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Algebraic structures of fixed point Floer homology of Dehn twists

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by

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Abstract

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This dissertation discusses the algebraic structures of fixed point Floer homologies. The dissertation is divided into three chapters, and is adapted from two joint papers by the author. Chapter 1 gives a brief review of the fixed point Floer homology. Chapter 2 gives a detailed computation of the product and the coproduct structures on the fixed point Floer homology of iterations of a single Dehn twist on a surface. Chapter 3 gives a direct verification of the closed-string mirror symmetry for nodal curves based on the computations from Chapter 2.

To my parents, Ms. Bo He and Mr. Jian Zhao.

Contents

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Chapter 1

Preliminaries

1.1 Structure of the Dissertation

The dissertation is divided into three parts. The first part gives a brief overview of the fixed point Floer (co)homology. The second part is adapted from joint work with Yao [YZ22], which gives computations of the product and coproduct structures on the fixed point Floer homology of iterations on a single Dehn twist of a surface. The third part is adapted from joint work with Jeffs and Yao [JYZ23], which defines and computes the symplectic cohomology of a nodal algebraic curve, and explains the connection to closed-string mirror symmetry for nodal curves.

1.2 Fixed point Floer homology of a surface

In this section, we briefly review the definition of the fixed point Floer homology of a symplectomorphism. For the purpose of this dissertation, We will restrict our attention to compact symplectic surfaces, although the construction applies to much broader situations.

Let (Σ, ω_0) be a connected, compact symplectic surface (possibly with boundary) and $\phi : \Sigma \to \Sigma$ a symplectomorphism. If $\partial\Sigma$ is nonempty, we further assume that near each boundary component of $\partial\Sigma$, we can identify an open neighborhood with[1] $((-\epsilon_i, 0]_{x_i} \times S^1_{y_i}, dx_i \wedge dy_i)$, such that ϕ is the time-1 map of the Hamiltonian $H_i(x_i, y_i) = \theta_i x_i$ for a small irrational number θ_i. Notice that under such assumptions, there are no fixed points of ϕ in a neighborhood of $\partial\Sigma$. We assume that ϕ is nondegenerate, that is, for every fixed point x of ϕ, the linearization $d\phi_x$ does not have 1 as an eigenvalue. The fixed point Floer homology is the homology of the chain complex $(\mathrm{CF}_*(\Sigma, \phi), \partial)$, whose underlying module is generated over[2] $\mathbb{Z}/2$ by all

[1] Here and throughout the chapter, we use notations like S^1_y and $[0, 1]_t$ to indicate the coordinates we choose.

[2] For simplicity, we use the $\mathbb{Z}/2$ coefficient here. Fixed point Floer homology can be defined over other rings. In subsequent chapters, we will make use of the $\mathbb{C}$ coefficient, and explain what necessary changes need to be made.

fixed points of ϕ.

The differential of $\mathrm{CF}_*(\Sigma,\phi)$ is defined by counting $J-$holomorphic cylinders in the symplectization of the mapping torus of ϕ. More precisely, for any symplectomorphism $\phi:\Sigma\to\Sigma$, the *mapping torus* Y_ϕ is defined as

$$Y_\phi = [0,1]_t \times \Sigma/((1,p)\sim(0,\phi(p))). \tag{1.1}$$

The mapping torus comes with a projection $\pi: Y_\phi \to S^1_t$, and the symplectic form ω_0 induces a closed 2-form $\omega_\phi \in \Omega^2(Y_\phi)$ which restricts to ω_0 on each fiber (to be more precise, ω_0 pulls back to a closed 2-form on the product $[0,1]\times\Sigma$, and ω_ϕ is the induced 2-form on the quotient space Y_ϕ). The vector field ∂_t on $[0,1]_t\times\Sigma$ descends to a vector field on Y_ϕ, which we still denote by ∂_t. Notice that there is a one-to-one correspondence between fixed points of ϕ and closed orbits of ∂_t that cover S^1_t once. We will denote by γ_x the closed orbit associated to a fixed point x .

The projection π now extends to $\mathbb{R}\times Y_\phi \to \mathbb{R}\times S^1$, and the fiberwise symplectic form ω_ϕ extends to the symplectization as well. The *symplectization* of Y_ϕ is the 4-manifold $\mathbb{R}_s\times Y_\phi$ together with the symplectic form $ds\wedge dt+\omega_\phi$. An almost complex structure J on $\mathbb{R}\times Y_\phi$ is called *ϕ-compatible*, if it is invariant under the natural $\mathbb{R}$-action, sends ∂_s to ∂_t, sends ker $d\pi$ to itself, and that $\omega_\phi(\cdot, J\cdot)$ is a Riemannian metric on ker $d\pi$. Given 2 fixed points x, y of ϕ, we define the moduli space $\mathcal{M}^J_{x,y}$ to be

$$\mathcal{M}^J_{x,y} := \{u:\mathbb{R}_s\times S^1_t \to \mathbb{R}\times Y_\phi | \partial_s u + J\partial_t u = 0; \lim_{s\to\infty} u(s,\cdot)=\gamma_x, \lim_{s\to-\infty} u(s,\cdot)=\gamma_y\}. \tag{1.2}$$

Now if ϕ is monotone (we will clarify this notion in the next chapter in Definition 2.5.1), the differential on $\mathrm{CF}_*(\Sigma,\phi)$ is defined by

$$\langle \partial x, y\rangle := \#_{\mathbb{Z}/2}(\mathcal{M}_{x,y}/\mathbb{R}). \tag{1.3}$$

Here $\mathcal{M}_{x,y}/\mathbb{R}$ is the natural quotient of $\mathcal{M}_{x,y}$ induced by the $\mathbb{R}$-translation on $\mathbb{R}\times Y_\phi$. Recall that every J-holomorphic section u has a *Fredholm index*:

$$\mathrm{ind}(u) = 2c_1^\tau(u) + \mathrm{CZ}_\tau(\gamma_x) - \mathrm{CZ}_\tau(\gamma_y) \tag{1.4}$$

where τ is a trivialization of $\ker d\pi$ over the periodic orbits γ_x and γ_y, $c_1^\tau(u)$ is the relative first Chern number[3], and CZ_τ is the Conley-Zehnder index of the Reeb orbits with respect to τ. For a more detailed explanation on these terms, see Section 2.5. Under the monotonicity assumption, for a generic almost complex structure J, the set of Fredholm index one J-holomorphic sections modulo the natural $\mathbb{R}$ action is a compact 0-dimensional manifold, and $\#_{\mathbb{Z}/2}(\mathcal{M}^J_{x,y}/\mathbb{R})$ denotes the mod 2 count of points in the moduli space. For a generic

[3]The relative first Chern number $c_1^\tau(u)$ is defined as follows: $u^*(\ker d\pi)$ is a symplectic bundle over the domain of u. One chooses a generic section ξ of this bundle which on each end is non-vanishing and constant with respect to the trivialization τ. $c_1^\tau(u)$ is defined to be the algebraic count of zeroes of ξ.

choice of the compatible almost complex structure J, we have $\partial^2 = 0$, and we will denote by $\mathrm{HF}_*(\Sigma, \phi)$ the homology of $(\mathrm{CF}_*(\Sigma, \phi), \partial)$. The homology $\mathrm{HF}_*(\Sigma, \phi)$ is invariant under symplectic isotopies of ϕ, see for example [Sei02].

Fixed point Floer homology for a symplectomorphism of a surface has been computed in various cases, see for example [DS94; Poź94; Sei96; Gau02; Eft04; Cot09; Ped]. Fixed point Floer homology can also be viewed as a special case of periodic Floer homology, which was calculated for iterations of a Dehn twist in [HS05]. In this paper, we will focus our attention to the case where the symplectomorphisms are (perturbed) iterations of a positive Dehn twist. We will review the specific setup in the next chapter.

In chapter 3, we will also consider the fixed point Floer cohomology of a symplectomorphism. We use the following convention: under the same assumptions on Σ, ϕ and J, the fixed point Floer cochain complex $\mathrm{CF}^*(\Sigma, \phi)$ is defined by $\mathrm{Hom}(\mathrm{CF}_*(\Sigma, \phi), \mathbb{Z}/2)$, which can be thought of as the cochain complex generated by the fixed points of ϕ, with the differential d the dual of ∂.

1.3 Product and coproduct structures

Under suitable monotonicity assumptions (see Definition 2.5.3), fixed point Floer homology is functorial, in the sense that fiberwise symplectic cobordisms with cylindrical ends induce morphisms between fixed point Floer homologies. We will later see that, as special examples, the product and coproduct structures arise in this way. We begin with a review of the concept of symplectic fiber bundles.

