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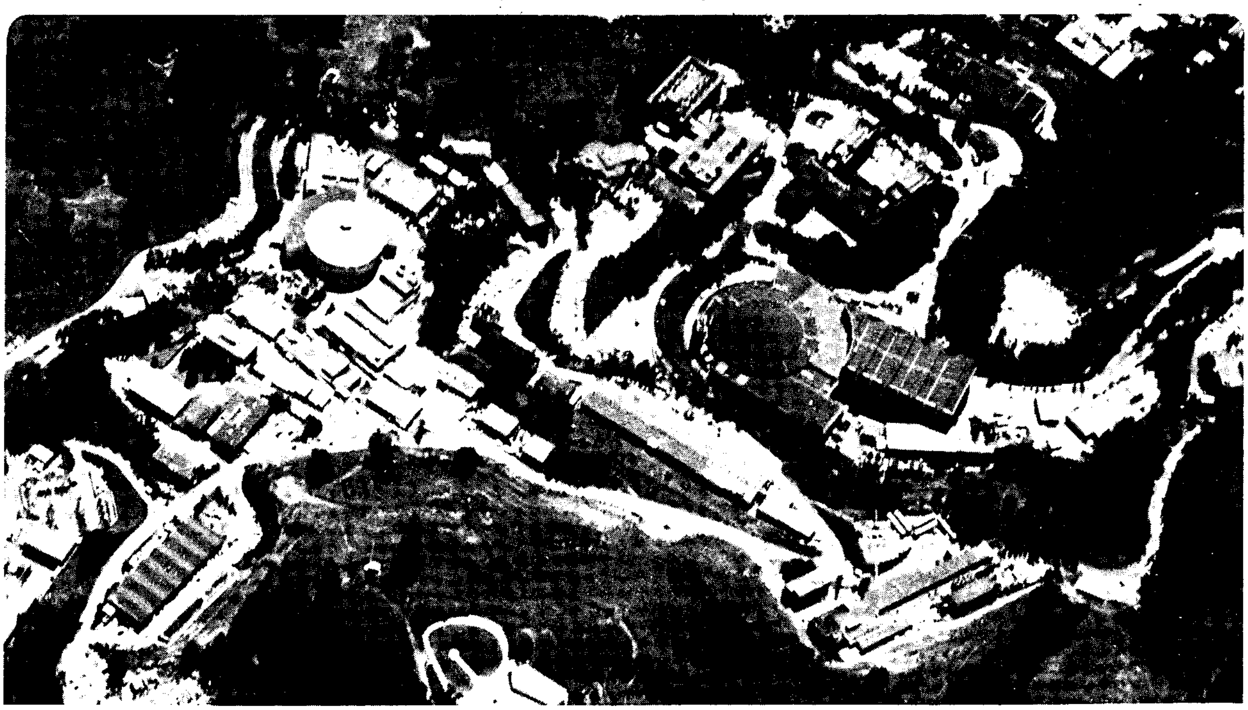

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DISCONTINUOUS INVERSE STURM-LIOUVILLE PROBLEMS WITH SYMMETRIC POTENTIALS[1]

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Ph.D. Thesis

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Abstract

In this paper we study the Inverse Sturm-Liouville problem on a finite interval with a symmetric potential function with two interior discontinuities. In the introductory chapter we survey previous results on the existence and uniqueness of solutions to inverse Sturm-Liouville problems and discuss earlier numerical methods. In chapter 1 we present a uniqueness proof for the inverse Sturm-Liouville problem on a finite interval with a symmetric potential having two interior jump discontinuities. In chapter 2 we show that any absolutely continuous function can be expanded in terms of the eigenfunctions of a Sturm-Liouville problem with two discontinuities. In chapter 3 we consider two Sturm-Liouville problems with different symmetric potentials with two discontinuities satisfying symmetric boundary conditions and symmetric jump conditions. We find that if only a finite number of eigenvalues differ then a simple expression for the difference of the potentials can be established. In addition, the locations of the discontinuities are uniquely determined. Finally, in chapter 4 we derive an algorithm for solving the discontinuous inverse Sturm-Liouville problem numerically and present the results of numerical experiments.

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Contents

Chapter 0

Introduction

0.1 Inverse Problems

The problem considered in my thesis lies in a very active area of research known as inverse problems. This branch of science has recently experienced an increased amount of attention; a new journal, Inverse Problems, begun in February of 1985, celebrates a distinguished list of contributing authors including Barcilon, Grunbaum, Santosa, Symes and Talenti. And numerous conferences have been held throughout the world [3], [5], [13], [21], [23], [59], [65], [80]. All inverse problems are associated with a forward problem which, ordinarily, is considerably easier to solve. To illustrate the concept of forward and inverse problems we consider a simple system studied by Euler and Bernoulli. Given the density, length and tension of a plucked string, determine the tones produced. The opposite or inverse problem is to determine the density of the string from its tones, its length and its tension. For an elementary presentation, see Durran [25]. Inverse problems appear in a wide variety of scientific areas [14] , [53] , [74]. Other examples include image reconstruction problems in X-ray tomography [30], [29], [75], the determination of the shape of flaws or cavities in metal castings [12], the modelling of groundwater [24], [44], potential flow studies [45], [46], heat conduction problems [66], the determination of material properties of a beam from its vibrational modes [11], [8], [9], [28], [60], [61], [62], multiplicative problems in molecular spectroscopy [20], [68], [78], the recovery of a cross-sectional area of the vocal tract from measured data [76], the determination of the density inside the earth from seismographic data [16], [15], [17], [19], [39], [37], [38], [64], problems in optics [7], [6], scattering problems in physics [1], [22], [13], [47], and mathematical inverse problems [52],

[58], [69], [71], [72], [81].

A large class of mathematical inverse problems known as inverse eigenvalue problems are described by the equation

$$L \ f_i = \lambda_i \ f_i \qquad ; \quad i = 0, 1, 2, \ldots$$

where L is an operator and the f_i are eigenfunctions corresponding to the eigenvalues λ_i. The forward problem is to determine the λ_i and f_i for a given operator L. The inverse problem is to determine L given the eigenvalues λ_i and some additional information about the f_i (e.g. boundary conditions). Examples of operators to be reconstructed include matrices acting on vectors and differential operators acting on functions. Recent articles by Barcilon [10] and McLaughlin [63] survey methods and properties of methods for recovering coefficients in differential equations from spectral data.

0.2 Inverse Sturm-Liouville Problems

This thesis examines the Sturm-Liouville equation

$$-y'' + q(x)\ y = \lambda y$$

on the bounded interval $[0, \pi]$. In the inverse Sturm-Liouville problem we measure the frequencies of a vibrating system and try to infer some physical properties of the system. For a complete historical development of works on the existence and uniqueness of solutions to the continuous and discrete inverse Sturm-Liouville problems up to 1972, see Hald [32]. There are five well-known versions of the continuous inverse Sturm-Liouville problem on a finite interval. The first four presented below do not directly pertain to the work in this thesis. Gelfand and Levitan [27], Marcenko [57], Krein [48] and Zikov [92] study an inverse Sturm-Liouville problem in which the potential, $q(x)$ and the boundary conditions are uniquely determined by the spectral function. Marcenko [57], Levitan [55], Gasymov and Levitan [26] and Zikov [92] examine a second case in which the potential and the boundary conditions are uniquely determined by two spectra. They show that this second version can be reduced to the previous one. Borg [18], Levinson [54] and Hochstadt [42] investigate a third variation in which the potential is uniquely determined by the boundary conditions and two reduced spectra. In a fourth version Borg [18], Levinson [54] and Hochstadt [42] show that if the boundary conditions and one reduced spectrum are given, then the potential is uniquely determined provided its is an even function with respect to the midpoint of the interval. This thesis uses the work from a fifth version by Hald [35], "The Inverse Sturm-Liouville Problem with Symmetric Potentials".

In the first theorem of [35] Hald considers two Sturm-Liouville problems with different potentials and different boundary conditions. If the potentials are even functions around the middle of the interval and the sum of the absolute value of the differences of the eigenvalues of the two problems is finite, then the potentials differ by a continuous function.

Theorem (Hald 1978) *Consider the eigenvalue problems*

$$-u'' + q(x)\, u = \lambda\, u$$
$$hu(0) - u'(0) = 0 \quad , \quad hu(\pi) + u'(\pi) = 0 \tag{0.1}$$

$$-u'' + \tilde{q}(x)\, u = \tilde{\lambda}\, u$$
$$\tilde{h}u(0) - u'(0) = 0 \quad , \quad \tilde{h}u(\pi) + u'(\pi) = 0 \tag{0.2}$$

where q and $\tilde{q}$ are integrable on $[0,\pi]$ and satisfy the symmetry conditions $q(x) = q(\pi - x)$ and $\tilde{q}(x) = \tilde{q}(\pi - x)$ almost everywhere in the interval $0 \le x \le \pi$. Let λ_j and $\tilde{\lambda}_j$ be the eigenvalues of (0.1) and (0.2). Let $\tilde{u}_j$ and $\tilde{v}_j$ be the solutions of

$$u'' + (\lambda - \tilde{q})u = 0 \tag{0.3}$$

$$u(0) = 1 \quad , \quad u'(0) = \tilde{h} \tag{0.4}$$
$$v(\pi) = 1 \quad , \quad v'(\pi) = -\tilde{h} \tag{0.5}$$

with $\lambda = \lambda_j$. Define the functions $\tilde{y}$ by

$$\tilde{y}_j = 2 \cdot \frac{\tilde{v}_j - k_j \tilde{u}_j}{\omega'(\lambda_j)} \quad . \tag{0.6}$$

Here $k_j/\omega'(\lambda_j) = 1/\int_0^\pi u_j^2\, dx$ where $k_j = (-1)^j$ and $u_j(x)$ are the eigenfunctions of (0.1) normalized such that $u_j(0) = 1$. If $\sum_j |\lambda_j - \tilde{\lambda}_j| < \infty$ then

$$h - \tilde{h} = \frac{1}{2}\sum_j \tilde{y}_j(0) \tag{0.7}$$

$$q - \tilde{q} = \sum_j (\tilde{y}_j u_j)' \quad a.e. \quad . \tag{0.8}$$

If two eigenvalue problems have the same eigenvalues then $\tilde{v}_j = k_j \tilde{u}_j$, all $\tilde{y}_j$ vanish identically, and the right-hand side of equations (0.7) and (0.8) are zero. In this manner Hald proves a uniqueness result for the potential and the boundary conditions in a corollary to the theorem above.

Corollary (Hald 1978) *Consider the eigenvalue problem (0.1) where q is integrable in $[0, \pi]$. If $q(x) = q(\pi - x)$ almost everywhere in $0 < x < \pi$ then $q(x)$ and h are uniquely determined by the spectrum.*

Furthermore Hald shows that the potential is uniquely determined almost everywhere by the reduced spectrum. The lowest eigenvalue plays a special role in the inverse eigenvalue problem; omitting any other eigenvalue from the spectrum fails to give the uniqueness results presented above. Hald's work is an extension of ideas presented by Hochstadt [41]. This thesis further extends the theorem by Hald to discontinuous inverse Sturm-Liouville problems with symmetric potentials. Formulae (0.7) and (0.8) are valid in the discontinuous problem, however we must derive additional formulae to take into account the jumps in the eigenfunctions and their derivatives. For details see chapter 3. The corollary by Hald can be extended to the discontinuous case by the same argument as the one given above. The other corollaries and the existence proof cannot be extended in a straightforward manner to the discontinuous problem.

The significance of Hald's theorem is not fully realized in considering only the corollaries; it is indispensible for constructing an algorithm for numerically solving the inverse Sturm-Liouville problem with symmetric potentials. In the problem described in Theorem 1 (See Chapter 1.), Hochstadt assumes that $h = \tilde{h}$ and only a finite number of the eigenvalues λ_j and $\tilde{\lambda}_j$ are different. (i.e. $\lambda_j = \tilde{\lambda}_j$ for $j > n$.) Let Λ_0 denote the index set for the first n eigenvalues. In this case equation (0.7) is trivially satisfied and the summation in equation (0.8) is only over those j for which $\lambda_j \neq \tilde{\lambda}_j$. Attempts by Hald and the author to numerically determine the potential function using Hochstadt's assumptions fail; the reconstructed potential is unsymmetric and in some cases diverges. Hald's realization that h and $\tilde{h}$ must differ is crucial for the success of the algorithm. In addition Hald realizes that $\omega'(\lambda_j)$ should not be evaluated from the definition. The derivative of the Wronskian may be calculated more simply by applying the Hadamard Factorization Theorem to each of the factors in the ratio $\omega(\lambda)/\tilde{\omega}(\lambda)$ and differentiating. Then

$$\omega'(\lambda_j) = \frac{\prod_{i\neq j}(\lambda_j - \lambda_i)}{\prod_{i\neq j}(\lambda_j - \tilde{\lambda}_i)} \; \frac{\tilde{\omega}(\lambda_j) - \tilde{\omega}(\tilde{\lambda}_j)}{\lambda_j - \tilde{\lambda}_j} \quad . \tag{0.9}$$

Let z_j be the eigenfunction of (0.2) corresponding to $\tilde{\lambda}_j$. Hald notes that the last quotient in (0.9) is equal to $-\tilde{h}w_j(\pi) - w_j'(\pi)$ where w_j satisfies the differential equation

$$w_j'' + (\tilde{\lambda}_j - \tilde{q})w_j = -\tilde{u}_j$$
$$w_j(0) = w_j'(0) = 0$$

Now we are ready to present Hald's recipe for solving the inverse Sturm-Liouville problem with symmetric potentials.

Algorithm (Hald 1978)

Step 1°: For each j in Λ_0 determine a k in Λ_0 such that

$$|\lambda_j - \tilde{\lambda}_k| = \min_{i\in\Lambda_0} |\lambda_j - \tilde{\lambda}_i|$$

Step 2°: For each j in Λ_0 solve the system

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & \tilde{q} - \tilde{\lambda}_k & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}$$

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}_j' \\ w_j \\ w_j' \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ \tilde{h} \\ 0 \\ 0 \end{bmatrix}.$$

Step 3°: For each j in Λ_0 compute

$$\omega'(\lambda_j) = \frac{\prod_{i\neq j}(\lambda_j - \lambda_i)}{\prod_{i\neq k}(\lambda_j - \tilde{\lambda}_i)} \; [\, -\tilde{h}w_j(\pi) - w_j'(\pi) \,] \ .$$

Step 4°: Set

$$h = \tilde{h} \; + \; \sum_{\Lambda_0} (\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \ .$$

Step 4^o**:** Solve the system

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q}-\lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{q}+\sum_{\Lambda_0}(\tilde{y}'_i u_i + \tilde{y}_i u'_i) - \lambda_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 2(\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \\ -2(\tilde{u}'_j(\pi) + (-1)^j\,\tilde{h})/\omega'(\lambda_j) \\ 1 \\ h \end{bmatrix}.$$

Step 5**:** Set

$$q = \tilde{q} \;+\; \sum_{\Lambda_0} (\tilde{y}'_j u_j + \tilde{y}_j u'_j)\ .$$

In chapter 4 we extend Hald's results for the continuous problem to the discontinuous case. The major modification we must make is to solve the systems given in Steps 2 and 5 on the three disjoint intervals $0 < x < d$, $d < x < (\pi - d)$ and $(\pi - d) < x < \pi$ taking into account the jumps in the eigenfunctions and in the derivatives of the eigenfunctions at the symmetrically located discontinuities $x = d$ and $x = \pi - d$.

Finally we briefly mention that at least one other technique has been developed to solve the inverse Sturm-Liouville problem numerically. The Gelfand-Levitan technique has been studied by Hald [36] among others. We have chosen to describe the algorithm given above [35] in detail since it requires $10Nn$ operations whereas the Gelfand-Levitan technique requires $N^3/3$ operations. Here we assume that the potential is wanted at N points in the interval $[0, \pi]$ and n is the number of perturbed eigenvalues. Thus if only a few eigenvalues are perturbed and the potential is wanted at many points, then the algorithm of Hald [35] is the most economical of the two.

0.3 Discontinuous Inverse Sturm-Liouville Problems

Many techniques developed for studying inverse problems assume the absence of jump discontinuities in the material properties of the medium. This assumption is frequently ill-founded as is shown by comparing the works of Krueger [49], [50] and Weston [84]. And as such several researchers have recently begun an investigation of the discontinuous inverse Sturm-Liouville problem. Several different approaches have been used for this purpose; the formulation of the problem depends on the physical system considered by the investigator. Krueger [49], [50] and Weston [84] present a general model for wave propagation as described below.

$$u_{xx} - u_{tt} + A(x)u_x + B(x)u_t + C(x)u = 0 \ , \quad -\infty < x < \infty \quad , \quad -\infty < t < \infty \quad , \qquad (0.10)$$

where $0 = x_0 < x_1 < \ \ldots \ < x_n = l$, with the hypotheses

(i) Support A, B, $C \subseteq [0, l]$,

(ii) A, B, C are piecewise continuous on $[0, l]$ with discontinuities at x_i,

(iii) A' and B' are continuous on the subintervals (x_i, x_{i+1}),

(iv) $u(x, t)$ is everywhere continuous and piecewise C^2,

(v) There exist nonzero constants c_i such that $c_i u_x(x_i+, t) = u_x(x_i-, t)$

for $i = 0, 1, \ldots, n$. They assume that the coefficients A, B, C, the locations and the number of discontinuities, i.e. the points x_i and the integer n, are unknown on $(0, l)$. The inverse problem is to determine information about these coefficients from information regarding u in the regions $x < 0$ and $x > l$. To explain further, a plane wave $u^i(x - t)$ propagating in the $+x$ direction gives rise to a reflected wave u^r propagating in the $-x$ direction for $x < 0$ and a transmitted wave u^t propagating in the $+x$ direction for $x > l$ so that

$$u(x,t) = u^i(x - t) + u^r(x + t) \ , \quad x < 0 \qquad (0.11)$$

$$= u^t(x - t) \ , \quad x > l \ , \qquad (0.12)$$

$u^i(s) = u^t(s) = 0$ for $s \geq 0$, and these functions are continuous and piecewise C^2. With the additional hypotheses

(vi) $u''(0-) \neq 0$,

(vii) If S is the set of all quantities $(x_i - x_j)$, where $0 \leq j < i \leq n$ then no element of S can be expressed as a linear combination (with positive, integral coefficients) of other elements of S,

(viii) A finite upper bound for x_n is known,

knowledge of the scattering data $u^i(s), u^t(s), u^r(-s)$ for $-4l < s < 0$ is sufficient to determine the coefficient $B(x)$ and the combination of coefficients $C - \frac{1}{2}A' - \frac{1}{4}A^2$ on $(0, l)$ as well as the points x_i and the integer n. Even if the hypotheses (vi) - (viii) are not satisfied the inverse problem may still be solvable but certain data may have to be obtained by means other than observing scattered waves. Krueger [51] and Weston [84] choose appropriate functions for the coefficients $A(x)$, $B(x)$, $C(x)$ and change variables to obtain a special case of problem (0.10) – (0.12) to describe one-dimensional electromagnatic wave propagation in nonmagnetic, nonabsorbing media. Similarly Krueger [51], Pao, Santosa and Symes [67] study a specific case of (0.10) – (0.12) to simulate wave propagation in elastic media, and Symes [79] examines yet another particular case of (0.10) – (0.12) with different boundary conditions to model acoustic waves.

Bolt [16], [15], [17], [19], Hald [39], [37], [38], [33] and Willis [88], [90] , [89], [91] investigate earthquake waves in the Earth's mantle. The inverse problem for the earth amounts to determining the material properties of the Earth from seismological data. At first Hald [33] considers the simpler inverse Sturm-Liouville problem for a cylinder. This model problem has several properties in common with the inverse problem for a sphere. However the theory involves smoothness assumptions which are not normally made in the case of the Earth, and the model lacks features corresponding to gravity and the rotation and ellipticity of the earth. Later he considers a simple, spherically symmetric, non-rotating model of the earth (See Alterman, Jarosh and Pekeris [2].). In [37] and [38] Hald presents the continuous inverse Sturm-Liouville problem from a geophysical perspective with the assumption that the Earth's mantle consists of homogeneous material. The homogeneity assumption implies that the Sturm-Liouville equation is smooth with respect to the radial parameter. A more refined model of the earth would recognize the distinct, concentric layers of the mantle. With this seismological model in mind Hald examines discontinuous inverse eigenvalue problems.

A uniqueness result by Hald for the discontinuous inverse Sturm-Liouville problem shows that if the eigenvalues and one of the boundary conditions are fixed and the potential q is given over half of the interval, then the potential and the other boundary condition are uniquely determined even if the differential equation has an interior discontinuity. This result is a generalization of a theorem due to Hochstadt and Liebermann [43] who assume that $a = 1$, $b = 0$ and $h = \tilde{h}$. We note that the last restriction is actually unnecessary (See Hald [37].).

Theorem (Hald 1984) *Consider the eigenvalue problem:*

$$-u'' + q(x)\ u = \lambda\ u \tag{0.13}$$

on the interval $0 < x < \pi$ *and with the boundary conditions:*

$$u'(0) - h\ u(0) = u'(\pi) + H\ u(\pi) = 0 \tag{0.14}$$

and with jump conditions:

$$u(d+) = au(d-)\ , \qquad u'(d+) = a^{-1}u(d-) + b\ u(d-) \tag{0.15}$$

where q *is an integrable function,* $0 < d < \frac{1}{2}\pi$, $a > 0$, *and* $|a-1| + |b| > 0$. *Let* $\lambda_0, \lambda_1, \cdots$ *be the eigenvalues. Consider the eigenvalue problem with* a, b, d, h, H, λ *and* q *replaced by* $\tilde{a}$, $\tilde{b}$, $\tilde{d}$, $\tilde{h}$, $\tilde{H}$, $\tilde{\lambda}$ *and* $\tilde{q}$. *If* $\lambda_j = \tilde{\lambda}_j$ *for* $j \geq 0$, $H = \tilde{H}$ *and* $q = \tilde{q}$ *almost everywhere in* $(\frac{1}{2}\pi, \pi)$, *then* $a = \tilde{a}$, $b = \tilde{b}$, $d = \tilde{d}$, $h = \tilde{h}$ *and* $q = \tilde{q}$ *almost everywhere.*

In the course of the proof Hald shows several interesting results; Volterra integral equations are derived, upper bounds for the eigenfunctions and higher order terms of the eigenfunctions are found, properties of the Wronskian are explored (Formulae for the Wronskian, ω, the leading order term, ω_0, the roots of ω_0, and a lower bound for ω are determined.), the jump constants a and d are shown to be uniquely determined provided $|a-1| + |b| > 0$, and integral equations for $b - \tilde{b}$ and $h - \tilde{h}$ are used to show that $q - \tilde{q} \equiv 0$ a.e. on $[0, \frac{\pi}{2}]$.

Willis [90] extends Hald's uniqueness theorem to inverse Sturm-Liouville problems with two discontinuities.

Theorem (Willis 1985) *Consider the eigenvalue problem:*

$$-\ u'' + q(x)\ u = \lambda\ u \tag{0.16}$$

on the interval $0 < x < \pi$ *and with the boundary conditions:*

$$u'(0) - h\ u(0) = u'(\pi) + H\ u(\pi) = 0 \tag{0.17}$$

and with jump conditions:

$$u(d_1+) = a_1 u(d_1-)\ , \qquad u'(d_1+) = a_1^{-1}u'(d_1-) + b_1\ u(d_1-) \tag{0.18}$$

$$u(d_2+) = a_2 u(d_2-)\ , \qquad u'(d_2+) = a_2^{-1}u'(d_2-) + b_2\ u(d_2-) \tag{0.19}$$

where q *is an integrable function,* $0 < d_1 < d_2 < \frac{1}{2}\pi$, $a_1, a_2 > 0$, $|a_1 - 1| + |b_1| > 0$, *and* $|a_2 - 1| + |b_2| > 0$. *Let* $\lambda_0, \lambda_1, \cdots$ *be the eigenvalues. Consider the eigenvalue problem with* a_1, b_1, d_1, a_2, b_2, d_2, h, H, λ *and* q *replaced by* $\tilde{a}_1$, $\tilde{b}_1$, $\tilde{d}_1$, $\tilde{a}_2$, $\tilde{b}_2$, $\tilde{d}_2$, $\tilde{h}$, $\tilde{H}$, $\tilde{\lambda}$ *and* $\tilde{q}$. *Let*

$\alpha_1 = (a_1 - a_1^{-1})/(a_1 + a_1^{-1})$ and $\alpha_2 = (a_2 - a_2^{-1})/(a_2 + a_2^{-1})$. If $\alpha = |\alpha_1| + |\alpha_2| + |\alpha_1\alpha_2| < 1$, $\tilde{\alpha} = |\tilde{\alpha}_1| + |\tilde{\alpha}_2| + |\tilde{\alpha}_1\tilde{\alpha}_2| < 1$, $\lambda_j = \tilde{\lambda}_j$ for $j \geq 0$, $H = \tilde{H}$ and $q = \tilde{q}$ almost everywhere in $(\frac{1}{2}\pi, \pi)$ then $a_1 = \tilde{a}_1$, $b_1 = \tilde{b}_1$, $d_1 = \tilde{d}_1$, $a_2 = \tilde{a}_2$, $b_2 = \tilde{b}_2$, $d_2 = \tilde{d}_2$, $h = \tilde{h}$ and $q = \tilde{q}$ almost everywhere.

Willis [90] follows the techniques of Hald [39] so that the structure of the equations she derives are similar, however many more terms are involved. In the case of two discontinuities the uniqueness proof for the constants a_1, d_1, a_2 and d_2 forces us to impose the restrictions $|a_1 - 1| + |b_1| > 0$ and $|a_2 - 1| + |b_2| > 0$, as in the single discontinuity case, and additionally $\alpha = |\alpha_1| + |\alpha_2| + |\alpha_1\alpha_2| < 1$. Willis' uniqueness proof for the a_i and d_i depends on the position of d_1 and d_2. 30 different possible cases must be considered with each case requiring a different proof. Similarly, to show that $q - \tilde{q} \equiv 0$ a.e. on $[0, \frac{\pi}{2}]$, 16 different cases must be considered in manipulating the integral equations for $b_1 - \tilde{b}_1$, $b_2 - \tilde{b}_2$ and $h - \tilde{h}$. It appears that a uniqueness proof for problems with even more discontinuities may be constructed by using arguments from Hald [39] and Willis [90]. We anticipate that the equations appearing in the proof will have structure similar to those in the one and two discontinuity cases, but will involve many more terms in the region $d_3 < x < \pi$ where d_3 is the third discontinuity. The hypotheses $|a_i - 1| + |b_i| > 0$ and $\alpha < 1$ where α is a sum of the absolute values of the α_i's and products of the α_i's will be required. Even more possible positions of the discontinuities $0 < d_i < \frac{\pi}{2}$ will lead to even more possible cases to be considered in the uniqueness proof.

Andersson [4] studies the inverse Sturm-Liouville system

$$(p^2(x)\, u'(x))' + \lambda\, p^2(x)\, u(x) \quad , \quad 0 \leq x \leq 1 \quad , \quad p > 0 \tag{0.20}$$

with suitable boundary conditions e.g.

$$u'(0) = u'(1) = 0 \ . \tag{0.21}$$

Hald and Willis assume that p is stepwise twice continuously differentiable and has at most one or two jump discontinuities. Andersson's aim is to treat the inverse eigenvalue problem (0.13) under more general conditions on p than earlier. The regularity conditions imposed on p are that $\ln p$ should be of bounded variation or that $(\ln p)' \in L^r(0,1)$ for some r with $1 \leq r \leq \infty$. The spectral data to be considered are the eigenvalues $(\lambda_{2k})_{k=0}^{\infty}$ of (0.13) and (0.14) and $(\lambda_{2k+1})_{k=0}^{\infty}$ of (0.13) together with the boundary conditions

$$u'(0) = u(1) = 0 \ . \tag{0.22}$$

Chapter 1

A Uniqueness Proof

A uniqueness proof for the inverse Sturm-Liouville problem on a bounded interval with a symmetric potential having two interior jump discontinuities is presented in this chapter. We derive several lemmas in order to complete this proof; the asymptotic form of the eigenvalues and eigenfunctions are determined, upper bounds for the eigenfunctions and their derivatives are established and a uniqueness proof for the jump constant a and discontinuity d is given. These results will be of consequence in later chapters where we build an algorithm to reconstruct the potential function. An alternate uniqueness proof will be given in chapter 3. Some of the lemmas from chapter 1 will also be needed in this second uniqueness proof. The techniques we use follow those of Hald [39] and Willis [88], [90].

1.1 Main Result

Theorem 1: *Consider the eigenvalue problem:*

$$-u'' + q(x)\ u = \lambda\ u \tag{1.1}$$

on the interval $0 \leq x \leq \pi$ *with the boundary conditions:*

$$h\ u(0) - u'(0) = h\ u(\pi) + u'(\pi) = 0 \tag{1.2}$$

and symmetric discontinuities at $x = d$ *and* $x = (\pi - d)$ *satisfying the symmetric jump conditions:*

$$u(d+) = au(d-)\ , \qquad u'(d+) = a^{-1}u'(d-) + b\ u(d-) \tag{1.3}$$

$$u((\pi - d)-) = a\ u((\pi - d)+)\ , \quad u'((\pi - d)-) = a^{-1}u'((\pi - d)+) - b\ u((\pi - d)+) \quad (1.4)$$

where $|a-1| + |b| > 0$ *and* $0 \le d < \pi/2$. *Consider also the eigenvalue problem with* u, q, λ, h, a, b *and* d *replaced by* $\tilde{u}$, $\tilde{q}$, $\tilde{\lambda}$, $\tilde{h}$, $\tilde{a}$, $\tilde{b}$ *and* $\tilde{d}$. *Assume* $q(x)$ *and* $\tilde{q}(x)$ *are integrable and satisfy the symmetry conditions* $q(x) = q(\pi - x)$ *and* $\tilde{q}(x) = \tilde{q}(\pi - x)$ *almost everywhere on* $[0, \pi]$. *Let* $\{\lambda_j\}$ *and* $\{\tilde{\lambda_j}\}$ *be the eigenvalues of the first and second eigenvalue problems. If* $\lambda_j = \tilde{\lambda_j}$ *for* $j \ge 0$, *then* $a = \tilde{a}$, $b = \tilde{b}$, $d = \tilde{d}$, $h = \tilde{h}$ *and* $q = \tilde{q}$ *almost everywhere on* $[0, \pi]$.

Remarks: Hald [39] and Willis [88], [90] proved a uniqueness result for the inverse Sturm-Liouville problem with one and two discontinuities; if the eigenvalues and one of the boundary conditions are fixed and the potential q is given in one half of the interval, then the potential and the other boundary condition are uniquely determined if the differential equation has one or two interior discontinuities. Their results is generalize a theorem due to Hochstadt and Liebermann [43] who assume that $a = 1$, $b = 0$ and $h = \tilde{h}$. (Actually, this last restriction is unnecessary, see Hald [37] .) Note that $0 \le d < \frac{\pi}{2}$ so that $d \ne \frac{\pi}{2}$. Hald [39] has shown that if $d = \frac{\pi}{2}$, uniqueness cannot be guaranteed by all the eigenvalues and jump condition h at $x = 0$.

Beginning of the Proof: Let u be the solution of equations (1.1), (1.3) - (1.4) satisfying the initial conditions $u(0) = 1$ and $u'(0) = 0$. We do not define u at the discontinuities $x = d$ and $x = (\pi - d)$. It is well known that the solution of a Sturm-Liouville problem satisfies a Volterra integral equation of the second kind [83], [39], [88], [90]. In this section we use this equation to estimate the solution and its derivative and show that u is an entire function in λ of order $\frac{1}{2}$. Then we consider the Wronskian $\omega(\lambda) = -u'(\pi) - h\ u(\pi)$. Its roots are real and simple and by using the estimates of u and u' we can give crude upper and lower bounds for the eigenvalues of the differential equation.

