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SCHUBERT POLYNOMIALS AS PROJECTIONS OF MINKOWSKI SUMS OF GELFAND–TSETLIN POLYTOPES

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Abstract. Gelfand–Tsetlin polytopes are classical objects in algebraic combinatorics arising in the representation theory of $\mathfrak{gl}_n(\mathbb{C})$. The integer point transform of the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ projects to the Schur function s_λ. Schur functions form a distinguished basis of the ring of symmetric functions; they are also special cases of Schubert polynomials $\mathfrak{S}_w$ corresponding to Grassmannian permutations.

For any permutation $w \in S_n$ with column-convex Rothe diagram, we construct a polytope $\mathcal{P}_w$ whose integer point transform projects to the Schubert polynomial $\mathfrak{S}_w$. Such a construction has been sought after at least since the construction of twisted cubes by Grossberg and Karshon in 1994, whose integer point transforms project to Schubert polynomials $\mathfrak{S}_w$ for all $w \in S_n$. However, twisted cubes are not honest polytopes; rather one can think of them as signed polytopal complexes. Our polytope $\mathcal{P}_w$ is a convex polytope, namely it is a Minkowski sum of Gelfand–Tsetlin polytopes of varying sizes. When the permutation w is Grassmannian, the Gelfand–Tsetlin polytope is recovered. We conclude by showing that the Gelfand–Tsetlin polytope is a flow polytope.

Keywords. Schubert polynomials, Gelfand–Tsetlin polytopes, flow polytopes
Mathematics Subject Classifications. 05E05

1. Introduction

Schubert polynomials, introduced by Lascoux and Schützenberger in 1982 [LS82], are extensively studied in algebraic combinatorics [BJS93, FK96, BB93, FS94, LLS21, WY18, Len04,

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Man01, KM05, KM04, BS98]. They represent cohomology classes of Schubert cycles in flag varieties, and they generalize Schur functions, a distinguished basis of the ring of symmetric functions.

A well-known property of the Schur function s_λ is that it is a projection of the integer point transform of the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$. This has inspired the following natural question for Schubert polynomials:

Question 1. *For $w \in S_n$, is there a natural polytope $\mathcal{P}_w$ and a projection map π_w such that the projection of the integer point transform of $\mathcal{P}_w$ under the map π_w equals the Schubert polynomial $\mathfrak{S}_w$?*

The construction of twisted cubes by Grossberg and Karshon in 1994 [GK94] is the first attempt at an answer to the above question. The integer point transforms of twisted cubes project to any Schubert polynomial. Indeed, Grossberg and Karshon show that for both flag and Schubert varieties, their (virtual) characters are projections of integer point transforms of twisted cubes. The one catch with twisted cubes is that they are not always honest polytopes; intuitively one can think of them as signed polytopal complexes due to self-intersection (see [Kir16] for some discussion). For the Grassmannian case they do not yield the Gelfand–Tsetlin polytope. Kiritchenko's beautiful work [Kir16] explains how to make certain corrections to the Grossberg–Karshon twisted cubes in order to obtain the Gelfand–Tsetlin polytope for Grassmannian permutations.

Recall that given a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geqslant 0}$, the **Gelfand–Tsetlin polytope** $\mathrm{GT}(\lambda)$ is the set of all nonnegative triangular arrays

$$\begin{array}{ccccccccc} x_{11} & & x_{12} & & & \cdots & & x_{1n} \\ & x_{22} & & & x_{23} & \cdots & x_{2n} & \\ & & \cdots & & & \cdots & & \\ & & x_{n-1,n-1} & & & x_{n-1,n} & & \\ & & & & x_{nn} & & & \end{array} \tag{$\dagger$}$$

such that

$$x_{1i} = \lambda_i \text{ for all } 1 \leqslant i \leqslant n, \tag{1.1}$$

$$x_{i-1,j-1} \geqslant x_{ij} \geqslant x_{i-1,j} \text{ for all } 1 \leqslant i \leqslant j \leqslant n. \tag{1.2}$$

To state our main result, which is a partial answer to Question 1, we need to consider the Minkowski sums of Gelfand–Tsetlin polytopes of partitions with different lengths.

Fix n, and for each $k \in [n]$, let $\lambda^{(k)}$ be a partition with k parts (with empty parts allowed). We wish to study the Minkowski sum

$$\mathrm{GT}(\lambda^{(1)}) + \mathrm{GT}(\lambda^{(2)}) + \cdots + \mathrm{GT}(\lambda^{(n)}).$$

To make this Minkowski sum well-defined, we embed $\mathbb{R}^{\binom{k+1}{2}}$ into $\mathbb{R}^{\binom{n+1}{2}}$ for each k. To do this, let y_{ij} be coordinates of $\mathbb{R}^{\binom{k+1}{2}}$ and x_{ij} be coordinates of $\mathbb{R}^{\binom{n+1}{2}}$ as in the definition of the Gelfand–Tsetlin polytope. The embedding is given by

$$y_{ij} \mapsto x_{i,j+n-k} \text{ for all } i + j \leqslant k + 1.$$

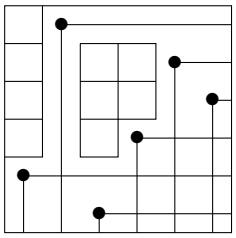


Figure 1.1: The permutation $w = 256413$ is column-convex and has Rothe diagram $D(w) = \{(1,1),(2,1),(3,1),(4,1),(2,3),(3,3),(4,3),(2,4),(3,4)\}$. We also have $\mathrm{Par}_D = (\varnothing,\varnothing,(1,1,0),(2,2,2,1),\varnothing,\varnothing)$.

In other words, when computing the Minkowski sum, we align the upper right corners of each array and sum coordinatewise.

To any permutation $w \in S_n$, we associate its **Rothe diagram** $D(w)$, which is the subset of $\mathbb{N} \times \mathbb{N}$ given by

$$D(w) = \{(i,j) \mid 1 \leqslant i,j \leqslant n,\, w(i) > j,\, w^{-1}(j) > i\}.$$

We can visualize $D(w)$ as the set of boxes remaining in the $n \times n$ grid after crossing out all boxes below or to the right of $(i, w(i))$ for each $i \in [n]$. We say $D(w)$ is **column-convex** if, for each j, the set $\{i \mid (i,j) \in D(w)\}$ is an interval in $\mathbb{N}$. (This will occur whenever w avoids the patterns 3142 and 4132.)

In the case when $D = D(w)$ is column-convex, we associate to w a family of partitions $\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$ in the following way. The shape $\lambda^{(i)}$, $i \in [n]$, has i parts and is obtained from D by ordering the columns of D whose lowest box is in the ith row in decreasing fashion and reading off $\lambda^{(i)}$ according to the French notation. In other words, $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_i^{(i)})$, where $\lambda_j^{(i)}$ is the number of columns c of D such that

$$(i-j+1, c), \ldots, (i,c) \in D,\ (i+1,c) \notin D.$$

Our main result can then be stated as follows.

Theorem 1.1. *Let $w \in S_n$ be a (3142 and 4132-avoiding) permutation with column-convex Rothe diagram D, and let $\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$ be defined as above. Then the Schubert polynomial $\mathfrak{S}_w$ is a projection of the integer point transform of*

$$\mathcal{P}_D \coloneqq \mathrm{GT}(\lambda^{(1)}) + \mathrm{GT}(\lambda^{(2)}) + \cdots + \mathrm{GT}(\lambda^{(n)})$$

(using the embedding specified above). We obtain $\mathfrak{S}_w(x_i)$ from the integer point transform $\sigma_{\mathcal{P}_D}(x_{ij})$ via the specialization

$$x_{ij} \mapsto \begin{cases} x_1 & \text{when } i = 1, \\ x_{i-1}^{-1} x_i & \text{when } i > 1. \end{cases}$$

For instance, it follows immediately in this case that $\mathfrak{S}_w(1,1,\ldots,1)$ is the number of lattice points in $\mathcal{P}_D$.

In the special case when w is a Grassmannian permutation, $\mathfrak{S}_w$ is a Schur polynomial $s_\lambda(x_1,\ldots,x_k)$. All of the $\lambda^{(i)}$ are empty except for $\lambda^{(k)} = \lambda$, so we recover that $s_\lambda(x_1,\ldots,x_k)$ is a projection of the integer point transform of the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$.

In fact, we will prove in Theorem 2.7 that for any column-convex diagram D, the character $\mathfrak{s}_D$ of the **flagged Schur module** associated to D can be obtained as a projection of the integer point transform of a Minkowski sum of Gelfand–Tsetlin polytopes. This will immediately imply Theorem 1.1 since $\mathfrak{S}_w$ is the character of the flagged Schur module associated to $D(w)$.

As a corollary, we deduce a combinatorial interpretation for $\mathfrak{S}_w$ or $\mathfrak{s}_D$ in the column-convex case in terms of column-strict, row-flagged fillings of D up to rearranging entries within rows—see Corollary 3.10.

It is interesting to note that the Newton polytope of a Schubert polynomial is a generalized permutahedron [FMSD18, MTY19]; thus, the affine projection specified in Theorem 2.7 maps $\mathcal{P}_{D(w)}$ to a generalized permutahedron for column-convex $D(w)$.

In [EM18], flow polytopes are connected to certain Grothendieck polynomials, a K-theoretic analogue of Schubert polynomials. The implications of this connection for the corresponding Schubert polynomials are explored in [MD17] (by the second and third author of the present paper). One implication is the existence of flow polytopes whose integer point transforms project to a family of Schubert polynomials. Analogously, we prove that Gelfand–Tsetlin polytopes can be realized as flow polytopes.

Theorem 1.2. *$\mathrm{GT}(\lambda)$ is integrally equivalent to the flow polytope $\mathcal{F}_{G_\lambda}$.*

We then investigate some consequences of Theorem 2.7 in the context of flow polytopes. Further discussion along these lines can be found in [LMSD19], also by the current authors.

This paper is organized as follows. After covering necessary background material, the proof of Theorem 2.7 can be found in Section 2. Section 4 contains background material on flow polytopes as well as the proof of Theorem 1.2, followed by some corollaries and discussion.

2. Diagrams, Schur modules, and Gelfand–Tsetlin polytopes

This section is devoted to introducing background about diagrams, flagged Schur modules and their characters, and Gelfand–Tsetlin polytopes. We will state our main theorem (Theorem 2.7) as well as determine the inequalities that define a Minkowski sum of Gelfand–Tsetlin polytopes.

2.1. Diagrams

A **diagram** is a finite subset of $\mathbb{N}\times\mathbb{N}$. Its elements $(i,j)\in D$ are called **boxes**. We will think of $\mathbb{N}\times\mathbb{N}$ as a grid of boxes in matrix notation, so $(1,1)$ is the topmost and leftmost box. We will sometimes implicitly associate to D a number of rows n, where $D\subseteq[n]\times\mathbb{N}$. (Some rows may be empty.)

Canonically associated to each permutation is its Rothe diagram.

