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MATHEMATICAL MODELS FOR VERBAL LEARNING[1/]

by

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The use of language is perhaps the most distinctive feature of human behavior. At an early age we learn the appropriate words for the objects and events which surround us, as well as how to communicate our needs and feelings to other people. As we grow older we develop associative relations of varying complexity among the words in our vocabulary, as for example in the use of grammatical rules to form sentences. To illustrate a simple type of verbal association, suppose someone asks you to respond with the first word that comes to mind when he says "cat"; your response will probably be "dog" or perhaps "rat". If he says "black", undoubtedly you will say "white". The problem facing a french student, on the other hand, is to learn to respond to "dog" with "le chien" and to "black" with "noir". The laboratory study of how such verbal associations are formed, in addition to having practical implications, has played an important role in testing theoretical ideas about the nature of the learning process. It is this last matter which will chiefly concern us in this paper, and we will concentrate our attention on a particular kind of verbal learning problem known as <u>paired-associate learning.</u>

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In paired-associate learning, the subject learns to give the correct response as each stimulus from a list of stimulus items is presented. In the experiments which will be considered in this paper, the subject is informed in advance of what responses he may use. He is then shown the stimuli one at a time in some random order, and is asked to guess which of the responses has been designated as the correct answer for that particular stimulus. After the response is made, the subject is told the correct answer and then the next stimulus is presented. After the entire list of stimulus items has been presented, the experimenter re-arranges the items in a new random order and again presents the list to the subject. As each item is shown to him, the subject attempts to anticipate the correct response, following which he is informed of the right answer. Each run through the list constitutes a *trial*, and when the subject is told the correct answer, we will speak of this event as a *reinforcement*.

An example of one such paired-associate study is an experiment by Atkinson and Crothers (1964), in which the stimulus items were 18 Greek letters and the responses were three nonsense syllables, RIX, FUB, and GED. Each response was paired with six stimuli, so that the three responses were used equally often as the correct answer. Fig. 1 presents the proportion of correct anticipations on each trial for this study. On the first trial, the proportion of successes is very close to the value of .33 to be expected if the subject simply chose one of the responses at random as each stimulus was presented. The curve rises exponentially and gradually approaches an asymptotic value of 1, i.e., eventually only correct anticipations occur.

One of the first theoretical attempts to account for data of this sort assumed that the effect of each reinforcement was to add an increment to the strength of the association between the stimulus and the correct response. Suppose that the probability of a correct anticipation on trial n, which will be denoted $Pr(c_n)$, is taken as an estimate of the associative strength on trial n. The probability of an error on trial n, $Pr(e_n)$, is an indication of how much remains to be learned. The basic assumption that is made in the "incremental" theory is that the effect of the reinforcement on trial n is to increase the probability of a correct response by an amount which is a constant proportion θ of the amount remaining to be learned, i.e.,

$$Pr(c_{n+1}) = Pr(c_n) + \theta Pr(e_n) \quad . \tag{1a}$$

Thus, every time a subject is told the correct answer to a stimulus item, there is an increase in the probability that the correct answer will be given when the item is presented again. Notice that this increase does not depend upon whether the correct or incorrect answer is given. Using the fact that $Pr(e_{n+1}) = 1 - Pr(c_{n+1})$, Eq. 1a may be rewritten as

$$Pr(c_{n+1}) = (1-\theta)Pr(c_n) + \theta \quad . \tag{1b}$$

In this form it is easy to see that the probability of a correct response on trial $n+1$ is assumed to be a linear function of the probability on the preceding trial, and hence this model frequently is referred to as a linear model. The properties of this model have been extensively investigated (Bush & Mosteller, 1955; Estes & Suppes, 1959; Sternberg, 1963). In particular, it can be shown that $Pr(c_n)$ may be written as a function of the parameter θ and g, (the guessing probability on trial 1 which will be the reciprocal of the number of responses), namely

$$Pr(c_n) = 1 - (1-g)(1-\theta)^{n-1} . \tag{2}$$

A derivation of Eq. 2 as the solution of the linear difference equation given in Eq. 1b may be found in any of the references above.

The theoretical curve in Fig. 1 was obtained from Eq. 2 with θ equal to .42, and it agrees very closely with the observed values. It is important to realize that the learning process for each individual item in the list is represented by Eq. 2. That is, if the probability of a correct response for a given stimulus item could be measured by some hypothetical "probability meter", the course of learning would resemble measurements from an analogue device such as a variable resistor operating in the following manner. On trial 1, the probability measurement would be equal to the guessing rate g, and on each succeeding trial the probability value would gradually move upward by some amount, as if the knob of the resistor were being turned in the same direction on each trial by an exponentially decreasing amount.

There have been objections to this type of representation of the learning process on several grounds. For example, some psychologists have argued that while very simple organisms might behave in this fashion, higher animals, especially when confronted with more complex problems, show learning of an all-or-none sort. It is not our intention to go into the history of the controversy concerning the relative merits of continuous and discontinuous characterizations of the learning process. (For some recent contributions to the issue see Bower, 1962; Estes, 1964; and Underwood and Keppel, 1962.) Rather, we want to consider a model that assumes that learning is all-or-none, and then look at the kinds of differential predictions made by the two types of models.

Following the analogy between the linear model and a variable resistor, the all-or-none model may be represented by a two-position switch which operates in this manner: initially the switch is in the "unlearned" position and responses are made at random from the available response set. After each reinforcement the switch is turned from the "unlearned" to the "learned" position with probability a, whereas with probability $1 - a$ the switch remains in the "unlearned" position. Once the switch has been turned to the "learned" position, it remains there, and the correct response is always given. More specifically, the model may be formulated as a two-state Markov process in which an item is assumed to be in the unlearned state U at the start of the experiment. When the subject is informed of the correct response to be associated with an item, then with probability a learning occurs, and there is a transition to the learned state L, whereas with probability $1 - a$, the item remains in state U. If an item is in state U, then the probability of a correct response is g, the guessing probability. Once an item is learned, however, then there will be no subsequent errors. These assumptions are incorporated in the matrix below, which specifies the transition probabilities between the two states U and L from trial n to trial n+1, and the response vector which gives the probability of a correct response in each of the states:

$$\begin{array}{c} \\ L_n \\ U_n \end{array} \begin{array}{c} \begin{array}{cc} L_{n+1} & U_{n+1} \end{array} \\ \begin{bmatrix} 1 & 0 \\ a & 1-a \end{bmatrix} \end{array} \quad \begin{array}{c} Pr(\text{Success}) \\ \begin{bmatrix} 1 \\ g \end{bmatrix} \end{array} . \tag{5}$$

For a detailed analysis of this model, see Bower (1961).

