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# MICROLOCAL STUDY OF S-MATRIX SINGULARITY STRUCTURE



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# MICROLOCAL STUDY OF S-MATRIX SINGULARITY STRUCTURE



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Abstract

Support is adduced for two related conjectures of simplicity of the analytic structure of the S-matrix and related function; namely, Sato's conjecture that the S-matrix is a solution of a maximally over-determined system of pseudo-differential equations, and our conjecture that the singularity spectrum of any bubble diagram function has the conormal structure with respect to a canonical decomposition of the solutions of the relevant Landau equations. This latter conjecture eliminates the open sets of allowed singularities that existing procedures permit.

## 1. Introduction

Two topics concerning the S-matrix singularity structure are discussed. The first is Sato's conjecture (1) that the S-matrix is a solution of a maximally over-determined system of pseudo-differential equations whose characteristic variety is given by the Landau equations. This property has already been established for a large class of Feynman integrals, and was used to investigate the character of the singularities associated with contracted diagrams (2). Sato's conjecture, if true, would provide a powerful tool for the determination of the analytic properties of scattering amplitudes.

Sato proposed, as a first test of his conjecture, a check of its consistency with the S-matrix discontinuity formulas. Some positive results along this line are reported here.

The second topic concerns the singularity structure of bubble diagram functions. These functions arise in the derivation of the S-matrix discontinuity formulas from unitarity and analyticity; both unitarity and the discontinuity formulas are expressed in terms of them. Bubble diagram functions are represented by bubble dia-

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grams, and are formed by integrating products of scattering amplitudes and complex conjugate amplitudes over the mass-shell variables corresponding to the internal lines of these diagrams.

The singularities of bubble diagram functions are controlled by the structure theorem (3). This theorem places conditions on the complete singularity spectrum of bubble diagram functions. That is, it limits the allowed real positions of singularities, and for each allowed real position it specifies an allowed set of imaginary directions from which the real point can be approached, staying in the domain of analyticity. This information combined with unitarity yields the S-matrix discontinuity formulas (4).

The structure theorem derived in (3) allows singularities corresponding to a certain degenerate case. This case is defined by an equation $u = 0$, and the corresponding allowed locations of singularities are called $u = 0$ points. These points cannot occur when the positive-$\alpha$ condition is imposed. But to study nonpositive-$\alpha$ singularities the $u = 0$ points must be considered.

We show here that in all cases known to us where $u = 0$ points cover open sets the actual singularities are confined to sets of lower dimension. This general result does not, however, specify the locations of these sets of lower dimension. This poses the problem of formulating and proving a generalized version of the structure theorem that restricts the allowed singularities to specified surfaces of codimension one or more.

A general theorem of this kind is not proved here. However, a number of special cases have been examined, and these conform to the following rule: A bubble diagram function $F^B(p)$ is singular at $p$ only if $p$ lies on a codimension-one or more component of the Landau surface associated with $F^B$. The components of this Landau surface are obtained by first considering for each Landau diagram $D$ associated with $F^B$ the solutions $(p,k)$ of the corresponding Landau equations. Here $k$ is the set of variables associated with the internal lines of $D$. These solutions define a variety in $(p,k)$ space, which can be canonically decomposed into a set of analytic manifolds. The p-space images of these manifolds can then be canonically decomposed into analytic manifolds. These latter manifolds are the components of the Landau surface. In the cases we have examined the components of the Landau surface that cover open sets in $p$ space are devoid of singularities, and the singularity spectrum has the conormal structure with respect to the remaining manifolds. That is, the cotangential component of the singularity spectrum at a point $p$ lying on component $N_i$ is confined to the conormal space to $N_i$ at $p$. (The cotangential component of the singularity spectrum at $p$ determines the allowed directions of approach to $p$.)

In the course of the analysis of $u = 0$ points another problem is encountered and resolved. This is the problem of $\mathcal{M}_0$ points. The set $\mathcal{M}_0$ is the part of the mass shell where two or more initial momentum-energy vectors are parallel or two or more final momentum-energy vectors are parallel. The information on the singularity structure of scattering functions near $\mathcal{M}_0$ points obtained by Iagolnitzer and Stapp (5) from macrocausality is insufficient for the present work. By introducing also the constraints of Lorentz invariance we obtain a more detailed characterization of the singularity structure at $\mathcal{M}_0$ points. The singularity spectrum has, in particular, the conormal structure demanded by Sato's conjecture and needed in our work on $u = 0$ points.