Definition 2.1. The **Rothe diagram** of a permutation $w \in S_n$ is the collection of boxes

$$D(w) = \{(i,j) \mid 1 \leqslant i,j \leqslant n,\ w(i) > j,\ w^{-1}(j) > i\}.$$

Definition 2.2. A diagram D is **column-convex** if, for each j, the set $\{i \mid (i,j) \in D\}$ is an interval in $\mathbb{N}$.

The Rothe diagram for any *dominant* or 132-avoiding permutation is the Young diagram of a partition (in English notation), which is column-convex. Permutations with column-convex Rothe diagrams can be characterized by a pattern avoidance condition.

Proposition 2.3. *The Rothe diagram $D(w)$ is column-convex if and only if w avoids the patterns* 3142 *and* 4132.

Proof. If $D = D(w)$ is not column-convex, then there exists some column l' and rows i, j, and k with $i < j < k$ such that $(i,l'), (k,l') \in D$ but $(j,l') \notin D$. Let $l = w^{-1}(l')$, so that $(k,l') \in D$ implies $k < l$. Then from the boxes appearing in column l', we find that $w(i), w(k) > l'$ but $w(j) < l'$. It follows that the subsequence $w(i)w(j)w(k)w(l)$ of w forms either a 3142 or 4132 pattern in w. The reverse direction is similar. □

Such permutations are counted by the (large) Schröder numbers—see [Kre00, Corollary 9].

To any column-convex diagram D with n rows, we associate a sequence of partitions

$$\begin{aligned}\mathrm{Par}_D &= \{\lambda^{(1)}, \ldots, \lambda^{(n)}\} \\ &= \{(\lambda_1^{(1)}), \quad (\lambda_1^{(2)}, \lambda_2^{(2)}), \quad \ldots, \quad (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)})\},\end{aligned}$$

where $\lambda_j^{(i)}$ is the number of columns c of D such that

$$(i-j+1, c), \ldots, (i,c) \in D,\ (i+1,c) \notin D.$$

In other words, $\lambda^{(i)}$ is the partition with i parts (empty parts allowed) whose Young diagram in French notation is obtained by considering only the columns of D whose lowest box is in the ith row and ordering them according to decreasing size.

2.2. Flagged Schur modules

Given a diagram D with n rows, we can construct a $GL_n(\mathbb{C})$-representation called a **Schur module** as follows. (See also, for instance, [ABW82, KP87].) Denote by Σ_D the symmetric group on the boxes in D. Let $\mathrm{Col}(D)$ be the subgroup of Σ_D permuting the boxes of D within each column, and define $\mathrm{Row}(D)$ similarly for rows. Let $\mathcal{T}_D$ denote the $\mathbb{C}$-vector space with basis indexed by fillings $T\colon D \to [n]$ of D. Observe that Σ_D, $\mathrm{Col}(D)$, and $\mathrm{Row}(D)$ act on $\mathcal{T}_D$ on the right by permuting the filled boxes.

Define idempotents α_D, β_D in the group algebra $\mathbb{C}[\Sigma_D]$ by

$$\alpha_D = \frac{1}{|\mathrm{Row}(D)|} \sum_{w \in \mathrm{Row}(D)} w, \qquad \beta_D = \frac{1}{|\mathrm{Col}(D)|} \sum_{w \in \mathrm{Col}(D)} \mathrm{sgn}(w) w,$$

where $\mathrm{sgn}(w)$ is the sign of the permutation w. Given a filling $T \in \mathcal{T}_D$, define $e_T \in \mathcal{T}_D$ to be the linear combination

$$e_T = T \cdot \alpha_D \beta_D.$$

Identify $\mathcal{T}_D$ with the tensor product $V^{\otimes N}$, where $V = \mathbb{C}^n$ and N is the number of boxes of D, in the following manner. First, fix an order on the boxes of D. Then read each filling T in this order to obtain a word $i_1, \ldots, i_N$ on $[n]$, and identify this word with the tensor $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \in V^{\otimes N}$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. As $GL_n(\mathbb{C})$ acts on V, it acts diagonally on $V^{\otimes N}$ by acting on each component. This left action of $GL_n(\mathbb{C})$ on $\mathcal{T}_D$ commutes with the right action of Σ_D. Thus, the subspace of $\mathcal{T}_D$ spanned by all elements e_T is a submodule, called the Schur module of D.

Call a filling T of D **row-flagged** if $T(i,j) \leqslant i$ for all i, j. Let B_n be the subgroup of $GL_n(\mathbb{C})$ consisting of upper triangular matrices.

Definition 2.4. The **flagged Schur module** $\mathcal{S}_D$ of a diagram D is the B_n-submodule of $\mathcal{T}_D$ spanned by

$$\{e_T \mid T \text{ is a row-flagged filling of } D\}.$$

The **formal character** $\mathrm{char}(\mathcal{S}_D)$, denoted by $\mathfrak{s}_D$, is the polynomial

$$\mathfrak{s}_D = \mathrm{char}(\mathcal{S}_D)(x_1, \ldots, x_n) = \mathrm{Trace}(X \colon \mathcal{S}_D \to \mathcal{S}_D),$$

where X is the diagonal matrix in B_n with diagonal entries $x_1, \ldots, x_n$.

A particularly important subclass of characters of flagged Schur modules is that of Schubert polynomials as shown by Kraśkiewicz-Pragacz [KP87] and explained in Theorem 2.6 below. Schubert polynomials are associated to permutations, and they admit various combinatorial and algebraic definitions. For a permutation $w \in S_n$, we will define the Schubert polynomial $\mathfrak{S}_w$ via divided difference operators ∂_i on polynomials.

Definition 2.5. The Schubert polynomial of the long word $w_0 \in S_n$ ($w_0(i) = n - i + 1$ for $1 \leqslant i \leqslant n$) is defined as

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \neq w_0$, there exists $i \in [n-1]$ such that $w(i) < w(i+1)$. For any such i, the **Schubert polynomial** $\mathfrak{S}_w$ is defined by

$$\mathfrak{S}_w := \partial_i \mathfrak{S}_{w s_i},$$

where

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}} = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}},$$

and s_i is the transposition swapping i and $i+1$. The operators ∂_i can be shown to satisfy the braid relations, so the Schubert polynomials $\mathfrak{S}_w$ are well-defined.

Schubert polynomials appear as the characters of flagged Schur modules of Rothe diagrams.

Theorem 2.6 ([KP87])**.** *Let $w \in S_n$ be a permutation, $D(w)$ be the Rothe diagram of w, and $\mathfrak{s}_{D(w)}$ be the character of the associated flagged Schur module $\mathcal{S}_{D(w)}$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{s}_{D(w)}(x_1, \ldots, x_n).$$

We can now state our main theorem, which relates the character $\mathfrak{s}_D$ for any column-convex diagram D with the Minkowski sum of Gelfand–Tsetlin polytopes. (See the introduction for the definition of Gelfand–Tsetlin polytopes.)

Theorem 2.7. *The character $\mathfrak{s}_D$ of the flagged Schur module associated to a column-convex diagram D with n rows and $\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$ is a projection of the integer point transform of*

$$\mathcal{P}_D := \mathrm{GT}(\lambda^{(1)}) + \mathrm{GT}(\lambda^{(2)}) + \cdots + \mathrm{GT}(\lambda^{(n)}). \tag{2.1}$$

We obtain $\mathfrak{s}_D(x_1, \ldots, x_n)$ from the integer point transform $\sigma_{\mathcal{P}_D}(x_{ij})$ via the specialization

$$x_{ij} \mapsto \begin{cases} x_1 & \text{when } i = 1, \\ x_{i-1}^{-1} x_i & \text{when } i > 1. \end{cases}$$

Recall that in order to take the Minkowski sum, we embed $\mathrm{GT}(\lambda^{(k)}) \subseteq \mathbb{R}^{\binom{k+1}{2}}$ into $\mathbb{R}^{\binom{n+1}{2}}$ by

$$y_{ij} \mapsto x_{i,j+n-k} \text{ for all } i + j \leqslant k + 1.$$

Example 2.8. If D is the diagram in Figure 1.1, then we find that

$$\mathcal{P}_D = \mathrm{GT}(1,1,0) + \mathrm{GT}(2,2,2,1).$$

These two Gelfand–Tsetlin polytopes have the following lattice points:

$$\mathrm{GT}(1,1,0): \quad \begin{array}{ccccc} 1 & & 1 & & 0 \\ & 1 & & a & \\ & & b & & \end{array} \qquad \text{where } 0 \leqslant a \leqslant b \leqslant 1,$$

$$\mathrm{GT}(2,2,2,1): \quad \begin{array}{ccccccc} 2 & & 2 & & 2 & & 1 \\ & 2 & & 2 & & c & \\ & & 2 & & d & & \\ & & & e & & & \end{array} \qquad \text{where } 1 \leqslant c \leqslant d \leqslant e \leqslant 2.$$

Summing these polytopes using the embedding described above gives the polytope $\mathcal{P}_D$ with lattice points

$$\begin{array}{ccccccc} 2 & & 3 & & 3 & & 1 \\ & 2 & & 3 & & a+c & \\ & & 2 & & b+d & & \\ & & & e & & & \end{array} \qquad \text{where } 0 \leqslant a \leqslant b \leqslant 1 \text{ and } 1 \leqslant c \leqslant d \leqslant e \leqslant 2.$$

The twelve possible choices for a through e only yield nine distinct possibilities for $(a + c, b + d, e)$:

$$(1,1,1),\ (1,2,1),\ (2,2,1),\ (1,1,2),\ (1,2,2),\ (2,2,2),\ (1,3,2),\ (2,3,2),\ (3,3,2).$$

Under the specialization given in Theorem 2.7, these triples give

$$\begin{aligned} \mathfrak{s}_D = {} & x_1^3x_2^3x_3^2x_4 + x_1^3x_2^2x_3^3x_4 + x_1^2x_3^3x_3^3x_4 + x_1^3x_2^3x_3x_4^2 + x_1^3x_2^2x_3^2x_4^2 \\ & + x_1^2x_2^3x_3^2x_4^2 + x_1^3x_2x_3^3x_4^2 + x_1^2x_2^2x_3^3x_4^2 + x_1x_2^3x_3^3x_4^2. \end{aligned}$$

We will prove Theorem 2.7 in Section 3.

2.3. Minkowski sums of Gelfand–Tsetlin polytopes

As a first step towards Theorem 2.7, we will describe the inequalities defining the Minkowski sum of Gelfand–Tsetlin polytopes.

Proposition 2.9. *Let $\lambda^{(1)}, \dots, \lambda^{(n)}$ be partitions such that $\lambda^{(i)}$ has i (possibly empty) parts. The Minkowski sum $\mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)})$ is defined by the following inequalities:*

- *for all $1 \leqslant i \leqslant j \leqslant n$, $x_{i-1,j-1} \geqslant x_{ij}$; and*
- *for any positive integer k and nonempty sequence I of even length $0 \leqslant i_k < i_{k-1} < \cdots < i_1 < j_1 < j_2 < \cdots < j_k \leqslant n$,*

$$\sum_{s=1}^{k} x_{j_s - i_s, j_s} - \sum_{s=1}^{k-1} x_{j_{s+1} - i_s, j_{s+1}} \geqslant \sum_{s=0}^{i_k} \lambda^{(n-s)}_{j_1 - s}, \tag{$*$}$$

with equality when $k = 1$ and $j_1 = i_1 + 1$.