The probability of a correct response on trial n, $Pr(c_n)$, for the all-or-none model is readily derived by considering the probability of an error on trial n. In order for an error to occur on trial n, (1) an item must remain in state U for n-1 trials, this with probability $(1-\underline{a})^{n-1}$, and (2) an incorrect guess must be made when this item is presented on trial n, this with probability 1-$\underline{g}$. Thus, $Pr(e_n)$ is $(1-\underline{g})(1-\underline{a})^{n-1}$, and so

$$Pr(c_n) = 1 - Pr(e_n) = 1 - (1-g)(1-a)^{n-1} . \tag{6}$$

It is evident that when an identification is made between θ and $\underline{a}$, the two models, though based on very different premises about the underlying learning process, predict the same mean learning curve.

As an example of a statistic that does differentiate the two models, consider for a particular stimulus-response pair the conditional probability of an error on trial n given an error on trial n-1, i.e.,

$Pr(e_{n+1}|e_n)$. In the linear model the probability of an error on trial $n+1$ does not depend upon whether the preceding response was right or wrong, and so

$$Pr(e_{n+1}|e_n) = Pr(e_{n+1}) = (1-g)(1-\theta)^n . \qquad (7)$$

Thus, for this model, the conditional probability of an error is an exponentially decreasing function of the trial number.

For the all-or-none models, however, the fact that an error occurs on trial n furnishes an important piece of information, viz., the item must have been in the unlearned state at the beginning of trial n, since no errors can occur once the item becomes learned. In order for an error to occur on trial $n+1$, therefore (1) learning must not have occurred following the reinforcement on trial n, that with probability $1-a$, and (2) an incorrect response must be made on trial $n+1$, that with probability $1-g$; therefore

$$Pr(e_{n+1}|e_n) = (1-g)(1-a) . \qquad (8)$$

Thus, the linear model predicts that $Pr(e_{n+1}|e_n)$ will decrease exponentially over trials, whereas the all-or-none model predicts that this probability will remain constant over trials. In Fig. 2 the conditional probability from Atkinson and Crothers' experiment is presented, along with the predictions from the all-or-none and linear models based on the same parameter value used to fit the mean learning curve. Although the conditional probability does tend to decrease over trials, the data are more in agreement with the constancy prediction of the all-or-none model, than with the decrease predicted by the linear model. Nevertheless, the noticeable decline over trials in the data of Fig. 2 has been found to characterize many paired-associate studies, and when appropriate statistical tests are applied, this decline has proven to differ significantly from the constancy predicted by the all-or-none model. (For similar experimental results see Atkinson and Crothers, 1964; Estes, 1960; and Suppes and Ginsberg, 1963.)

Consequently, consideration has been given to ways in which the basic models described above may be modified so as to yield a more adequate account of paired-associate learning. We shall not attempt to deal with all the variations that have been proposed, but rather will restrict our attention to an extension of the all-or-none model suggested by Atkinson and Crothers (1964). As these authors point out, the inability of simpler models to account for all the details of the data indicates that one or more important psychological processes have been disregarded. For example, in paired-associate learning, it has been shown that considerable forgetting may result because the subject is trying to learn a number of stimulus-response pairs simultaneously (Melton, 1963; Murdock, 1961; Peterson & Peterson, 1959; Tulving, 1964). One way in which forgetting may affect the learning of paired-associates is suggested in the following analysis. Suppose we consider the course of learning for a single item $\underline{i}$ from a list. The item is presented to the subject, and following his response, he is told the correct answer. Now if item $\underline{i}$ is presented again immediately, it is very likely that the correct answer will be given. However, if other items from the list are interpolated between the two presentations of item *i*, the subject will be less likely to answer correctly on the second presentation of item *i*. The interpolated items are said to interfere with the retention of item *i*, or, more commonly, the subject forgets the association to item *i*. In general, as the number of interpolated items between the nth and (n+1)st presentation of item *i* is increased, the amount of forgetting increases.

The two complementary processes — learning due to reinforcement and forgetting due to interference — are both incorporated in the model that will be considered now. It will be assumed that each item in a list may be in one of three states: (1) state U is an unlearned state, in which the subject guesses at random from the set of response alternatives, (2) state S is a short-term memory state, and (3) state L is a long-term memory state. The subject will always give a correct response to an item in the short-term state, but it is possible for an item in state S to be forgotten, i.e., to return to state U. Once an item moves to state L it is completely learned, in the sense that it will remain in state L and the correct response will always be given on subsequent presentations of the item.

The associative effect of a reinforcement is described by matrix $\underline{\underline{A}}$ below:

$$\underline{\underline{A}} = \begin{array}{c} \\ L \\ S \\ U \end{array} \begin{array}{c} \begin{array}{ccc} L & S & U \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ a & 1-a & 0 \\ a & 1-a & 0 \end{bmatrix} \end{array} \qquad (8)$$

This matrix gives the probabilities of transitions between states for an item immediately after reinforcement. Thus, if an item is in the unlearned state, and the correct answer is told to the subject, then with probability a the item is learned (i.e., it moves to state L), whereas with probability $1-a$ it moves to short-term memory. Thus, immediately following a reinforcement, an item will be either in long-term or short-term memory, and if the item is immediately presented again, the subject will give the correct response.

The effect of an interpolated unlearned stimulus-response pair on the learning state of a particular item is described by matrix $\underline{\underline{F}}$,

$$\underline{\underline{F}} = \begin{array}{c} \\ L \\ S \\ U \end{array} \begin{array}{c} \begin{array}{ccc} L & S & U \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-f & f \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{9}$$

If an item is in short-term memory and an unlearned stimulus-response pair is presented, then the interference produced by the unlearned pair results in forgetting of the item (i.e., transition to state U) with probability f, whereas with probability $1-f$ the item remains in short-term memory. If an item is in long-term memory, the interference has no effect, and if an item is in the unlearned state then again the interference will have no effect.

