The "Figure of Merit", $Q/\mu_{max}^{2/3}$, for Beam Transport Through Periodic Focussing Systems

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Recent investigations[1] of the stability of high intensity beams (Kapchinskij-Vladimirskij distribution) have indicated that it may be prudent to design and operate transport systems in such a manner that

(i) the zero-intensity phase advance of individual particle betatron oscillations per period of the structure (σ_0) not exceed 90 degrees, and

(ii) the intensity be limited to values such that σ not be depressed below 40% or 50% of σ_0.

These two conditions may impose substantial restrictions on the beam current that one can plan to transport. The implications of these conditions can first be examined by reference to the scaled envelope equations[2] and then by de-scaling in accordance with additional restrictions (e.g., maximum pole-tip field for a quadrupole transport system). In this case the de-scaling procedure indicated the significance of the "figure of merit", defined as $Q/\mu_{max}^{2/3}$ in terms of the scaled variables. [Thus

$$I = C_2 \left(\frac{A}{q}\right)^{1/3} B_Q^{2/3} (\beta\gamma)^{5/3} \varepsilon_N^{2/3} \quad \frac{Q}{\mu_{max}^{2/3}}$$

where

$$C_2 = \frac{1}{4} \left(\frac{4\pi}{\mu_0}\right)^{5/6} \left(\frac{m_p c^2}{r_p}\right)^{1/6} = 3.66 \times 10^6 \text{ MKS·A units}$$

-- as given by the second of Eqns. (3) of Ref. 2.]

We accordingly present below a table of this quantity $Q/u_{max}^{2/3}$ for σ_0 = 90 deg. and for the slightly less marginal σ_0 = 80 deg., and for the tune depressions σ/σ_0 = 0.50 and σ/σ_0 = 0.40 (the latter value σ/σ_0 perhaps being close to marginal). The lattice designated by p = 0 is a symmetrical FODO structure with <u>no gaps</u> between the individual quadrupole elements, while that designated by p = 2 employs inter-quadrupole gaps of length equal to the length of the individual quadrupole elements themselves. These data thus serve to supplement results recorded earlier in Ref. 3. [The quantity Θ (= $\sqrt{R}$ L) is the half-period of the focusing structure, expressed in scaled units.]

For $\sigma/\sigma_0 = 0.50$					For $\sigma/\sigma_0 = 0.40$				
σ	Q	u_0	v_0	$Q/U_0^{2/3}$	σ	Q	u_0	v_0	$Q/U_0^{2/3}$
p = 0, Θ = 1.570796326795, σ_0 = 90°									
45.°	1.457	2.902_5	1.490	0.716	36.°	2.035_5	3.223	1.674	0.933
p = 0. Θ = 1.49668994341, σ_0 = 80°									
40°	1.375	2.879	1.586	0.679	32°	1.922	3.204	1.780	0.884
p = 2, Θ = 1.86361748098, σ_0 = 90°									
45°	1.228	3.234_6	1.599	0.562	36°	1.716	3.592	1.797	0.731
p = 2, Θ = 1.77643752505, σ_0 = 80°									
40°	1.158	3.201	1.705	0.533	32°	1.619	3.562	1.913	0.694

Somewhat more detailed data are recorded as an Addendum.

Limits for Solenoidal Focusing Systems

The quantity $(u^2 - \frac{1}{u^2})$ can play the role of a figure of merit for a focusing system formed by a continuous solenoid -- thus, from the second of Eqns. (4) in Ref. 2,

$$I = C_4 B_s (\beta\gamma) \varepsilon_N (u^2 - \frac{1}{u^2}),$$

where

$$C_4 = \frac{1}{4}\left(\frac{4\pi}{\mu_0}\right) = 2.5 \times 10^6$$ MKS-A units, or more simply

[e.g., from the definition of Q]:

$$I = \frac{1}{8}\left(\frac{4\pi}{\mu_0}\right) B_s\,(\beta\gamma)\,\varepsilon_N\,Q = 1.25 \times 10^6\, B_s(\beta\gamma)\,\varepsilon_N\,Q.$$

A limit to transportable intensity in such a system (and probably also for an interrupted-solenoid transport system) again must lie close to a value such that $\sigma/\sigma_0 \cong 0.4$.

For the continuous solenoid, one obtains (from the envelope equation scaled variables)

$$u^2 = \sqrt{1 + (\frac{Q}{4})^2} + \frac{Q}{4} \qquad \frac{1}{u^2} = \sqrt{1 + (\frac{Q}{4})^2} - \frac{Q}{4}$$

$$u^2 - \frac{1}{u^2} = \frac{Q}{2}$$

and from the single-particle equation

$$\omega/\omega_0 = \sqrt{1 - \frac{Q}{2u^2}}\ .$$

For $\omega/\omega_0 = 0.50$, one obtains $Q = 3.0$ and $u^2 - \frac{1}{u^2} = 1.5$ $(u^2 = 2.0)$

For $\omega/\omega_o = 0.40$, one obtains $Q = 4.2$ and $u^2 - \frac{1}{u^2} = 2.1$ ($u^2 = 2.5$).