Remark 2.10. A simple calculation shows that if, for instance, $i_{s+1} = i_s$ for some s, then neither side of ($*$) would change if we simply remove i_{s+1} and j_{s+1} from the sequence. Likewise, if $j_s = j_{s+1}$ for some s, then neither side would change if we remove i_s and j_s from the sequence. Therefore we may equivalently take the inequalities ($*$) for sequences $0 \leqslant i_k \leqslant \cdots \leqslant i_1 < j_1 \leqslant \cdots \leqslant j_k \leqslant n$.

One should observe that the entries occurring on the left side of ($*$) lie at the corners of a path that zigzags southeast and southwest inside the triangular array, starting at $x_{j_1 - i_1, j_1}$ and ending at $x_{j_k - i_k, j_k}$, where the lengths of the southeast steps are given by the differences $j_{s+1} - j_s$ and the lengths of the southwest steps are given by the differences $i_{s+1} - i_s$.

Example 2.11. Suppose $n = 3$. We first have inequalities $x_{11} \geqslant x_{22} \geqslant x_{33}$ and $x_{12} \geqslant x_{23}$ as with ordinary Gelfand–Tsetlin patterns. Then for $k = 1$, we get equalities

$$x_{11} = \lambda^{(3)}_1, \qquad x_{12} = \lambda^{(2)}_1 + \lambda^{(3)}_2, \qquad x_{13} = \lambda^{(1)}_1 + \lambda^{(2)}_2 + \lambda^{(3)}_3,$$

as well as inequalities

$$x_{22} \geqslant \lambda^{(3)}_2, \qquad x_{23} \geqslant \lambda^{(2)}_2 + \lambda^{(3)}_3, \qquad \text{and} \qquad x_{33} \geqslant \lambda^{(3)}_3.$$

Finally, for $k = 2$, there is one more inequality, namely

$$x_{12} - x_{23} + x_{33} \geqslant \lambda^{(3)}_2.$$

One direction of Proposition 2.9 is given by the following lemma.

Lemma 2.12. *Let*

$$P = P(\lambda^{(1)}, \dots, \lambda^{(n)}) = \mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)}),$$

and let $Q = Q(\lambda^{(1)}, \dots, \lambda^{(n)})$ be the polytope given by the inequalities in Proposition 2.9. Then $P \subset Q$.

Proof. For any point $(x_{ij})_{1\leqslant i\leqslant j\leqslant n} \in P$, choose, for each $0 \leqslant m < n$, points $(y_{ij}^{(n-m)})_{1\leqslant i\leqslant j\leqslant m} \in \mathrm{GT}(\lambda^{(n-m)})$ summing to it, so that $x_{ij} = \sum_{s=0}^{j-i} y_{i,j-s}^{(n-s)}$. In particular, $\mathrm{GT}(\lambda^{(n-m)})$ will contribute to a coordinate of the form $x_{j-i,j}$ if and only if $m \leqslant i$.

Inequalities of the form $x_{i-1,j-1} \geqslant x_{ij}$ are derived by summing the respective inequalities $y_{i-1,j-1-s}^{(n-s)} \geqslant y_{i,j-s}^{(n-s)}$ over all $0 \leqslant s \leqslant j-i$. For inequalities of type ($*$), consider a sequence I, and suppose first that $0 \leqslant m \leqslant i_k$. Then

$$\sum_{s=1}^{k} y_{j_s-i_s,j_s-m}^{(n-m)} - \sum_{s=1}^{k-1} y_{j_{s+1}-i_s,j_{s+1}-m}^{(n-m)} = y_{j_1-i_1,j_1-m}^{(n-m)} + \sum_{s=1}^{k-1} (y_{j_{s+1}-i_{s+1},j_{s+1}-m}^{(n-m)} - y_{j_{s+1}-i_s,j_{s+1}-m}^{(n-m)}) \geqslant \lambda_{j_1-m}^{(n-m)},$$

since each term in the sum is nonnegative by the defining inequalities of $\mathrm{GT}(\lambda^{(n-m)})$. If instead $m > i_k$, then let $k' < k$ be the minimum value such that $m \leqslant i_{k'}$. Then

$$\sum_{s=1}^{k'} y_{j_s-i_s,j_s-m}^{(n-m)} - \sum_{s=1}^{k'} y_{j_{s+1}-i_s,j_{s+1}-m}^{(n-m)} = \sum_{s=1}^{k'} (y_{j_s-i_s,j_s-m}^{(n-m)} - y_{j_{s+1}-i_s,j_{s+1}-m}^{(n-m)}) \geqslant 0$$

since again each term in the sum is nonnegative. Summing these inequalities over all m then gives the desired inequality. In the case that $k = 1$ and $j_1 = i_1 + 1$, we get equality since

$$x_{1,j_1} = \sum_{s=0}^{j_1-1} y_{1,j_1-s}^{(n-s)} = \sum_{s=0}^{i_1} \lambda_{j_1-s}^{(n-s)}. \qquad \square$$

To complete the proof of Proposition 2.9, we will need the following Lemma 2.13, which is proved in Section 4.

Lemma 2.13. *If λ has n parts, then the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ decomposes as a Minkowski sum:*

$$\mathrm{GT}(\lambda) = \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) \mathrm{GT}(1^k 0^{n-k}).$$

Armed with this, we are now prepared to finish the proof of Proposition 2.9.

Proof of Proposition 2.9. By Lemma 2.12 (and using the notation defined there), it suffices to show that $Q \subseteq P$. We induct on n and the size of $\lambda^{(n)}$. First suppose $\lambda^{(n)} = \varnothing$. The inequalities involving x_{jj} are $x_{11} \geqslant x_{22} \geqslant \cdots \geqslant x_{nn}$, and, when $i_k = 0$,

$$\sum_{s=1}^{k-1} (x_{j_s-i_s,j_s} - x_{j_{s+1}-i_s,j_{s+1}}) + x_{j_k,j_k} \geqslant \lambda_{j_1}^{(n)} = 0$$

with equality if also $k = 1$ and $j_1 = 1$. These imply that $x_{jj} = 0$ for all $1 \leqslant j \leqslant n$ and impose no additional constraints on the other entries. Removing the diagonal of entries x_{jj} then

yields a triangular array that satisfies the inequalities defining $Q(\lambda^{(1)}, \cdots, \lambda^{(n-1)})$. Therefore by induction

$$Q(\lambda^{(1)}, \ldots, \lambda^{(n-1)}, \varnothing) = Q(\lambda^{(1)}, \ldots, \lambda^{(n-1)}) = P(\lambda^{(1)}, \ldots, \lambda^{(n-1)}) = P(\lambda^{(1)}, \ldots, \lambda^{(n-1)}, \varnothing).$$

If $\lambda^{(n)} \neq \varnothing$, then let $m = \ell(\lambda^{(n)})$ be the number of nonzero parts. We will prove that $Q \subseteq \mathrm{GT}(1^m 0^{n-m}) + Q'$, where we let $Q' = Q(\lambda^{(1)}, \ldots, \lambda^{(n-1)}, \mu^{(n)})$ for $\mu^{(n)} = (\lambda_1^{(n)} - 1, \ldots, \lambda_m^{(n)} - 1, 0, \ldots, 0)$. This will prove the result by induction on the size of λ using Lemma 2.13 since then $\mathrm{GT}(1^m 0^{n-m}) + \mathrm{GT}(\mu^{(n)}) = \mathrm{GT}(\lambda^{(n)})$.

Recall that Gelfand–Tsetlin polytopes are integral polytopes. Given any integer point $(x_{ij}) \in Q$, set $t_j = 1$ for $1 \leqslant j \leqslant m$, while for $m < j \leqslant n$, set t_j to be the minimum value such that $t_j > t_{j-1}$ and $x_{t_j-1,j-1} = x_{t_j,j}$ (if such an index exists, otherwise set $t_j = \infty$). Then define the point $(z_{ij})_{1 \leqslant i \leqslant j \leqslant n} \in \mathrm{GT}(1^m 0^{n-m})$ by $z_{ij} = 1$ if $i \geqslant t_j$, otherwise $z_{ij} = 0$.

We claim that $(x'_{ij}) = (x_{ij} - z_{ij}) \in Q'$. Our choice of t_j guarantees that $x_{i-1,j-1} - x_{i,j} \geqslant 1$ whenever $z_{i-1,j-1} - z_{i,j} = 1$, which ensures that $x'_{i-1,j-1} \geqslant x'_{ij}$ for all $1 \leqslant i \leqslant j \leqslant n$. Therefore it suffices to show inequalities of type (∗) for (x'_{ij}).

Given any sequence I, suppose that for some s, $z_{j_s - i_{s-1}, j_s} = 0$ but $z_{j_s - i_s, j_s} = 1$. Consider what happens to the left hand side of (∗) if we insert $j' = j_s - 1$ between j_{s-1} and j_s, and we insert $i' = j_s - t_{j_s}$ between i_s and i_{s-1} to get a new sequence I'. (Note that $j_{s-1} \leqslant j' < j_s$ and $i_s \leqslant i' < i_{s-1}$.) This reduces the left hand side of (∗) by

$$\begin{aligned}(x'_{j'-i_{s-1},j'} - x'_{j_s - i_{s-1}, j_s}) - (x'_{j'-i',j'} - x'_{j_s - i', j_s}) &= (x'_{j_s - 1 - i_{s-1}, j_s - 1} - x'_{j_s - i_{s-1}, j_s}) \\ &\quad - (x'_{t_{j_s}-1, j_s - 1} - x'_{t_{j_s}, j_s}) \\ &= x'_{j_s - 1 - i_{s-1}, j_s - 1} - x'_{j_s - i_{s-1}, j_s} \\ &\geqslant 0,\end{aligned}$$

while the right hand side of (∗) is unchanged. Thus (∗) for the sequence I is implied by (∗) for the new sequence I'. Since $z_{j_s - i', j_s} = z_{t_{j_s}, j_s} = 1$, by iteratively applying this procedure to the new sequence, we will eventually arrive at a sequence for which such an s does not exist.