To show that the equations for u can be written as a Volterra integral equation, we follow the convention of Hald [39] and Willis [88], [90] and write the eigenfunctions as follows.

$$u_1(x) \quad = \quad g_1(x) \ + \ \int_0^x G_{11}(x,t)q(t)u_1(t)\ dt \qquad\qquad (0 \le x < d) \qquad (1.5)$$

$$u_2(x) \quad = \quad g_2(x) \ + \ \int_0^d G_{21}(x,t)q(t)u_1(t)\ dt$$

$$+ \int_d^x G_{22}(x,t)q(t)u_2(t)\,dt \qquad (d < x < \pi - d) \tag{1.6}$$

$$\begin{aligned} u_3(x) &= g_3(x) + \int_0^d G_{31}(x,t)q(t)u_1(t)\,dt \\ &+ \int_d^{\pi-d} G_{32}(x,t)q(t)u_2(t)\,dt \\ &+ \int_{\pi-d}^x G_{33}(x,t)q(t)u_3(t)\,dt \qquad (\pi - d < x \leq \pi) \end{aligned} \tag{1.7}$$

where:

$$g_1(x) = \cos kx + \frac{h}{k}\sin kx \tag{1.8}$$

$$\begin{aligned} g_2(x) &= a\,[\,\cos kd + \frac{h}{k}\sin kd\,]\,\cos k(x-d) \\ &+ a^{-1}\,[\,-\sin kd + \frac{h}{k}\cos kd\,]\,\sin k(x-d) \\ &+ \frac{b}{k}\,[\,\cos kd + \frac{h}{k}\sin kd\,]\,\sin k(x-d) \end{aligned} \tag{1.9}$$

$$\begin{aligned} g_3(x) &= a^{-1}\,[\,a\cos k(\pi-2d)\cos kd - a^{-1}\sin k(\pi-2d)\sin kd \\ &+ \frac{b}{k}\sin k(\pi-2d)\cos kd + \frac{h}{k}\,\{\,a^{-1}\sin k(\pi-2d)\cos kd \\ &+ a\,\cos k(\pi-2d)\sin kd + \frac{b}{k}\sin k(\pi-2d)\sin kd\,\}\,]\,\cos k(x-\pi+d) \\ &+ a\,[\,-a\sin k(\pi-2d)\cos kd - a^{-1}\cos k(\pi-2d)\sin kd \\ &+ \frac{b}{k}\cos k(\pi-2d)\cos kd + \frac{h}{k}\,\{\,a^{-1}\cos k(\pi-2d)\cos kd \\ &- a\sin k(\pi-2d)\sin kd + \frac{b}{k}\cos k(\pi-2d)\sin kd\,\}\,]\,\sin k(x-\pi+d) \\ &+ \frac{b}{k}\,[\,a\cos k(\pi-2d)\cos kd - a^{-1}\sin k(\pi-2d)\sin kd \\ &+ \frac{b}{k}\sin k(\pi-2d)\cos kd + \frac{h}{k}\{\,a^{-1}\sin k(\pi-2d)\cos kd \\ &+ a\cos k(\pi-2d)\sin kd + \frac{b}{k}\sin k(\pi-2d)\sin kd\,\}\,]\,\sin k(x-\pi+d) \end{aligned} \tag{1.10}$$

and

$$G_{11}(x,t) = G_{22}(x,t) = G_{33}(x,t) = \frac{\sin k(x-t)}{k} \tag{1.11}$$

$$
\begin{aligned}
G_{21}(x,t) \;=\; & \frac{1}{k}\{\; a \sin k(d-t) \cos k(x-d) \\
& + a^{-1} \cos k(d-t) \sin k(x-d) \\
& + \frac{b}{k} \sin k(d-t) \sin k(x-d) \;\}
\end{aligned} \tag{1.12}
$$

$$
\begin{aligned}
G_{31}(x,t) \;=\; & \frac{1}{k}\{\; a^{-1}[\; a \cos k(\pi-2d) \sin k(d-t) \\
& + a^{-1} \sin k(\pi-2d) \cos k(d-t) \\
& + \frac{b}{k} \sin k(\pi-2d) \sin k(d-t) \;]\; \cos k(x-\pi+d) \\
& + a\,[\; -a \sin k(\pi-2d) \sin k(d-t) \\
& + a^{-1} \cos k(\pi-2d) \cos k(d-t) \\
& + \frac{b}{k} \cos k(\pi-2d) \sin k(d-t) \;]\; \sin k(x-\pi+d) \\
& + \frac{b}{k}\,[\; a \cos k(\pi-2d) \sin k(d-t) \\
& + a^{-1} \sin k(\pi-2d) \cos k(d-t) \\
& + \frac{b}{k} \sin k(\pi-2d) \sin k(d-t) \;]\; \sin k(x-\pi+d) \;\}
\end{aligned} \tag{1.13}
$$

$$
\begin{aligned}
G_{32}(x,t) \;=\; & \frac{1}{k}\{\; a^{-1} \sin k(\pi-d-t) \cos k(x-\pi+d) \\
& + a \cos k(\pi-d-t) \sin k(x-\pi+d) \\
& + \frac{b}{k} \sin k(\pi-d-t) \sin k(x-\pi+d) \;\}
\end{aligned} \tag{1.14}
$$

These formulae were derived by Hald [39] and Willis [88], [90]. We may write u(x) more concisely as

$$u(x) = g(x) + \int_0^x G(x,t)q(t)u(t)\,dt \tag{1.15}$$

on the three disjoint intervals $[0,d)$, $(d,\pi-d)$ and $(\pi-d,\pi]$. As in earlier uniqueness proofs of Hald [39] and Willis [88], [90], the leading order terms of $g(x)$ play an important part in the proof. We denote this term by $\varphi(x)$ and let

$$\varphi_1(x) = \cos kx \tag{1.16}$$

$$(0 \le x < d)$$

$$\varphi_2(x) = \frac{A}{2}[\cos kx + \alpha \cos k(x-2d)] \tag{1.17}$$

$$(d < x < \pi - d)$$

$$\varphi_3(x) = \frac{A^2}{4}[\cos kx + \alpha \cos k(x - 2d) - \alpha \cos k(x - 2\pi + 2d) - \alpha^2 \cos k(x - 2\pi + 4d)] \tag{1.18}$$

$$(\pi - d < x \leq \pi)$$

where $A = (a + a^{-1})$ and $\alpha = (a - a^{-1})/(a + a^{-1})$, so that $|\alpha| < 1$. $\varphi = \varphi_1, \varphi_2, \varphi_3$ on $[0, d)$, $(d, \pi - d)$ and $(\pi - d, \pi]$ respectively. We note that $g(x)$ satisfies (1.1), (1.3) - (1.4) when $q \equiv 0$.

To determine specific bounds for u_1, u_2 and u_3 we use the following lemmas of Hald [39] and Willis [88], [90].

Lemma 1.1: (Hald 1984) *Consider the integral equation:*

$$u(x) - \int_a^x K(x,t)q(t)u(t)\ dt = f(x) \tag{1.19}$$

where f and K are continuous and q is integrable. This equation has a unique solution u which is continuous and satisfies:

$$|u(x)| \leq M(x)e^{L(x)\rho(x)} \tag{1.20}$$

where

$$M(x) = \max_{a \leq t \leq x} |f(t)|\ , \qquad L(x) = \max_{a \leq t \leq x} |K(x,t)| \quad \textit{and} \quad \rho(x) = \int_a^x |q(t)|\ dt\ .$$

Remarks: Lemma 1.1 shows how specific bounds for the eigenfunctions, their derivatives and leading order terms can be obtained from a Volterra Integral equation. We must find bounds for the maximum norms of the inhomogeneous term $f(t)$ and the kernel $K(x,t)$ and the L_1 norm of the potential. For a proof of Lemma 1.1 see Hald [39] .

Lemma 1.2: (Hald 1984, Willis 1985) *Let u_1, u_2, and u_3 be the solutions of equations (1.5), (1.6) and (1.7) respectively. And let $\sqrt{\lambda} = \sigma + i\tau$, $c = \max(|b|, |h|, \int_0^\pi |q(t)|dt)$ and $A = a + a^{-1}$. Then u_1, u_2 and u_3 are entire functions of λ of order $\frac{1}{2}$ and*

$$|u_1(x,\lambda)| \leq (1 + c\pi)e^{cx+|\tau|x} \qquad (0 \leq x < d) \tag{1.21}$$

$$|u_2(x,\lambda)| \leq A(1 + c\pi)^3 e^{cx+|\tau|x} \qquad (d < x < \pi - d) \tag{1.22}$$

$$|u_3(x,\lambda)| \le A^2(1+c\pi)^5 e^{cx+|\tau|x} \qquad (\pi - d < x \le \pi) \tag{1.23}$$

Remarks: The first two bounds (1.21) and (1.22) have been established by Hald [39] using Lemma 1.1, the trigonometric inequalities

$$|\cos kx|, |\sin kx|, \left|\frac{\sin kx}{kx}\right| \le e^{|\tau|x} \tag{1.24}$$

and the definition of c. The third bound (1.23) was established by Willis [90] in the inverse Sturm-Liouville problem on $[0,\pi]$ with two discontinuities $0 < d_1 < d_2 < \pi/2$ and jump constants a_1, b_1, a_2 and b_2. Willis follows the techniques of Hald. By exploiting the symmetry of our problem we can derive simpler expressions than those of Willis. In the symmetric problem we replace the second discontinuity d_2 by $(\pi - d)$ and note that the jump constants in the eigenfunction are related; $a_1 = a_2^{-1}$ so that $A_1 = a_1 + a_1^{-1}$ equals $A_2 = a_2 + a_2^{-1}$. We set $A = a_1 + a_1^{-1} = a_2 + a_2^{-1}$.

Lemma 1.3: (Hald 1984, Willis 1985) *Let u_1, u_2 and u_3 be the solutions of the integral equations (1.5), (1.6) - (1.7). Let $k = \sqrt{\lambda} = \sigma + i\tau$, $A = a + a^{-1}$ and $c = \max(|b|, |h|, \int_0^\pi |q| dt)$. If $|k| \ge 3c$, then*

$$|u_1(x)| \le 2e^{|\tau|x} \tag{1.25}$$

$$|u_1(x) - \varphi_1(x)| \le 3\frac{c}{|k|}e^{|\tau|x} \tag{1.26}$$

$$(0 \le x < d)$$

$$|u_2(x)| \le 3Ae^{|\tau|x} \tag{1.27}$$

$$|u_2(x) - \varphi_2(x)| \le 5\frac{c}{|k|}Ae^{|\tau|x} \tag{1.28}$$

$$|u_2'(x) - \varphi_2'(x)| \le 5cAe^{|\tau|x} \tag{1.29}$$

$$(d < x < \pi - d)$$

$$|u_3(x)| \le 11A^2e^{|\tau|x} \tag{1.30}$$

$$|u_3(x) - \varphi_3(x)| \le 28\frac{c}{|k|}A^2e^{|\tau|x} \tag{1.31}$$

$$|u_3'(x) - \varphi_3'(x)| \leq 24cA^2e^{|\tau|x} \tag{1.32}$$

$$(\pi - d < x \leq \pi)$$

Remarks: Lemma 1.3 is proved by using the Volterra integral equations for the eigenfunctions, the estimates from Lemma 1.2 and the results from Lemma 1.1. This approach has been used by Liouville [56] , Hobson [40], Borg [18], Hald [39] and Willis [88], [90]. We can obtain slightly better bounds than Willis for $|u_3|$, $|u_3 - \varphi_3|$ and $|u_3' - \varphi_3'|$ by using the symmetry in our problem. In deriving the bounds (1.24) - (1.31) Willis uses the estimates

$$\int_0^d \mid q(t) \mid \leq \int_0^\pi \mid q(t) \mid \leq c \tag{1.33}$$

and

$$\int_{\pi-d}^\pi \mid q(t) \mid \leq \int_0^\pi \mid q(t) \mid \leq c\ . \tag{1.34}$$

In the symmetric problem $0 < d < \pi/2 < \pi - d < \pi$ so that

$$\int_0^{d_1} \mid q(t) \mid = \int_{d_2}^\pi \mid q(t) \mid \leq \frac{1}{2} \int_0^\pi \mid q(t) \mid \leq \frac{1}{2}\, c\ \ . \tag{1.35}$$

This factor of $\frac{1}{2}$ gives us the improved upper bounds

$$|u_3(x)| \leq 5A^2e^{|\tau|x} \tag{1.36}$$

$$|u_3(x) - \varphi_3(x)| \leq 10\frac{|c|}{|k|}\ A^2e^{|\tau|x} \tag{1.37}$$

$$|u_3'(x) - \varphi_3'(x)| \leq 9cA^2e^{|\tau|x}\ \ . \tag{1.38}$$

The next lemma determines a lower bound for the Wronskian ω. We will prove the existence of the eigenvalues using the Cauchy integral technique. Appropriate contours will be defined for this purpose. Let $\sqrt{\lambda} = k$ and let R_n be the rectangle in the k-plane with vertices at $\pm(n - \frac{1}{2}) + i0$ and $\pm(n - \frac{1}{2}) + i(n - \frac{1}{2})$. And let Γ_n be the contour in the λ-plane that corresponding to the points of R_n for which $Im\ k > 0$.

Lemma 1.4: (Willis 1985) *Let u be the solution of (1.1), (1.3) - (1.4) satisfying $u(0) = 1$ and $u'(0) = h$, and let $\omega(\lambda) = -u'(\pi) - h\ u(\pi)$. Then ω is an entire function of λ of order*

$\frac{1}{2}$. *Its roots* $\lambda_0 < \lambda_1 < \ldots$ *are real and simple. Let* $\bar{\alpha} = 2\,|\alpha| + |\alpha^2|$ *and* $k = \sqrt{\lambda} = \sigma + i\tau$. *If* $\bar{\alpha} < 1$ *and* $n > \max\{(1+\bar{\alpha})/(1-\bar{\alpha}), (840c)/(1-\bar{\alpha})\}$ *where* $c = \max(|b|, |h|, \int_0^\pi |q| dt)$, *then*

$$|\omega(\lambda)| \geq \frac{A^2|k|}{24} e^{|\tau|\pi}(1-\bar{\alpha}) \tag{1.39}$$

for all points λ *on the countour* Γ_n *and* $|\sqrt{\lambda_n} - n| < \frac{1}{2}$

Remarks: For a proof of Lemma 1.4 see Willis [88]. Willis' approach follows a presentation in Titchmarsch [83], page 13 and Hald [39]. By using the symmetry in our problem and the bounds (1.36), (1.37) and (1.38) we can obtain the slightly improved bound

$$|\omega(\lambda)| \geq \frac{A^2|k|}{13} e^{|\tau|\pi}(1-\bar{\alpha}) \ . \tag{1.40}$$

1.2 Integral Representation of the Eigenfunctions

In this section we examine the eigenfunctions of a Sturm-Liouville problem with two symmetrically placed discontinuities. In the lemma below the eigenfunctions are rewritten as the sum of the first order terms φ and an integral of the product of a kernel and a cosine function.

Lemma 1.5: (Hald 1984, Willis 1985) *Let* u *be the solution (1.1) - (1.4) where* $u(0) = 1$ *and* $u'(0) = h$. *Let* $\lambda = k^2$ *and* φ *be the first order term defined by equations (1.16) - (1.18). Then* u *may be alternatively expressed as*

$$u(x,k) = \varphi(x,k^2) + \int_0^x K(x,t)\cos kt \ dt \ . \tag{1.41}$$

Where $K(x,t)$ *is a bounded function such that* $K(x,t) = 0$ *if* $t < 0$ *or* $t > x$.

Remarks: Povzner and Levitan were the first to realize the significance of rewriting u in the form (1.34). Hald [39] used the same form for the discontinuous problem. His proof for one discontinuity was extended to the two discontinuities by Willis [88], [90]. The restriction $0 \leq d_1, d_2 \leq \frac{\pi}{2}$ is not used in the proof by Willis [88], [90] so that her arguments also cover our formulation.

1.3 Uniqueness of Position and Sizes of Discontinuities

In this section we show that if two different eigenvalue problems of the form (1.1) - (1.4) have the same eigenvalues, then the discontinuities in the problems and the jump constants for the eigenfunctions are identical. No relationship between the jump constants for the derivatives of the eigenfunctions is given.

Lemma 1.6: *The jump constant a and the discontinuity d in the eigenvalue problem (1.1) - (1.4) are uniquely determined by the eigenvalues provided* $|a-1|+|b|>0$, $\bar{\alpha} = 2|\alpha|+|\alpha^2|<1$ *and* $0<d<\frac{\pi}{2}$.

Remarks: The condition $|a-1|+|b|>0$ guarantees that either u or u' is discontinuous at d and $\pi-d$. Without this restriction u and u' would be continuous and d could be anywhere.

Proof: Lemma 1.4 established that the Wronskian $\omega(\lambda)$ is an entire function of λ of order $\frac{1}{2}$ with simple roots. Let ω and $\tilde{\omega}$ be the Wronskians for two different eigenvalue problems with identical eigenvalues. We apply Hadamard's theorem [82] to find that $\omega = C\tilde{\omega}$ for all λ, where $C \neq 0$. We will show that ω and $\tilde{\omega}$ are equal, i.e. $C=1$. Let ω_0 be the leading term in the Wronskian ω. By using the definition of φ_3 and the equations (1.31) and (1.32) we see that

$$\omega_0(\lambda) = \frac{1}{4}kA^2[\sin k\pi + 2\alpha\sin k(\pi-2d) + \alpha^2\sin k(\pi-4d)] \;.. \tag{1.42}$$

We rewrite $\omega_0 - C\tilde{\omega}_0$ as $C(\tilde{\omega}-\tilde{\omega}_0)-(\omega-\omega_0)$ to find

$$\begin{aligned} \omega_0 - C\tilde{\omega}_0 &= C(\tilde{\omega}-\tilde{\omega}_0)-(\omega-\omega_0) \\ &= \frac{1}{4}k\ [\ (A^2 - C\tilde{A}^2)\sin k\pi + 2\alpha A^2\sin k(\pi-2d) \\ &\qquad - 2\tilde{\alpha}C\tilde{A}^2\sin k(\pi-2\tilde{d}) + \alpha^2A^2\sin k(\pi-4d) \\ &\qquad - \tilde{\alpha}^2C\tilde{A}^2\sin k(\pi-4\tilde{d})\ ]\ . \end{aligned} \tag{1.43}$$

Let $\bar{c} = \max(c,\tilde{c})$, where $c = \max(|b|,|h|,\int_0^\pi|q(t)|dt)$ and $\tilde{c} = \max(|\tilde{b}|,|\tilde{h}|,\int_0^\pi|\tilde{q}(t)|dt)$. We multiply equation (1.43) by $T^{-2}\sin k\pi$ and integrate with respect to k from $3\bar{c}$ to T. Note that $\omega-\omega_0 = -u_3'(\pi)-hu_3(\pi)+\varphi_3'(\pi)$, and use the bounds from Lemma 1.3 to derive the bound

$$|\ C(\tilde{\omega}-\tilde{\omega}_0)-(\omega-\omega_0)\ | \leq max(1,C)\ \cdot \frac{1861}{144}(\ cA^2+\tilde{c}\tilde{A}^2\ ) \tag{1.44}$$

for $k > 3\bar{c}$. Thus the integration yields

$$\frac{1}{4}(A^2 - C\tilde{A}^2)\ \left[\ \frac{1}{4} + O(T^{-1})\ \right] + O(T^{-1}) = O(T^{-1}) \tag{1.45}$$

The proof requires that $d \in (0, \frac{\pi}{2})$. We let T tend to infinity to find that $A^2 = C\tilde{A}^2$.

To show that $a = \tilde{a}$, two distinct cases must be considered. In the first case $d = \tilde{d}$, and in the second $d \neq \tilde{d}$. The proof in the first case is straightforward. The second proof is by contradiction.

CASE 1: There are two subcases to be considered when $d = \tilde{d}$; slightly different proofs are needed when $\pi = 3d$ and $\pi \neq 3d$.

First let $d = \tilde{d}$ and $(\pi - 2d) \neq (-\pi + 4d)$, i.e. $\pi \neq 3d$. We multiply equation (1.43) by $T^{-2}\sin k(\pi - 2d)$, integrate with respect to k from $3\bar{c}$ to T and consider the limit as T goes to infinity. Then $2\alpha A^2 = 2\tilde{\alpha}C\tilde{A}^2$. Since $A^2 = C\tilde{A}^2$, we find that $\alpha = \tilde{\alpha}$. And from the definition of α and $\tilde{\alpha}$, it follows that $a = \tilde{a}$.

Next let $d = \tilde{d}$ and $(\pi - 2d) = (-\pi + 4d)$, i.e. $\pi = 3d$. We multiply equation (1.43) by $T^{-2}\sin k(\pi - 2d)$, integrate with respect to k from $3\bar{c}$ to T and consider the limit as T goes to infinity. Then $(2\alpha - \alpha^2)A^2 = (2\tilde{\alpha} - \tilde{\alpha}^2)C\tilde{A}^2$ or $2\alpha - \alpha^2 = 2\tilde{\alpha} - \tilde{\alpha}^2$. We use the quadratic formula to find that $\alpha = -\tilde{\alpha} + 2$ or $\alpha = \tilde{\alpha}$. Since we assume that $\bar{\alpha} = 2|\alpha| + |\alpha^2| < 1$ and $\hat{\alpha} = 2|\tilde{\alpha}| + |\tilde{\alpha}^2| < 1$, the first equation cannot be true. Hence α equals $\tilde{\alpha}$ and $a = \tilde{a}$.

CASE 2: Consider the second case $d \neq \tilde{d}$. We show that if the assumption $d \neq \tilde{d}$ leads to an eigenvalue problem with no discontinuties. 8 subcases must be considered. The technique of multiplying by a sine function times T^{-2}, integrating with respect to k and letting T tend to infinity is used repeatedly to obtain equations for a and $\tilde{a}$ and relationships between a and $\tilde{a}$. In addition to the above solving of a quadratic equation may be required. One of the roots of the quadratic can be eliminated by the assumption $\bar{\alpha} < 1$ or $\hat{\alpha} < 1$. We rewrite equation (1.43) as

$$\begin{aligned} \omega_0 - C\tilde{\omega}_0 &= C(\tilde{\omega} - \tilde{\omega}_0) - (\omega - \omega_0) \\ &= \frac{1}{4}kA^2[\ 2\alpha\sin k(\pi - 2d)\ - 2\tilde{\alpha}\sin k(\pi - 2\tilde{d}) \\ &\qquad -\alpha^2\sin k(-\pi + 4d) + \tilde{\alpha}^2\sin k(-\pi + 4\tilde{d})\ ] \end{aligned} \tag{1.46}$$

and consider the following subcases

SUBCASE 1 $\quad d \neq \tilde{d} \quad d \neq \frac{\pi}{3} \quad \pi - 2d \neq -\pi + 4\tilde{d} \quad d \neq 2\tilde{d}$

SUBCASE 2	$d \neq \tilde{d}$	$\tilde{d} \neq \frac{\pi}{3}$	$\pi - 2\tilde{d} \neq -\pi + 4d$	$\tilde{d} \neq 2d$
SUBCASE 3	$d = \frac{\pi}{5}$	$\tilde{d} = \frac{2\pi}{5}$		
SUBCASE 4	$d = \frac{2\pi}{5}$	$\tilde{d} = \frac{\pi}{5}$		
SUBCASE 5	$d \neq \tilde{d}$	$d = \frac{\pi}{3}$	$\tilde{d} \neq \frac{\pi}{6}$	
SUBCASE 6	$d \neq \tilde{d}$	$\tilde{d} = \frac{\pi}{3}$	$d \neq \frac{\pi}{6}$	
SUBCASE 7	$d \neq \tilde{d}$	$d = \frac{\pi}{3}$	$\tilde{d} = \frac{\pi}{6}$	
SUBCASE 8	$d \neq \tilde{d}$	$\tilde{d} = \frac{\pi}{3}$	$d = \frac{\pi}{6}$	

Multiply equation (1.46) by T^{-2} times a sine function, integrate and let T tend to infinity. The chart below summarizes out results. We emphasize that the order in which the multiplications take place is crucial. However, the order given in the chart below may not necessarily be the unique sequence which leads to the desired answer.

SUBCASE	MULTIPLY (1.63) BY	CONCLUDE
1	$T^{-2}\sin k(\pi - 2d)$	$\alpha = 0$
	$T^{-2}\sin k(\pi - 2\tilde{d})$	$\tilde{\alpha} = 0$
2	$T^{-2}\sin k(\pi - 2\tilde{d})$	$\tilde{\alpha} = 0$
	$T^{-2}\sin k(\pi - 2d)$	$\alpha = 0$
3	$T^{-2}\sin k(\pi - 2d)$	$2\alpha + \tilde{\alpha}^2 = 0$
	$T^{-2}\sin k(\pi - 2\tilde{d})$	$2\tilde{\alpha} - \alpha^2 = 0$
4	$T^{-2}\sin k(\pi - 2\tilde{d})$	$2\tilde{\alpha} + \alpha^2 = 0$
	$T^{-2}\sin k(\pi - 2d)$	$2\alpha - \tilde{\alpha}^2 = 0$
5	$T^{-2}\sin k(\pi - 2d)$	$2\alpha - \alpha^2 = 0$
	$T^{-2}\sin k(\pi - 2\tilde{d})$	$\tilde{\alpha} = 0$
6	$T^{-2}\sin k(\pi - 2\tilde{d})$	$2\tilde{\alpha} - \tilde{\alpha}^2 = 0$
	$T^{-2}\sin k(\pi - 2d)$	$\alpha = 0$
7	$T^{-2}\sin k(\pi - 2\tilde{d})$	$\tilde{\alpha} = 0$
	$T^{-2}\sin k(\pi - 2d)$	$2\alpha - \alpha^2 = 0$
8	$T^{-2}\sin k(\pi - 2d)$	$\alpha = 0$
	$T^{-2}\sin k(\pi - 2\tilde{d})$	$2\tilde{\alpha} - \tilde{\alpha}^2 = 0$

We see that in all of the above cases $\alpha = \tilde{\alpha} = 0$ and $a = \tilde{a}$ so that $A^2 = C\tilde{A}^2$ implies $C = 1$, and we conclude that $\omega_0 = \tilde{\omega}_0$.

Next we further study the Wronskian

$$\omega(k^2) = -(u_3'(\pi) + h\ u_3(\pi)) \ . \tag{1.47}$$

Willis [88], [90] uses the integral equations for u_3 and trigonometric identities to find

$$\begin{aligned}
\omega(k^2) \quad = \quad & \frac{kA^2}{4}\{ \sin k\pi + 2\alpha \sin k(\pi - 2d) + \alpha^2 \sin k(\pi - 4d) \} \\
& + \frac{1}{4} \{ A^2[-2h + \frac{1}{2}\int_0^\pi q(s)\ ds\ ] \ - 2Ab \} \ \cos k\pi \\
& + \frac{1}{4} \{ -2Ab - A^2\alpha \int_0^d q(s)\ ds \} \ \cos k(\pi - 2d) \\
& + \frac{1}{4}\{ -A^2\alpha^2\ [-2h - \frac{1}{2}\int_0^{\pi-d} q(s)\ ds \\
& \qquad + \frac{1}{2}\int_{\pi-d}^{\pi} q(s)\ ds\ ] \ - \ 2A\alpha b \} \ \cos k(-\pi + 4d) \\
& + \int_0^\pi V_1(t)\cos kt\ dt \ + \ E
\end{aligned} \tag{1.48}$$

The term $\int_0^\pi V_1(t)\cos kt\ dt$ consists of a sum of integrals of the form $\int_{t_1}^{t_2} \cos kt\ q(s(t))\ dt$, where $-\pi \le t_1 \le t_2 \le \pi$, and E consists of all terms of ω which are $O(k^{-1})$. We have simplified the formula given in [88], [90] by using the symmetry properties of the jump and boundary constants and the potential function. In particular

$$\int_0^d q\ ds = \int_{\pi-d}^{\pi} q\ ds \ . \tag{1.49}$$

Let $k = \sigma + i\tau$. We use the inequalities (1.24) and the bounds from Lemmas 1.2 and 1.3 to find that $|E(k^2)| \le |k|^{-1}Ce^{|\tau|\pi}$. If k is real, then E is real, and furthermore E is even in k. Using the Paley-Wiener Theorem we rewrite E as $E(k^2) = \int_0^\pi V_2(t)\cos kt\ dt$, where V_2 is a square integrable function. By combining the arguments above, the Wronskian can be expressed as

$$\begin{aligned}
\omega(k^2) \quad = \quad & \omega_0 + C_0\cos k\pi + C_1 \cos k(\pi - 2d) \\
& + C_2\cos k(-\pi + 4d) + \int_0^\pi V(t)\cos kt\ dt \quad .
\end{aligned} \tag{1.50}$$

And since $\omega_0 = \tilde{\omega}_0$ and $C = 1$,

$$\begin{aligned}
\omega - \tilde{\omega} \quad = \quad & (C_0 - \tilde{C}_0)\cos k\pi + C_1\cos k(\pi - 2d) - \tilde{C}_1 \cos k(\pi - 2\tilde{d}) \\
& + C_2\cos k(-\pi + 4d) - \tilde{C}_2\cos k(-\pi + 4\tilde{d}) + \int_0^\pi (V - \tilde{V})\cos kt\ dt \ .
\end{aligned} \tag{1.51}$$

To complete the proof we will multiply equation (1.51) by a term of the form $T^{-1}\cos k\beta$ and integrate by parts with respect to k from $3\bar{c}$ to T to show that $b = 0$. An analogous argument is used to show that $\tilde{b} = 0$. Four cases must be considered for each of the proofs. We present case 1 in detail to show the reader the method.

CASE 1: $d \neq \frac{\pi}{3}$, $d \neq 2\tilde{d}$ Multiply the equation above by $T^{-1}\cos k(\pi - 2d)$, integrate with respect to k from $3\bar{c}$ to T and arrive at

$$C_1 \left[\frac{1}{2} + O(T^{-1}) \right] + T^{-1} \int_0^\pi (V - \tilde{V}) \int_{3\bar{c}}^T \cos kt \cos k\beta \, dk \, dt = 0 \ , \qquad (1.52)$$

where we have used Fubini's Theorem to interchange the order of integration. We let T tend to infinity and find $C_1 = 0$. Since $\alpha = 0$ we have $C_1 = -\frac{1}{2}Ab = 0$. Therefore $b = 0$.

TO SHOW $b = 0$

case	assume	multiply by	conclude
1	$d \neq \frac{\pi}{3}$, $d \neq 2\tilde{d}$	$T^{-1}\cos k(\pi - d)$	$C_1 = 0$
2	$d \neq \frac{\pi}{3}$, $d = 2\tilde{d}$	$T^{-1}\cos k(\pi - d)$	$C_1 - \tilde{C}_2 = 0$
3	$d = \frac{\pi}{3}$, $\tilde{d} \neq \frac{\pi}{6}$	$T^{-1}\cos k(\pi - d)$	$C_1 + C_2 = 0$
4	$d = \frac{\pi}{3}$, $\tilde{d} = \frac{\pi}{6}$	$T^{-1}\cos k(\pi - d)$	$C_1 + C_2 - \tilde{C}_2 = 0$

TO SHOW $\tilde{b} = 0$

case	assume	multiply by	conclude
1	$\tilde{d} \neq \frac{\pi}{3}$, $\tilde{d} \neq 2d$	$T^{-1}\cos k(\pi - \tilde{d})$	$\tilde{C}_1 = 0$
2	$\tilde{d} \neq \frac{\pi}{3}$, $\tilde{d} = 2d$	$T^{-1}\cos k(\pi - \tilde{d})$	$-C_2 + \tilde{C}_1 = 0$
3	$\tilde{d} = \frac{\pi}{3}$, $d \neq \frac{\pi}{6}$	$T^{-1}\cos k(\pi - \tilde{d})$	$\tilde{C}_1 + \tilde{C}_2 = 0$
4	$\tilde{d} = \frac{\pi}{3}$, $d = \frac{\pi}{6}$	$T^{-1}\cos k(\pi - \tilde{d})$	$-C_2 + \tilde{C}_1 + \tilde{C}_2 = 0$

$\alpha = \tilde{\alpha} = 0$ implies $C_2 = \tilde{C}_2 = 0$ so that all the above cases reduce to $C_1 = -\frac{1}{2}Ab = 0$ and $\tilde{C}_1 = -\frac{1}{2}A\tilde{b} = 0$. Consequently $a = \tilde{a} = 1$ and $b = \tilde{b} = 0$ which contradicts the hypothesis of our lemma. Hence $d = \tilde{d}$.

In the following corollary we show that the conclusion in Lemma 1.6 is valid even if a finite number of eigenvalues are not known.

Corollary 1.6.1: *The constants a and d in the eigenvalue problem (1.1) - (1.4) are uniquely determined by the eigenvalues* $\{\lambda_j\}$; $j > n$, *provided* $|a - 1| + |b| > 0$, $\bar{\alpha} = 2|\alpha| + |\alpha^2| < 1$ *and* $0 < d < \frac{\pi}{2}$.

Remarks: Hald [39] proved the corresponding result for the inverse eigenvalue problem with one discontinuity. His work can also be extended to the inverse eigenvalue problem considered by Willis [88], [90] with two discontinuities and a potential that is known over half of the interval. We discuss the discontinuous problem with symmetric potentials. Hald's proof begins by considering the expression $\omega - C\tilde{\omega}$.

$$\omega - C\tilde{\omega} = C \left(\prod_{j=0}^{n} [\, 1 + \frac{\tilde{\lambda}_j - \lambda_j}{\lambda - \tilde{\lambda}_j} \,] - 1 \right) \tilde{\omega} \tag{1.53}$$

Techniques such as bounding the Wronskian, rewriting products as exponentials, noting that $|\log(1+x)| \leq 2|x|$ for all $|x| \leq \frac{1}{2}$, multiplying by appropriate sine and cosine functions, integrating and taking limits show that when only a finite number of eigenvalues differ, the RHS of equation (1.53) is zero, and the rest of the proof follows from the proof of Lemma 1.6. The proof for the discontinuous problem differs only in the specific bounds derived for the Wronskian.

Corollary 1.6.1 will be useful for constructing a modified Sturm-Liouville expansion. (See Corollary 2.1.) This expansion will be used to establish an algorithm to reconstruct the potential function in an inverse Sturm-Liouville problem with symmetric potentials and symmetric discontinuities.

1.4 Completion of Proof

In the next lemma we consider two eigenvalue problems whose eigenvalues are equal. We derive integral equations for the difference between the two potentials. By using these equations we complete the proof of our main theorem and show that the two potentials must be equal.

