trace on graphs without certain supercritical tangles, we bypass this complexity by starting from a simpler computation of an expansion for all irreducible closed walks up to any order, and naturally separate the expansion into a good part and a bad part with small probability of the bad part occurring. Consequently, the analysis of the good part and the bad part is easier without involving B-selective trace.

Table 1.1 summarizes the major ingredients of Friedman's [Fri08], Puder's [Puder12], and our approach.

More precisely, an outline of our proof is as follows.

Step 1 The Trace method. Let A be the adjacency matrix of a random $2d$-regular graph G on n vertices. We estimate the expected number of closed walks of length k in the graph using the trace formula

$$\sum_{i=1}^{n} \lambda_i^k = tr(A^k) = \#\{\text{closed walks of length } k\} = \sum_{i=1}^{n} \sum_{\omega \in \Pi^k} I[\omega(i) = i] \,,$$

where $I[\omega(i) = i]$ is the indicator of the event that a length k word $\omega \in \Pi^k$ corresponds to a closed walk from vertex i to itself on the graph G, and $\Pi = \{\pi_1^{\pm 1}, \cdots, \pi_d^{\pm 1}\}$. For a closed walk ω from vertex $t_1 = i$ to itself with sequentially visited vertices $\mathbf{t} = \{t_2, \cdots, t_k\}$, Friedman [Fri91]

Table 1.1: Comparing between Friedman's, Puder's, and our approach.

Friedman's [Fri08]	Under the permutation model, constructing B-selective trace, then computing the expansion of B-selective trace to **any order**; next, further computing the expansion of B-selective trace on subgraphs without containing certain small configurations to **any order**; after estimating the loss probability, Friedman's Lemma 2.1.12 of complex random variables, and a Markov type of argument lead to Friedman's Theorem 1.0.1.
Puder's [Puder12]	Applying Puder and Parzanchevski's Theorem 2.2.6 to count closed walks in random n lifts of a banquet with d self loops; after cancellation among certain core graphs, computing the sizes of critical groups by a purely counting argument; next, applying co-growth formula to obtain leading terms of the expansion of all walks **up to order 2d**; finally a Markov type of argument leads to Puder's Theorem 1.0.2.
Our approach	Under the permutation model, the connection between generalized forms and core graphs (Lemma 2.3.1) makes Puder and Parzanchevski's Theorem 2.2.6 still applicable in our setting; Friedman's expansion method [Fri91] is applied to study the result from Theorem 2.2.6 to **any order**; then applying a pivotal trick of separating high order terms into a good part and a bad part, and showing the probability of bad part occurring is small; finally using Bartholdi identity (recalled in Lemma 2.3.10), Friedman's Lemma 2.1.12 of complex random variables, and a Markov type of argument lead to our Theorem 1.0.3.

defined a generalized form $\Gamma_{\omega,i,\mathbf{t}}$ to be a subgraph on all distinct vertices among $\{i, t_2, \cdots, t_k\}$, and the edge set of $\Gamma_{\omega,i,\mathbf{t}}$ contains all of generic free choice steps and coincidence steps (see Definition 2.1.1). If ignoring a particular $(\omega, i, \mathbf{t})$ and maintaining the same shape, one has an abstract generalized form Γ. Given a length k word ω, a labeling l, and an abstract generalized form Γ, the expected total number of closed walks for various $(i, \mathbf{t})$ can be easily calculated as,

$$E[\Gamma]_\omega = \sum_{\{(i,\mathbf{t})|\Gamma_{\omega,i,\mathbf{t}}=\Gamma\}} EI[\omega(i) = i] = \frac{n!}{(n-v)!} \prod_{j=1}^{d} \frac{(n-\alpha_j)!}{n!}, \tag{1.7}$$

where v is the size of vertex set of Γ, and α_j is the number of appearance of π_j and π_j^{-1} labeled edges in Γ for $j = 1, \cdots, d$.

Additionally, Friedman showed that the Taylor expansion of $E[\Gamma]_\omega$ up to any order r (see Lemma 2.1.3) is,

$$E[\Gamma]_\omega = n^{v-e}(p_0 + \frac{p_1}{n} + \cdots + \frac{p_{r-1}}{n^{r-1}} + \frac{\text{error}}{n^r}), \tag{1.8}$$

with error term $\leq \exp(\frac{rk}{n-k})k^{2r}$, where $e = |E(\Gamma)|$, $v = |V(\Gamma)|$, and k is the length of walk. Here p_i is a polynomial $p_i = p_i(v, \alpha_1, \cdots, \alpha_d)$ for $i = 0, \cdots, r-1$.

Step 2 New machinery of closed walk counting in random n lifts. Linial and Puder first [LP10] associated a word ω with a universal core graph $\Gamma_\Pi(<\omega>)$, and defined its core graph quotients $\Gamma_\Pi(H)$ with $<\omega> \overset{\Pi}{\twoheadrightarrow} H \leq F_d(\Pi)$. The universal graph of ω together with all its quotient graphs is a set denoted as Q_ω. Further, Puder and Parzanchevski ([Puder11], [PP12]) defined the primitive rank $\pi(\omega)$ and the critical group $Crit(\omega)$ of a word ω (see Definition 2.2.4). For a fixed ω, in random n lifts, Puder and Parzanchevski provided a crucial estimation (see Theorem 2.2.6) of the expected number of fixed points of ω,

$$E[\Phi_{\omega,n}] = \sum_{\Gamma \in Q_\omega} E(\Gamma) = 1 + \frac{|Crit(\omega)|}{n^{\pi(\omega)-1}} + O(\frac{1}{n^{\pi(\omega)}}), \tag{1.9}$$

where the error term can be re-estimated to be an uniform bound $\frac{Ck^{2\pi(\omega)+2}}{n^{\pi(\omega)-1}(n-k^2)}$ for all ω of length k (see Lemma 2.2.10). We provide an alternative proof of the above uniform error term in Lemma 2.3.3.

For random n lifts of a graph Ω, the expected number of fixed points of ω is the same as the expected number of closed walks in the lifts of Ω under ω , and the expected value is fully determined by ω. Therefore, the expected number of fixed points of ω under realization $\Gamma_\Pi(H) \in Q_\omega$ is the expected number of closed walks on $\Gamma_\Pi(H)$.

Step 3 A connection between generalized forms and core graphs. We observe that there is a one-to-one correspondence between generalized forms and core graphs (see Lemma 2.3.1). Obviously, the expected number of closed walks compatible with $\Gamma_{\omega,i,\mathbf{t}}$ on a $2d$-regular graph with n vertices is the same as the expected number of fixed points of $\Gamma_\Pi(H)$ in random n lifts of a graph with d edges. That is,

$$E[\Gamma]_\omega = \frac{n!}{(n-v)!}\prod_{j=1}^{d}\frac{(n-\alpha_j(\Gamma))!}{n!} = E(\Gamma_\Pi(H)). \tag{1.10}$$

The above one-to-one correspondence and Equation (1.10) enable Puder and Parzanchevski's result (Equation 1.9) to be applicable in the permutation model. Namely, by grouping closed walks according to their primitive ranks and the quotient relation, and using Puder and Parzanchevski's result (Equation 1.9), we have the summation of irreducible closed walks of length k as follows (see Equation 2.22),

$$\begin{aligned}
&\sum_i \sum_{\omega\in Irred_k} I[\omega(i)=i] \qquad (1.11)\\
\leq\ & \sum_{j=1}^{d}\sum_{\{\omega\in Irred_k|\pi(\omega)=j\}}\left(1+\frac{|Crit(\omega)|}{n^{j-1}}+\frac{Ck^{2j+2}}{n^{j-1}(n-k^2)}\right)\\
\leq\ & 2d(2d-1)^{k-1}+\sum_{j=1}^{d}\sum_{\substack{\Gamma \text{ with}\\ Coin(\Gamma)=j}}\left(E[\Gamma]_\omega I(\text{multiplicity of each edge}\geq 2)+\frac{Ck^{2j+2}}{n^{j-1}(n-k^2)}\right),
\end{aligned}$$

where $Coin(\Gamma)$ denotes the coincidence of Γ. Here we apply a useful observation by Linial and Puder (see Lemma 2.2.8) of the following equivalence:

1. H is a proper algebraic extension of $<\omega>$;

2. The closed walk ω in the core graph $\Gamma_\Pi(H)$ traces every edge at least twice.

Step 4 Friedman's expansion method. For each primitive rank $j = 1,\cdots,d$, the resulting irreducible closed walks from Equation 1.11 can be re-grouped as,

$$\begin{aligned}&\sum_{\substack{\Gamma \text{ with} \\ Coin(\Gamma)=j}} E[\Gamma]_\omega I(m_i \geq 2) \\ = &\sum_{Coin(T)=j} \sum_{\substack{m_1,\cdots,m_t \\ \text{with } m_i\geq 2}} N(T,\overrightarrow{m}) \sum_{\substack{k_1,\cdots k_t\geq 1 \\ and\ \sum_{i=1}^t m_i k_i = k}} \sum_{l\in L_{T,k_1,\cdots,k_t}} E[T,l],\end{aligned} \tag{1.12}$$

where a type T (i.e. a type is an abstract generalized form by ignoring all degree two vertices.) has edge set $\{e_1,\cdots,e_t\}$, with edge multiplicities $\overrightarrow{m} = \{m_1,\cdots,m_t\}$ and edge lengths $\overrightarrow{k} = \{k_1,\cdots,k_t\}$. k is the length of closed walks, $l \in L_{T,k_1,\cdots,k_t}$ is a labeling compatible with T of fixed edge lengths $\overrightarrow{k} = \{k_1,\cdots,k_t\}$, and $N(T,\overrightarrow{m})$ is the number of length k irreducible closed walks compatible with T of edge multiplicities $\overrightarrow{m}$.

When a type T is given, we need specify the multiplicities, the lengths, and the labels of all edges. The starting point is the following result by Friedman (see Lemma 2.1.4),

$$\sum_{l\in L_{T,k_1,\cdots,k_t}} p_i(T,l) = \sum_{K_1,K_2,K_3} (2d-1)^{|K_1|}(-1)^{|K_2|} Q_{K_1,K_2,K_3}(\overrightarrow{k}), \tag{1.13}$$

for a fixed type T, with fixed multiplicities $\overrightarrow{m}$, fixed edge lengths $\overrightarrow{k}$, and fixed coincidence $r \leq d$. And p_i is the polynomial from the Taylor expansion of $E[\Gamma_\omega]$ in Equation 1.8 with $i \leq r-1$. K_1,K_2,K_3 is a partition of the set $\overrightarrow{k} = \{k_1,\cdots,k_t\}$, with size $|K_s| = \sum_{k_j\in K_s} k_j$ for $s = 1,2,3$, and Q_{K_1,K_2,K_3} is a polynomial of degree at most $2i$, whose coefficients are bounded by $(cd)^{cd^2}$.

In our case, after cancellation, since we are only left with irreducible closed walks with edge multiplicities being larger than or equal to two, the computation of the expansion is easier than Friedman's original version [Fri91]. Essentially, the multiplicity condition reduces the chance of a closed walk passing through certain bad configurations.

Proceeding along the lines of Friedman's expansion method [Fri91], we finally obtain the following result (see Proposition 2.3.5):

$$\sum_{\substack{\Gamma \text{ with} \\ Coin(\Gamma)=j}} E[\Gamma]_\omega I(m_i \geq 2) = \frac{f_{j-1}}{n^{j-1}} + \cdots + \frac{f_{r_{exp}-1}}{n^{r_{exp}-1}} + \frac{\varepsilon_j}{n^{r_{exp}}}, \tag{1.14}$$

where the expansion order r_{exp} will be chosen later. Thus, we have

$$\begin{aligned} & f_{j-1}, \cdots, f_{r_{exp}-1} \\ \leq & \begin{cases} (2d-1)^{\frac{k}{2}} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } 1 \leq j \leq \lfloor (\sqrt{2d-1}+1)/2 \rfloor \\ (2d-1)^{\frac{k}{2}} (\frac{2j-1}{\sqrt{2d-1}})^{k/2} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } \lfloor (\sqrt{2d-1}+1)/2 \rfloor < j \leq d \end{cases} \end{aligned} \tag{1.15}$$

with error $\varepsilon_j \leq C 2d (2d-1)^{k-1} k^{4j+2r_{exp}} n^{1-j-r_{exp}}$.

Step 5 Controlling high order terms from expansion. The expansion (Equation 1.14) is true for any expansion order r_{exp}. However, when r_{exp} is larger than $\lfloor (\sqrt{2d-1}+1)/2 \rfloor$, we have contributions of type $(2d-1)^{\frac{k}{2}} (\frac{2l-1}{\sqrt{2d-1}})^{k/2} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}}$. It is difficult to control all of those high order contributions simultaneously and directly. Through separating those high order terms into a good part and a bad part, and showing the probability of the bad part occurring is small, we solve this key issue. Roughly,

$$\begin{aligned} & (2d-1)^{\frac{k}{2}} (\frac{2l-1}{\sqrt{2d-1}})^{k/2} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}} \\ = & \underbrace{(2d-1)^{\frac{k}{2}} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}}}_{good} + \underbrace{(2d-1)^{\frac{k}{2}} ((\frac{2l-1}{\sqrt{2d-1}})^{k/2} - 1) k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}}}_{bad} \end{aligned} \tag{1.16}$$

Using the spectral method, the number of all length k closed walks in a type T with edge multiplicities $\overrightarrow{m}$ can be easily computed to be $O(k^c(\rho_T)^{m/2})$, where $m = \sum_i m_i$ and ρ_T is the spectral radius of T. Comparing this result with the above good part, the proof of Equation 1.14 and Equation 1.16 tells that we need restrict all irreducible closed walks to be those not tracing out a subgraph H with $\lambda_{irred}(H) \geq \sqrt{2d-1}$. As a result, irreducible closed walks with the above property will be our good part, and the rest is the bad part. After determining the good part and the bad part combinatorially, we will estimate the probability of the bad part occurring in the next step.

Step 6 Bartholdi identity. A very useful tool to study the spectrum of a regular graph is Bartholdi identity ([OS09], recalled in Lemma 2.3.10),

$$\det(I^{2|E|} - s(B-J)) = (1-s^2)^{|E|-|V|}\det((1+(2d-1)s^2)I^{|V|} - sA), \tag{1.17}$$

where A is the adjacency matrix of a $2d$-regular graph $G = (V, E)$, B is a $2E \times 2E$ matrix describing the length 2 irreducible walks of the graph G in terms of its directed edges, and J is a $2E \times 2E$ matrix describing the length 2 back-tracking walks.

Furthermore, the spectrum of $B - J$ can be read out as,

$$\begin{aligned}\sigma(B-J) \quad = \quad & \Big\{2d-1, 1, 1\times(|E|-|V|), -1\times(|E|-|V|), \\ & \sqrt{2d-1}e^{i\phi_i}, \sqrt{2d-1}e^{-i\phi_i}, \text{ with } \phi_i = \arccos(\frac{\lambda_i}{2\sqrt{2d-1}}) \text{ for } i = 2, \cdots, |V|\Big\},\end{aligned} \tag{1.18}$$

where $2d = \lambda_1 \geq \lambda_2 \cdots \geq \lambda_{|V|}$ are all the eigenvalues of the graph G.

From Bartholdi identity, we are able to read out the spectrum of directional $G_{irred} = B - J$. We realize that those bad part could be a banquet with $\lfloor(\sqrt{2d-1}+1)/2\rfloor$ self loops. Luckily, banquet with $\lfloor(\sqrt{2d-1}+1)/2\rfloor$ self loops occurs with maximal probability

$n^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor}$ in all of those bad part subgraphs. Also, it is easy to check that the probability of more than two occurrences of bad subgraph is smaller than $n^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor}$, and there are only finite many bad subgraphs. Therefore, the total loss probability is bounded by $O(n^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor})$.

Again, with help of Bartholdi identity, we are able to rewrite the eigenvalues of G into a new form

$$\mu_{1,2}(\lambda_i) = \frac{\lambda_i \pm \sqrt{\lambda_i - 4(2d-1)}}{2} \text{ for } i = 2, \cdots, |V| = n. \tag{1.19}$$

The new eigenvalue forms $\mu_{1,2}(\lambda_i)$ enable us to separate exceptional eigenvalues (i.e. $\lambda_i \geq 2\sqrt{2d-1}$) and well-tempered eigenvalues (i.e. $\lambda_i < 2\sqrt{2d-1}$).

Step 7 A sidestepping lemma of complex random variables. The very last tool is a lemma of complex random variables from Friedman [Fri08], see Lemma 2.1.12. Very roughly, for complex random variables $\theta_1, \cdots, \theta_{cn}$, from the binomial expansion of order r, one would expect

$$E(\sum_{i=1}^{cn}(1-\theta_i)^k) = \sum_{j=0}^{r-1} p_j(k)n^{-j} + O(k^{r'} n^{-r}) \text{ for some polynomial } p_j(k), \tag{1.20}$$

where $r' = r'(r)$ is a constant. The lemma tells that there is a similar expansion for $E(\sum_{i=1}^{cn} \chi_{|\theta_i|>\log^{-2} n}(1-\theta_i)^k)$, where χ is the indicator function.

By the lemma of complex random variables (Lemma 2.1.12), and a Markov inequality type standard argument, we obtain the main Theorem 1.0.3.

The second part of the dissertation focus on the connection between the spectra of random matrices and the spectra of random d-regular graphs/random Hecke operators. Jakobson, Miller and Rivin [JMR96] indicated that the level spacing distribution of a generic regular graph approaches that of the Gaussian orthogonal ensemble of random matrix theory. Later, Miller, Novikoff and Sabelli [MNS08] pointed out that the edge spectra of families of random regular graphs could be well modeled by $\beta = 1$ Tracy-Widom distribution. However, due to the lack of proper normalization, the connection between the spectra of random regular graphs and the spectra of random matrices is still not fully understood.

In Chapter 3, we provide a normalization factor by matching the first four moments of general β ensembles with the first four moments of random regular graphs/random Hecke operators. It is found that the normalization factor does not depend on the value of β, and the dimension of a random matrix depends linearly on the dimension of a regular graph/Hecke operator. Random d-regular graphs/random Hecke operators can be normalized in a similar manner as random matrices, but with an extra factor of $\sqrt{\frac{n}{d-1}}$.

Conjecture 1.0.4 (Conjecture 3.0.17)**.** *For a random d-regular graph on N vertices, or a random Hecke operator with $d/2$ independent and random generators under N-dimensional irreducible representation, its edge spectrum λ_1, λ_N, and $\lambda_\pm = \max(\lambda_1, |\lambda_N|)$ can be normalized as*

$$\widetilde{\lambda_{edge}} = n^{1/6}(\lambda_{edge} - \mu_{sample})\sqrt{\frac{n}{d-1}}, \tag{1.21}$$

where $n \sim \frac{2d}{2d-1}N$, and μ_{sample} is the sample mean of the edge spectrum.

Besides random d-regular graphs, random Cayley graphs over $SL_2(\mathbb{F}_p)$ are another important type of random Hecke operators. Bourgain and Gamburd [BGSL08] proved the existence of the uniform spectral gaps for Cayley graphs over $SL_2(\mathbb{F}_p)$, and the existence of the spectral gaps of random Hecke operators over $SU(2)$ [BGSU08] and $SU(N)$ [BG12]. We carry out numerical studies of the edge spectra of random Cayley graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ and random Hecke operators over $SU(2)$ in Chapter 4, 5. These numerical studies conclude that the normalization factor is valid and explore the connection between the edge spectra of random Hecke operators and random matrices.

In Chapter 4, all irreducible representations of $SL_2(\mathbb{F}_p)$ are reviewed at the beginning. There are $\frac{p+5}{2}$ principal irreducible representations and $\frac{p+3}{2}$ discrete irreducible representations. The adjacency matrix of a Cayley graph $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ can be decomposed into $p+4$ diagonal blocks. The normalized (Chapter 3) edge spectra of the Cayley graph blocks are shown to fit with certain Tracy-Widom distributions through numerical experiments. In order to obtain a more conclusive evidence, the Kolmogorov-Smirnov tests are performed, and the P-values are computed. When the dimension is large enough, a supportive conclusion is obtained. Also for all prime number p, sample means of the edge spectra of all blocks are shown on the left side of the Ramanujan bound, which converge to the Ramanujan bound as dimension increases.

In Chapter 5, similar numerical experiments of random Hecke operators over $SU(2)$ are carried out. $SU(2)$ only has one type of irreducible representations $Sym^N V$ (with V to be standard two dimensional representation). With the normalization factor (Chapter 3), edge spectra are shown to fit with certain Tracy-Widom distributions. The sample mean is also shown on the left of the Ramanujan bound and converges to the Ramanujan bound.

Chapter 2

Expansion of Random Regular Graphs

In this chapter, we provide an alternative proof of Alon's conjecture for $2d$-regular graphs in a simpler way. Our approach contains five major ingredients: Linial and Puder's new machinery [LP10] along with the crucial cancellation result by Puder and Parzanchevski ([Puder11], [PP12]) of counting the expected number of closed walks in random n lifts; our observation of a connection between core graphs and generalized forms (see Lemma 2.3.1); Friedman's earlier expansion method on random regular graphs [Fri91]; our key trick (see Equation 2.36) of controlling all high order terms in the expansion indirectly; and Friedman's sidestepping lemma of complex random variables [Fri08].

We start this chapter by reviewing Friedman's expansion method of regular graphs under the permutation model in Section 2.1. Then we revisit Linial, Puder and Parzanchevski's new framework of counting closed walks in random n lifts in Section 2.2. We complete the proof of $\lambda(G) \leq 2\sqrt{2d-1}+\varepsilon$ with probability $1-O(n^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor})$ in Section 2.3.

2.1 A Brief Review of Friedman's Approach

2.1.1 Expansion Method for Random Regular Graphs by Friedman

Friedman's earlier approach [Fri91] is based on the trace method, where he was working under the permutation model of random $2d$-regular graphs on n vertices. A $2d$-regular graph under the permutation model can be constructed as follows: pick d permutations $\pi_1, \cdots, \pi_d$ independently and randomly, then the edge set is defined to be

$$E = \{(i, \pi_j(i)), (i, \pi_j^{-1}(i)) | j = 1, \cdots, d \;\; i = 1, \cdots, n\} .$$

In the permutation model, any regular graph automatically carries with labels and directions. Namely, edge $(i, \pi_j(i))$ from vertex i to vertex $\pi_j(i)$ is labeled by π_i, and edge $(i, \pi_j^{-1}(i))$ from vertex i to vertex $\pi_j^{-1}(i)$ is labeled by π_i^{-1} (One has the freedom of reversing the direction of π^{-1} labeled edge by changing the label to be π). Thus, any length k walk on graph G will give rise to a length k formal word ω from a rank d free group $F_d(\Pi)$ with letters $\Pi = \{\pi_1^{\pm}, \cdots, \pi_d^{\pm}\}$.