It therefore suffices to prove inequality (∗) for (x'_{ij}) in the case that there exists some s' such that $z_{j_s - i_{s-1}, j_s} = 1$ and $z_{j_s - i_s, j_s} = 1$ exactly when $s \leqslant s'$. If $j_1 \leqslant m$, then the left hand side of (∗) is

$$\begin{aligned}\sum_{s=1}^{k} x'_{j_s - i_s, j_s} - \sum_{s=1}^{k-1} x'_{j_{s+1} - i_s, j_{s+1}} &= \left(\sum_{s=1}^{k} x_{j_s - i_s, j_s} - s'\right) - \left(\sum_{s=1}^{k-1} x_{j_{s+1} - i_s, j_{s+1}} - s' + 1\right) \\ &= \sum_{s=1}^{k} x_{j_s - i_s, j_s} - \sum_{s=1}^{k-1} x_{j_{s+1} - i_s, j_{s+1}} - 1,\end{aligned}$$

while the right hand side is

$$\mu_{j_1}^{(n)} + \sum_{s=1}^{i_k} \lambda_{j_1 - s}^{(n-s)} = \sum_{s=0}^{i_k} \lambda_{j_1 - s}^{(n-s)} - 1,$$

so this inequality follows from the corresponding inequality for $(x_{ij}) \in Q$. If $j_1 > m$, then consider the sequence obtained by inserting $m, m+1, \ldots, j_1 - 1$ before j_1, and $j_1 - t_{j_1}, j_1 - 1 - t_{j_1-1}, \ldots, m+1-t_{m+1}$ after i_1 in the sequence. For $(x_{ij}) \in Q$, this yields the inequality

$$\left(\sum_{j=m}^{j_1-1} x_{t_{j+1}-1,j} + \sum_{s=1}^{k} x_{j_s-i_s,j_s}\right) - \left(\sum_{j=m}^{j_1-1} x_{t_{j+1},j+1} + \sum_{s=1}^{k-1} x_{j_{s+1}-i_s,j_{s+1}}\right) \geqslant \sum_{s=0}^{i_k} \lambda_{m-s}^{(n-s)}.$$

But $x_{t_{j+1}-1,j} = x_{t_{j+1},j+1}$, and the right side is strictly greater than $\sum_{s=0}^{i_k} \lambda_{j_1-s}^{(n-s)}$ (since $\lambda_m^{(n)} > 0 = \lambda_{j_1}^{(n)}$). Thus

$$\sum_{s=1}^{k} x_{j_s-i_s,j_s} - \sum_{s=1}^{k-1} x_{j_{s+1}-i_s,j_{s+1}} \geqslant \sum_{s=0}^{i_k} \lambda_{j_1-s}^{(n-s)} + 1,$$

or equivalently,

$$\left(\sum_{s=1}^{k} x_{j_s-i_s,j_s} - s'\right) - \left(\sum_{s=1}^{k-1} x_{j_{s+1}-i_s,j_{s+1}} - s' + 1\right) \geqslant \sum_{s=0}^{i_k} \lambda_{j_1-s}^{(n-s)} = \mu_{j_1}^{(n)} + \sum_{s=1}^{i_k} \lambda_{j_1-s}^{(n-s)},$$

which is the inequality (∗) for $(x'_{ij}) \in Q'$. This completes the proof. □

We can give the following combinatorial interpretation for the integer points in

$$\mathcal{P}_D = \mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)}).$$

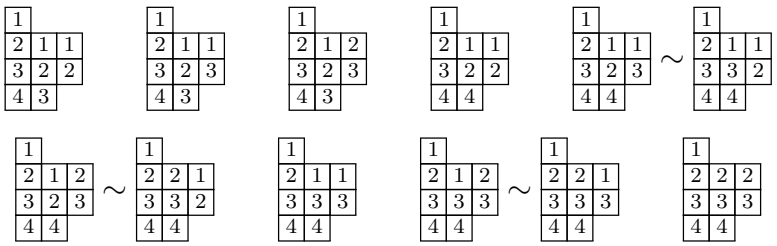
Call a filling $T\colon D \to [n]$ **column-strict** if the entries in each column increase from top to bottom. Call two fillings T and T' **row equivalent** if one can be obtained from the other by permuting elements within each row. (In other words, the number of occurrences of i in row j is the same for T and T' for all i and j.)

Corollary 2.14. *Let D be a column-convex diagram and $\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$. The lattice points of $\mathcal{P}_D = \mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)})$ are in bijection with column-strict, row-flagged fillings of D up to row equivalence. Specifically, a point $(x_{ij}) \in \mathcal{P}_D$ corresponds to a filling in which x_{ij} is the number of entries in row $n+i-j$ that are at least i.*

Proof. Let D have columns $C_1, C_2, \ldots$. Our inductive proof of Proposition 2.9 actually showed that any integer point of $\mathcal{P}_D$ is a sum of integer points, each lying in some $\mathcal{P}_{C_s} = \mathrm{GT}(1^m 0^{k-m})$, where C_s has m boxes, the lowest of which lies in row k.

The standard bijection between Gelfand–Tsetlin patterns and (reverse) semistandard Young tableaux sends each integer point $(y_{ij}) \in \mathrm{GT}(1^m 0^{k-m}) \subseteq \mathbb{R}^{\binom{k+1}{2}}$ to the column-strict filling of C_s such that y_{ij} is the number of entries in row $k+i-j$ that are at least i. After embedding $\mathbb{R}^{\binom{k+1}{2}}$ into $\mathbb{R}^{\binom{n+1}{2}}$, the image $(x_{ij}) \in \mathbb{R}^{\binom{n+1}{2}}$ then has x_{ij} equal to the number of entries in row $n+i-j$ that are at least i. Combining the fillings for each column C_s therefore has the effect of adding the corresponding coordinates x_{ij}, so the result follows. □

Example 2.15. Using the diagram D from Figure 1.1, the twelve column-strict, row-flagged fillings are given below with row equivalences indicated. The nine equivalence classes correspond to the lattice points in $\mathcal{P}_D$ as well as the terms in $\mathfrak{s}_D$ as found in Example 2.8.



In the next section, we will show how the facet description of $\mathcal{P}_D$ is related to the characters of flagged Schur modules.

3. Demazure operators and parapolytopes

To prove Theorem 2.7, we will need a formula for the character $\mathfrak{s}_D$. The following formula is essentially a particular case of one due to Magyar [Mag98]. (See also Reiner–Shimozono [RS98].) We first define the **isobaric divided difference operator** (or **Demazure operator**) π_i acting on polynomials $f(x_1, \ldots, x_n)$ by

$$\pi_i f(x_1, \ldots, x_n) = \partial_i(x_i f) = \frac{x_i f - x_{i+1} s_i f}{x_i - x_{i+1}},$$

where $s_i f$ is the polynomial obtained from f by switching x_i and x_{i+1}. Note that $\pi_i f = f$ if f is symmetric in x_i and x_{i+1}.

Proposition 3.1. *Let D be a column-convex diagram with n rows with $\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$. Define $\widetilde{D}$ to be the diagram with $n-1$ rows such that $\mathrm{Par}_{\widetilde{D}} = (\widetilde{\lambda}^{(1)}, \ldots, \widetilde{\lambda}^{(n-1)})$, where $\widetilde{\lambda}^{(i)}_j = \lambda^{(i+1)}_j - \lambda^{(i+1)}_{i+1}$. (Here, $\widetilde{D}$ is obtained from D by removing any column with a box in the first row and then shifting all remaining boxes up by one row.) Also let*

$$\mu = (\lambda^{(1)}_1 + \lambda^{(2)}_2 + \cdots + \lambda^{(n)}_n, \lambda^{(2)}_2 + \cdots + \lambda^{(n)}_n, \ldots, \lambda^{(n)}_n),$$

the partition formed from all columns of D with a box in the first row. Then

$$\mathfrak{s}_D = x_1^{\mu_1} \cdots x_n^{\mu_n} \pi_1 \pi_2 \cdots \pi_{n-1}(\mathfrak{s}_{\widetilde{D}}).$$

Proof. Note that D can be obtained from $\widetilde{D}$ by switching the ith and $(i+1)$st row for $i = n-1, n-2, \ldots, 1$, and then adding μ_i columns with boxes in rows $\{1, \ldots, i\}$ for each $i = 1, \ldots, n$. The result then follows immediately from [Mag98] (see, for instance, Proposition 15). □

We now show that the polytope for D can be constructed iteratively in a way that mimics the application of the operator π_i. This geometric operation is the same as the operator D_i given by Kiritchenko in [Kir16] specialized for our current situation.

The key lemma is the following calculation.

Lemma 3.2. *Choose nonnegative integers N_1, N_2 and $\mu_1 \leqslant \nu_1, \ldots, \mu_k \leqslant \nu_k$ such that*

$$\sum_{i=1}^{k}(\mu_i + \nu_i) \leqslant N_1 + N_2.$$

Define the polynomial

$$f(x_1, x_2) = \sum_{c_1=\mu_1}^{\nu_1} \cdots \sum_{c_k=\mu_k}^{\nu_k} x_1^{N_1-c_1-\cdots-c_k} x_2^{c_1+\cdots+c_k-N_2}.$$

Then

$$\pi_1 f(x_1, x_2) = \sum_{c_1=\mu_1}^{\nu_1} \cdots \sum_{c_k=\mu_k}^{\nu_k} \sum_{c_{k+1}=0}^{\nu_{k+1}} x_1^{N_1-c_1-\cdots-c_k-c_{k+1}} x_2^{c_1+\cdots+c_k+c_{k+1}-N_2},$$

where $\nu_{k+1} = N_1 + N_2 - \sum_{i=1}^{k}(\mu_i + \nu_i)$.

Proof. Note that reversing the order of each of the summations in the expression for f gives

$$f = \sum_{c_1=\mu_1}^{\nu_1} \cdots \sum_{c_k=\mu_k}^{\nu_k} x_1^{\nu_{k+1}-N_2+c_1+\cdots+c_k} x_2^{N_1-\nu_{k+1}-c_1-\cdots-c_k} = (\tfrac{x_1}{x_2})^{\nu_{k+1}} \cdot s_1 f.$$

Hence

$$\pi_1 f = \frac{x_1 f - x_2 s_1 f}{x_1 - x_2} = f \cdot \frac{1 - \left(\frac{x_2}{x_1}\right)^{\nu_{k+1}+1}}{1 - \frac{x_2}{x_1}} = f \cdot \sum_{c_{k+1}=0}^{\nu_{k+1}} x_1^{-c_{k+1}} x_2^{c_{k+1}},$$

as desired. □

Consider $\mathbb{R}^{\binom{n+1}{2}}$ with coordinates x_{ij} for $1 \leqslant i \leqslant j \leqslant n$. Let $\varphi_k \colon \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{\binom{n+1}{2}-n-1+k}$ be the projection onto the coordinates x_{ij} for all $i \neq k$.

Definition 3.3 ([Kir16])**.** A **parapolytope** $P \subset \mathbb{R}^{\binom{n+1}{2}}$ is a convex polytope such that, for all k, every fiber of the projection φ_k on P is a coordinate parallelepiped.

In other words, for every k and every set of constants c_{ij} ($i \neq k$), there exist constants μ_j and ν_j (depending on the c_{ij}) such that $(x_{ij}) \in P$ with $x_{ij} = c_{ij}$ for $i \neq k$ if and only if $\mu_j \leqslant x_{kj} \leqslant \nu_j$.