Lemma 1.7: *Let $u = u(x,\lambda)$ be the solution of equation (1.1) that satisfies the condition $u = 1$, $u' = h$ at $x = 0$ and the jump conditions (1.3) and (1.4). Let $\tilde{u}$ be defined similarly with a, b, d, h and q replaced by $\tilde{a}$, $\tilde{b}$, $\tilde{d}$, $\tilde{h}$ and $\tilde{q}$. Let $\bar{\alpha} = 2|\alpha|+|\alpha^2| < 1$ and $\hat{\alpha} = 2|\tilde{\alpha}| + |\tilde{\alpha}^2| < 1$. Set $u_{d-} = u(d-)$, $\tilde{u}_{d-} = \tilde{u}(\tilde{d}-)$, $u_{(\pi-d)-} = u((\pi-d)-)$ and $\tilde{u}_{(\pi-d)-} = \tilde{u}((\pi-\tilde{d})-)$. If $\lambda_j = \tilde{\lambda}_j$ for $j \geq 0$, then $a = \tilde{a}$, $d = \tilde{d}$ and*

$$b - \tilde{b} \;=\; -\frac{a^2 - a^{-2}}{2a} \int_d^{\pi} (q - \tilde{q})(t)\, dt \tag{1.54}$$

$$h - \tilde{h} \;=\; -\frac{1}{2}\int_0^d (q-\tilde{q})(t)\,dt \;-\; \frac{1}{2a^2}\int_d^{\frac{\pi}{2}} (q-\tilde{q})(t)\,dt \tag{1.55}$$

$$0 \;=\; \int_0^d (q-\tilde{q})(t)\,[\,u\tilde{u}(t) - \frac{1}{2}\,]\,dt$$

$$+\int_d^{\frac{\pi}{2}} (q-\tilde{q})(t)\,[\,u\tilde{u}(t) - \frac{1}{2a^2} - \frac{1}{2}\,(a^2 - a^{-2})\,u_{d-}\tilde{u}_{d-}\,]\,dt \tag{1.56}$$

Remarks: If $q = \tilde{q}$ almost everywhere on $[0, \frac{\pi}{2}]$, then it follows from equations (1.54) and (1.55) that $b = \tilde{b}$ and $h = \tilde{h}$. The equations above are consistent with those for the continuous problem. If we set $a = 1$ and $b = 0$ our equations reduce to those of Hochstadt and Liebermann [43] and Hald [37]. Our equations differ slightly from those of Willis since she assumes that in the 2 discontinuity case $0 < d_1 < d_2 < \frac{\pi}{2}$ and $q = \tilde{q}$ a.e. on $(\frac{\pi}{2}, \pi)$. We assume that the discontinuities d_1, d_2 satisfy $0 < d_1 < \frac{\pi}{2}$ and $d_2 = \pi - d_1$. Symmetry assumptions simplify many of the expressions to reduce our problem to the single discontinuity case considered by Hald [39]. However we do not assume $q = \tilde{q}$ a.e. on $(\frac{\pi}{2}, \pi)$ as is done in Hald [39].

Proof: Our proof uses the techniques and arguments given in Hald [39] and Willis [88], [90]. Integral equations (1.54) - (1.56) will be derived by following the argument given in Willis [88], Lemma 7. Since the eigenvalues of the two Sturm-Liouville problems are identical $a = \tilde{a}$ and $d = \tilde{d}$. (See Lemma 1.6). And furthermore

$$(u\tilde{u}' - \tilde{u}u')' + (q - \tilde{q})u\tilde{u} = 0$$

at the eigenvalues. Denote d by d_1 and $\pi - d$ by d_2. We integrate the equation above by parts.

$$2\,(h-\tilde{h}) + a(b-\tilde{b})u(d_1-)\tilde{u}(d_1-) + a^{-1}(b-\tilde{b})u_{d_2-}\tilde{u}_{d_2-} + \int_0^{\pi} (q-\tilde{q})\;u\tilde{u}dt = 0 \tag{1.57}$$

From the symmetry of our problem we have that

$$a^{-1}\;u(d_2-) \;=\; u(d_2+) \;=\; u(d_1-)$$
$$a^{-1}\;\tilde{u}(d_2-) \;=\; \tilde{u}(d_2+) \;=\; \tilde{u}(d_1-)$$

and

$$\int_0^{\frac{\pi}{2}} (q-\tilde{q})(t)\ u\tilde{u}(t)\ dt \ = \ \int_{\frac{\pi}{2}}^{\pi} (q-\tilde{q})(t)\ u\tilde{u}(t)\ dt$$

so that equation (1.57) reduces to

$$(h-\tilde{h}) + a(b-\tilde{b})u(d_1-)\tilde{u}(d_1-) + \int_0^{\frac{\pi}{2}} (q-\tilde{q})(t)u\tilde{u}(t)dt = 0 \tag{1.58}$$

at the eigenvalues. We note that the equation above is identical to that found by Hald [39]. Define the function Φ to be the LHS of the resulting expression. Note that $\Phi(\lambda) = 0$ at any of the eigenvalues of the Sturm-Liouville equations. We will show that $\Phi \equiv 0$. u and $\tilde{u}$ are entire functions with respect to λ so that Φ is entire. Now consider the function $\Psi = \Phi/\omega$ where ω is the Wronskian of the two eigenvalue problems. (Recall from Lemma 1.6 that the eigenvalues determines the Wronskian.) Let Γ_n be the contour described in Lemma 1.4. Then

$$|\omega| > \frac{A^2(1-\bar{\alpha})}{24} \cdot \sqrt{|\lambda|} e^{|\tau|\pi}$$

for n sufficiently large. Let $\bar{c} = \max(c, \tilde{c})$, where $c = \max(|b|, |h|, \int_0^\pi |q(t)|dt)$ and $\tilde{c} = \max(|\tilde{b}|, |\tilde{h}|, \int_0^\pi |\tilde{q}(t)|dt)$. Use the estimates from Lemma 3 and Hald [39] to find that Φ is bounded by

$$|\Phi| \leq 18\bar{c}Ae^{|\tau|\pi}$$

where $A = (a + a^{-1})$. Since Φ and ω are entire functions with respect to λ and the zeros of ω are simple and are also the zeros of Φ, Ψ is an entire function of λ and the estimates from Φ and ω give the bound

$$|\Psi| \leq \frac{432\ \bar{c}}{A\ (1-\bar{\alpha})\ \sqrt{|\lambda|}}$$

on the curve Γ_n for n sufficiently large. From the maximum principle follows that $\Psi$is bounded. As $n \to \infty$, $\Psi \to 0$ so that $\Psi \equiv 0$.

We follow the work of Hald [39] and express Φ as

$$\Phi(\lambda) = h - \tilde{h} \ + \ a\ (b-\tilde{b})\ \varphi_1^2(d) + \int_0^{\frac{\pi}{2}} (q-\tilde{q})(t)\ \varphi^2(t)\ dt \ + \ \tilde{E}$$

where $\varphi = \varphi_1$ or φ_2 is defined by equations (1.16) - (1.17) and $\tilde{E}$ is

$$\tilde{E} \ = \ a\ (b-\tilde{b})\ \{\ (u_{d-} - \varphi_1(d))\ \tilde{u}_{d-} \ + \ (\tilde{u}_{d-} - \varphi_1(d))\ u_{d-}\}$$

$$+ \int_0^{\frac{\pi}{2}} (q - \tilde{q})(t) \{ (u(t) - \varphi(t))\, \tilde{u}(t) + (\tilde{u}(t) - \varphi(t))\, u(t) \}\, dt \ . \tag{1.59}$$

Using the bounds from Lemma 3 we find that

$$\tilde{E} \leq \frac{620\ \bar{c}^2\ A^4}{|k|} \ .$$

Next note that Φ may be written

$$\Phi(\lambda) = A + B \cos 2kd + I + E = 0$$

where

$$A = h - \tilde{h} + \frac{a(b - \tilde{b})}{2} + \frac{1}{2}\int_0^d (q - \tilde{q})(t)\, dt + \frac{(a^2 + a^{-2})}{4} \int_d^{\frac{\pi}{2}} (q - \tilde{q})(t)\, dt$$

$$B = \frac{a(b - \tilde{b})}{2} + \frac{(a^2 + a^{-2})}{4} \int_d^{\frac{\pi}{2}} (q - \tilde{q})(t)\, dt$$

and the term I is of the form

$$I = \int_0^{\frac{\pi}{2}} V(t) \cos 2kt\, dt \ .$$

The bounds from Lemma 3 and Hald [39] show that E is bounded by

$$|E| \leq \frac{49\bar{c}^2 A^2}{k} \ .$$

To derive the equations for $b - \tilde{b}$ and $h - \tilde{h}$, Hald mulitplies by $T^{-1} \cos 2kd$, integrates with respect to k from $3\bar{c}$ to T, considers the limit as $t \to \infty$, and finds that $B = 0$. Next Hald shows that the integral I equals zero as $k \to \infty$ by using the Riemann Lebesgue Lemma, and $E \to 0$ as k increases since it is $O(\frac{1}{k})$. Finally we find that $A = 0$, and this concludes our proof of Lemma 1.7.

The next lemma considers the integral equations for the product of two eigenfunctions of two different eigenproblems. These equations will be substituted into equation (1.56) of Lemma 1.7 to complete our proof.

Lemma 1.8: (Hald 1984) *Let u, $\tilde{u}$, u_{d-} and $\tilde{u}_{d-}$ be defined as in Lemma 1.7 and assume that $d = \tilde{d}$ and $a = \tilde{a}$. Let $k = \sqrt{\lambda}$. Then there exists a bounded function $\bar{K}(x,t)$ such that for all k*

$$u\tilde{u} - \frac{1}{2} = \frac{1}{2}\cos 2kx + \frac{1}{2}\int_0^x \bar{K}(x,t)\cos 2kt\, dt \tag{1.60}$$

when $0 \le x < d$, and

$$\begin{aligned} u\tilde{u} - \frac{1}{2a^2} - \frac{1}{2}(a^2 - a^{-2})u_{d-}\tilde{u}_{d-} &= \frac{1}{2}\,[\, A\cos 2kx + B\cos 2k(x-d) + C\cos 2k(x-2d)\,] \\ &+ \frac{1}{2}\int_0^x \bar{K}(x,t)\cos 2kt\, dt \end{aligned} \tag{1.61}$$

when $d < x < \frac{\pi}{2}$. Here $4A = (a + a^{-1})^{-2}$, $2B = a^2 - a^{-2}$ and $4C = (a - a^{-1})^2$.

Remark: Equations (1.60) and (1.61) are derived in Hald [39].

To complete the proof of the main theorem, let $Q = q - \tilde{q}$. By combining equation (1.56) with Lemma 1.8 we find that

$$\begin{aligned} 0 &= \int_0^d Q(x)\,\{\,\cos 2kx + \int_0^x \bar{K}(x,t)\cos 2kt\, dt\,\}\, dx \\ &+ \int_d^{\frac{\pi}{2}} Q(x)\,\{\, A\cos 2kx + B\cos 2k(x-d) \\ &\qquad + C\cos 2k(x-2d) + \int_0^x \bar{K}(x,t)\cos 2kt\, dt\,\}\, dx \end{aligned} \tag{1.62}$$

for all k. For the integral equation given above, Hald [39] has shown that $Q \equiv 0$ almost everywhere on $[0, \frac{\pi}{2}]$. Since the potential is symmetric, $Q \equiv 0$ almost everywhere on the whole interval $[0, \pi]$. We return to the equations given in Lemma 1.7 to find that $b - \tilde{b} = h - \tilde{h} = 0$. And this completes the proof of our main result, Theorem 1.

Chapter 2

Eigenfunction Expansions

In this chapter we show that any absolutely continuous function can be expanded in terms of the eigenfunctions of a Sturm-Liouville problem with two discontinuities. Next the orthogonality property of the eigenfunctions is proved and a relationship between the derivative of the Wronskian and the L^2-norm of the eigenfunctions is established. In the first of two corollaries a variation of the eigenfunction expansion is presented. This result is used to construct an algorithm for the discontinuous inverse Sturm-Liouville problem with symmetric potentials and symmetric discontinuities. (See Chapters 3 and 4.) In the second corollary we give the Sturm-Liouville expansion in a form which may be of more use to the general reader. This expression is analogous to the standard Fourier Series expansion. Finally we derive a relationship between the Wronskians of two distinct eigenvalue problems when only a finite number of their eigenvalues differ.

2.1 Main Theorem

Theorem 2: *Consider the eigenvalue problem*

$$-u'' + q(x)\ u = \lambda\ u \tag{2.1}$$

on the interval $0 < x < \pi$ *satisfying the symmetric boundary conditions:*

$$h\ u(0) - u'(0) = h\ u(\pi) + u'(\pi) = 0 \tag{2.2}$$

with two symmetric discontinuities at $x = d_1 = d$ *and* $x = d_2 = \pi - d$, $0 < d_1 < \frac{\pi}{2}$ *satisfying the jump conditions:*

$$u(d_1+) = \ a\ u(d_1-)\ , \quad u'(d_1+) = a^{-1}\ u'(d_1-) + b\ u(d_1-) \tag{2.3}$$

$$u(d_2-) = \ a\ u(d_2+)\ , \quad u'(d_2-) = a^{-1}\ u'(d_2+) - b\ u(d_2+)\ . \tag{2.4}$$

Here q *is integrable on* $0 \leq x \leq \pi$, $a > 0$ *and* $|a-1| + |b| > 0$. *Let* $A = a + a^{-1}$ *and* $\alpha = (a - a^{-1})/(a + a^{-1})$. *Assume* $\bar{\alpha} = 2|\alpha| + |\alpha^2| < 1$. *Let* $\{\lambda_j\}$ *be the eigenvalues of (2.1) and let* u_j *and* v_j *be the solutions of equation (2.1) satisfying the boundary conditions*

$$u(0) = 1, \qquad u'(0) = h \tag{2.5}$$

and

$$v(\pi) = 1, \qquad v'(\pi) = -h\ . \tag{2.6}$$

Let f *be a sectionally* C^1 *function on* $[0,\pi]$ *with sections* $(0,d)$, $(d,\pi-d)$ *and* $(\pi-d,\pi)$. *Then*

$$f(x) \ = \ \sum_{j=0}^{\infty} \frac{v_j \int_0^x u_j f\ dy \ + \ u_j \int_x^\pi v_j f\ dy}{\omega'(\lambda_j)} \tag{2.7}$$

on $(0,d_1)$, (d_1,d_2) *and* (d_2,π).

Remarks: " The idea of expanding an arbitrary function in terms of the solutions of a second-order differential equation goes back to the time of Sturm and Liouville, more than a hundred years ago. The first satisfactory proofs were constructed by various authors early in the twentieth century. Later, a general theory of the 'singular' cases was given by Weyl [85], [87], [86] , who based it on the theory of integral equations. An alternative method, proceeding via the general theory of linear operators in Hilbert space, is to be found in the treatise by Stone [77] on this subject. " (from Titchmarsch [83]) Poincaré suggested yet another method, the Cauchy integral technique, which involves contour integration and the calculus of residues. However other scientists, including Knesner, Birkhoff and Tamarkin, are responsible for implementing this technique. The motivation for our work comes from Hochstadt and Hald [35], who studied the inverse Sturm-Liouville problem with continuous symmetric potentials.

Beginning of Proof: The outline and techniques we use follow from Titchmarsch [83]; our proof is merely a discontinuous variation of that given in Titchmarsch's treatise on eigenfunction expansions. We begin by examining the solutions to the Sturm-Liouville equation and their asymptotic expansions. Here u satisfies the Volterra integral equations (1.5) - (1.16) presented in Hald [39] and Willis [88], [90]. Note that Hald and Willis assume that $0 < d_1 < d_2 < \frac{\pi}{2}$. We assume instead that $0 < d < \pi/2$. The equation for v is found by replacing x by $\pi - x$ and replacing by suitable and jump and boundary conditions. We set $\lambda = k^2$ where $k = \sigma + i\tau$. From the Volterra integral equations (1.5) - (1.16) follows that for each x, $u(x,\lambda)$ is an entire function of λ of order $\frac{1}{2}$. In addition, we obtain the asymptotic expansions for u and v from these equations.

$$u(x,\lambda) = \cos kx + O\{|k|^{-1}e^{|\tau|x}\}$$

$$\begin{aligned} v(x,\lambda) = \frac{1}{4}A^2 [\cos k(\pi - x) - \alpha \cos k(\pi + x - 2d) + \alpha \cos k(\pi - x - 2d) \\ - \alpha^2 \cos k(\pi + x - 4d)] + O\{|k|^{-1}e^{|\tau|(\pi - x)}\} \end{aligned}$$

$$(0 < x < d)$$

$$u(x,\lambda) = \frac{1}{2}A [\cos kx + \alpha \cos k(x - 2d)] + O\{|k|^{-1}e^{|\tau|x}\}$$

$$v(x,\lambda) = \frac{1}{2}A [\cos k(\pi - x) + \alpha \cos k(\pi - x - 2d)] + O\{|k|^{-1}e^{|\tau|(\pi - x)}\}$$

$$(d < x < \pi - d)$$

$$\begin{aligned} u(x,\lambda) = \frac{1}{4}A^2 [\cos kx + \alpha \cos k(x - 2d) - \alpha \cos k(x - 2\pi + 2d) \\ - \alpha^2 \cos k(x - 2\pi + 4d)] + O\{|k|^{-1}e^{|\tau|x}\} \end{aligned}$$

$$v(x,\lambda) = \cos k(\pi - x) + O\{|k|^{-1}e^{|\tau|(\pi - x)}\}$$

$$(\pi - d < x < \pi)$$

The functions u and v may be abbreviated as follows

$$\begin{aligned} u(x,\lambda) &= u_0 + O\{|k|^{-1}e^{|\tau|x}\} \\ v(x,\lambda) &= v_0 + O\{|k|^{-1}e^{|\tau|(\pi - x)}\} . \end{aligned}$$

where u_0 is the leading order term of u and v_0 is the leading order term of v. We now introduce the Wronskian

$$\omega(\lambda) = -h\ u(\pi,\lambda) - u'(\pi,\lambda) \tag{2.8}$$

and note that λ is an eigenvalue of (2.1) iff $\omega(\lambda) = 0$. The asymptotic expansion for ω is obtained by differentiating the Volterra integral equations for u. We specialize the results of Willis [88], [90] to obtain

$$\begin{aligned} \omega(\lambda) &= \frac{1}{4}kA^2\ [\ \sin k\pi + 2\alpha \sin k(\pi - 2d) - \ \alpha^2 \sin k(-\pi + 4d)\ ] \\ &\qquad + O\{e^{|\tau|\pi}\} \end{aligned} \tag{2.9}$$

Next, consider the function

$$\Phi(x,\lambda) = \frac{v\int_0^x uf\ dy\ +\ u\int_x^\pi vf\ dy}{\omega(\lambda)}\ . \tag{2.10}$$

Here u and v are solutions of (2.1) with boundary conditions (2.5) and (2.6) and have the asymptotic expansions given above. We integrate Φ along a large contour Γ in the λ-plane. Let $k = \sqrt{\lambda}$ and consider the contour in the k-plane consisting of the lines

$$\begin{aligned} L_1 &: \ (n+\frac{1}{2}) + it \\ L_2 &: \ (n+\frac{1}{2}) - 2t + i\,(n+\frac{1}{2}) \\ L_3 &: \ -(n+\frac{1}{2}) + i\,[\,(n+\frac{1}{2}) - t\,] \end{aligned}$$

where $0 \le t \le (n+\frac{1}{2})$. Let Γ be the corresponding contour on the λ-plane. First we note that

$$\frac{1}{2\pi i}\int_\Gamma \Phi(x,\lambda)\,d\lambda \ \rightarrow\ \sum_{j=0}^{\infty} \frac{v_j\int_0^x u_jf\ dy\ +\ u_j\int_x^\pi v_jf\ dy}{\omega'(\lambda_j)} \tag{2.11}$$

on $(0,d)$, $(d,\pi-d)$, $(\pi-d,\pi)$ as $n\to\infty$ from the residue theorem. We will show that

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \Phi(x,\lambda)\,d\lambda \ &\rightarrow\ \frac{1}{2\pi i}\int_\Gamma\int_0^x \frac{v_0(x,\lambda)\,u_0(y,\lambda)}{\omega_0(\lambda)}\ f(y)\ dy\ d\lambda \\ &\qquad +\ \frac{1}{2\pi i}\int_\Gamma\int_x^\pi \frac{u_0(x,\lambda)\,v_0(y,\lambda)}{\omega_0(\lambda)}\ f(y)\ dy\ d\lambda \\ &\rightarrow\ f(x) \end{aligned} \tag{2.12}$$

on $(0,d)$, $(d,\pi-d)$, $(\pi-d,\pi)$ as $n\to\infty$. Here $\omega_0(\lambda)$ denotes the leading order term of the Wronskian ω.

$$\omega_0(\lambda) = \frac{1}{4}kA^2\ [\ \sin k\pi + 2\alpha \sin k(\pi-2d) +\ \alpha^2 \sin k(\pi-4d)\ ] \tag{2.13}$$

And v_0 and u_0 denote the leading order terms of v and u.

We will prove the first expression in (2.12). From the definition of Φ we have that

$$\frac{1}{2\pi i}\int_\Gamma \Phi(x,\lambda)\,d\lambda \;=\; \frac{1}{2\pi i}\int_\Gamma\int_0^x \frac{v(x,\lambda)\,u(y,\lambda)}{\omega(\lambda)}\,f(y)\,dy\,d\lambda$$

$$+\;\frac{1}{2\pi i}\int_\Gamma\int_x^\pi \frac{u(x,\lambda)\,v(y,\lambda)}{\omega(\lambda)}\,f(y)\,dy\,d\lambda$$

on $(0,d)$, $(d,\pi-d)$, $(\pi-d,\pi)$. Write u, v and ω as the sum of the leading order term and the lower order terms, i.e. $u = u_0 + u_L$, $v = v_0 + v_L$ and $\omega = \omega_0 + \omega_L$. Here the subindex 0 is used to denote the leading order term and the subindex L is used to denote the lower order term. We consider the quotient

$$\begin{aligned}
\frac{u\cdot v}{\omega} &= \frac{u_0\cdot v_0}{\omega} + \frac{u_0\cdot v_L}{\omega} + \frac{u_L\cdot v}{\omega}\\
&= (\ u_0\cdot v_0\ )\left(\frac{1}{\omega} - \frac{1}{\omega_0}\right) + \frac{u_0\cdot v_0}{\omega_0} + \frac{u_0\cdot v_L\ +\ u_L\cdot v}{\omega}\\
&= \frac{u_0\cdot v_0}{\omega_0} - \frac{u_0\cdot v_0\ (\ \omega-\omega_0)}{\omega\cdot\omega_0} + \frac{u_0\cdot v_L\ +\ u_L\cdot v}{\omega}\\
&= \frac{u_0\cdot v_0}{\omega_0} - \frac{u_0\cdot v_0\cdot\omega_L}{\omega\cdot\omega_0} + \frac{u_0\cdot v_L\ +\ u_L\cdot v}{\omega}\\
&= \frac{u_0\cdot v_0}{\omega_0} + E
\end{aligned}$$

where E is defined by

$$E = -\frac{u_0\cdot v_0\cdot\omega_L}{\omega\cdot\omega_0} + \frac{u_0\cdot v_L\ +\ u_L\cdot v}{\omega}\quad.$$

We will show that

$$\frac{1}{2\pi i}\int_\Gamma\int_0^x E\ f(y)\,dy\,d\lambda \to 0 \tag{2.14}$$

and

$$\frac{1}{2\pi i} \int_\Gamma \int_x^\pi E \; f(y) \; dy \; d\lambda \to 0 \tag{2.15}$$

as $n \to \infty$. The proof consists of three distinct cases; the first case is when $0 < x < d$, the second when $d < x < \pi - d$ and the third when $\pi - d < x < \pi$. Since our problem is symmetric the proof for the first interval $0 < x < d$ gives us the proof for the third interval $\pi - d < x < \pi$. Only the first two cases must therefore be considered. We present only the first case in detail since the proof for the second case is the same except for the constants in the bounds.

Consider the integral (2.14) when $0 < x < d$. We analyze the integral on two separate intervals $(0, x - \delta)$ and $(x - \delta, x)$ where $x + \delta < d$ and $x - \delta > 0$. We use the bounds from Lemma 1.3 and the bound for the Wronskian from Lemma 1.4 to find that on $(0, x - \delta)$

$$|E| \;\leq\; \frac{432984c}{|k|^2} \cdot \frac{e^{|\tau| y} e^{|\tau|(\pi - x)}}{e^{|\tau|\pi}(1 - \bar{\alpha})^2}$$

for large n. Here we assume that $c/|k| < 1/3$. Note that f is piecewise C^1 so that $|f(x)|$, $|f'(x)| \leq M$ for some constant M on $(0, d)$, $(d, \pi - d)$ and $(\pi - d, \pi)$. Therefore

$$\left| \frac{1}{2\pi i} \int_\Gamma \int_0^{x-\delta} E \; f(y) \; dy \; d\lambda \right| \leq \frac{1}{2\pi i} \int_\Gamma \frac{432984 \, cM}{|k|^2} \cdot \frac{(e^{|\tau|(x-\delta)} - 1) e^{|\tau|(\pi - x)}}{e^{|\tau|\pi}(1 - \bar{\alpha})^2} \; |d\lambda|$$

for large n, and the integral converges to 0 as $n \to \infty$. On $(x - \delta, x)$ the integral is bounded by

$$\frac{1}{2\pi} \int_\Gamma \int_{x-\delta}^{x} \frac{432984 \, cM}{|k|^2 (1 - \bar{\alpha})^2} \; |dy| \; |d\lambda| \;\; .$$

This expression can be made arbitrarily small by choosing δ to be small enough. The proof for the second integral (2.15) is constructed by analyzing the integral on the two intervals $(x, x + \delta)$ and $(x + \delta, \pi)$ and by following the the same techniques as above.

Again the proof of the second relation in (2.12) consists of three distinct cases; the first case is when $0 < x < d$, the second when $d < x < \pi - d$ and the third when $\pi - d < x < \pi$. Since our problem is symmetric the proof for the first interval $0 < x < d$ gives us the proof for the third interval $\pi - d < x < \pi$. Only the first two cases must therefore be considered. We only present the details of the first case since techniques used in the proof of both cases are the same.

Consider the integral

$$\frac{1}{2\pi i} \int_\Gamma \int_0^x \frac{v_0(x,\lambda)\, u_0(y,\lambda)}{\omega_0(\lambda)}\, f(y)\, dy\, d\lambda \quad . \tag{2.16}$$

When $0 < y < d$, the term

$$\frac{v_0(x,\lambda)\, u_0(y,\lambda)}{\omega_0(\lambda)}$$

$$= \left[\, \cos k(\pi - x) - \alpha \cos k(\pi + x - 2d) + \alpha \cos k(\pi - x - 2d) - \alpha^2 \cos k(\pi - 4d + x)\, \right]$$

$$\times \frac{\cos ky}{k \cdot [\, \sin k\pi + 2\alpha \sin k(\pi - 2d) + \alpha^2 \sin k(\pi - 4d)\,]} \quad .$$

Note that $|\pi + x - 2d|$, $|\pi - x - 2d|$, $|\pi - 4d + x| < (\pi - x)$ and $|\pi - 2d|$, $|\pi - 4d| < \pi$. We analyze the integral (2.16) on two separate intervals $(0, x - \delta)$ and $(x - \delta, x)$ where $x + \delta < d$ and $x - \delta > 0$. f is sectionally continuous on $[0, \pi]$ where the sections are intervals $(0, d)$, $(d, \pi - d)$ and $(\pi - d, \pi)$. There exists a constant M such that $|f(x)|$, $|f\,'(x)| \le M$ on each of these intervals. Let $k = \sigma + i\tau$. Integrate by parts to find

$$\int_0^{x-\delta} \cos ky\; f(y)\; dy \;=\; \frac{1}{k}\, \sin ky\; f(y)\, |_0^{x-\delta} - \int_0^{x-\delta} \frac{\sin ky}{k}\, f'(y)\, dy \quad ,$$

and use the trigonometric inequalities (1.24) to find that the expression above is bounded by

$$\frac{M}{|k|}\, e^{-\tau(x-\delta)}\, [\, 1 \;+\; (x - \delta)\,] \quad .$$

Since v_0 is bounded above by

$$|v_0(x,\lambda)| \le \frac{A^2}{4}\, (1 + \bar{\alpha})\, e^{|\tau|(\pi - x)}$$

and the Wronskian is bounded below by

$$|\omega_0(\lambda)| \ge \frac{A^2(1 - \bar{\alpha})}{24|k|}\, e^{|\tau|\pi}$$

(See Lemma 1.4), on $(0, x - \delta)$ the integral (2.16) is bounded by

$$\frac{3\, M \cdot [\, 1 \;+\; (x - \delta)\,]\, (1 + \bar{\alpha})}{\pi(1 - \bar{\alpha})} \int_\Gamma \frac{1}{|k^2|}\, e^{-\tau\delta}\, |d\lambda| \quad .$$

For fixed δ and fixed x this integral tends to 0 as $n \to \infty$. For details see Titchmarsch [83]. Next consider the integral (2.16) on the second interval $(x-\delta, x)$.

$$\int_{x-\delta}^{x} \frac{v_0(x,\lambda)\, u_0(y,\lambda)}{\omega_0(\lambda)}\, f(y)\, dy$$

Since f is C^1 on $(x-\delta, x)$, we may replace $f(y)$ by $f(x)$. Then

$$\begin{aligned} f(x) \cdot \frac{v_0(x,\lambda)}{\omega_0(\lambda)} \int_{x-\delta}^{x} u_0(y,\lambda)\, dy &= f(x) \cdot \frac{v_0(x,\lambda)}{\omega_0(\lambda)} \int_{x-\delta}^{x} \cos\, ky\, dy \\ &= f(x) \cdot \frac{v_0(x,\lambda)}{\omega_0(\lambda)} \frac{1}{k} \left[\, \sin\, kx \;-\; \sin\, k(x-\delta)\, \right] . \end{aligned}$$

Rewrite v_0 as

$$v_0 = \cos\; k(\pi - x) \;+\; \sum \beta_i \cos\; k\eta_i$$

where $\sum |\beta_i| = \bar{\alpha} < 1$ and $|\eta_i| < \pi - x$. Then

$$\begin{aligned} v_0 \cdot \int_{x-\delta}^{x} u_0\, dy &\doteq \frac{1}{4A^2 k} \left[\sin\; kx \cdot \cos\; k(\pi - x) + \sum \tilde{\beta}_i\; \sin\; k\tilde{\eta}_i \right] \\ &= \frac{1}{4A^2 k} \left[\frac{1}{2}\; \sin\; k\pi + \sum \tilde{\beta}_i\; \sin\; k\tilde{\eta}_i \right] \end{aligned}$$

where $\sum |\tilde{\beta}_i|$ is bounded and $|\tilde{\eta}_i| < \pi$. Here we use the trigonometric identity

$$\sin\theta \;\cdot\; \cos\tilde{\theta} \;=\; \frac{1}{2} \left[\, \sin(\theta + \tilde{\theta}) \;+\; \sin(\theta - \tilde{\theta})\, \right] .$$

Next we rewrite $1/\omega_0$ as

$$\begin{aligned} \frac{1}{\omega_0} &= \frac{1}{4A^2} \left[\frac{1}{k\; \sin\; k\pi} + \frac{1}{\omega_0} - \frac{1}{k\; \sin\; k\pi} \right] \\ &= \frac{1}{4A^2} \left[\frac{1}{k\; \sin\; k\pi} + \frac{-2\alpha \sin\; k(\pi - 2d) - \alpha^2 \sin\; k(\pi - 4d)}{\omega_0\; \sin\; k\pi} \right] \\ &= \frac{1}{4A^2} \left[\frac{1}{k\; \sin\; k\pi} + E_L \right] \end{aligned}$$

where $|E_L|$ is bounded by

$$|E_L| < \frac{\bar{\alpha}}{4A^2\; |\omega_0|}\; e^{\tau(\hat{\eta} - \pi)}$$

and $\hat{\eta} = \max\{(\pi - 2d), (\pi - 4d)\}$. We combine our results from above to find that

$$f(x) \cdot \frac{v_0}{\omega_0} \int_{x-\delta}^{x} u_0 \, dy \;=\; \frac{f(x)}{k^2} \cdot \frac{\frac{1}{2}\sin\, k\pi \;+\; \sum \tilde{\beta}_i \,\sin\, k\tilde{\eta}_i}{\sin\, k\pi} \;+\; \tilde{E}_L$$

$$= \frac{1}{2}\frac{f(x)}{k^2} \;+\; \hat{E}_L$$

where $|\tilde{E}_L|$ is derived from multiplying the terms associated with E_L and the term

$$f(x) \cdot \; v_0(x, \lambda) \int_{x-\delta}^{x} u_0(y, \lambda) \, dy \;\; .$$

And $|\tilde{E}_L|$ and $|\hat{E}_L|$ are bounded by

$$|\tilde{E}_L| \;\le\; \frac{|f(x)|}{|k^2|} \,\bar{\alpha}\, e^{\tau(\hat{\eta}-\pi)} [\; \frac{1}{2} \;+\; \sum |\tilde{\beta}_i| \; e^{\tau(\tilde{\eta}_i - \pi)} \;]$$

$$|\hat{E}_L| \;\le\; |\tilde{E}_L| \;+\; \frac{|f(x)|}{|k^2|} \cdot \sum |\tilde{\beta}_i| \; e^{\tau(\tilde{\eta}_i - \pi)} \;\; .$$

Here we use the lower bound for the Wronskian from Lemma 1.4 again. Note that

$$\int_\Gamma \frac{|f(x)|}{|k^2|} \; e^{\tau(\tilde{\eta}_i - \pi)} \; |d\lambda| \; \to 0$$

and

$$\int_\Gamma \frac{|f(x)|}{|k^2|} \; e^{\tau(\hat{\eta} - \pi)} \; |d\lambda| \; \to 0$$

as $n \to \infty$ since $0 < \tilde{\eta}_i \; , \; \hat{\eta} < \pi$ for fixed x and fixed δ. And

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(x)}{2k^2} \, d\lambda \; = \frac{1}{2} f(x) \;\; .$$

Therefore on $(x - \delta, x)$ integral (2.16) converges to $\frac{1}{2}f(x)$.