Definition 1.3.1 ([Sei97] Definition 7.1)**.** *Let B be a smooth manifold. A symplectic fiber bundle (E, π, ω) over B is a smooth proper submersion $\pi : E \to B$ together with a closed 2-form $\omega \in \Omega^2(E)$ such that the restriction of ω to any fiber is nondegenerate.*

The mapping torus Y_ϕ, together with the natural projection $\pi : Y_\phi \to S^1$ and the 2-form ω_ϕ, is an example of a symplectic fiber bundle. If $\phi, \psi : \Sigma \to \Sigma$ are two symplectomorphisms, then there is a symplectic fiber bundle (X, π_X, ω_X) over the thrice punctured sphere B_0, which, near the three punctures, is symplectomorphic to $[0, \infty) \times Y_\phi$, $[0, \infty) \times Y_\psi$ and $(-\infty, 0] \times Y_{\psi\circ\phi}$ respectively. A more precise description for the cases we are interested in will appear in the next chapter, Section 2.1. For now, let us observe that (under suitable assumptions on monotonicity, see Definition 2.5.3), such a bundle induces a morphism

$$\bullet : \mathrm{HF}_*(\Sigma, \phi) \otimes \mathrm{HF}_*(\Sigma, \psi) \longrightarrow \mathrm{HF}_*(\Sigma, \psi \circ \phi) \tag{1.5}$$

This is what we call the *product structure* of the fixed-point Floer homology. In particular, if we assume that ψ is isotopic to the identity, then the product structure gives a $H_*(\Sigma; \mathbb{Z}_2)$ module structure on $\mathrm{HF}_*(\Sigma, \phi)$. For computations of this module structure, see [Sei96; Eft04; Gau02; Cot09]. Similarly, one can define, under suitable conditions, the *coproduct structure*:

$$\Delta : \mathrm{HF}_*(\Sigma, \psi \circ \phi) \longrightarrow \mathrm{HF}_*(\Sigma, \phi) \otimes \mathrm{HF}_*(\Sigma, \psi). \tag{1.6}$$

Like the definition of the differential, the product and coproduct structures have geometric descriptions, this time by counting rigid pseudo holomorphic sections of the bundle $X \to B_0$ with appropriate asymptotes. Namely, if x, y, z are fixed points of ϕ, ψ and $\psi \circ \phi$ respectively, and J is a generic tame almost complex structure (see Section 2.1 for the definition) then the moduli space $\mathcal{M}^J_{x,y;z}$ is defined by:

$$\mathcal{M}^J_{x,y;z} = \left\{ u : B_0 \to X \,\middle|\, \begin{array}{l} \pi_X \circ u = \text{id},\ u \text{ is } J\text{-holomorphic, and} \\ u \text{ is asymptotic to } \gamma_x,\ \gamma_y \text{ and } \gamma_z \text{ over the} \\ \text{three appropriate punctures.} \end{array} \right\} \tag{1.7}$$

The product (under suitable monotonicity assumptions, see Definition 2.5.3) on the chain level is now defined as:

$$\langle x \bullet y, z \rangle = \#_{\mathbb{Z}/2} \mathcal{M}^J_{x,y;z} \tag{1.8}$$

where $\#_{\mathbb{Z}/2} \mathcal{M}^J_{x,y;z}$ denotes the mod 2 count of Fredholm index 0 sections for a generic almost complex structure. The coproduct structure is defined in a similar way, and the same construction applies to fix point Floer cohomologies.

Chapter 2

Product and coproduct on fixed point Floer homology of positive Dehn twists

In this chapter, we compute the product and coproduct structures on fixed point Floer homologies of iterations of a single positive Dehn twist on a surface. We show that the resulting product and coproduct structures are determined by the product and coproduct on Morse homology of the complement of the twist region, together with certain sectors of product and coproduct structures on the symplectic homology of T^*S^1. The computation is done via a direct enumeration of $J-$holomorphic sections: we use a local energy inequality to show that some of the putative holomorphic sections do not exist, and we use a gluing construction plus some Morse-Bott theory (see, for instance, [Yao22]) to construct the sections we are unable to rule out.

2.1 Setup

Iterations of positive Dehn twists

We fix a connected, compact symplectic surface (Σ, ω_0), and a homologically nontrivial simple closed curve $\gamma \subset \Sigma$. We choose a tubular neighborhood N of γ with coordinates $x \in (-\epsilon, 1+\epsilon)$ and $y \in S^1 = \mathbb{R}/\mathbb{Z}$, and $\omega_0 = dx \wedge dy$. The *(unperturbed) positive Dehn twist* along γ is a symplectomorphism of Σ, which has the form

$$\phi_0 : (x, y) \mapsto (x, y - x) \tag{2.1}$$

inside N, and is the time-1 map of a Hamiltonian H_0 outside of $N' = [\epsilon, 1-\epsilon] \times S^1 \subset N$. We require H_0 takes the following form on $N \setminus N'$:

1. $H_0(x, y) = \frac{1}{2}x^2$ in $(-\epsilon, \epsilon) \times S^1 \subset N$, and

2. $H_0(x,y) = \frac{1}{2}(x-1)^2$ in $(1-\epsilon, 1+\epsilon) \times S^1 \subset N$.

We assume that on $\Sigma \setminus N$, the function H_0 is a C^2 small Morse function, so that the associated time-1 map is non-degenerate outside of N. We further assume that near each boundary component of Σ, there are tubular coordinates $x_i \in (-\epsilon_i, 0]$, $y_i \in S^1$ where $\omega_0 = dx_i \wedge dy_i$, and a small real number θ_i such that $H_0(x_i, y_i) = \theta_i x_i$.

We note the unperturbed positive Dehn twist ϕ_0 is non-degenerate, except for the Morse-Bott S^1 family of periodic orbits corresponding to $x = 0$ and $x = 1$. We shall later consider iterations of ϕ_0, which we denote by ϕ_0^n. By the above, on N, the map ϕ_0^n takes the form

$$(x, y) \mapsto (x, y - nx),$$

and looks like the time-1 map of nH_0 outside of N'. We assume both nH_0 and $n\theta_i$ are small (in the C^2 norm).

We note that in order to define the fixed point symplectic homology, we need the symplectomorphisms to be nondegenerate (equivalently, that the Reeb orbits are cut out transversely). Since the symplectomorphism ϕ_0^n on $\Sigma - N$ is the time-1 map of a Hamiltonian nH_0, this is achieved outside of N by requiring that H_0 be a C^2-small Morse function. Inside the tubular region N, Reeb orbits come in Morse-Bott S^1 families. Following [HS05], we overcome this technical difficulty by perturbing ϕ_0 (in a small neighborhood of finitely many values of x over which Reeb orbits exist) in a Hamiltonian way, which amounts to adding a Hamiltonian perturbation term.

To be more specific, near $x = 0$, we modify H_0 to be $(\frac{1}{2}x^2 + \lambda(x)h(y)))$, where $\lambda(x)$ is a cutoff function supported in $(-\delta, \delta)_x$ with $\lambda(0) = 1$ as a non-degenerate local max, and $h : S^1_y \to \mathbb{R}$ is a small perfect Morse function. We perform this kind of perturbation for each S^1 family of fixed points in N. We always assume that the Hamiltonian perturbation only takes place in the union of all intervals $(x_i - \delta, x_i + \delta)$ (where x_i's are the x-coordinates for all possible Morse-Bott S^1-families) for some positive real number δ much smaller than ϵ (we'll later call the complement of these intervals the *unperturbed range*). Once this is done, viewed from the perspective of the mapping torus $Y_{\phi_0^n}$, the S^1 family of the Reeb orbits over $x = \frac{i}{n}$ become perturbed to a pair of Reeb orbits (one elliptic and one hyperbolic).

The symplectic fiber bundle $X_{m,n}$

With the above *perturbed positive Dehn twist*, which we denote by ϕ^n, we can define its fixed point Floer homology $\mathrm{HF}_*(\Sigma, \phi^n)$ after we pick a generic ϕ^n compatible almost complex structure J on $\mathbb{R} \times Y_{\phi^n}$. We next describe the symplectic fiber bundle that allows us to define product and coproduct structures on $\mathrm{HF}_*(\Sigma, \phi^n)$. We first describe the construction for the unperturbed positive Dehn twists, then discuss the perturbation needed to break the Morse-Bott degeneracy. In the computations to come, we will first enumerate J-holomorphic sections in the unperturbed setup, then use a correspondence result to find the count of sections in the perturbed setting.

Recall that, given two symplectomorphisms, there is a symplectic fiber bundle (X, π_X, ω_X) over the thrice punctured sphere B_0, which is modelled by the symplectizations of mapping tori over the punctures. We now describe in more details what the bundle $X_{m,n}$ used in computing the product structure

$$\bullet : \mathrm{HF}_*(\phi^m) \otimes \mathrm{HF}_*(\phi^n) \longrightarrow \mathrm{HF}_*(\phi^{m+n}) \tag{2.2}$$

looks like. The description for the bundle $X^{m,n}$ used to compute the coproduct structure is almost identical, and we will mention at the end of this section what changes need to be made.

We designate two of the punctures of B_0 as "positive", and the other as "negative". Choose local conformal coordinates $s_i \in [0, \infty)$ and $t_i \in S^1 (i = 1, 2)$ near the 2 positive punctures of B_0, and local conformal coordinates $s_{-\infty} \in (-\infty, 0]$, $t_{-\infty} \in S^1$ near the negative puncture. Fix also a smooth map [1] $g_{m,n} : B_0 \to S^1$ such that $dg_{m,n} = m dt_1$ near the first positive puncture, $dg_{m,n} = n dt_2$ near the second positive puncture and $dg_{m,n} = (m+n)dt_{-\infty}$ near the negative puncture. We further assmume that $g_{m,n} = d \cdot g'_{m,n}$ where $d = gcd(m, n)$ (the primitive $g'_{m,n}$ will be used in the proof of Theorem 2.2.1). Define the closed one-form $\beta_{m,n} = dg_{m,n}$.