We shall now describe these results in more detail, together with some new technical results upon which they are based. The proofs will be given elsewhere.

The results are based on the general theory of microfunctions discussed by Sato, Kawai, and Kashiwara (6). Thus the singularity spectrum of a hyperfunction defined on a real analytic manifold $M$ of dimension $n$ is regarded as the support in $\sqrt{-1}\, S^*M$ of the hyperfunction regarded as a microfunction. The space $\sqrt{-1}\, S^*M$ consists of points $(p, \sqrt{-1}\, u)$ where $p$ lies in $M$ and $u$ is a nonzero real $n$ vector defined modulo real positive (multiplicative) factors. The vector $u$ lies in the cotangent space and determines allowed directions in the dual tangent space.

## 2. Structure Theorem, $\mathcal{M}_0$ Points, and Generalized Landau Equations

An important concept is that of the space-time Landau equations. Consider a diagram D. This is a topological structure consisting of $n$ external lines $L_r$ and $n'$ internal lines $L_\ell$ connected at $n''$ vertices $V_j$. Each line has a direction, and the incidence numbers $[j,\ell]$ (or $[j,r]$) are $+1,-1$, or $0$, according to whether $V_j$ lies on the front-end, back-end, or no end of $L_\ell$ (or $L_r$). The indices $j^+(\ell)$ and $j^-(\ell)$ are uniquely defined: $[j^\pm(\ell),\ell] = \pm 1$. They label the vertices lying on the ends of internal line $L_\ell$. Each $L_\ell$ (or $L_r$) is associated with a mass $\mu_\ell$ (or $m_r$), and each line $L_\ell$ of some (possibly empty) subset of the internal lines carries a sign $\sigma_\ell$.

<u>Definition 1</u>. A set $(p_1,\cdots,p_n; u_1,\cdots,u_n) \equiv (p,u)$ consisting of $n$ real four-vectors $p_r$ and $n$ real four vectors $u_r$ is said to be a solution of the Landau equations corresponding to Landau diagram D if and only if there are a four vector $a$, sets of real four vectors $k_\ell$ $(\ell = 1,\cdots,n')$ and $v_j$ $(j = 1,\cdots,n'')$, and scalars $\alpha_\ell$ $(\ell = 1,\cdots,n')$ and $\beta_r$ $(r = 1,\cdots,n)$ such that the following equations hold:

$$(1)\begin{cases} \sum_r [j:r]p_r + \sum_\ell [j:\ell]k_\ell = 0 & \text{for } j = 1,\cdots,n'' & (1.a) \\ p_r^2 = m_r^2, \quad p_r^0 > 0 & \text{for } r = 1,\cdots,n & (1.b) \\ k_\ell^2 = \mu_\ell^2 \quad k_\ell^0 > 0 & \text{for } \ell = 1,\cdots,n' & (1.c) \\ v_{j^+(\ell)} - v_{j^-(\ell)} = \alpha_\ell k_\ell & \text{for } \ell = 1,\cdots,n' & (1.d) \\ u_r + \beta_r p_j = -[j(r):r]\left(v_{j(r)} + a\right) & \text{for } r = 1,\cdots,n & (1.e) \end{cases}$$

$$(1)\begin{cases} \sigma_\ell \alpha_\ell \geqq 0 & \text{for all signed lines } \ell \qquad (1.f) \\ \alpha_\ell \neq 0 & \text{for some } \ell \qquad (1.g) \end{cases}$$

Note that the vector $u \equiv (u_1,\cdots,u_n)$ is not uniquely determined by these equations: given one solution another is generated by a common positive scale change of the $v_j$'s, $\alpha_\ell$'s, $\beta_r$'s, and $u_r$'s. Also, if $(p,u)$ is a solution then so is $(p, u+u_o(p))$ where $u_o(p)$ is any $4n$ vector of the form $u_o = (\beta_r' p_r + a, \beta_2' p_2 + a, \cdots, \beta_n' p_n + a)$. Thus $u$ at $p$ is defined modulo vectors of the form $u_o(p)$, and modulo positive scale changes.