Let $\varepsilon_1, \cdots, \varepsilon_k = \pm 1$. A length k formal word $\omega = \pi_1^{\varepsilon_1} \cdots \pi_k^{\varepsilon_k} \in \Pi^k$ in free group $F_d(\Pi)$ with letters $\Pi = \{\pi_1^{\pm 1}, \cdots, \pi_d^{\pm 1}\}$ will correspond to a walk of length k on graph G from vertex i_0 to vertex i_k if there exists a sequence of vertices $i_0, \cdots, i_k$ (the vertices may not be distinct) such that $\pi_s(i_{s-1}) = i_s$ for all $s = 1, \cdots, k$,

$$i_0 \xrightarrow{\pi_1} i_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_k} i_k ,$$

where the indicator of the above event is denoted by $I[\omega(i_0) = i_k]$.

Moreover, if $i_0 = i_k$, the walk is a closed walk. If a word ω is reduced without any consecutive pair of form $\pi\pi^{-1}$ for a letter $\pi \in \Pi$, this word is called irreducible. The set of irreducible words of length k is denoted by $Irred_k$. Now, one has the trace formula for k-th power adjacency matrix to be

$$\sum_{i=1}^{n}\lambda_i^k = tr(A^k) = \#\{\text{closed walks of length } k\} = \sum_{i=1}^{n}\sum_{\omega\in\Pi^k} I[\omega(i)=i], \tag{2.1}$$
$$\sum_{i=1}^{n} q_k(\lambda_i) = tr(q_k(A)) = \#\{\text{irreducible closed walks of length } k\} = \sum_{i=1}^{n}\sum_{\omega\in Irred_k} I[\omega(i)=i],$$

where $q_k(2\sqrt{2d-1}\cos\theta) = (2d-1)^k\left(\frac{2}{2d-1}\cos k\theta + \frac{2d-2}{2d-1}\frac{\sin(k+1)\theta}{\sin\theta}\right)$, see [LPS86], or [LPS88].

The choices and the outcomes of $\pi_s(i_{s-1}) = i_s$ are divided into three cases [Fri91]:

Definition 2.1.1. • *If i_s has already been determined by previous steps such that there is no freedom to choose, we call this s-step a* ***forced choice****;*

- *If i_s has not been determined, we call this s-step a free choice; For a free choice, there are still two cases. If i_s appears previously (i.e., $\pi_s(i_{s-1})$ takes one of previous values $i_0,\cdots,i_{s-1}$), we call it a* ***coincidence****; otherwise, we call it a* ***generic free choice****, where i_s dose not appear previously (i.e., $\pi_s(i_{s-1})$ takes one of $n-m$ new values, where $m \le s-1$);*

It's easy to see that step s will be a coincidence with probability less than or equal to $\frac{s}{n-s+1} \le \frac{k}{n-k}$.

For a closed walk from vertex i to itself on a labeled graph, its edges' labels give rise to a word $\omega = \pi_1\cdots\pi_k$ and a sequence of vertices $\mathbf{t} = (t_2,\cdots,t_k)$. Associated with this walk, one can define a directed and labeled graph, a **generalized form** $\Gamma_{\omega,i,\mathbf{t}}$. The vertex set of $\Gamma_{\omega,i,\mathbf{t}}$ contains all distinct vertices among $\mathbf{t}\cup\{i\}$, denoted as $V_{\Gamma_{\omega,i,\mathbf{t}}} = \{i_1,\cdots,i_{|V_{\Gamma_{\omega,i,\mathbf{t}}}|}\}$. And the edge set E_Γ contains all directed free choice steps (including coincidence steps), with edges labeled by the word. If ignoring the particular $(\omega,i,\mathbf{t})$ and grouping all generalized forms with the same shape, one has an **abstract generalized form**, $\Gamma = (V_\Gamma, E_\Gamma)$. Given an abstract generalized form Γ, a fixed compatible word ω, and a fixed compatible label l, one has $n(n-1)\cdots(n-|V_\Gamma|)$

ways to pick different vertices. After vertices are fixed, there are $\alpha_j(\Gamma)$ conditions on π_j and π_j^{-1} labeled edges, where $\alpha_j(\Gamma)$ is the number of the appearances of π_j and π_j^{-1} labeled edges. Therefore, it's easy to see that the expected number of closed walks which are compatible with Γ, ω and label l is,

$$E[\Gamma]_\omega = \sum_{\{(i,\mathbf{t})|\Gamma_{\omega,i,\mathbf{t}}=\Gamma\}} EI[\omega(i)=i] = \frac{n!}{(n-v)!}\prod_{j=1}^{d}\frac{(n-\alpha_j)!}{n!}. \tag{2.2}$$

The abstract generalized forms can be further grouped based on their "topological shapes". One obtains a **type** $T=(V_T,E_T)$ by ignoring all degree two vertices in the abstract generalized form. For each $e_i \in E_T$, the length $k_i \geq 1$ is defined to be one plus the number of the erased degree 2 vertices on this edge, and the label is a length k_i word $\omega_i \in \Pi^{k_i}$. When a walk goes through the type, it might cross an edge multiple times. One can denote the multiplicity to be $m_i \geq 1$ for edge e_i. Thus, when given a type T, lengths $\overrightarrow{k}=(k_1,\cdots,k_{|E(T)|})$ and a label l, $E[\Gamma_\omega]$ of Equation (2.2) is fully determined and can be denoted by $E[T,l]$ alternatively.

If $N(T,\overrightarrow{m})$ is the number of words which are compatible with a type T with multiplicity $\overrightarrow{m}=\{m_1,\cdots,m_t\}$, and if $L_{T,k_1,\cdots,k_t}$ contains all the labelings which are compatible with the type T of fixed edge lengths $k_1,\cdots,k_t$, Friedman had the following decomposition [Fri91],

$$\sum_i\sum_{\omega\in\Pi^k} I[\omega(i)=i] = \sum_T\sum_{m_1,\cdots,m_t} N(T,\overrightarrow{m}) \sum_{\substack{k_1,\cdots k_t\geq 1\\ and\ \sum m_i k_i=k}}\ \sum_{l\in L_{T,k_1,\cdots,k_t}} E[T,l]. \tag{2.3}$$

We list below some essential properties (Lemma 2.1-2.6 of [Fri91]) of generalized forms Γ and types T. For any $\Gamma \in T$ with $T=(V,E)$,

- Coincidence number is $Coin(\Gamma)=Coin(T)=|E|-|V|+1$;
- If T has coincidence r, then $|V|\leq 2r$ and $|E|\leq 3r-1$;
- The number of types of coincidence r is less than $(2r)^{6r-2}$;

- The maximal degree of a type with coincidence r is $2r$ if no leaf is connected to a banquet of r self loops; The minimal degree of a type is 2 except possible degree one at leaves;
- For a fixed type T of coincidence r, there are at most $(2r)^m$ walks which go through this type T with multiplicity $\overrightarrow{m} = \{m_1, \cdots, m_t\}$, where $m = \sum_i m_i$.

Here one more lemma from the above properties,

Lemma 2.1.2 (A claim in the proof of Theorem 2.18 of [Fri91]). *For a given word ω of length k, the probability of coincidence $\geq r$ is bounded by $n\binom{k}{r}(\frac{k}{n-k})^r \leq k^{2r}n^{-r+1}$. On the other hand, for a given ω of length k, there are at most $2k^{2r}$ compatible Γ with coincidence $\leq r-1$.*

Friedman analyzed $E[\Gamma]_\omega$ by its Taylor expansion.

Lemma 2.1.3 (Lemma 2.7 of [Fri91]). *$E[\Gamma]_\omega$ of Equation (2.2) has expansion for any integer $r \geq 0$, denoting $v = |V(\Gamma)|$ and $e = |E(\Gamma)|$,*

$$E[\Gamma]_\omega = n^{v-e}(p_0 + \frac{p_1}{n} + \cdots + \frac{p_{r-1}}{n^{r-1}} + \frac{error}{n^r}), \tag{2.4}$$

where $error \leq \exp(\frac{rk}{n-k})k^{2r}$, and p_i is polynomial $p_i = p_i(v, \alpha_1, \cdots, \alpha_d)$ with $\alpha_j = \#|\{\pi_j, \pi_j^{-1}$ labeled edge in $\Gamma\}|$ for $j = 1, \cdots, d$.

If fixing a type $T = (V_T, E_T)$ with $E_T = (e_1, \cdots, e_t)$, fixing the multiplicities $\overrightarrow{m} = (m_1, \cdots, m_t)$ and the edge lengths $\overrightarrow{k} = (k_1, \cdots, k_t)$, then one is left with the only freedom of the labeling l. The set of all possible labelings compatible with the fixed type T with edge lengths $\overrightarrow{k}$ is denoted by $L_{T,k_1,\cdots,k_t}$. Then,

Lemma 2.1.4 (Lemma 2.13 of [Fri91]). *For a fixed type T, fixed multiplicities $\overrightarrow{m}$, fixed edge lengths $\overrightarrow{k}$, and fixed coincidence $r \leq d$, for p_i from the expansion of $E[\Gamma]_\omega$ in Lemma 2.1.3 with* *any $i \leq r-1$, one has,*

$$\sum_{l \in L_{T,k_1,\cdots,k_t}} p_i(T,l) = \sum_{K_1,K_2,K_3} (2d-1)^{|K_1|}(-1)^{|K_2|} Q_{K_1,K_2,K_3}(\overrightarrow{k}), \tag{2.5}$$

where K_1, K_2, K_3 is a partition of $\overrightarrow{k} = \{k_1, \cdots, k_t\}$, with size $|K_s| = \sum_{k_j \in K_s} k_j$ for $s = 1,2,3$. And Q_{K_1,K_2,K_3} is a polynomial of degree at most $2i$, whose coefficients are bounded by $(cd)^{cd^2}$.

Remark 2.1.5. *The proof of Lemma 2.1.4 has a tiny error of bounding of the coefficients, original bound $(cdr)^{cr}$ should be $(cdr)^{cr^2}$ as above, though the error does not affect Friedman's final result in [Fri91]. On the other hand, from Friedman's proof, we realize that the lemma is still true for any integer i which is larger than d. But in this case, the coefficients will be bounded by $(cdi)^{cdi}$.*

With Lemma 2.1.4, the following result is obtained by relaxing the condition on the lengths of all edges in a type T.

Lemma 2.1.6 (Lemma 2.14 of [Fri91])**.** *For a fixed type T with coincidence $\leq r$, fixed multiplicities $\overrightarrow{m}$, and fixed $k = \sum_1^t k_i m_i$, for any $i \leq r-1$, one has*

$$\sum_{\substack{\sum_1^t k_i m_i = k \text{ with} \\ k_i \geq 1 \text{ and } k \geq \sum_1^t m_i}} \sum_{l \in L_{T,k_1,\cdots,k_t}} p_i(T,l) = (2d-1)^{k+t-m} P_i(k) + \varepsilon, \tag{2.6}$$

where $\varepsilon \leq (2d-1)^{\frac{k-m}{2}} k^{t+2i} (cdr+m)^{cr^2}$, and $P_i(k)$ is a polynomial of degree $t+2i$ with coefficients bounded by $(cdr+m)^{cr^2}$ for some c, and $m = \sum_1^t m_i$.

Remark 2.1.7. *We will develop a similar result as Lemma 2.1.6 on irreducible closed walks with $m_i \geq 2$, and the result will be presented in Lemma 2.3.4.*

With Lemma 2.1.6, one still needs to sum over all possible multiplicities and all types. After that, to achieve the result in [Fri91], another two ingredients are needed, magnification inequalities and equation $A^k = \sum N_{k,s} q_s(A)$. Here $N_{k,s}$ is the number of length k words which are reduced to a given irreducible word of length s, q is a polynomial in Equation 2.1. On the other hand, magnification inequalities are

Lemma 2.1.8 (Theorem 3.1 of [Fri91])**.** *For any $2d$-regular graph G, there exists a $\alpha > 0$ such that*

$$Pr(|\lambda(G)| \leq 2d - \alpha) = 1 - \frac{1}{n^{d-1}} + O(\frac{1}{n^{2d-2}}), \tag{2.7}$$
$$Pr(\lambda(G) = 2d) = \frac{1}{n^{d-1}} + O(\frac{1}{n^{2d-2}}).$$

The magnification inequalities show that expansion polynomials vanish except the zero-th order f_0. Those ingredients lead to Friedman's earlier bound [Fri91] of Alon's conjecture $\lambda(G) \leq 2\sqrt{2d-1}(1 + 2\log d + c)$ for $2d$-regular graphs.

Remark 2.1.9. *The first magnification inequality was refined in [Fri08] as $Pr(|\lambda(G)| \leq 2d - \alpha) = 1 - O(n^{\lfloor (\sqrt{2d-1}+1)/2 \rfloor})$.*

2.1.2 Strong Form of Alon's Conjecture by Friedman

Friedman [Fri08] proved a strong form of Alon's conjecture $\lambda(G) \leq 2\sqrt{2d-1} + \varepsilon$ for random $2d$-regular graphs with probability $1 - O_{d,\varepsilon}(n^{-\lfloor (\sqrt{2d-1}+1)/2 \rfloor})$. In order to compare his new approach with the earlier version (See Subsection 2.1.1), the basic ideas are very briefly reviewed in this subsection. The interested reader can refer to [Fri08] for more details.

Fact 2.1.10 (See [SW49], or Theorem 3.5 of [Fri08])**.** *Given a graph $G = (V,E)$, a graph G_{irred} is defined on vertex set $E(G)$, where an edge from vertex e_1 to e_2 exists if and only if $e_1 e_2$ is a length 2 irreducible walk in G. Friedman defined supercritical tangles to be graphs*

H with $\lambda_{irred} \geq \sqrt{d-1}$*, where* λ_{irred} *is the largest eigenvalue of* H_{irred}*. Moreover, if* $f(u,v) = \sum_0^\infty c_G(u,v;k)z^k$ *in which* $c_G(u,v;k)$ *denotes the number of length k irreducible walks from u to v on graph G, then* $1/\lambda_{irred}(G)$ *is equal to the radius of convergence of f.*

With the notation of G_{irred} and with respect to inclusions, a minimum set $\Psi(r)$ of supercritical tangles (i.e. subgraphs with $\lambda_{irred} \geq \sqrt{2d-1}$) with coincidences at most r exists, and the size of $\Psi(r)$ is finite. A type T is modified to be a B-**new type** $\tilde{T} = (T, E_{long}, E_{fixed}, \overrightarrow{k}^{fixed})$ (see Chapter 5 of [Fri08]), with the lengths of its edges from E_{fixed} fixed and $0 \leq k_i^{fixed} \leq B$, while the edges from E_{long} have lengths $k_i^{long} > B$. In a B-new type selective trace, Friedman required irreducible closed walks not to have any length $< B$ subwalk which traces out a supercritical tangle. However, this selective trace is not equal to the summation of normal irreducible closed walks unless the graph is free of supercritical tangles. Consequently, subgraphs traced out by B-selective walks are additionally required not to contain a minimal set of supercritical tangles $\Psi(r)$. Let $\chi_{\Psi(r)}$ be the indicator of a B-selective trace on a subgraph which contains elements from $\Psi(r)$. Thus, $E(1-\chi_{\Psi_{min}(r)})SSIT_{S,\Psi}(G,k) = E(1-\chi_{\Psi(r)})SIT(G,k)$, where $SIT(G,K)$ denotes all strong irreducible walks of length k (i.e. irreducible closed walks with $e_1 \neq e_k^{-1}$) and $SSIT$ denotes selective strong irreducible walks. Under this new framework, B-selective trace has a similar expansion (Chapter 8 and Chapter 9 of [Fri08]) as in [Fri91]. Denotes $M_1 = m_1 + \cdots + m_t$ (i.e. M_1 is the summation of multiplicities of all long edges), $M_2 = m_{t+1} + \cdots + m_b$ (i.e. M_2 is the summation of multiplicities of all fixed length edges), and $m = M_1 + M_2$ (i.e. m is the summation of multiplicities of all edges). Since $m = M_1 + M_2 \leq BM_1 + M_2$, the contribution from B-selective traces would have a gain from $(2d-1)^{\frac{k-m}{2}}$ to $(2d-1)^{\frac{k-BM_1-M_2}{2}}$. In fact, this is the motivation of Friedman's construction of B-selective trace. However, the expansion of a B-selective trace is tedious, and the expansion of the B-selective trace on a graph that is free of a minimal set of supercritical tangles is even more complicated.

Meanwhile, Friedman used the following fact to bound the number of irreducible walks with given edge multiplicities in a B-new type.

Fact 2.1.11 (A general fact used in Theorem 6.6 of [Fri08]). *For power series $f(z)=\sum_{k=0}^{\infty} a_k z^k$ and $g(z)=\sum_{k=0}^{\infty} b_k z^k$ with all coefficients $a_k, b_k \geq 0$. $f(z)$ majorizes $g(z)$ if and only if $\sum_{k=0}^{j} a_k \geq \sum_{k=0}^{j} b_k$ for any $j \geq 0$. Therefore, $b_k \leq f(z_0) z_0^{-k}$ for $z_0 \in (0,1)$.*

With a lemma of complex random variables (Chapter 11 of [Fri08]), he successfully controlled the behavior of the eigenvalues which are not close to $2d$.

Lemma 2.1.12. *(Lemma 11.1 of [Fri08]) Fix integer r, r', $d \geq 2$, polynomials $p_0, \cdots, p_r$, a constant c, and integer D. Assume there are $m = nD$ complex valued random variables, $\theta_1, \cdots, \theta_m$ with $\max_i |1-\theta_i| \leq 1$, and assume $1-\theta_i$ is real if and only if $|\theta_i| \geq (2d-1)^{-1/2}$. Further if all θ_i satisfy*

$$E\left(\sum_{i=1}^{m} (1-\theta_i)^k\right) = \sum_{j=0}^{r-1} p_j(k) n^{-j} + O\left(k^{r'} n^{-r} + k^c (2d-1)^{-k/2}\right) \tag{2.8}$$

then for a sufficiently large n, one has

$$E\left(\sum_{i=1}^{m} \chi_{|\theta_i| > \log^{-2} n} (1-\theta_i)^k\right) = O_{d,r}\left(D n^{1-(r/3)} + k^c (2d-1)^{-k/2}\right) \tag{2.9}$$

After applying the above sidestepping Lemma 2.1.12 to the expansion result, a standard Markov inequality type argument leads to Friedman's final conclusion, for a $2d$-regular graph, for any $\varepsilon > 0$, there exists a $c = c(d,\varepsilon) > 0$, such that $\lambda(G) \leq 2\sqrt{2d-1} + \varepsilon$ with probability $1 - cn^{-\lfloor(\sqrt{2d-1}+1)/2\rfloor}$.

2.2 New Framework on Closed Walks Counting in Random n Lifts by Linial, Puder and Parzanchevski

In studying the spectra of random graph lifts, Linial and Puder [LP10] first use the notation of core graphs of subgroups of a free group to study closed walks in random n lifts. In the following, we review some basic facts about their new machinery. The interested reader can refer to [LP10] for details.

Given a finite connected graph Ω with d labeled edges $x_1, \cdots, x_d$, let random n lifts of Ω be Υ as follows: first define the vertex set of Υ to be $V(\Omega) \times \{1, \cdots, n\}$; then one can independently and uniformly pick d permutations $\sigma_1, \cdots, \sigma_d$ from the symmetric group S_n; for each edge $e = (u, v) \in E(\Omega)$, one can define n edges in the lifts Υ of form $((u,i),(v,\sigma_e(i)))$ for $i = 1, \cdots, n$. If denote $\varepsilon_1, \cdots, \varepsilon_k = \pm 1$, any length k closed walk in Ω of form $x_{j_i}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k}$ will correspond to a permutation $\omega(\sigma_1, \cdots, \sigma_d)$ by substituting x_j with σ_j for $j = 1, \cdots, d$. A closed walk in Ω might not be lifted as a closed walk unless there is a fixed point of $\omega(\sigma_1, \cdots, \sigma_d)$. Thus, if denoting $\phi_{\omega,n}$ to be the number of fixed points of $\omega(\sigma_1, \cdots, \sigma_d)$, one has

$$E(\phi_{\omega,n}) = E(\text{the number of closed walks in random n lifts}). \tag{2.10}$$

Further, Linial and Puder [LP10] associated a word ω with a **core graph**. The original concept of core graphs applies to every subgroup of a free group $H \leq F_d(\Pi)$. A word ω corresponds a special case $H =< \omega >$. In general, the core graph of H can be constructed as follows ([Puder11], [PP12]). At first one constructs Schreier right coset graph $\overline{\Gamma}_\Pi(H)$ of H with respect to some basis Π of the free group $F_d(\Pi)$. The vertices of $\overline{\Gamma}_\Pi(H)$ are all right cosets Hu with a base point $H1$. For any two vertices Hu and Hv, if there is a $x_i \in \Pi$ such that $Hu = Hvx_i$, then an edge pointing from Hv to Hu is added. This edge is labeled by the letter x_i. We fix the orientation by reversing the direction of x^{-1} labeled edge and changing the label to be x. As a result, the **core graph** $\Gamma_{\Pi}(H)$ is the union of all irreducible (non-backtracking) walks from base point H to itself.

Moreover, if H is a finitely generated group $H = < h_1, \cdots, h_r >$ where h_i is non-identity and reduced with respect to the basis Π, $\Gamma_{\Pi}(H)$ can be effectively constructed as follows ([KM02], [MVW07], etc): pick a base point and construct r simple closed walks from the base point to itself; then each closed walk is labeled by a word h_i, and its orientation is fixed by reversing the direction of x^{-1} labeled edge and changing label to x. Roughly, one will have a bouquet shape graph. Then possibly Stallings folding is needed by identifying two edges e_1 and e_2 to be a single edge, if they have the same label x_j, and if they have either the same starting point or the same ending point. The folded new edge inherits the same label and the same direction. One can show that this process terminates in finite steps, and the resulting graph is a core graph $\Gamma_{\Pi}(H)$ such that $\pi_1(\Gamma_{\Pi}(H)) = H$, and the core graph dose not depend on the order of the foldings ([KM02], [MVW07], etc).

We list several important properties of core graphs as follows.

Fact 2.2.1. *(Lemma 3.3, Lemma 6.1, and Lemma 8.2 of [KM02])*

1. *There is a way to construct a free generating set of H on $\Gamma_{\Pi}(H)$ as follows: at first, one can pick a spanning tree T for $\Gamma_{\Pi}(H)$. Assume edges $e_1, \cdots, e_s$ are outside the spanning tree T. For one of such edge e, there is an unique irreducible walk p_1 on the spanning tree from the base point to the starting point of e, as well as an unique irreducible walk p_2 on the spanning tree from the base point to the ending point of e. Thus, the whole closed walk $p_1 e p_2^{-1}$ is an irreducible walk from the base point to itself. It is not hard to*

check that H is generated by $\{p_{j_1}e_jp_{j_2}^{-1}$, for $j = 1,\cdots,s\}$, and $H = \pi_1(\Gamma_\Pi(H))$ is a free subgroup of F_d. Also H is independent of the choice of a spanning tree.