Let $\Sigma_0 = \Sigma - N'$. We now describe the fiberwise symplectic cobordism $X_{m,n}$ as the union of two fiberwise symplectic cobordisms X_D and X_H as follows. Topologically, $X_D = B_0 \times N$ and $X_H = B_0 \times \Sigma_0$. In order to describe ϕ_0 as the time-1 map of the Hamiltonian H_0 near the two ends of the tubular region N we choose coordinates $(p, x_L, y_L) \in B_0 \times (-\epsilon, \epsilon) \times S^1$ and $(p, x_R, y_R) \in B_0 \times (1-\epsilon, 1+\epsilon) \times S^1$ for the two ends of $B_0 \times (N - N') \subset X_H$ and impose that

1. $\omega_0 = dx_L \wedge dy_L$ or $dx_R \wedge dy_R$ in the two components of $N - N'$,
2. $H_0(x_L, y_L) = \frac{1}{2}x_L^2$ in $(-\epsilon, \epsilon) \times S^1 \subset N$, and
3. $H_0(x_R, y_R) = \frac{1}{2}(x_R - 1)^2$ in $(1 - \epsilon, 1 + \epsilon) \times S^1 \subset N$.

Topologically, the 4-manifold X is defined to be $X = X_H \bigcup X_D / \sim$, where we identify points $(p, x, y) \in B_0 \times (-\epsilon, \epsilon)_x \times S^1_y \subset X_D$ with $(p, x, y) \in B_0 \times (-\epsilon, \epsilon)_{x_L} \times S^1_{y_L} \subset X_H$, and $(p, x, y) \in B_0 \times (1-\epsilon, 1+\epsilon)_x \times S^1_y \subset X_D$ with $(p, x, y + g_{m,n}(p)) \in B_0 \times (1-\epsilon, 1+\epsilon)_{x_R} \times S^1_{y_R} \subset X_H$.

Now we define the fiberwise symplectic 2-form $\omega_{X,0}$ to be

$$dx \wedge dy + d(\frac{1}{2}x^2 \beta_{m,n})$$

in X_D, and

$$\omega_0 + d(H_0 \beta_{m,n})$$

[1] To see such a map exists, we can choose a degree $m + n$ branched covering map $\varphi : B_0 \to \mathbb{R} \times S^1$ such that near the three punctures φ has the standard local form $(s, t_*) \mapsto (s, kt_*)$ (where $k = m, n, m + n$ respectively), and let g be φ followed by the projection map to S^1.

in X_H. It is not difficult to see that the two definitions agree in $X_D \supset B_0 \times (-\epsilon, \epsilon)_x \times S^1_y = B_0 \times (-\epsilon, \epsilon)_{x_L} \times S^1_{y_L} \subset X_H$. To see that the two definitions agree in $X_D \supset B_0 \times (1-\epsilon, 1+\epsilon)_x \times S^1_y = B_0 \times (1-\epsilon, 1+\epsilon)_{x_R} \times S^1_{y_R} \subset X_H$, we calculate

$$\begin{aligned} dx_R \wedge dy_R + d(\frac{1}{2}(x_R-1)^2\beta_{m,n}) &= dx \wedge (dy + \beta_{m,n}) + d(\frac{1}{2}(x-1)^2\beta_{m,n}) \\ &= dx \wedge dy + dx \wedge \beta_{m,n} + (x-1)dx \wedge \beta_{m,n} \\ &= dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n}). \end{aligned} \tag{2.3}$$

We remark that over the positive punctures, the symplectic fiber bundle defined above are isomorphic to $[0,\infty)$ times the mapping tori $Y_{\phi_0^m}$ and $Y_{\phi_0^n}$, and over the negative puncture, the above fiber bundle is modelled by $(-\infty, 0]$ times the mapping torus $Y_{\phi_0^{m+n}}$.

As noted in the previous section, the above fiber bundle have Morse-Bott degeneracies in its Reeb orbits at each of its punctures. To arrive at the definition of $X_{m,n}$, we perturb the Reeb orbits to be non-degenerate as before. We will achieve this by adding a Hamiltonian perturbation term to $\omega_{X,0} = dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n})$, described below.

As before, near $x = 0$, we can modify $\omega_{X,0}$ to be

$$\omega_X = dx \wedge dy + d((\frac{1}{2}x^2 + \lambda(x)h(y))\beta_{m,n}),$$

where $\lambda(x)$ is a cutoff function supported in $(-\delta, \delta)_x$ with $\lambda(0) = 1$ as a nondegenerate local max, and $h : S^1_y \to \mathbb{R}$ is a small perfect Morse function. We assume as before that the Hamiltonian perturbation only takes place in the union of all intervals $(x_i - \delta, x_i + \delta)$ (where x_i's are the x-coordinates for all possible Morse-Bott S^1-families) for some positive real number δ much smaller than ϵ (we'll later call the complement of these intervals the *unperturbed range*). Once this is done, near the first (resp. the second) positive puncture, $x = \frac{i}{m}$ (resp. $x = \frac{j}{n}$) each correspond to a pair of Reeb orbits (one elliptic and one hyperbolic), and near the negative puncture $x = \frac{k}{m+n}$ each correspond to a pair of Reeb orbits (one elliptic and one hyperbolic).

This defines the fiberwise symplectic 2-form ω_X, and we now describe the symplectic structure on $X_{m,n}$. It's illustrated in e.g. [Che21] that from the fiberwise symplectic cobordism (X, π_X, ω_X) one can construct a symplectic form $\Omega_X = \omega_X + K\pi_X^*\omega_{B_0}$ where K is a large positive number, and ω_{B_0} is an area form on B_0. Without loss of generality, we assume from now on that $K\pi_X^*\omega_{B_0} = ds_i \wedge dt_i$ near the punctures.

This concludes the definition of $X_{m,n}$. We now equip it with a tame almost complex structure J, and conclude the description of the product and coproduct structures.

Tame almost complex structures and the product

Definition 2.1.1. *An almost complex structure on $(X_{m,n}, \omega_X, \pi)$ is called tame if the following conditions are satisfied*

1. *Near the punctures of B_0 where the symplectic fiber bundle is isomorphic to $[0,\infty)\times Y_{\phi^n}$ (resp. $[0,\infty)\times Y_{\phi^m}$ or $(-\infty,0]\times Y_{\phi^{m+n}}$), the almost complex structure is given by the restriction of a ϕ^n (resp. ϕ^m, or ϕ^{m+n}) compatible almost complex structure.*

2. *Away from the cylindrical neighborhoods around the punctures of B_0, the almost complex structure J is tamed by the symplectic form Ω_X.*

Then for generic tame J, if x, y, z are fixed points of ϕ^n, ϕ^m and ϕ^{m+n} respectively, under suitable topological assumptions (e.g. when the bundle $X_{m,n}$ is weakly monotone; see Section 2.5) the moduli space $\mathcal{M}^J_{x,y;z}$, defined by

$$\mathcal{M}^J_{x,y;z} = \left\{ u : B_0 \to X \;\middle|\; \begin{array}{l} \pi_X \circ u = \text{id},\ u \text{ is } J\text{- holomorphic, and} \\ u \text{ is asymptotic to } \gamma_x,\ \gamma_y \text{ and } \gamma_z \text{ over the} \\ \text{three appropriate punctures.} \end{array} \right\} \tag{2.4}$$

is a manifold whose dimension is given by the Fredholm index formula

$$\text{ind}(u) = 1 + 2\langle c_1^\tau(TX_{m,n}), [u]\rangle + \text{CZ}_\tau(\gamma_x) + \text{CZ}_\tau(\gamma_y) - \text{CZ}_\tau(\gamma_z). \tag{2.5}$$

Here τ denotes a choice of fixed trivializations around each Reeb orbit, and CZ_τ denotes the Conley-Zehnder indices of Reeb orbits with respect to this trivialization. Likewise, the relative first Chern class c_1^τ is also determined by this choice of trivialization. See Section 2.5 for our specific choices of τ. The product on the chain level is now defined as:

$$\langle x \bullet y, z\rangle = \#_{\mathbb{Z}/2}\mathcal{M}^J_{x,y;z} \tag{2.6}$$

where $\#_{\mathbb{Z}/2}\mathcal{M}^J_{x,y;z}$ denotes the mod 2 count of Fredholm index 0 sections (we will explain in Section 2.5 that the monotonicity condition ensures that the moduli space is compact). See Section 2.5 for the details of this computation.