References:

1. Lloyd Smith, HI-FAN-13; Ingo Hofmann and L. Jackson Laslett, HI-FAN-15 (Lawrence Berkeley Laboratory; October 1977).
2. G. R. Lambertson, L.J. Laslett, and L. Smith, IEEE Trans. Nucl. Sci., p. 993 (June 1977) -- LBL-5552.
3. Victor O. Brady and L. Jackson Laslett, "Figure of Merit", $Q'/u_m^{2/3}$, for Beam Transport Through a Periodic Quadrupole Lens System" (L.B.L. Internal Notes).

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ADDENDUM

Background Data

σ_0 (deg.)	$\Theta = \sqrt{K}\, L$	
	p = 0	p = 2
90.	1.570796326795	1.86361748098
80.	1.49668994341	1.77643752505
60.	1.31844943075	1.56609291708
30.	0.947331104711	1.12622633002

Computational Results

p = 0, $\Theta = 1.570796326795$

Q'	u_0	v_0	σ, deg.	$Q/u_0^{2/3}$
0.	2.1932800_5	1.	90.000000	(0.)
1.45742	2.9025023	1.4898640	45.000057	0.7162583
2.03553	3.22339166	1.67445715	36.000003	0.9328292

p = 2 $\Theta = 1.86361748098$

Q'	u_0	v_0	σ, deg.	$Q/u_0^{2/3}$
0.	2.4447567	1.0729641	90.000000	(0.)
1.2284	3.2345569	1.5991859	45.000056	0.5616468
1.7157	3.5920809^-	1.7974079	35.999510	0.7314937

p = 0 $\Theta = 1.49668994341$

Q'	u_0	v_0	σ, deg.	$Q/u_0^{2/3}$
0.	2.1282038	1.0826523	80.000000	(0.)
1.37491	2.8790852	1.5864476	40.000082	0.6793679
1.9218	3.2039376	1.7803018	32.000141	0.8842712

p = 0 $\Theta = 1.77643752505$

Q'	u_0	v_0	σ, deg.	$Q/u_0^{2/3}$
0.	2.3667256	1.1631666	80.000000	(0.)
1.158387	3.2012124	1.7049021	40.000000	0.5333071
1.619155	3.5623385^-	1.9132891	32.000027	0.6941686

Implication Concerning the Maschke-Courant Formula

For quadrupole focusing in a FODO lattice, descaling of the results obtained from study of the scaled equations leads to a transportable beam current given by[2]

$$I = 3.66 \times 10^6 \left(\frac{A}{q}\right)^{1/3} B_Q^{2/3} (\beta\gamma)^{5/3} \varepsilon_N^{5/3} \frac{Q}{u_{max.}^{2/3}} \quad \text{[MKS-A units]*}$$

if the "pole-tip field" limitation $B'(a_x)_m = K[B\rho](a_x)_m \leq B_Q$ is imposed. Expressed as beam power, through introduction of the factors $(\gamma-1)\frac{A}{q}\frac{M_p c^2}{e}$,

$$P = 3.43 \times 10^{15} \frac{Q}{u_{max}^{2/3}} \left(\frac{A}{q}\right)^{4/3} B_Q^{2/3} (\beta\gamma)^{5/3} \varepsilon_N^{2/3} (\gamma-1).$$

The factor $\frac{Q}{u_{max.}^{2/3}}$ has been seen to assume values in the range 0.53 - 0.93 if the limitations suggested earlier are adopted. One thus obtains

$$P = C \left(\frac{A}{q}\right)^{4/3} B_Q^{2/3} (\beta\gamma)^{5/3} \varepsilon_N^{2/3} (\gamma-1),$$

where the coefficient C falls in the range (2 to 3) x 10^{15}. For $Q/u_{max.}^{2/3} = 0.7$, $C = 2.4 \times 10^{15}$ and thus is 1.5 times the value (1.67×10^{15}) suggested by Maschke.

It is gratifying that the result obtained here indicates a coefficient quite close to that proposed by Maschke in 1976. It will be recalled that Courant indicated at that time that Maschke's formula appeared to be conservative.

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* Note: $\pi\varepsilon_N$ is the normalized emittance, so that the actual emittance is $\pi\varepsilon_N/(\beta\gamma)$.