To justify the substitution of $f(y)$ by $f(x)$ note that f is C^1 on $(0, d)$ so that $f(y) - f(x)$ is also C^1, and it may be written

$$f(y) - f(x) = g(y) - h(y)$$

where $g(y)$ and $h(y)$ are positive monotone functions that tend to zero as $y \to x$. (See Royden [73], page 100.) By the second mean-value theorem

$$\int_{x-\delta}^{x} \cos ky \;\, g(y) \; dy \;=\; g(x - \delta) \int_{x-\delta}^{\xi} \cos ky \; dy$$

$$= \; O\left(\frac{g(x - \delta)}{|k|} e^{\tau x}\right) \;\; .$$

This term contributes

$$\int O \left| \frac{g(x-\delta)}{\lambda} \right| |d\lambda| = O\{g(x-\delta)\}$$

to integral (2.16), and it tends to zero as $\delta \to 0$. A similar argument can be used for the term involving h. We remark that the convergence we have proven is pointwise.

Next consider the integral

$$\frac{1}{2\pi i} \int_\Gamma \int_x^\pi \frac{u_0(x,\lambda)\, v_0(y,\lambda)}{\omega_0(\lambda)}\, f(y)\, dy\, d\lambda \tag{2.17}$$

When $0 < y < d$, the term

$$\frac{u_0(x,\lambda)\, v_0(y,\lambda)}{\omega_0(\lambda)}$$

$$= [\, \cos k(\pi - y) - \alpha \cos k(\pi + y - 2d) + \alpha \cos k(\pi - y - 2d) - \alpha^2 \cos k(\pi - 4d + y)\,]$$

$$\times \frac{\cos kx}{k \cdot [\, \sin k\pi + 2\alpha \sin k(\pi - 2d) + \alpha^2 \sin k(\pi - 4d)\,]}$$

Note that $|\pi + y - 2d|, |\pi - y - 2d|, \ |\pi - 4d + y| < (\pi - y)$ and $|\pi - 2d|, |\pi - 4d| < \pi$.
When $d < y < \pi - d$, the term

$$\frac{u_0(x,\lambda)\, v_0(y,\lambda)}{\omega_0(\lambda)}$$

$$= \frac{\cos kx + \alpha \cos k(x - 2d)}{k} \times \frac{\cos k(\pi - y) + \alpha \cos k(\pi - y - 2d)}{\sin k\pi + 2\alpha \sin k(\pi - 2d) + \alpha^2 \sin k(\pi - 4d)}$$

Note that $|x - 2d| < x, \ |\pi - y - 2d| < (\pi - y)$ and $|\pi - 2d|, |\pi - 4d| < \pi$.
When $\pi - d < y < \pi$, the term

$$\frac{u_0(x,\lambda)\, v_0(y,\lambda)}{\omega_0(\lambda)}$$

$$= [\, \cos kx + \alpha \cos k(x - 2d) - \alpha \cos k(x - 2\pi + 2d) - \alpha^2 \cos k(x - 2\pi + 4d)\,]$$

$$\times \frac{\cos k(\pi - y)}{k\, [\, \sin k\pi + 2\alpha \sin k(\pi - 2d) + \alpha^2 \sin k(\pi - 4d)\,]}$$

We note that $|x-2d|$, $|x-2\pi+2d|$, $|x-2\pi+4d|<|x|$, $|\pi-y-2d|<(\pi-y)$ and $|\pi-2d|$, $|\pi-4d|<\pi$. To show that integral (2.18) converges to $\frac{1}{2}f(x)$ as $n\to\infty$, we examine the integral on two separate intervals $(x,x+\delta)$ and $(x+\delta,\pi)$ and use techniques similar to those used in the analysis of integral (2.16). The sum of integrals (2.16) and (2.18) approaches $f(x)$ as $n\to\infty$.

2.2 Three Lemmas and Two Corollaries

In this first lemma we show that the eigenfunctions of problem (2.1) - (2.4) are orthogonal.

Lemma 2.1 (Orthogonality): *Consider the eigenvalue problem:*

$$-\;u''+q(x)\;u=\lambda\;u \tag{2.18}$$

on the interval $0\le x\le\pi$ *with the boundary conditions:*

$$h\;u(0)-u'(0)=h\;u(\pi)+u'(\pi)=0 \tag{2.19}$$

and discontinuities at $x=d_1$ *and* $x=d_2$ *satisfying the jump conditions:*

$$u(d_1+)=a_1u(d_1-)\;,\qquad u'(d_1+)=a_1^{-1}u'(d_1-)+b_1\;u(d_1-) \tag{2.20}$$

$$u(d_2+)=a_2u(d_2-)\;,\qquad u'(d_2+)=a_2^{-1}u'(d_2-)+b_2\;u(d_2-) \tag{2.21}$$

Let u_j *and* v_j *be the eigenfunctions of (2.17) - (2.20) corresponding to eigenvalues* λ_i *and* λ_j *respectively. Then*

$$\int_0^\pi u_i\;u_j=0\;\;;\;\;i\neq j\;\;.$$

Remark: This lemma may be reduced to the single discontinuity case and may be extended to multiple discontinuity cases in the obvious manner.

Proof: Let L denote the operator

$$L\equiv-\frac{d^2}{dx^2}+q(x)\;\;.$$

We will show

$$\int_0^\pi u_i L u_j = \int_0^\pi u_j L u_i \quad . \tag{2.22}$$

Consider the difference

$$\begin{aligned}\int_0^\pi u_i L u_j - \int_0^\pi u_j L u_i &= -\int_0^\pi (u_i u_j'' - u_i'' u_j) \\ &= -\int_0^\pi (u_i' u_j - u_i u_j')' \quad .\end{aligned}$$

Then

$$\begin{aligned}-\int_0^\pi (u_i' u_j - u_i u_j')' &= (-u_i u_j' + u_i' u_j)|_0^{d_1-} + (-u_i u_j' + u_i' u_j)|_{d_1+}^{d_2-} \\ &\quad + (-u_i u_j' + u_i' u_j)|_{d_2+}^{\pi}\end{aligned}$$

The last integral term equals zero. Substituting the boundary and jump conditions shows that the first three terms vanish, and we are led to the identity (2.22). In the Sturm-Liouville problem (2.18) - (2.21), $L\ u_i = \lambda_i\ u_i$ and $L\ u_j = \lambda_j\ u_j$ so we may write equation (2.22) as

$$(\lambda_i - \lambda_j) \int_0^\pi u_i u_j = 0 \quad . \tag{2.23}$$

So that

$$\int_0^\pi u_i u_j = 0 \qquad ; \qquad i \neq j \quad . \tag{2.24}$$

Using Theorem 2 and Lemma 2.1 we determine a relation between the derivative of the Wronskian and the L^2 - norm of the eigenfunctions.

Lemma 2.2: *Consider the eigenvalue problem (2.1) - (2.4) given in Theorem 2 with a symmetric potential $q(x)$. Let $k_j = (-1)^j$. Then*

$$k_j / \omega'(\lambda_j) = 1/\|u_j\|_2^2 \quad . \tag{2.25}$$

Proof: Using Theorem 2 we set $f = u_i$. Then

$$u_i = \sum_{j=0}^{\infty} \frac{v_j \int_0^x u_j u_i \, dy \ + \ u_j \int_x^\pi v_j u_i \, dy}{\omega'(\lambda_j)} \quad .$$

From the symmetry of our problem we have $v_j = (-1)^j u_j$, and the eigenfunctions are orthogonal. By Lemma 2.1, the expression above equals

$$u_i = \sum_{j=0}^{\infty} \frac{(-1)^j u_i \|u_j\|_2^2}{\omega'(\lambda_j)} \cdot \delta_{ij} .$$

Hence $k_j/\omega'(\lambda_j) = 1/\|u_j\|_2^2$.

In the proof of Theorem 2 note that the expansion (2.7) for f is determined by the leading order terms of the eigenfunctions u_j and v_j. As a consequence, we have the following corollary.

Corollary 2.1: *Let f, u_j and v_j be defined as in Theorem 2. And let $\tilde{u}_j$ and $\tilde{v}_j$ be the the solutions of (2.1) with a, b, q, h and $\{\lambda_j\}$ replaced by $\tilde{a}$, $\tilde{b}$, $\tilde{q}$, $\tilde{h}$ and $\{\tilde{\lambda}_j\}$. If $\lambda_j = \tilde{\lambda}_j$ for $j > n$, then f has an expansion*

$$f(x) = \sum_{j=0}^{\infty} \frac{\tilde{v}_j \int_0^x u_j f \, dy + \tilde{u}_j \int_x^\pi v_j f \, dy}{\omega'(\lambda_j)} \tag{2.26}$$

on compact subsets of $(0, d_1)$, (d_1, d_2), (d_2, π).

Proof: Corollary 1.6.1 establishes that the conclusion in Lemma 1.6 is valid even if a finite number of eigenvalues is not known. Therefore $a = \tilde{a}$ and $d = \tilde{d}$. The potential $\tilde{q}$ does not appear in the asymptotic expansion for u. The jump constant b and the boundary constant $\tilde{h}$ do appear in the leading order terms. Hence the proof of this corollary follows immediately from that of Theorem 2.

The expansions for f we have presented thus far are not in a form convenient for the general reader. In the corollary below we present an eigenfunction expansion analogous to the Fourier Series expansion.

Corollary 2.2: *Let u_j be an eigenfunction of (2.1) - (2.6), and let f and f' be sectionally continuous on* $[0, \pi]$ *with sections* $(0, d)$, $(d, \pi - d)$, $(\pi - d, \pi)$. *Then*

$$f(x) = \sum_{j=0}^{\infty} \left(f, \frac{u_j}{\|u_j\|_2} \right) \cdot \frac{u_j(x)}{\|u_j\|_2} \tag{2.27}$$

on $(0, d_1)$, (d_1, d_2), (d_2, π).

Proof: From the definition of u_j and v_j follows that $u_j = k_j v_j$. Then

$$\begin{aligned} f(x) &= \sum_{j=0}^{\infty} \frac{k_j u_j \int_0^x u_j f\, dy \;+\; u_j \int_x^\pi k_j u_j f\, dy}{\omega'(\lambda_j)} \\ &= \sum_{j=0}^{\infty} \frac{k_j u_j \int_0^\pi u_j f\, dy}{\omega'(\lambda_j)} \\ &= \sum_{j=0}^{\infty} (f, u_j) \cdot \frac{u_j(x)}{\|u_j\|_2^2} \\ &= \sum_{j=0}^{\infty} \left(f, \frac{u_j}{\|u_j\|_2}\right) \cdot \frac{u_j(x)}{\|u_j\|_2} \end{aligned}$$

where we have used $k_j/\omega'(\lambda_j) = 1/\|u_j\|_2^2 = 1$ from Lemma 2.2.

In this final lemma we study the ratio of Wronskians for two different eigenvalue problems. We will use Lemma 2.3 in Chapter 4 to construct the Hochstadt-Hald algorithm.

Lemma 2.3: *Consider the eigenvalue problem (2.1) - (2.4) given in Theorem 2. Let ω denote the Wronskian. Consider a second eigenvalue problem where a, b, q, h, ω and $\{\lambda_j\}$ are replaced by $\tilde{a}$, $\tilde{b}$, $\tilde{q}$, $\tilde{h}$, $\tilde{\omega}$ and $\{\tilde{\lambda}_j\}$. If $\lambda_j = \tilde{\lambda}_j$ for $j > n$, then*

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} \to 1$$

as $j \to \infty$.

Proof: From Lemma 2.2 we have that

$$k_j/\omega'(\lambda_j) = 1/\|u_j\|_2^2 = 1$$

and

$$k_j/\tilde{\omega}'(\lambda_j) = 1/\|\tilde{u}_j\|_2^2 = 1$$

so that

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} = \frac{\|u_j\|_2^2}{\|\tilde{u}_j\|_2^2}$$

$$= \frac{\|\varphi_j + u_j - \varphi_j\|^2}{\|\tilde{\varphi}_j + \tilde{u}_j - \tilde{\varphi}_j\|^2} - \frac{\|\varphi_j\|^2}{\|\tilde{\varphi}_j\|^2} + \frac{\|\varphi_j\|^2}{\|\tilde{\varphi}_j\|^2}$$

$$= \frac{\|\varphi_j\|^2}{\|\tilde{\varphi}_j\|^2} + E$$

where E is equal to the first two terms of the equation above. And

$$\begin{aligned} E &= \frac{\|\varphi_j + u_j - \varphi_j\|^2}{\|\tilde{\varphi}_j + \tilde{u}_j - \tilde{\varphi}_j\|^2} - \frac{\|\varphi_j\|^2}{\|\tilde{\varphi}_j\|^2} \\ &\leq \frac{2\|\varphi_j\| \cdot \|\tilde{\varphi}_j\|^2 \cdot \|u_j - \varphi_j\| + \|\tilde{\varphi}_j\|^2 \cdot \|u_j - \varphi_j\|^2}{\|\tilde{\varphi}_j\|^2 \cdot \|\tilde{u}_j\|^2} \\ &\quad + \frac{2\|\varphi_j\|^2 \cdot \|\tilde{\varphi}_j\| \cdot \|\tilde{u}_j - \tilde{\varphi}_j\| + \|\varphi_j\|^2 \cdot \|\tilde{u}_j - \tilde{\varphi}_j\|^2}{\|\tilde{\varphi}_j\|^2 \cdot \|\tilde{u}_j\|^2} \end{aligned}$$

Where we have used the Cauchy-Schwarz inequality. Here the norm $\|\cdot\|$ is the L^2 - norm. We use the bounds from Lemma 1.3

$$\|\varphi\|, \ \|\tilde{\varphi}\| \leq 11A^2\pi$$

$$\|\varphi\|^2, \ \|\tilde{\varphi}\|^2 \leq 121A^4\pi^2$$

$$\|u - \varphi\|, \ \|\tilde{u} - \tilde{\varphi}\| \leq 28A^2\pi\frac{c}{|k|}$$

to obtain a bound for $|E|$.

$$|E| \leq \left(\frac{75152A^8c}{|k|} + \frac{23912A^8c^2}{|k|^2} \right) \cdot \frac{\pi}{\|\tilde{\varphi}_j\|^2 \cdot \|\tilde{u}_j\|^2}$$

We will show that $\|\tilde{\varphi}_j\|^2, \|\tilde{u}_j\|^2 > constant > 0$ for all $\sqrt{\lambda_j} > M$. Then $E \to 0$ as $|j| \to \infty$. We will now determine M.

$$\begin{aligned} \|\tilde{\varphi}_j\| \ , \ \|\tilde{u}_j\| &\geq \|\tilde{\varphi}_j\| - \|\tilde{u}_j - \tilde{\varphi}_j\| \\ &\geq \sqrt{\int_0^d \varphi_1^2} - \sqrt{\int_0^\pi (u - \varphi)^2} \\ &= \sqrt{\frac{d}{2} + \frac{\sin 2kd}{4k}} - \frac{28\pi\sqrt{\pi}cA^2}{|k|} \end{aligned}$$

Choose k so that

$$\sqrt{\frac{d}{2}+\frac{\sin\ 2kd}{4k}}>\sqrt{\frac{d}{4}}$$

and

$$\sqrt{\frac{d}{4}}-\frac{28\pi\sqrt{\pi}cA^2}{|k|}>\frac{\sqrt{d}}{4}\ ,$$

that is $|k|>1/d$ and $|k|>112\pi cA^2/\sqrt{d}$. Since $k=\sqrt{\lambda_j}$ and $|\sqrt{\lambda_j}-j|<\frac{1}{2}$ by Lemma 1.4 we conclude

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)}\rightarrow\frac{||\varphi_j||^2}{||\tilde{\varphi}_j||^2}$$

as $j\rightarrow\infty$. By Lemma 1.6 we have that $a=\tilde{a}$ and $d=\tilde{d}$ so that $\varphi_j=\tilde{\varphi}_j$. Therefore

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)}\rightarrow 1$$

as $j\rightarrow\infty$.

Chapter 3

The Difference of Two Potentials

In this chapter we consider two Sturm-Liouville problems with different symmetric potentials with symmetric discontinuities satisfying different symmetric boundary and jump conditions. The main result is that if only a finite number of eigenvalues differ then a simple expression for the difference of the potentials can be established.

Theorem 3: *Consider the following eigenvalue problems with symmetric discontinuities at* $x = d_1$ *and* $x = d_2 = \pi - d_1$

$$\begin{aligned} -u'' + q(x)\, u &= \lambda\, u \\ h\, u(0) - u'(0) = 0 \quad &, \quad h\, u(\pi) + u'(\pi) = 0 \\ u(d_1+) = a\, u(d_1-) \quad &, \quad u'(d_1+) = a^{-1} u'(d_1-) + b\, u(d_1-) \\ u(d_2-) = a\, u(d_2+) \quad &, \quad u'(d_2-) = a^{-1} u'(d_2+) - b\, u(d_2+) \end{aligned} \tag{3.1}$$

and symmetric discontinuities at $x = \tilde{d}_1$ *and* $x = \tilde{d}_2 = \pi - \tilde{d}_1$

$$\begin{aligned} -u'' + \tilde{q}(x)\, u &= \tilde{\lambda}\, u \\ \tilde{h}\, u(0) - u'(0) = 0 \quad &, \quad \tilde{h}\, u(\pi) + u'(\pi) = 0 \\ u(\tilde{d}_1+) = a\, u(\tilde{d}_1-) \quad &, \quad u'(\tilde{d}_1+) = a^{-1} u'(\tilde{d}_1-) + \tilde{b}\, u(\tilde{d}_1-) \\ u(\tilde{d}_2-) = a\, u(\tilde{d}_2+) \quad &, \quad u'(\tilde{d}_2-) = a^{-1} u'(\tilde{d}_2+) - \tilde{b}\, u(\tilde{d}_2+)\ . \end{aligned} \tag{3.2}$$

Here q *and* $\tilde{q}$ *are integrable on* $[0, \pi]$ *and satisfy the symmetry conditions* $q(x) = q(\pi - x)$

and $\tilde{q}(x) = \tilde{q}(\pi - x)$ *almost everywhere on the interval* $0 \leq x \leq \pi$. *The jump constants satisfy* $|a-1| + |b| > 0$. *Finally,* λ_j *and* $\tilde{\lambda}_j$ *are the eigenvalues of (3.1) and (3.2). Let* $\tilde{u}_j$ *and* $\tilde{v}_j$ *be the solutions of*

$$-u'' + \tilde{q}(x)\, u = \lambda\, u$$

$$u(\tilde{d}_1+) = \tilde{a}\, u(\tilde{d}_1-) \quad , \quad u'(\tilde{d}_1+) = \tilde{a}^{-1} u'(\tilde{d}_1-) + \tilde{b}\, u(\tilde{d}_1-)$$

$$u(\tilde{d}_2-) = \tilde{a}\, u(\tilde{d}_2+) \quad , \quad u'(\tilde{d}_2-) = \tilde{a}^{-1} u'(\tilde{d}_2+) - \tilde{b}\, u(\tilde{d}_2+) \tag{3.3}$$

$$u(0) = 1 \quad , \quad u'(0) = \tilde{h} \tag{3.4}$$

$$v(\pi) = 1 \quad , \quad v'(\pi) = -\tilde{h} \tag{3.5}$$

with $\lambda = \lambda_j$. *Define the functions* $\tilde{y}$ *by*

$$\tilde{y}_j \;=\; 2\, \frac{\tilde{v}_j - k_j \tilde{u}_j}{\omega'(\lambda_j)} \quad . \tag{3.6}$$

Here $k_j/\omega'(\lambda_j) = 1/\int_0^\pi u_j^2\, dx$ *where* $k_j = (-1)^j$ *and* $u_j(x)$ *are the eigenfunctions of (3.1) normalized such that* $u_j(0) = 1$. *If* $\lambda_j = \tilde{\lambda}_j$ *for* $j > n$, *then*

$$h - \tilde{h} \;=\; \frac{1}{2} \sum_{j=0}^{n} \tilde{y}_j(0) \tag{3.7}$$

$$b - \tilde{b} \;=\; \frac{1}{2} (a^3 - a^{-1}) \sum_{j=0}^{n} \tilde{y}_j(d_1-)\, u_j(d_1-) \tag{3.8}$$

$$q - \tilde{q} \;=\; \sum_{j=0}^{n} (\tilde{y}_j\, u_j)' \quad a.e. \tag{3.9}$$

Remarks: In [35] Hald remarks that a work by Hochstadt [42] inspired him to examine the possibility of constructing an algorithm to solve the continuous symmetric inverse Sturm Liouville problem numerically. In the continuous problem Hald assumes that an infinite number of eigenvalues could differ so long as the sum $\sum_j |\lambda_j - \tilde{\lambda}_j|$ converges. This always holds for perturbation of finitely many eigenvalues. Hald's result is significant in that it allows him to determine an algorithm to reconstruct the potential function. Under certain assumptions he can prove that the algorithm has a solution and that this solution is unique. Here we extend Hald's characterization results to the discontinuous symmetric inverse Sturm-Liouville problem where a finite number of eigenvalues differ. The techniques of Hald [35] and Hochstadt [42] are used to derive expressions for the difference between the boundary constants and the difference between the potential functions. Since we consider the discontinuous problem, we also obtain a formula for the difference between the jump constants. This is our main contribution. These formulae are used in chapter 4 to construct an algorithm to determine the potential function. The proof below uses the Sturm-Liouville expansion derived in chapter 3.

Beginning of Proof: We begin by noting that $a = \tilde{a}$ and $d = \tilde{d}$ since $\{\lambda_j\}$ are identical for $j > n$. (See Corollary 1.6.1.) For the remainder of this chapter a and d will be used in the place of $\tilde{a}$ and $\tilde{d}$. Let $u(x,\lambda)$ and $v(x,\lambda)$ be the solutions of equation (3.1), (3.4) - (3.5), where $\tilde{h}$ is replaced by h.We let $\tilde{u}$ and $\tilde{v}$ be defined as above in equations (3.3) - (3.5). If f and f' are sectionally continuous on $[0,\pi]$ with sections $(0,d)$, $(d,\pi-d)$ and $(\pi-d)$ then by Corollary 2.2 f has an expansion

$$f(x) \doteq \sum_{j=0}^{\infty} \frac{\tilde{v}_j \int_0^x u_j f\ dy \ + \ \tilde{u}_j \int_x^\pi v_j f\ dy}{\omega'(\lambda_j)} \tag{3.10}$$

for $0 \leq x \leq \pi$, $x \neq d_1, d_2$.

We note that u_j and v_j represent the same eigenfunction whereas $\tilde{u}_j$ and $\tilde{v}_j$ are not necessarily eigenfunctions. q is symmetric so that $v_j = k_j u_j$ where $k_j = (-1)^j$. When $q = \tilde{q}$ and $h = \tilde{h}$ then (3.10) reduces to the Sturm-Liouville expansion and consequently $k_j/\omega'(\lambda_j) = 1/\int_0^\pi u_j^2\ dx$. (See Corollary 2.2 and Lemma 2.2.) Let f be the eigenfunction u_0 of (3.1), and substitute into equation (3.10). Then

$$u_0 = \tilde{u}_0 + \frac{1}{2} \sum_{j=0}^{n} \tilde{y}_j \int_0^x u_j u_0\ dt \quad , \tag{3.11}$$

where we have used that $\tilde{y}_j = 0$ for $j > n$. Formally differentiate equation (3.11). Let $f_j = \tilde{y}_j \int_0^x u_j u_0\ dt$. Then $f_j(0) = 0$ and $f_j'(0) = \tilde{y}_j(0)$. Note that u_j and u_0 are eigenfunctions of (3.1) and $\tilde{y}_j$ is a solution of (3.3) with $\lambda = \lambda_j$. We will differentiate f_j twice and use integration by parts to show that

$$f_j'' + (\lambda_0 - \tilde{q}) f_j = 2(\tilde{y}_j u_j)' u_0 \quad . \tag{3.12}$$

The expressions for f_j, f_j' and f_j'' are

$$f_j \quad = \quad \tilde{y}_j \int_0^x u_j u_0$$

$$f_j' = \tilde{y}_j u_j u_0 + \tilde{y}_j' \int_0^x u_j u_0$$

$$f_j'' = 2\tilde{y}_j' u_j u_0 + \tilde{y}_j u_j' u_0 + \tilde{y}_j u_j u_0' + \tilde{y}_j'' \int_0^x u_j u_0$$

Substitute into the LHS of equation (3.12) to find

$$\begin{aligned} & f_j'' + (\lambda_0 - \tilde{q}) f_j \\ = \; & 2\,(\tilde{y}_j u_j)'\, u_0 + (\lambda_0 - \lambda_j)\, \tilde{y}_j \int_0^x u_j u_0 + [\, u_j u_0' - u_j' u_0 \,]\, \tilde{y}_j \;. \end{aligned} \tag{3.13}$$

Consider the integral term

$$\begin{aligned} I &= (\lambda_0 - \lambda_j) \int_0^x u_j u_0 \\ &= \int_0^x u_j \lambda_0 u_0 - \int_0^x \lambda_j u_j u_0 \\ &= \int_0^x u_j\, [\, \tilde{q} u_0 - u_0'' \,] - \int_0^x [\, \tilde{q} u_0 - u_0'' \,]\, u_0 \\ &= \int_0^x (u_j'' u_0 - u_0'' u_j) \\ &= \int_0^x (u_j' u_0 - u_0' u_j)' \end{aligned}$$

which equals

$$[\, u_j' u_0 - u_0' u_j \,]|_0^x$$

for $0 < x < d$,

$$[\, u_j' u_0 - u_0' u_j \,]|_0^{d-} + [\, u_j' u_0 - u_0' u_j \,]|_{d+}^x$$

for $d < x < \pi - d$, and

$$[\, u_j' u_0 - u_0' u_j \,]|_0^{d-} + [\, u_j' u_0 - u_0' u_j \,]|_{d+}^{(\pi-d)-} + [\, u_j' u_0 - u_0' u_j \,]|_{(\pi-d)+}^x$$

for $\pi - d < x < \pi$. In all three cases we use the boundary and jump conditions to show that I equals $[\, u_j' u_0 - u_0' u_j \,](x)$. Since

$$u_0 - \tilde{u}_0 = \frac{1}{2} \sum_{j=0}^{n} f_j \;,$$

it follows that

$$u_0' - \tilde{u}_0' \;=\; \frac{1}{2}\sum_{j=0}^{n} f_j'$$

$$u_0'' - \tilde{u}_0'' \;=\; (\tilde{q} - \lambda_0)(u_0 - \tilde{u}_0) \;+\; \sum_{j=0}^{n} (\tilde{y}_j u_j)' u_0 \;\; .$$

To derive equation (3.7) let $x = 0$ in the first equation. To derive (3.9) use the second equation, the relations $u_0'' = (q-\lambda_0)\, u_0$ and $\tilde{u}_0'' = (\tilde{q}-\lambda_0)\, \tilde{u}_0$ and note that the eigenfunction u_0 is positive in the whole interval. Finally we will use equations (3.11) and (3.13) to determine formula (3.8) for $b - \tilde{b}$. To simplify the notation we will write $(+)$ for (d_1+) and $(-)$ for (d_1-). Let $x = d-$ in formula (3.11) then

$$u_0(-) \;=\; \tilde{u}_0(-) \;+\; \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(-)\int_0^d u_j u_0 \;\; .$$

We differentiate (3.11).

$$u_0'(x) \;=\; \tilde{u}_0'(x) \;+\; \frac{1}{2}\sum_{j=0}^{n} \left[\, \tilde{y}_j'(x)\int_0^x u_j u_0 + \; \tilde{y}_j(x)\; u_j(x) u_0(x) \,\right] \tag{3.14}$$

Let $x = d-$ and let $x = d+$ then

$$u_0'(-) \;=\; \tilde{u}_0'(-) \;+\; \frac{1}{2}\sum_{j=0}^{n} \left[\, \tilde{y}_j'(-)\int_0^d u_j u_0 + \; \tilde{y}_j(-)\; u_j(-) u_0(-) \,\right] \;, \tag{3.15}$$

and

$$u_0'(+) \;=\; \tilde{u}_0'(+) \;+\; \frac{1}{2}\sum_{j=0}^{n} \left[\, \tilde{y}_j'(+)\int_0^d u_j u_0 + \; \tilde{y}_j(+)\; u_j(+) u_0(+) \,\right] \;. \tag{3.16}$$

Next substitute in the jump conditions for u into equation (3.16) above then

$$a^{-1} u_0'(-) \;+\; b\; u_0(-)$$

$$= \;\; a^{-1}\tilde{u}_0'(-) \;+\; \tilde{b}\; \tilde{u}_0(-) \;+\; \frac{1}{2}\sum_{j=0}^{n} \left[\, a^{-1}\tilde{y}_j'(-) \;+\; \tilde{b}\; \tilde{y}_j(-) \,\right] \int_0^d u_j u_0$$

$$+\, \frac{1}{2}\sum_{j=0}^{n} a \cdot \tilde{y}_j(-) \;\cdot\; a \cdot\; u_j(-) \;\cdot a \cdot u_0(-) \;\; .$$

Multiply equation (3.15) by a^{-1} use the equation to cancel terms in the expression above then

$$\begin{aligned} b\, u_0(-) &= \tilde{b}\, \tilde{u}_0(-) + \tilde{b}\, \frac{1}{2} \sum_{j=0}^{n} \tilde{y}_j'(-) \int_0^d u_j u_0 \\ &\quad + \frac{1}{2} \sum_{j=0}^{n} (\, a^3 - a^{-1}\,)\, \tilde{y}_j(-)\, u_j(-)\, u_0(-) \\ &= \tilde{b}\, u_0(-) + \frac{1}{2} \sum_{j=0}^{n} (\, a^3 - a^{-1}\,)\, \tilde{y}_j(-)\, u_j(-)\, u_0(-)\ . \end{aligned}$$

Finally divide by $u_0(-)$.

$$b = \tilde{b} + \frac{1}{2}\,(a^3 - a^{-1}) \sum_{j=0}^{n} \tilde{y}_j(-) u_j(-)$$

This completes the proof.

The formulae derived in Theorem 3 enable us to present a more elegant uniqueness proof for discontinuous symmetric inverse Sturm-Liouville problems. The symmetries of the eigenfunctions and potential functions are fully exploited to give a concise and clear proof.

Corollary 3.1: (A Second Uniqueness Proof) *Consider the eigenvalue problem (3.1) where q is integrable on $0 \le x < \pi$. If $q(x) = q(\pi - x)$ almost everywhere in $0 < x < \pi$ and $|a - 1| + |b| > 0$ then $q(x)$, a, b and h are uniquely determined by the spectrum.*

Proof: Assume that we have two Sturm-Liouville problems with the same eigenvalues $\lambda_j = \tilde{\lambda}_j$. By Corollary 1.6.1 a equals $\tilde{a}$. From equations (3.3) and (3.4) follows that $\tilde{u}_j$ is an eigenfunction, and since the potential $\tilde{q}$ is symmetric we conclude that $\tilde{v}_j = k_j \tilde{u}_j$. This shows that all $\tilde{y}_j$ vanish identically and the right hand sides of equations (3.7), (3.8) and (3.9) are zero.

Chapter 4

The Hochstadt-Hald Algorithm

In this chapter we derive and implement an algorithm for solving the discontinuous symmetric inverse Sturm-Liouville problem numerically. The idea of constructing an algorithm was originated by Hochstadt [42]. It was then refined and successfully implemented by Hald [35] in the continuous inverse Sturm-Liouville problem with a symmetric potential. We extend Hald's ideas to the discontinuous inverse Sturm-Liouville problems with a symmetric potential. The results of numerical experiments using the new algorithm are given and errors in the examples are discussed.

4.1 The Algorithm

The Hochstadt-Hald algorithm is based on the eigenvalue problems (3.1) and (3.2). The problem is to determine q and h of equation (3.1) when a, $\tilde{b}$, $\tilde{h}$, $\tilde{q}(x)$, $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$ are given and $\lambda_j = \tilde{\lambda}_j$ for $j > n$. Here λ_j are the eigenvalues of equation (3.1) and $\tilde{\lambda}_j$ are the eigenvalues of equation (3.2). Note that only a finite number of eigenvalues differ. We use equation (3.9) to determine the relationship between q and the three terms $\tilde{q}$, $\tilde{y}_j$ and u_j. The denominator $\omega'(\lambda_j)$ can be computed using our knowledge of the eigenvalues λ_j, however a more computationally suitable method has been suggested by Hald. His ideas are presented below. To determine $\tilde{y}_j$ we solve the system (3.3) - (3.5). At $x = d$ and $x = \pi - d$ adjust u, u', w and w' using the jump conditions given in (3.2) and (3.3). We can now determine the boundary constant h using equation (3.7).

$$h = \tilde{h} + \frac{1}{2} \sum_{j=0}^{n} \tilde{y}_j(0) \ . \tag{4.1}$$

To determine the potential $q(x)$ solve the system below for u_i

$$u_i'' + [\ \lambda_i - \tilde{q}(x) - \sum_{j=0}^{n} (\tilde{y}_j u_j)'\ ]\ u_i = 0 \tag{4.2}$$

$$u_i(0) = 1 \qquad , \qquad u_i'(0) = h \tag{4.3}$$

with discontinuities at $x = d_1$ and $x = d_2 = \pi - d_1$ satisfying the jump conditions

$$u(\tilde{d}_1+) = a\ u'(\tilde{d}_1-) \qquad , \qquad u'(\tilde{d}_1+) = a^{-1}u'(\tilde{d}_1-) + \tilde{b}\ u(\tilde{d}_1-)$$

$$u(\tilde{d}_2-) = a\ u(\tilde{d}_2+) \qquad , \qquad u'(\tilde{d}_2-) = a^{-1}u'(\tilde{d}_2+) - \tilde{b}\ u(\tilde{d}_2+)$$

for $i = 0,\ 1,\ \ldots\ n$. Here we use equations (3.7), (3.8) and (3.9) to determine h, b and q. The technique we have outlined yields the solution of the inverse Sturm-Liouville problem with symmetric potentials and symmetric discontinuities. In order to understand the construction of the algorithm, a brief discussion of the history of the problem must be given. Hochstadt [42] examined the continuous inverse Sturm-Liouville problem with two spectra. He constructed an algorithm based on a representation theorem in which $h = \tilde{h}$. Thus Hochstadt uses $\tilde{h}$ in equation (4.3) instead of h. Numerical investigations by Hald and later by the author show that a straightforward implementation of Hochstadt's algorithm to the continuous inverse Sturm-Liouville problem with symmetric potentials yields poor results; non-symmetric potentials and eigenfunctions which do not satisfy the right-hand boundary condition are found. And if the eigenvalues are sufficiently perturbed then the solution of (4.2) may go to infinity during the calculation. In [35] Hald modifies Hochstadt's algorithm by realizing that h cannot be equal to $\tilde{h}$ and setting h to be equal to $\tilde{h} + \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(0)$. His changes are critical for the success of the algorithm. In the symmetric inverse Sturm-Liouville problem with jump discontinuities we follow the Hochstadt-Hald algorithm and set h equal to $\tilde{h} + \frac{1}{2}\sum_{j=0}^{n} \tilde{y}_j(0)$. In addition the constant b must be determined. Set $b = \tilde{b} + \frac{1}{2}(a^3 - a^{-1})\sum_{j=0}^{n} \tilde{y}_j(d-)u_j(d-)$ by Theorem 3. That b is not equal to $\tilde{b}$ at the jumps is an analog of Hald's observation that h cannot equal $\tilde{h}$ at the boundaries. Thus our algorithm is a natural extension of the Hochstadt-Hald algorithm.