We denote this parallelepiped (which depends on k and c_{ij} for $i \neq k$) by

$$\Pi(\mu_k, \ldots, \mu_n; \nu_k, \ldots, \nu_n) = \Pi(\mu, \nu) = \{(x_{kj})_{j=k}^{n} \mid \mu_j \leqslant x_{kj} \leqslant \nu_j\} \subset \mathbb{R}^{n+1-k}.$$

Given a polytope $P \subset \mathbb{R}^{\binom{n+1}{2}}$, let σ_P be its **integer point transform**

$$\sigma_P(x_{ij}) = \sum_{(c_{ij}) \in P \cap \mathbb{Z}^{\binom{n+1}{2}}} \prod_{1 \leqslant i \leqslant j \leqslant n} x_{ij}^{c_{ij}},$$

and define $s_P(x_i)$ to be the image of $\sigma_P(x_{ij})$ under the specialization sending

$$x_{ij} \mapsto \begin{cases} x_1 & \text{when } i = 1, \\ x_{i-1}^{-1} x_i & \text{when } i > 1. \end{cases}$$

In other words, the point $(c_{ij}) \in P \cap \mathbb{Z}^{\binom{n+1}{2}}$ corresponds to the monomial in which the exponent of x_i is $C_i - C_{i+1}$, where $C_i = \sum_{j=i}^{n} c_{ij}$.

Lemma 3.4. *Fix $2 \leqslant k \leqslant n$, and let $P, Q \subset \mathbb{R}^{\binom{n+1}{2}}$ be parapolytopes. Suppose that for any fixed integer point $c = (c_{ij})_{i \neq k}$, the fiber over c of the projection φ_k on P is the (integer) parallelepiped*

$$\Pi_P = \Pi(\mu_k, \ldots, \mu_{n-1}, 0; \nu_k, \ldots, \nu_{n-1}, 0),$$

while the fiber over c of φ_k on Q is

$$\Pi_Q = \Pi(\mu_k, \ldots, \mu_{n-1}, \mu_n; \nu_k, \ldots, \nu_{n-1}, \nu_n),$$

where $\mu_n = 0$ and

$$\nu_n = \sum_{j=k-1}^{n} c_{k-1,j} + \sum_{j=k+1}^{n} c_{k+1,j} - \sum_{j=k}^{n-1} (\mu_j + \nu_j) \geqslant 0.$$

Then $s_Q = \pi_{k-1} s_P$.

Proof. For fixed c, the contribution to s_P of the fiber over c has the form

$$M \cdot \sum_{(c_{kk}, \ldots, c_{k,n-1}) \in \Pi_P} x_{k-1}^{C_{k-1} - \sum_j c_{kj}} x_k^{\sum_j c_{kj} - C_{k+1}},$$

where M is a monomial that does not contain x_{k-1} nor x_k, and $C_i = \sum_{j=i}^{n} c_{ij}$ only depends on c for $i \neq k$. This summation has the same form as the one in Lemma 3.2, so applying π_{k-1} as per the lemma immediately gives the result. □

Remark 3.5. The operator that produces Q from P is denoted by D_{k-1} in [Kir16]. However, it is important to note that the operator D_{k-1} will not in general yield a parapolytope (or even necessarily a polytope) from a general parapolytope P. Indeed, the inequality $x_{kn} \leqslant \nu_n$ satisfied by Q can cause $\varphi_{k-1}(Q)$ and $\varphi_{k+1}(Q)$ to have fibers that are not parallelepipeds due to the sums $\sum c_{k-1,j}$ and $\sum c_{k+1,j}$ appearing in ν_n. In the application below, we will see that this issue does not occur since most of the terms in $\sum c_{k-1,j} + \sum c_{k+1,j}$ will be canceled out by $-\sum(\mu_j + \nu_j)$.

We are now ready to prove our main theorem.

Proof of Theorem 2.7. Let D be a column-convex diagram with n rows with

$$\mathrm{Par}_D = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\},$$

and let

$$\mathcal{P}_D = \mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)}).$$

We first claim that we can reduce to the case when D does not contain any boxes in the first row. Indeed, adding a column with boxes in rows $1, 2, \ldots, k$ to D serves to add 1 to each part of $\lambda^{(k)}$, which, by Lemma 2.13, translates $\mathrm{GT}(\lambda^{(k)})$ by the single point $\mathrm{GT}(1^k)$ and hence does the same to $\mathcal{P}_D$. This translation adds $k+1-i$ to the sum of row i for $i = 1, \ldots, k$, so it multiplies $s_{\mathcal{P}_D}$ by $x_1 x_2 \cdots x_k$. Since Proposition 3.1 shows that adding this column also multiplies $\mathfrak{s}_D$ by $x_1 x_2 \cdots x_k$, the claim follows.

Therefore, we may assume that D has no boxes in the first row, so that $\lambda_k^{(k)} = 0$ for all k, which implies that $\mathcal{P}_D$ is contained in the hyperplane $x_{1n} = 0$. Denote by $\mathcal{P}_D^{(m)}$ the intersection of $\mathcal{P}_D$ with the subspace $x_{1n} = x_{2n} = \cdots = x_{mn} = 0$. In fact, $\mathcal{P}_D^{(m)}$ is also the orthogonal projection of $\mathcal{P}_D$ onto this subspace. To see this, note that for any Gelfand–Tsetlin pattern $(y_{ij})_{1 \leqslant i \leqslant j \leqslant k} \in \mathrm{GT}(\lambda^{(k)})$, setting $y_{in} = 0$ for any $i \leqslant m$ again yields a valid Gelfand–Tsetlin pattern. Thus for any $(x_{ij}) \in \mathcal{P}_D$, setting $x_{in} = 0$ for all $i \leqslant m$ will again yield an element of $\mathcal{P}_D$.

Note also that $\mathcal{P}_D^{(n)}$ is just a translate of $\mathcal{P}_{\widetilde{D}}$ where $\widetilde{D}$ is the diagram obtained from D by shifting each box up by one row. (Any Gelfand–Tsetlin pattern for $\lambda^{(k)}$ in which the last entry in each row is 0 is just a Gelfand–Tsetlin pattern for $\lambda^{(k)}$, thought of as a partition of length $k-1$.) It follows that $\mathfrak{s}_{\widetilde{D}} = s_{\mathcal{P}_{\widetilde{D}}} = s_{\mathcal{P}_D^{(n)}}$.

We first show that the inequalities defining $\mathcal{P}_D^{(m)}$ are precisely the inequalities for $\mathcal{P}_D$ described in Proposition 2.9 that do not involve any x_{in} for $i \leqslant m$ (together with $x_{1n} = x_{2n} = \cdots = x_{mn} = 0$). Clearly any inequality of the form $x_{i-1,n-1} \geqslant x_{in}$ for $i \leqslant m$ is redundant since it is implied by inequality (∗) for $i_1 = n - i$ and $j_1 = n - 1$. Then consider any inequality (∗) for a sequence I with $j_k = n$ and $i_{k-1} \geqslant n - m$:

$$\sum_{s=1}^{k-1} x_{j_s - i_s, j_s} - \sum_{s=1}^{k-2} x_{j_{s+1} - i_s, j_{s+1}} + x_{n-i_k,n} - x_{n-i_{k-1},n} \geqslant \sum_{s=0}^{i_k} \lambda_{j_1 - s}^{(n-s)}.$$

Let I' be the sequence obtained from I by removing i_k and j_k. The corresponding inequality is

$$\sum_{s=1}^{k-1} x_{j_s - i_s, j_s} - \sum_{s=1}^{k-2} x_{j_{s+1} - i_s, j_{s+1}} \geqslant \sum_{s=0}^{i_{k-1}} \lambda_{j_1 - s}^{(n-s)}.$$

Since $x_{n-i_k,n} \geqslant 0$, $x_{n-i_{k-1},n} = 0$, and $i_{k-1} > i_k$, we see that the inequality for I follows immediately from that for I'.

Since none of the inequalities defining $\mathcal{P}_D^{(m)}$ involve two coordinates in the same row, $\mathcal{P}_D^{(m)}$ is a parapolytope. It therefore suffices to show that $\mathcal{P}_D^{(m)}$ and $\mathcal{P}_D^{(m-1)}$ are related as in Lemma 3.4, for it will then follow that $s_{\mathcal{P}_D^{(m-1)}} = \pi_{m-1} s_{\mathcal{P}_D^{(m)}}$, which combined with $\mathfrak{s}_{\widetilde{D}} = s_{\mathcal{P}_D^{(n)}}$ and $s_{\mathcal{P}_D} = s_{\mathcal{P}_D^{(1)}}$ will imply that $s_{\mathcal{P}_D} = \pi_1 \pi_2 \cdots \pi_{n-1}(\mathfrak{s}_{\widetilde{D}}) = \mathfrak{s}_D$ by Proposition 3.1, as desired.

Therefore, fix c_{ij} for $i \neq m$, with $c_{in} = 0$ for $i < m$, and define $\mu_m, \ldots, \mu_n, \nu_m, \ldots, \nu_n$ as in Definition 3.3 for $\mathcal{P}_D^{(m-1)}$. We claim that $\nu_j + \mu_{j-1} = c_{m-1,j-1} + c_{m+1,j}$. It will then follow by

summing over all j that

$$\nu_n = \sum_{j=m-1}^{n-1} c_{m-1,j} + \sum_{j=m+1}^{n} c_{m+1,j} - \sum_{j=m}^{n-1} (\mu_j + \nu_j).$$

Together with noting that the only lower bound on x_{mn} is 0, this will complete the proof by Lemma 3.4.

Consider the upper bounds on x_{mj} in $\mathcal{P}_D^{(m-1)}$. We need to show that if $x_{mj} \leqslant C$ (where C is some function of c_{ij} for $i \neq m$), then $x_{m,j-1} \geqslant c_{m-1,j-1} + c_{m+1,j} - C$. This is immediate for the inequality $x_{mj} \leqslant c_{m-1,j-1}$ since $x_{m,j-1} \geqslant c_{m+1,j}$. Then consider a sequence I such that $j_{s'+1} - i_{s'} = m$ and $j_{s'+1} = j$ for some s', so that $-x_{mj}$ appears on the left side of $(*)$. Thus $C - x_{mj} \geqslant 0$, where

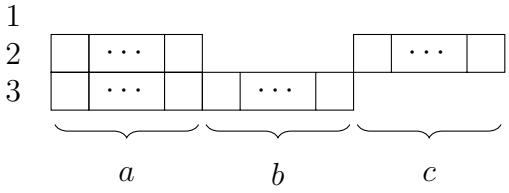
$$C = \sum_{s=1}^{k} c_{j_s - i_s, j_s} - \sum_{\substack{1 \leqslant s \leqslant k-1 \\ s \neq s'}} c_{j_{s+1} - i_s, j_{s+1}} - \sum_{s=0}^{i_k} \lambda_{j_1 - s}^{(n-s)}.$$

Inserting $j_{s'+1} - 1 = j - 1$ before $j_{s'+1} = j$ as well as $i_{s'} - 1 = j - m - 1$ before $i_{s'} = j - m$ in the sequence I yields a new sequence I'. The left side of $(*)$ for I' differs from the left side of $(*)$ for I by $x_{m,j-1} + x_{m,j} - c_{m-1,j-1} - c_{m+1,j}$. Therefore the inequality $(*)$ for I' is equivalent to

$$C + x_{m,j-1} - c_{m-1,j-1} - c_{m+1,j} \geqslant 0,$$

or $x_{m,j-1} \geqslant c_{m-1,j-1} + c_{m+1,j} - C$, as desired. A similar argument shows that any lower bound $x_{m,j-1} \geqslant C'$ yields an upper bound $x_{mj} \leqslant c_{m-1,j-1} + c_{m+1,j} - C'$, which completes the proof. □

Example 3.6. Let $n = 3$, and let D be the column-convex diagram shown below with $\lambda^{(3)} = (a + b, a, 0)$, $\lambda^{(2)} = (c, 0)$, and $\lambda^{(1)} = (0)$.