2. *Item [1.] is true for any folded labeled directed finite graph.*

3. *From the construction, the rank $rk(H) = |E(\Gamma_\Pi(H))| - |E(T)| = |E(\Gamma_\Pi)| - |V(\Gamma_\Pi)| + 1$.*

There is also a one-to-one correspondence between finitely generated subgroups of $F_d(\Pi)$ and finite core graphs labeled by Π (see [MVW07]):

$$\{\text{(finitely) generated subgroups of } F_d(\Pi)\} \rightleftharpoons \{\text{(finite) core graphs labeled by } \Pi\} \tag{2.11}$$

With the notation of core graph morphism, Puder and Parzanchevski ([Puder11], [PP12]) grouped certain core graphs together. A core graph morphism is a graph morphism between two core graphs, if it preserves incidence relation, the base point, and the labels. We list some core graph morphism properties as follows (see [Sta83], [KM02], [MVW07], or [Puder11]):

- A morphism $\eta : \Gamma_\Pi(H_1) \to \Gamma_\Pi(H_2)$ exists if and only if $H_1 \leq H_2$;
- A surjective morphism is called covering, denoted by $\Gamma_1 \twoheadrightarrow \Gamma_2$ or $H_1 \overset{\Pi}{\twoheadrightarrow} H_2$; Γ_2 is called quotient of Γ_1, and H_1 is called cover of H_2;
- A finite core graph only has finite number of quotients.

In [LP10], for a given word ω, Linial and Puder defined the universal graph of ω to be $\Gamma_\Pi(<\omega>)$. Then they grouped together all quotient graphs $\Gamma_\Pi(H)$ of core graphs $\Gamma_\Pi(<\omega>)$ for all $<w> \overset{\Pi}{\twoheadrightarrow} H \leq F_d(\Pi)$. This set is denoted by Q_ω.

Remark 2.2.2. *Linial and Puder [LP10] also defined quotient graphs in a combinatorial way. They noticed all quotient graphs can be generated as follows. First, one can divide the vertex set* $i_0, \cdots, i_k$ *of a length k walk* ω *into blocks, and then identify all vertices in the same block on the universal graph. After possible finite steps Stallings foldings, this procedure generates a quotient core graph of the universal core graph. A quotient graph* H_1 *is also called an immediate quotient of* H_2 *if* H_1 *can be obtained by identifying a single pair of vertices in* H_2.

By the construction from Remark 2.2.2, an easy observation tells,

Lemma 2.2.3 (Lemma 18 of [LP10])**.** $|\{\Gamma \in Q_\omega | rk(\Gamma) = i\}| \leq |\omega|^{2i}$.

A word $\omega \in F_d(\Pi)$ is called to be primitive if it belongs to a basis of $F_d(\Pi)$. In general, a subgroup H of a free group J is called to be a free factor of J, if every basis of H can be extended to be a basis of J. On the other hand, J is called an algebraic extension of $<\omega>$ if $<\omega> \leq J$ and $<\omega>$ is not contained in any proper free factor of J. To study primitive property of a word, Puder and Parzanchevski gave the following definitions.

Definition 2.2.4 (Definition 1 of [Puder11], or Definition 1.4 of [PP12])**.** *The primitive rank of a word* $\omega \in F_d(\Pi)$ *is*

$$\begin{aligned} \pi(\omega) &= \min\{rk(J) | \omega \textit{ is not primitive in } J\} \qquad (2.12) \\ &= \min\{rk(J) | <\omega> \overset{\Pi}{\twoheadrightarrow} J \textit{ and } <\omega> \textit{ is not a free factor of } J\} \end{aligned}$$

And the critical group of ω *is*

$$Crit(\omega) = \{J | <\omega> \overset{\Pi}{\twoheadrightarrow} J \textit{ with } rk(J) = \pi(\omega) \textit{ and } <\omega> \textit{ is not a free factor of } J\}. \qquad (2.13)$$

Remark 2.2.5. *By the definition of algebraic extension,* $Crit(\omega)$ *consists of all algebraic extensions of* $<\omega>$ *with minimal rank* $\pi(\omega)$ *besides* $<\omega>$ *itself.*

Puder and Parzanchevski [Puder11], [PP12] grouped all core graphs from Q_ω together, investigated their total contributions of the expected number of fixed points of ω, and denoted the total contribution by $\Phi_{\omega,n}$. After a complicated analysis, a crucial theorem is obtained as follow,

Theorem 2.2.6 (Table 1 of [Puder11], Theorem 1.5 of [PP12])**.** *For any $\omega \in F_d(\Pi)$, the expectation of $\Phi_{\omega,n}$ is*

$$E[\Phi_{\omega,n}] = \sum_{\Gamma \in Q_\omega} E(\Gamma) = 1 + \frac{|Crit(\omega)|}{n^{\pi(\omega)-1}} + O(\frac{1}{n^{\pi(\omega)}}) \qquad (2.14)$$

Remark 2.2.7. *Roughly, this theorem says that the contribution of $\frac{1}{n^{\pi(\omega)-1}}$ term of the universal core graph $\Gamma_\Pi(<\omega>)$ of a word ω is offset by the contribution of all non-algebraic extension quotient core graphs of characteristic $\leq \pi(\omega)$, and the resulting $\frac{|Crit(\omega)|}{n^{\pi(\omega)-1}}$ term is from critical quotient core graphs.*

A useful observation to analyze algebraic extension:

Lemma 2.2.8 (Lemma 10 of [LP10], Lemma 4.1 of [Puder12])**.** *Let $\omega \in F_d(\Pi)$ and let $<\omega> \overset{\Pi}{\twoheadrightarrow} H$, if H is a proper algebraic extension of $<\omega>$, then the walk ω in core graph $\Gamma_\Pi(H)$ traces every edge at least twice.*

Remark 2.2.9. *We notice that the converse of the lemma is still true. Proof is as follows: To be a basis, ω must start from the base point, and go through a spanning tree T, arrive at the starting point of an edge e (which is outside of the spanning tree), then cross the edge e only once, and return to the base point. Since $e \in \Gamma_\Pi(H) - T$, e will be traced only once. Because ω traces every edge more than twice, ω can not be a basis of H.*

Using Theorem 2.2.6 and Lemma 2.2.8, Puder provided a new result $\lambda(G) \leq 2\sqrt{d-1} + 0.84$ for d-regular graph in [Puder12] very recently. In that paper, a banquet of $d/2$ self loops (for d even) is lifted and Theorem 2.2.6 is used to obtain the cancellation between proper grouped closed walks in the lifts. Using Lemma 2.2.8 together with a purely counting argument on the sizes of critical groups and a "cogrowth formula", an asymptotic result is obtained (Corollary of [Puder12]),

$$\limsup_{t\to\infty}\left[\sum_{\substack{\omega\in(X\cup X^{-1})^t\\ \pi(\omega)=m}}|Crit(\omega)|\right]^{1/t}=g(2m-1)\leq\begin{cases}2\sqrt{d-1} & \text{if } 1\leq m\leq\sqrt{d-1},\\ \frac{2d-1}{2m-1}+2m-1 & \text{if } \sqrt{d-1}<m\leq d.\end{cases} \tag{2.15}$$

Further, Puder re-estimated an uniform bound for the error term in Theorem 2.2.6 to be

Lemma 2.2.10 (Lemma 19 of [LP10], Proposition 5.1 of [Puder12])**.** *Uniform bound for the error term for all primitive rank m words is* $\frac{t^{2+2m}}{(n-t^2)(n^{m-1})}$.

Therefore, Puder had an expansion up to order d as follows,

$$\begin{aligned} E\lambda(G)^k &\leq \sum_{m=0}^{d/2}\sum_{\substack{\omega\in(X\cup X^{-1})^k\\ \pi(\omega)=m}}\left(|Crit(\omega)|+\frac{k^{2m+2}}{n^{m-1}(n-k^2)}\right) && (2.16)\\ &\leq \left(1+\frac{k^{d+2}}{n-k^2}\right)(d/2+1)\left[\max\left\{n^{1/k}(g(0+\varepsilon)),g(1)+\varepsilon,\cdots,\frac{g(d-3)}{n^{(d/2-2)/k}},\frac{d+\varepsilon}{n^{(d/2-1)/k}}\right\}\right]^k \end{aligned}$$

With a carefully picked $n^{1/k}=e^{\frac{2}{5\sqrt{d-1}}}$, Puder reached his result $\lambda(G)\leq 2\sqrt{d-1}+0.84$ *a.s.* for d-regular graphs.

2.3 An Alternative Proof of Alon's Conjecture of Random $2d$-Regular Graphs In A Simpler Way

As we have seen in Section 2.2, Puder's approach in random n lifts of a banquet with d self loops has an expansion to order d. Our approach will have an advantage of choosing any desired expansion order, which benefits from applying Friedman's expansion method under the permutation model. On the other hand, as we have seen in Subsection 2.1.2, Friedman's approach has to compute tedious expansions two times, for B-selective trace and for B-selective trace on graphs without containing certain supercritical tangles. We will bypass this issue by an easier computation of an expansion for all irreducible closed walks, then separating the expansion into good parts and bad parts naturally, and showing the probability of bad parts occurring is small. Furthermore, Puder and Parzanchevski's cancellation theorem (see Theorem 2.2.6) helps simplify the computation of the expansion.

Connection Between Generalized Forms And Core Graphs

Our approach is motivated by Puder and Parzanchevski's cancellation Theorem 2.2.6 in random lifts. Lemma 2.3.1 enable us to work under the permutation model of random $2d$-regular graphs, where not only Friedman's expansion method [Fri91], but also Puder and Parzanchevski's Theorem 2.2.6 are applicable.

Lemma 2.3.1. *There is a one-to-one correspondence between the following generalized forms and reduced core graphs,*

1. *A generalized form $\Gamma_{\omega,i,\mathbf{t}}$ traced out by a length k irreducible closed walk ω on a $2d$-regular graph with n-vertices, and with a given label l compatible with ω;*

2. *A finite reduced core graph $\Gamma_{\Pi}(H)$ traced by the same length k irreducible closed walk ω in random n lifts of a finite connected graph with d edges, with some subgroup H satisfying $<\omega> \overset{\Pi}{\twoheadrightarrow} H \leq F_d(\Pi)$.*

Remark 2.3.2. • *Here we require ω to trace out both $\Gamma_{\omega,i,\mathbf{t}}$ and $\Gamma_{\Pi}(H)$. If this is not true, both cases will give zero contribution, which is not of our interest.*

- *Given $\Gamma_{\omega,i,\mathbf{t}}$ and a fixed ω, the label of $\Gamma_{\omega,i,\mathbf{t}}$ is not fully fixed by ω. For example, barbell shape generalized form for walk $x_1x_2x_1^{-1}x_2^{-1}$ has two different labels depending on the choice of label on the handlebar. Different labels will give rise to different quotient core graphs of ω.*

Proof. $\Longrightarrow$. Based on the permutation model, the edges of a $2d$-regular graph automatically carry labels and directions. We fix the orientation by reverting the direction of π^{-1} labeled edges and changing the label to be π. Noticing at every vertex, there are at most one edge labeled by some letter x and at most one edge labeled by x^{-1}, thus $\Gamma_{\omega,i,\mathbf{t}}$ is folded. Now we pick a spanning tree T on $\Gamma_{\omega,i,\mathbf{t}}$. For each edge $e_j \in \Gamma_{\omega,i,\mathbf{t}} - T$, one has an unique irreducible walk p_{j_1} from vertex i to the starting point of e_j through the spanning tree, and another unique irreducible walk p_{j_2} from vertex i to the ending point of e_j through the spanning tree. Since e_j does not belong to the spanning tree, the closed walk $p_{j_1}e_jp_{j_2}^{-1}$ is irreducible. Therefore the set $\{p_{j_1}e_jp_{j_2}^{-1}\}$ forms a free and reduced basis for a subgroup $H \leq F_d(\Pi)$. Since $\Gamma_{\omega,i,\mathbf{t}}$ is folded, by Fact 2.2.1, $H = \pi_1(\Gamma_{\omega,i,\mathbf{t}})$ is independent of the choice of the spanning tree. H corresponds to an unique reduced core graph $\Gamma_\Pi(H)$, and $\pi_1(\Gamma_\Pi(H)) = H$. Next we want to show that $\Gamma_{\omega,i,\mathbf{t}}$ is a core graph. If it is true, then by the one-to-one correspondence between finitely generated subgroups and core graphs, $\Gamma_{\omega,i,\mathbf{t}} = \Gamma_\Pi(H)$. Therefore, ω trace out $\Gamma_\Pi(H)$. Consequently, $\Gamma_{\omega,i,\mathbf{t}}$ must be obtained by identifying certain vertices. Linial and Puder's construction of quotient core graph (Remark 2.2.2) shows $< \omega > \overset{\Pi}{\twoheadrightarrow} H$.

If $\Gamma_{\omega,i,\mathbf{t}}$ is not a core graph, by definition of core graphs, we have

$$\Gamma_{\omega,i,\mathbf{t}} \supsetneq \cup\{\text{all irreducible closed walks on } \Gamma_{\omega,i,\mathbf{t}} \text{ from vertex i to itself}\}. \tag{2.17}$$

Case 1, If there is a vertex $v \neq i$ not visited by any irreducible closed walk on $\Gamma_{\omega,i,\mathbf{t}}$ from vertex i to itself, then either v is disconnected to vertex i, or v will be visited by back-tracking closed walks from vertex i to itself. Since $\Gamma_{\omega,i,\mathbf{t}}$ is connected, we avoid the first case. Thus, the closed walk visiting v must have a leaf other than the possible one at the starting point i. Since ω is irreducible, then $\Gamma_{\omega,i,\mathbf{t}}$ has at most one possible leaf at the starting vertex i, which is a contradiction.

Case 2, Assume there is an edge $e = (u, v)$ that can not be visited by irreducible closed walks from vertex i to itself. As discussed earlier, any vertex will be visited. So we have irreducible closed walks p_1 and p_2 from i to itself, such that $p_1 = p_1^1 u p_1^2$ and $p_2 = p_2^1 v p_2^2$ with at least one part p_i^j not to be empty for $i, j = 1, 2$. Assume $p_1^1 \neq Id$, we have a new closed walk of form $p_1^1 e p_2^1$ from vertex i to itself, which is irreducible, contradiction.

Therefore, $\Gamma_{\omega,i,\mathbf{t}}$ must be a core graph, done.

$\Longleftarrow$. Given a fixed finite core graph $\Gamma_\Pi(H)$ traced out by some length k irreducible closed walk ω, we want to show all edges traced out by ω must be free choice (include coincidence). If this is not true, then there is at least one forced step $e_1 = (u, v)$ with label π. By the definition of forced choice, $\pi(u) = v$ is determined previously, either by $e_0 : \pi(u) = v$ or by $e_0 : \pi^{-1}(v) = u$. But under both cases, these two edges e_0 and e_1 will be folded, which is a contradiction. Since ω traces out the core graph $\Gamma_\Pi(H)$, $\Gamma_\Pi(H)$ is a generalized form $\Gamma_{\omega,i,\mathbf{t}}$ with vertex i being the base point, and the labels on the edges are compatible with ω.

Finally, we want to embed the core graph $\Gamma_\Pi(H)$ in a $2d$-regular graph. This can be done by adding self loops to every vertex. If the vertex v is even degree, we just add $\frac{2d - deg(v)}{2}$ self loops to the vertex. Notice odd degree vertices appear in pairs. For any pair of odd degree vertices, we add an edge to connect them, then add $\frac{2d - deg(v) - 1}{2}$ self loops to each of the two odd degree vertices. And since there are at most d letters, there obviously exists a compatible labeling on the $2d$-regular graph. □

Meanwhile, we know

$$E[\Gamma]_{\omega} = \frac{n!}{(n-v)!}\prod_{j=1}^{d}\frac{(n-\alpha_j(\Gamma))!}{n!} = E(\Gamma_{\Pi}(H)), \tag{2.18}$$

where for an unique H satisfying $<\omega> \overset{\Pi}{\twoheadrightarrow} H$. Notice on the left hand side, n is the number of vertices of a $2d$-regular graph; while on the right hand side, n is the number of lifts of some connected graph with d edges.

Therefore, by Lemma 2.3.1 and Equation 2.18, instead of estimating the irreducible closed walks in a $2d$-regular graph with n vertices, we can estimate irreducible closed walks in random n lifts of **any** finite connected d edges graph[1]. Consequently, Puder and Parzanchevski's Theorem 2.2.6 is applicable to our random $2d$-regular graph permutation model.

Applying Puder and Parzanchevski's Cancellation Theorem

Before using Puder and Parzanchevski's Theorem 2.2.6, we need to estimate an uniform bound for the error term in their theorem. The following Lemma 2.3.3 is similar as Lemma 2.2.10 , but we provide a simpler proof by using Taylor expansion (see Lemma 2.1.3).

Lemma 2.3.3. *For a given word ω with primitive rank $\pi(\omega) = m$, the error term $O(\frac{1}{n^m})$ in Puder and Parzanchevski's Theorem 2.2.6 can be re-estimated to be $C\frac{k^{2m+2}}{n^{m-1}(n-k^2)}$, where $k \leq n/2$, and C is an absolute constant.*

Proof. The error term comes from the contribution of finite many quotient graphs of ω with coincidence $Coin \leq m+1$. Each quotient core graph with coincidence $Coin = m+1$ contributes at most $\frac{1}{n^m}$. By Lemma 2.2.3, there are at most $k^{2(m+1)}$ quotients. Applying Lemma

[1] In [Puder12], Puder worked in random n lifts of a banquet with d self loops.

2.1.3 to each quotient core graph of coincidence $Coin = m - l$ with expansion order $r = l$, then the contribution is bounded by $\exp((l+1)k/(n-k))k^{2l+2}$. Again there are at most $k^{2(m-l)}$ quotients. When $k \leq n/2$, $\exp((l+1)k/(n-k))$ are all bounded by a constant C for $l = 1, \cdots, d$. Then the m-th error term can be bounded by

$$\text{error} \leq C(1 + k^2 + \cdots + k^{2m+2})/n^m \leq Ck^{2m+2}/n^m. \tag{2.19}$$

To get an uniform bound, we sum over all terms from m to ∞.

$$\sum_{m}^{\infty} C\frac{k^{2m+2}}{n^m} = \frac{Ck^{2m+2}}{n^{m-1}(n-k^2)}. \tag{2.20}$$

□

By using Linial and Puder's Lemma 2.2.8, for $j = 1, \cdots, d$, we have

$$\begin{aligned}
& \sum_{\substack{\pi(\omega)=j \text{ and} \\ |\omega|=k}} |Crit(\omega)|/n^{j-1} && (2.21)\\
= & \sum_{\substack{\pi(\omega)=j \text{ and} \\ |\omega|=k}} \sum_{\substack{<\omega>\overset{\Pi}{\twoheadrightarrow}H \text{ and} \\ rk(H)=j}} I_{Crit(\omega)}(H)/n^{j-1} \\
= & \sum_{\substack{<\omega>\overset{\Pi}{\twoheadrightarrow}H \text{ and} \\ rk(H)=j}} \frac{|\{\omega \in F_d(\Pi) | |\omega| = k \text{ and } H \in Crit(\omega)\}|}{n^{j-1}} \\
= & \sum_{\substack{<\omega>\overset{\Pi}{\twoheadrightarrow}H \text{ and} \\ rk(H)=j}} \frac{|\{\omega \in F_d(\Pi) | |\omega| = k \text{ and } <\omega> \leq_{alg} H\}|}{n^{j-1}} \\
= & \sum_{\substack{<\omega>\overset{\Pi}{\twoheadrightarrow}H \text{ and} \\ rk(H)=j}} \frac{|\{\omega \in F_d(\Pi) | \omega \text{ trace every edge of } \Gamma_\Pi(H) \text{ at least twice}\}|}{n^{j-1}} \\
= & \sum_{\substack{\Gamma \\ \text{with } Coin(\Gamma)=j}} \frac{|\{\omega \in F_d(\Pi) | \omega \text{ trace every edge of } \Gamma \text{ at least twice}\}|}{n^{j-1}} \\
\leq & \sum_{\substack{\Gamma \\ \text{with } Coin(\Gamma)=j}} E[\Gamma]_\omega I(\text{multiplicity of each edge} \geq 2)
\end{aligned}$$

Here, the second "=" sign is due to changing the order of summation, the third "=" sign is by the definition of critical groups, the fourth "=" sign comes from Linial and Puder's Lemma 2.2.8, and the fifth "=" sign is because of the one-to-one correspondence between finitely generated subgroups of $F_d(\Pi)$ and finite core graphs labeled by Π, see Correspondence 2.11.

Using Puder and Parzanchevski's theorem 2.2.6, we group closed walks by core graph quotient relation. Then when $k \leq n^{1/2}$, the summation of irreducible closed walks becomes:

$$\begin{aligned} & \sum_{i=1}^{n} \sum_{\omega \in Irred_k} I[\omega(i) = i] \qquad (2.22) \\ \leq \quad & \sum_{j=1}^{d} \sum_{\{\omega \in Irred_k | \pi(\omega) = j\}} \left(1 + \frac{|Crit(\omega)|}{n^{j-1}} + \frac{Ck^{2j+2}}{n^{j-1}(n-k^2)}\right) \\ \leq \quad & 2d(2d-1)^{k-1} + \sum_{j=1}^{d} \sum_{\substack{\Gamma \text{ with} \\ Coin(\Gamma)=j}} \Big(E[\Gamma]_\omega I(m_i \geq 2) \Big) + 2d(2d-1)^{k-1} \frac{Ck^4}{(n-k^2)}, \end{aligned}$$

where we do not have irreducible words of length k with $\pi(\omega) = 0$, since those words are all reduced to identity 1. Recall $I(m_i \geq 2)$ denotes the event of each edge multiplicity large than or equal to two.

Applying Friedman's early Expansion Method

The middle term of Equation 2.22 can be studied by Friedman's early expansion method. Starting from Friedman's Lemma 2.1.4, and summing over all irreducible closed walks with fixed type T and fixed multiplicities≥ 2, we arrive at the following Lemma 2.3.4, which is an analog of Friedman's Lemma 2.1.6.

Lemma 2.3.4. *For a fixed type T with t edges, fixed multiplicities $m_s \geq 2$ for $s = 1, \cdots, t$, any coincidence $r \leq d$, and any expansion order r_{exp} which will be chosen later, we have*

$$\sum_{\substack{\sum_1^t k_i m_i = k \text{ with} \\ k_i \geq 1 \text{ and } k \geq \sum_1^t m_i}} \sum_{l \in L_{T,k_1,\cdots,k_t}} p_i(T,l) \leq (2d-1)^{\frac{k-m}{2}} k^{t+2r_{exp}-1} (cdr_{exp})^{cdr_{exp}}, \qquad (2.23)$$

where $m = m_1 + \cdots + m_t$, and $i \leq r_{exp} - 1$.