In order to give a precise and efficient algorithm to solve the Sturm-Liouville problem we note that $\tilde{v}_j(x) = \tilde{u}_j(\pi - x)$ for all x so that $\tilde{v}_j$ does not have to be calculated. In addition we must find a suitable method for calculating $\omega'(\lambda_j)$. From the Hadamard factorization theorem we have that

$$\frac{\omega(\lambda)}{\tilde{\omega}(\lambda)} = \frac{C}{\tilde{C}} \prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j}\ . \tag{4.4}$$

Here we assume that the eigenvalues $\tilde{\lambda}_j$ are nonzero. We will show that the ratio $C/\tilde{C}$ equals one. Rewrite equation (4.4) as

$$\omega(\lambda) = \frac{C}{\tilde{C}} \prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \cdot \tilde{\omega}(\lambda)$$

and differentiate. Then

$$\omega'(\lambda) = \frac{C}{\tilde{C}} \left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right)' \cdot \tilde{\omega}(\lambda) + \frac{C}{\tilde{C}} \left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right) \cdot \tilde{\omega}'(\lambda) \ . \tag{4.5}$$

We now use that $\lambda_j = \tilde{\lambda}_j$ for all $j > n$ and consider the limit as $j \to \infty$. $\tilde{\omega}(\lambda_j) = 0$ and the term

$$\left(\prod_{j=0}^{n} \frac{\lambda - \lambda_j}{\lambda - \tilde{\lambda}_j} \right) \to 1$$

as $j \to \infty$ so that

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} \to \frac{C}{\tilde{C}}$$

as $j \to \infty$. From Lemma 2.3 we have that

$$\frac{\omega'(\lambda_j)}{\tilde{\omega}'(\lambda_j)} \to 1$$

as $j \to \infty$. Therefore $C/\tilde{C} = 1$. Now return to equation (4.5) to find that if λ_j is not an eigenvalue of equation (3.2) then

$$\omega'(\lambda_j) = \prod_{i \neq j} \frac{\lambda_j - \lambda_i}{\lambda_j - \tilde{\lambda}_i} \cdot \tilde{\omega}(\lambda_j) \ . \tag{4.6}$$

Here we assume that λ_j is not an eigenvalue of (3.2). Let z_j be the eigenfunction of (3.2) corresponding to $\tilde{\lambda}_j$, and let w_j be the function $w_j = (\tilde{u}_j - z_j)/(\lambda_j - \tilde{\lambda}_j)$. Since $\tilde{\omega}(\lambda) = -\tilde{h}\tilde{u}(\pi) - \tilde{u}'(\pi)$ the last term in (4.6) is equal to $-\tilde{h}w_j(\pi) - w_j'(\pi)$ where w_j satisfies the differential equation

$$w_j'' + (\tilde{\lambda}_j - \tilde{q})w_j = -\tilde{u}_j$$

with boundary conditions

$$w_j(0) = w_j'(0) = 0$$

and jump conditions

$$w(d_1+) = a\ w(d_1-) \quad , \quad w'(d_1+) = a^{-1}w'(d_1-) + b\ w(d_1-)$$
$$w(d_2-) = a\ w(d_2+) \quad , \quad w'(d_2-) = a^{-1}w'(d_2+) - b\ w(d_2+) \ .$$

See also Chapter 3. If $\tilde{\lambda}_j \to \lambda_k$ with $k \neq j$ then we replace $\tilde{\lambda}_j$ and z_j in the above arguments by $\tilde{\lambda}_k$ and z_k.

An algorithm for solving the discontinuous, symmetric inverse Sturm-Liouville problem with symmetric potentials is given below.

Step 1°: For $j = 0, 1, \ldots, n$ determine a k where $0 \leq k \leq n$ such that

$$|\lambda_j - \tilde{\lambda}_k| = \min_{i \in \{0,1,\ldots,n\}} |\lambda_j - \tilde{\lambda}_i|$$

Step 2°: For each $j = 0, 1, \ldots, n$ solve the system given below on the intervals $0 < x < d_1$, $d_1 < x < d_2$ and $d_2 < x < \pi$

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & \tilde{q} - \tilde{\lambda}_k & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ \tilde{h} \\ 0 \\ 0 \end{bmatrix}$$

and the jump conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_1+} = \begin{bmatrix} a & 0 & 0 & 0 \\ \tilde{b} & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & \tilde{b} & a^{-1} \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_1-}$$

and

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_2+} = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ \tilde{b} & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & \tilde{b} & a \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \\ w_j \\ w'_j \end{bmatrix}_{x=d_2-} .$$

Step 3°: For each $j = 0, 1, \ldots, n$ compute

$$\omega'(\lambda_j) = \frac{\prod_{i \neq j}(\lambda_j - \lambda_i)}{\prod_{i \neq k}(\lambda_j - \tilde{\lambda}_i)} \; [\, -\tilde{h} w_j(\pi) - w'_j(\pi) \,] \; .$$

Step 4°: Set

$$h = \tilde{h} \; + \; \sum_{j=0}^{n} (\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \; .$$

Step 5°A: Solve the system given below on the intervals $0 < x < d_1-$, $d_1+ < x < d_2-$ and $d_2+ < x < \pi$

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \tilde{q} - \lambda_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{q} + \sum_{j=0}^{n} (\tilde{y}'_i u_i + \tilde{y}_i u'_i) - \lambda_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 2(\tilde{u}_j(\pi) - (-1)^j)/\omega'(\lambda_j) \\ -2(\tilde{u}'_j(\pi) + (-1)^j \, \tilde{h})/\omega'(\lambda_j) \\ 1 \\ h \end{bmatrix}$$

and the jump conditions

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_1+} = \begin{bmatrix} a & 0 & 0 & 0 \\ \tilde{b} & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & b & a^{-1} \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_1-}$$

and

$$\begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_2+} = \begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ \tilde{b} & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & b & a \end{bmatrix} \begin{bmatrix} \tilde{y}_j \\ \tilde{y}'_j \\ u_j \\ u'_j \end{bmatrix}_{x=d_2-} .$$

Step $5^o B$**:** At $x = d_1$ calculate

$$b = \tilde{b} + \frac{1}{2}(a^3 - a^{-1}) \sum_{j=0}^{n} \tilde{y}_j(d-)u_j(d-) .$$

Step 6**:** Set

$$q = \tilde{q} + \sum_{j=0}^{n} (\tilde{y}'_j u_j + \tilde{y}_j u'_j) .$$

Note that Step $5^o B$ takes place during Step $5^o A$ since b must be calculated at $x = d_1$ before the jumps in u_j and u'_j are determined. In addition we remark that Step 6 takes place during Step 5 ; q is evaluated as we solve for $\tilde{y}_j$, $\tilde{y}'_j$, u_j and u'_j. Finally note that $\tilde{y}_j$ is computed in Step $5^o A$ even though it can be expressed in terms of the $\tilde{u}_j$ from Step 2^o to avoid storing $\tilde{u}_j$ and $\tilde{u}'_j$ for all $j = 0, 1, \ldots n$.

4.2 Example 1: Matthieu's Equation

Using the Hochstadt-Hald algorithm we have tried to reconstruct the potential of the Matthieu equation with discontinuities from its first fifteen eigenvalues. Begin with the Sturm-Liouville system with potential $q \equiv 0$:

SYSTEM 1:

$$-u'' = \lambda\ u \quad ,$$

with boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric jump conditions:

$$u(d_1+) = au(d_1-)\ , \qquad u'(d_1+) = a^{-1}u'(d_1-) + b\ u(d_1-)$$

$$u(d_2-) = au(d_2+)\ , \qquad u'(d_2-) = a^{-1}u'(d_2+) - b\ u(d_2-)$$

where $d_2 = \pi - d_1$. We determine the eigenvalues for system 1 from formulae (1.7) and (1.10). Since $q \equiv 0$, $u_3(\pi) = g_3(\pi)$. The IMSL subroutine ZBRENT is then used to determine the zeros of $u_3'(\pi)$ where λ is considered to be the variable. The eigenvalues are the points λ_i where $u_3'(\pi, \lambda_i) = 0$. Determining the eigenvalues of Matthieu's equation:

SYSTEM 2:

$$-u'' + (\ 2\ \cos 2x\ )\ u = \lambda\ u \quad ,$$

with the boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric jump conditions:

$$u(d_1+) = au(d_1-)\ , \qquad u'(d_1+) = a^{-1}u'(d_1-) + b\ u(d_1-)$$

$$u(d_2-) = au(d_2+)\ , \qquad u'(d_2-) = a^{-1}u'(d_2+) - b\ u(d_2-)$$

requires a different technique since $\tilde{q} = 2\cos 2x$ so that the integral terms in formulae (1.7) and (1.10) do not vanish. We solve the system

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \tilde{q} - \lambda_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the jump conditions

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}_{x=d_1+} = \begin{bmatrix} a & 0 \\ \tilde{b} & a^{-1} \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}_{x=d_1-}$$

and

$$\begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}_{x=d_2+} = \begin{bmatrix} a^{-1} & 0 \\ \tilde{b} & a \end{bmatrix} \begin{bmatrix} \tilde{u}_j \\ \tilde{u}'_j \end{bmatrix}_{x=d_2-} .$$

using the classical fourth order Runge-Kutta method. The IMSL subroutine ZBRENT is used to find the points λ_i such that $u'_3(\pi, \lambda_i) = 0$. The λ_i are the eigenvalues. To determine the fifteenth eigenvalue to eleven decimal places, we use a gridsize of at most $\pi/10000$ where π is the length of the interval. Tables 1, 2 and 3 illustrate how the accuracy of the eigenvalues is determined. As the gridsize is halved we gain one to two decimal places of accuracy. In the continuous problem this rate of convergence is expected. This experiment shows that the method is also a fourth order method in the discontinuous problem. The expression for the calculated eigenvalues using the classical fourth order method is

$$\lambda_{j\ calculated} = \lambda_{j\ exact} + C_j \cdot h^4 + \ldots$$

where h is the gridsize. We have determined C_j for system 2 when $a = 1.5$, $b = 0.5$ and $d = \pi/20$. Table 3 shows our calculations for the eleventh eigenvalue with varying meshsizes. C_{11} varies between 21.99 and 22.85 for meshsizes $\pi/320$ to $\pi/20480$. Table 4 shows our calculations for various eigenvalues for meshsize $\pi/640$. From our experiments we find that C_j is proportional to λ_j^p where $2 < p < 3$ and $p \approx 2.5$. We note that other higher order methods may be used to find the eigenvalues [31]. The Prince-Dormand Runge-Kutta order 7-8 method [70] was used by the author. Extra work is required in feeding the coefficients into the routine, and it is not clear whether there is a significant savings in computation time. In the fourth order Runge-Kutta method a very small step size is needed to achieve high accuracy, whereas in the Prince-Dormand routine a large number of sums and products must be computed for each step.

Eigenvalue data for systems 1 and 2 obtained using the methods described are given in tables 5 and 6. To test the accuracy of the eigenvalues we graph the eigenfunctions of the corresponding differential equation. The fifth eigenfunction for system 1 is displayed in figure 1. In figure 2 we present the fifth eigenfunction for system 2.

TABLE 1 : EIGENVALUES OF SYSTEM 2
CLASSICAL R-K ORDER 4
($a = 1.5$, $b = 0.5$, $d = \pi/20$)

eig	grid=$\pi/20$	grid=$\pi/40$	grid=$\pi/80$
0	-0.4306981276051	-0.4307275152390	-1.4307294013086
1	2.1620272873399	2.1618894642776	2.1618804810992
2	5.2036771147733	5.2024435207584	5.2023626935021
3	10.498869429917	10.488873232651	10.488197487002
4	18.160254395653	18.112534683690	18.109136174826
5	27.980549911953	27.823872752543	27.811890417745
6	39.840315750298	39.446098258952	39.412995401478
7	53.565002739420	52.759540612320	52.683044154636
8	68.875951670387	67.502030929479	67.347870466487
9	85.437445968856	83.473336816661	83.193728062165
10	102.99929230914	100.73379139013	100.26410072243
11	121.41037323180	119.70858008250	118.95911602247
12	140.30542254682	140.99570123950	139.84086479332
13	158.84829470501	165.05042300555	163.32253012725
14	176.03108004145	192.06764387893	189.56098997198

eig	grid=$\pi/160$	grid=$\pi/320$	grid=$\pi/640$
0	-0.4307295199687	-0.4307295273972	-0.4307295278617
1	2.1618799137973	2.1618798782492	2.1618798760260
2	5.2023575821418	5.2023572617439	5.2023572417043
3	10.488154421860	10.488151717153	10.488151547903
4	18.108916823793	18.108903003728	18.108902138236
5	27.811104094706	27.811054347561	27.811051228887
6	39.410779027054	39.410638106747	39.410629261370
7	52.677802211172	52.677467012048	52.677445941997
8	67.337028572576	67.336330886805	67.336286962173
9	83.173492820124	83.172181724313	83.172099039754
10	100.22901598900	100.22672587499	100.22658118178
11	118.90108682012	118.89726808220	118.89702631875
12	139.74762542877	139.74143338192	139.74104047572
13	163.17591181294	163.16607420631	163.16544838565
14	189.33523405173	189.31991102384	189.31893347385

TABLE 1 (CONTINUED)

eig	grid=$\pi/1280$	grid=$\pi/2560$	grid=$\pi/5120$
0	-0.4307295278907	-0.4307295278925	-0.4307295278926
1	2.1618798758870	2.1618798758783	2.1618798758778
2	5.2023572404516	5.2023572403733	5.2023572403684
3	10.488151537321	10.488151536660	10.488151536619
4	18.108902084116	18.108902080733	18.108902080522
5	27.811051033821	27.811051021627	27.811051020865
6	39.410628707940	39.410628673342	39.410628671179
7	52.677444623236	52.67744452.784	52.677444535630
8	67.336284211871	67.336284039899	67.336284029149
9	83.172093860323	83.1720935364269	83.172093516180
10	100.22657211390	100.22657154677	100.22657151132
11	118.89701115982	118.89701021163	118.89701015235
12	139.74101582595	139.74101428388	139.7410139.748
13	163.16540909852	163.16540664036	163.16540648668
14	189.31887206261	189.31886821948	189.31886797921

eig	grid=$\pi/10240$	grid=$\pi/20480$	grid=$\pi/40960$
0	-0.4307295278926	-0.4307295278926	-0.4307295278926
1	2.1618798758777	2.1618798758777	2.1618798758777
2	5.2023572403681	5.2023572403681	5.2023572403681
3	10.488151536616	10.488151536616	10.488151536616
4	18.108902080508	18.108902080508	18.108902080508
5	27.811051020818	27.811051020815	27.811051020815
6	39.410628671044	39.410628671036	39.410628671035
7	52.677444535308	52.677444535288	52.677444535286
8	67.336284028478	67.336284028436	67.336284028433
9	83.172093514914	83.172093514835	83.172093514830
10	100.22657150910	100.22657150896	100.22657150895
11	118.89701014865	118.89701014842	118.89701014840
12	139.74101418146	139.74101418108	139.74101418106
13	163.16540647707	163.16540647647	163.16540647644
14	189.31886796419	189.31886796325	189.31886796319

TABLE 1 (CONTINUED)

eig	grid=$\pi/81920$
0	-0.4307295278926
1	2.1618798758777
2	5.2023572403681
3	10.488151536616
4	18.108902080508
5	27.811051020815
6	39.410628671035
7	52.677444535286
8	67.336284028433
9	83.172093514830
10	100.22657150895
11	118.89701014840
12	139.74101418105
13	163.16540647643
14	189.31886796319

TABLE 2 : RELATIVE ERRORS IN EIGENVALUES OF SYSTEM 2

($a = 1.5$, $b = 0.5$, $d = \pi/20$)

eig	$grid = \pi/20$	$grid = \pi/40$	$grid = \pi/80$
0	-0.000072900244	-0.000004672662	-0.000000293883
1	0.000068186704	0.000004435214	0.000000279951
2	0.000253706992	0.000016584865	0.000001048204
3	0.001021904886	0.000068810603	0.000004381171
4	0.002835749783	0.000200597649	0.000012927030
5	0.006094659674	0.000461030103	0.000030182136
6	0.010902822253	0.000900000561	0.000060053100
7	0.016848922949	0.001558467343	0.000106300133
8	0.022865349108	0.002461479772	0.000172068272
9	0.027236929579	0.003621927609	0.000260117864
10	0.027664528063	0.005060732634	0.000374443752
11	0.021138993153	0.006825822896	0.000522350175
12	0.004038959994	0.008978660029	0.000714540487
13	-0.026458499168	0.011552795227	0.000962971589
14	-0.070187340885	0.014519291951	0.001278911138

eig	grid=$\pi/160$	grid=$\pi/320$	grid=$\pi/640$
0	-0.000000018396	-0.000000001150	-0.000000000072
1	0.000000017540	0.000000001097	0.000000000069
2	0.000000065696	0.000000004109	0.000000000257
3	0.000000275096	0.000000017213	0.000000001076
4	0.000000814146	0.000000050982	0.000000003188
5	0.000001908374	0.000000119620	0.000000007482
6	0.000003815113	0.000000239420	0.000000014979
7	0.000006789925	0.000000426687	0.000000026704
8	0.000011057102	0.000000695886	0.000000043568
9	0.000016824216	0.000001060566	0.000000066428
10	0.000024389541	0.000001540171	0.000000096510
11	0.000034287420	0.000002169388	0.000000136003
12	0.000047310718	0.000002999841	0.000000188167
13	0.000064384582	0.000004092350	0.000000256851
14	0.000086447213	0.000005509544	0.000000346033

TABLE 2 (CONTINUED)

eig	grid=$\pi/1280$	grid=$\pi/2560$	grid=$\pi/5120$
0	-0.000000000004	0.000000000000	0.000000000000
1	0.000000000004	0.000000000000	0.000000000000
2	0.000000000016	0.000000000001	0.000000000000
3	0.000000000067	0.000000000004	0.000000000000
4	0.000000000199	0.000000000012	0.000000000001
5	0.000000000468	0.000000000029	0.000000000002
6	0.000000000936	0.000000000059	0.000000000004
7	0.000000001670	0.000000000104	0.000000000007
8	0.000000002724	0.000000000170	0.000000000011
9	0.000000004154	0.000000000260	0.000000000016
10	0.000000006036	0.000000000377	0.000000000024
11	0.000000008507	0.000000000532	0.000000000033
12	0.000000011771	0.000000000736	0.000000000046
13	0.000000016070	0.000000001005	0.000000000063
14	0.000000021654	0.000000001354	0.000000000085

eig	grid=$\pi/10240$	grid=$\pi/20480$	grid=$\pi/40960$
0	0.000000000000	0.000000000000	0.000000000000
1	0.000000000000	0.000000000000	0.000000000000
2	0.000000000000	0.000000000000	0.000000000000
3	0.000000000000	0.000000000000	0.000000000000
4	0.000000000000	0.000000000000	0.000000000000
5	0.000000000000	0.000000000000	0.000000000000
6	0.000000000000	0.000000000000	0.000000000000
7	0.000000000000	0.000000000000	0.000000000000
8	0.000000000001	0.000000000000	0.000000000000
9	0.000000000001	0.000000000000	0.000000000000
10	0.000000000001	0.000000000000	0.000000000000
11	0.000000000002	0.000000000000	0.000000000000
12	0.000000000003	0.000000000000	0.000000000000
13	0.000000000004	0.000000000000	0.000000000000
14	0.000000000005	0.000000000000	0.000000000000

TABLE 3 : C_{11} OF SYSTEM 2

($a = 1.5$, $b = 0.5$, $d = \pi/20$)

gridsize	constant
$\pi/20$	3.382238904480
$\pi/40$	17.47410661376
$\pi/160$	21.39546316800
$\pi/320$	22.47060357120
$\pi/640$	22.74768191488
$\pi/1280$	22.81751707648
$\pi/2560$	22.83580424192
$\pi/5120$	22.84922601472
$\pi/10240$	22.67742732288
$\pi/20480$	21.99023255552

TABLE 4 : CONSTANT C_j OF SYSTEM 2

($a = 1.5$, $b = 0.5$, $d = \pi/20$, $grid = \pi/640$)

eig	constant
0	-0.01205862400
1	0.01150287872
2	0.04308598784
3	0.18049138688
4	0.53458501632
5	1.25430661120
6	2.51050065920
7	4.47413747712
8	7.29689358336
9	11.12084054016
10	16.14986346496
11	22.74768191488
12	31.45561276416
13	42.91139993600
14	57.77175609344

TABLE 5 : EIGENVALUES OF SYSTEM 1

eigenvalue	a=1.5, b=0.5, d=π/20	a=1.5, b=0.5, d=π/5
0	0.19580682310081	0.25453518163183
1	1.5331849949472	1.8966688083167
2	4.9048225382094	4.7043829746856
3	10.446349508605	8.1565250534583
4	18.114377373458	14.594351841884
5	27.810687688601	25.542353551058
6	39.378433068084	39.235588639987
7	52.592723485642	51.254197254521
8	67.186365762428	61.607223406172
9	82.965139598281	77.279468280663
10	100.000000000000	100.54482961176
11	118.70156388250	126.55437680324
12	139.61053224029	147.80341478801
13	163.10531200978	165.05804583091
14	189.31737666681	189.96170701152

TABLE 6 : EIGENVALUES OF SYSTEM 2

eigenvalue	a=1.5, b=0.5, d=π/20	a=1.5, b=0.5, d=π/5
0	-0.4307295278926	-0.5463115991602
1	2.1618798758777	2.4261144985530
2	5.2023572403681	5.5130953402817
3	10.488151536616	8.709182622613
4	18.108902080508	14.478817137700
5	27.811051020815	25.154929353908
6	39.410628671035	39.087497504161
7	52.677444535286	51.677732641333
8	67.336284028433	62.019486164317
9	83.172093514830	77.107025223585
10	100.22657150895	100.12334482531
11	118.89701014840	126.38349426037
12	139.74101418105	148.21141027459
13	163.16540647643	165.46099066982
14	189.31886796319	189.78248770538

FIGURE 1: FIFTH EIGENFUNCTION FOR SYSTEM 1

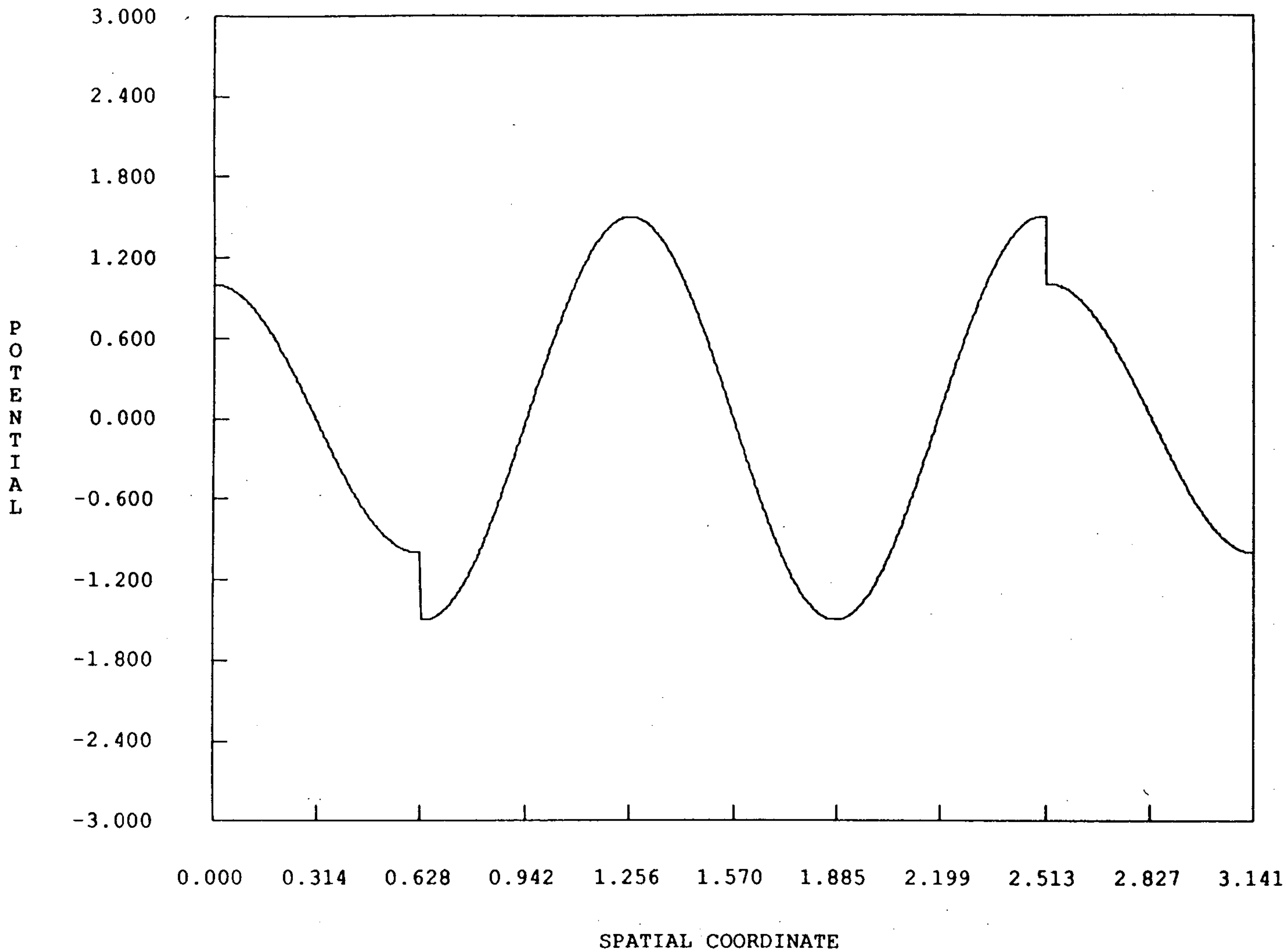
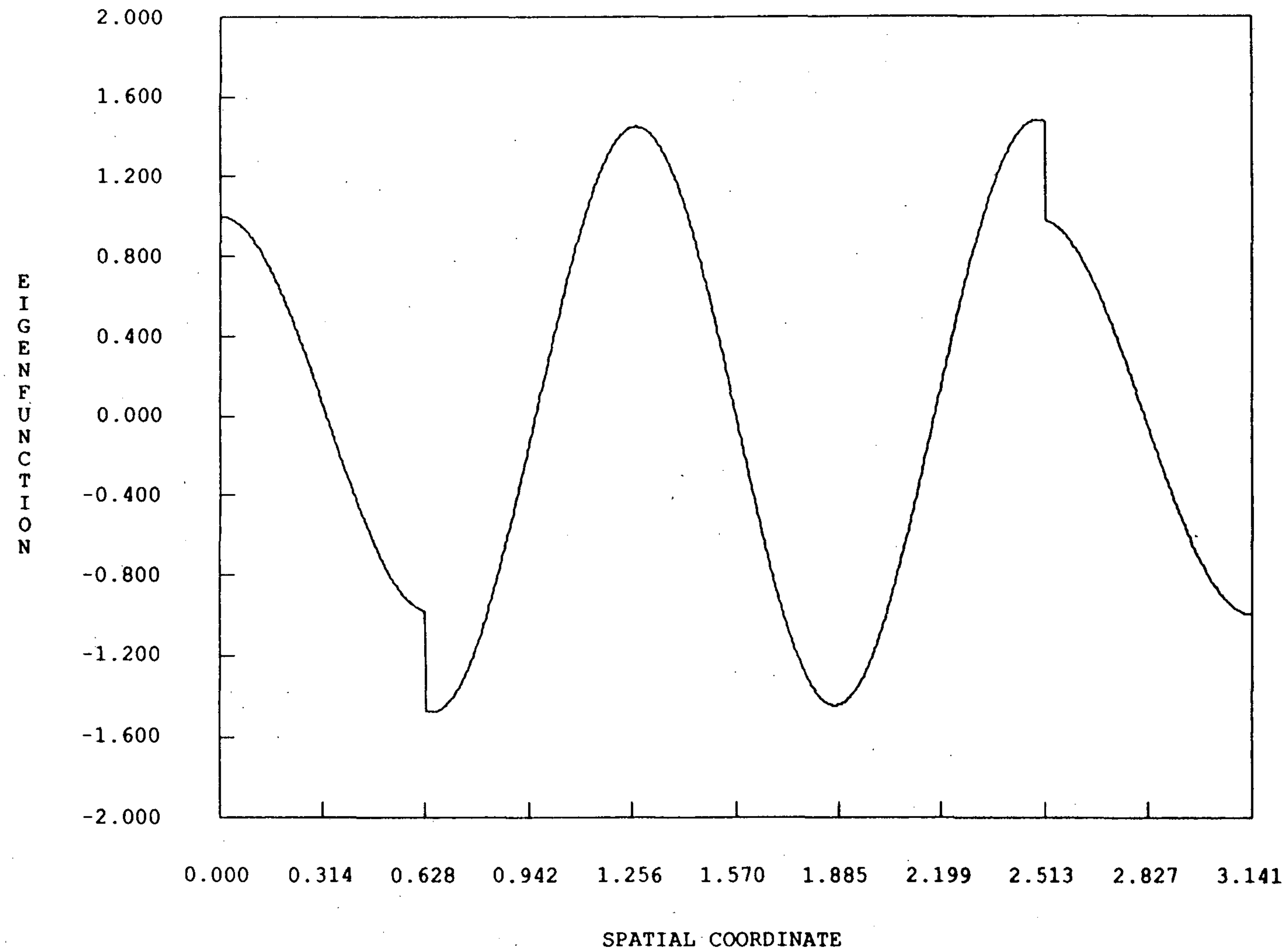


FIGURE 2: FIFTH EIGENFUNCTION FOR SYSTEM 2



To illustrate the dependence of the eigenvalues on the jump constants a and b, we select the second eigenvalue from system 2 and show how it varies as a and b range from 0.1 to 2.1 and -1.0 to 1.0 respectively. (See figure 3.) It is clear from the graph that for a given value of λ_i there is an associated level set of pairs $\{(a,b)\}$, i.e. the value of λ_2 does not uniquely determine (a,b). In addition we note that the eigenvalue increases with an increase in either or both a and b.

We have tried to reconstruct the potential function $q = 2 \cos 2x$ (figure 4) using the data from tables 5 and 6 and the Hochstadt-Hald Algorithm. The results are given on the next several pages in figures 5 - 7 and 12 - 14. The value of b we calculate for the Matthieu system using the zeroth through the fourteenth eigenvalues is given in tables 5 and 7. In addition the L_1-error, L_2-error and L_∞-error is also given along with graphs to show the change in b and the error with the number of eigenvalues used in the reconstruction. (See tables 8 and 10 and figures 8 - 11 and 15 - 18.)