Using the notation in the proof of Theorem 2.7, all the polytopes $\mathcal{P}_D^{(m)}$ for $m = 1, 2, 3$ have $x_{11} = a + b$, $x_{12} = a + c$, and $x_{13} = 0$.

- For $m = 3$, $\mathcal{P}_D^{(3)}$ is a segment since we have $a \leqslant x_{22} \leqslant a + b$.
- For $m = 2$, the fiber of $\mathcal{P}_D^{(2)}$ above a point of $\mathcal{P}_D^{(3)}$ is defined by $0 \leqslant x_{33} \leqslant x_{22}$, making $\mathcal{P}_D^{(2)}$ a trapezoid. Note that for fixed x_{33}, the condition on x_{22} is that $\max\{a, x_{33}\} \leqslant x_{22} \leqslant a+b$.

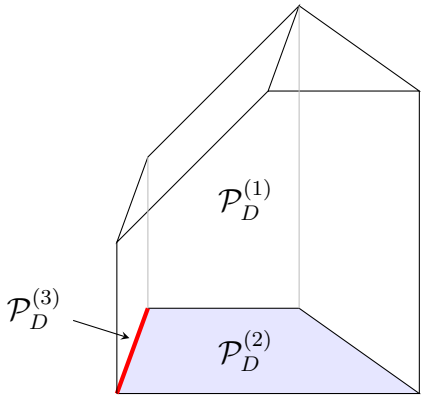


Figure 3.1: $\mathcal{P}_D = \mathcal{P}_D^{(1)}$ with faces $\mathcal{P}_D^{(2)}$ and $\mathcal{P}_D^{(3)}$ as in Example 3.6. See also Example 2.11. Note that $\mathcal{P}_D^{(1)}$ is a parapolytope (as are $\mathcal{P}_D^{(2)}$ and $\mathcal{P}_D^{(3)}$): for instance, it intersects any plane $x_{33} = c$ (which is parallel to the leftmost face) in a coordinate rectangle.

- For $m = 1$, the fiber of $\mathcal{P}_D = \mathcal{P}_D^{(1)}$ above a point of $\mathcal{P}_D^{(2)}$ is defined by

$$\begin{aligned} 0 \leqslant x_{23} &\leqslant x_{11} + x_{12} + x_{13} + x_{33} - (\mu_2 + \nu_2) \\ &= (a+b) + (a+c) + 0 + x_{33} - (\max\{a, x_{33}\} + a + b) \\ &= c + \min\{a, x_{33}\}. \end{aligned}$$

  This is equivalent to the inequalities on x_{23} given in Example 2.11:

$$\lambda_2^{(2)} + \lambda_3^{(3)} = 0 \leqslant x_{23} \leqslant c + a = x_{12}, \qquad x_{23} \leqslant c + x_{33} = x_{12} - \lambda_2^{(3)} + x_{33}.$$

See Figure 3.1 for a depiction of $\mathcal{P}_D^{(m)}$ for $m = 3, 2, 1$.

Note that in the special case when w is a Grassmannian permutation, we recover the fact that the integer point transforms of Gelfand–Tsetlin polytopes project to Schur polynomials.

Corollary 3.7. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ is a projection of the integer point transform of* $\mathrm{GT}(\lambda)$.

Proof. If w is the Grassmannian permutation (with descent at k) such that $\mathfrak{S}_w = s_\lambda(x_1, \ldots, x_k)$, then in Par_D, $\lambda^{(k)} = \lambda$ and all other $\lambda^{(i)}$ are empty. Theorem 2.7 then gives $\mathcal{P}_D = \mathrm{GT}(\lambda)$, implying the result. □

In this case, the construction of $\mathcal{P}_D = \mathrm{GT}(\lambda)$ from the proof of Theorem 2.7 in terms of the operators D_i specializes to the description given by Kiritchenko [Kir16, Theorem 3.4].

Remark 3.8. The results of Magyar [Mag98] allow one to compute the character of the flagged Schur module for any diagram whose columns form a so-called *strongly separated family* (or

equivalently, for any *percentage-avoiding diagram* [RS98]), which includes all Rothe diagrams of permutations. The technique above can be used to find suitable polytopes for a somewhat more general class of diagrams and permutations as Minkowski sums of faces of Gelfand–Tsetlin polytopes (such as the intermediate steps $\mathcal{P}_D^{(m)}$ in the proof of Theorem 2.7), but it does not apply in full generality to all Schubert polynomials due to the ill behavior of general parapolytopes. For example, attempting to use these geometric operators to mimic the calculation $\pi_1\pi_2(x_1x_2 \cdot \pi_3\pi_2(x_1x_2))$ does not yield a parapolytope for the reason described in Remark 3.5.

Remark 3.9. It was shown in [FMSD18] (see also [MTY19]) that the Newton polytope of a Schubert polynomial is a generalized permutahedron, so it follows that the polytopes $\mathcal{P}_D$ in Theorem 2.7 project onto generalized permutahedra. One can also prove this directly from the definition of $\mathcal{P}_D$ using the fact that each Minkowski summand is a Gelfand–Tsetlin polytope that projects onto a permutahedron.

Combining Theorem 2.7 and Corollary 2.14 gives the following combinatorial interpretation for $\mathfrak{s}_D$. For a filling T of D, let x^T denote the product of variables x_i over all entries i appearing in T.

Corollary 3.10. *Let D be a column-convex diagram. Then*

$$\mathfrak{s}_D = \sum_T x^T,$$

where T ranges over all column-strict, row-flagged labelings of T up to row equivalence.

Proof. After specializing (as in Theorem 2.7) the monomial of the integer point transform of $\mathcal{P}_D$ corresponding to a filling T (as in Corollary 2.14), the exponent of x_i appearing is the number of entries at least i appearing in any row of T minus the number of entries at least $i-1$ appearing in any row of T, which is the number of entries equal to i in T. □

It would be interesting to investigate how this combinatorial interpretation is related to others, such as the one given by Reiner–Shimozono [RS98] or, in the case that $D = D(w)$ for a permutation w, other known interpretations for Schubert polynomials (such as in [BJS93]) which are typically not polytopal in nature.

4. Gelfand–Tsetlin polytopes as flow polytopes

In this section we show that the Gelfand–Tsetlin polytope is integrally equivalent to a flow polytope and give alternative proofs of several known results using flow polytopes. This will allow us to view our earlier results from the perspective of flow polytopes. We start by defining flow polytopes and providing the necessary background on them.

4.1. Background on flow polytopes

Let G be a loopless directed acyclic connected (multi-)graph on the vertex set $[n+1]$ with m edges. An integer vector $a = (a_1, \ldots, a_n, -\sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$ is called a **netflow vector**.

A pair (G, a) will be referred to as a **flow network**. To minimize notational complexity, we will typically omit the netflow a when referring to a flow network G, describing it only when defining G. When not explicitly stated, we will always assume vertices of G are labeled so that $(i, j) \in E(G)$ implies $i < j$.

To each edge (i, j) of G, associate the type A positive root $e_i - e_j \in \mathbb{R}^n$. Let M_G be the incidence matrix of G, the matrix whose columns are the multiset of vectors $e_i - e_j$ for $(i, j) \in E(G)$. A **flow** on a flow network G with netflow a is a vector $f = (f(e))_{e\in E(G)}$ in $\mathbb{R}_{\geqslant 0}^{E(G)}$ such that $M_G f = a$. Equivalently, for all $1 \leqslant i \leqslant n$, we have

$$\sum_{e=(k,i)\in E(G)} f(e) + a_i = \sum_{e=(i,k)\in E(G)} f(e).$$

The fact that the netflow of vertex $n + 1$ is $-\sum_{i=1}^{n} a_i$ is implied by these equations.

Define the **flow polytope** $\mathcal{F}_G(a)$ of a graph G with netflow a to be the set of all flows on G:

$$\mathcal{F}_G = \mathcal{F}_G(a) = \{f \in \mathbb{R}_{\geqslant 0}^{E(G)} \mid M_G f = a\}.$$

Remark 4.1. When G is a flow network (G, a), we will write $\mathcal{F}_G$ for $\mathcal{F}_G(a)$.

4.2. The Gelfand–Tsetlin polytope as a flow polytope

In this section, we will show that the Gelfand–Tsetlin polytope can be realized as a flow polytope.

Recall that two integral polytopes $\mathcal{P}$ in $\mathbb{R}^d$ and $\mathcal{Q}$ in $\mathbb{R}^m$ are **integrally equivalent** if there is an affine transformation $\varphi\colon \mathbb{R}^d \to \mathbb{R}^m$ whose restriction to $\mathcal{P}$ is a bijection $\varphi\colon \mathcal{P} \to \mathcal{Q}$ that preserves the lattice, i.e., φ is a bijection between $\mathbb{Z}^d \cap \operatorname{aff}(\mathcal{P})$ and $\mathbb{Z}^m \cap \operatorname{aff}(\mathcal{Q})$, where $\operatorname{aff}(\cdot)$ denotes affine span. The map φ is called an **integral equivalence**. Note that integrally equivalent polytopes have the same Ehrhart polynomials and therefore the same volume.

We now define the flow network G_λ, describing the graph and its associated netflow (see Remark 4.1). For an illustration of G_λ, see Figure 4.1.

Definition 4.2. For a partition $\lambda \in \mathbb{Z}_{\geqslant 0}^n$ with $n \geqslant 2$, let G_λ be defined as follows.

If $n = 1$, let G_λ be a single vertex v_{22} defined to have flow polytope consisting of one point, 0. Otherwise, let G_λ have vertices

$$V(G_\lambda) = \{v_{ij} \mid 2 \leqslant i \leqslant j \leqslant n\} \cup \{v_{i,i-1} \mid 3 \leqslant i \leqslant n + 2\} \cup \{v_{i,n+1} \mid 3 \leqslant i \leqslant n + 1\}$$

and edges

$$\begin{aligned} E(G_\lambda) = {} & \{(v_{ij}, v_{i+1,j}) \mid 2 \leqslant i \leqslant j \leqslant n\} \cup \{(v_{i,n+1}, v_{i+1,n+1}) \mid 3 \leqslant i \leqslant n + 1\} \\ & \cup \{(v_{ij}, v_{i+1,j+1}) \mid 2 \leqslant i \leqslant j \leqslant n\} \cup \{(v_{i,i-1}, v_{i+1,i}) \mid 3 \leqslant i \leqslant n + 1\}. \end{aligned}$$

The default netflow vector on G_λ is as follows:

- To vertex v_{2j} for $2 \leqslant j \leqslant n$, assign netflow $\lambda_{j-1} - \lambda_j$.
- To vertex $v_{n+2,n+1}$, assign netflow $\lambda_n - \lambda_1$.