Proof. Recall K_1, K_2, K_3 is a partition of the set $\overrightarrow{k} = \{k_1, \cdots, k_t\}$, with size $|K_s| = \sum_{k_j \in K_s} k_j$ for $s = 1,2,3$. Assume $K_1 = (k_1, \cdots, k_u)$. By Friedman's Lemma 2.1.4 on summing over all labels, and by switching the summation order, we have

$$\sum_{K_1,K_2,K_3} \sum_{\sum k_i m_i = k} (2d-1)^{k_1+\cdots+k_u} (-1)^{|K_2|} Q_{K_1,K_2,K_3}. \tag{2.24}$$

Notice Q_{K_1,K_2,K_3} is a polynomial of degree at most $2r_{exp}$, whose coefficients are bounded by $(c_1 d r_{exp})^{c_1 d r_{exp}}$. Thus $|Q| \leq (c_1 d r_{exp})^{c_1 d r_{exp}} (\sum k_i)^{2r_{exp}} \leq (c_1 d r_{exp})^{c_1 d r_{exp}} (k/2)^{2r_{exp}}$.

On the other hand,

$$\begin{aligned} (k_1 + \cdots + k_u) &= \frac{1}{2}(k - \sum_{i=1}^{t} m_i k_i + 2(k_1 + \cdots + k_u)) \qquad (2.25) \\ &= \frac{1}{2}(k - (m_1 - 2)k_1 - \cdots - (m_u - 2)k_u - m_{u+1}k_{u+1} - \cdots - m_t k_t) \\ &\leq \frac{1}{2}(k - (m_1 - 2) - \cdots - (m_u - 2) - m_{u+1} - \cdots - m_t) \\ &\leq \frac{1}{2}(k - m + 2u). \end{aligned}$$

Also note there are at most $\binom{k+t-1}{t-1} \leq k^{t-1}$ solutions of $\sum_{i=1}^{t} k_i m_i = k$. In addition, $u \leq t \leq 3r - 1$, and then $(2d-1)^u \leq (c_2 d)^{c_2 d}$. Thus

$$\sum_{\sum k_i m_i = k} (2d-1)^{k_1+\cdots+k_u} Q_{K_1,K_2,K_3} \leq (2d-1)^{\frac{k-m}{2}} k^{t+2r_{exp}-1} (c d r_{exp})^{c d r_{exp}}. \tag{2.26}$$

Finally, we need sum over all possible K_1, K_2, K_3 of the partition of $\overrightarrow{k}$. Since there are no more than $3^t \leq (c d r_{exp})^{c d r_{exp}}$ partitions, Lemma 2.3.4 is proved. □

Further, by summing over all types, and by summing over all possible multiplicities $m_i \geq 2$, we have the following Proposition 2.3.5.

Proposition 2.3.5. *Summing over all possible types of fixed coincidence l satisfying $1 \leq Coin(T) = j \leq d$, and summing over all possible multiplicities $\overrightarrow{m}$ with $m_i \geq 2$ for $i = 1, \cdots, t$, one has the* *following expansion,*

$$\sum_{\substack{\Gamma_{\omega,i,\mathbf{t}}\in T \text{ with} \\ Coin(T)=j}} E[\Gamma]_\omega I(m_i\geq 2) = \frac{f_{j-1}}{n^{j-1}} + \cdots + \frac{f_{r_{exp}-1}}{n^{r_{exp}-1}} + \frac{\varepsilon_j}{n^{r_{exp}}}, \tag{2.27}$$

where

$$\begin{aligned} & f_{j-1} = \cdots = f_{r_{exp}-1} \\ \leq & \begin{cases} (2d-1)^{\frac{k}{2}} k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } 1\leq j\leq \lfloor(\sqrt{2d-1}+1)/2\rfloor \\ (2d-1)^{\frac{k}{2}}(\frac{2j-1}{\sqrt{2d-1}})^{k/2} k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } \lfloor(\sqrt{2d-1}+1)/2\rfloor < j\leq d \end{cases} \end{aligned} \tag{2.28}$$

and error $\varepsilon_j \leq C2d(2d-1)^{k-1}k^{4j+2r_{exp}}n^{1-j-r_{exp}}$. *Here the expansion order* r_{exp} *is a constant to be chosen later, and* $c_1 = c_1(r_{exp})$, c_2, *and* C *are constants.*

Proof. Denote $N(T,\overrightarrow{m})$ is the number of length k irreducible closed walks compatible with T of edge multiplicities $\overrightarrow{m}$, then

$$\begin{aligned} & \sum_{\substack{\Gamma \text{ with} \\ Coin(\Gamma)=j}} E[\Gamma]_\omega I(\omega\in Irred_k \text{ and } m_i\geq 2) \\ = & \sum_{\substack{T \text{ with} \\ Coin(\Gamma)=j}} \sum_{\overrightarrow{m}} \left[N(T,\overrightarrow{m}) \sum_{\substack{\sum k_i m_i = k \\ \text{with } k_i\geq 1}} \sum_{l\in L_{T,k_1,\cdots,k_t}} E[T,l]\right], \end{aligned} \tag{2.29}$$

where the third summation has non-zero contributions only when $k\geq m=\sum m_i$. Given a word ω of length k and $Coin=j$, the total number of possible walks compatible with $\Gamma\in T$ is $\binom{k}{j}k^j \leq k^{2j}$. By Lemma 2.1.3, $E[T,l]$ has the following expansion for any r,

$$E[T,l] = \frac{p_{Coin(T)-1}}{n^{Coin(T)-1}} + \cdots + \frac{p_{r-1}}{n^{r-1}} + \frac{error}{n^r} \tag{2.30}$$

with $error \leq \exp(\frac{rk}{n-k})k^{2r}$. For $r=r_{exp}$, one has

$$\sum_{\substack{T \text{ with} \\ Coin(\Gamma)=j}} \sum_{\overrightarrow{m}} \left[N(T,\overrightarrow{m}) \sum_{\substack{\sum k_i m_i = k \\ \text{with } k_i\geq 1}} \sum_{l\in L_{T,k_1,\cdots,k_t}} E[T,l]\right] = \frac{f_{j-1}}{n^{j-1}} + \cdots + \frac{f_{r_{exp}-1}}{n^{r_{exp}-1}} + \frac{\varepsilon_j}{n^{r_{exp}}} \tag{2.31}$$

with $\varepsilon_j \leq (2d)(2d-1)^{k-1}k^{2j}n^{-j+1}k^{2j}\exp(\frac{r_{exp}k}{n-k})k^{2r_{exp}}n^{-r_{exp}} \leq C2d(2d-1)^{k-1}k^{4j+2r_{exp}}n^{-j-r_{exp}+1}$ for $k \leq n/2$. The $k^{2j}n^{-j+1}$ term comes from Lemma 2.1.2, the probability of a given word with j coincidence.

Therefore, for $i = j-1, \cdots, r_{exp}-1$, we have

$$f_i = \sum_{\substack{T \text{ with} \\ Coin(\Gamma)=j}} \sum_{\overrightarrow{m}} \left[N(T, \overrightarrow{m}) \sum_{\substack{\sum k_i m_i = k \\ \text{with } k_i \geq 1}} \sum_{l \in L_{T,k_1,\cdots,k_t}} p_i[T,l] \right]. \tag{2.32}$$

Notice that any type T with $Coin(T) = j$ has the number of edges $t \leq 3j-1$, the total number of coincidence j types is at most $(2j)^{6j-2}$, and the total number of irreducible walks on a given T with $Coin(T) = j$ and multiplicities $\overrightarrow{m}$ is at most $(2j)(2j-1)^{m-1}$.

Also

$$\begin{aligned} (m_1 + \cdots + m_t) &= \frac{1}{2}(k - \sum_{i=1}^{t} m_i k_i + 2(m_1 + \cdots + m_t)) \qquad (2.33) \\ &= \frac{1}{2}(k - (k_1-2)m_1 - \cdots - (k_t-2)m_t) \\ &\leq \frac{1}{2}(k - 2(k_1-2) - \cdots - 2(k_t-2)) \\ &\leq \frac{k}{2} - t + 2t \leq \frac{k}{2} + t \leq \frac{k}{2} + 3d - 1. \end{aligned}$$

Therefore, by Lemma 2.3.4, one has

$$\begin{aligned} & f_{j-1}, \cdots, f_{r_{exp}-1} \qquad (2.34) \\ \leq \;& (2j)^{6j-2}(2j)(2j-1)^{m-1}(2d-1)^{\frac{k-m}{2}}k^{2r_{exp}+t-1}(c_2 d r_{exp})^{c_2 d r_{exp}} \\ \leq \;& \begin{cases} (2d-1)^{\frac{k}{2}} k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } 1 \leq j \leq \lfloor (\sqrt{2d-1}+1)/2 \rfloor \\ (2d-1)^{\frac{k}{2}} (\frac{2j-1}{\sqrt{2d-1}})^{k/2} k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}} & \text{if } \lfloor (\sqrt{2d-1}+1)/2 \rfloor < j \leq d \end{cases} \end{aligned}$$

□

Remark 2.3.6. *All f_i do not depend on the index $i = j-1, \cdots, r_{exp}-1$. When $1 \leq j \leq \lfloor (\sqrt{2d-1}+$*

$1)/2\rfloor$, we denote it to be $f=(2d-1)^{\frac{k}{2}}k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}}$, and when $\lfloor(\sqrt{2d-1}+1)/2\rfloor < j \leq d$, we denote it to be $f(\frac{2j-1}{\sqrt{2d-1}})^k$. Also we denote $rk_c = \lfloor\frac{\sqrt{2d-1}+1}{2}\rfloor$.

Remark 2.3.7. *From the proof of our Lemma 2.3.3, for any primitive rank j, the uniform error bound is $\delta_j \leq 2d(2d-1)^{k-1}\sum_{i=j}^{\infty}\frac{Ck^{2i+2}}{n^{i-1}(n-k^2)}$. Thus, the expansion error ε_j from the above Proposition 2.3.5 is already included in δ_j, once the expansion order $r_{exp} > j$ and $k < n^{1/2}$.*

Thus, let r_{exp} be large enough (to be chosen later), by applying Proposition 2.3.5, we have

$$
\begin{aligned}
& Irred(G,k) = \sum_{i=1}^{n}\sum_{\omega\in Irred_k} I[\omega(i)=i] \qquad (2.35)\\
\leq\ & 2d(2d-1)^{k-1} + f + \frac{f}{n} + \cdots + \frac{f}{n^{r_{exp}-1}} + \delta_1\\
& +\frac{f}{n} + \cdots + \frac{f}{n^{r_{exp}-1}} + \delta_2\\
& +\cdots\\
& +\frac{f(\frac{2rk_c+1}{\sqrt{2d-1}})^{k/2}}{n^{rk_c+1}}\cdots + \frac{f(\frac{2rk_c+1}{\sqrt{2d-1}})^{k/2}}{n^{r_{exp}-1}} + \delta_{rk_c+1}\\
& +\cdots\\
& +\frac{f(\frac{2d-1}{\sqrt{2d-1}})^{k/2}}{n^{d-1}} + \cdots + + \frac{f(\frac{2d-1}{\sqrt{2d-1}})^{k/2}}{n^{r_{exp}-1}} + \delta_d\\
\leq\ & 2d(2d-1)^{k-1} + f + \frac{2f}{n} + \cdots + \frac{(rk_c)f}{n^{rk_c-1}} + \frac{(rk_c + (\frac{2rk_c+1}{\sqrt{2d-1}})^{k/2})f}{n^{rk_c}} + \cdots\\
& +\frac{(rk_c + \sum_{j=rk_c+1}^{d}(\frac{2j-1}{\sqrt{2d-1}})^{k/2})f}{n^{d-1}} + \cdots + \frac{(rk_c + \sum_{j=rk_c+1}^{d}(\frac{2j-1}{\sqrt{2d-1}})^{k/2})f}{n^{r_{exp}-1}} + 2d(2d-1)^{k-1}\frac{ck^4}{n-k^2}
\end{aligned}
$$

Controlling High Order Terms

To get a finer estimation of $\lambda(G)$, we have to choose a sufficient large expansion order. When the expansion order $r_{exp} \geq rk_c + 1$, we need deal with terms of form $(2d-$

$1)^{\frac{k}{2}}(\frac{2j-1}{\sqrt{2d-1}})^{k/2}k^{c_1 d}(c_2 d r_{exp})^{c_2 d r_{exp}}$, where $\lfloor(\sqrt{2d-1}+1)/2\rfloor < j \leq d$. However, it is hopeless to control all such terms simultaneously and directly. This is the essential reason that the expansion order was only selected to be $\lfloor(\sqrt{2d-1}+1)/2\rfloor$ in [Fri91]. We find a trick to estimate each high order term indirectly through separating it into a good part and a bad part, and showing the probability of bad part occurring is small. As a result, when $rk_c + 1 \leq r \leq d$, we have

$$\frac{(rk_c + \sum_{j=rk_c+1}^{r}(\frac{2j-1}{\sqrt{2d-1}})^{k/2})f}{n^{r-1}} = \underbrace{\frac{rf}{n^{r-1}}}_{good} + \underbrace{\frac{(\sum_{j=rk_c+1}^{r}(\frac{2j-1}{\sqrt{2d-1}})^{k/2} - (r - rk_c))f}{n^{r-1}}}_{bad}. \tag{2.36}$$

Accordingly, we have $Irred(G,k) = Irred(G,k)\chi_{good} + Irred(G,k)(1-\chi_{good})$.

From the proof of Proposition 2.3.5, to achieve the contribution of form $\frac{rf}{n^{r-1}}$,the number of irreducible closed walks on a given type T with multiplicities $\overrightarrow{m}$ must be no more than $O((2d-1)^{m/2})$.

When estimating the number of irreducible closed walks on a given type T with given edge multiplicities, the edge length and the corresponding label are irrelevant. Thus, we need estimate the number of irreducible closed walk with length $m = \sum m_i$ on a given type T. Notice, a type T with coincidence r has at most $2r$ vertices and $3r-1$ edges. Since the walk will cross each edge at least twice, the walk can be decomposed into at most $2(3r-1)$ parts with $2(3r-1)-1$ spacers. By the spectral method (the idea originated in [Buck86], and used in [Fri03], also in [LP10], etc), the number of all length k closed walks can be bounded by

$$k \sum_{l_1+\cdots+l_{6r-3}=k-2t} (A_T\delta_x)(y)(A_T^{l_1}\delta_x)(y)\cdots(A_T\delta_x)(y) \leq O(k^{6r-3}(\rho_T)^k), \tag{2.37}$$

where A_T is the adjacency matrix of T, and ρ_T is the spectral radius of T. This inspires us to restrict T with $\lambda_{irred}(T) \leq \sqrt{2d-1}$ to match the desired number of irreducible closed walks of type $O((2d-1)^{m/2})$. (Recall, $\lambda_{irred}(G)$ is the largest eigenvalue of G_{irred}. And given G, G_{irred}

is a derived graph on vertex set $V(G_{irred}) = E(G)$, with an edge between a vertex $e \in E(G)$ and another vertex $e^{'} \in E(G)$ if e and $e^{'}$ form a length two irreducible walk in G.)

Turns out, the above observation is true, and we obtain the following lemma.

Lemma 2.3.8. *Given a fixed type T of coincidence r with t edges, if multiplicities $\overrightarrow{m} = (m_1, \cdots, m_t)$ is given with $m_i \geq 2$, and if T does not contain any subgraph H with $\lambda_{irred}(H) > \sqrt{2d-1}$, then the number of irreducible closed walks of length k from a fixed starting point is at most $c(2d-1)^{-m/2}$, where $m = \sum_{i=1}^{t} m_i$ and c is a constant.*

Remark 2.3.9. *Recall Fact 2.1.11, for power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with all coefficients $a_k, b_k \geq 0$. $f(z)$ majorizes $g(z)$ if and only if $\sum_{k=0}^{j} a_k \geq \sum_{k=0}^{j} b_k$ for any $j \geq 0$. Therefore, $b_k \leq f(z_0) z_0^{-k}$ for $z_0 \in (0,1)$. We will use the above fact to estimate the number of irreducible closed walks over restricted T.*

Proof. Since we are estimating the number of irreducible closed walks on topological T, the edge lengths and labels are irrelevant. Therefore, we only need estimate the number of irreducible closed walks with length $m = \sum_{i=1}^{t} m_i$, tracing out T topologically.

Any type T is a finite graph. Since it has coincidence r, we have at most $2r$ vertices and $t \leq 3r-1$ edges. As the walk will trace out each edge at least twice, the whole walk can be decomposed into $6r-2+6r-3$ parts, where $6r-3$ is the number of spacers between two edges being traced out consecutively. Each part will be a subgraph H_i for $i = 1, \cdots, 12r-5$, and its corresponding type $T(H_i)$ will be a subgroup of our fixed type T. From the assumption, we know $\lambda_{irred}(H_i) < \sqrt{2d-1}$. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a generating function with c_k being the number of irreducible walks we are interested. Let $h_i = \sum_{k=0}^{\infty} c_{i,k} z^k$ with $c_{i,k}$ being the number of irreducible closed walks on H_i. Then $h_i(z)$ has convergence radius at least $(2d-1)^{-1/2}$ by

Fact 2.1.10. If define $\hat{c}_{i,k} = c_{1,k} + \cdots + c_{12r-5,k}$, $\hat{c}_{i,k}$ denote the number of all irreducible walks with length at most k in H_i. Therefore, one has generating function,

$$\hat{h}_i(z) = \sum_{k=0}^{\infty} \hat{c}_{i,k} z^k = \frac{1}{1-z} h_i(z) \quad \text{with convergence radius } > (2d-1)^{-1/2}. \tag{2.38}$$

Denote $h(z) = \sum_{i=1}^{12r-5} h_i(z)$ and $\hat{h}(z) = \sum_{i=1}^{12r-5} \hat{h}_i(z)$. Then we claim $(1-d\hat{h}(z))^{-1}\hat{h}(z)$ majorizes $f(z)$. Starting from a vertex v, the walk must first trace out a H_{i_1} in $j_i < m$ steps, then the walk has at most d choices to continue, and keeps tracing out another H_{i_2} in $j_2 < m$ steps, ... until trace out the very last part to go back the starting vertex v. As a result, $f(z)$ must be majorized by $(1-d\hat{h}(z))^{-1}\hat{h}(z)$.

If we pick $z_0 = (2d-1)^{-1/2}$ inside convergence radius, then we have

$$c_k \leq (1-d\hat{h}(z_0))^{-1}\hat{h}(z_0) z_0^m \leq c(2d-1)^{-m/2} \text{ for some c.}$$

Applying Bartholdi Identity

We found out the bad part combinatorially in previous Subsection 2.3. Now we will use Bartholdi identity to estimate the loss probability from throwing away the bad part. First of all, let's recall Bartholdi identity,

Lemma 2.3.10. *(see Equation 2.1 of [OS09]) For a $2d$-regular graph with adjacency matrix A, one can define that B is a $2E \times 2E$ matrix describing irreducible length 2 walks, $B_{e,e'} = \delta_{t(e),o(e')}$ with $t(e)$ denoting the ending point of e and $o(e)$ denoting the starting point of e; J is a $2E \times 2E$ matrix describing back-tracking length 2 walks, $J_{e,e'} = \delta_{\overline{e},e'}$ with $\overline{e}$ denoting changing the direction of the edge e.*

Therefore,

$$\det(I^{2|E|} - s(B-J)) = (1-s^2)^{|E|-|V|} \det((1+(2d-1)s^2)I^{|V|} - sA). \tag{2.39}$$

Furthermore, the spectrum of $B-J$ can be read out as,

$$\sigma(B-J) = \Big\{2d-1, 1, 1\times(|E|-|V|), -1\times(|E|-|V|), \sqrt{2d-1}e^{i\phi_i}, \sqrt{2d-1}e^{-i\phi_i} \text{ with } \phi_i = \arccos(\frac{\lambda_i}{2\sqrt{2d-1}}), \text{ for } i = 2, \cdots, |V|\Big\}, \tag{2.40}$$

where $2d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}$ are all eigenvalues of the adjacency matrix A.

Remark 2.3.11. *It's easy to see that $B-J$ is actually the adjacency matrix of a directional graph G_{irred}.*

Easy observations give the following two simple results,

Lemma 2.3.12.

- *If H is a banquet of rk_c self loops, by Bartholdi identity, $\lambda_{irred}(H) = \sqrt{2d-1}$;*

- *By the one-to-one correspondence between generalized form and core graphs, also the one-to-one correspondence between core graphs and finitely generated subgroups of $F_d(\Pi)$, there are only finite many types with coincidence l (actually, this number can be bounded by $(2l)^{6l-2}$). Among all of those types, banquet of l self loops happens with maximal probability n^{-l}. For all the other types, we must have at least one edge label occurs more than once, $\alpha_i \geq 2$. Thus, we will have a factor $\frac{1}{n(n-1)\cdots(n-\alpha_i+1)}$ in probability of the occurrence of the type, which is smaller than $\frac{1}{n}$ for a banquet.*

Lemma 2.3.12 tells us that a banquet with l self loops occurs with maximal probability in all coincidence l types, and we know its value is λ_{irred}. Now, we still need analyze λ_{irred} of all other types with the same coincidence l.

Since a banquet G_1 can be obtained by contracting all edges and identifying all vertices of a graph G_2 with the same coincidence, any length k irreducible closed walk on G_2 from a base vertex v_o to itself will correspond to an irreducible closed walk on the banquet G_1 with length less than or equal to k. Thus, the number of irreducible closed walks with length $\leq k$ is less than or equal to the number of irreducible closed walks on the banquet G_1. Therefore, $\lambda_{irred}(G_1) \geq \lambda_{irred}(G_2)$. On the other hand, if a graph G_3 with $\lambda_{irred}(G_3) > \lambda_{irred}(G_1)$, by the same reason as the above, we know that G_3 must have a higher coincidence. Then the probability of occurring of G_3 will have at least one additional factor $\frac{k}{n-k} < 1$, compared with the probability of occurring of the banquet G_1. As a result, by Lemma 2.3.12, we know that in all types of coincidence rk_c, banquet with rk_c self loops has the maximal eigenvalue $\lambda_{irred} = \sqrt{2d-1}$, and the maximal probability of occurring n^{-rk_c}. Further, if there are more than one occurrence of subgraphs H_1 and H_2 with $\lambda_{irred}(H_{1,2}) \geq \sqrt{2d-1}$, there are two cases. If $H_1 \cap H_2 = \emptyset$, then the probability of occurrence of H_1 and H_2 at the same time is the product of probabilities of each occurrence, of course smaller than n^{-rk_c}. If their intersection is non-empty, their union $H_1 \cup H_2$ must has coincidence at least rk_c, thus the probability of occurrence is also bounded by n^{-rk_c}. Therefore, the probability of occurrences of at least one bad subgraph H is bounded by n^{-rk_c}. Since the maximum primitive rank is d, after Puder's cancellation Theorem 2.2.6, the maximum coincidence of T is thus d (see the summation index of the middle term in Equation 2.22), as well as any subgraph of T. As a result, there are at most finite many bad events, $\sum_{rk_c}^{d}(2l)^{6l-2} = O(1)$. Finally, we find out the probability of bad part occurring is bounded by $O(n^{-rk_c})$.