The symplectic fiber bundle $X^{m,n}$ and the coproduct

For the coproduct structure, the symplectic fiber bundle $X^{m,n}$ is defined almost verbatim, so we only highlight the minor changes that need to be made. We again begin with the thrice punctured sphere B_0, but this time choose one of the three punctures as the "positive puncutre" with a local conformal coordinate $(s_\infty, t_\infty) \in [0,\infty)\times S^1$, and choose the other two punctures as the two "negative" punctures with local coordinates $(s_i, t_i) \in (-\infty, 0]\times S^1$. Fix a smooth function $g^{m,n} : B_0 \to S^1$ such that $dg^{m,n} = (m+n)dt_\infty$ near the positive puncture and $dg^{m,n} = mdt_1$ and ndt_2 near the two negative punctures respectively. We again define $X^{m,n}$ by gluing two trivial fiber bundles $X_H = B_0 \times \Sigma_0$ and $X_D = B_0 \times N$, but this time the gluing map for the right side of X_D is

$$B_0 \times (1-\epsilon, 1+\epsilon)_x \times S^1_y \ni (p,x,y) \sim (p,x,y+g^{m,n}(p)) \in B_0 \times (1-\epsilon,1+\epsilon)_{x_R} \times S^1_{y_R} \tag{2.7}$$

Let $\beta^{m,n} = dg^{m,n}$. We similarly define the (unperturbed) fiberwise symplectic 2-form $\omega_{X,0}$ to be

$$dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n})$$

in X_D and

$$\omega_0 + d(H_0\beta_{m,n})$$

in X_H. As before, we perturb $\omega_{X,0}$ to be ω_X in order to break the Morse-Bott degeneracy, and we always assume that such perturbation is supported in a δ neighborhood of Reeb orbits inside X_D.

We also define the symplectic form Ω_X on $X^{m,n}$, and the notion of tame almost complex structures with respect to Ω_X. Then the coproduct is defined by considering the moduli space of J-holomorphic sections $\mathcal{M}^J_{z;x,y}$ where z is a fixed point of ϕ^{m+n}, and x and y are fixed points of ϕ^n and ϕ^m respectively. For generic choice of tame J, this moduli space is a manifold and the coproduct is defined by the mod 2 count of Fredholm index 0 J-holomorphic sections.

2.2 Main results

In this section, we summarize the main results of this chapter.

Let Σ_0 denote $\Sigma - N'$. It was shown, for example in [Sei96] or [HS05], that for all positive integers m,

$$\mathrm{HF}_*(\phi^m) \cong H_*(\Sigma_0; \mathbb{Z}_2) \oplus (\oplus_{i=1}^{m-1} H_*(S^1)). \tag{2.8}$$

The isomorphism can be understood as follows. With our assumption on ϕ, the chain complex $CF_*(\phi^m)$ is generated by two types of fixed points: those corresponding to the critical points of H_0 on Σ_0 and those from the breaking of the Morse-Bott S^1-family of the fixed points of ϕ_0^m inside N'. As it turns out, the two types of fixed points generate subcomplexes of $CF(\phi^m)$, whose homologies correspond to the summands of the right-hand side of equation (2.8).

Let m and n be two positive integers. With the previously described isomorphism understood, let proj denote the projection map $\mathrm{HF}_*(\phi^m) \to H_*(\Sigma_0; \mathbb{Z}_2)$, let $\cap$ denote the intersection product on $H_*(\Sigma_0; \mathbb{Z}_2)$, and let ι denote the inclusion map $H_*(\Sigma_0; \mathbb{Z}_2) \to \mathrm{HF}_*(\phi^{m+n})$. The first main result of this chapter is the following:

Theorem 2.2.1. *Suppose*

- *If γ is non-separating, then $\partial\Sigma \neq \emptyset$ or Σ is closed with genus at least 2;*
- *If γ is separating, then each component of $\Sigma - \gamma$ either contains a component of $\partial\Sigma$ or has genus at least 2.*

Then the product of $\mathrm{HF}(\phi^m)$ *and* $\mathrm{HF}(\phi^n)$ *is the composition of*

$$\mathrm{HF}_*(\phi^m)\otimes\mathrm{HF}_*(\phi^n) \xrightarrow{\mathrm{proj}\otimes\mathrm{proj}} H_*(\Sigma_0;\mathbb{Z}_2)\otimes H_*(\Sigma_0;\mathbb{Z}_2) \xrightarrow{\ \cap\ } H_*(\Sigma_0;\mathbb{Z}_2) \xrightarrow{\ \iota\ } \mathrm{HF}_*(\phi^{m+n}). \tag{2.9}$$

The counterpart for the coproduct structure is more involved. Let us first make the following remark on the decomposition (2.8). The i-th component of $\oplus_{i=1}^{m-1}H_*(S^1)$ has an explicit description as follows: let e_i^m (resp. h_i^m) be the elliptic (resp. hyperbolic) orbit of ϕ^m over the tubular coordinate $x=\frac{i}{m}$ $(i=0,1,\cdots,m)$ that arises from perturbing the Morse-Bott degenerate ϕ to ϕ_0 (see section 2.1). Then e_i^m and h_i^m are cycles and the i-th component of $\oplus_{i=1}^{m-1}H_*(S^1)$ is spanned by the two homology classes $[e_i^m]$ and $[h_i^m]$. Let us denote by $[e_j^n]$, $[h_j^n]$, $[e_k^{m+n}]$, $[h_k^{m+n}]$ the homology classes appearing in the similar decomposition for $\mathrm{HF}_*(\phi^n)$ and $\mathrm{HF}_*(\phi^{m+n})$ respectively. Finally, let us recall that for any space M, a coproduct structure Δ_0 on $H_*(M;\mathbb{Z}/2)$ is defined as the composition of $\mathrm{diag}_*: H_*(M;\mathbb{Z}/2)\to H_*(M\times M;\mathbb{Z}/2)$ and $H_*(M\times M;\mathbb{Z}/2)\cong H_*(M;\mathbb{Z}/2)\otimes H_*(M;\mathbb{Z}/2)$. The second main result of this chapter is the following:

Theorem 2.2.2. *Suppose*

- *If* γ *is non-separating, then* $\partial\Sigma\neq\emptyset$ *or* Σ *is closed with genus at least 2;*
- *If* γ *is separating, then each component of* $\Sigma-\gamma$ *either contains a component of* $\partial\Sigma$ *or has genus at least 2.*

Then the coproduct $\Delta:\mathrm{HF}_*(\phi^{m+n})\to\mathrm{HF}_*(\phi^m)\otimes\mathrm{HF}_*(\phi^n)$ *described in the previous section is completely determined by the following:*

1. *When restricted to* $H_*(\Sigma_0;\mathbb{Z}_2)\subset\mathrm{HF}_*(\phi^{m+n})$, Δ *is equal to*

$$H_*(\Sigma_0;\mathbb{Z}_2)\xrightarrow{\ \Delta_0\ } H_*(\Sigma_0;\mathbb{Z}_2)\otimes H_*(\Sigma_0;\mathbb{Z}_2)\longrightarrow \mathrm{HF}_*(\phi^m)\otimes\mathrm{HF}_*(\phi^n). \tag{2.10}$$

2. *For each* $[e_k^{m+n}]\in\oplus_{i=1}^{m+n-1}H_*(S^1)$,

$$\Delta([e_k^{m+n}]) = \sum_{i\in\{0,1,\cdots,m\},\ k-i\in\{0,1,\cdots,n\}} [e_i^m]\otimes[e_{k-i}^n]. \tag{2.11}$$

3. *For each* $[h_k^{m+n}]\in\oplus_{i=1}^{m+n-1}H_*(S^1)$,

$$\Delta([h_k^{m+n}]) = \sum_{i\in\{0,1,\cdots,m\},\ k-i\in\{0,1,\cdots,n\}} [e_i^m]\otimes[h_{k-i}^n]+[h_i^m]\otimes[e_{k-i}^n]. \tag{2.12}$$

Remark 2.2.3. It was asked in [Cot09] how one could get an understanding of the ring structure of $\oplus_{n=0}^{\infty}\mathrm{HF}_*(\Sigma,\phi^n)$ for a general symplectomorphism ϕ. Our results compute the algebra and co-algebra structure of $\oplus_{n>0}^{\infty}\mathrm{HF}_*(\Sigma, \phi^n)$ when ϕ is the positive Dehn twist. One can use the computation in [Sei96] to calculate the case for $n = 0$.

The coproduct structure on $\oplus_{n\geq 0}\mathrm{HF}_*(\phi^n)$ can be interpreted as the dual of the product structure on the ring $\oplus_{n\leq 0}\mathrm{HF}_*(\phi^n)$ associated with the negative Dehn twist. To see this, we observe that the curves counted in computing the coproduct structure $\Delta : \mathrm{HF}_*(\phi^{m+n}) \to \mathrm{HF}_*(\phi^m)\otimes\mathrm{HF}_*(\phi^n)$ are in one-to-one correspondence with the curves counted in the product structure $\mathrm{HF}^*(\phi^{-m}) \otimes \mathrm{HF}^*(\phi^{-n}) \to \mathrm{HF}^*(\phi^{-m-n})$, where $\mathrm{HF}^*(\phi^{-m})$ can be naturally viewed as the dual of $\mathrm{HF}_*(\phi^{-m})$ since we are using field coefficients. Likewise, the product structure on $\oplus_{n\geq 0}\mathrm{HF}_*(\phi^n)$ can be interpreted as the dual of the coproduct structure on $\oplus_{n\leq 0}\mathrm{HF}_*(\phi^n)$.