The matrix describing the transitions between states from trial n to trial $n+1$ for a given item, which will be denoted $\underline{\underline{T}}_n$, is found by taking the product of $\underline{\underline{A}}$ and the Z_nth power of $\underline{\underline{F}}$, where Z_n is the number of unlearned pairs which intervene between the nth and (n+1)st presentations of the particular item. The association matrix $\underline{\underline{A}}$ represents the nth reinforced presentation of the item, and the forgetting matrix $\underline{\underline{F}}$ is applied Z_n times, once for each of the intervening unlearned pairs. Performing the matrix multiplication yields

$$\underline{\underline{T}}_n = \begin{array}{c} \\ L_n \\ S_n \\ U_n \end{array} \begin{array}{c} \begin{array}{ccc} L_{n+1} & S_{n+1} & U_{n+1} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ a & (1-a)(1-F_n) & (1-a)F_n \\ a & (1-a)(1-F_n) & (1-a)F_n \end{bmatrix} \end{array} \qquad (10)$$

where $F_n = 1 - (1-f)^{Z_n}$.

Unfortunately, there is no way of extracting from the data the exact value of Z_n, the number of interpolated pairs which are not in state L. If an incorrect response is given to an intervening stimulus-response pair, then the pair must be in the unlearned state, but if a correct response occurs, then the pair may be in either long-term or short-term memory, or it may even be that the intervening pair is in the unlearned state and the correct response occurred by chance. Since the exact value of Z_n is indeterminate, as an <u>approximation</u> we will use the <u>expected</u> number of unlearned items intervening between the nth and (n+1)st presentation of an item. Suppose that there are X+1 items in the list being learned. On the average, X items will be interpolated between any two consecutive presentations of a particular item. Since the items are arranged in random order, the average position of a particular item will be in the middle of a trial. Thus, for half the interpolated items, (those which follow item $\underline{i}$ on trial n), the probability of being either in state U or state S will be $(1-\underline{a})^{n-1}$. Similarly, for the other half of the interpolated items, (those which precede item $\underline{i}$ on trial n+1), the probability that learning has not taken place is $(1-\underline{a})^{n}$. Combining these results, the expected number of unlearned items intervening between the nth and (n+1)st presentation of item $\underline{i}$ will be $X(1-\underline{a}/2)(1-\underline{a})^{n-1}$, and it is this value which will be used as an approximation to Z_n in Eq. 10.

Next we turn to the derivation of several statistics of interest for this model. In these derivations, it should be kept in mind that F_n is a function of Z_n, for which the approximation just discussed will be used. The mean learning curve may be obtained by noting that for an error to occur on trial n+1 (1) an item must have failed to move to the long-term state on n preceding trials, which has probability $(1-\underline{a})^n$, (2) the item must change from state S to state U between the nth and (n+1)st presentations, which occurs with probability F_n, and (3) while in state U an incorrect guess must be made with probability $1-\underline{g}$; hence

$$\Pr(c_{n+1}) = 1 - (1-g)(1-a)^n F_n \tag{12}$$

For fixed values of $\underline{a}$ and $\underline{f}$, as the length of the list is increased (i.e., as X becomes larger) F_n increases and therefore $\Pr(c_{n+1})$ will decrease. In other words, the model predicts that the longer lists will be more difficult to learn, which of course is in agreement with empirical findings.

The probability of an error conditional on an error, $Pr(e_{n+1}|e_n)$, is also found by noting that if an error occurs on trial n, then the item must have been in the unlearned state. Thus the probability of an error on the next trial is

$$Pr(e_{n+1}|e_n) = (1-g)(1-a)F_n \tag{13}$$

since (1) learning must fail, with probability $1-\underline{a}$, to result in transition to state L, (2) forgetting must occur with probability F_n, and (3) an incorrect guess must be made. Since F_n decreases over trials, $Pr(e_{n+1}|e_n)$ will also decrease over trials.

Because the amount of forgetting is a function of the trial number, this model will be referred to as the trial-dependent-forgetting (TDF) model. In the remainder of this paper we will present the results of a paired-associate study in which list length was varied, and each of the three models discussed here will be applied to the data in order to determine their relative merits.

The subjects for the experiment were three groups of 25 college students, each of whom learned a single paired-associate list. The stimulus member of each pair consisted of a two-digit number, and the response member was one of three nonsense syllables RIX, FUB or GED. A set of 21 stimulus items was chosen on the basis of low inter-item association value, and for Groups 9, 15 and 21, the experimental list consisted of a selection of 9, 15 or 21 items, respectively, from this set. For Group 21, the entire set of stimulus items was used, whereas for the other groups a different subset was randomly selected for each subject. Each of the three response alternatives was correct equally often for each subject. The list was learned to a criterion of two consecutive errorless trials or ten trials, whichever was shorter. The subject was first given instructions about the nature of the task and then ran through a practice list of four items to minimize warmup effects. The subject was asked if he had any questions, and then the experimental run was begun. In order to reduce primacy effects (Peterson & Peterson, 1962), the first three stimulus items shown to the subject were two-digit numbers which were not in the set of 21 experimental items; these three items did not reoccur on later trials. Then, without interruption, the experimental list, arranged in a random order, was presented to the subject, and for each item, the subject was required to choose one of the three responses, following which he was informed of the correct answer. After the entire list had been presented in this fashion, the second trial then proceeded without interruption in the same manner with the items arranged in a new random order. Thus, the procedure involved the continuous presentations of items with no breaks between trials.

