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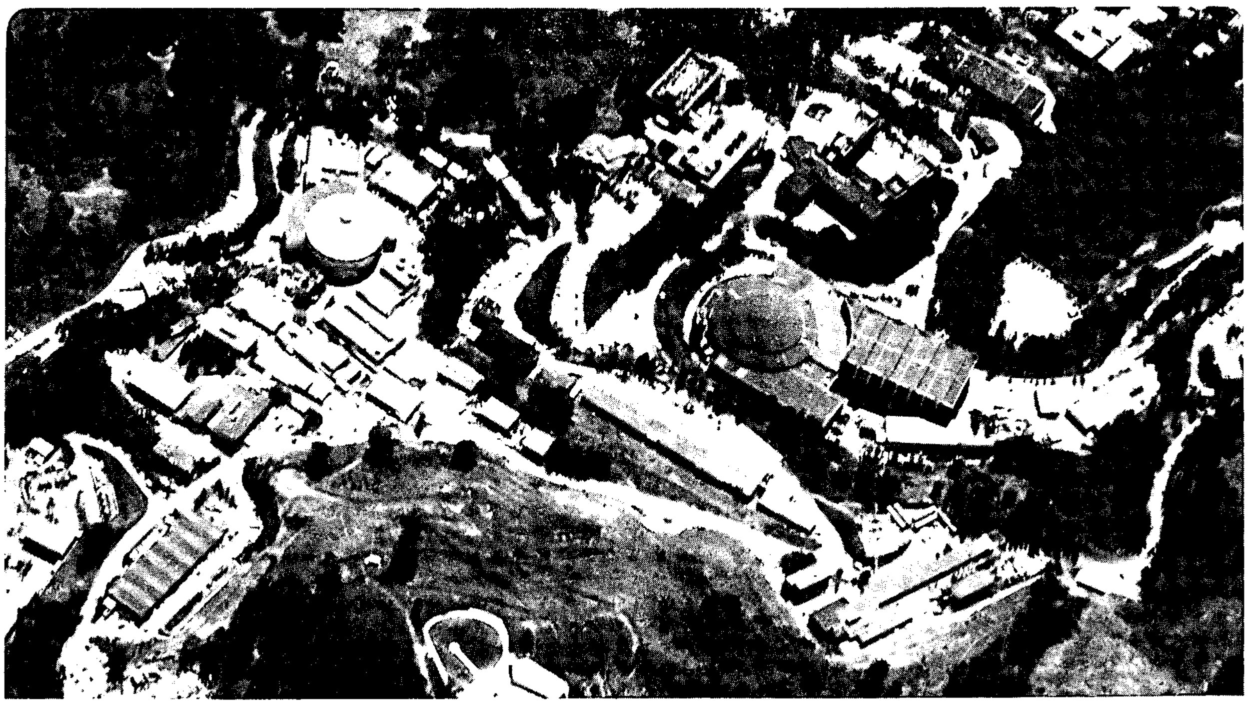

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INSTABILITY OF CERTAIN CAPILLARY SURFACES[1]

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INSTABILITY OF CERTAIN CAPILLARY SURFACES

Paul Concus and Robert Finn

A family of capillary surface configurations, shown in an earlier paper to be unstable for sufficiently small Bond number B, is here shown to be unstable also for B sufficiently large. Numerical evaluations indicate that it is unstable for all B. Some numerical calculations of the corresponding container shapes are included.

In the reference [1], the existence was shown of an "exotic " container $\mathcal{C}$ yielding a continuum $\mathcal{F}$ of rotationally symmetric capillary surfaces, all of which meet $\mathcal{C}$ in the same contact angle γ, bound with $\mathcal{C}$ the same volume V of fluid, and yield identical mechanical energy. These surfaces included the horizontal disk $u \equiv 0$, $x^2 + y^2 < r_0^2$, for any gravity field, as measured by a parameter κ, see [1] for definition. It was proved in [1] that if the "Bond number" $B = \kappa r_0^2 < 8$, then the energy of this configuration can be decreased by an asymmetric deformation; thus it cannot be expected that the family $\mathcal{F}$ would be seen physically, as the particular surface of $\mathcal{F}$ considered is unstable.