Lemma 2.3.13. *$Pr(Bad\ parts\ occurring) = O(n^{-rk_c})$.*

Now, Proposition 2.3.5 can be modified as follows. By Lemma 2.3.8, when $\lfloor(\sqrt{2d-1}+1)/2\rfloor < l$, if T does not contain any subgroup H with $\lambda_{irred}(H) > \sqrt{2d-1}$, we can replace the original bound $2l(2l-1)^{m-1}$ by $c(2d-1)^{m/2}$. As a result, we have $f_0 = \cdots = f_{r_{exp}-1} = (2d-1)^{\frac{k}{2}} k^{c_1 d} (c_2 d r_{exp})^{c_2 d r_{exp}}$.

Accordingly, we have the good part of length k irreducible walks as,

$$\begin{aligned} Irred(G,k)\chi_{good} \leq\ & 2d(2d-1)^{k-1} + f + \frac{2f}{n} + \cdots + \frac{(rk_c)f}{n^{rk_c-1}} + \frac{(rk_c+1)f}{n^{rk_c}} \\ & + \cdots + \frac{r_{exp}f}{n^{r_{exp}-1}} + (2d)(2d-1)^{k-1}\frac{Ck^4}{n-k^2}. \end{aligned} \quad (2.41)$$

Applying A Sidestepping Lemma of Complex Random Variables

From Bartholdi identity 2.3.10, we have

$$\begin{aligned} Irred(G,k) \geq tr((B-J)^k) &= (2d-1)^k + 1 + (1+(-1)^k)(nd-n) \\ &+ \sum_{i=2}^{n} (\sqrt{2d-1}e^{i\phi_i})^k + (\sqrt{2d-1}e^{-i\phi_i})^k. \end{aligned} \quad (2.42)$$

If combine pair $\sqrt{2d-1}e^{i\phi_i}, \sqrt{2d-1}e^{-i\phi_i}$ for subscript $i = 2, \cdots, n$, we can write the pair of eigenvalues into another form, solutions of the following quadratic equation,

$$\begin{aligned} & \mu^2 - (\sqrt{2d-1}e^{i\phi_i} + \sqrt{2d-1}e^{-i\phi_i})\mu + \sqrt{2d-1}e^{i\phi_i}\sqrt{2d-1}e^{-i\phi_i} \quad (2.43) \\ = \ & \mu^2 - \lambda_i\mu + (2d-1) = 0. \end{aligned}$$

Therefore, we can define $\mu_{1,2}(\lambda_i) = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4(2d-1)}}{2}$ for $i = 2, \cdots, n$. Now we can easily separate exceptional eigenvalues (i.e. $\lambda_i \geq 2\sqrt{2d-1}$) with well-tempered eigenvalues (i.e. $\lambda_i < 2\sqrt{2d-1}$) in this new form, since exceptional eigenvalue λ_i will make $\mu_{1,2}(\lambda_i)$ be real, and well-tempered λ_j will make $\mu_{1,2}(\lambda_i)$ be complex with norm $\sqrt{2d-1}$.

We denote all the eigenvalues of $B-J$ by $\{\nu_1 \cdots, \nu_{2nd}\}$, ν_s is complex if it is of absolute value $\sqrt{2d-1}$ for $s = 1, \cdots, 2nd$. Thus, if let $\theta_s = 1 - \nu_s/(2d-1)$, from the contribution of the good part of irreducible walks of length k (Inequality 2.41), the condition of Friedman's sidestepping Lemma 2.1.12 is satisfied. More precisely, in our case, for $k = O(\log n)$, p_0 is a constant polynomial, $p_1 = k^4$, and all other p_i vanish, and the rest part of $Irred(G,k)\chi_{good}$ is bounded by $f(2d-1)^{-k} = O_{d,e}(k^c(2d-1)^{-k/2})$.

On the other hand, since $\mu_{1,2}(\lambda_i)$ is complex if and only if it is of absolute value $\sqrt{2d-1}$. As a result, for even k, we have

$$\sum_{i \text{ s.t. } \mu_{1,2}(\lambda_i)\notin\mathbb{R}} \sum_{j=1}^{2} \mu_j^k(\lambda_i) \geq -2(n-1)(2d-1)^{k/2} \tag{2.44}$$

Since the bipartite graphs are rare, and with probability roughly $O(n^{1/2}e^{-cn}) \leq n^{-rk_c}$ [Wormald99] in $G_{n,d}$, they are excluded from our discussion. Also by the second magnification inequality 2.1.8, $Pr(\lambda(G) = 2d) = \frac{1}{n^{d-1}} \leq n^{-rk_c}$ when $2d \geq 6$. Thus, we can assume $\lambda_i < 2d$, and the corresponding $\mu_{1,2}(\lambda_i) < 2d-1$ for $i = 2,\cdots,n$, at a loss probability $O(n^{-rk_c})$.

A Markov Type Argument Leads To The Proof

Let A be the event $\mu_j(\lambda_i) \geq e^{\eta}\sqrt{2d-1}$ for some $i \neq 1$ and $j = 1,2$. In terms of λ, event A will correspond to $\lambda_i = 2\sqrt{2d-1} + (e^{\eta}-1)(\sqrt{2d-1}+1)$ for some i.

Therefore, by following a Markov inequality type standard argument, and applying Friedman's sidestepping Lemma 2.1.12 ([Fri08]), we have

$$Pr(A)(e^{\eta}\sqrt{2d-1})^k \leq E\left(\sum_{i \text{ s.t } \mu_j(\lambda_i)\in\mathbb{R}} \sum_{j=1}^{2} \mu_j^k(\lambda_i)\right) \tag{2.45}$$

$$\leq \quad E(\sum_{i=2}^{n}\sum_{j=1}^{2}\mu_j^k(\lambda_i)) - E(\sum_{i \text{ s.t } \mu_j(\lambda_i)\notin\mathbb{R}} \sum_{j=1}^{2}\mu_j^k(\lambda_i))$$

$$\leq \quad O_{d,r_{exp}}(Dn^{1-r_{exp}/3}(2d-1)^k + k^c(2d-1)^{k/2}) + 2(n-1)(2d-1)^{k/2}$$

If picking $k = 2\lceil \frac{r_{exp} \log n}{3\log(2d-1)} \rceil$, we can combine all terms together

$$\begin{aligned} Pr(A) &\leq O_{d,r_{exp}}(ne^{-\eta k} + k^c e^{-\eta k} + 2(n-1)e^{-\eta k}) \\ &\leq cne^{-k\eta} = cn^{1-\alpha r_{exp}}, \quad \text{with} \quad \alpha = \frac{2/3\eta}{\log(2d-1)}. \end{aligned} \tag{2.46}$$

Choosing r_{exp} satisfying $\alpha r_{exp} - 1 \geq rk_c$, thus $Pr(A) = O(n^{-rk_c})$.

Consequently, for any fixed small ε, we can solve η from $\lambda_i = 2\sqrt{2d-1} + (e^{\eta} - 1)(\sqrt{2d-1}+1)$. And such η together with α determines the expansion order r_{exp} satisfying $\alpha r_{exp} - 1 \geq rk_c$ (r_{exp} depends on d and ε.). Finally, for such expansion order r_{exp}, we have a loss probability $O(n^{-rk_c})$ from ruling out the bad part, and another loss probability $O(n^{-rk_c})$ from requiring $\lambda(G) < 2d$. Therefore, we have $P(\lambda(G) \geq 2\sqrt{2d-1}+\varepsilon) = O_{d,e}(n^{-rk_c}) + O(n^{-rk_c}) = O_{d,e}(n^{-rk_c})$.

Theorem 2.3.14. *For a random $2d$-regular graph G with $2d \geq 6$ and any $\varepsilon > 0$, there exists an expansion order r_{exp} such that $P(\lambda(G) \geq 2\sqrt{2d-1}+\varepsilon) = O_{d,r_{exp}}(n^{-rk_c})$.*

Remark 2.3.15. *If apply Remark 2.1.9, $2d \geq 6$ condition in the above theorem can be improved to $2d \geq 4$.*

Remark 2.3.16. *We will carry out two numerical studies on the edge spectra of random Cayley graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ in Chapter 4 and random Hecke operators over $SU(2)$ in Chapter 5. In the both numerical experiments, it is shown that the sample mean of $\lambda(G)$ is on the left side of the Ramanujan bound, and the distance to the Ramanujan bound is about $\varepsilon(n) \sim -n^{-c}$ for some $c \in (0,1)$. Thus, can one improve Alon's conjecture of error term $\varepsilon = 0$ or even $\varepsilon < 0$?*

Chapter 3

Edge Spectrum Normalization

In Section 2.3, the largest non-trivial eigenvalue of a random $2d$-regular graph was studied, $\lambda(G) \leq 2\sqrt{2d-1}+\varepsilon$ with probability $1-O(n^{-rk_c})$. On the other hand, Jakobson, Miller, and Rivin ([JMR96]) indicated that the level spacing distribution of a generic k-regular graph approaches that of the Gaussian orthogonal ensemble of random matrix theory. Later, Miller, Novikoff and Sabelli ([MNS08]) pointed out that the edge spectra of a family of random regular graphs could be well modeled by $\beta = 1$ Tracy-Widom distribution. However, due to the lack of a proper normalization, the connection between the spectra of regular graphs and the spectra of random matrices is still not fully understood.

In this chapter, a normalization method for the edge spectra of regular graphs is provided. And it is numerically proven to be correct for random Cayley graphs and random Hecke operators in Chapter 4 and Chapter 5.

Let the eigenvalues of a $2d$-regular graph on N vertices be sorted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.

Recall ([AGZ10]), Gaussian orthogonal ensemble (GOE, $\beta = 1$) is defined as symmetric matrices $M_n = (\xi_{jk})_{1\le j,k\le n}$ with $\xi_{ij} = \xi_{ji}$, where their $\frac{n(n+1)}{2}$ entries $\{\xi_{ij}, 1 \le i \le j \le n\}$ are independent Gaussian random variables with mean zero and variance $(1+\delta_{ij})\sigma^2$.

Gaussian unitary ensemble (GUE, $\beta = 2$) is defined as Hermitian matrices $M_n = (\xi_{jk})_{1\le j,k\le n}$ with $\xi_{ij} = \overline{\xi_{ji}}$, where n^2 elements $\{\Re\xi_{ij}, 1 \le i \le j \le n; \Im\xi_{ij}, 1 \le i < j \le n\}$ are independent Gaussian random variables with mean zero and variance $(\frac{1+\delta_{ij}}{2})\sigma^2$.

Gaussian symplectic ensemble (GSE, $\beta = 4$) is defined as quaternion self-dual Hermitian matrices $M_n = (\xi_{jk})_{1\le j,k\le n}$ with $\xi_{ij} = \xi_{ij}^{(0)} + \xi_{ij}^{(1)}e_1 + \xi_{ij}^{(2)}e_2 + \xi_{ij}^{(3)}e_3 = \xi_{ji}^{(0)} - \xi_{ji}^{(1)}e_1 - \xi_{ji}^{(2)}e_2 - \xi_{ji}^{(3)}e_3 = \overline{\xi_{ji}}$, where their $n(2n-1)$ elements $\{\xi_{ij}^{(0)}, 1 \le i \le j \le n; \xi_{ij}^{(k)}, 1 \le i < j \le n \text{ for } k = 1,2,3\}$ are independent Gaussian random variables with mean zero. For $i < j$, $\xi_{ij}^{(k)}$ has variance $\sigma^2/4$ for each $k = 0,1,2,3$ and $\xi_{ii}^{(0)}$ has variance $\sigma^2/2$.

By applying Householder transform, the above three models can be described in generalized β ensemble ([DE02]) as a real symmetric tridiagonal matrix H_n with independent entries $\xi \sim N(0,\sqrt{2})$ and positive roots of Chi square random variables $\chi_{k\beta}$, $k = 1\cdots,n-1$,

$$M_n \sim H_n = \frac{\sigma}{\sqrt{\beta}} \begin{pmatrix} N(0,\sqrt{2}) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,\sqrt{2}) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,\sqrt{2}) & \chi_{\beta} \\ & & & \chi_{\beta} & N(0,\sqrt{2}) \end{pmatrix}. \tag{3.1}$$

Using this tridiagonal β ensemble, we can easily compute the moments of the matrix M_n.

Let $\mu_1,\cdots,\mu_n$ be the eigenvalues of a $n\times n$ Gaussian random matrix M_n. Noticing $E\xi^2 = 2$, $E\xi^4 = 12$, $E\xi^6 = 120$, $E\chi_k^2 = k$, $E\chi_k^4 = k(k+2)$, $E\chi_k^6 = k(k+2)(k+4)$, and $tr(A^2) =$

$\sum_{i,j} a_{ij}^2$ for a symmetric matrix $A = (a_{ij})$, we have

$$E\sum_1^n \mu_i^2 = tr(M_n^2) = tr(H_n^2) = \{\beta n^2 + (2-\beta)n\}\sigma^2/\beta. \tag{3.2}$$

Similarly, H_n^2 is a 5-diagonal real symmetric matrix, and we have

$$E\sum_1^n \mu_i^4 = tr((H_n^2)^2) = \{2\beta^2 n^3 + (-5\beta^2 + 10\beta)n^2 + (3\beta^2 - 10\beta + 12)n\}\sigma^4/\beta^2. \tag{3.3}$$

On the other hand, for a $2d$-regular graph, Solé's trace formula ([Sole92]) states that

$$\frac{1}{N}\sum_1^N \lambda_i^m = \int_{-2\sqrt{2d-1}}^{2\sqrt{2d-1}} u^m d\rho_{KM}(u) \;\; for\; m = 0, 1, \cdots, g-1, \tag{3.4}$$

where g is the girth of a $2d$-regular graph, and ρ_{KM} is Kesten-Makcy law of the form,

$$\rho_{KM}(u) = \frac{2d\sqrt{4(2d-1)-u^2}}{2\pi((2d)^2-u^2)} \;\; for\; u \in [-2\sqrt{2d-1}, 2\sqrt{2d-1}]. \tag{3.5}$$

By matching the first four moments of the spectrum of a $2d$-regular graph with M_n, we have

$$2dN = \sum_1^N \lambda_i^2 = \sum_1^n \mu_j^2 = \{\beta n^2 + (2-\beta)n\}\sigma^2/\beta, \tag{3.6}$$

$$\begin{aligned} 2d(4d-1)N &= \sum_1^N \lambda_i^4 = \sum_1^n \mu_j^4 \\ &= \{2\beta^2 n^3 + (-5\beta^2 + 10\beta)n^2 + (3\beta^2 - 10\beta + 12)n\}\sigma^4/\beta^2. \end{aligned} \tag{3.7}$$

Numerically, we find that for a fixed number of vertices N, when β increases, the dimension n of a random matrix M_n decreases extremely slow. Similarly, the variance of the matrix entries σ increases very slow, see Figure (3.1).

Therefore, we could assume $\beta = 2$ when computing the dimension and the variance of the corresponding Gaussian random matrix. Solving equation (3.6) and (3.7), n and σ can be written in terms of N. Asymptotically, we will see $n \sim \frac{4d}{4d-1}N$ for $2d$-regular graphs. Then

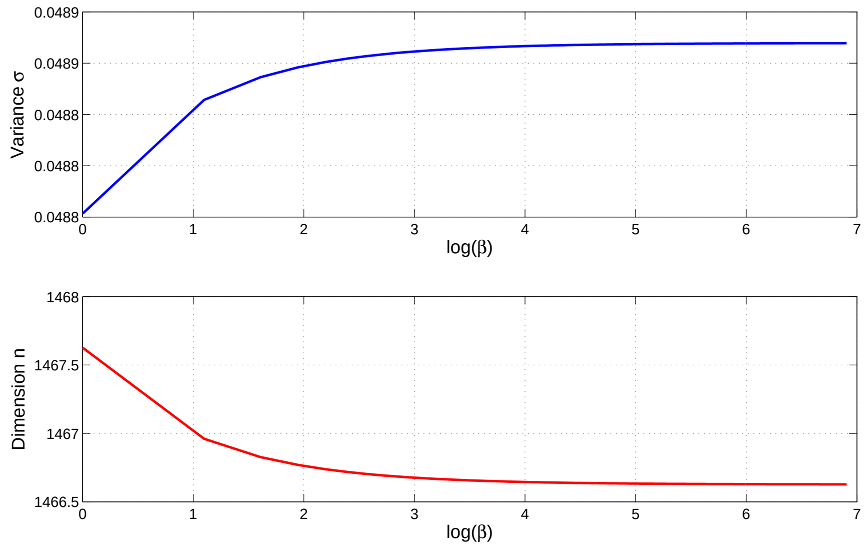


Figure 3.1: Corresponding Gaussian random matrix dimension $n = n(N)$ and variance σ with respect to β for a diagonal block of a random Cayley graph with random pair generators at principal presentation ρ of $\dim(\rho) = 1283$.

the edge eigenvalues of $2d$-regular graphs can be normalized in a similar manner as random matrices. For random matrices, we know

$$\widetilde{\mu_{max}} = n^{1/6}(\mu_{max}(M_n) - 2\sigma\sqrt{n}). \tag{3.8}$$

Thus, we conjecture the following normalization for $2d$-regular graphs,

Conjecture 3.0.17.

$$\widetilde{\lambda_{max}} = n^{1/6}(\lambda_{max} - \mu_{sample})\sqrt{\frac{n}{2d-1}}, \tag{3.9}$$

where $n = n(N)$ is the solution of Equation (3.6) and (3.7). the expectation of edge spectrum of a random matrix $2\sigma\sqrt{n}$ is replaced by sample mean μ_{sample}, and an extra factor $\sqrt{\frac{n}{2d-1}}$ is added.

Chapter 4

Random Cayley Graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$

Cayley graph is a special case of Schreier right coset graph with respect to $H = \{1_G\}$. Namely, every two elements g_1, g_2 are connected if there exists $s \in S \cup S^{-1}$ such that $g_1 = g_2 s$. The resulting graph is a $2|S|$-regular graph. If elements of $S \subseteq G$ are selected independently and randomly, a random Cayley graph is constructed. (Random) Cayley graph received a lot of attention. Lubotzky, Phillips and Sarnak [LPS88], and Margulis [Margulis88] provided a celebrated construction of a family of Cayley graphs, which are so called Ramuanjuan graphs $X^{p,q}$. $X^{p,q}$ are Cayley graphs over $PSL_2(\mathbb{F}_q)$ with respect to a very special choice of a generating set of size $p+1$. The largest eigenvalue of $X^{p,q}$ satisfies the Ramanujan bound (i.e. less than or equal to $2\sqrt{2p+1}$). Recently, Bourgain and Gamburd [BGSL08] fully established the spectral gap property (the first non-trivial eigenvalue $\lambda_1 < 2d$) of a random $2d$-Cayley graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$, where S is a generic symmetric generating set, and S_p is the set obtained by reducing S by mod p.

In this chapter, the distribution of the edge spectrum of each diagonal block of the adjacency matrix of a random Cayley graph $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ under all non-trivial irreducible representations is fully investigated. Using the normalization factor in Chapter 3, the edge spectrum is shown to be $\beta = 2$ or $\beta = 4$ Tracy-Widom distributions up to an extra $\pm$ sign.

4.1 The Fourier Transform of Random Cayley Graphs

Fourier analysis provides a useful tool to study the spectrum of $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$. Recall, the Fourier transform of a complex valued function f on a group G at any irreducible representation ρ of G can be defined to be,

$$\hat{f}(\rho) = \sum_{g \in G} f(g)\rho(g). \tag{4.1}$$

Since the size of $SL_2(\mathbb{F}_p)$ is $p(p^2-1)$, the adjacency matrix of $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ is a $p(p^2-1)$ by $p(p^2-1)$ matrix. This adjacency matrix can be thought as the Fourier transform of δ_{S_p} at the regular representation π_R, where $S_p = \{s_1, s_1^{-1}, \cdots, s_k, s_k^{-1}\}$. Then the adjacency matrix can be written as

$$A(\mathcal{G}(SL_2(\mathbb{F}_p)), S_p) = \hat{\delta}_{S_p}(\pi_R) = \pi_R(s_1) + \pi_R(s_1^{-1}) + \cdots + \pi_R(s_k) + \pi_R(s_k^{-1}). \tag{4.2}$$

From representation theory, we know that any irreducible representation $\rho \in \hat{G}$ appears in π_R with multiplicity $\dim \rho$. That is,

$$\pi_R = \rho_0 \bigoplus_{\rho \in \hat{G}, \rho \neq Id} \underbrace{\rho \oplus \cdots \oplus \rho}_{\dim \rho}. \tag{4.3}$$

Therefore, the adjacency matrix is of a diagonal form,

$$\hat{\delta}_{S_p}(\pi_R) \sim diag(\underbrace{B_1, B_1 \cdots B_1}_{\dim B_1}, \cdots, \underbrace{B_r, B_r \cdots B_r}_{\dim B_r}). \tag{4.4}$$

Instead of studying the whole adjacency matrix, we study each diagonal block matrix B_i, which is the Fourier transform of $\hat{\delta}_{S_p}(\rho)$ at irreducible representation $\rho \in \hat{SL}_2(\mathbb{F}_p)$. We denote the block submatrix B_i as $\hat{z}_\rho = \rho(s_1) + \rho(s_1^{-1}) + \cdots + \rho(s_k) + \rho(s_k^{-1})$.

The irreducible representations of $SL_2(\mathbb{F}_p)$ occur in two types, principal representations and discrete representations. The difference between these two types depends on their restrictions to Borel subgroup B of upper triangular matrices. The restrictions of the principal representations to B contain the trivial representation, while the restrictions of the discrete representation do not.

Briefly, the principal representations are constructed by inducing characters from B to $SL_2(\mathbb{F}_p)$, denoted as $\rho_\phi = \phi \uparrow SL_2(\mathbb{F}_p)$, where ϕ is any character of $\mathbb{F}_p^*$, and $\widetilde{\phi}$ is the associated character of Borel subgroup defined by $\widetilde{\phi}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \phi(a)$. Then

$$(\rho_\phi(g)f)(g^{'}) = f(g^{'}g) \text{ with } g \in SL_2(\mathbb{F}_p), \tag{4.5}$$

where f is a complex function on $SL_2(\mathbb{F}_p)$ such that

$$f(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g) = \phi(a)f(g). \tag{4.6}$$

From [NS82], all irreducible principal representations contain:

Theorem 4.1.1.