The restricted mass shell $\mathcal{M}^r$ is defined by $\{p;\ p_r^{\,2} = m_r^{\,2}$ for $r = 1,\cdots,n$, $\sum_r [j(r),r] p_r = 0$, at least two $p_r$ are not parallel$\}$. The solutions $(p, \sqrt{-1}\,u)$ of (1) define a variety $\mathcal{L}(D)$ in $\sqrt{-1}\, S^*\mathcal{M}^r$. This variety is called the Landau variety and its projection to the base space $\mathcal{M}^r$ is the Landau surface $L(D)$.

The Landau equations associated with a bubble diagram function $F_r^{\,B}$ (which is $F^B$ with the conservation law $\delta$-function factor removed) is the set of Landau equations (1) corresponding to all diagrams $D \subset \bar{B}$. The set $\bar{B}$ is the set of Landau diagrams that can be constructed by replacing each bubble $b$ of $B$ by some Landau diagram $D_b$, where the internal lines of $D_b$ all carry the sign of $b$. {See (3), (4), or (5) for further details.}

The functions $F^B$ are defined on the (unrestricted) mass shell $\{p;\ p_r^{\,2} = m_r^{\,2}$, $r = 1,2,\cdots,n\}$. The corresponding equations are (1) with $a = 0$.

<u>Definition 2</u>. A point $p$ is said to be a $u = 0$ point if there is a solution $(p,u) = (p,0)$ of the Landau equations (1).

<u>Theorem 1</u>. If the singularity spectrums of the bubble diagram functions $F_r^{\,B_1}(p_1,\cdots,p_s,p_{s+1},\cdots,p_{n_1})$ and $F_r^{\,B_2}(p_{s+1},\cdots,p_{n_1},p_{n_1+1},\cdots,p_n)$ are confined to solutions of the associated space-time Landau equations except possibly at $u = 0$ points, then the bubble diagram function $F_r^{\,B}$ corresponding to the bubble diagram $B$ obtained by joining $B_1$ and $B_2$ with respect to $p_{s+1},\cdots,p_{n_1}$ has the same property.

The usual structure theorem (3) follows from a repeated application of Theorem 1 starting from bubble diagrams $B_1$ and $B_2$ consisting of single bubbles. For these simplest bubble diagrams the only $u = 0$ points are the $\mathcal{M}_o$ points ((7)).

To prove Theorem 1 we first prove a corresponding theorem for the (non-reduced) bubble diagram functions, i.e., bubble diagram functions which contain the overall $\delta$-function. This is easily done by the successive application of Corollary 2.4.2 and Theorem 2.3.1 in Chapter I of (6). In fact, Corollary 2.4.2 guarantees that the integrand appearing in the definition of $F^B$ is well defined under the $u \neq 0$ assumption and estimates its singularity spectrum. Therefore Theorem 2.3.1 immediately applies to estimate the singularity spectrum of $F^B$. Next we apply Theorem 2.1.8 in Chapter III of (6) to estimate the singularity spectrum of $F_r^{\,B}$, the function obtained by factorizing out the overall $\delta$ function from $F^B$.

That the theory of microfunctions would yield a simple proof of the structure theorem was noted by Professor Pham several years ago. (Private communication from Professor Iagolnitzer to HPS.)

Theorem 1 estimates the singularity spectrums of bubble diagram functions and is used in the derivation of the S-matrix discontinuity formula. ((4)) However, the limitation to $u \neq 0$ points is quite serious, since, unlike $u \neq 0$ points, the $u = 0$ points may cover open sets.

To overcome this difficulty we do two things. The first is to obtain more information on the singularity spectrum of the S matrix at $\mathcal{M}_0$ points, by making use of the Lorentz invariance property of the S matrix. The second is to take account of the specific form of the singularities. In fact, the troubles at $u = 0$ points come from the fact that the multiplication procedure needed to define the integrand at such points cannot be legitimate unless the singularities enjoy special properties. On the other hand, solutions of maximally overdetermined systems enjoy properties that allow their products to be defined even at $u = 0$ points.

In connection with the $\mathcal{M}_0$ problem we introduce a generalized version of the Landau equations.