Strategy of the proof

In this subsection, we summarize the key ideas behind the proof of the two main theorems of this chapter. As explained in Section 2.1, the symplectic fiber bundle X computing the product or the coproduct can be decomposed into two pieces, which we call X_D and X_H. The first key observation is that, under mild assumptions on the almost complex structure J, any J-holomorphic section of X that has wrapping number (see Section 2.3) 0 must be completely contained in either X_D or X_H. A key technical lemma used in the argument is the "local energy inequality" (Lemma 2.3.2) that was inspired by Lemma 3.11 of [HS05], which is reproved in our setting in Section 2.3. We next observe that for J-holomorphic sections, Fredholm index being 0 implies that the wrapping number is 0. Thus we only need to focus on sections contained in one of the two pieces.

J-holomorphic sections that are contained in X_H are relatively easy to understand: by results of [PSS96; FS07; Lan16] (also known as the PSS isomorphism), these sections contribute to the intersection product or the coproduct of $H_*(\Sigma_0; \mathbb{Z}_2)$.

J-holomorphic sections that are contained in X_D are more interesting. When computing the product structure, we are again able to rule out most of them using the local energy inequality. For the remaining sections, we use a translation trick (equation (2.71)) together with the PSS isomorphism to conclude that they contribute zero.

For the coproduct structure, sections contained in X_D do make contributions. To understand the contributions of these sections, we give a concrete description of the (unperturbed) moduli spaces, see Proposition 2.6.14, whose proof involves a deformation argument and a concrete construction. Finally, a Morse-Bott correspondence theorem (explained in Section 2.6) finishes the calculation.

2.3 The "no crossing" results for unperturbed J

In this section, we establish one of the main technical results that lead to the computations of the product and coproduct structures: the "no crossing" lemmas. Throughout this section, the symplectic fiber bundle X refers to either $X_{m,n}$ or $X^{m,n}$. As explained in Chapter 1, to define the cobordism map, we count J-holomorphic sections that asymptote to appropriate

Reeb orbits. We'll prove some key properties about such J-holomorphic sections for some particularly nice almost complex structures. Before doing that, let us introduce some terminologies. Following [MS12], the *vertical distribution* Ver is the kernel of $d\pi_X : TX \to TB_0$. The *horizontal distribution* Hor is defined as $\mathrm{Hor}_x := \{u \in T_xX \mid \omega_X(u, v) = 0 \ \forall v \in \mathrm{Ver}_x\}$.

Definition 2.3.1 ([MS12] Definition 8.2.6)**.** *An almost complex structure on (X, ω_X) is called fibration-compatible if the following holds:*

1. *The projection π_X is holomorphic: $d\pi \circ J = j_0 \circ d\pi$.*
2. *For every $p \in B_0$, the restriction J_p of J to $\pi_X^{-1}(p)$ is tamed by $\omega_X|_{\pi_X^{-1}(p)}$.*
3. *The horizontal distribution* Hor *is preserved by J.*

Note that by definition, there is a one-to-one correspondence between fibration-compatible almost complex structures and ω_X-tame almost complex structures on the vertical distribution.

In this section we only consider fibration-compatible almost complex structures. For a fibration-compatible J, all the horizontal sections are J-holomorphic (a section $u : B_0 \to X$ is horizontal, if $du(TB_0) \subset \mathrm{Hor}$).

Following [HS05] Lemma 3.11, we now establish a local energy inequality for J-holomorphic sections. To state the inequality, for any $x \in (-\epsilon, 1 + \epsilon)$ we let F_x denote the 3-manifold $B_0 \times \{x\} \times S^1_y \subset X_D$. Likewise, let $F_{[x_1,x_2]}$ denote the 4- manifold $B_0 \times [x_1, x_2]_x \times S^1_y \subset X_D$. The first homology group of $X_D = B_0 \times (-\epsilon, 1 + \epsilon)_x \times S^1_y$ is $\mathbb{Z}^3$, generated by $[S^1_{t_1}]$, $[S^1_{t_2}]$ and $[S^1_y]$. In the following, we identify $p[S^1_y] + q_1[S^1_{t_1}] + q_2[S^1_{t_2}] \in H_1(X_D)$ with a tuple $(p, q_1, q_2) \in \mathbb{Z}^3$.

Lemma 2.3.2 (Local energy inequality)**.** *Let C be a J-holomorphic section $u : B_0 \to X$ which is not horizontal. Assume that C intersects F_x transversely and that $C \cap F_x \neq \emptyset$ for some x in the unperturbed range. Orient each circle in $C \cap F_x$ using the boundary orientation of $C \cap F_{[x-\epsilon',x]}$ (for a small ϵ') induced by j_0. Under this orientation, let (p, q_1, q_2) denote the homology class of $C \cap F_x$, then we have*

$$p + x(mq_1 + nq_2) > 0 \tag{2.13}$$

Before proving the lemma, here are some observations on the vertical energy of J-holomorphic sections.

Definition 2.3.3. *Let $u : B_0 \to X$ be a smooth section of the bundle $X \to B_0$, J be an almost complex structure and g_J the metric induced by ω_X and J. Let (s, t) be a local conformal coordinate on B_0. The* vertical energy *of u is defined to be*

$$E(u) = \frac{1}{2} \int_{B_0} |\partial_s u - \partial_s^{\#}|^2_{g_J} + |\partial_t u - \partial_t^{\#}|^2_{g_J} ds \wedge dt \tag{2.14}$$

where $\partial_s^{\#}$ and $\partial_t^{\#}$ are the horizontal lifts of the vector fields ∂_s and ∂_t, respectively.

Remark 2.3.4. Our definition, written in local conformal coordinates, coincides with that of [MS12] equation 8.1.8. It's also clear from the definition that a smooth map has zero vertical energy if and only if it is a horizontal section. In Definition 2.6.7 we will generalize the notion of vertical energy to include more examples of symplectic fiber bundles that will be useful later. For now, the main observation is the following:

Lemma 2.3.5. *Let u be a $J-$ holomorphic section of (X, ω_X, π_X) described in Section 2.1. Let (s,t) be a local conformal coordinate on B_0 and $\partial_s^{\#}$ and $\partial_t^{\#}$ be the horizontal lifts of the vector fields ∂_s and ∂_t. The the two-form*

$$\frac{1}{2}(|\partial_s u - \partial_s^{\#}|_{g_J}^2 + |\partial_t u - \partial_t^{\#}|_{g_J}^2) ds \wedge dt$$

can be rewritten as

$$u^*\omega_X - \omega_X(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt$$

which is equal to $u^*\omega_X$ in X_D minus the perturbed region.

Proof of Lemma 2.3.5. If u is $J-$holomorphic, we have

$$\begin{aligned}
&\frac{1}{2}(|\partial_s u - \partial_s^{\#}|_{g_J}^2 + |\partial_t u - \partial_t^{\#}|_{g_J}^2) ds \wedge dt \\
&= \omega_X(\partial_s u - \partial_s^{\#}, \partial_t u - \partial_t^{\#}) ds \wedge dt \\
&= \omega_X(\partial_s u, \partial_t u) - \omega_X(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt \\
&= u^*\omega_X - \omega_X(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt
\end{aligned}$$

If we write the two-form ω_X as

$$dx \wedge dy + F(x,y,s,t)ds + G(x,y,s,t)dt$$

inside X_D, then the term $\omega_X(\partial_s^{\#}, \partial_t^{\#})$ is equal to

$$\frac{\partial G}{\partial s} - \frac{\partial F}{\partial t} + \frac{\partial G}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial F}{\partial x},$$

which is equal to zero outside of the perturbed region, where $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial y} = 0$, and $\frac{\partial G}{\partial s} = \frac{\partial F}{\partial t}$ by construction. □

It follows that if u is a $J-$holomorphic section of X, then the part of u in X_D minus the perturbed region satisfies

$$u^*\omega_X(v, j_0 v) \geq 0 \tag{2.15}$$

for any $v \in TB_0$, and the equality holds if and only if $du(v) \in \text{Hor}$. Now we are ready to prove Lemma 2.3.2.

Proof of Lemma 2.3.2. Choose $x_1 < x < x_2$ such that

1. C intersects both F_{x_1} and F_{x_2} transversely,

2. $x - x_1 = x_2 - x$, and

3. $[x_1, x_2]$ is contained in the unperturbed range.

We orient $C \cap F_{x_1}$ and $C \cap F_{x_2}$ in the same way as stated in the lemma, so we have $\partial(C \cap F_{[x_1,x_2]}) = C \cap F_{x_2} - C \cap F_{x_1}$. Also notice that with the specified orientations, $C \cap F_{x_1}$, $C \cap F_{x_2}$ and $C \cap F_x$ all have the same homology class in $H_1(X_D)$. Now we note that $C \cap F_{[x_1,x_2]}$ is not horizontal, otherwise C has to be horizontal everywhere by unique continuation (recall that all horizontal sections are J-holomorphic).