- *Any ρ_ϕ corresponding to a character ϕ of $\mathbb{F}_p^*$ with $\phi^2 \neq 1$ is irreducible with dimension $p+1$;*
- *Two representations ρ_{ϕ_i} and ρ_{ϕ_j} are equivalent if and only if $\phi_i = \phi_j$, or $\phi_i = \phi_j^{-1}$;*
- *$\rho_1 = 1 + \widetilde{\rho_1}$, where 1 is the trivial representation, and $\widetilde{\rho_1}$ is an irreducible representation with dimension p;*
- *If ϕ_0 assumes the value 1 at all squares in $\mathbb{F}_p^*$, and -1 at all non-squares $\mathbb{F}_p^* \diagdown \mathbb{F}_p^{*^2}$, then $\rho_{\phi_0} = \rho_{\phi_0}^+ + \rho_{\phi_0}^-$, where both $\rho_{\phi_0}^+$ and $\rho_{\phi_0}^-$ are irreducible representations with dimension $\frac{p+1}{2}$.*

- *There are totally $\frac{p+5}{2}$ irreducible principal representations. One might sort them in the order of $\rho_{\phi_0} = 1 \oplus \widetilde{\rho}_{\phi_0}, \rho_{\phi_1}, \cdots, \rho_{\phi_{\frac{p-3}{2}}}, \rho_{\phi_{\frac{p-1}{2}}} = \rho^+_{\phi_{\frac{p-1}{2}}} \oplus \rho^-_{\phi_{\frac{p-1}{2}}}$, where ϕ_i are inequivalent characters of $\mathbb{F}_p^*$.*

Numerically, we fix the coset representatives of B to be

$$s_x = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix}, \cdots, s_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.7}$$

where $x \in \mathbb{F}_p$. Under a basis $f_x(s_y) = \delta_x(y)$, we have

$$\begin{aligned}
&\rho_\phi(u_a)f_\infty = f_\infty, \quad \rho_\phi(u_a)f_x = f_{x-a}, \\
&\rho_\phi(g_\alpha)f_\infty = \phi(\alpha)f_\infty, \quad \rho_\phi(g_\alpha)f_x = \phi(\alpha^{-1})f_{\alpha^2 x}, \\
&\rho_\phi(\omega)f_\infty = \phi(-1)f_0, \quad \rho_\phi(\omega)f_0 = f_\infty, \quad \rho_\phi(\omega)f_x = \phi(u)f_{-x^{-1}},
\end{aligned} \tag{4.8}$$

where $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $g_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $e_x = \sum_{y\in\mathbb{F}_p} \chi(xy)f_y$ and $e_\infty = f_\infty$, where $x \in \mathbb{F}_p$ and χ are non-identity additive characters of $\mathbb{F}_p$. When calculating $\rho^\pm_{\phi_0}$, we need to change basis from $\{f_x\}$ to $\{e_x, \Gamma e_\infty \pm e_0\}$, where $x \in \mathbb{F}_p^*$ and $\Gamma = \sum_{v\neq 0} \chi(v)\phi_0(v)$. After re-grouping the basis in the following order, $\Gamma e_\infty + e_0, e_{x^2}, x \in \mathbb{F}_p^*$ and $\Gamma e_\infty - e_0, e_x, x \in \mathbb{F}_p^* \diagdown \mathbb{F}_p^{*2}$, ρ_{ϕ_0} becomes a 2 by 2 diagonal block form with $\rho^\pm_{\phi_0}$ on diagonal.

On the other hand, discrete representations can be constructed from non-decomposable characters of quadratic extension $\mathbb{F}_p(\sqrt{\varepsilon})/\mathbb{F}_p$. Let $U = \{t \in \mathbb{F}_p(\sqrt{\varepsilon}) | t\bar{t} = 1\}$ be the subgroup of norm 1 elements, where $\bar{t} = x - y\sqrt{\varepsilon}$ for $t = x + y\sqrt{\varepsilon}$. The character ψ of $\mathbb{F}_p(\sqrt{\varepsilon})$ is non-decomposable if $\psi(t) \neq 1$ for $t \in U$. Discrete representations can be realized through $SL_2(\mathbb{F}_p)$

group action on the vector space of complex functions over $\mathbb{F}_p^*$ by

$$(\rho_\psi(g)f)(x) = \sum_{v\in\mathbb{F}_p^*} K_\psi(x,y;g)f(y) \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_p), \tag{4.9}$$

where

$$K_\psi(u,v;g) = \begin{cases} \frac{1}{p}\chi(\frac{dy+ax}{c})\sum_{t\bar{t}=xy^{-1}}\chi(-\frac{yt+x^{-1}\bar{t}}{c})\psi(t) & \text{if } c \neq 0; \\ \psi(d)\chi(dbu)\delta(d^2u-v) & \text{if } c = 0. \end{cases}$$

Here χ is a fixed additive character of $\mathbb{F}_p$, which is not identically equal to 1.

From [NS82], all discrete representations contain:

Theorem 4.1.2. *Denote the restriction of ψ of $\mathbb{F}_p(\sqrt{\varepsilon})$ on U to be π.*

- *Any ρ_π corresponding to π with $\pi^2 \neq 1$ is irreducible with dimension $p-1$;*
- *Two representations ρ_{π_i} and ρ_{π_j} are equivalent if and only if $\pi_i = \pi_j$, or $\pi_i = \pi_j^{-1}$;*
- *If π_1 assumes the value 1 at all squares in U^2, and -1 at all non-squares $U\setminus U^2$, then $\rho_{\pi_1} = \rho_{\pi_1}^+ + \rho_{\pi_1}^-$, where both $\rho_{\pi_1}^+$ and $\rho_{\phi_1}^-$ are irreducible representations with dimension $\frac{p-1}{2}$.*
- *There are totally $\frac{p+3}{2}$ irreducible discrete representations. One might sort them in the order of $\rho_{\pi_1}, \rho_{\pi_2}, \cdots, \rho_{\pi_{\frac{p-1}{2}}}, \rho_{\pi_{\frac{p+1}{2}}} = \rho^+_{\pi_{\frac{p+1}{2}}} \oplus \rho^-_{\pi_{\frac{p+1}{2}}}$. where $\pi_i = \psi_i|_U$ are inequivalent restrictions of characters ψ of $\mathbb{F}_p(\sqrt{\varepsilon})$.*

Under a basis of $f_x(y) = \delta_x(y)$, we have

$$\rho_\psi(u_a)f_x = \chi(ax)f_x,$$

$$\rho_\psi(g_\alpha)f_x = \psi(\alpha^{-1})f_{\alpha^{-2}x},$$

$$\rho_\psi(\omega)f_x = K_\psi(x,y;g)f_y. \tag{4.10}$$

where $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $g_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Theorem 4.1.1 and 4.1.2 provide all $p+4$ irreducible representations of $SL_2(\mathbb{F}_p)$.

4.2 Generating Random Cayley Graphs

This section focuses on the study of 4-regular random Cayley graphs, whose generating set is a random pair and its inverse. We follow the approach from [LR92].

Using Bruhat decomposition, group $SL_2(\mathbb{F}_p)$ can be parameterized as:

$$SL_2(\mathbb{F}_p) = TU\omega U \bigcup TU, \tag{4.11}$$

where T denotes diagonal matrices subgroup of the form $T = \{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} | \alpha \in \mathbb{F}_p^*\}$, and U denotes unipotent matrices subgroup of the form $\{U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} | u \in \mathbb{F}\}$, and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

To uniformly generate an element of $SL_2(\mathbb{F}_p)$, all elements of $SL_2(\mathbb{F}_p)$ are ordered naturally. First, we generate a random integer $1 \le r \le p(p^2-1)$. If $r \le p(p-1)$, then we uniformly pick $\alpha \in \mathbb{F}_p^*$ and $u \in \mathbb{F}_p$, and construct $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in TU$; otherwise, we uniformly pick $\alpha \in \mathbb{F}_p^*$ and $u, v \in \mathbb{F}_p$, and construct

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in TU\omega U \tag{4.12}$$

.

For the uniformly generated random pair from $SL_2(\mathbb{F}_p)$, we still need to check that the random pair does not generate any proper subgroup of $SL_2(\mathbb{F}_p)$, or equivalently the pair does not generate any proper subgroup of $PSL_2(\mathbb{F}_p)$. The subgroups of latter are clearly known, which are the following six types,

Theorem 4.2.1. *[Su82] All possible subgroups of $PSL_2(\mathbb{F}_p)$ are*

- *Abelian groups;*
- *Dihedral groups of order $2n$ with $n|\frac{p\pm1}{2}$;*
- *Alternating group A_4;*
- *Noncommutative subgroups of upper triangular subgroup, and their conjugates;*
- *Symmetric group S_4 when $p^2-1=0 \mod 16$;*
- *Alternating group A_5 when $p=5$ or $p^2-1=0 \mod 5$.*

If the random pair does not generate any subgroup listed in Theorem 4.2.1, one can keep doing the Fourier transform of its indicator function, and compute the spectrum.

4.3 Discussion of Numerical Results of Random Cayley Graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$

In this section, several numerical results of the edge spectra of random Cayley graphs of $SL_2(\mathbb{F}_p)$ under both principal representations and discrete representations are presented. The spectrum for a random Cayley graph is ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, with $N = \dim\rho$. The edge spectrum is normalized as in Equation (3.9) in Chapter 3.

$$\widetilde{\lambda_{max}} = n^{1/6}(\lambda_{max} - \mu_{sample})\sqrt{\frac{n}{2d-1}}, \tag{4.13}$$

where $n = n(N)$ is the solution of Equation (3.6) and (3.7). Then the distributions of λ_1, $|\lambda_N|$, and $\lambda_\pm = \max(\lambda_1, |\lambda_N|)$ are analyzed. It is shown that under any non-identity irreducible representation, the edge spectrum of Fourier transform $\hat{\delta}_{S_p}(\rho)$ fits a properly shifted Tracy-Widom $\beta = 2$ or $\beta = 4$ distribution quite well.

4.3.1 Edge Spectra of Principal Series

The largest integer datatype in MATLAB is 32 bits and $1300 \times (1300^2 - 1) > 2^{31}$. Therefore, in order to keep the uniformity of random pairs, we only allow prime numbers $p \leq 1300$. Here p =83, 163, 307, 587, 643, 733, 877, 997, 1187, 1237, 1283 cases are investigated, and the results are summarized in Table (4.1). For each p, there are four different types of principal representations, $\widetilde{\rho_{\phi_0}}$, $\rho^{\pm}_{\phi_{\frac{p-1}{2}}}$, $\rho_{even\ index}$ and $\rho_{odd\ index}$, while $\rho_{odd\ index}$ and $\rho^{\pm}_{\phi_{\frac{p-1}{2}}}$ are further divided into two subcases depending on $1 \equiv k \mod 4$ or $3 \equiv k \mod 4$. The extra "-" sign comes from the asymmetric mass distribution of the edge spectrum of $\hat{z}_\rho$.

When $p = 1237 \equiv 1 \mod 4$, it is shown in Figures (4.1), (4.2), (4.3), (4.4), that the probability density function of the normalized λ_1 fits well with a shifted $\beta = 2$ Tracy-Widom distribution for $\hat{z}_{\widetilde{\rho}_{\phi_0}}, -\hat{z}_{\rho_{\phi_1}}, \hat{z}_{\rho_{\phi_2}}, \cdots, (-1)^{\frac{p-3}{2}}\hat{z}_{\rho_{\phi_{\frac{p-3}{2}}}}$, and $(-1)^{\frac{p-1}{2}}\hat{z}_{\rho^+_{\phi_{\frac{p-1}{2}}}}$, $(-1)^{\frac{p-1}{2}}\hat{z}_{\rho^-_{\phi_{\frac{p-1}{2}}}}$. In all figures, dark blue color represents the edge spectrum of $\hat{z}_\rho$, and $\beta = 1$, $\beta = 2$, and $\beta = 4$ Tracy-Widom distributions are denoted with red, green, and cyan colors, respectively.

Similarly, the normalized $\lambda_\pm$ of $(-1)^k\hat{z}_{\rho_k}$ when $p = 1237$ are displayed in Figures (4.5). For even index k, $\lambda_\pm$ fits $\beta = 2$ Tracy-Widom distribution. For odd index k, $-\lambda_\pm$ fits $\beta = 4$ Tracy-Widom distribution. When $k = \frac{p-1}{2}$ it fits $\beta = 4$ Tracy-Widom distribution.

Also, for the same $p = 1237$, the normalized λ_N of $(-1)^k\hat{z}_{\rho_k}$ of any index fits $\beta = 2$

Table 4.1: The edge spectrum of a random pair under non-identity principal irreducible representations fits with different Tracy-Widom distributions.

$k = 1, 2, \cdots, \frac{p-3}{2}$	λ_1	λ_N	$\lambda_\pm$
$\hat{z}_{\widetilde{\rho}_{\phi_0}}$	$\beta = 2$	$\beta = 2$	$\beta = 2$
$\hat{z}_{\rho_{\phi_k}}(k \text{ even})$	$\beta = 2$	$\beta = 2$	$\beta = 2$
$-\hat{z}_{\rho_{\phi_k}}(1 \equiv k \mod 4)$	$\beta = 2$	$\beta = 2$	$\beta = 4$
$-\hat{z}_{\rho_{\phi_k}}(3 \equiv k \mod 4)$	$\beta = 2$	$\beta = 4$	$\beta = 4$
$\hat{z}_{\rho^+_{\phi_{\frac{p-1}{2}}}}(1 \equiv p \mod 4)$	$\beta = 2$	$\beta = 2$	$\beta = 4$
$-\hat{z}_{\rho^+_{\phi_{\frac{p-1}{2}}}}(3 \equiv p \mod 4)$	$\beta = 4$	$\beta = 4$	$\beta = 4$

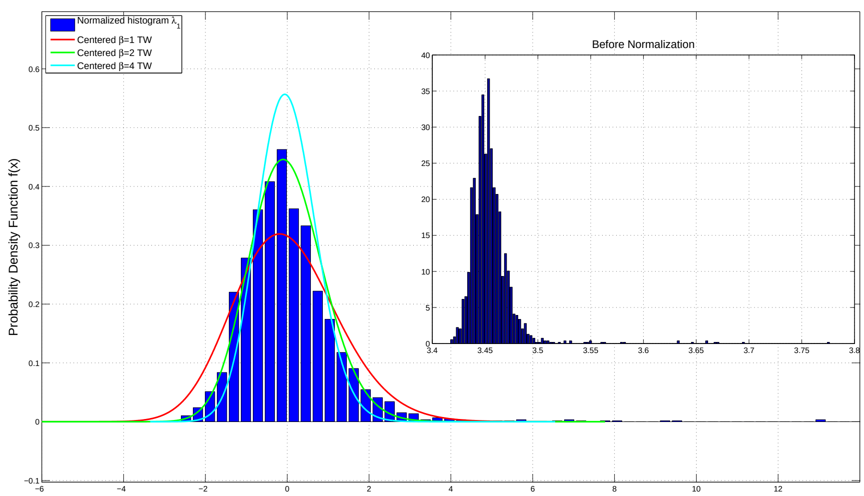


Figure 4.1: Probability density of the normalized λ_1 of $\hat{z}_{\widetilde{\rho}_{\phi_0}}$ of $p = 1237$ with 2000 samples, which fits $\beta = 2$ Tracy-Widom distribution. Recall $\rho_{\phi_0} = 1 \oplus \widetilde{\rho}_{\phi_0}$. So λ_1 actually is the second largest eigenvalue of $\hat{z}_{\rho_{\phi_0}}$.

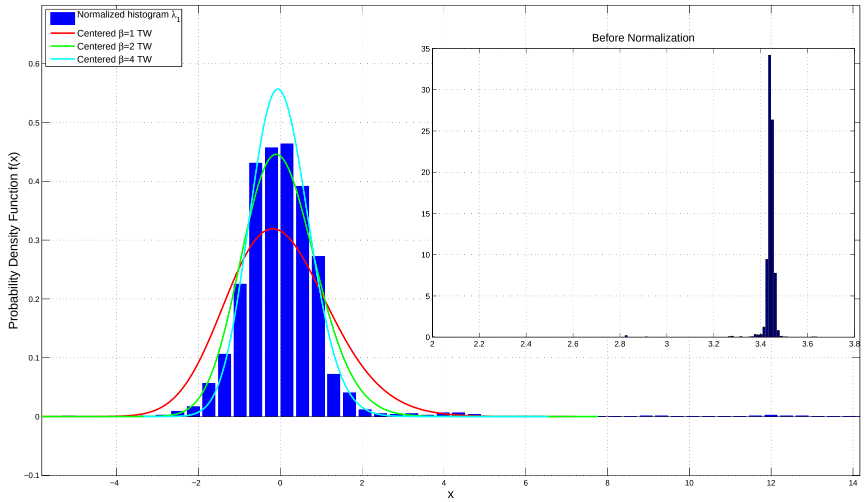


Figure 4.2: Probability density of the normalized $-\lambda_1$ of $\hat{z}_{\rho_{\phi_1}}$ of $p = 1237$ with 2000 samples, which fits $\beta = 2$ Tracy-Widom distribution.

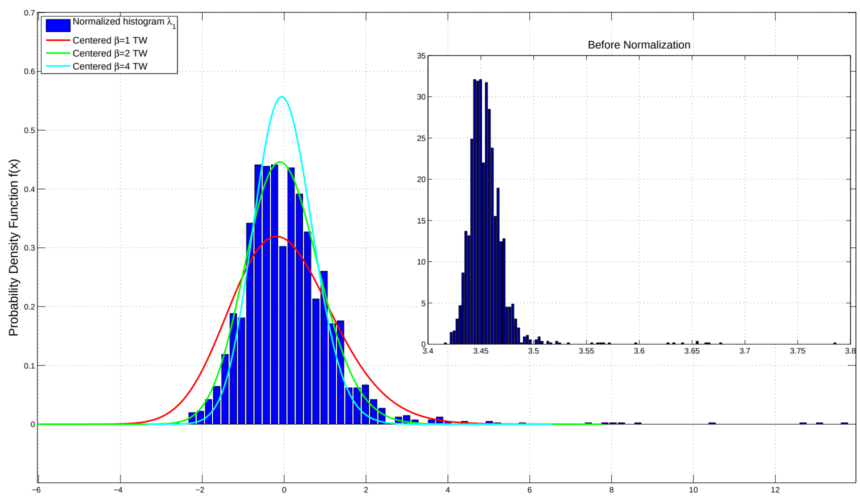


Figure 4.3: Probability density of the normalized λ_1 of $\hat{z}_{\rho_{\phi_2}}$ of $p = 1237$ with 2000 samples, which fits $\beta = 2$ Tracy-Widom distribution.

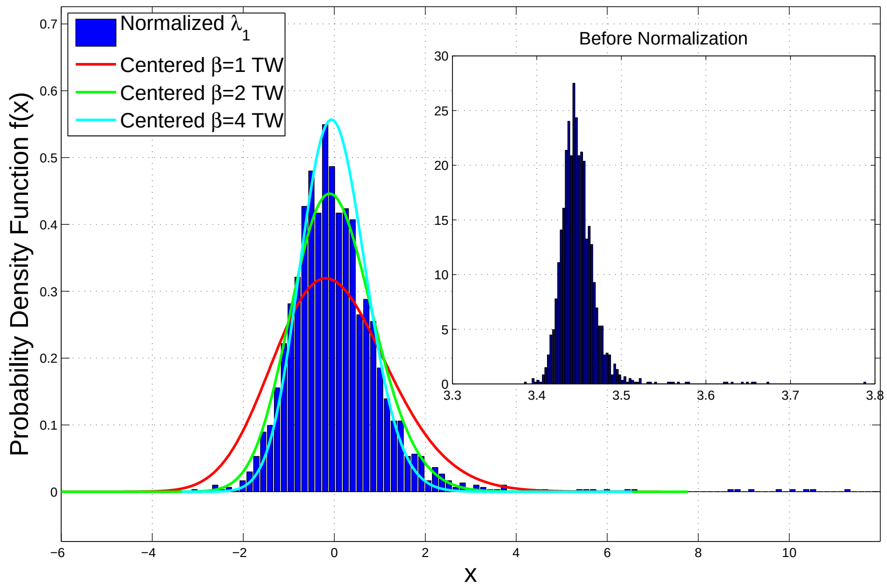


Figure 4.4: Probability density of the normalized λ_1 of $\hat{z}_{\rho^+_{\phi_{618}}}$ of $p = 1237$ with 2000 samples, which fits $\beta = 2$ Tracy-Widom distribution. Recall $\rho_{\phi_{\frac{p-1}{2}}} = \rho^+_{\phi_{\frac{p-1}{2}}} \oplus \rho^-_{\phi_{\frac{p-1}{2}}}$.

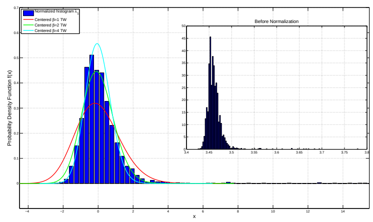
Tracy-Widom distribution, see Figures (4.6).

In all cases, a small discrepancy is observed at the right tail of the distribution, where the sample distribution has a longer tail than Tracy-Widom distribution. This is amplified by the normalization factor $N^{2/3}/\sqrt{3}$. Also the edge spectrum of $\hat{z}_{\rho_{2k+1}}$ of odd index is shown to concentrate more sharply than $\hat{z}_{2k}$ with even index.

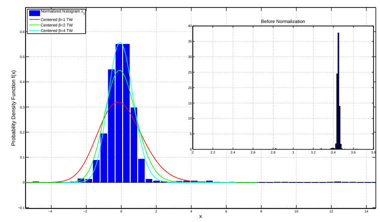
4.3.1.1 Kolmogorov-Smirnov test (KS Test) of λ_1 of Principal Series

Kolmogorov-Smirnov test is used here to verify the fitting between the normalized edge spectrum and Tracy-Widom distribution.

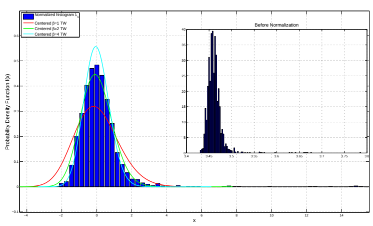
Let $F_n(x) = \frac{1}{n}\sum_1^n I_{X_k \le x}$ be the empirical distribution function for n independently identical distributed samples X_k, and let $F(x)$ be the cumulative distribution function of the theoret-



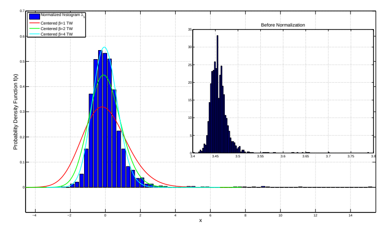
(a) $\lambda_{\pm}$ of $\hat{z}_{\widetilde{\rho}_{\phi_0}}$ fits $\beta = 2$.