The mean learning curves for the three groups are presented in Fig. 3. As may be seen, the three curves are ordered according to list length, i.e., as the number of items in the list is increased, there is a concomitant decrease in the mean proportion of successes on trial n. The curves for $Pr(e_{n+1}|e_n)$ are shown in Fig. 4. The three curves are again ordered by list length, and there is a decrease in the conditional probability over trials for each of three groups. (The numerals by

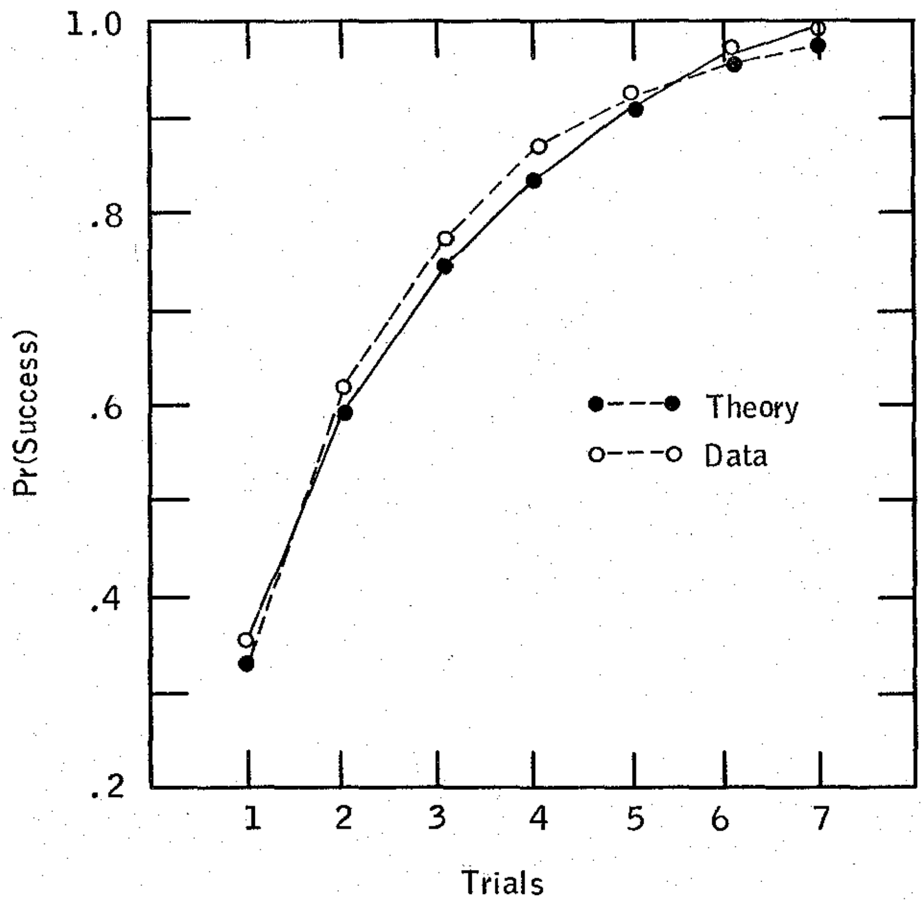

Fig. 1. The average probability of a success on trial n in Atkinson and Crothers' experiment.

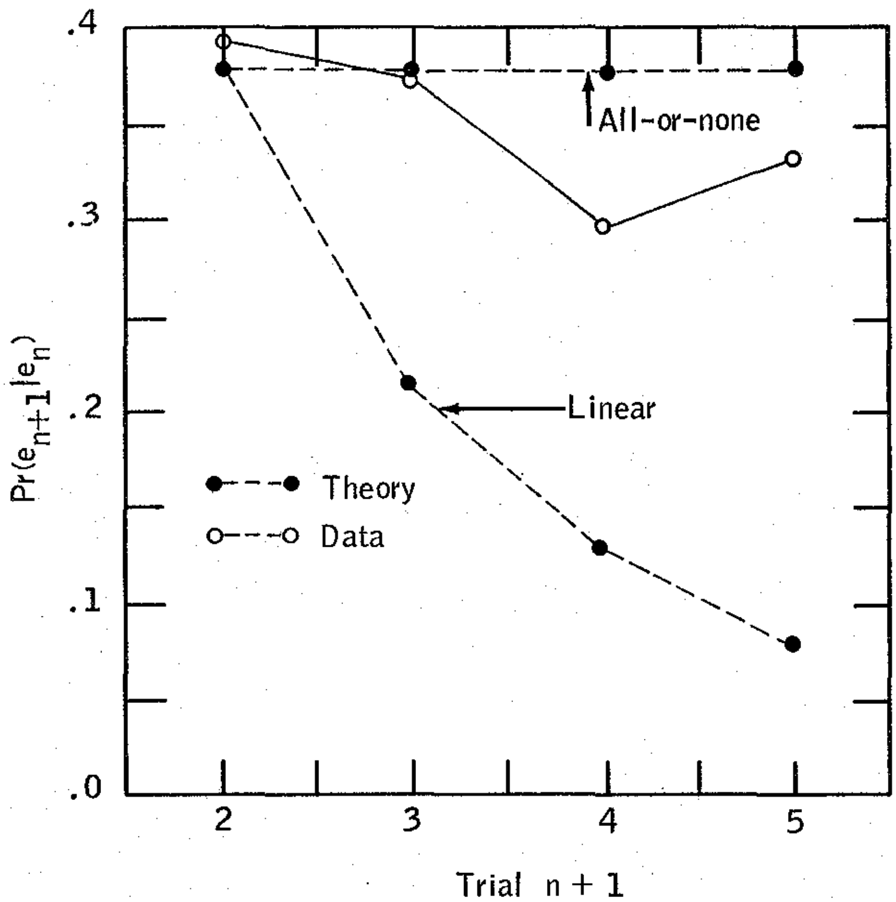

Fig. 2. Average probability of an error on trial $n+1$, given an error on trial n for Atkinson and Crothers' experiment.