Using the inequality (2.15), we have (in the following the one-form β refers to either $\beta_{m,n}$ or $\beta^{m,n}$)

$$\begin{aligned}
0 &< \int_{C\cap F_{[x_1,x_2]}} u^*\omega_X = \int_{C\cap F_{[x_1,x_2]}} u^*(dx \wedge dy + d(\frac{1}{2}x^2\beta)) \\
&= \int_{\partial(C\cap F_{[x_1,x_2]})} u^*(xdy + \frac{1}{2}x^2\beta) \\
&= x_2 \int_{C\cap F_{x_2}} u^*dy + \frac{1}{2}x_2^2 \int_{C\cap F_{x_2}} u^*\beta - x_1 \int_{C\cap F_{x_1}} u^*dy - \frac{1}{2}x_1^2 \int_{C\cap F_{x_1}} u^*\beta \\
&= x_2 p + \frac{1}{2}x_2^2(mq_1 + nq_2) - x_1 p - \frac{1}{2}x_1^2(mq_1 + nq_2) \\
&= (x_2 - x_1)(p + x(mq_1 + nq_2)).
\end{aligned} \tag{2.16}$$

□

Remark 2.3.6. It is clear from the proof that if u is a horizontal section, then the equality $p + x(mq_1 + nq_2) = 0$ holds.

Let's assume for the moment that $\Sigma_0 = \Sigma - N'$ is connected. Following [HS05], we define the *wrapping number* of J-holomorphic sections with cylindrical asymptotes:

Definition 2.3.7. *The wrapping number of a J-holomorphic section C with cylindrical asymptotes is $\eta(C) = \#C \bigcap (B_0 \times \{P_0\})$, where $P_0 \in \Sigma_0$ is not a critical point of H_0.*

The algebraic intersection number does not depend on the choice of P_0, so the wrapping number is well-defined. We also note that $\eta(C)$ is identically zero if $\partial\Sigma \neq \emptyset$ and under the additional assumption that near each boundary component, J is induced from the vertical almost complex structure that sends ∂_{x_i} to ∂_{y_i}. The reason is that once such a J is chosen, no J-holomorphic sections can enter the boundary region by the following maximum principle, so one can choose P_0 inside one of the boundary region and easily see that $C \bigcap (B_0 \times \{P_0\}) = \emptyset$.

Lemma 2.3.8 (Maximum principle)**.** *Let J be a fibration-compatible almost complex structure on X that sends ∂_{x_i} to ∂_{y_i} near each boundary component of Σ. Let V denote an open subset of B_0 with local conformal coordinates (s,t). Let $\tilde{u} : V \to X_H$ denote a J-holomorphic section, which in coordinates takes the form $(s,t) \mapsto (s,t,x_i(s,t),y_i(s,t))$. We further assume for $(s,t) \in V$, the pair $(x_i(s,t),y_i(s,t))$ is in a neighborhood of the ith boundary component of Σ. Then $x_i(s,t)$ is a harmonic function.*

Proof. The setup is almost identical to that of Lemma 2.6.10 (to be stated later), except that the Hamiltonian function is $\theta_i x_i$ instead of $x_i^2/2$. In particular, the horizontal lifts are

$$\partial_s^{\#} = \partial_s - \theta_i F \partial_y,\ \partial_t^{\#} = \partial_t - \theta_i G \partial_y \tag{2.17}$$

and we have a similar equation:

$$\begin{cases} \frac{\partial x_i}{\partial t} + \frac{\partial y_i}{\partial s} + \theta_i F = 0 \\ \frac{\partial y_i}{\partial t} - \frac{\partial x_i}{\partial s} + \theta_i G = 0. \end{cases} \tag{2.18}$$

Notice that $\beta = F(s,t)ds + G(s,t)dt$ being closed tells us that $\frac{\partial F}{\partial t} = \frac{\partial G}{\partial s}$, so the conclusion follows by a simple calculation. □

In particular, the above lemma implies that as long as J is chosen in a neighborhood of each of the boundary components of Σ to be fibration compatible and sends ∂_{x_i} to ∂_{y_i}, no J-holomorphic section may approach the boundary components of Σ.

Remark 2.3.9. If $\Sigma_0 = \Sigma - N'$ is not connected, i.e. γ is separating, we can define two wrapping numbers η_1 and η_2 for each of the connected components of Σ_0. It is clear from the above arguments that if each connected components of Σ_0 contains part of $\partial\Sigma$, then all the wrapping numbers vanish automatically.

Remark 2.3.10. Following a similar idea in [HS05] Lemma 4.3, we will show in the following (Remark 2.3.12) that the wrapping numbers of any J-holomorphic sections are non-negative.

The first main result of this section is the following "no crossing" lemma:

Lemma 2.3.11. *Assume J is a fibration-compatible almost complex structure on $X_{m,n}$. If C is a J-holomorphic section of $X_{m,n}$ such that all wrapping numbers are zero, then C is either contained in X_H or contained in X_D.*

Proof of Lemma 2.3.11. We first consider the case where C is not horizontal. Suppose there is a nonhorizontal J-holomorphic section C that is neither contained in X_H nor in X_D, then since C is connected, we can either find some $\epsilon_1 \in (\delta,\epsilon)$ such that C intersects both F_{ϵ_1} and $F_{-\epsilon_1}$ transversely and $C \bigcap F_{\pm\epsilon_1} \neq \emptyset$, or some $\epsilon_1 \in (\delta,\epsilon)$ such that C intersects both $F_{1+\epsilon_1}$ and $F_{1-\epsilon_1}$ transversely and $C \bigcap F_{1\pm\epsilon_1} \neq \emptyset$. Without loss of generality, let us assume the first situation happens. (If the second situation happens, the following proof works almost verbatim; the only change one needs to make is that, if we use $(p^{\pm}, q_1^{\pm}, q_2^{\pm})$ to denote the homology classes of $C \bigcap F_{1\pm\epsilon_1}$, then the condition $\eta = 0$ translates to $p^{\pm} + mq_1^{\pm} + nq_2^{\pm} = 0$.)

Let $(p^{\pm}, q_1^{\pm}, q_2^{\pm})$ denote the homology classes of $C \bigcap F_{\pm\epsilon_1}$. Since all the wrapping numbers vanish, we observe that $p^{\pm} = 0$. To see this fact, notice that we can choose P_0 to be $(\pm\epsilon_1, y_0) \in \Sigma_0$ for some fixed $y_0 \in S^1$, then $\#C \bigcap (B_0 \times \{P_0\})$ is precisely the number of times $C \bigcap F_{\pm\epsilon_1}$ passes through y_0, which equals $p^{\pm}$.

Lemma 2.3.2 tells us that

$$\epsilon_1(mq_1^+ + nq_2^+) > 0 > \epsilon_1(mq_1^- + nq_2^-) \tag{2.19}$$

which implies that

$$mq_1^+ + nq_2^+ \geq 1,\ mq_1^- + nq_2^- \leq -1. \tag{2.20}$$

Now we consider $C_{[-\epsilon_1,\epsilon_1]} := C \bigcap F_{[-\epsilon_1,\epsilon_1]}$ which is a surface with boundary $C \bigcap F_{\epsilon_1} - C \bigcap F_{-\epsilon_1}$, possibly with positive and negative punctures at $x = 0$. Let d_1(resp. d_2, $d_{-\infty}$) $\in \{0, 1\}$ denote the number of punctures of $C_{[-\epsilon_1,\epsilon_1]}$ that project to the first positive puncture (resp. the second positive puncture, the negative puncture) of B_0. Notice that the two Reeb orbits [2] over $x = 0$ at the first positive puncture (resp. the second puncture, the negative puncture) have the homology class $(0, 1, 0)$ (resp. $(0, 0, 1)$ and $(0, 1, 1)$), so we have

$$d_1(0, 1, 0) + d_2(0, 0, 1) + (0, q_1^+, q_2^+) = d_{-\infty}(0, 1, 1) + (0, q_1^-, q_2^-) \tag{2.21}$$

and hence

$$2 \leq (mq_1^+ + nq_2^+) - (mq_1^- + nq_2^-) = (m + n)d_{-\infty} - md_1 - nd_2 \tag{2.22}$$

which implies that

$$d_{-\infty} = 1. \tag{2.23}$$

The above equation implies that C has no other negative punctures. So for any $\epsilon_2 \in (\delta, \epsilon)$, the section C cannot intersect both $F_{1-\epsilon_2}$ and $F_{1+\epsilon_2}$, because otherwise, the same argument as above would imply that C has another negative puncture asymptotic to one of the Reeb orbits over $x = 1$, a contradiction. So there are two remaining possibilities:

1. $C \bigcap F_{1-\epsilon_2} = \emptyset$. Let us consider $C \bigcap F_{[\epsilon_1, 1-\epsilon_2]}$. For this part of C, there are no negative punctures or positive punctures, so we conclude that $\partial(C \bigcap F_{[\epsilon_1,1-\epsilon_2]}) = -C \bigcap F_{\epsilon_1}$ is null homologous in $H_1(X_D)$, which contradicts the fact that $mq_1^+ + nq_2^+ \geq 1$.

2. $C \bigcap F_{1+\epsilon_2} = \emptyset$. Let us consider $C \bigcap F_{[\epsilon_1,1+\epsilon_2]}$. For this part of C, let a (resp. b)$\in \{0, 1\}$ denote the number of punctures C has at $x = 1$ that project to the first (resp. second) positive puncture of B_0. Observe that the Reeb orbits at $x = 1$ near the first (resp. second) positive puncture have the homology class $(-1, 1, 0)$ (resp. $(-1, 0, 1)$) $\in H_1(X_D)$, so we conclude that

$$a(-m, 1, 0) + b(-n, 0, 1) = (0, q_1^+, q_2^+) \tag{2.24}$$

which in turn implies that $a = b = 0$. But then it follows that $q_1^+ = q_2^+ = 0$, contradicting $mq_1^- + nq_2^- \leq -1$.