FIGURE 3 : SECOND EIGENVALUE OF MATTHIEU'S EQUATION

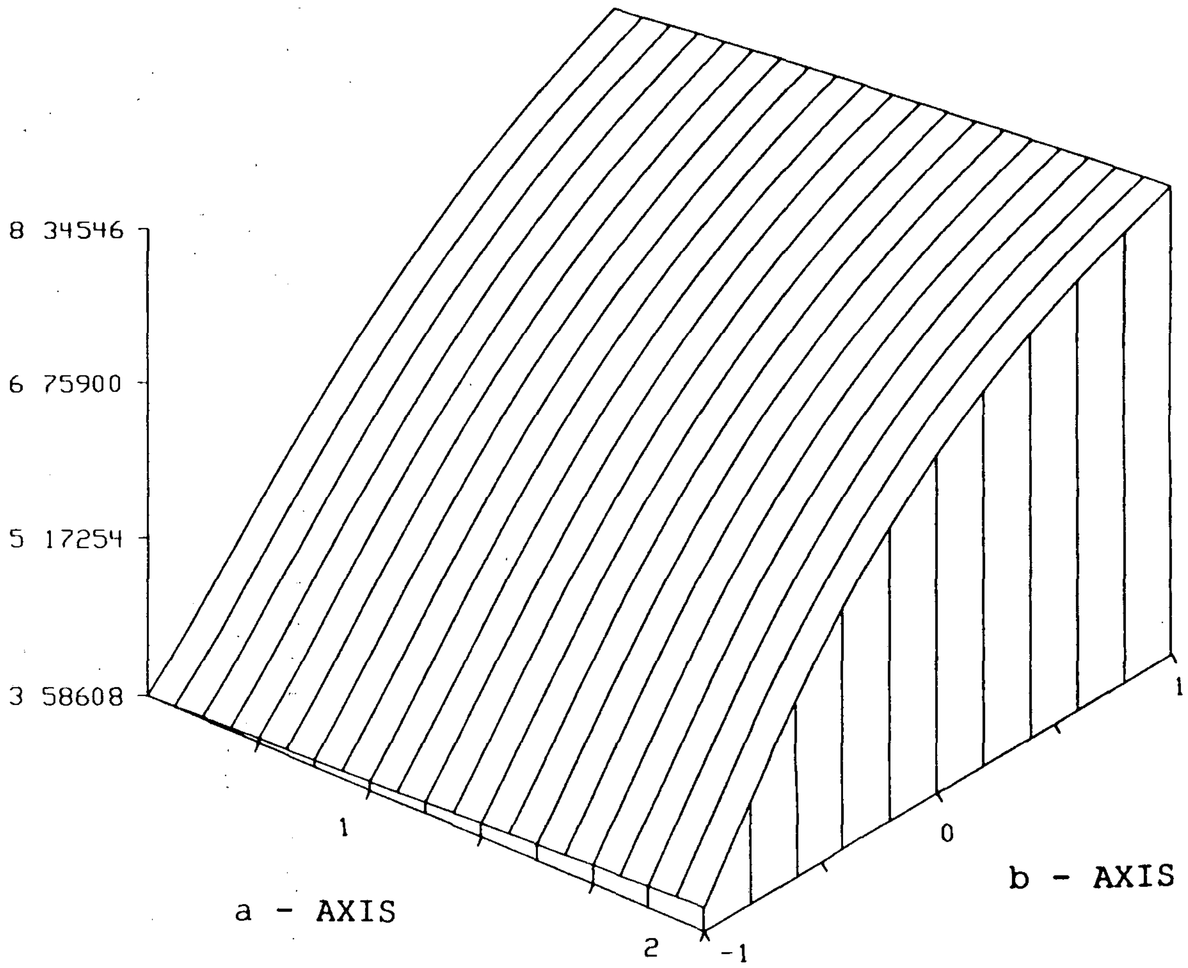


FIGURE 4 : 2 COS 2X

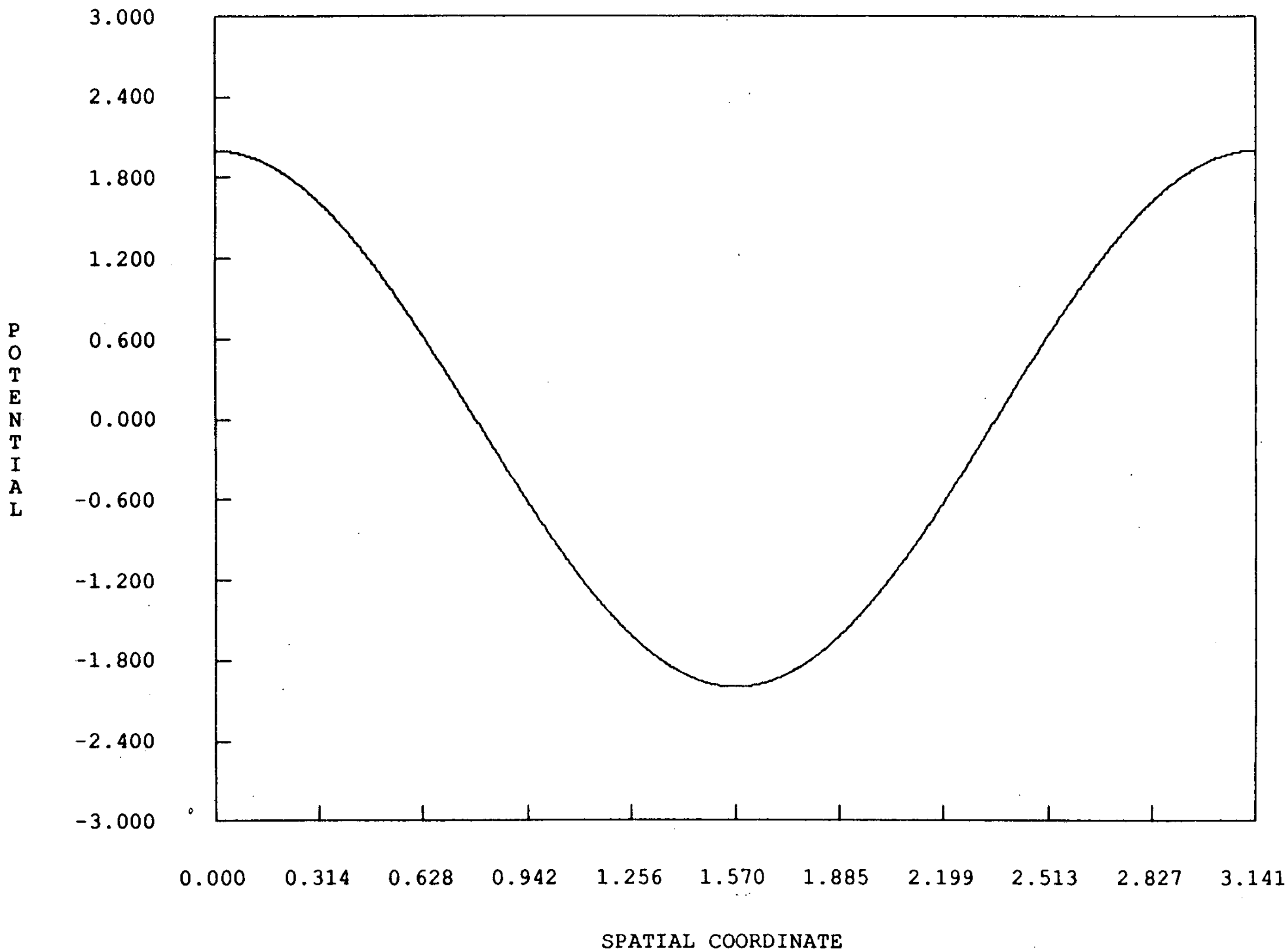


FIGURE 5 : FIVE EIGENVALUES, a = 1.5, b = 0.5, d = pi/5

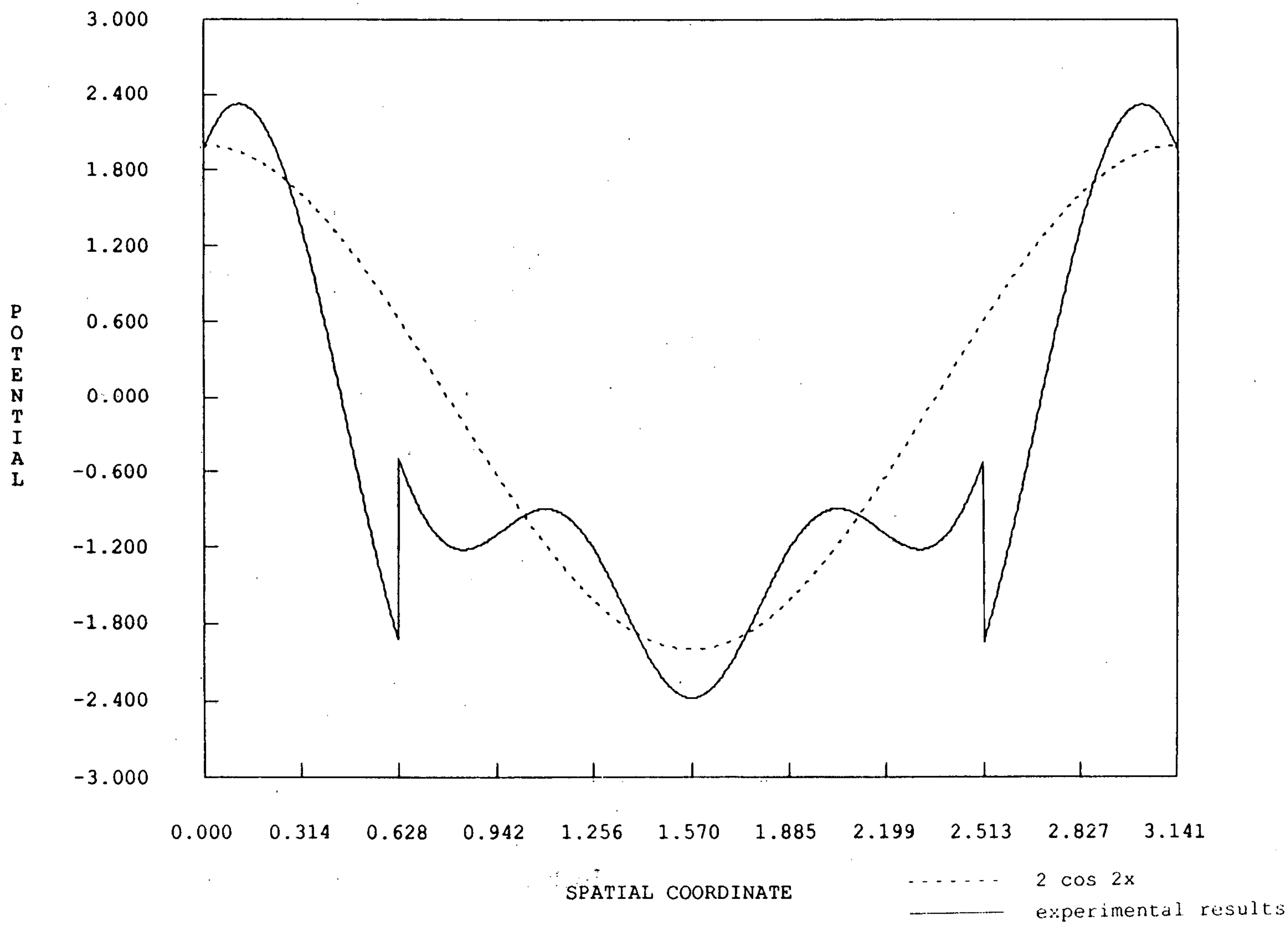


FIGURE 6 : TEN EIGENVALUES, a = 1.5, b = 0.5, d = pi/5

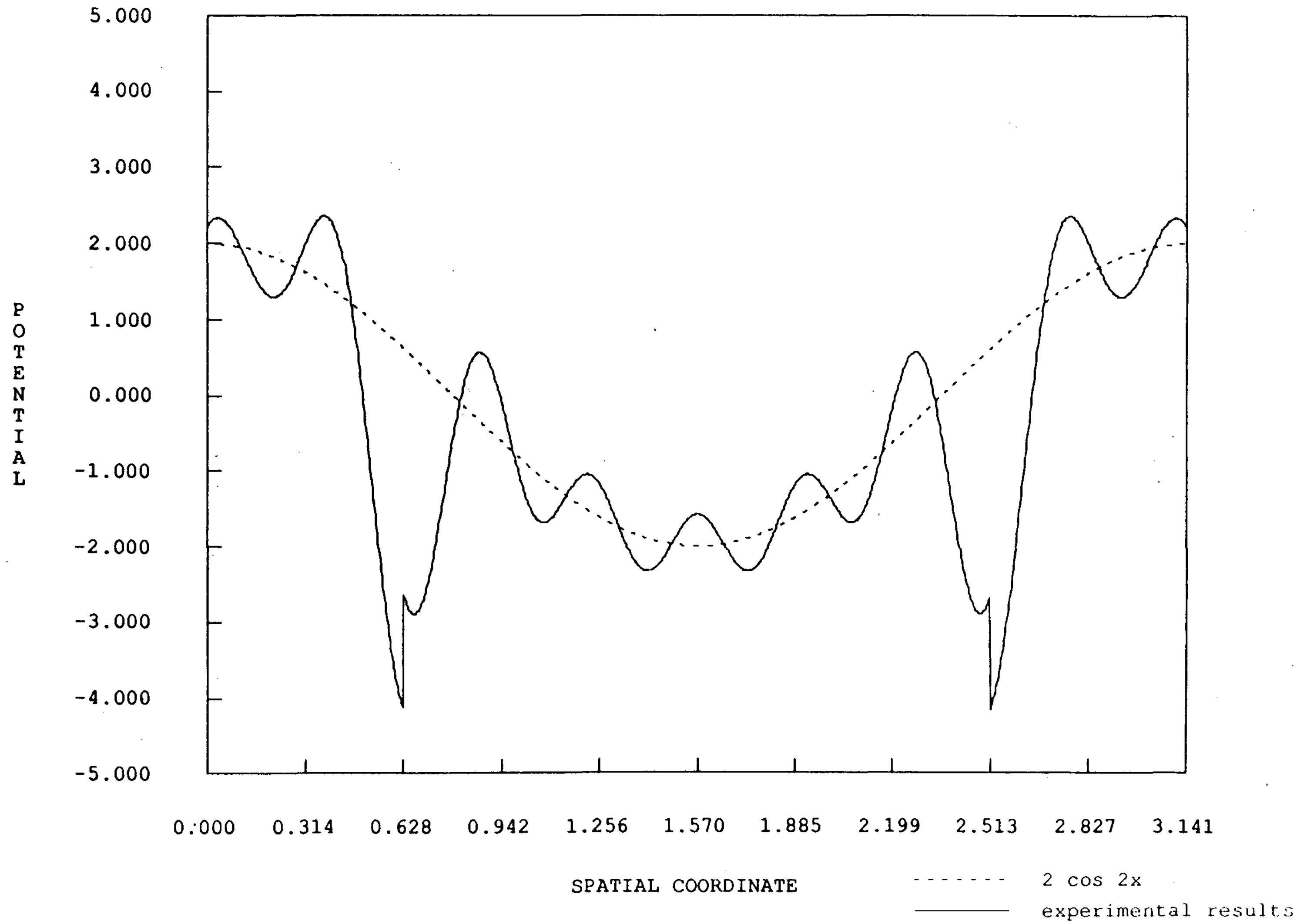


FIGURE 7 : FIFTEEN EIGENVALUES, a = 1.5, b = 0.5, d = pi/5

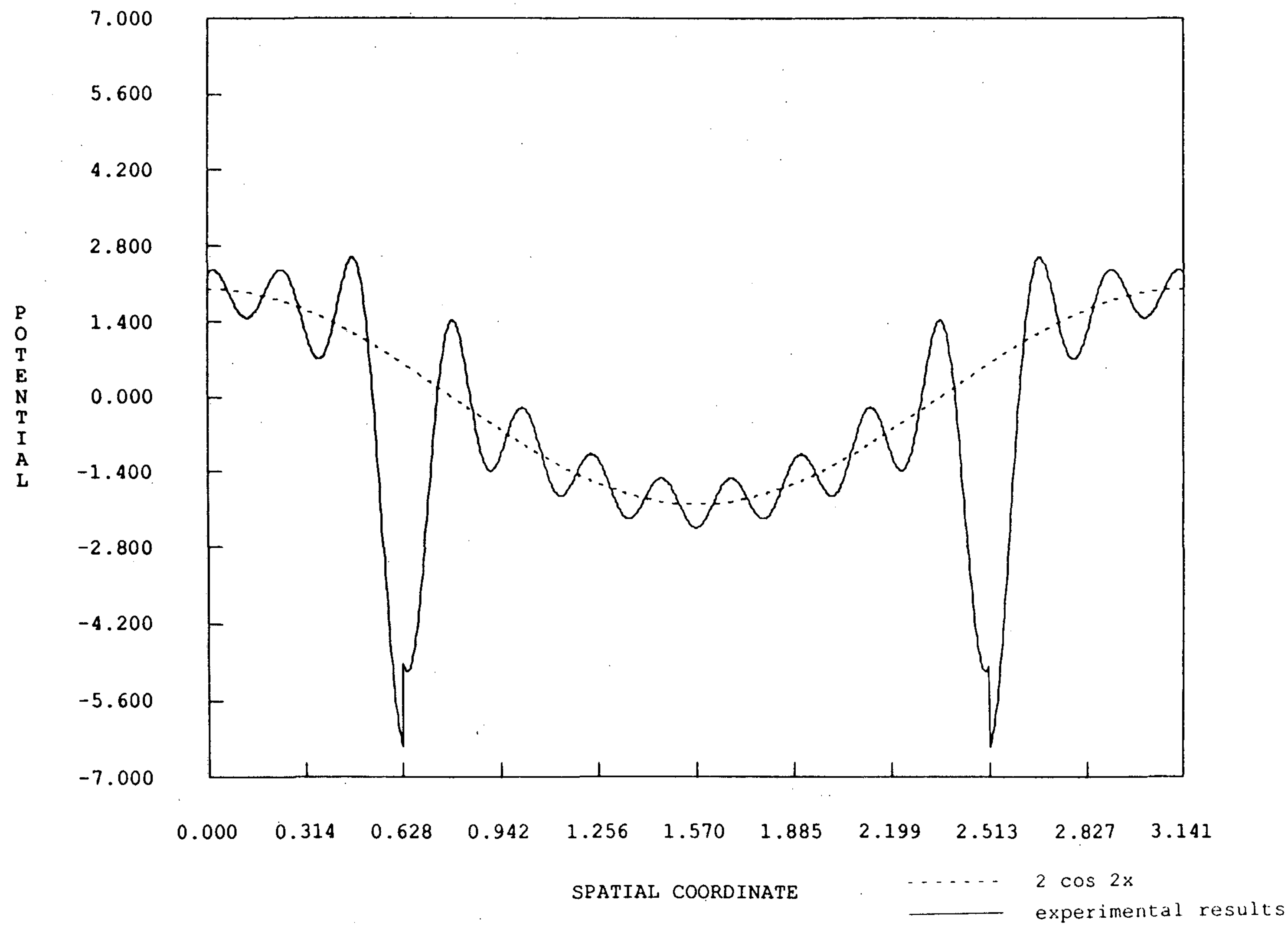


FIGURE 8 : NEW VALUE OF b (a = 1.5 , b = 0.5 , d = pi/5)

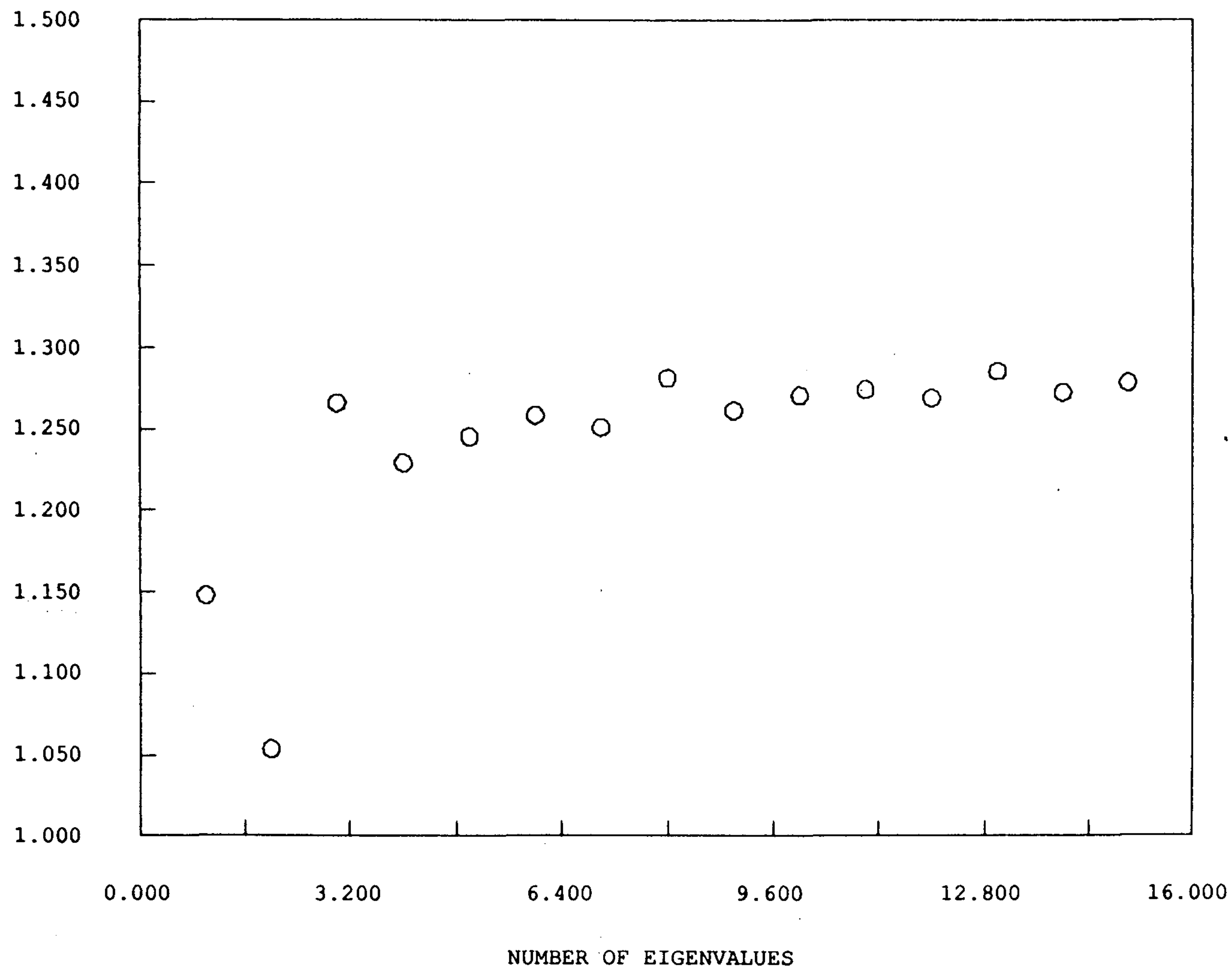


FIGURE 9 : L1-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/5)

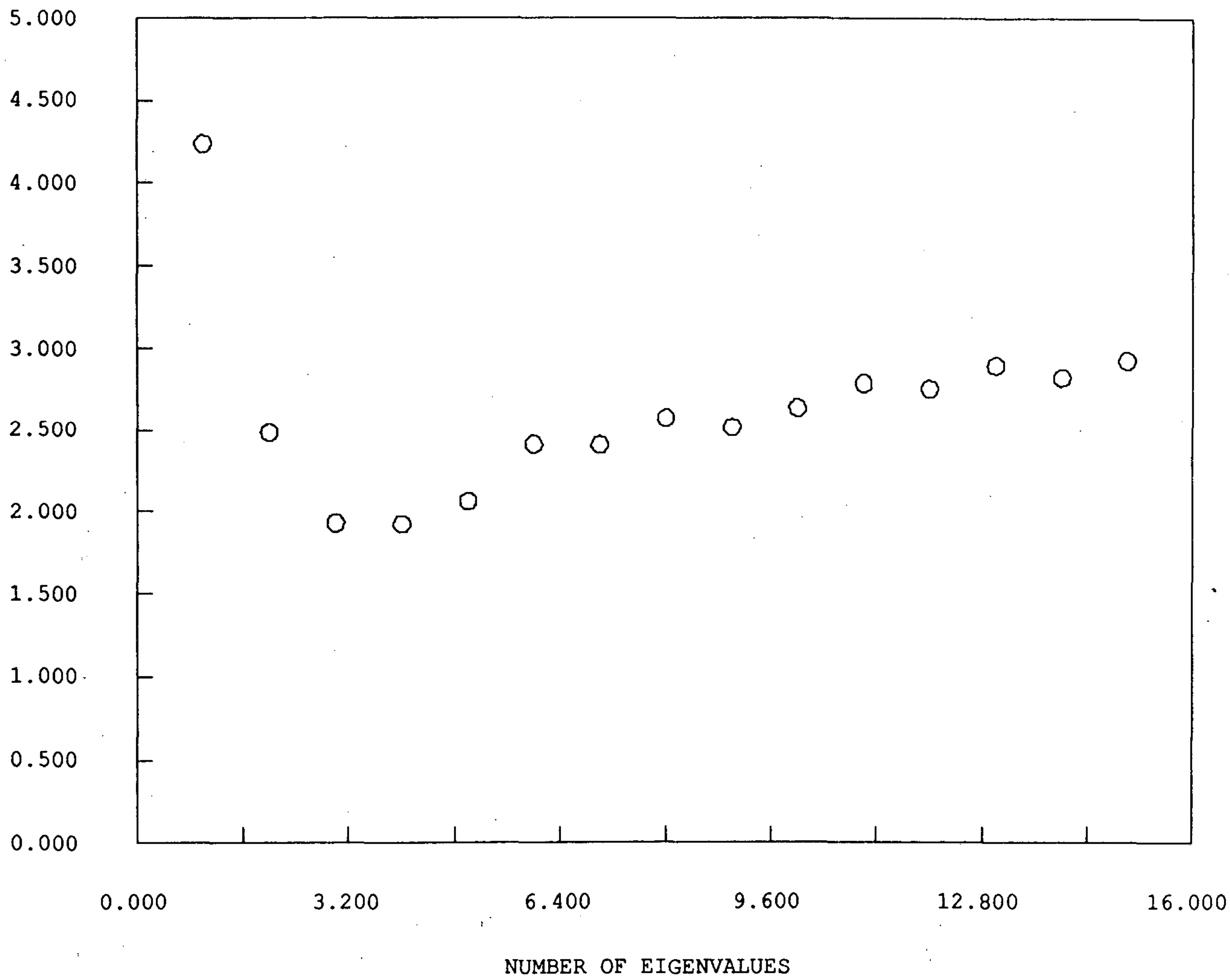


FIGURE 10 : L2-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/5)

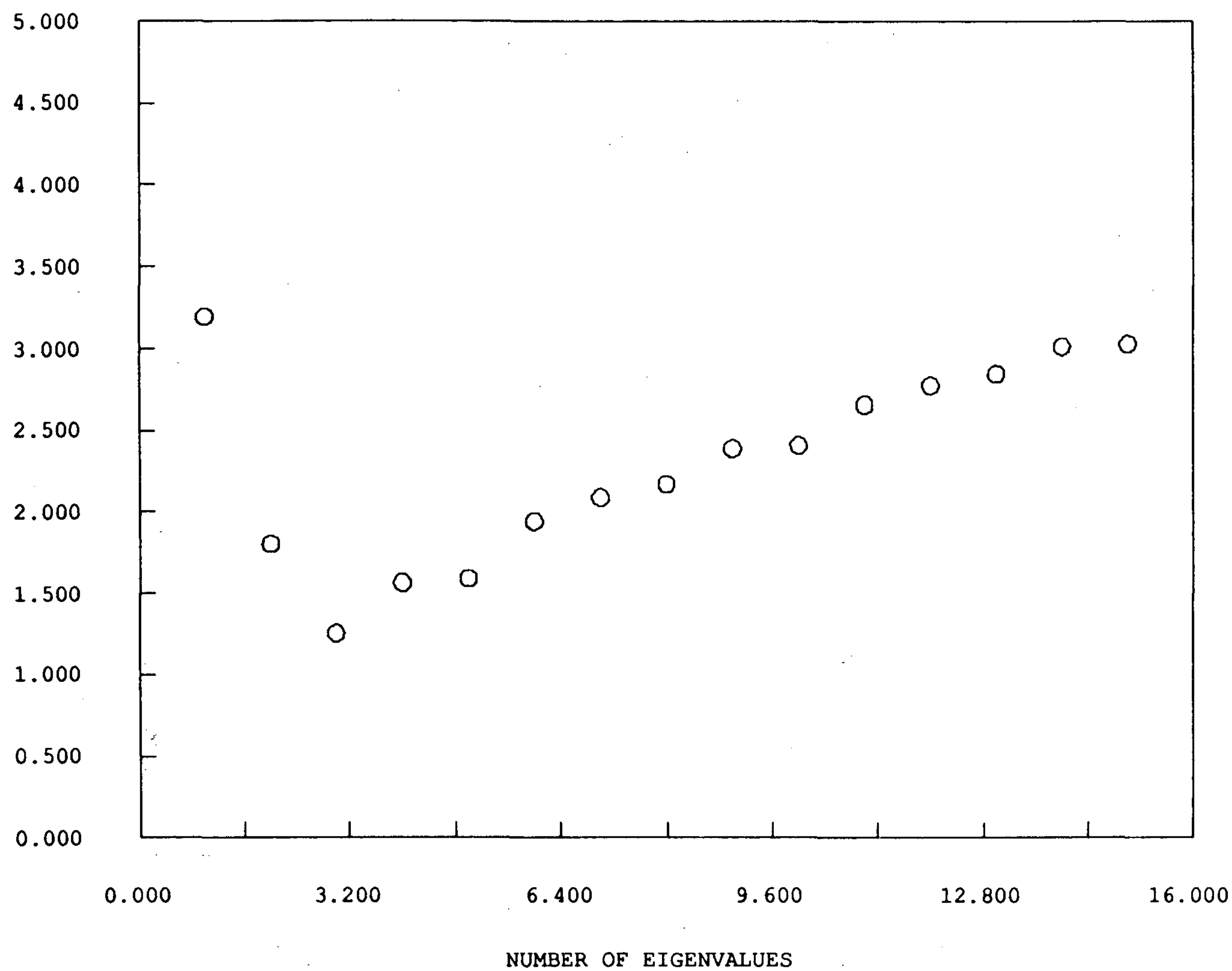


FIGURE 11 : L_∞-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/5)

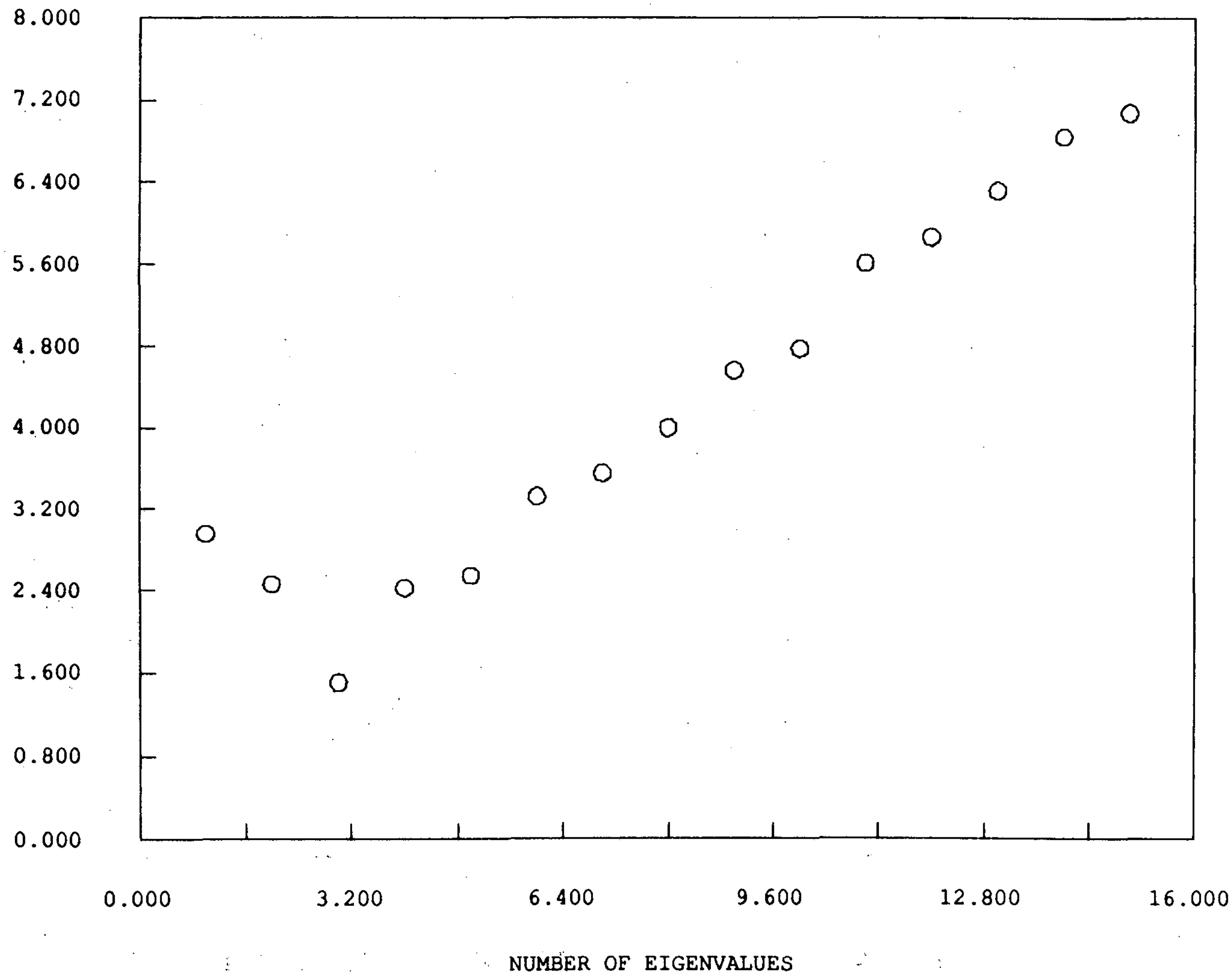


TABLE 7 : RECONSTRUCTION OF : Q = 2 COS 2X

($a = 1.5$, $b = 0.5$, $d = \pi/5$, $grid = \pi/10000$)

eigenvalue	new b
0	1.148217718713521
1	1.054319356185690
2	1.266420407635518
3	1.229750494977116
4	1.245866928565473
5	1.258908029382008
6	1.251501892241937
7	1.281778867173037
8	1.261721409028434
9	1.271127050503889
10	1.274724254989807
11	1.269138274868165
12	1.285815580635267
13	1.272696825343071
14	1.278732751555410

TABLE 8 : RECONSTRUCTION OF : Q = 2 COS 2X

($a = 1.5$, $b = 0.5$, $d = \pi/5$, $grid = \pi/10000$)

eig	L_1 error	L_2 error	L_∞ error
0	4.2412249084	3.1999062911	2.9630268875
1	2.4891398366	1.8090956878	2.4695365235
2	1.9347812003	1.2585929849	1.5128579849
3	1.9292193811	1.5708523916	2.4320546089
4	2.0663421053	1.5963579680	2.5562631898
5	2.4148064438	1.9474294406	3.3303886153
6	2.4111971300	2.0922577725	3.5588045112
7	2.5737685278	2.1753883996	4.0036474012
8	2.5185152781	2.3927573743	4.5634338555
9	2.6397378124	2.4132399886	4.7770698411
10	2.7898023155	2.6621228419	5.6196240319
11	2.7541517856	2.7762631510	5.8696647082
12	2.8982030253	2.8498528424	6.3254219521
13	2.8252136117	3.0159666159	6.8492081226
14	2.9292647401	3.0326486338	7.0794535669