A reasoning was given in [1] to support the view that instability continues for all $B < 23.09$, but the question of what happens for larger B was left open.

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In the present note we describe an explicit deformation, which we prove decreases energy for all small enough B and also for all large enough B. Further, we provide a numerically computed graph of the second variation of energy induced by the deformation; this graph provides convincing evidence that energy is actually decreased regardless of B.

In what follows we use the notation of [1], with minor changes. We replace the deformation height ζ and radial coordinate r by normalized coordinates ζ/r_0, r/r_0, which we again denote by ζ, r. The relations (30), (35) and the relation preceding (43) in [1] then yield the (nondimensional) expression

$$\frac{1}{\pi}\ddot{E}(0) = \int_0^{2\pi}\int_0^1 |\nabla\zeta|^2 \rho d\rho d\theta + B\int_0^{2\pi}\int_0^1 \zeta^2 \rho d\rho d\theta$$

$$+\frac{B}{2-\sqrt{B}\,\frac{I_0(\sqrt{B})}{I_1(\sqrt{B})}}\oint_0^{2\pi}\zeta^2 d\theta \tag{1}$$

for the second variation of energy.

We choose the particular variation

$$\zeta = \sqrt{r}\,\frac{\sinh\sqrt{B}r}{\sinh\sqrt{B}}\sin\theta\ . \tag{2}$$

Setting $\tau = \sqrt{B}$, we calculate formally

$$\frac{1}{\pi}\ddot{E}(0) = \tau\coth\tau + \frac{7}{8}\frac{\coth\tau}{\tau} - \frac{1}{2}$$

$$-\frac{7}{8\sinh^2\tau} - \frac{\tau^2}{\tau\frac{I_0}{I_1} - 2}\ . \tag{3}$$

Consider first the case $\tau \sim 0$. We obtain, using power expansions for I_0, I_1 and for the hyperbolic functions (see, e.g., [2])

$$\frac{1}{\pi}\ddot{E}(0) = -\frac{35}{12} + 0(\tau^2)$$

which is negative for small enough τ, approaching the limit $-35/12$.

For large τ we find (cf [2] p.377)

$$I_0(\tau) \sim \frac{e^\tau}{\sqrt{2\pi\tau}} \left(1 + \frac{1}{8\tau}\right)$$
$$I_1(\tau) \sim \frac{e^\tau}{\sqrt{2\pi\tau}} \left(1 - \frac{3}{8\tau}\right) .$$

and thus

$$\frac{I_0(\tau)}{I_1(\tau)} \sim 1 + \frac{1}{2\tau} .$$

From (3) now follows

$$\frac{1}{\pi}\ddot{E}(0) \sim \tau \coth\tau + \frac{7}{8}\frac{\coth\tau}{\tau} - \frac{1}{2}$$
$$- \frac{7}{8\sinh^2\tau} - \left(1 + \frac{3}{2\tau}\right)\tau$$

which is again negative, for large enough τ, and approaches the limit -2.

Thus, the indicated deformation is energy decreasing, both for small and for large B. For intermediate values, the expression (3) is easily evaluated numerically. The result, shown in Fig. 1, should leave little doubt as to the physical behavior to be expected.

In [1] some "exotic" container shapes corresponding to $B = 0$ were calculated, for varying contact angles γ. In Fig. 2 we set $\gamma = \frac{\pi}{2}$ and indicate the change of shape for varying B. The case $B = 0$ is known explicity [3]; the other shapes were obtained by integration of (a parametric form of) equation (11) in [1].