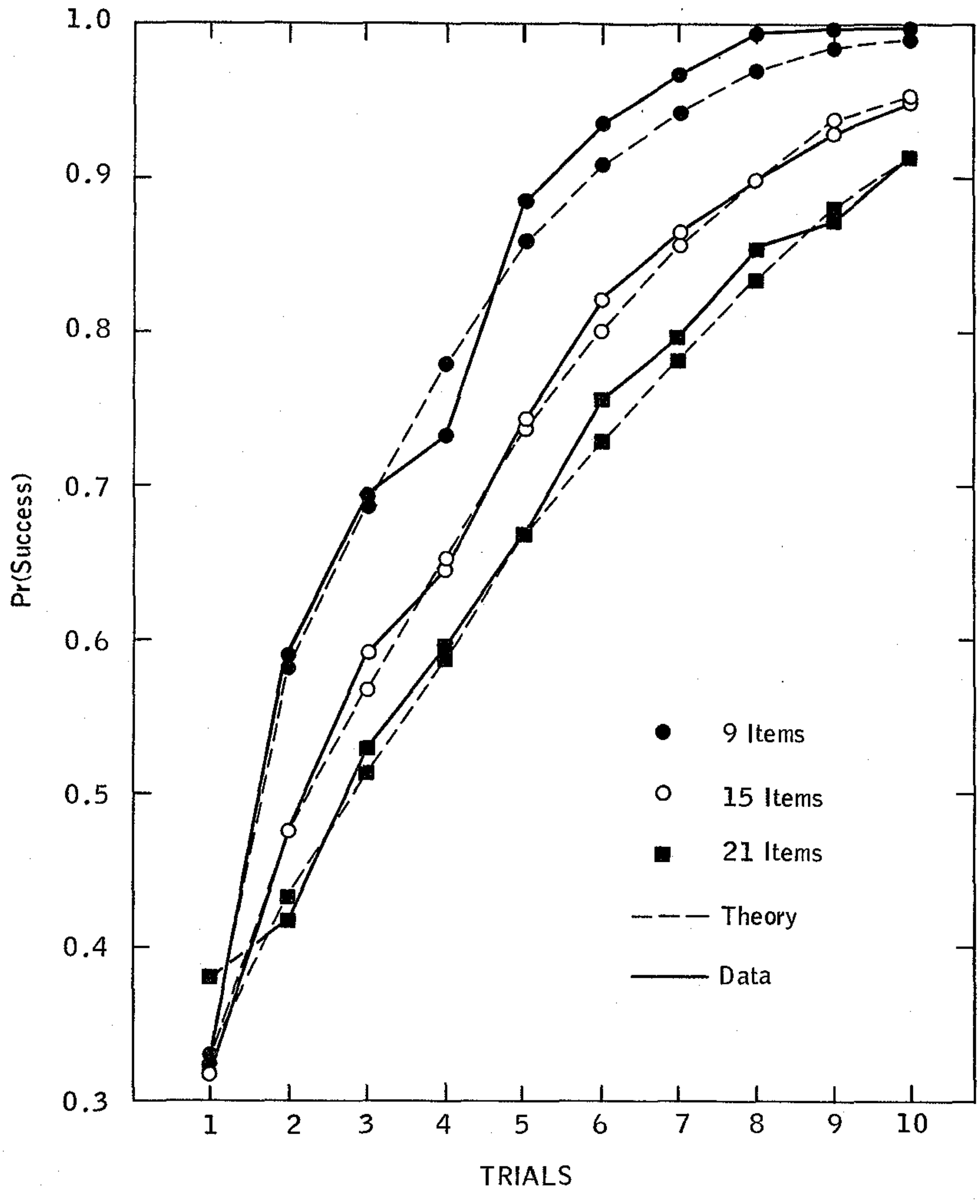

Fig. 3. Average probability of a success on trial n for three groups with different list lengths. See text for description of theoretical curves.

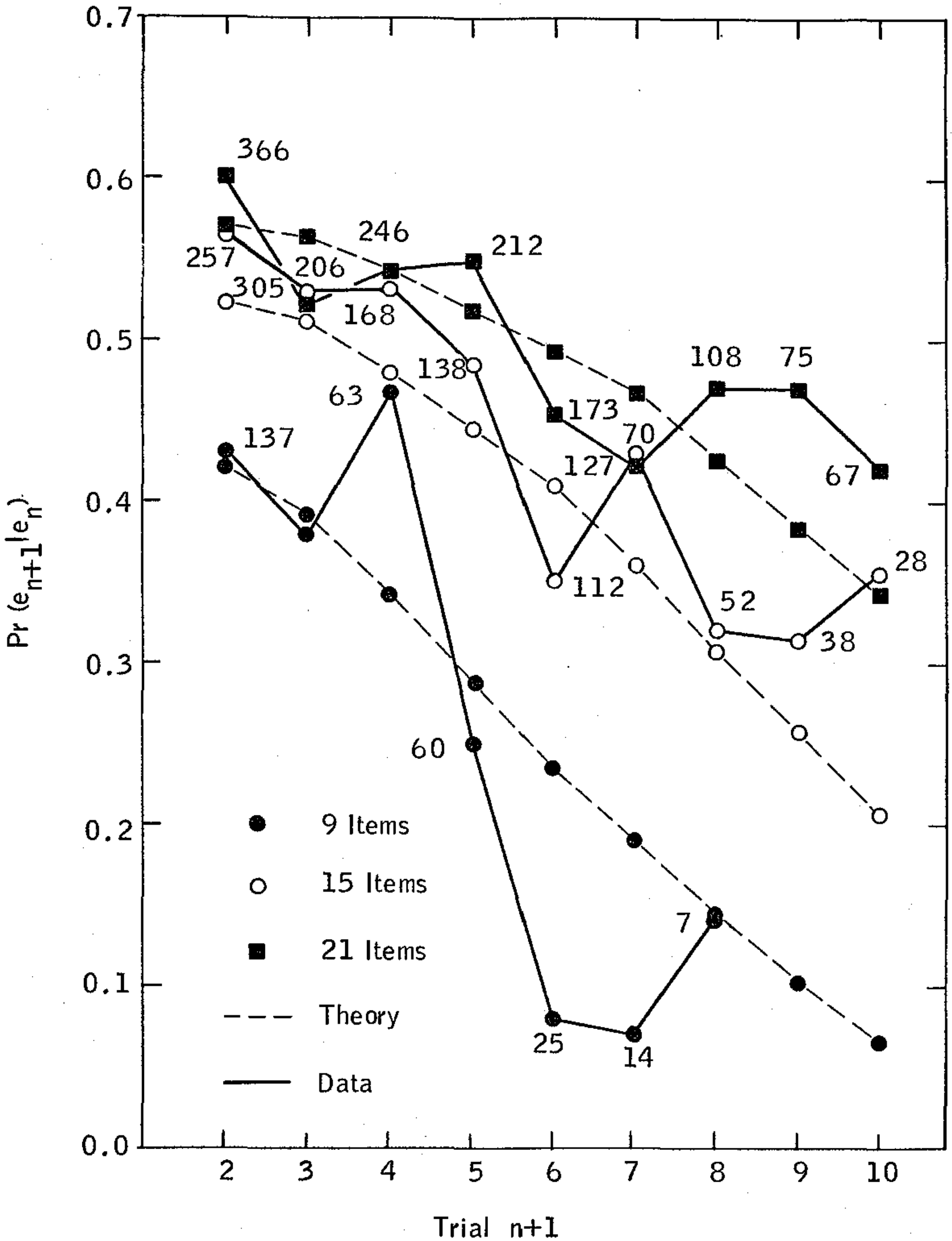

Fig. 4. Average probability of an error on trial n+1, given an error on trial n for three groups with different list lengths.

each of the data points represent the number of observations on which the point is based.)