Definition 3. The generalized space-time Landau equations corresponding to a scattering function $S_r(p)$ or its conjugate $S_r^\dagger(p)$ are the same as the original space-time Landau equations (1) for these functions except that for every set $\{L_r; r\varepsilon\Gamma\}$ of external lines that all originate (or all terminate) on a single vertex, there is an alternative to the set of equations (1.d) and (1.e) associated with these lines. This alternative set consists of the equations

$$(2)\left\{\begin{array}{lll} \dfrac{p_r}{p_r^{\,0}} = \dfrac{p_{r'}}{p_{r'}^{\,0}} & \text{if } r\varepsilon\Gamma \text{ and } r'\varepsilon\Gamma & (2.a) \\ \text{and} & & \\ u_r + \beta_r p_r = -[j(r),r](v_r + \eta_r) & \text{for } r\varepsilon\Gamma & (2.b) \\ \text{where the } \eta_r \text{ satisfy} & & \\ \sum_{r\varepsilon\Gamma}(p_r^{\,0}\eta_r^{\,\nu} - p_r^{\,\nu}\eta_r^{\,0}) = 0 \,. & & (2.c) \end{array}\right.$$

Also, in place of (1.g) one imposes on the complete solution the condition

$$(3) \quad u \neq 0 \pmod{u_0} .$$

All solutions $(p; u)$ of the form $(p_1, p_2, \cdots, p_n;\ \beta_r' p_1 + a, \beta_2' p_2 + a, \cdots, \beta_n' p_n + a)$ are eliminated by (3).

Theorem 2. The macroscopic causality and Lorentz invariance properties of the S matrix entail that the singularity spectrums of $S_r(p)$ and $S_r^\dagger(p)$ be confined to solutions of the corresponding generalized Landau equations.

For a general bubble diagram function $F_r^B$ the generalized Landau equations are obtained from the ordinary equations (1) by allowing for the variables associated with each individual bubble b of B the options allowed by (2) and (3). That is, if $D_b$ is the part of some $D \subset \tilde{B}$ that is associated with a bubble b of B, then for each vertex v of $D_b$ that is an external vertex of $D_b$ considered alone, one allows the option described by (2), but excludes, for the variables $u_i$ associated with all the external lines of $D_b$ considered alone, all solutions excluded by (3). {Geometrically, the option (2) allows certain parallel displacements of the space-time trajectories corresponding to a set of explicit lines of B that all begin or end on a common vertex v of D, provided these trajectories are all parallel. Condition (3) excludes any solution in which the trajectories associated with the external lines of any single $D_b$ considered alone, all pass through a common point. The geometrical interpretation of the Landau equations as a classical space-time scattering diagram is discussed in (3), (4), and (5). The generalized Landau equations allow the vertices to go, in effect, to infinity, subject to conservation-law constraints.}

Definition 4. A point p is said to be a generalized u = 0 point of $F_r^B$ if (p,u) = (p,0) satisfies the corresponding generalized Landau equations.

Theorem 3. The singularity spectrum of $F^B(p)$ is confined to solutions of the generalized Landau equations except possibly at generalized u = 0 points.

A simple case covered by Theorem 3, but not by Theorem 1, is illustrated in Fig. 1.

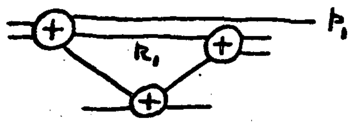


Fig. 1. Bubble diagram for a case covered by Theorem 3.

If for some p the integration region contains a point k where $p_1$ is parallel to $k_1$ then the original Landau equations have a u = 0 solution. Such points p cover an open set, and Theorem 1 gives no information there. However, the parallelness of $k_1$ and $p_1$ does not lead to a u = 0 solution of the generalized Landau equations. The generalized u = 0 points do not cover open sets, in this case, and the singularities allowed by Theorem 3 are confined to sets of lower dimension.

## 3. Sato's Conjecture

Our further results depend on the form of the singularity itself, not merely its location, and are restricted to simple cases where we have determined this form. A principal limitation arises from the exclusion of three-particle thresholds; the form of the singularity at such thresholds is very complicated and has not yet been determined.

Two-particle threshold points are analyzable.

Definition 5. A two-particle threshold point is an argument $p = (p_1, \cdots, p_n)$ of $S_r(p)$ such that: (1) no three initial $p_i$ are parallel, (2) no three final $p_i$

are parallel, (3) there is a (multisheeted) complex neighborhood $\omega$ of $p$ such that $S_r(p)$ has no singularities in $\omega$ except those that coincide with the two-particle thresholds corresponding to the various pairs of parallel initial and final $p_i$, and (4) each of these two-particle thresholds lies below the lowest three-particle threshold in its channel.