We remark that if $\gamma \neq \frac{\pi}{2}$ then the variation (2) is not volume preserving, and formally a correction term should be added. This term is however of negligible order and disappears in the variational procedure, as in the discussion following equation (30) of [1].

References

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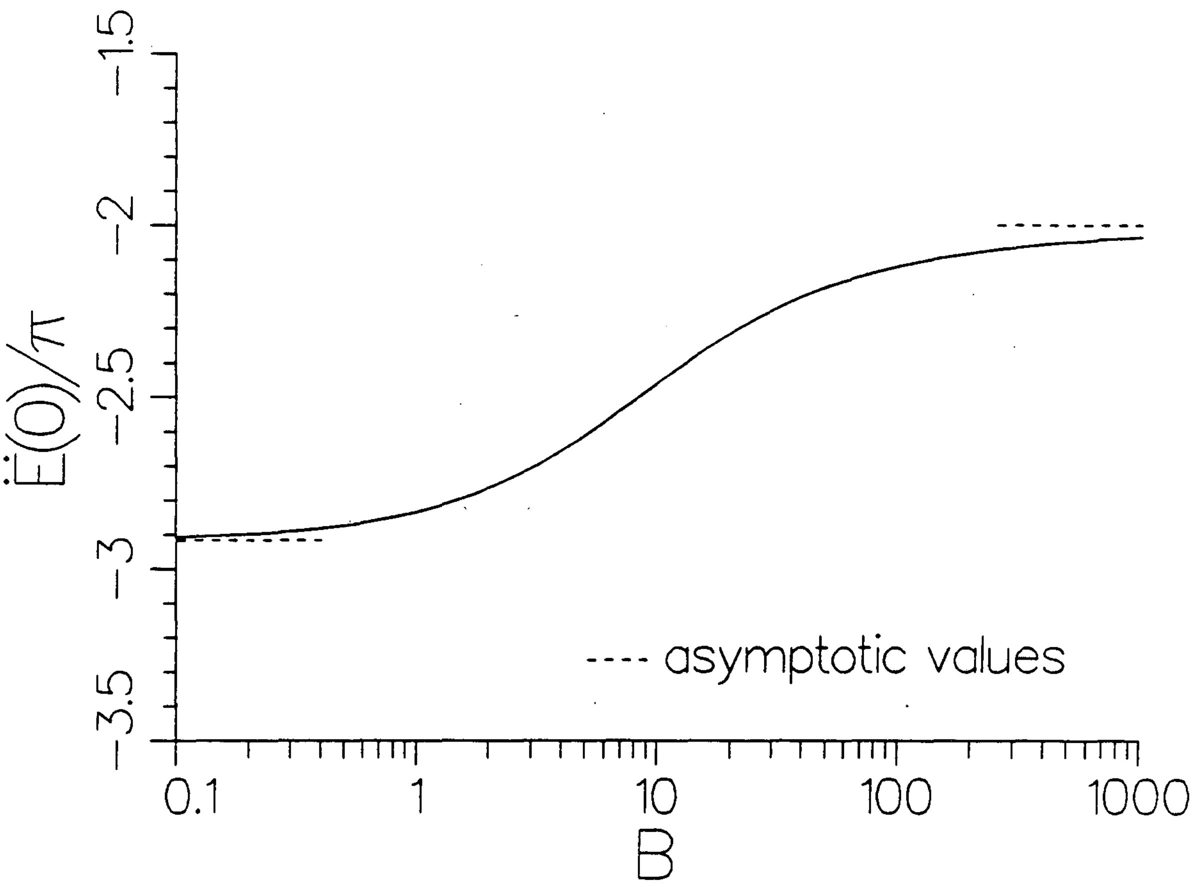