In order to determine the quantitative accuracy of the various models that have been presented, it is necessary to obtain parameter estimates. There are a number of alternative procedures for estimation; we will use the technique of chi-square minimization on specific response sequences suggested by Atkinson and Crothers (1964). This method yields parameter estimates having certain desirable properties and also provides a goodness-of-fit test. We begin by looking at the sequence of responses of a single subject for one stimulus item. This response is rewritten as a string of *c*'s representing correct responses and *e*'s representing errors. For example, the response sequence for a particular subject-item, beginning at trial 1 and ending at trial 10, might be written eceececccc. For parameter estimation, those portions of the sequence from trials 2 to 5 and from trials 6 to 9 will be used; in the example above, these subsequences are ceec and eccc, respectively. The 2^4 or 16 combinations of *c*'s and *e*'s possible in each four-trial block are listed in Table 1. Also in Table 1 are the observed frequencies with which each combination occurred for the three experimental groups from trials 2 to 5; the data for trials 6 to 9 are in Table 2. For example, the sequence cccc, no errors on trials 2 through 5, was observed in 83 out of a total of 225 subject-items in Group *9*.

For notational purposes, let $O_{i,j,n}$ denote the *i*th sequence listed in the table for experimental group j, where the sequence starts at trial n, and let $N(O_{i,j,n})$ be the observed frequency of this sequence. The predicted relative frequency of each of the specific response sequences is derived for the models presented above. For the all-or-none model, for example, the probability of the sequence, no successes from trial 2 through trial 5, is found to be the probability that the item is not learned by trial 5, and that four incorrect guesses occur, which is $(1-g)^4(1-a)^4$. The same sequence is predicted by the linear model to occur with probability $(1-g)^4(1-\theta)^{10}$, since on each trial $Pr(e_n)$ is equal to $(1-g)(1-\theta)^{n-1}$ in the model. The derivation of the theoretical expressions for these sequences is very lengthy, and the interested reader is referred to the Atkinson and Crothers paper (1964) for further details.

Suppose that for an arbitrary model, $Pr(O_{i,j,n}; p)$ is the predicted probability of the _i_th sequence for group j starting at trial n and ending at trial $n+3$, where the prediction depends on a particular choice of the parameter(s), p, of the model. Further, let the total number of subject-item combinations in a given block of four trials for group j be denoted by T_j. Then we define the function

$$\chi^2_{i,j,n} = \frac{\left[T_j Pr(O_{i,j,n}; p) - N(O_{i,j,n})\right]^2}{T_j Pr(O_{i,j,n}; p)} . \qquad (14a)$$

A measure of the discrepancy between a model and the data from group j is found by taking the sum of Eq. 14a over the sixteen sequences and two blocks of four trials,

$$\chi^2_j = \sum_{i=1}^{16} \chi^2_{i,j,2} + \sum_{i=1}^{16} \chi_{i,j,6} \qquad (14b)$$

TABLE 1

Observed and predicted frequencies for specific response sequences from trials 2 through 5.

Trial				9 Items					15 Items					21 Items				
2	3	4	5	Observed	Linear	All-or-none	TDF	TDF-Revised	Observed	Linear	All-or-None	TDF	TDF-Revised	Observed	Linear	All-or-none	TDF	TDF-Revised
c	c	c	c	83	59.0	88.4	69.3	79.0	98	39.9	103.7	94.6	88.0	97	45.4	112.6	124.8	102.0
c	c	c	e	3	9.5	1.3	6.0	4.2	10	17.8	3.8	6.4	5.6	11	24.2	6.8	7.3	6.6
c	c	e	c	10	15.2	3.0	9.5	8.7	13	23.9	6.6	10.5	11.3	14	31.5	10.3	11.9	13.3
c	c	e	e	4	2.4	2.7	4.8	3.6	10	10.7	7.6	8.4	9.0	12	16.8	13.5	12.0	14.3
c	e	c	c	18	25.7	10.4	17.2	18.1	25	33.1	17.3	21.6	22.9	35	42.2	23.0	26.2	26.4
c	e	c	e	2	4.1	2.7	5.7	4.3	4	14.8	7.6	9.6	10.0	14	22.5	13.5	13.0	15.1
c	e	e	c	10	6.6	6.1	9.0	8.2	7	19.8	13.3	15.7	16.7	17	29.3	20.7	21.3	23.3
c	e	e	e	3	1.1	5.3	4.5	3.4	12	8.9	15.2	12.6	13.2	20	15.6	27.1	21.4	25.1
e	c	c	c	40	48.3	41.9	36.3	40.0	58	48.7	57.3	55.7	54.7	78	59.4	67.6	74.3	66.2
e	c	c	e	3	7.8	2.7	6.7	4.2	6	21.8	7.6	10.6	8.4	15	31.7	13.5	13.7	11.7
e	c	e	c	12	12.5	6.1	10.5	9.4	16	29.2	13.3	17.3	18.0	22	41.2	20.7	22.5	24.3
e	c	e	e	2	2.0	5.3	5.3	3.9	12	13.0	15.2	13.9	14.3	30	22.0	27.1	22.6	26.1
e	e	c	c	14	21.1	20.8	19.1	19.7	31	40.5	34.6	35.6	36.3	47	55.2	46.0	49.3	48.3
e	e	c	e	2	3.4	5.3	6.3	4.7	11	18.1	15.2	15.8	15.8	16	29.5	27.1	24.4	27.7
e	e	e	c	13	5.4	12.2	10.0	9.0	32	24.2	26.5	25.9	26.4	42	38.3	41.4	40.1	42.6
e	e	e	e	6	.9	10.7	5.0	3.7	30	10.8	30.3	20.8	21.0	55	20.4	54.1	40.4	45.8
x^2					73.5	42.5	17.5	10.9		173.2	30.3	21.7	23.2		180.5	21.8	23.6	19.8