From Zimmermann's result (8) on the square-root nature of singularities lying below the lowest three-particle threshold in elastic scattering amplitudes, combined with the S-matrix discontinuity formulas (4), one may obtain the following result:

Theorem 4. In a neighborhood of a two-particle threshold point $p$ the S-matrix $S_r(p)$ is simply a product of normal-threshold factors

$$\left[\left((p_i + p_j)^2 - (m_i + m_j)^2 + i0\right)^{\frac{1}{2}} \phi_{ij} + \psi_{ij}\right] \quad , \tag{4}$$

where the $\phi_{ij}$ and $\psi_{ij}$ are analytic.

This result validates Sato's conjecture in the neighborhood of any two-particle threshold point. It is used in conjunction with the discontinuity formulas to validate Sato's conjecture at points where the discontinuity functions involve scattering amplitudes evaluated at two-particle threshold points.

A positive-$\alpha$ diagram is a Landau diagram each line of which carries a positive sign $\sigma_\ell$. A diagram subject to this condition is written $D^+$. The union of the $\mathcal{L}(D^+)$ is $\mathcal{L}^+$, the union of the $L(D^+)$ is $L^+$. Nonbasic $D^+$ are ignored (4).

Definition 6. A point $(p; \sqrt{-1}\, u)$ of $\mathcal{L}^+$ is invertible if and only if the following three conditions hold:

(5) There is a unique $D^+$ such that $(p; \sqrt{-1}\, u)$ lies in $\mathcal{L}(D^+)$.

(6) The complexification of $\mathcal{L}(D^+)$ is an analytic submanifold near $(p; \sqrt{-1}\, u)$.

(7) If $x$ represents a set of local coordinates of $\mathcal{L}(D^+)$, then there is a unique set of analytic functions $k_\ell(x)$, $\alpha_\ell(x)$, and $v_j(x)$ such that for each point $x$ in some complex neighborhood of the point $x_0$ = image of $(p; \sqrt{-1}\, u)$ the unique solution of the Landau equations that define $\mathcal{L}(D^+)$ is $p = p(x)$, $u = u(x)$, $k = k(x)$, $\alpha = \alpha(x)$, and $v = v(x)$.

Theorem 5. Suppose $(p; \sqrt{-1}\, u)$ of $\mathcal{L}^+$ is invertible. Suppose the corresponding $D^+$ has the property that at most two lines connect any pair of vertices. Suppose each of the scattering functions that occurs in the discontinuity formula at $p$ is evaluated at a two-particle threshold point. Then the S-matrix $S_r(p)$ satisfies near $(p; \sqrt{-1}\, u)$ a maximally over-determined system of pseudo-differential equations. This system is simple, in the sense of (6) Chapter II §4, except for some analytic varieties. Its order is $\alpha = 2n'' - \frac{3}{2}n'$, where $n'$ is the number of internal lines and $n''$ is the number of vertices of $D^+$.

Remark 1. The excluded subvarieties in Theorem 6 correspond to the zeros of the scattering amplitude. In other words, the S matrix is a solution of a simple maximally overdetermined system except for a multiplicative factor, possibly vanishing.

Remark 2. To obtain these results we have used a micro-local version of the discontinuity formula (4). This version makes sense even when $L^+$ is not a hypersurface. The arguments in (4) establish, in effect, also this generalization.

Remark 3. The result stated in Theorem 5 arises from a very special property of the S-matrix discontinuity formula. It would not hold if that function were replaced by the similar bubble diagram function $F^B$ corresponding to the $B$ obtained by simply replacing the vertices of $D^+$ by plus bubbles. The S-matrix is thus particularly simple, from the point of view of maximally over-determined systems.

Definition 7. A point $p\varepsilon L^+$ is simple if and only if

(8) $L^+$ near $p$ is a codimension-one analytic submanifold $\{p\varepsilon\mathcal{M}; \phi(p) = 0\}$, and

(9) The point $(p; \sqrt{-1}\, u) = \left(p; \sqrt{-1}\, \mathrm{grad}_p\, \phi(p)\right)$ in $\mathcal{L}^+$ is invertible.