(b) $\lambda_{\pm}$ of $-\hat{z}_{\rho_{\phi_1}}$ fits $\beta = 4$.



(c) $\lambda_{\pm}$ of $\hat{z}_{\rho_{\phi_2}}$ fits $\beta = 2$.



(d) $\lambda_{\pm}$ of $\hat{z}_{\rho^{+}_{\phi_{618}}}$ fits $\beta = 4$.

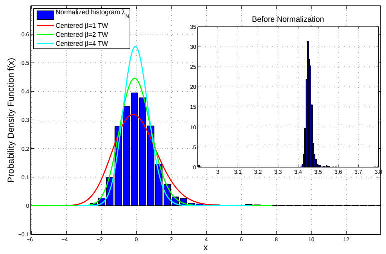
Figure 4.5: Probability density function of the normalized $\lambda_{\pm}$ for $p = 1237$ with 2000 samples.

ical prediction. Discrepancy can be defined as $D_n = \sup_x |F_n(x) - F(x)|$. If D_n is small, then the samples fit theoretical prediction well. If the samples follow theoretical prediction $F(x)$, as $n \to \infty$, the normalized discrepancy converges $\sqrt{n}D_n \to \sup_{t\in[0,1]} |B(F(t))|$, where $B(t)$ is Brownian Bridge. Meanwhile $\sup_{t\in[0,1]} |B(F(t))|$ has a limit distribution, called Kolomogorov Law,

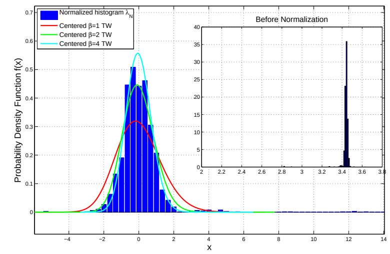
$$Pr(\sup_{t\in[0,1]} |B(t)| \leq z) = \sum_{-\infty}^{\infty}(-1)^k e^{-2k^2z^2}. \tag{4.14}$$

For a given α, one can define K_α to satisfy $Pr(K \leq K_\alpha) = 1 - \alpha$ of Kolomogorov Law. If the normalized discrepancy $\sqrt{n}D_n \leq K_\alpha$, this fitting is good at level of α. Statisticians also use p-value to characterize the fitting, which is the probability of obtaining the actually observed, by assuming that it follows theoretical prediction.

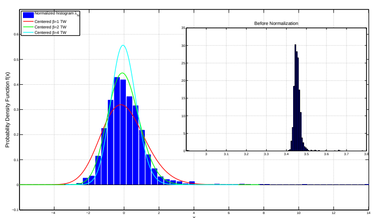
Given $\alpha = 0.05$, KS test results of the normalized edge spectrum are summarized in



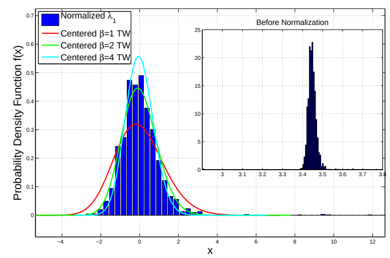
(a) λ_N of $\hat{z}_{\tilde{\rho}_{\phi_0}}$ fits $\beta = 2$.



(b) λ_N of $-\hat{z}_{\rho_{\phi_1}}$ fits $\beta = 2$.



(c) λ_N of $\hat{z}_{\rho_{\phi_2}}$ fits $\beta = 2$.



(d) λ_N of $\hat{z}_{\rho^+_{\phi_{618}}}$ fits $\beta = 2$.

Figure 4.6: Probability density function of the normalized λ_N for $p = 1237$ with 2000 samples.

Table (4.2) and (4.3). $\hat{z}_{\rho_{\phi_k}}$ with even k case is skipped, since its edge spectrum behaviors is found to be nearly identical to $\hat{z}_{\tilde{\rho}_{\phi_0}}$ case, see Figures (4.1) and (4.3). In all KS tests, $KS(x,y) = 0$ denotes that null hypothesis of x and y are from the same continuous distribution, while $KS(x,y) = 1$ denotes that null hypothesis is rejected.

We notice that the fitting and the p-values are roughly improved with the increase of the dimension. In Table (4.3), we compute KS test between λ_1 with both $\beta = 2$ and $\beta = 4$ Tracy-Widom distributions. If both tests accept null hypothesis, their P-values are further compared to make a conclusion. In Table (4.3), $p = 307$ case is the only exception, which dose not follow the results summarized in Table (4.1). And $p = 83$ case rejects both KS tests.

Table 4.2: Kolomogorov test between the shifted $\beta = 2$ Tracy-Widom distribution and the normalized λ_1.

(a) λ_1 of $\hat{z}_{\tilde{\rho}_{\phi_0}}$ with 2000 Samples.

Dimension	KS Test	P-value
$p = 1283$	0	0.1441
$p = 1237$	0	0.2816
$p = 1187$	0	0.3072
$p = 997$	0	0.1298
$p = 877$	0	0.1298
$p = 733$	0	0.1763
$p = 643$	0	0.1763
$p = 587$	1	0.0412
$p = 307$	0	0.1048
$p = 163$	0	0.1441
$p = 83$	0	0.0666

(b) λ_1 of $-\hat{z}_{\rho_{\phi_1}}$ with 2000 Samples.

Dimension	KS Test	P-value
$p = 1283$	0	0.3632
$p = 1237$	0	0.1441
$p = 1187$	0	0.3072
$p = 997$	0	0.4586
$p = 877$	0	0.2575
$p = 733$	0	0.1048
$p = 643$	0	0.5654
$p = 587$	0	0.2575
$p = 307$	0	0.4586
$p = 163$	0	0.1168
$p = 83$	1	0.0122

Table 4.3: Kolomogorov test between the shifted $\beta = 4$ Tracy-Widom distribution and the normalized λ_1 of $(-1)^{\frac{p-1}{2}} \hat{z}_{\rho_{\phi_{\frac{p-1}{2}}}^{+}}$ with 2000 Samples.

Dimension	KS Test($\beta = 2$)	P-value	KS test($\beta = 4$)	P-value	Result
$p = 1283$	1	0.0036	0	0.1441	$\beta = 4$
$p = 1237$	0	0.1168	0	0.0939	$\beta = 2$
$p = 1187$	1	0.0043	0	0.3344	$\beta = 4$
$p = 997$	0	0.3344	1	0.0141	$\beta = 2$
$p = 877$	0	0.1944	1	0.0141	$\beta = 2$
$p = 733$	0	0.5654	1	0.0141	$\beta = 2$
$p = 643$	1	0.0122	0	0.3344	$\beta = 4$
$p = 587$	1	0.0244	0	0.2350	$\beta = 4$
$p = 307$	0	0.1048	0	0.0666	$\beta = 2$
$p = 163$	0	0.1168	0	0.1298	$\beta = 4$
$p = 83$	1	0.0079	1	0.0036	NA

4.3.2 Edge Spectra of Discrete Series

In practice, the computation of Fourier transform under discrete representations is more time consuming than principal representations. So only p =37, 43, 53, 67, 107, 157 cases are investigated due to computer limitations. For each p, there are three types of irreducible discrete representations, $\rho_{\pi_{keven}}$, $\rho_{\pi_{kodd}}$ and $\rho^{\pm}_{\pi_{\frac{p+1}{2}}}$, where $k = 1, 2, \cdots, \frac{p-1}{2}$. The probability density functions of 1000 samples of the normalized λ_1 for $p = 157$ are displayed in Figure (4.7), (4.8) and (4.9). More computations are needed to draw a confirmed answer for $\hat{z}_\pi$ at discrete representations.

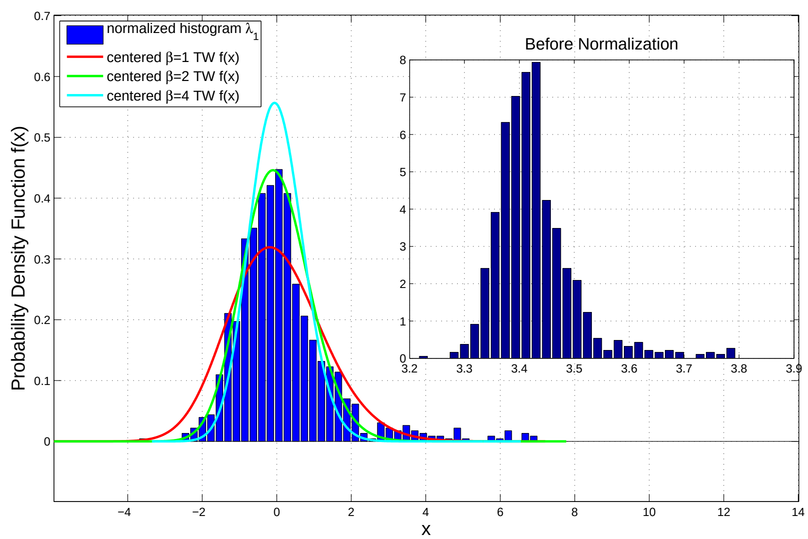


Figure 4.7: Probability density function of the normalized λ_1 of $\hat{z}_{\rho_{\pi_2}}$ of $p = 157$ with 1000 samples, which fits $\beta = 2$ centered Tracy-Widom distribution.

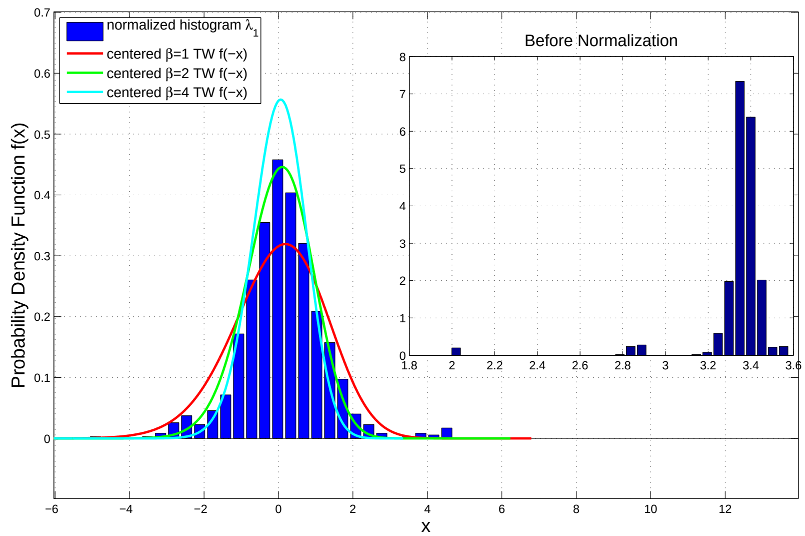


Figure 4.8: Probability density function of the normalized λ_1 of $-\hat{z}_{\rho_{\pi_3}}$ of $p = 157$ with 1000 samples, which fits $\beta = 2$ centered Tracy-Widom distribution.

4.3.3 Correlations Between Edge Spectra of Different Irreducible Representations

A numerical study of the distribution of the edge spectrum of each irreducible representation (i.e each block inside the adjacency matrix of Cayley graphs) is carried out in Section 4.3.1 and 4.3.2. Though the adjacency matrix is block diagonalized, blocks are not independent of each other. Intuitively, for the same generator set, if one block shows large non-trivial eigenvalue, the other blocks under different irreducible representations should have higher probability to behavior similarly. To confirm this intuition, edge spectrum correlations between different irreducible representations are studied here.

Recall for given samples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, the correlation

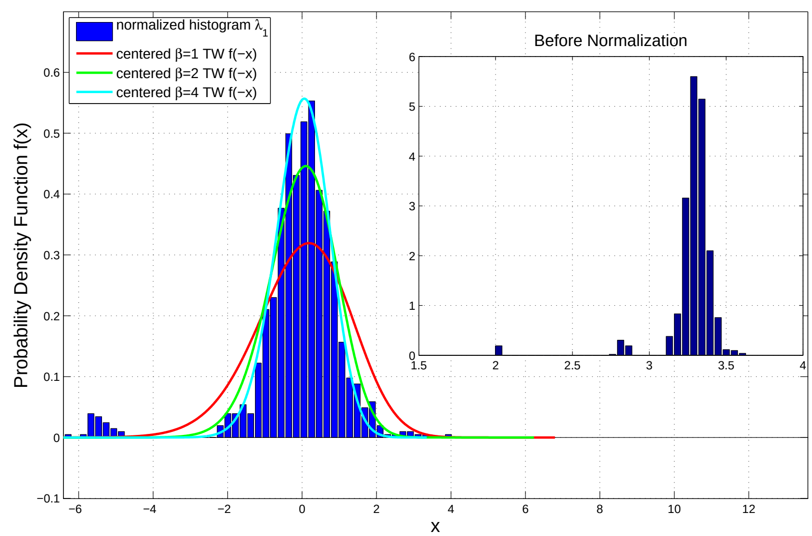


Figure 4.9: Probability density function of the normalized λ_1 of $(-1)^{\frac{157+1}{2}}\hat{z}_{\rho_{\pi_{79}}}$ of $p = 157$ with 1000 samples, which fits $\beta = 4$ centered Tracy-Widom distribution. Recall $\rho_{\pi_{\frac{p+1}{2}}} = \rho^+_{\pi_{\frac{p+1}{2}}} \oplus \rho^-_{\pi_{\frac{p+1}{2}}}$.

coefficient is defined as

$$r_{xy} = \frac{\sum_1^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_1^n (x_i - \bar{x})^2 \sum_1^n (y_i - \bar{y})^2}}$$

Our numerical experiments show that for all three cases λ_1, λ_N, $\lambda_\pm$, they are either almost fully correlated with correlation coefficient ~ 0.95 or strong correlated with correlation coefficient $\sim \pm 0.60$, where $\pm$ sign depends on index even or odd, no matter under principal representations or discrete representations. More important, there are exact half of them with positive correlation coefficients, and half of them with negative coefficients. Correlations of $\lambda_\pm$ between principal representations and discrete representations with 1000 samples for $p = 67$ are shown in Figure (4.10)

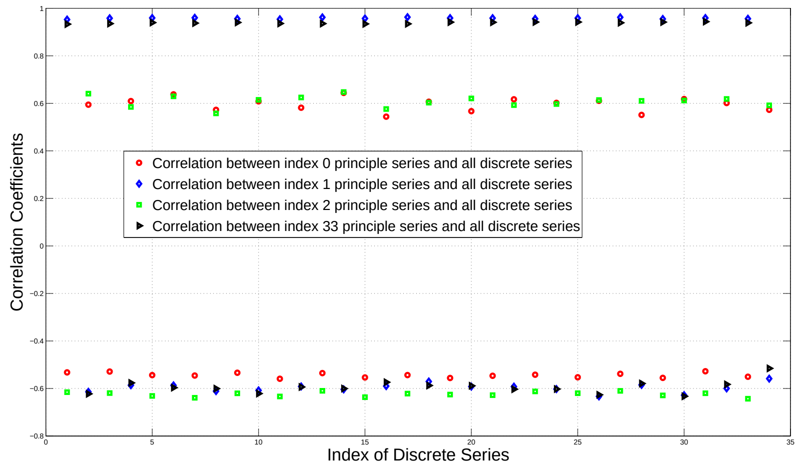


Figure 4.10: Correlation coefficients of $\lambda_{\pm}$ between all principal irreducible representations and discrete irreducible representations for the same 1000 random generating pairs when $p = 67$. Recall we order principal representations using index $0 : \frac{p-1}{2}$, and discrete representations using index $1 : \frac{p+1}{2}$.

4.3.4 Sample Mean and Sample Variance

Sample mean has its own interest of connection with the Ramuanjuan bound. To study the difference between sample mean and the Ramanujan bound $2\sqrt{d-1}$, edge eigenvalues are standardized with mean 0 and variance 1 in the following form,

$$\widetilde{\lambda_1} = \frac{\lambda_1 - 2\sqrt{2k-1} + c_1 N^m}{c_2 N^s}. \tag{4.15}$$

The numerical experiments of random pairs show that $\log c_1 N^m$ and $\log c_2 N^s$ are approximate to linear functions of $\log N$. For example, for $\lambda_\pm$ case one obtain followings,

$$
\begin{aligned}
\log c_1 N^m &\sim -0.70\log N + 2.00 \text{ for } \hat{z}_{\tilde{\rho}_{\phi_0}}, \\
\log c_2 N^s &\sim -0.56\log N + 0.30; \\
\log c_1 N^m &\sim -0.75\log N + 1.65 \text{ for } \hat{z}_{\rho_{\phi_1}}, \\
\log c_2 N^s &\sim -0.59\log N + 1.23; \\
\log c_1 N^m &\sim -0.78\log N + 1.79 \text{ for } \hat{z}_{\rho^+_{\phi_{\frac{p-1}{2}}}} \text{ when } p \equiv 3 \mod 4, \\
\log c_2 N^s &\sim -0.56\log N + 1.19; \\
\log c_1 N^m &\sim -0.47\log N - 0.83 \text{ for } \hat{z}_{\rho^+_{\phi_{\frac{p-1}{2}}}} \text{ when } p \equiv 1 \mod 4, \\
\log c_2 N^s &\sim -0.77\log N + 1.75. \qquad (4.16)
\end{aligned}
$$

For all cases, we have $c_1 > 0$, and $c_2 > 0$. This tells that the sample mean μ_{sample} of the edge spectrum is less than the Ramanujan bound $2\sqrt{3}$, and its distance decays at a rate of N^{-m}. Also the edge spectrum will concentrate at a rate of N^{-s}. Interestedly, we have $s > m$ except $\hat{z}_{\rho^\pm_{\phi_{\frac{p-1}{2}}}}$ when $p \equiv 1 \mod 4$. Similar results hold for λ_1 and $|\lambda_N|$ under any non-identity irreducible principal representation. While under discrete irreducible representations, the edge spectrum also has similar results with no exception. Results of $\lambda_\pm$ of $\hat{z}_{\rho_\pi}$ are listed below in

Equation (4.17).

$$
\begin{aligned}
\log c_1 N^m &\sim -0.78 \log N + 0.57 \text{ for } \hat{z}_{\rho_{\pi_2}}, \\
\log c_2 N^s &\sim -0.54 \log N + 0.16; \\
\log c_1 N^m &\sim -0.74 \log N + 1.62 \text{ for } \hat{z}_{\rho_{\pi 3}}, \\
\log c_2 N^s &\sim -0.41 \log N + 0.33; \\
\log c_1 N^m &\sim -0.77 \log N + 1.41 \text{ for } \hat{z}_{\rho^+_{\pi_{\frac{p+1}{2}}}} \text{ when } p \equiv 3 \mod 4, \\
\log c_2 N^s &\sim -0.67 \log N + 0.83; \\
\log c_1 N^m &\sim -0.77 \log N - 2.17 \text{ for } \hat{z}_{\rho^+_{\pi_{\frac{p+1}{2}}}} \text{ when } p \equiv 1 \mod 4, \\
\log c_2 N^s &\sim -0.44 \log N + 0.41.
\end{aligned} \tag{4.17}
$$

Since $s > m$, $c_1 > 0$ and $c_1 > 0$, Equation (4.18) tells that the Ramanujan bound $2\sqrt{3}$ has zero standard deviation to the right of the sample mean μ_{sample} when $p \to \infty$ and $p \not\equiv 1 \mod 4$ for any non-identity irreducible representation. While $p \equiv 1 \mod 4$, to make Equation (4.18) still hold, σ_{sample} must decay at speed of N^{s-m}. Also sample mean μ_{sample} is "far" to the left side of the Ramanujan bound $2\sqrt{3}$, which provides a better chance to be Ramanujan graphs.

$$2\sqrt{3} \approx \mu_{sample} + \frac{c_1}{c_2} N^{m-s} \sigma_{sample}, \tag{4.18}$$

Conjecture 4.3.1. *Random Cayley graphs $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ have better chance to be Ramanujan for $p \equiv 1 \mod 4$ than other prime numbers.*

Chapter 5

Random Hecke Operators of $SU(2)$

In Chapter 4, the distribution of the edge spectra of random Cayley graphs of $\mathcal{G}(SL_2(\mathbb{F}_p), S_p)$ has been shown numerically to be certain Tracy-Widom distribution. Similar behaviors for a type of random Hecke operators of $SU(2)$ are shown in this chapter.

Lubotzky, Philips and Sarnak [LPS86] first introduced those arithmetic Hecke operators on S^2, which evenly distribute points $\{p, Hp, \cdots, H^n p \cdots\}$ over S^2. Gamburd and Bourgain [BGSU08], fully established the spectral gap property $\lambda_1 < 2k$ for any random Hecke type of operators with form,

$$z_{g_1,\cdots,g_k} f(x) = \sum_1^k (f(g_i x) + f(g_i^{-1} x)) \text{ for } k \geq 2, \text{ and } f(x) \in L^2(SU(2)), \tag{5.1}$$

where $g_1, \cdots, g_k$ is a finite set of elements in $G = SU(2)$ generating a free group and satisfying non-commutative diophantine property. Gamburd, Jakobson and Sarnak ([GJS99]) also found that the bulk spectrum of $\hat{z}_{\pi_N}$ converges to GOE/GSE bulk spectrum for dimension even/odd, and the level spacing of $\hat{z}_{\pi_N}$ fits very well with GOE/GSE statistics for dimension even/odd. This connection with random matrices also suggests that the properly normalized edge spectra of random Hecke operators as in Equation (5.1) would follow $\beta = 1/\beta = 2$ Tracy-Widom distributions. This result is shown in this Chapter numerically.

5.1 Generating of $SU(2)$ Random Hecke Operators

The irreducible representation of $G = SU(2)$ is $\pi_N = sym^N V$ with $N \geq 0$, where V is the standard two dimensional representation of G. It can be represented by linear action as

$$(x,y) \longrightarrow (\alpha x + \gamma y, \beta x + \delta y), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G \tag{5.2}$$

on W_{N+1}, the space of homogeneous polynomials in (x,y) of degree N. Thus the dimension of π_N is $N+1$.

To generate $SU(2)$ elements uniformly with respect to Haar measure, we use Euler angle to parametrize $SU(2)$. Let $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, then the generic element of $SU(2)$ can be written as

$$g = e^{\phi\sigma_1} e^{\theta\sigma_3} e^{\psi\sigma_2}, \tag{5.3}$$

with $\phi \in [0,\pi]$, $\psi \in [0,2\pi]$, $\theta \in [0,\pi/2]$. Here ϕ, ψ, θ are so-called Euler angles of $SU(2)$. Hence, Haar measure can be written in terms of ϕ, ψ, θ as $dg \sim \sin\theta d\theta d\phi d\psi$.