FIGURE 12 : FIVE EIGENVALUES, a = 1.5, b = 0.5, d = pi/20

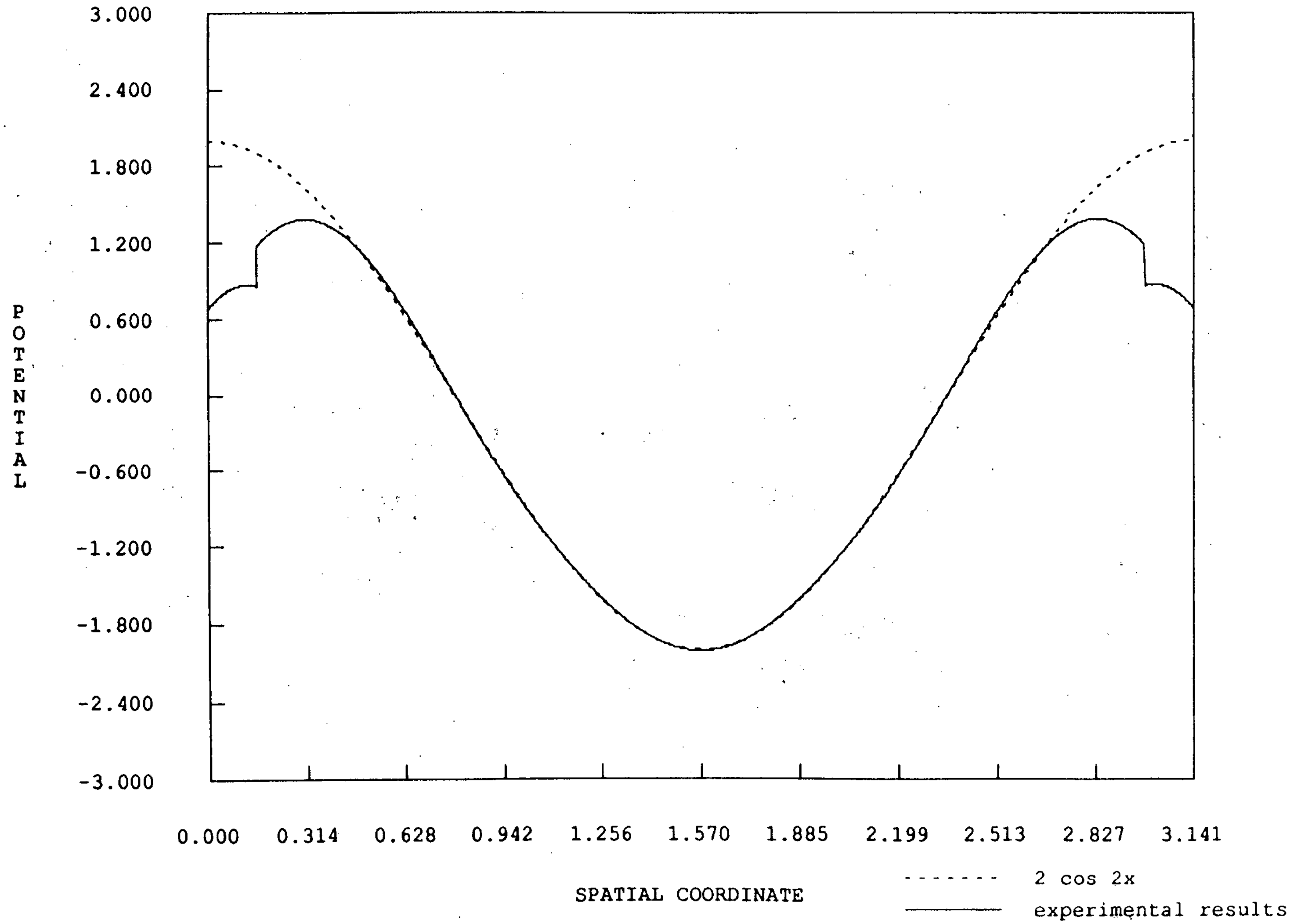


FIGURE 13 : TEN EIGENVALUES, a = 1.5, b = 0.5, d = pi/20

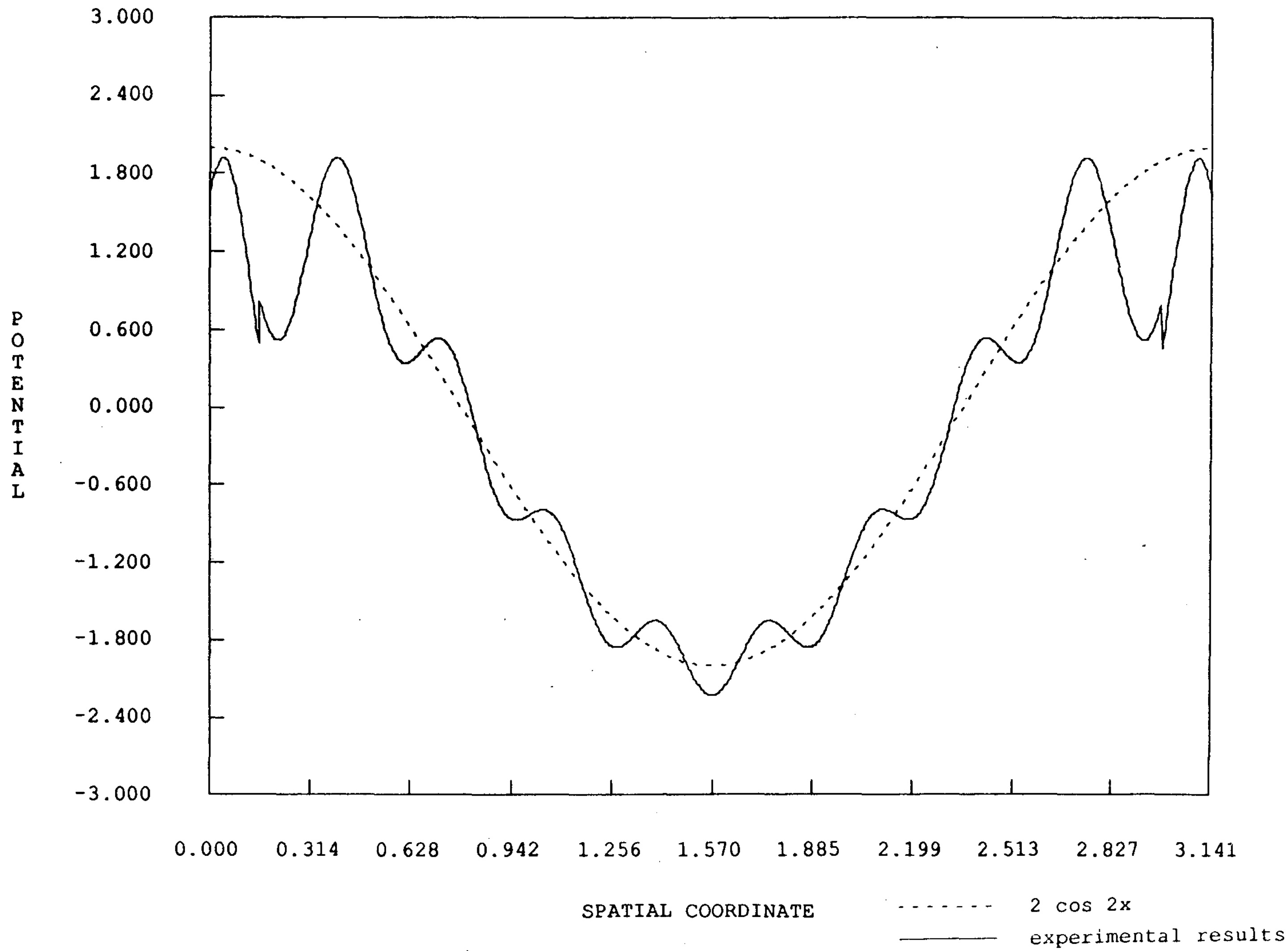


FIGURE 14 : FIFTEEN EIGENVALUES, a = 1.5, b = 0.5, d = pi/20

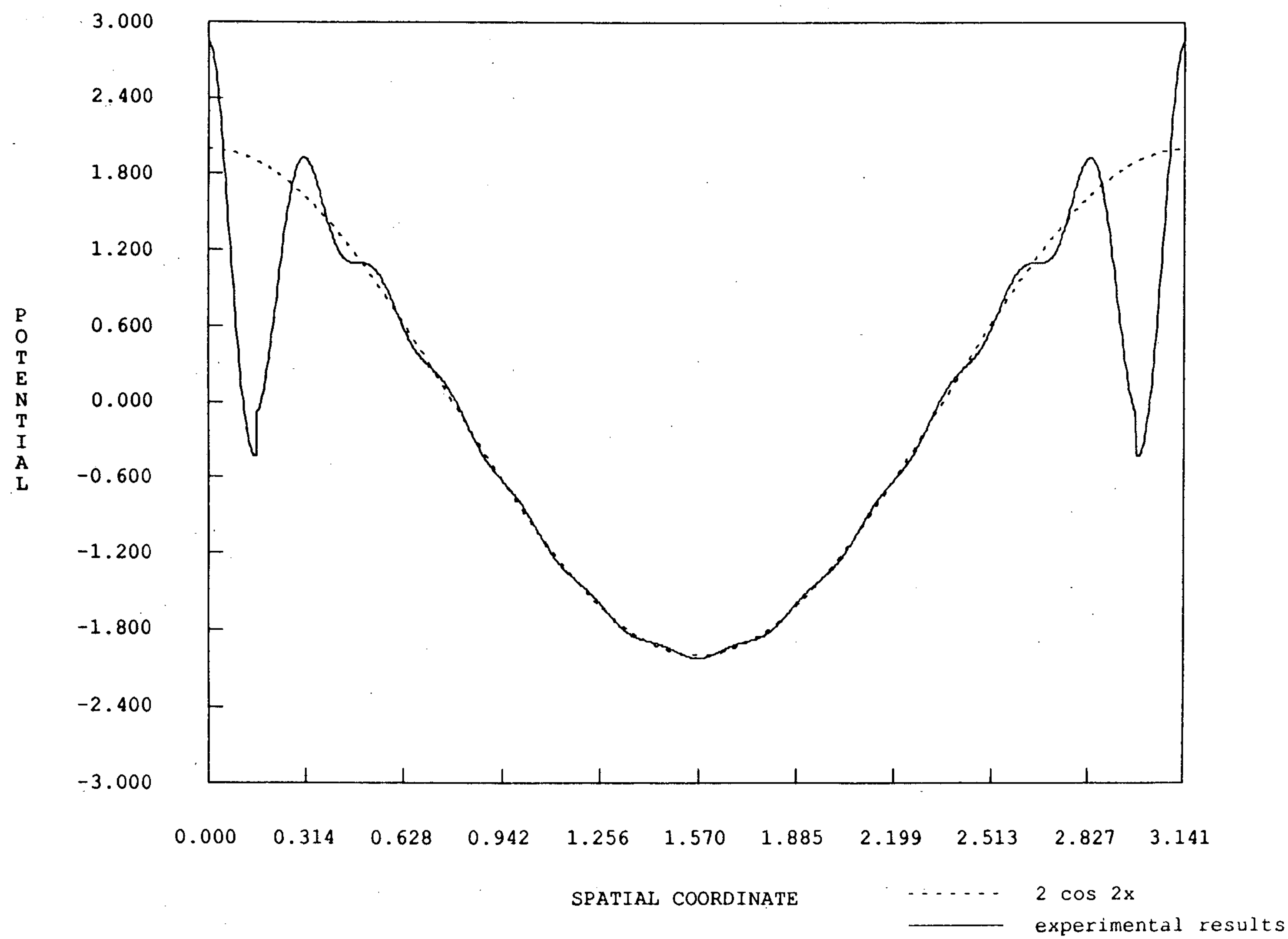


FIGURE 15 : NEW VALUE OF b (a = 1.5 , b = 0.5 , d = pi/20)

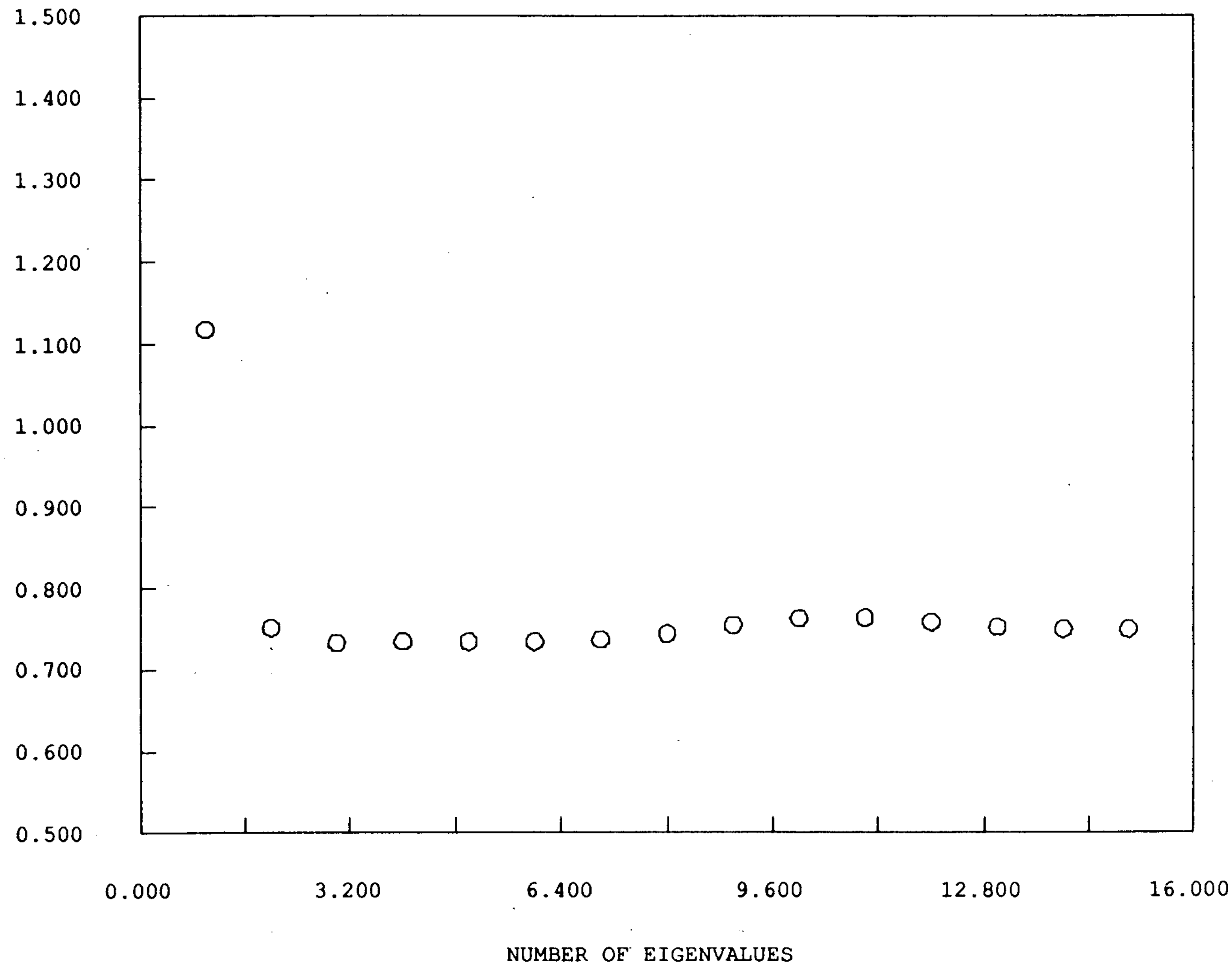


FIGURE 16 : L1-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/20)

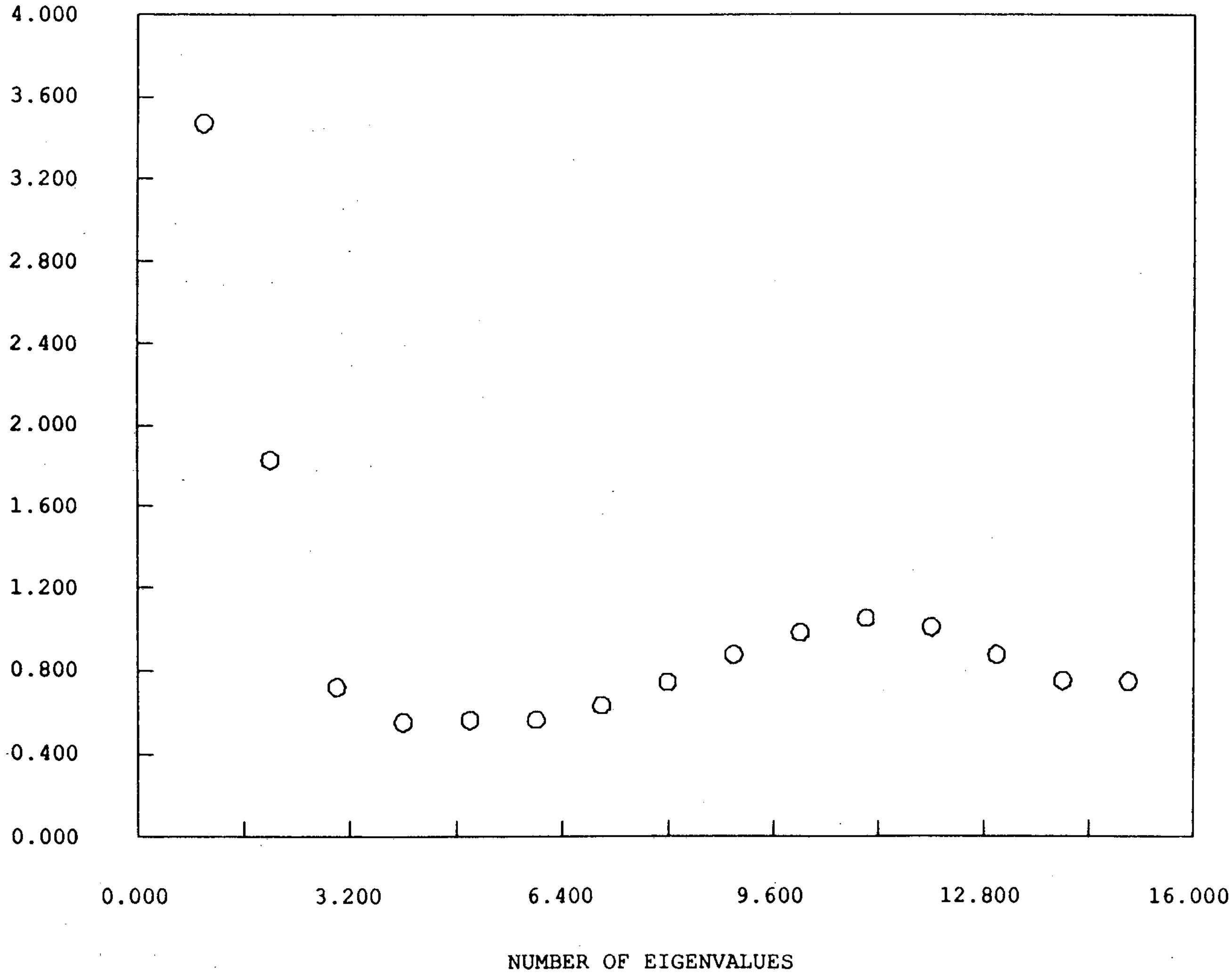


FIGURE 17 : L2-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/20)

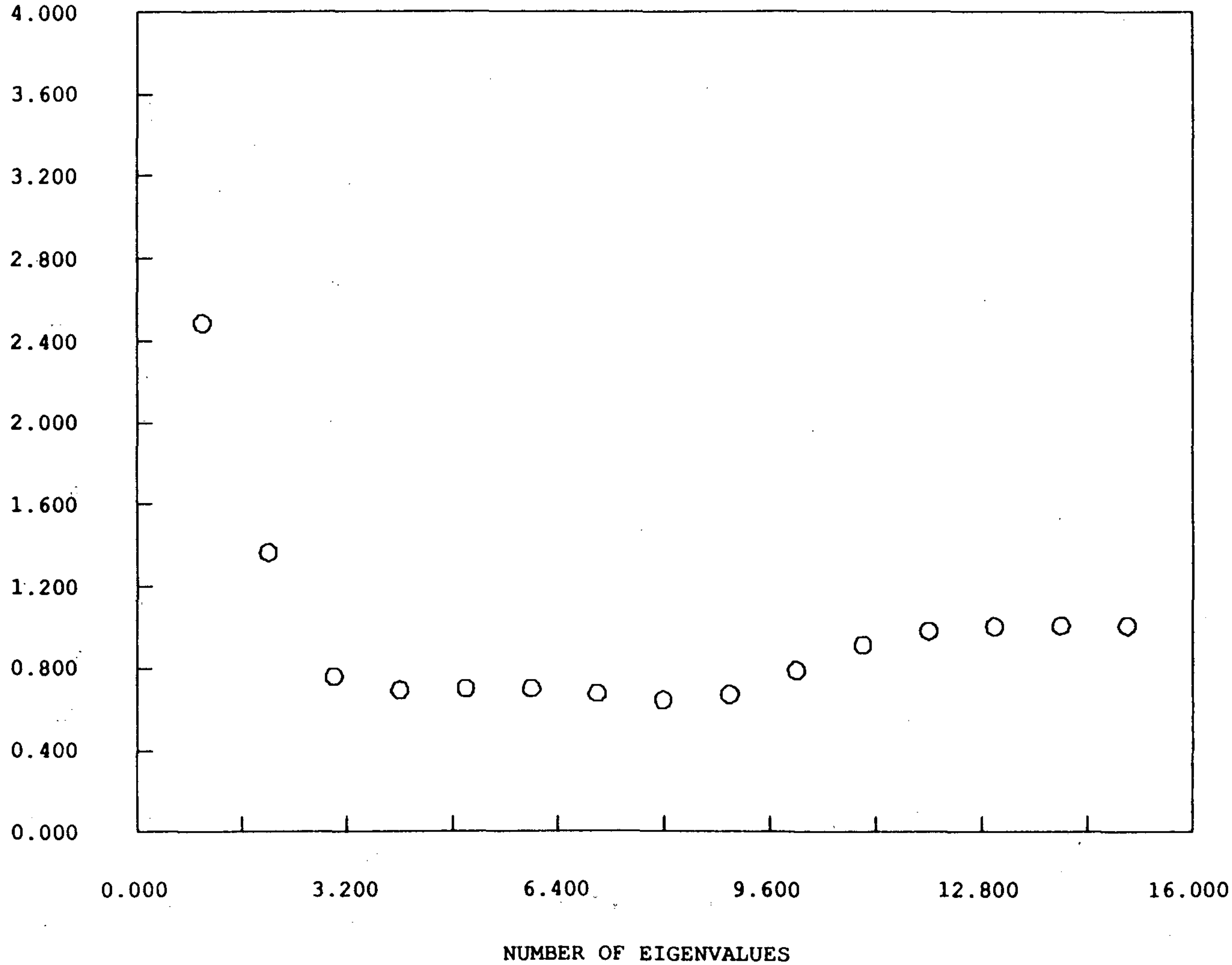


FIGURE 18 : L_{∞}-ERROR IN Q = 2 COS 2X (a = 1.5 , b = 0.5 , d = pi/20)

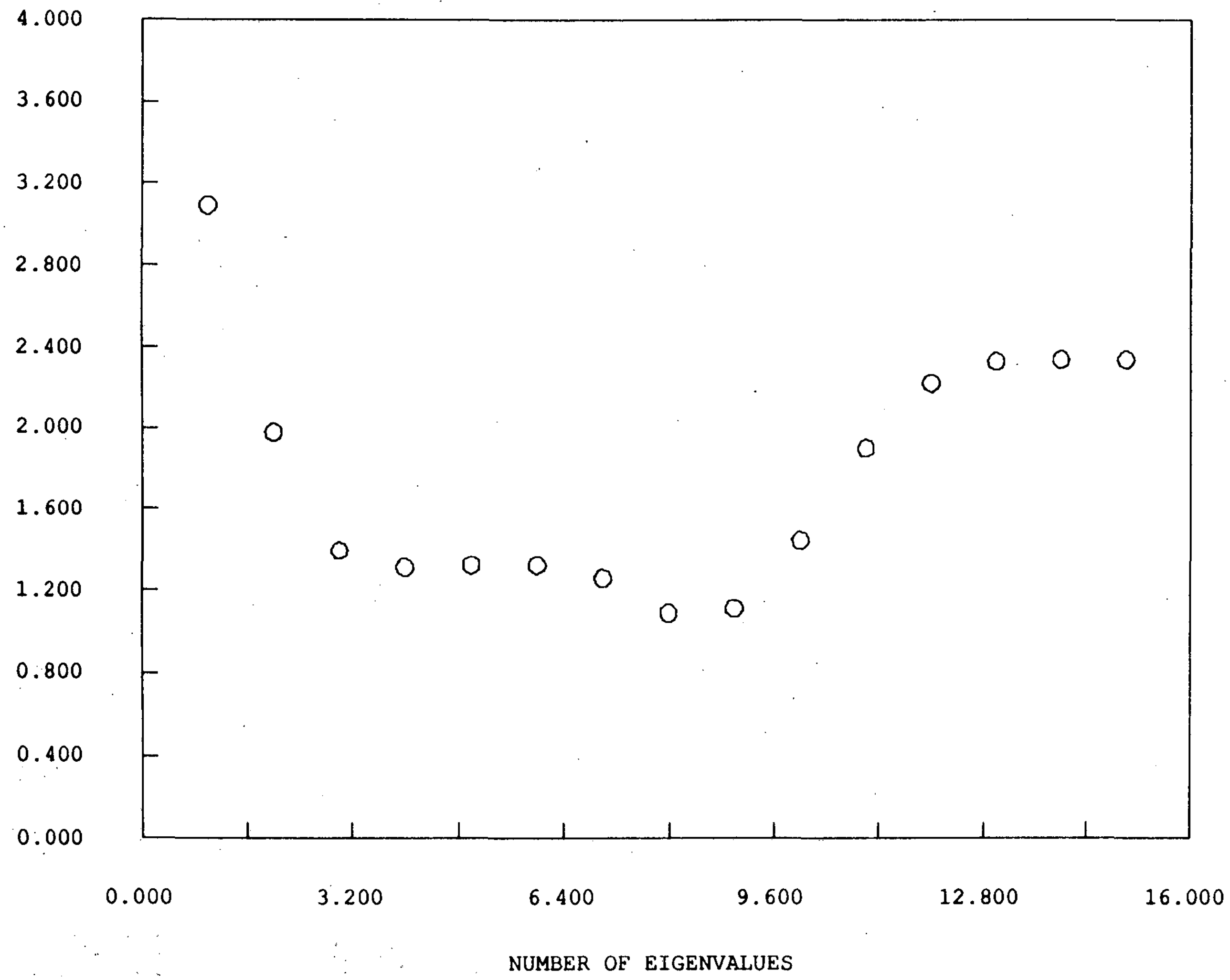


TABLE 9 : RECONSTRUCTION OF : Q = 2 COS 2X

($a = 1.5$, $b = 0.5$, $d = \pi/20$, $grid = \pi/10000$)

eig	new b
0	1.118571048747403
1	0.7524805275676702
2	0.7339579460273298
3	0.7355974229924883
4	0.7352377473681646
5	0.7352661758504616
6	0.7379721177627896
7	0.7449614016301399
8	0.7553770197577960
9	0.7639349390603760
10	0.7646762538516538
11	0.7589453110904047
12	0.7530256920980140
13	0.7502084130985343
14	0.7501471938792797

TABLE 10 : RECONSTRUCTION OF : Q = 2 COS 2X

($a = 1.5$, $b = 0.5$, $d = \pi/20$, $grid = \pi/10000$)

eig	L_1 error	L_2 error	L_∞ error
0	3.4737003836	2.4886824712	3.0970768929
1	1.8314569658	1.3694847299	1.9815521240
2	0.7241262909	0.7641748136	1.3990083933
3	0.5552553031	0.6968590175	1.3160430193
4	0.5655100463	0.7039719143	1.3269224167
5	0.5663492667	0.7035855987	1.3262004852
6	0.6349092528	0.6780527495	1.2621757984
7	0.7463757911	0.6459631670	1.0932894945
8	0.8784633734	0.6745447187	1.1151049137
9	0.9856518433	0.7943381214	1.4504045248
10	1.0537147652	0.9183167991	1.9040108919
11	1.0091627495	0.9850470136	2.2257614136
12	0.8793629339	1.0076081324	2.3328120708
13	0.7536904888	1.0106726120	2.3412837982
14	0.7489268648	1.0105892396	2.3408544064

As the number of eigenvalues used in the reconstruction increases to 4 or 5, the experimental results appear to converge toward the potential $q = 2 \cos 2x$. However as we pass to 6 or more eigenvalues the experimental results begin to oscillate about $q = 2 \cos 2x$. The cause for this behaviour is unclear, and several explanations have been suggested . First we note that the eigenvalues we use are generated assuming that $b = 0.5$. In the reconstruction algorithm we determine a new value for b which neither equals 0.5 nor converges towards 0.5. Next we note that the oscillations we observe for six or more eigenvalues appear to be related to the Gibbs phenomenon which has been observed in other similar numerical experiments [34].

Two tall, thin spikes are observed in the reconstructed potential function. These spikes occur at the points where the eigenfunctions are discontinuous and become more pronounced as the number of eigenvalues used in the reconstruction increases. This observation led us to search for an inherent error in our experiment. Ching-ju Lee at the University of California, Berkeley generated the first 100 eigenvalues for systems 1 and 2 to five or six digit accuracy and calculated the difference between the corresponding eigenvalues of the two systems. This difference did not converge to zero; it oscillated in a periodic manner about zero. The locations of the discontinuities and the jump constants were varied to see how the oscillatory patterns were changed. C. J. Lee's results are presented in tables 11 and 12 and figures 19 and 20. These findings point to a fundamental error in our assumptions. The Hochstadt-Hald algorithm is based on the assumption that the eigenvalues $\{\lambda_j\}$ and $\{\tilde{\lambda}_j\}$ for problems (3.1) and (3.2) are equal for $j > n$. In example 1 the corresponding eigenvalues for systems 1 and 2 are not equal for $j > n$, and furthermore their difference does not converge to zero.

FIGURE 19: a = 1.5 , b = - 0.5 , d = pi/5

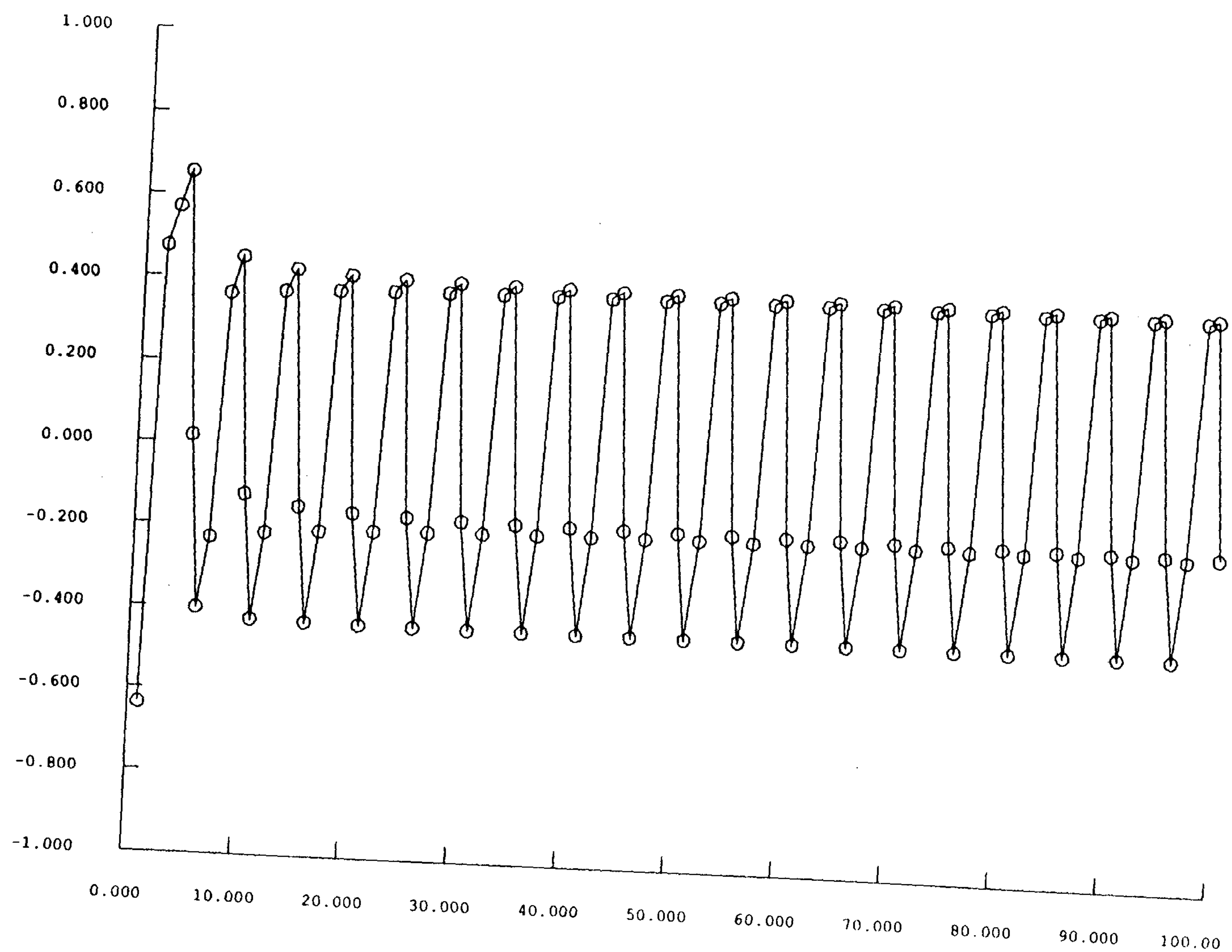


TABLE 11

Comparison Between Spectra

q1 = 2.0 * cos(2.0 * x)

q2 = 0.0

First spectrum is obtained from Runge-Kutta method of order four.