Figure 1

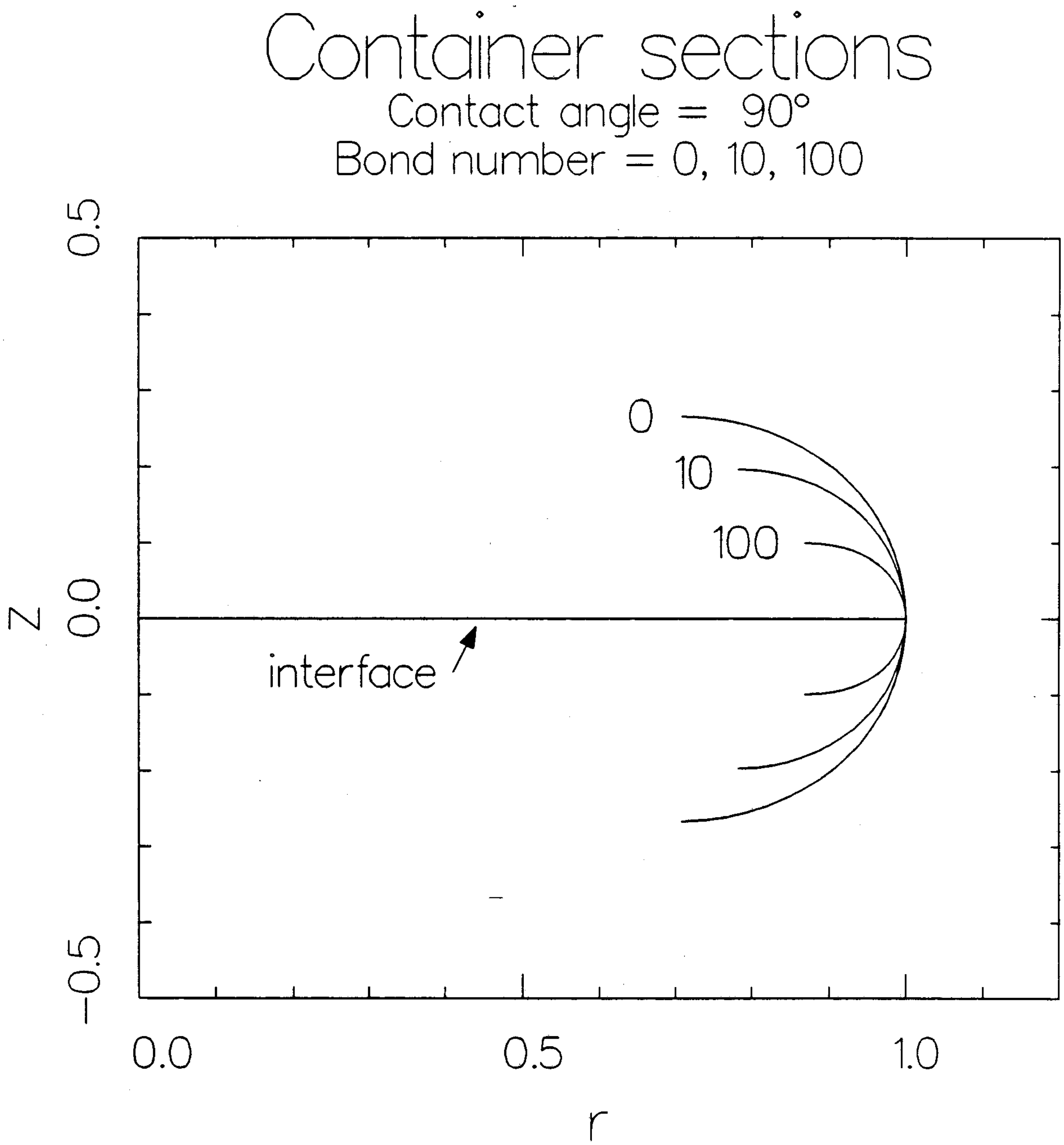
Container sections
Contact angle = 90°
Bond number = 0, 10, 100
0
10
100
interface
Z
0.5
0.0
−0.5
0.0
0.5
1.0
r

Figure 2

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