Theorem 6. If $p$ is a simple point of $L^{+p}$ then the scattering amplitude near $p$ has the following form (15):

$$(10)\quad \begin{cases} h_1(p)\left(\phi(p) + i0\right)^{-\alpha+3/2} + h_2(p) & \text{if } -\alpha + \frac{3}{2} \text{ is neither a positive integer nor zero.} \\ \text{or} & \\ h_1(p)\, \phi(p)^{-\alpha+3/2} \log\left(\phi(p) + i0\right) + h_2(p) & \text{if } -\alpha + \frac{3}{2} \text{ is a positive integer or zero.} \end{cases}$$

Here $h_1(p)$ and $h_2(p)$ are analytic.

To fully analyze the S-matrix singularity structure we must study it also at singular points of the Landau variety. In view of Theorem 1 of (9), and the remark following it, the most important singular points are the points where two irreducible components of the Landau variety cross normally along some subvariety of codim 1 in the Landau variety. This is the situation which appears if some single internal line is contracted. To delineate this case we introduce the following definition.

Definition 8. A point $(p; \sqrt{-1}\, u)$ of $\mathcal{L}^+$ is a simple contraction point if and only if the following conditions are satisfied:

(11) There are exactly two basic diagrams $D_1^+$ and $D_2^+$ that satisfy $p\varepsilon\mathcal{L}(D_1^+)$.

(12) $D_2^+$ is obtained by contracting exactly one line of $D_1^+$.

(13) The complexifications $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}(D_1^+)$ and $\mathcal{L}(D_2^+)$ are both analytic submanifolds near $(p; \sqrt{-1}\, u)$, and each satisfies the invertibility condition (7).

(14) $\mathcal{L}_1 \cap \mathcal{L}_2$ is an analytic submanifold near $(p; \sqrt{-1}\, u)$, and $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect transversally at $(p; \sqrt{-1}\, u)$.

Theorem 7. Suppose $(p; \sqrt{-1}\, u)$ in $\mathcal{L}^+$ is a simple contraction point. Suppose both $D_1^+$ and $D_2^+$ have the properties required in Theorem 5 of $D^+$, except for the pole singularity associated with the contracted line. Then the conclusions of Theorem 5 still hold, except for the simplicity of the system.

The explicit form of the scattering amplitude can also be given in this case under some moderate conditions on $D_1$, by making use of a canonical form of the system of equations it must satisfy.

The results stated in Theorems 5 and 7 verify Sato's conjecture in nontrivial cases.

4. Analysis of Generalized $u = 0$ Points

All open sets of generalized $u = 0$ points known to us arise from the occurrence within a diagram D of a symmetric hammock part. A hammock part is a part that is connected to the rest of the diagram at precisely two vertices, has other vertices, and has no external lines. Two examples are shown in Fig. 2.

Fig. 2. Two hammock parts.

The dark vertices at the ends are the only vertices connected to the rest of the diagram D, which is not shown. If a hammock diagram is symmetric with respect to a vertical center-line, apart from a reversal of the signs $\sigma_\ell$, then for every solution of the Landau equations for the half diagram there is a corresponding solution for the whole diagram. This solution is symmetric, in the sense that corresponding vertices from the two half diagrams coincide. In particular the two end points coincide. Hence a solution of the Landau equations for the entire diagram D can be obtained by setting to zero all $\alpha_\ell$'s corresponding to lines outside the hammock part. This solution is a $u = 0$ solution, and it usually generates open sets of generalized $u = 0$ points. Any bubble diagram with some pair of bubbles connected by more than two lines has open sets of generalized $u = 0$ points of this kind.

Consider, for example, the bubble diagram B of Fig. 3.

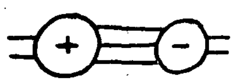

Fig. 3. The bubble diagram B.

Both diagrams of Fig. 2 are contained in $\bar{B}$, and each gives an open set of generalized $u = 0$ points that covers the entire region where $F^B$ is nonzero. Thus the (generalized) structure theorem gives no information about the singularity spectrum of $F^B$. Special assumptions were needed in (4) to exclude effects of $u = 0$ contributions.