TABLE 2

Observed and predicted frequencies for specific response sequences from trials 6 through 9

Trial 6	7	8	9	9 Items Observed	Linear	All-or-none	TDF	TDF-Revised	15 Items Observed	Linear	All-or-none	TDF	TDF-Revised	21 Items Observed	Linear	All-or-none	TDF	TDF-Revised
c	c	c	c	205	177.7	192.2	164.1	192.1	271	156.3	263.9	251.6	259.3	319	178.1	309.7	339.5	310.4
c	c	c	e	0	5.3	.3	4.2	1.2	6	26.1	1.6	5.5	3.5	8	39.5	3.5	5.9	4.9
c	c	e	c	0	7.9	.7	6.3	2.8	8	32.8	2.7	8.6	7.3	13	48.4	5.4	9.4	9.9
c	c	e	e	0	.2	.6	1.6	.3	2	5.5	3.1	3.7	2.5	4	10.7	7.1	5.7	6.2
c	e	c	c	12	5.0	2.5	10.2	6.4	13	41.6	7.1	15.0	15.4	27	59.8	12.0	17.8	20.2
c	e	c	e	0	.4	.6	1.9	.5	1	6.9	3.1	4.5	3.1	6	13.3	7.1	6.7	7.1
c	e	e	c	1	.5	1.5	2.9	1.0	2	8.7	5.4	7.0	6.1	11	16.3	10.8	10.7	12.5
c	e	e	e	0	.0	1.3	.7	.1	5	1.5	6.2	3.1	2.1	10	3.6	14.1	6.5	7.8
e	c	c	c	13	18.3	10.1	17.6	14.4	24	53.5	23.5	30.2	34.7	55	74.8	35.3	40.5	47.4
e	c	c	e	0	.5	.6	2.3	.6	2	8.9	3.1	5.3	3.3	10	16.6	7.1	7.7	6.5
e	c	e	c	0	.8	1.5	3.5	1.4	11	11.2	5.4	8.3	7.3	5	20.3	10.8	12.4	14.0
e	c	e	e	0	.0	1.3	.9	.2	1	1.9	6.2	3.6	2.5	3	4.5	14.1	7.5	8.7
e	e	c	c	1	1.2	5.0	5.6	3.1	15	14.2	14.2	14.6	15.5	17	25.1	24.0	23.4	28.5
e	e	c	e	0	.0	1.3	1.1	.2	5	2.4	6.2	4.3	3.1	7	5.6	14.1	8.8	10.0
e	e	e	c	0	.1	2.9	1.6	.5	5	3.0	10.9	6.8	6.2	11	6.8	21.6	14.1	17.6
e	e	e	e	0	.0	2.6	.4	.1	4	.5	12.4	3.0	2.1	19	1.5	28.3	8.5	11.0
x^2					25.5	21.3	43.7	10.6		210.0	52.0	17.2	20.5		428.9	76.0	39.2	33.7

TABLE 3

Parameter estimates for various models and

total χ^2 values over groups

Model	Parameter	9 Item	15 Item	21 Item	χ^2 Value		
					Trials 2-5	Trials 6-9	Total
Linear	θ	.32	.17	.15	427.2	664.4	1091.6
All-or-none	c	.30	.20	.15	94.6	149.3	243.9
TDF	a	.16	-	-	62.8	100.1	162.9
	f	.22	-	-			
TDF Revised	a	.37	-	-			
	b	.11	-	-	53.9	64.8	118.7
	f	.15	-	-			

In the case of the all-or-none and linear models, estimates of the parameters $\underline{a}$ and θ were found which minimized Eq. 14b for each group. This minimization is not readily performed by analytic means, and so a high-speed computer was programmed to find parameter estimates by a search procedure on the parameter space. If the set of subject-items is homogeneous and stochastically independent, then under the null hypothesis, it can be shown that the χ^2 from Eq. 14b has the usual limiting distribution with 29 degrees of freedom for each group; one degree of freedom is subtracted from each of the two sets of sixteen frequencies because of the requirement that each set sum to the total observed subject-item combinations, and one degree of freedom is subtracted for the parameter estimate. The total χ^2 value over the three experimental groups will have 87 degrees of freedom.

Since the TDF model is formulated in a fashion that takes list length into account, values of $\underline{a}$ and $\underline{f}$ were found for this model which jointly minimized the χ^2 function for all three groups. That is, we define the function

$$\chi^2 = \sum_{j=1}^{3} \chi_j^2 \qquad (14c)$$

and find the values of $\underline{a}$ and $\underline{f}$ which minimize Eq. 14c. Since two parameters are estimated over the three groups, the χ^2 from Eq. 14c will have 88 degrees of freedom.

The response-sequence frequencies predicted by each of the models for trials 2 through 5 are listed in Table 1, and in Table 2 we present the predictions for trials 6 through 9. In Table 3 the parameter estimates obtained by the minimization procedure are presented, as well as the minimum χ^2 values for each of the models over all groups. The linear model is definitely inferior to the all-or-none model, and the TDF model does a better job than either of the other two, although one less parameter has been estimated from the data for this model.

In spite of the fact that the TDF model provides a more adequate account of the data than the other models, there is some cause to be dissatisfied with this formulation. For one thing, the overall χ^2 value is about 163, which with 88 degrees of freedom far exceeds the .001 level of significance. More importantly, there is evidence that the association parameter, $\underline{a}$, is not independent of list length. It will be recalled that in the analysis above, parameter estimation for the TDF model was carried out under the assumption that the parameters $\underline{a}$ and $\underline{f}$ are invariant over list lengths. The appropriateness of this assumption was evaluated by finding the best estimate of the two parameters separately for each experimental group; i.e., estimates of $\underline{a}$ and $\underline{f}$ were obtained using Eq. 14b for each list length. Good agreement was found among the three estimates of the forgetting parameter $\underline{f}$; the estimates were .25, .25 and .21 for groups 9, 15 and 21, respectively. However, the separate estimates of the association parameter $\underline{a}$ were ordered according to the number of items in the list; for groups 9, 15 and 21, the estimates were .20, .17 and .14.