Mesh sizes: h1 = 0.998916583017e-03, h2 = 0.999446231259e-03

Minimum and maximum entered: -2.000000000000000 2.000000000000000

a = 1.500000000000000 b = -.5000000000000000 d = 0.6283185307179587

Interval = [0. , 3.14159265]

n	First Spectrum	Second Spectrum	Difference
1	-.9295801923411719	-.2930773628517348	-.6365028294894371
2	1.570792626286364	1.095027685899663	0.4757649403867010
3	5.152098628598501	4.581179930442522	0.5709186981559790
4	8.690132732812006	8.033294753852839	0.6568379789591670
5	13.81396187721764	13.79502184136381	0.1894003585383008e-01
6	24.05029412566355	24.45171515639147	-.4014210307279202
7	38.20790052204555	38.43583519899387	-.2279346769483199
8	51.49741662055895	51.13089713636646	0.3665194841924899
9	61.93901039399759	61.48392213969305	0.4550882543045400
10	76.35846775175746	76.47951353575121	-.1210457839937487
11	99.02815676030265	99.45368324953430	-.4255264892316504
12	125.5412009661729	125.7543598209524	-.2131588547795005
13	148.0561518878900	147.6801072941558	0.3760445937341999
14	165.3646379712037	164.9347380868170	0.4298998843866997
15	189.0150431035434	189.1616463201021	-.1466032165586988
16	224.0245978134954	224.4540410894718	-.4294432759764000
17	262.8675558068356	263.0726993914605	-.2051435846249063
18	294.6108220131425	294.2292634070325	0.3815586061100049
19	318.8057485370657	318.3855891300604	0.4201594070053005
20	351.6862750718177	351.8434728628709	-.1571977910531999
21	399.0233862567019	399.4541658562524	-.4307795995505046
22	450.1906440123594	450.3910326441212	-.2003886317618040
23	491.1633650466546	490.7784027513201	0.3849622953344962
24	522.2514796464600	521.8364527261802	0.4150269202798000
25	564.3622329843479	564.5252061962439	-.1629732118960021
26	624.0228312639235	624.4542235222155	-.4313922582919929
27	687.5121059001909	687.7093723311625	-.1972664309715952
28	737.7147851747341	737.3275347352466	0.3872504394875023
29	775.6991881771625	775.2873222296883	0.4118659474741975
30	827.0402964882059	827.2068998281312	-.1666033399253024
31	899.0225313008136	899.4542548247964	-.4317235239827966
32	974.8326534438125	975.0277181177265	-.1950646739139899
33	1034.265551956898	1033.876662838531	0.3888891183670182
34	1079.147920934686	1078.738194981792	0.4097259528940072
35	1139.719478423813	1139.888573088138	-.1690946643250015
36	1224.022350944389	1224.454273691740	-.4319227473510239
37	1312.152638264284	1312.346068542294	-.1934302780099983
38	1380.815907491204	1380.425788646233	0.3901188449709991
39	1432.597251663583	1432.189069710338	0.4081819532449913
40	1502.399324777447	1502.570234570177	-.1709097927300149

41	1599.022234091430	1599.454285934133	-.4320518427030038
42	1699.472252802954	1699.664422398840	-.1921695958860141
43	1777.365988072403	1776.974912983513	0.3910750888900054
44	1836.046961517220	1835.639945731351	0.4070157858690209
45	1915.079597794059	1915.251888652352	-.1722908582929961
46	2024.022154070603	2024.454294326109	-.4321402555059990
47	2136.791610946276	2136.982778817676	-.1911678713999549
48	2223.915875991327	2223.524036322947	0.3918396683800438
49	2289.496926708361	2289.090822643513	0.4061040648480230
50	2377.760160989985	2377.933537791239	-.1733768012539940
51	2499.022096878266	2499.454300328292	-.4322034500260088
52	2624.110784304287	2624.301137182475	-.1903528781879800
53	2720.465623773602	2720.073158953562	0.3924648200400043
54	2792.947071996134	2792.541700195834	0.4053718003000313
55	2890.440930450016	2890.615183468712	-.1742530186959925
56	3024.022054587838	3024.454304768627	-.4322501807890262
57	3161.429820129303	3161.619497052819	-.1896769235160036
58	3267.015266484811	3266.622281062965	0.3929854218459923
59	3346.397349031841	3345.992578223865	0.4047708079759786
60	3453.121851751859	3453.296826629148	-.1749748772889461
61	3599.022022436409	3599.454308145824	-.4322857094149981
62	3748.748750846701	3748.937858108824	-.1891072621230023
63	3863.564828413811	3863.171402777634	0.3934256361769712
64	3949.847725337779	3949.443456614822	0.4042687229569992
65	4065.802888065633	4065.978467906713	-.1755798410799798
66	4224.021997423511	4224.454310773841	-.4323133503299914
67	4386.067599440880	4386.256220113166	-.1886206722859924
68	4510.114326907561	4509.720524186562	0.3938027209990196
69	4603.298178293211	4602.894335289014	0.4038430041970287
70	4728.484013570600	4728.660107734417	-.1760941638169697
71	4899.021977581816	4899.454312859064	-.4323352772479438
72	5073.386382655464	5073.574582887232	-.1882002317680644
73	5206.663774687417	5206.269645353627	0.3941293337900333
74	5306.748691660705	5306.345214188477	0.4034774722280190
75	5441.165209639076	5441.341746426381	-.1765367873049399
76	5624.021961577948	5624.454314541267	-.4323529633189764
77	5810.705112975082	5810.892946294057	-.1878333189749810
78	5953.213181288043	5952.818766326111	0.3944149619319433
79	6060.199253483649	6059.796093269552	0.4031602140970563
80	6203.846462489591	6204.023384210768	-.1769217211769956
81	6399.021948481355	6399.454315916540	-.4323674351850286
82	6598.023799897171	6598.211310226697	-.1875103295260487
83	6749.762553997519	6749.367887139736	0.3946668577830224
84	6863.649854765242	6863.246972500973	0.4028822642690102
85	7016.527761707851	7016.705021259049	-.1772595511979489
86	7224.021937630330	7224.454317058974	-.4323794286440261
87	7435.342450778325	7435.529674602969	-.1872238246439792
88	7596.311898481312	7595.917007822003	0.3948906593089987
89	7717.100488603278	7716.697851855869	0.4026367474090193
90	7879.209099281269	7879.386657702205	-.1775584209359522
91	8099.021928539090	8099.454318015119	-.4323894760290159
92	8322.661071394566	8322.848039352001	-.1869679574349448
93	8492.861219209553	8492.466128394052	0.3950908155009074
94	8620.551149616918	8620.148731314777	0.4024183021410863
95	8791.890468939592	8792.068293640532	-.1778247009399365
96	9024.021920842862	9024.454318824274	-.4323979814118957
97	9259.979666358875	9260.166404422426	-.1867380635510472
98	9439.410519756633	9439.015248873848	0.3952708827850984
99	9574.001833549787	9573.599610862342	0.4022226874451462
100	9754.571865709534	9754.749929154544	-.1780634450099114

Sum of the absolute difference = 32.62927924930268

FIGURE 20: a = 1.5 , b = 1.1 , d = pi/sqrt(24)

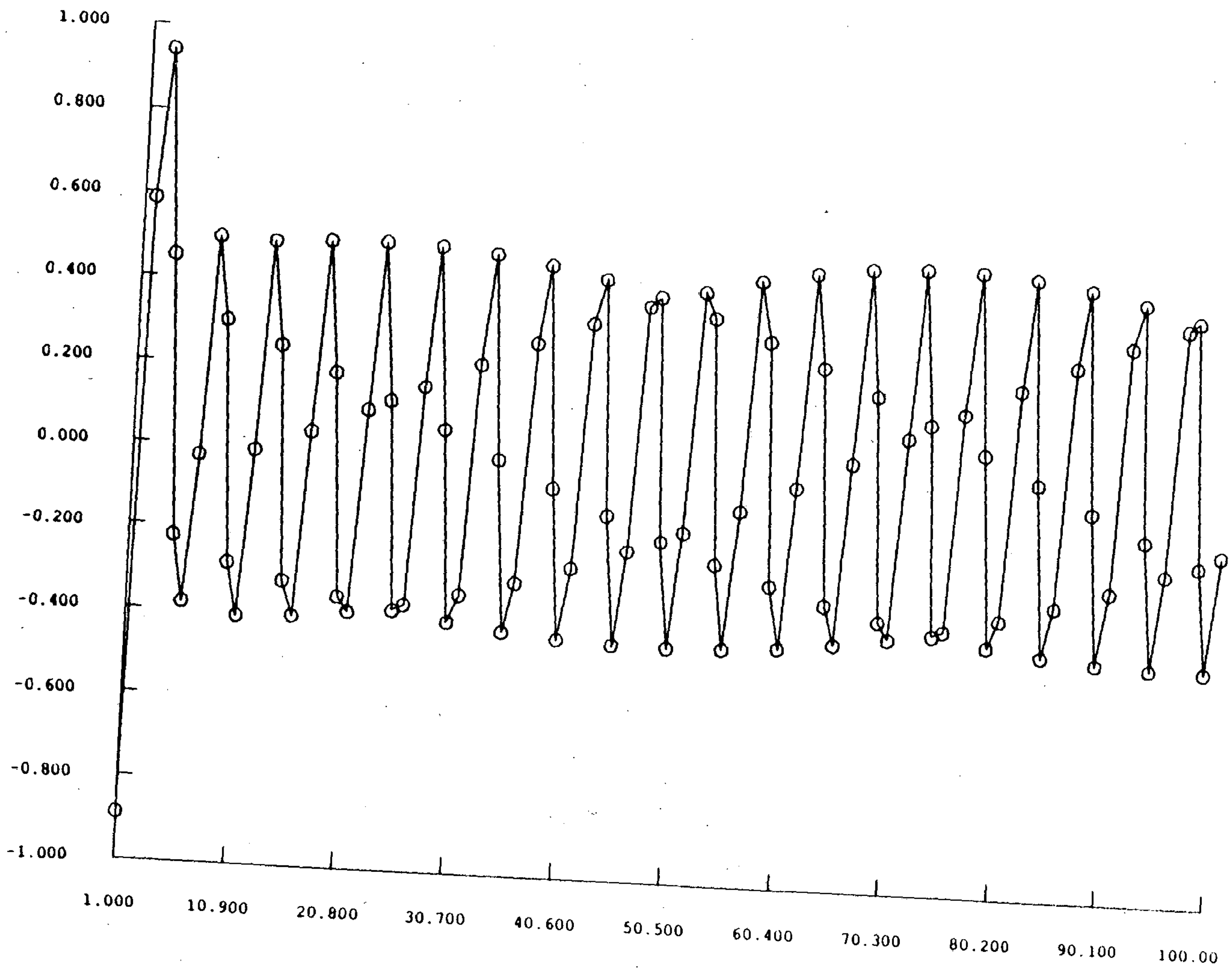


TABLE 12

Comparison Between Spectra

q1 = 2.0 * cos(2.0 * x)

q2 = 0.0

First spectrum is obtained from Runge-Kutta method of order four.

Mesh sizes: h1 = 0.998870584238e-03, h2 = 0.999485388940e-03

Minimum and maximum entered: -2.000000000000000 2.000000000000000

a = 1.500000000000000 b = 1.100000000000000 d = 0.6412749150809320

Interval = [0. , 3.14159265]

n	First Spectrum	Second Spectrum	Difference
1	-.3664927493121458	0.5211803082073549	-.8876730575195007
2	2.878868135507526	2.291593130998497	0.5872750045090290
3	5.644599605366601	4.701150151954369	0.9434494534122320
4	8.633428583962558	8.179229750212256	0.4541988337503020
5	14.96019088878085	15.18326355375406	-.2230726649732098
6	26.10364738635285	26.48508825342375	-.3814408670709000
7	39.75399364045155	39.77977390865279	-.2578026820124002e-01
8	51.17217577453723	50.67084377710769	0.5013319974295403
9	61.47837098487897	61.17516541816868	0.3032055667102904
10	78.05725216656841	78.34151723235411	-.2842650657857000
11	101.9614063979488	102.3713526292191	-.4099462312703004
12	127.2635202192312	127.2721739711376	-.8653751906399165e-02
13	146.5090715077965	146.0120125146837	0.4970589931127982
14	164.0479301101010	163.8000707778986	0.2478593322023990
15	191.5732216728066	191.8943967755407	-.3211751027340988
16	228.4015633378583	228.8053925860099	-.4038292481516024
17	264.7718234998748	264.7295232108106	0.4230028906420102e-01
18	291.1904775971442	290.6861765812088	0.5043010159354040
19	316.3628078240626	316.1737298244533	0.1890779996093030
20	355.5761813500234	355.9291777120939	-.3529963620705061
21	405.3727370455690	405.7605626916414	-.3878256460724003
22	452.1215571807402	452.0196642289608	0.1018929517793978
23	485.1838780153896	484.6760652519915	0.5078127633980998
24	518.5532976908120	518.4267941176987	0.1265035731132969
25	570.1256658147278	570.5055627799724	-.3798969652445976
26	632.8223682653530	633.1872965113805	-.3649282460274890
27	689.1723567894353	689.0083662328735	0.1639905565617994
28	728.4927097913052	727.9883125232344	0.5043972680708038
29	770.7570118020492	770.6951504476018	0.6186135444740160e-01
30	835.2649135845120	835.6666105678435	-.4016969833315045
31	910.6837258566056	911.0194676319943	-.3357417753886978
32	975.7922763646422	975.5668483571581	0.2254280074841120
33	1021.146642338021	1020.653623112021	0.4930192260000013
34	1073.110961893654	1073.113941040381	-.2979146727000170e-02
35	1151.021825142040	1151.440183587296	-.4183584452559899
36	1238.874159236950	1239.174547132744	-.3003878957940174
37	1311.861694719879	1311.577763214823	0.2839315050560174
38	1363.199890504854	1362.726581727984	0.4733087768699988
39	1425.745581735709	1425.811855989673	-.6627425396399644e-01
40	1517.409781295251	1517.839701955707	-.4299206604559913

41	1617.294599582759	1617.553516541543	-.2589169587839990
42	1697.278036150593	1696.940325573085	0.3377105775080054
43	1754.730315355047	1754.285104493508	0.4452108615390102
44	1828.779761716994	1828.906289579326	-.1265278623320114
45	1934.428200600786	1934.864638299859	-.4364376990729966
46	2045.829815550342	2046.041285023185	-.2114694728429924
47	2131.959642520906	2131.574274188752	0.3853683321540302
48	2195.838248596014	2195.429342428202	0.4089061678120061
49	2282.317218333629	2282.499780667620	-.1825623339909725
50	2402.062814691575	2402.500755023106	-.4379403315310242
51	2524.349636124392	2524.508014393329	-.1583782689369855
52	2615.848400131014	2615.422534936419	0.4258651945949623
53	2686.644578558532	2686.279759110493	0.3648194480389861
54	2786.444342224648	2786.677882936659	-.2335407120110062
55	2920.285646633907	2920.720061778961	-.4344151450540039
56	3052.711438302592	3052.811683730015	-.1002454274229763
57	3148.911176127881	3148.452701976889	0.4584741509920036
58	3227.287747365392	3226.974092808621	0.3136545567710414
59	3341.229398625765	3341.508343717325	-.2789450915599900
60	3489.054697504152	3489.480492853680	-.4257953495280162
61	3630.764115834489	3630.802107240317	-.3799140582799510e-01
62	3731.140342683384	3730.657621944722	0.4827207386620103
63	3817.919427122823	3817.663003850388	0.2564232724349722
64	3946.722764282403	3947.041288688360	-.3185244059570209
65	4108.313372967858	4108.725333590867	-.4119606230090085
66	4258.353536787093	4258.326400462326	0.2713632476707062e-01
67	4362.553734924556	4362.055417356402	0.4983175681539933
68	4458.698858934376	4458.504415302084	0.1944436322919501
69	4602.957811864928	4603.310039489100	-.3522276241719737
70	4777.989715025045	4778.382460363804	-.3927453387589139
71	4935.329193203493	4935.235587226777	0.9360597671593496e-01
72	5043.194340608494	5042.689234989021	0.5051056194730563
73	5149.786145200128	5149.656853392532	0.1292918075960188
74	5309.952076608204	5310.332212515082	-.3801359068779675
75	5497.995547925559	5498.363503801441	-.3679558758819894
76	5661.551426335824	5661.391730553499	0.1596957823250023
77	5773.129901337820	5772.626889481530	0.5030118562899588
78	5891.335107424118	5891.272402610021	0.6270481409694639e-01
79	6067.708417486248	6068.110819933524	-.4024024472760175
80	6268.225704748211	6268.563103268442	-.3373985202309768
81	6436.898396429515	6436.674771260080	0.2236251694349676
82	6552.452456298764	6551.960427438368	0.4920288603959762
83	6683.486538212919	6683.490090633467	-.3552420548089685e-02
84	6876.215977068736	6876.635181069625	-.4192040008889535
85	7088.557571998944	7088.858491560434	-.3009195614899909
86	7261.271964023482	7260.988268602267	0.2836954212150431
87	7381.277707088055	7380.805488314128	0.4722187739270112
88	7526.362680052825	7526.430508924682	-.6782887185704567e-01
89	7735.450825590266	7735.881530298662	-.4307047083959787
90	7958.851266329523	7959.109724558555	-.2584582290319304
91	8134.601886178004	8134.263477021864	0.3384091561399600
92	8259.743944825114	8259.300203918775	0.4437409063391442
93	8420.063527847777	8420.192235534255	-.1287076864778101
94	8645.376236269580	8645.813267239179	-.4370309695989363
95	8878.950821159089	8879.160931822208	-.2101106631191669
96	9056.848124483844	9056.461578615772	0.3865458680720621
97	9188.010177833326	9187.603278856415	0.4068989769109521
98	9364.665157859413	9364.850239954906	-.1850820954930441
99	9605.942581560461	9606.380837068598	-.4382555081372175
100	9848.686778514604	9848.842977507583	-.1561989929791707

Sum of the absolute difference = 32.02124743018982

4.3 Example 2: A Discontinuous Potential

In the previous section we tried to reconstruct the smooth potential $q(x) = 2\cos 2x$ from the zero potential. In this section we present an example to study how the Hochstadt-Hald algorithm reconstructs a discontinuous, symmetric potential function. Consider the two systems given below.

SYSTEM 3: Let u satisfy the equation:

$$-u'' = \lambda\, u \quad ,$$

with symmetric boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric discontinuities d_1, d_2 satisfying symmetric jump conditions:

$$u(d_1+) = au(d_1-)\ , \qquad u'(d_2+) = a^{-1}u'(d_1-) + b\ u(d_1-)$$

$$u(d_2-) = au(d_2+)\ , \qquad u'(d_2-) = a^{-1}u'(d_2+) - b\ u(d_2-)$$

SYSTEM 4: And let $\tilde{u}$ satisfy:

$$-u'' \ +\ q\ u = \lambda\, u \quad ,$$

with symmetric boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric discontinuities d_1, d_2 satisfying symmetric jump conditions:

$$u(d_1+) = au(d_1-)\ , \qquad u'(d_2+) = a^{-1}u'(d_1-) + b\ u(d_1-)$$

$$u(d_2-) = au(d_2+)\ , \qquad u'(d_2-) = a^{-1}u'(d_2+) - b\ u(d_2-)$$

where $0 < d_1 < \frac{\pi}{2} < d_2 < \pi$, $d_2 = \pi - d_1$, $0 < x < \pi$ and the potential to be reconstructed is described by

$$q = \begin{cases} -2 & \text{for } 0 \le x < \frac{\pi}{4} \\ +2 & \text{for } \frac{\pi}{4} < x < \frac{3\pi}{4} \\ -2 & \text{for } \frac{3\pi}{4} < x \le \pi \end{cases}$$

We choose eigenfunctions u, $\tilde{u}$ with discontinuities at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ and jump constants $a = 1.5$, $b = 0.5$. Note that the discontinuities in the eigenfunctions coincide with those of the desired potential $q(x)$. We investigate this problem to determine whether this match in the discontinuities reduces the error. To implement the algorithm, first generate the eigenvalues $\{\lambda_i\}_{i=0}^{i=14}$ for systems 3 and 4 respectively. The eigenvalues are given in tables 13 and 14. Then we implement the algorithm. The results are given in figures 21 - 23.

FIGURE 21: FIVE EIGENVALUES , a = 1.5 , b = 0.5 , d = pi/4

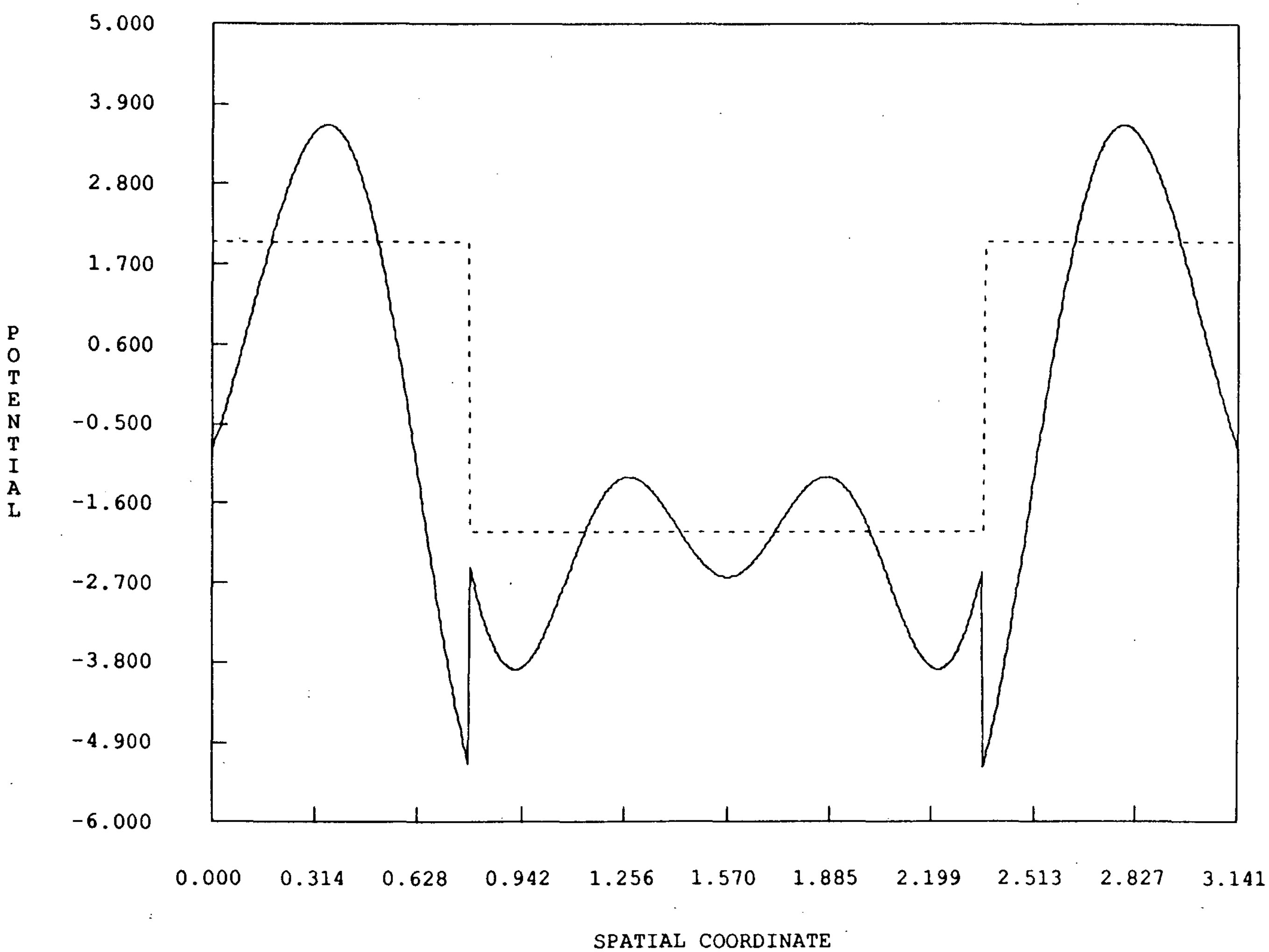


FIGURE 22: TEN EIGENVALUES , a = 1.5 , b = 0.5 , d = pi/4

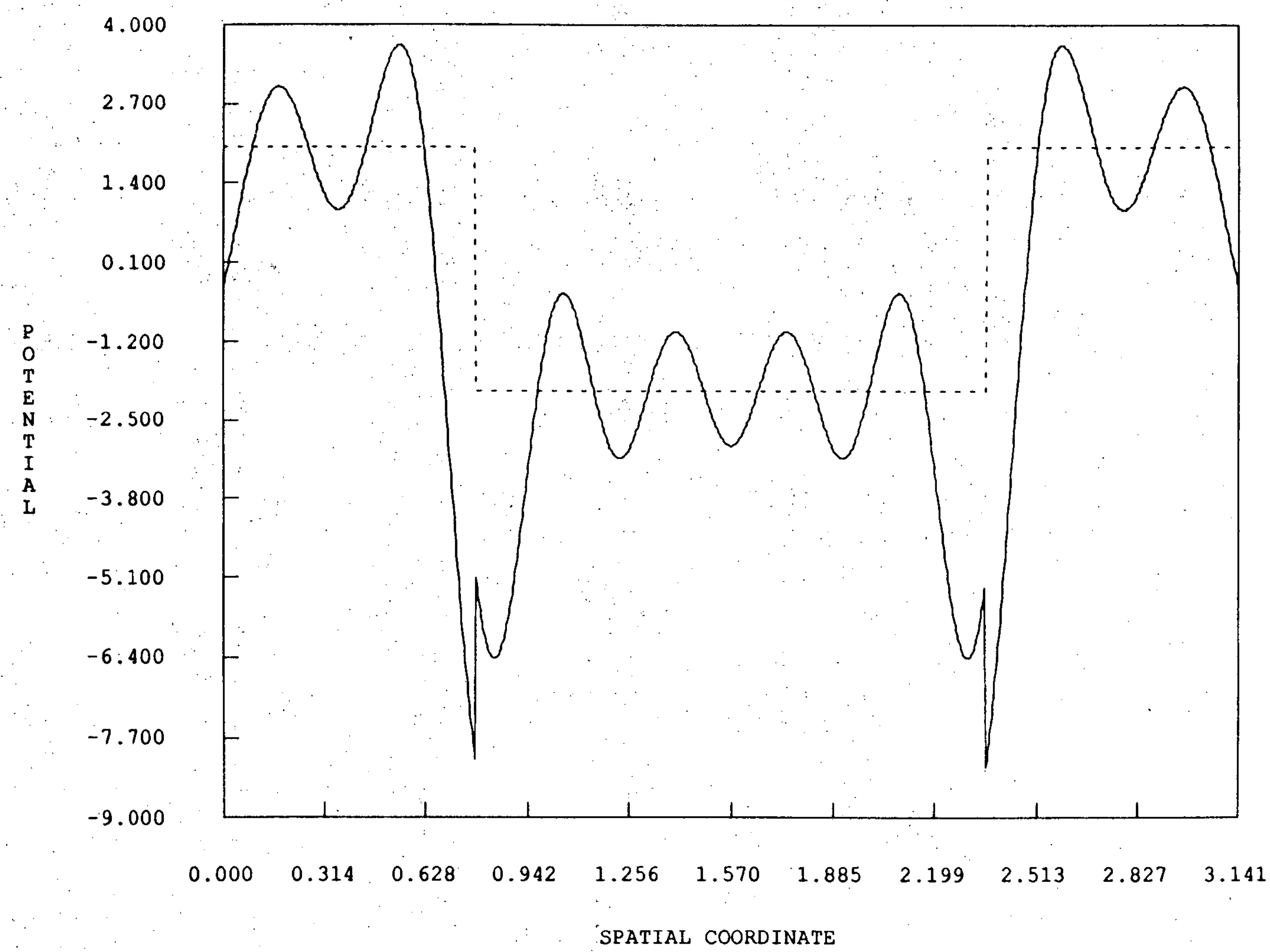


FIGURE 23: FIFTEEN EIGENVALUES , a = 1.5 , b = 0.5 , d = pi/4

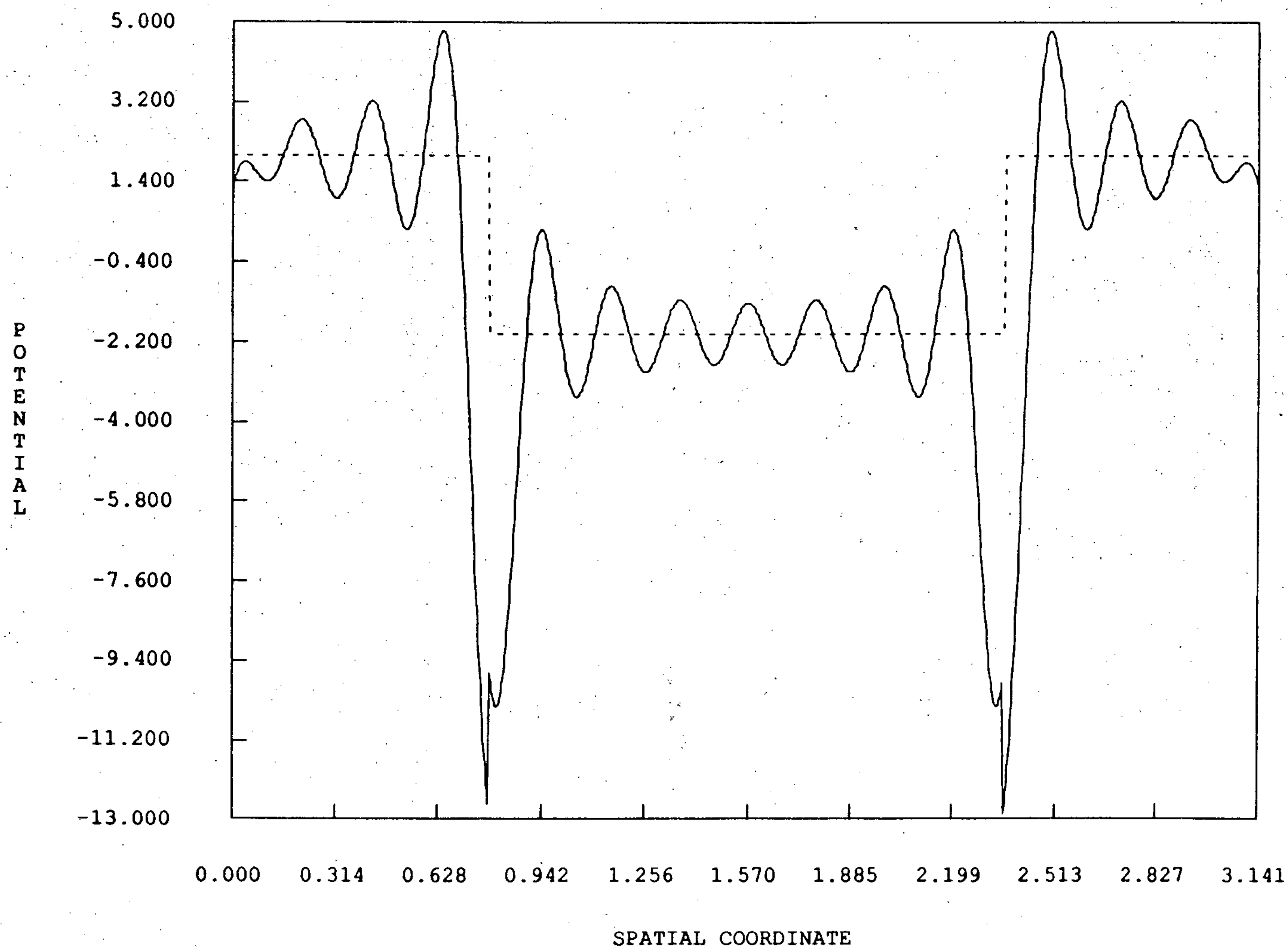


TABLE 13 : EIGENVALUES OF SYSTEM 3

CLASSICAL R-K ORDER 4

($a = 1.5$, $b = 0.5$, $d = \pi/4$, $grid = \pi/10000$)

	eigenvalue
0	0.27690177330005
1	1.8359318360150
2	4.0000000000000
3	7.8505228771748
4	16.581825775556
5	27.866692130150
6	36.000000000000
7	45.839840166183
8	64.586151611451
9	85.879132273410
10	100.000000000000
11	115.82877098730
12	144.58697993561
13	175.89040353128
14	196.00000000000

TABLE 14 : EIGENVALUES OF SYSTEM 4

CLASSICAL R-K ORDER 4

($a = 1.5$, $b = 0.5$, $d = \pi/4$, $grid = \pi/10000$)

	eigenvalue
0	-0.99398080410923
1	2.4245644136291
2	5.2111878170492
3	7.6931333247145
4	15.761516981583
5	28.077280870914
6	36.838221374377
7	45.715517405275
8	63.804345284600
9	85.997452926926
10	100.79478326893
11	115.74373722390
12	143.81271282430
13	175.97208353010
14	196.78236955350

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