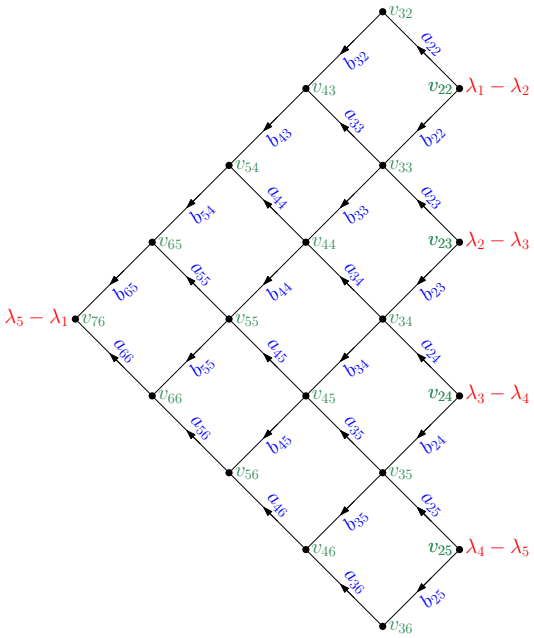


Figure 4.1: The flow network G_λ with $\ell(\lambda) = 5$.

- To all other vertices, assign netflow 0.

Given a flow on G_λ, denote the flow value on each edge $(v_{ij}, v_{i+1,j})$ by a_{ij}, and denote the flow value on each edge $(v_{ij}, v_{i+1,j+1})$ by b_{ij}.

Recall from the introduction the definition of the **Gelfand–Tsetlin polytope** $\mathrm{GT}(\lambda)$ in terms of triangular arrays (†) using inequalities (1.1) and (1.2). We are now ready to prove that $\mathrm{GT}(\lambda)$ can be realized as a flow polytope.

Proof of Theorem 1.2. To map a point $(x_{ij})_{i,j} \in \mathrm{GT}(\lambda)$ to $\mathcal{F}_{G_\lambda}$, use the map

$$\begin{aligned} a_{i\,j} &= x_{i-1,j-1} - x_{ij}, \\ b_{i\,j} &= x_{ij} - x_{i-1,j}. \end{aligned}$$

Conversely, to map a flow $f \in \mathcal{F}_{G_\lambda}$ to $\mathrm{GT}(\lambda)$, use either

$$x_{ij} = \lambda_j + \sum_{k=2}^{i} b_{kj} \quad \text{or} \quad x_{ij} = \lambda_{j-i+1} - \sum_{k=0}^{i-2} a_{i-k,j-k}.$$

It is easily checked these two maps are inverses of each other and are both integral, completing the proof. □

Example 4.3. For $n = 5$, the integral equivalences between $\mathrm{GT}(\lambda)$ and $\mathcal{F}_{G_\lambda}$ are shown in Figure 4.2.

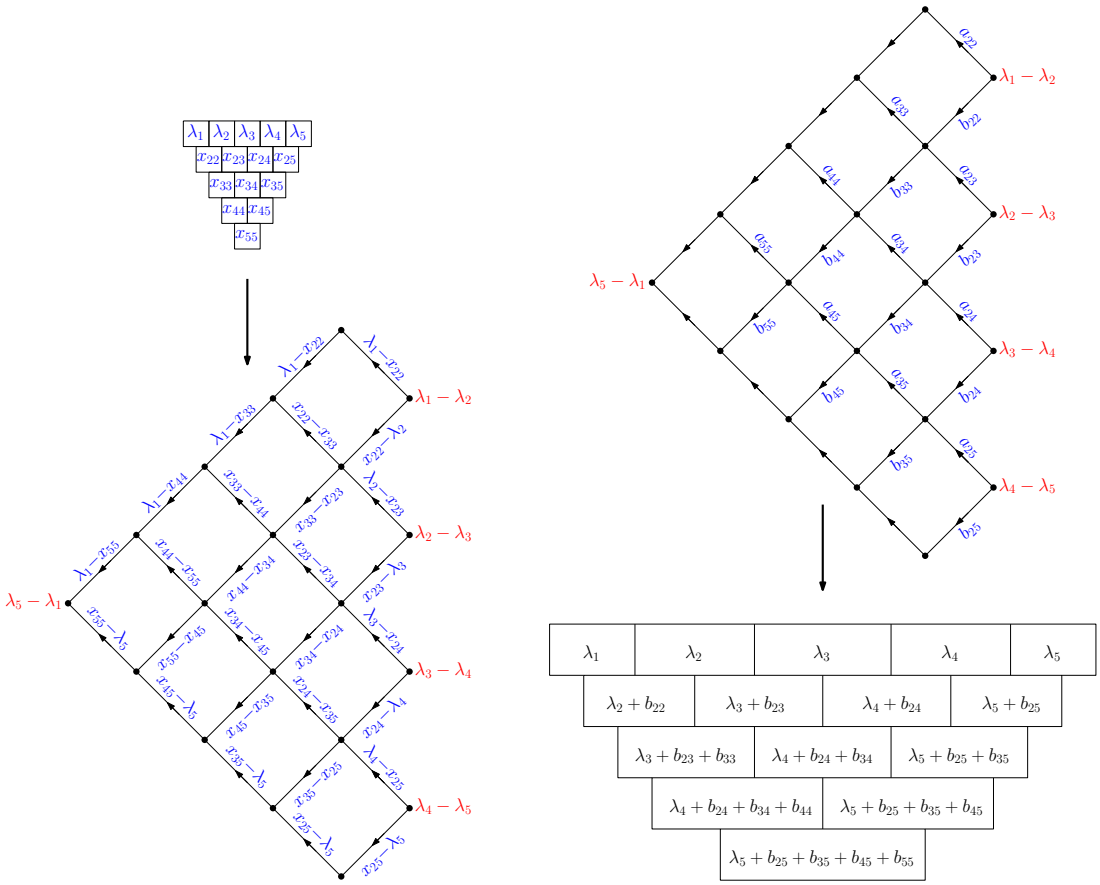


Figure 4.2: Integral equivalences between $\mathrm{GT}(\lambda)$ and $\mathcal{F}_{G_\lambda}$ when $n = 5$.

4.3. Consequences of the Gelfand–Tsetlin polytope being a flow polytope

Here we provide a few corollaries to the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda)$ being integrally equivalent to the flow polytope $\mathcal{F}_{G_\lambda}$. In [LMSD19] further applications of this result are given, particularly about the volume and Ehrhart polynomial of Gelfand–Tsetlin polytopes. The corollaries presented below are all well known; we include them here to demonstrate proofs via flow polytopes. We begin with two well-known results about flow polytopes, and then we give their applications to Gelfand–Tsetlin polytopes.

Lemma 4.4 ([BV08])**.** *For a graph G on $[n+1]$ and nonnegative integers $a_1, \ldots, a_n$,*

$$\mathcal{F}_G(a_1, \ldots, a_n, -\sum_{i=1}^{n} a_i) = a_1 \mathcal{F}_G(e_1 - e_{n+1}) + a_2 \mathcal{F}_G(e_2 - e_{n+1}) + \cdots + a_n \mathcal{F}_G(e_n - e_{n+1}).$$

Proof. One inclusion is proven by adding flows edgewise. The other is shown by induction on the number of nonzero a_i. □

Corollary 4.5. *If G is a graph on $[n+1]$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are nonnegative integers, then*

$$\mathcal{F}_G(a_1, \ldots, a_n, -\sum_{i=1}^{n} a_i) + \mathcal{F}_G(b_1, \ldots, b_n, -\sum_{i=1}^{n} b_i) = \mathcal{F}_G(a_1 + b_1, \ldots, a_n + b_n, -\sum_{i=1}^{n} a_i + b_i).$$

Proof. Induct on the number of nonzero b_i and use Lemma 4.4. □

As a consequence of the previous two results and the integral equivalence of $\mathrm{GT}(\lambda)$ and $\mathcal{F}_{G_\lambda}$, we obtain the following fact about Gelfand–Tsetlin polytopes.

Lemma 4.6. *If λ and μ are partitions with n parts, then*

$$\mathrm{GT}(\lambda) + \mathrm{GT}(\mu) = \mathrm{GT}(\lambda + \mu),$$

where $\lambda + \mu = (\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n)$.

Proof. Express each term as a flow polytope using Theorem 1.2 and apply Corollary 4.5. □

We can also easily derive a proof of Lemma 2.13.

Proof of Lemma 2.13. Either apply Lemma 4.6 repeatedly, or combine Theorem 1.2 with Lemma 4.4. □

Recall that the **Schur polynomial** s_λ can be expressed as

$$s_\lambda(x_1, \ldots, x_n) = \sum_{P \in \mathrm{GT}(\lambda) \cap \mathbb{Z}^{\binom{n+1}{2}}} x_1^{wt(P)_1} x_2^{wt(P)_2} \cdots x_n^{wt(P)_n}$$

where $wt\colon \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^n$ is the **weight map**, defined by

$$wt(P)_i = \sum_{j=i}^{n} x_{ij} - \sum_{j=i+1}^{n} x_{i+1,j}$$

for $P \in \mathrm{GT}(\lambda)$. We now introduce the flow polytopal analogue of wt and study it. Recall the variables $\{a_{ij}\}_{i,j} \cup \{b_{ij}\}_{i,j}$ of Definition 4.2: in $\mathcal{F}_{G_\lambda}$, a_{ij} represents the flow on the edge $(v_{ij}, v_{i+1,j})$ and b_{ij} represents the flow on the edge $(v_{ij}, v_{i+1,j+1})$.

Definition 4.7. Let λ be a partition with n parts. Define the **graphical weight map**

$$gwt\colon \mathbb{R}^{E(G_\lambda)} \to \mathbb{R}^n$$

by setting

$$gwt(x_{(v_{ij},v_{i+1,j})}) = e_{i-1} \quad \text{and} \quad gwt(x_{(v_{ij},v_{i+1,j+1})}) = \mathbf{0},$$

so in particular

$$gwt(a_{ij}) = e_{i-1} \quad \text{and} \quad gwt(b_{ij}) = \mathbf{0}.$$

Proposition 4.8. *For a partition λ with n parts, let $f \in \mathcal{F}_{G_\lambda}$ correspond to $P_f \in \mathrm{GT}(\lambda)$. Then, the maps gwt and wt are related by the translation*

$$wt(P_f) = gwt(f) + \lambda_n \mathbf{1}_n,$$

where $\mathbf{1}_n$ denotes the vector of all ones in $\mathbb{R}^n$.