[2]Recall that the unperturbed Reeb vector fields over x are $\partial_{t_1} - mx\partial_y$, $\partial_{t_2} - nx\partial_y$ and $\partial_{t_{-\infty}} - (m + n)x\partial_y$near the three punctures.

Finally, we consider the case where C is horizontal. Suppose there exists such a horizontal section C that is neither contained in X_D nor in X_H, then again without loss of generality we can assume that there exists some $\epsilon_1 \in (\delta, \epsilon)$ such that C intersects both F_{ϵ_1} and $F_{-\epsilon_1}$ transversely and $C \bigcap F_{\pm\epsilon_1} \neq \emptyset$. Recall that inside X_D apart from the perturbed region, $\omega_X = dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n})$. We show in the following that outside of the perturbed region, the x-coordinate of the section contained in X_D is locally constant, which then leads to a contradiction.

To see this fact, we write the one-form $\frac{1}{2}x^2\beta_{m,n}$ as

$$\frac{1}{2}x^2\beta_{m,n} = fds + gdt \tag{2.25}$$

where (s,t) is the local conformal coordinate for B_0. We next compute that the horizontal lifts $\partial_s^\#$, $\partial_t^\#$ of the two vector fields ∂_s, ∂_t are:

$$\partial_s^\# = \partial_s + \frac{\partial f}{\partial y}\partial_x - \frac{\partial f}{\partial x}\partial_y, \tag{2.26}$$

$$\partial_t^\# = \partial_t + \frac{\partial g}{\partial y}\partial_x - \frac{\partial g}{\partial x}\partial_y. \tag{2.27}$$

It follows that if u is horizontal, then the part of $u(s,t) = (s,t,x(s,t),y(s,t))$ in X_D outside of the perturbed region satisfies:

$$\frac{\partial x}{\partial s} = \frac{\partial f}{\partial y}, \ \frac{\partial y}{\partial s} = -\frac{\partial f}{\partial x} \tag{2.28}$$

$$\frac{\partial x}{\partial t} = \frac{\partial g}{\partial y}, \ \frac{\partial y}{\partial t} = -\frac{\partial g}{\partial x}. \tag{2.29}$$

Recall that by our assumption, away from the perturbed region inside X_D, we have $f_y = g_y = 0$. It follows that x is locally constant.

This concludes the proof of Lemma 2.3.11. □

Remark 2.3.12. If we do not assume that the section C has vanishing wrapping number(s), the above argument still shows that the wrapping number $\eta(C)$ is non-negative. To see this, using the same notation we have $p^+ + \epsilon(mq_1^+ + nq_2^+) \geq 0$ for all generic $\epsilon > 0$. Notice that the homology class (p^+, q_1^+, q_2^+) does not depend on generic $\epsilon > 0$, so we can let $\epsilon \to 0$ and conclude that $\eta(C) = p^+ \geq 0$.

A parallel result holds for $X^{m,n}$:

Lemma 2.3.13. *Assume J is a fibration-compatible almost complex structure on $X^{m,n}$. If C is a J-holomorphic section of $X^{m,n}$ such that all wrapping numbers are zero, then C is either contained in X_H or contained in X_D.*

The proof of this result, however, is different from the one described above, so we present the details here:

Proof of Lemma 2.3.13. As before, we only need to consider the case where C is not horizontal. Suppose there is some J-holomorphic section C that is neither contained in X_H nor X_D, without loss of generality we assume that there is some $\epsilon_1 \in (\delta, \epsilon)$ such that C intersects both $F_{\pm\epsilon_1}$ transversely and $C \bigcap F_{\pm\epsilon_1} \neq \emptyset$.

Let $(p^{\pm}, q_1^{\pm}, q_2^{\pm})$ denote the homology classes of $C \bigcap F_{\pm\epsilon_1}$. Since $\eta(C) = 0$, we again observe that $p^{\pm} = 0$. Now the local energy inequality implies that:

$$mq_1^+ + nq_2^+ \geq 1,\ mq_1^- + nq_2^- \leq -1. \tag{2.30}$$

Let d_∞(resp. d_1, d_2) $\in \{0, 1\}$ denote the number of punctures of $C_{[-\epsilon_1,\epsilon_1]}$ that project to the positive puncture (resp. the two negative punctures) of B_0. We have:

$$d_1(0,1,0) + d_2(0,0,1) + (0, q_1^-, q_2^-) = d_\infty(0,1,1) + (0, q_1^+, q_2^+) \tag{2.31}$$

and hence

$$2 \leq m(q_1^+ - q_1^-) + n(q_2^+ - q_2^-) = m(d_1 - d_\infty) + n(d_2 - d_\infty). \tag{2.32}$$

We conclude that $d_\infty = 0$ and that at least one of d_1 and d_2 is 1. There are two possibilities:

1. $d_1 = d_2 = 1$. If this is the case, then C does not have other outputs. We conclude that for any small enough ϵ_2, the section C cannot intersect both $F_{1\pm\epsilon_2}$, otherwise the exact same argument would tell us that C has at least another output over $x = 1$. Choose $l \in \{1 - \epsilon_2, 1 + \epsilon_2\}$ such that $C \bigcap F_l = \emptyset$. We now look at $C \bigcap F_{[\epsilon_1,l]}$. This part of C can only have a positive puncture (or no punctures at all) with homology class $(-k, 1, 1)$ for some $k \in \{1, 2, \cdots, m + n\}$, but the same homology class should match $(0, q_1^+, q_2^+)$, which means that there's no positive puncture. So we conclude that $C \bigcap F_{[\epsilon_1,l]}$ is a surface without puncture, whose boundary is $-C \bigcap F_{\epsilon_1}$, which implies that $q_1^+ = q_2^+ = 0$, contradicting $mq_1^+ + nq_2^+ \geq 1$.

2. We have either $d_1 = 1$ and $d_2 = 0$ or $d_1 = 0$ and $d_2 = 1$. Without loss of generality let us assume the first case happens. There are two sub-cases.

   Case 2.1 If there is some small $\epsilon_2 \in (\delta, \epsilon)$ such that $C \bigcap F_{1-\epsilon_2} = \emptyset$ or $C \bigcap F_{1+\epsilon_2} = \emptyset$, then as before we fix $l \in \{1 - \epsilon_2, 1 + \epsilon_2\}$ such that $C \bigcap F_l = \emptyset$, and look at $C \bigcap F_{[\epsilon_1,l]}$. This part of C can have at most one positive puncture with homology class $(-k, 1, 1)$ where $k \in \{1, 2, \cdots, m + n\}$ and at most one negative puncture with homology class $(-j, 0, 1)$ for some $j \in \{1, 2, \cdots, n\}$. If $C \bigcap F_{[\epsilon_1,l]}$ has no punctures, then we argue as before to show that $q_1^+ = q_2^+ = 0$, which leads to a contradiction. So $C \bigcap F_{[\epsilon_1,l]}$ has at least one puncture, but then again by homology considerations we conclude that $C \bigcap F_{[\epsilon_1,l]}$ has precisely two punctures, with homology classes $(-k, 1, 1)$ and $(-k, 0, 1)$ for some $k \in \{1, 2, \cdots, n\}$. Now we have:

$$(-k, 1, 1) = (0, q_1^+, q_2^+) + (-k, 0, 1) \tag{2.33}$$

which implies that $q_1^+ = 1$ and $q_2^+ = 0$. Now $d_1 = 1$ and $d_2 = 0$ tells us that $q_1^- = q_1^+ - 1 = 0$ and $q_2^- = q_2^+ = 0$, contradicting $mq_1^- + nq_2^- \leq -1$.

Case 2.2 The other possibility is that we can find some $\epsilon_2 \in (\delta, \epsilon)$ such that C intersects both $F_{1\pm\epsilon_2}$ transversely. We use $(p^{1\pm\epsilon_2}, q_1^{1\pm\epsilon_2}, q_2^{1\pm\epsilon_2})$ to denote the homology classes of $C \bigcap F_{1\pm\epsilon_2}$. The condition $\eta(C) = 0$ now translates to $p^{1\pm\epsilon_2} + mq_1^{1\pm\epsilon_2} + nq_2^{1\pm\epsilon_2} = 0$, because the wrapping number is now the integral of dy_R, which equals the integral of $dy + \beta^{m,n}$. The local energy inequality tells us that:

$$p^{1-\epsilon_2} + (1-\epsilon_2)(mq_1^{1-\epsilon_2} + nq_2^{1-\epsilon_2}) > 0 \tag{2.34}$$

$$p^{1+\epsilon_2} + (1+\epsilon_2)(mq_1^{1+\epsilon_2} + nq_2^{1+\epsilon_2}) > 0 \tag{2.35}$$

which simplifies to

$$mq_1^{1+\epsilon_2} + nq_2^{1+\epsilon_2} \geq 1,\ mq_1^{1-\epsilon_2} + nq_2^{1-\epsilon_2} \leq -1. \tag{2.36}$$

Let d'_∞(resp. d'_2) $\in \{0,1\}$ denote the number of punctures of $C_{[1-\epsilon_2,1+\epsilon_2]}$ that project to the positive puncture (resp. the second negative punctures) of B_0. We have:

$$d'_2(-n,0,1) + (p^{1-\epsilon_2}, q_1^{1-\epsilon_2}, q_2^{1-\epsilon_2}) = d'_\infty(-m-n,1,1) + (p^{1+\epsilon_2}, q_1^{1+\epsilon_2}, q_2^{1+\epsilon_2}) \tag{2.37}$$

which implies that

$$q_1^{1+\epsilon_2} = q_1^{1-\epsilon_2} - d'_\infty,\ q_2^{1+\epsilon_2} = q_2^{1-\epsilon_2} + d'_2 - d'_\infty. \tag{2.38}$$

Again

$$2 \leq m(q_1^{1+\epsilon_2} - q_1^{1-\epsilon_2}) + n(q_2^{1+\epsilon_2} - q_2^{1-\epsilon_2}), \tag{2.39}$$

which tells us that $d'_\infty = 0$ and $d'_2 = 1$, and hence $p^{1-\epsilon_2} - n = p^{1+\epsilon_2}$.