Consequently, consideration was given to modifications of the TDF model which would give a more adequate account of the data and also yield parameter values which would be relatively invariant over the list length variable. In the association phase of the model as originally formulated (Eq. 8), it was assumed that the probability of moving to long-term memory was the same whether an item was in the unlearned state or the short-term state; in both instances the transition probability was $\underline{a}$. In the revised TDF model which will now be described, it will be assumed that the effect of a reinforced presentation of an item will depend on the state of the item at the time of reinforcement. If the item has been forgotten and is in state U, then there is a transition to long-term memory with probability $\underline{b}$, whereas with probability $1-\underline{b}$ the item goes to short-term memory. If an item is in state S (i.e., it has not been learned, but neither has it been forgotten since the last reinforced presentation), then with probability $\underline{a}$ the item is learned and moves to long-term memory, whereas with probability $1-\underline{a}$ it remains in state S. Thus matrix $\underline{\underline{A}}$ (Eq. 8) is replaced by matrix $\underline{\underline{A}}'$ below,

$$\underline{\underline{A}}' = \begin{array}{c} \\ L \\ S \\ U \end{array} \begin{array}{c} \begin{array}{ccc} L & S & U \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ a & 1-a & 0 \\ b & 1-b & 0 \end{bmatrix} \end{array} \qquad (15)$$

In all other respects the revised model is unchanged from the original formulation, and, in particular, the expected value of Z_n will be used as an approximation to Z_n in deriving all statistics. That is, suppose that $Pr(L_n)$ is the probability of being in long-term memory on trial n which, for the revised TDF model, is a function of $\underline{a}$, $\underline{b}$,

f, and X. Then the expected number of unlearned stimulus-response pairs between the nth and (n+1)st presentations of an item will be

$$X \left\{ 1 - \frac{1}{2} \left[Pr(L_n) + Pr(L_{n+1}) \right] \right\}$$

The minimum χ^2 estimation procedure was used to obtain estimates of a, b and f for the three experimental groups jointly, using Eq. 14c. The results are presented in Tables 1, 2 and 3 as the revised TDF model. As may be seen, the introduction of the parameter b reduced the overall χ^2 value of the original TDF model by more than 25 per cent, from 163 to 119, which represents a considerable improvement. Moreover, when estimates of the parameters a, b and f were obtained separately for each of the experimental groups, it was found that the three estimates of f were quite consistent with one another, and that the variations in a and b were unrelated to list length.

It is of interest to note that the probability that an item is learned is more than three times larger if the item is in short-term memory than if it is in the unlearned state, i.e., the estimate of the parameter a is .37 while b is .11. This relation between a and b suggests an explanation for the dependency between the association parameter in the original TDF model and the length of the paired-associate list. The effect of increasing list length in these models is to make it more likely that an item is in state U. In the original model, where a and b are assumed to be equal, the estimate of the associative effect of a reinforcement will be some weighted average of a and b, which we will call $\bar{a}$. In the case of a list with very few stimulus-response pairs, the probability that an item is in state S is larger than when there are many pairs, since as the number of pairs becomes larger, more forgetting will occur and hence an item is more likely to be in state U. Thus the relative contribution of the parameter $\underline{a}$ to the average $\bar{\underline{a}}$ will decrease as list length increases, and as list length becomes very large, $\bar{\underline{a}}$ will approach $\underline{b}$. Since $\underline{a}$ is greater than $\underline{b}$, the finding that $\bar{\underline{a}}$ decreases with increasing list length in the original TDF model would be expected.

Fig. 3 presents the mean learning curves predicted by the revised model for each of the three list lengths. As may be seen, there is good agreement between the data and the theoretical curves. The model also was used to predict the curves for $Pr(e_{n+1}|e_n)$ which are shown in Fig. 4 along with the observed values. The data points are fairly variable, but overall the theoretical curves fit reasonably well. One way of directly testing the interference assumptions of the TDF model embodied in Eq. 9 would be to look at the probability of a correct response on trial n+1 as a function of the number of items interpolated between the nth and (n+1)st presentation of a particular stimulus-response pair. This probability should decrease exponentially according to the interference hypothesis of the TDF model. Unfortunately, when the experiment reported in this paper was conducted, no record was made of the specific presentation order within a trial, and so this direct test cannot be made. In an unreported study by Shelton, within-trial presentation order was recorded. Examination of these data clearly indicates that that the probability of a success was a decreasing function of the number of interpolated items.

It has been our goal in presenting this analysis of paired-associate learning to illustrate the manner in which mathematical representations of psychological processes may be used to test specific hypotheses about the way in which these processes operate. While the revised TDF model is sufficiently complex to present serious difficulties in carrying out mathematical derivations, it is unlikely that the behavioral situation is as simple as the model indicates. Indeed, although the revised model is the most adequate formulation that has been presented, it is not satisfactory in a number of respects. For one thing, the χ^2 value of 119 with 87 degrees of freedom would lead to rejection of the null hypothesis at the .05 level of significance. This is not a serious fault per se, since we would certainly not discard the model until a more adequate representation of paired-associate learning could be suggested.

A more serious type of criticism involves the application of the model to data from other experiments to determine its generality. Revisions will certainly be necessary to account for some aspects of paired-associate learning that are not brought out by the experiment reported in this paper. For example, suppose that when a subject makes an incorrect response he is given a second guess at the right answer. According to the TDF model, the second guess should be chosen at random from the reduced set of response alternatives. In fact, the probability of a correct response on the second guess is found to be somewhat greater than this chance expectation (Binford & Gettys, 1964).

While it is always possible to modify any particular model to account for empirical findings, some other formulation might be preferred on the basis of simplicity and parsimony. Within the limits of the study reported in this paper, however, the TDF model is successful in providing a relatively good account of the data, both qualitatively and in quantitative detail. Moreover, it does so by incorporating both association and forgetting processes known to be important in paired-associate learning, and these processes are represented in such a fashion that changes in the data produced by variations in list length are accounted for by the representation of the learning process, rather than by changes in the parameter values.

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