Proof. We have

$$\begin{aligned} gwt(f)_i &= a_{i+1,i+1} + \cdots + a_{i+1,n} + a_{i+1,n+1} \\ &= a_{i+1,i+1} + \cdots + a_{i+1,n} + b_{2n} + \cdots + b_{in}. \end{aligned}$$

Using the integral equivalence $x_{ij} = \lambda_{j-i+1} - \sum_{k=0}^{i-2} a_{i-k,j-k}$ between $\mathrm{GT}(\lambda)$ and $\mathcal{F}_{G_\lambda}$,

$$\begin{aligned} wt(P_f)_i &= \sum_{j=i}^{n} x_{ij} - \sum_{j=i+1}^{n} x_{i+1,j} \\ &= x_{in} + \sum_{j=i}^{n-1} (x_{ij} - x_{i+1,j+1}) = x_{in} + \sum_{j=i+1}^{n} a_{i+1,j}. \end{aligned}$$

Now, using the integral equivalence $x_{ij} = \lambda_j + \sum_{k=2}^{i} b_{kj}$, we have

$$\begin{aligned} (gwt(f) - wt(P_f))_i &= x_{in} - \sum_{k=2}^{i} b_{kn} \\ &= \left(\lambda_n + \sum_{k=2}^{i} b_{kn}\right) - \sum_{k=2}^{i} b_{kn} = \lambda_n. \end{aligned}$$

□

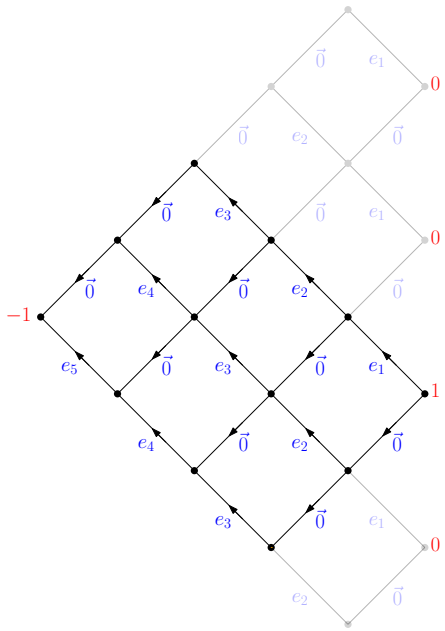


Figure 4.3: $\mathrm{GT}(1,1,1,0,0)$ and the associated map gwt.

Using the map gwt, we now describe the polytopes $\mathrm{GT}(1^k0^{n-k})$ and rederive a result of Postnikov from [Pos09].

Proposition 4.9. *If λ is of the form 1^k0^{n-k} with $1 \leqslant k \leqslant n$, then $gwt(\mathcal{F}_{G_\lambda})$ equals the hypersimplex $\Delta_{k,n} = \mathrm{Conv}(\{x \in [0,1]^n \mid x_1 + x_2 + \cdots + x_n = k\})$.*

Proof. If λ is of the form 1^k0^{n-k}, then G_λ will have a single source with netflow 1 and a single sink with netflow -1. Ignoring all edges and vertices not lying on path from the source to sink (which will carry zero flow), we are left with a rectangular grid as shown in Figure 4.3. A path from source to sink in the grid requires k NW steps and $n-k$ SW steps. Recall (cf. [Hil03], Lemma 3.1) that the vertices of a flow polytope with a single source and sink are exactly the flows that are nonzero only on a path from source to sink.

Thus, the vertices of $\mathcal{F}_{G_\lambda}$ are exactly the flows with support a path from source to sink in the grid. These paths are in bijection with length n words on $\{N,S\}$ having k N's (corresponding to NW steps in the path) and $n-k$ S's (corresponding to SW steps in the path). By definition, the map gwt takes a vertex of $\mathcal{F}_{G_\lambda}$ to the vector with ones in the positions of the N's in the corresponding string, and zero elsewhere. Thus,

$$gwt(V(\mathcal{F}_{G_\lambda})) = \{x \in \{0,1\}^n \mid x_1 + \cdots + x_n = k\} = V(\Delta_{k,n}),$$

so $gwt(\mathcal{F}_{G_\lambda}) = \Delta_{k,n}$. □

Corollary 4.10 ([Pos09])**.** *The permutahedron $\mathcal{P}_\lambda = \mathrm{Conv}(S_n \cdot \lambda)$ of λ equals the Minkowski sum of hypersimplices*

$$\mathcal{P}_\lambda = (\lambda_1 - \lambda_2)\Delta_{1,n} + (\lambda_2 - \lambda_3)\Delta_{2,n} + \cdots + (\lambda_{n-1} - \lambda_n)\Delta_{n-1,n} + \lambda_n\Delta_{n,n}.$$

Proof. Since $wt(\mathrm{GT}(\lambda)) = \mathcal{P}_\lambda$, applying gwt to both sides of

$$\mathcal{F}_{G_\lambda} = \sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) \mathcal{F}_{G_{(1^k 0^{n-k})}}$$

and using Propositions 4.9 and 4.8 yields

$$\mathcal{P}_\lambda - \lambda_n \mathbf{1}_n = (\lambda_1 - \lambda_2)\Delta_{1,n} + (\lambda_2 - \lambda_3)\Delta_{2,n} + \cdots + (\lambda_{n-1} - \lambda_n)\Delta_{n-1,n}. \qquad \square$$

4.4. The Minkowski sum of Gelfand–Tsetlin polytopes

In this section we observe that the Minkowski sum of Gelfand–Tsetlin polytopes $\mathcal{P}_D$ appearing in Theorem 2.7 can be viewed naturally as a subset of a larger Gelfand–Tsetlin polytope.

Recall the embedding of the Gelfand–Tsetlin polytopes in the sum

$$\mathcal{P}_D = \mathrm{GT}(\lambda^{(1)}) + \mathrm{GT}(\lambda^{(2)}) + \cdots + \mathrm{GT}(\lambda^{(n)})$$

from Section 1. In light of Theorem 1.2, $\mathcal{P}_D$ should be integrally equivalent to a sum of flow polytopes

$$\mathcal{F}_{G_{\lambda^{(1)}}} + \cdots + \mathcal{F}_{G_{\lambda^{(n)}}}.$$

Just like for the Gelfand–Tsetlin polytope sum, we must specify how the graphs $G_{\lambda^{(i)}}$, $i \in [n]$, are embedded. Let us embed $G_{\lambda^{(k)}}$, $k \in [n]$, into $G_{\lambda^{(n)}}$ by identifying v_{ij} (see Definition 4.2) in $G_{\lambda^{(k)}}$ with $v_{i,j+n-k}$ in $G_{\lambda^{(k)}}$. Note that the trivial case $G_{\lambda^{(1)}}$ is just a single vertex with netflow 0 and flow polytope defined to be the single point 0.

Lemmas 4.11 and 4.13 below follow readily by the definitions and the integral equivalence given in Theorem 1.2:

Lemma 4.11. *The Minkowski sum*

$$\mathrm{GT}(\lambda^{(1)}) + \cdots + \mathrm{GT}(\lambda^{(n)})$$

is integrally equivalent to

$$\mathcal{F}_{G_{\lambda^{(1)}}} + \cdots + \mathcal{F}_{G_{\lambda^{(n)}}}$$

with the embedding specified above.

Definition 4.12. Given partitions $\lambda^{(k)}$ of size k for $k \in [n]$, let $G(\lambda^{(1)}, \ldots, \lambda^{(n)})$ denote the flow network obtained by overlaying the flow networks $G_{\lambda^{(1)}}, \ldots, G_{\lambda^{(n)}}$ according to the embedding specified above and adding the corresponding netflows. Let $\widehat{G}(\lambda^{(1)}, \ldots, \lambda^{(n)})$ denote the flow network obtained from $G_{\lambda^{(1)}}, \ldots, G_{\lambda^{(n)}}$ by moving all negative netflows to $v_{n+2,n+1}$ and replacing them by zero netflows. The case $n = 4$ is demonstrated in Figure 4.4.

Lemma 4.13. *The following polytope inclusions hold:*

$$\mathcal{F}_{G_{\lambda^{(n)}}} + \cdots + \mathcal{F}_{G_{\lambda^{(1)}}} \subset \mathcal{F}_{G(\lambda^{(1)},\ldots,\lambda^{(n)})} \subset \mathcal{F}_{\widehat{G}(\lambda^{(1)},\ldots,\lambda^{(n)})},$$

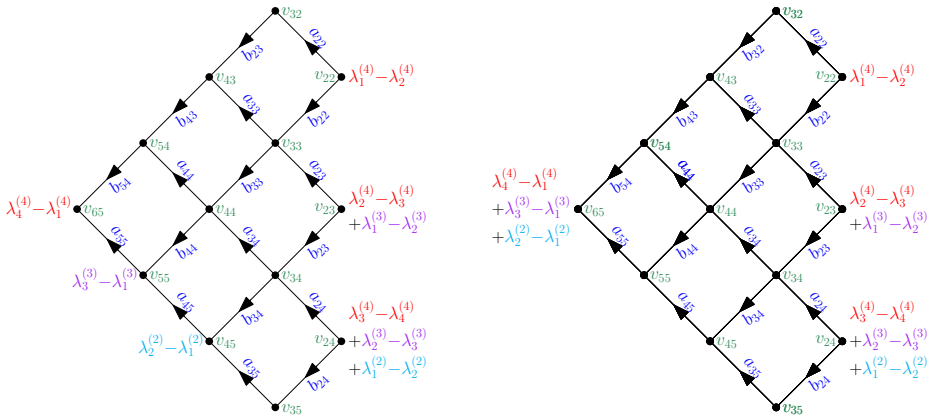


Figure 4.4: The flow networks $G(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)})$ (left) and $\widehat{G}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)})$ (right). Vertices without netflow indicated have netflow 0.

the latter being true up to an integral translation of $\mathcal{F}_{G(\lambda^{(1)},\ldots,\lambda^{(n)})}$.

In general, none of the above inclusions is an equality. The polytope $\mathcal{F}_{\widehat{G}(\lambda^{(1)},\ldots,\lambda^{(n)})}$ *is integrally equivalent to the Gelfand–Tsetlin polytope* $\mathrm{GT}(\mu)$ *where* μ_n *is arbitrary, and for* $k < n$,

$$\mu_k = \mu_{k+1} + \sum_{j=0}^{k-1} (\lambda_{k-j}^{(n-j)} - \lambda_{k-j+1}^{(n-j)}).$$

Thus, we conclude that for a column-convex diagram D the polytope $\mathcal{P}_D$ can be thought of as obtained from $\mathrm{GT}(\mu)$ specified in Lemma 4.13 via further hyperplane cuts. (Recall also Proposition 2.9, which gives another view on $\mathcal{P}_D$.)

Example 4.14. In Example 2.8, a description of the lattice points of $\mathcal{P}_D$ was found. By adding 1 to each entry x_{ii}, we arrive at a translation of $\mathcal{P}_D$ with lattice points

$$\begin{array}{ccccccc} 3 & & 3 & & & 3 & & & 1 \\ & 3 & & & 3 & & & a+c & \\ & & 3 & & & b+d & & & \\ & & & e+1 & & & & & \end{array}$$

where $0 \leqslant a \leqslant b \leqslant 1$ and $1 \leqslant c \leqslant d \leqslant e \leqslant 2$,

which form a proper subset of the lattice points in $\mathrm{GT}(\mu)$ for $\mu = (3, 3, 3, 1)$.

Acknowledgements

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