The matrix realization of a random Hecke operator z can be thought as the Fourier transformation of $\delta = \sum_1^k \delta_{g_i} + \delta_{g_i^{-1}}$ at an irreducible representation π_N. Thus,

$$\begin{aligned} \hat{z}_{\pi_N} &= \sum_{i=1}^{k} \pi(e^{\phi_i\sigma_1})\pi(e^{\theta_i\sigma_3})\pi(e^{\psi_i\sigma_2}) + (\pi(e^{\phi_i\sigma_1})\pi(e^{\theta_i\sigma_3})\pi(e^{\psi_i\sigma_2}))^{-1} \\ &= \sum_{i=1}^{k} e^{d\pi(\phi_i\sigma_1)} e^{d\pi(\theta_i\sigma_3)} e^{d\pi(\psi_i\sigma_2)} + \left(e^{d\pi(\phi_i\sigma_1)} e^{d\pi(\theta_i\sigma_3)} e^{d\pi(\psi_i\sigma_2)}\right)^{*T}. \end{aligned} \tag{5.4}$$

Here $d\pi$ is the induced representation on Lie algebra $su(2)$. Under a basis $e_j = \frac{x^j y^{N-j}}{\sqrt{j!(N-j)!}}$, the

$N+1$ dimensional induced representation has the forms,

$$d\pi(\sigma_1)(e_j) = i(2j-N)e_j, \tag{5.5}$$

$$d\pi(\sigma_2)(e_j) = -\sqrt{j(N-j+1)}e_{j-1} + \sqrt{(N-j)(j+1)}e_{j+1}, \tag{5.6}$$

$$d\pi(\sigma_3)(e_j) = i\sqrt{j(N-j+1)}e_{j-1} + i\sqrt{(N-j)(j+1)}e_{j+1}, \tag{5.7}$$

where $j = 0, 1, \cdots, N$.

By applying Equation (5.4), we can compute the matrix realization of random Heck operators. After that, the edge spectrum can be normalized as

$$\widetilde{\lambda_{max}} = n^{1/6}(\lambda_{max} - \mu_{sample})\sqrt{\frac{n}{2d-1}}, \tag{5.8}$$

where $n = n(N)$ is the solution of Equation (3.6) and (3.7) in Chapter 3.

5.2 Discussion of Numerical Results of $SU(2)$ Random Hecke Operators $\hat{z}_{\pi_N}$

As in Chapter 4, the spectra of $SU(2)$ random Hecke operators $\hat{z}_{\pi_N}$ are sorted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N+1}$. Results of the distributions of the normalized edge spectrum λ_1, $|\lambda_{N+1}|$, and $\lambda_{\pm} = \max(\lambda_1, |\lambda_{N+1}|)$ of $\hat{z}_{\pi_N}$ are illustrated for even N =50, 100, 200, 300, 500, 800, 1000, 1200, 1500, 2000, 2500, 3000, and for odd N =51, 101, 201, 301, 501, 801, 1001, 1201, 1501, 2001, 2501, 3001. It is showed that for even N the normalized λ_1 and $|\lambda_{N+1}|$ fit $\beta = 1$ Tracy-Widom distribution quite well, and the normalized $\lambda_{\pm}$ fits $\beta = 2$ Tracy-Widom distribution. While for odd N, the distributions of the normalized λ_1, $|\lambda_{N+1}|$, and $\lambda_{\pm}$ are all showed to be fitted with $\beta = 2$ Tracy-Widom distribution.

5.2.1 The Edge Spectra of Random Hecke Operators of SU(2)

The probability density function of the normalized largest eigenvalue λ_1 for $k = 3$ with $N = 3000$ (i.e. $dim = 3001$) is displayed in Figure (5.1). Similarly, with the same random samples as in Figure (5.1), $k = 3$ and $N = 3000$, the probability density function of the normalized smallest eigenvalue $|\lambda_{N+1}|$ and the normalized maximal eigenvalue $\lambda_{\pm}$ are shown in Figure (5.2) and (5.3). In all figures, dark blue color represents the edge spectrum of $\hat{z}_{\pi_N}$. $\beta = 1$, $\beta = 2$, and $\beta = 4$ Tracy-Widom distributions are denoted with red, green, and cyan colors, respectively. It is shown that the normalized edge spectrum fits well with certain Tracy-Widom distributions except at the right tail. The normalized $\lambda_{\pm}$ fits $\beta = 2$ instead of $\beta = 1$ Tracy-Widom distribution, which might be due to the large correlations between λ_1 and λ_{N+1}, see Figure (5.7).

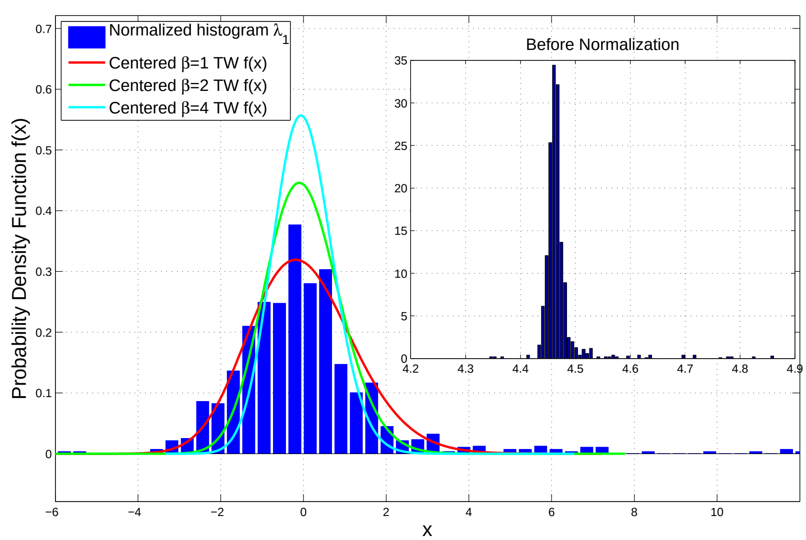


Figure 5.1: Probability density function of the normalized λ_1 of $\hat{z}_{\pi_{3000}}$ with 1000 samples, which fits $\beta = 1$ Tracy-Widom distribution.

When N is odd, [GJS99] shows that the level spacing of de-symmetrized (i.e. remov-

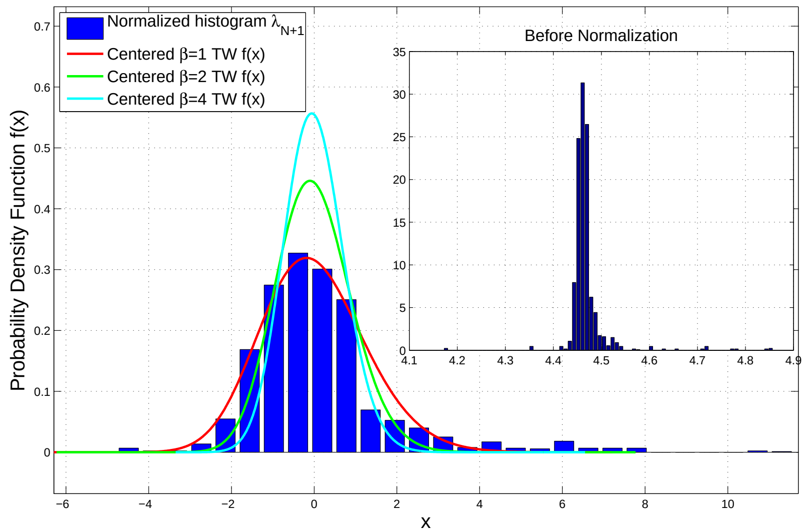


Figure 5.2: Probability density function of the normalized λ_{3001} of $\hat{z}_{\pi 3000}$ with 1000 samples, the same $\hat{z}_{\pi 3000}$ as in Figure 5.1, which fits $\beta = 1$ Tracy-Widom distribution.

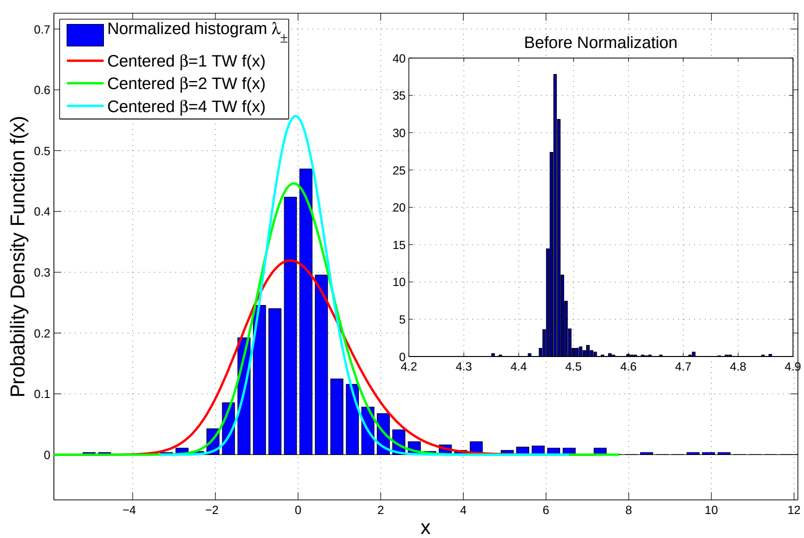


Figure 5.3: Probability density of the normalized $\lambda_{\pm}$ of $\hat{z}_{\pi 3000}$ with 1000 samples, the same $\hat{z}_{\pi 3000}$ as in Figure 5.1, which fits $\beta = 2$ Tracy-Widom distribution.

ing multiplicities) $\hat{z}_{\pi_N}$ follows $\beta = 4$ level spacing distribution. Since de-symmetry does not make any difference for the edge spectrum, the normalized λ_1, λ_{N+1}, and $\lambda_\pm$ of $-\hat{z}_{\pi_N}$ now all fit $\beta = 2$ Tracy-Widom distribution. $N = 3001$ (i.e. $dim = 3002$) cases are shown in Figure (5.4), Figure (5.5) and Figure (5.6). The extra "-"sign is due to the asymmetric mass distribution of the edge spectrum of both random Hecke operators and random matrices.

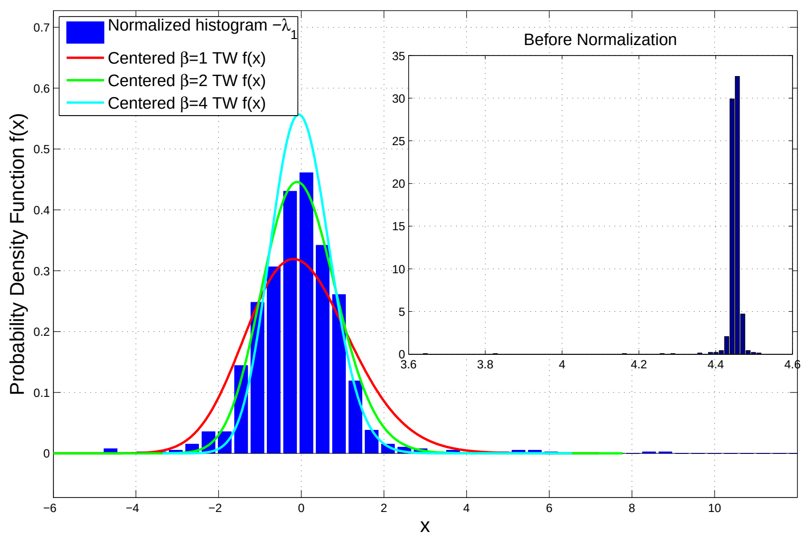


Figure 5.4: Probability density function of the normalized λ_1 of $-\hat{z}_{\pi_{3001}}$ with 1000 samples, which fits $\beta = 2$ Tracy-Widom distribution.

Recall for a given α, if discrepancy $\sqrt{n}D_n \leq K_\alpha$, where K_α satisfying Kolomogorov law $Pr(K \leq K_\alpha) = 1 - \alpha$, one says that the fitting is good at level α. And p-value is the probability of obtaining the actually observes, by assuming observes follow theoretical prediction. See Section 4.3.1.1 of Chapter 4.

Given $\alpha = 0.05$, the results for the normalized λ_1 and λ_{N+1} with even N are summarized in Table (5.1), and the results of $\lambda_\pm$ with even N are shown in Table (5.2). We notice the

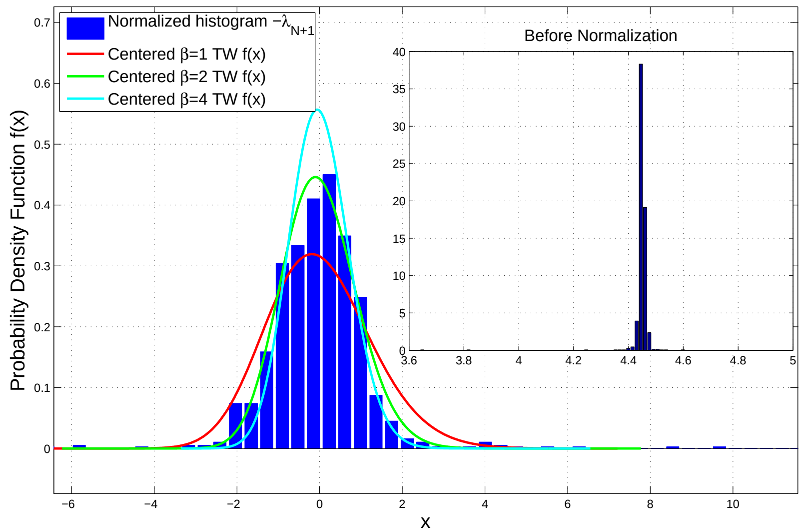


Figure 5.5: Probability density function of the normalized λ_{N+1} of $-\hat{z}_{\pi_{3001}}$ with 1000 samples, the same $\hat{z}_{\pi_{3001}}$ as in Figure 5.4, which fits $\beta = 2$ Tracy-Widom distribution.

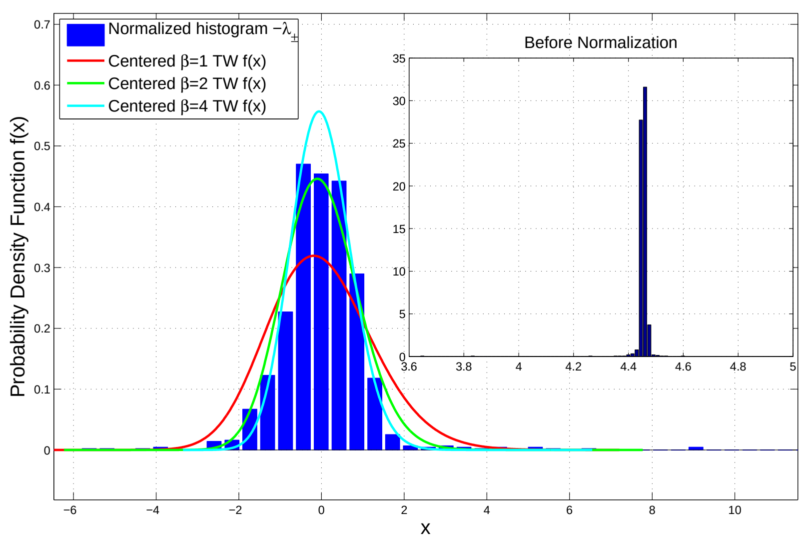


Figure 5.6: Probability density function of the normalized $\lambda_{\pm}$ of $-\hat{z}_{\pi_{3001}}$ with 1000 samples, the same $\hat{z}_{\pi_{3001}}$ as in Figure 5.4, which fits $\beta = 2$ Tracy-Widom distribution.

Table 5.1: Kolomogorov test between the shifted $\beta = 1$ Tracy-Widom distribution and the normalized λ_1 and $\lambda_{N+1}$of $\hat{z}_{\pi_N}$ with 1000 samples.

Dim $N+1$	KS (λ_1)	P-value	KS (λ_{N+1})	P-value
$N = 50$	0	0.5566	0	0.2419
$N = 100$	0	0.1238	0	0.2350
$N = 200$	0	0.2184	0	0.2184
$N = 300$	0	0.3599	0	0.4493
$N = 500$	0	0.1909	0	0.1056
$N = 800$	0	0.2184	0	0.1238
$N = 1000$	0	0.2829	0	0.1238
$N = 1200$	0	0.1326	0	0.1907
$N = 1500$	0	0.3584	0	0.1657
$N = 2000$	0	0.1863	0	0.1711
$N = 2500$	0	0.1774	0	0.1467
$N = 3000$	0	0.2208	0	0.1238

fit and the p-value are roughly improved with the increase of the dimension. Also, $\lambda_{\pm}$ rejects all KS test with $\beta = 1$ Tracy-Widom distribution with at level of $\alpha = 0.05$.

In all KS test tables, $KS(x,y) = 0$ denotes that null hypothesis of x and y are from the same continuous distribution, while $KS(x,y) = 1$ denotes that null hypothesis is rejected.

Shown in Figure (5.4), Figure (5.5), and Figure (5.6), when N is odd, the normalized edge spectrum of $-\hat{z}_{\pi_N}$ matches with $\beta = 2$ Tracy-Widom distribution. Table (5.3) summarizes Kolomogorov test results, which confirm the above observations.

Table 5.2: Kolomogorov test between the shifted $\beta = 2$ Tracy-Widom distribution and the normalized $\lambda_{\pm}$ of $\hat{z}_{\pi_N}$ with 1000 samples

Dim $N+1$	KS Test($\beta = 2$)	P-value	KS Test($\beta = 1$)	P-vale
$N = 50$	0	0.1436	1	0.0066
$N = 100$	0	0.2829	1	0.0002
$N = 200$	0	0.2491	1	0.0068
$N = 300$	0	0.1657	1	0.0079
$N = 500$	0	0.1063	1	0.0002
$N = 800$	0	0.4031	1	0.0013
$N = 1000$	0	0.2491	1	0.0009
$N = 1200$	0	0.1907	1	0.0002
$N = 1500$	0	0.1238	1	0.0022
$N = 2000$	0	0.4031	1	0.0022
$N = 2500$	0	0.1907	1	0.0011
$N = 3000$	0	0.4981	1	0.0007

Table 5.3: Kolomogorov test between the shifted $\beta = 2$ Tracy-Widom distribution and the normalized λ_1, λ_{N+1} and $\lambda_{\pm}$ of $-\hat{z}_{\pi_N}$ with 1000 samples. 0 denote KS test accepted, while 1 denote rejected.

Dim $N+1$	KS (λ_1)	P-value	KS (λ_{N+1})	P-value	KS ($\lambda_{\pm}$)	P-value
$N = 51$	1	0.0008	1	0.0007	1	0.0141
$N = 101$	1	0.0122	1	0.0244	0	0.1763
$N = 201$	1	0.0410	0	0.0666	0	0.1944
$N = 301$	0	0.1796	0	0.4493	0	0.2350
$N = 501$	0	0.4253	0	0.4586	0	0.1763
$N = 801$	0	0.5288	0	0.5654	0	0.1596
$N = 1001$	0	0.6785	0	0.2816	0	0.4586
$N = 1201$	0	0.3344	0	0.4931	0	0.1168
$N = 1501$	0	0.6785	0	0.3632	0	0.1441
$N = 2001$	0	0.3632	0	0.4253	0	0.1907
$N = 2501$	0	0.3344	0	0.4586	0	0.1763
$N = 3001$	0	0.3935	0	0.2816	0	0.1658

By computing the sample correlation coefficient $\frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ (see Equation 4.15), the numerical experiments indicate that there are strong correlations between λ_1 and λ_{N+1} for both N even and odd, see Figure (5.7).

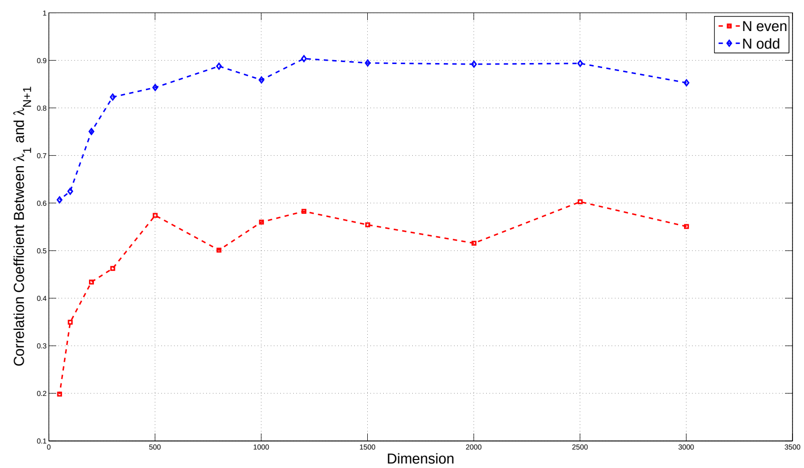


Figure 5.7: Correlation coefficient between λ_1 and $|\lambda_{N+1}|$ of random Hecke operators.

5.2.2 Sample Mean and The Ramanujan Bound

As in Section (4.3.4), edge eigenvalues are standardized with mean 0 and variance 1 as follows,

$$\widetilde{\lambda_1} = \frac{\lambda_1 - 2\sqrt{2k-1} + c_1 N^m}{c_2 N^s}. \tag{5.9}$$

The logarithm of the standard derivation $\log c_1 N^s$ and the logarithm of the difference to the Ramanujan bound $\log c_1 N^m$ are shown to be linear functions of $\log N$ approximately. In the case of λ_1 for $k = 3$ and $N = 3000$ even, we have,

$$\log c_2 N^s \sim -0.53 \log N + 0.40 \text{ for } \lambda_1, \tag{5.10}$$

$$\log c_1 N^m \sim -0.76 \log N + 0.85. \tag{5.11}$$

Similarly, when $k = 3$ and $N = 3000$ even, for λ_{N+1} and $\lambda_{\pm}$, we have

$$\begin{aligned}
\log c_2 N^s &\sim -0.52\log N + 0.35 \text{ for } \lambda_{N+1}, \\
\log c_1 N^m &\sim -0.78\log N + 0.80, \\
\log c_2 N^s &\sim -0.51\log N + 0.20 \text{ for } \lambda_{\pm}, \\
\log c_1 N^m &\sim -0.82\log N + 0.66.
\end{aligned} \tag{5.12}$$

While $N = 3001$ odd, for $k = 3$, we have

$$\begin{aligned}
\log c_2 N^s &\sim -0.43\log N + 0.27 \text{ for } \lambda_1, \\
\log c_1 N^m &\sim -0.68\log N + 1.74, \\
\log c_2 N^s &\sim -0.44\log N + 0.30 \text{ for } \lambda_{N+1}, \\
\log c_1 N^m &\sim -0.69\log N + 1.79, \\
\log c_2 N^s &\sim -0.43\log N + 0.22 \text{ for } \lambda_{\pm}, \\
\log c_1 N^m &\sim -0.68\log N + 1.50,
\end{aligned} \tag{5.13}$$

In all cases, the numerical experiments indicate that $c_1 > 0$, $c_2 > 0$ and $s > m$. This indicates that $2\sqrt{5} - \mu_{edgespectrum} > 0$, and it decays to 0 at a rate of N^{-m}. μ_{sample} will converge to the Ramanujan bound $2\sqrt{5}$. Also the variance decays at a rate of N^{-s}, which tells that the edge spectrum is getting more and more concentrated.

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