Points $k = (k_1, k_2, k_3)$ in the integration domain where the only singularities of the integrand are those associated with Fig. 2a lead to no singularities of $F^B$. The integrand contains a factor $(\phi + i0)^{\frac{1}{2}}(\phi - i0)^{\frac{1}{2}}$ which is not defined by the rules for products of microfunctions. (The product is too similar to $(\delta(\phi))^2$.) However, for products of representatives of Hilbert space kernels one has, for $\lambda \geq 0$,

$$(\phi + i0)^\lambda(\phi - i0)^\lambda = \phi_+^{2\lambda} + \phi_-^{2\lambda} \tag{15}$$

where $\phi$ is the distance below threshold, and $\phi_+^{2\lambda}$ and $\phi_-^{2\lambda}$ are the two Heaviside resolutions. Using (15) one can effectively reduce the diagram of Fig. 2a to a simple three-particle normal threshold diagram, which yields no singularities above the three-particle threshold.

The same argument eliminates most of the singularities of $F^B$ coming from Fig. 2b. However, there are two values of $(p_1 + p_2)^2 = (p_3 + p_4)^2$ where the singularities do not have the required form $(\phi \pm i0)^\lambda$, and the argument fails. These two values of $(p_1 + p_2)^2$ are those such that the triangle diagram singularity of the half diagram, considered as a surface in $k$ space that changes as $p$ is changed, either (1) touches the two-particle normal threshold surface corresponding to Fig. 2a, or (2) touches the point where $k_3$ is parallel to $p_1 + p_2$. (Here $k_3$ is the variable associated with the lowest line in Fig. 2b.) When either of these cases is reached the singularity surface in k-space suffers an abrupt change of topological structure, and a singularity of $F^B(p)$ is expected, barring cancellations.

The above argument rests on (15), and works at points $p$ such that all the points $(p,k)$ on the singularity surface of the half diagram are simple points, since the forms of the singularities are then $(\phi \pm i0)^\lambda$. The result, however, continues to hold when the singularity surface in $(p,k)$ space includes also simple contraction points:

Theorem 8. Consider any bubble diagram function $F^B(p)$ and an argument $\bar{p}$ that satisfies the following conditions:

(16) There is a unique $D$ such that $\bar{p}$ lies on $L(D')$ for $D' \subset \bar{B}$ only if $D'$ is $D$, or a diagram obtained from $D$ by contracting some signed lines.

(17) This unique diagram $D$ is a symmetric hammock diagram $D_h$ with half diagrams $D_r$ and $D_\ell$.

(18) Every solution $(p,k)$ of the Landau equations for any $D \subset \bar{B}$ is obtained from coincident points of $D_r$ and $D_\ell$, or from coincident simple points of contractions of $D_r$ and $D_\ell$, or from coincident simple contraction points of $D_r$ and $D_\ell$.

(19) $p$ lies on a codimension zero component of $L^B \equiv \bigcup_{D \subset \bar{B}} L(D)$.

Then $F^B$ is not singular at $\bar{p}$.

The components of $L^B$ are constructed by decomposing the union of the solutions of the corresponding Landau equations into analytic manifolds in the canonical fashion, then further decomposing these manifolds into manifolds that are analytic over $p$ space, in the sense that the local coordinates of the $p$ space manifold can be used as a subset of the local coordinates of the $(p,k)$ space manifold lying over it. This construction gives a well-defined decomposition of $L$ into analytic manifolds, which are called its components. If the analytic properties are controlled by the geometry of the Landau surfaces, as they are in the simple examples studied above, then any single codimension zero component would be expected to be completely free of singularities or completely filled with singularities, since the geometry is analytically uniform over each component.

That no codimension zero component is filled with singularities is suggested by the following result.

Theorem 9. Let the phase space functions $F^D$ be defined by the same rules as the bubble diagram functions $F^B$, except that whereas the bubbles of B are associated with scattering amplitudes, the vertices of D are associated with constants, which are all taken to be unity. Let $D_h$ be a hammock diagram. Then $F^{D_h}$ is analytic except on a locally finite set of analytic manifolds of codimension one or more. Moreover, its singularity spectrum enjoys the conormal structure.

This result is proved by first reducing out the delta functions in the phase space functions, and then applying the result of (10).

The singularities of the bubble diagram functions are, in a certain sense, generated by the singularities of phase space functions. Hence it seems unlikely that the former could give open sets of singularities if the latter do not.

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