We now look at $C \bigcap F_{[-\epsilon_1,1+\epsilon_2]}$. This part of C has two outputs with the homology classes $(0,1,0)$ and $(-n,0,1)$, and at most one puncture with homology class $(-k,1,1)$ for some $k \in \{1,2,\cdots,m+n-1\}$. We also have $\partial(C \bigcap F_{[-\epsilon_1,1+\epsilon_2]}) = C \bigcap F_{1+\epsilon_2} - C \bigcap F_{-\epsilon_1}$. We observe that $C \bigcap F_{[-\epsilon_1,1+\epsilon_2]}$ must contain a positive puncture, otherwise $p_1^{1+\epsilon_2} = -n$, and hence $p^{1-\epsilon_2} = 0$, so $\eta = 0$ implies that $mq_1^{1-\epsilon_2} + nq_2^{1-\epsilon_2} = 0$, contradicting the local energy inequality $mq_1^{1-\epsilon_2} + nq_2^{1-\epsilon_2} \leq -1$. Finally, we have:

$$(0, q_1^-, q_2^-) + (0,1,0) + (-n,0,1) = (p^{1+\epsilon_2}, q_1^{1+\epsilon_2}, q_2^{1+\epsilon_2}) + (-k,1,1) \tag{2.40}$$

which implies that $q_1^- = q_1^{1+\epsilon_2}$ and $q_2^- = q_2^{1+\epsilon_2}$, but then the local energy inequalities $mq_1^- + nq_2^- \leq -1$ and $mq_1^{1+\epsilon_2} + nq_2^{1+\epsilon_2} \geq 1$ cannot both be true. This concludes the proof of Lemma 2.3.13.

□

2.4 The "no crossing" results for general J

Although the fibration compatible almost complex structures in Definition 2.3.1 are convenient to work with, they are not suitable for defining the cobordism map. The reason is that for given fibration compatible J, not all J-holomorphic sections are cut out transversely, so there is not a well-defined count for the cobordism map as in Section 1.3. In this section, we use the SFT compactness theorem developed in [Bou+03] to show that we can always perturb the almost complex structure slightly to a tame almost complex structure - not necessarily fibration compatible, in such a way that the "no crossing" results Lemma 2.3.11 and Lemma 2.3.13 continue to hold.

Throughout this section, we let X denote either the bundle $X_{m,n}$ or $X^{m,n}$. We fix a fibration-compatible almost complex structure J on X, and denote by J_+^1, J_+^2, J_- its restrictions on the three cylindrical ends of X. In the case that $\partial\Sigma \neq \emptyset$, we choose coordinates (x_i, y_i) near each boundary component of $\partial\Sigma$, such that any almost complex structure we choose, even if it is not fibration compatible elsewhere, is fibration-compatible near the boundary and sends ∂_{x_i} to ∂_{y_i}.

Theorem 2.4.1. *Let $\{J_k\}$ be a sequence of tame almost complex structures that C^∞ converges to a fixed fibration-compatible almost complex structure J, and $\{C_k\}$ be a sequence of finite-energy J_k-holomorphic sections, which we view as maps $u_k : B_0 \to X$, that are asymptotic to fixed Reeb orbits in Y_{ϕ^m}, Y_{ϕ^n} and $Y_{\phi^{m+n}}$. If all wrapping numbers of $\{C_k\}$ vanish, then C_k is contained in X_H or X_D for sufficiently large k.*

The proof of Theorem 2.4.1 relies largely on a careful analysis of J-holomorphic sections in X and the symplectizations Y_{ϕ^m}, Y_{ϕ^n} and $Y_{\phi^{m+n}}$, which we take up in the following subsections. To begin the proof of Theorem 2.4.1, let us make the following simple observation. We can slightly shrink the two open subsets X_H and X_D to $X_{H,\tilde{\epsilon}}$ and $X_{D,\tilde{\epsilon}}$, where $\tilde{\epsilon} \in (\delta, \epsilon]$ and

$$X_{D,\tilde{\epsilon}} := B_0 \times (-\tilde{\epsilon}, 1 + \tilde{\epsilon})_x \times S^1_y \tag{2.41}$$

$$X_{H,\tilde{\epsilon}} := B_0 \times (\Sigma - (\tilde{\epsilon}, 1 - \tilde{\epsilon})_x \times S^1_y) \tag{2.42}$$

such that Lemma 2.3.11 and Lemma 2.3.13 still hold for the new cover $X = X_{D,\tilde{\epsilon}} \cup X_{H,\tilde{\epsilon}}$.

J-holomorphic cylinders in symplectizations

The next step is to analyze J-holomorphic cylinders in the symplectization $\mathbb{R} \times Y_{\phi^m}$. The analysis for the remaining cases of Y_{ϕ^n} and $Y_{\phi^{m+n}}$ are analogous. Similar to what we saw in Section 2.1, there is a decomposition of Y_{ϕ^m}:

$$Y_{m,D,\tilde{\epsilon}} := S^1_t \times (-\tilde{\epsilon}, 1 + \tilde{\epsilon})_x \times S^1_y \tag{2.43}$$

$$Y_{m,H,\tilde{\epsilon}} := S^1_t \times (\Sigma - (\tilde{\epsilon}, 1 - \tilde{\epsilon})_x \times S^1_y). \tag{2.44}$$

When $\tilde{\epsilon} = \epsilon$, without causing confusions, we will abbreviate the two components by Y_D and Y_H respectively. The gluing map of $Y_{m,D,\tilde{\epsilon}}$ and $Y_{m,H,\tilde{\epsilon}}$ is defined similarly as in Section 2.1. For J-holomorphic sections in $\mathbb{R} \times Y_{\phi^m}$, the wrapping numbers are defined similarly, see [HS05] Definition 4.2.

As in Section 2.3, by slightly abusing the notations, let us denote by F_x the three-dimensional manifold $\mathbb{R} \times S^1_t \times \{x\} \times S^1_y \subset Y_D$. Let $F_{(x_1,x_2)}$ denote the four-manifold $\mathbb{R} \times S^1_t \times (x_1, x_2)_x \times S^1_y \subset \mathbb{R} \times Y_D$. We also identify the first homology class in Y_D with a pair (p, q). Fix a symplectization compatible (and hence by Definition 2.3.1, a fibration compatible) almost complex structure J on $\mathbb{R} \times Y_{\phi^m}$. The local energy inequality for J-holomorphic sections in $\mathbb{R} \times Y_{\phi^m}$ is the following:

Lemma 2.4.2 ([HS05] Lemma 3.11)**.** *Let C be a J-holomorphic section, which we write as a map $u : \mathbb{R} \times S^1 \to \mathbb{R} \times Y_{\phi^m}$. Assume that C intersects F_x transversely for some x in the unperturbed range. We have:*

$$p + mxq \geq 0. \tag{2.45}$$

Furthermore, the equality holds if and only if $C \cap F_x = \emptyset$.

Proof. This is a straightforward generalization of Lemma 2.3.2. For a different proof, see [HS05]. □

The above inequality implies the following "no crossing" result for J-holomorphic cylinders in $\mathbb{R} \times Y_{\phi^m}$:

Lemma 2.4.3. *Let J be a symplectization-compatible almost complex structure on $\mathbb{R} \times Y_{\phi^m}$. If C is a J-holomorphic section of the bundle $\mathbb{R} \times Y_{\phi^m} \to \mathbb{R} \times S^1_t$ with vanishing wrapping numbers, then:*

1. *For any $\tilde{\epsilon} \in (\delta, \epsilon)$, the section C is either contained in $\mathbb{R} \times Y_{m,D,\tilde{\epsilon}}$ or $\mathbb{R} \times Y_{m,H,\tilde{\epsilon}}$;*
2. *For such a section, if the positive end is one of the two orbits over $x = 0$ (resp. $x = 1$), then for any $\tilde{\epsilon} \in (\delta, \epsilon)$, C is contained in $F_{(-\tilde{\epsilon},\tilde{\epsilon})}$ (resp. $F_{(1-\tilde{\epsilon},1+\tilde{\epsilon})}$);*
3. *For such a section, if the negative end is one of the two orbits over $x = 0$ (resp. $x = 1$), then for any $\tilde{\epsilon} \in (\delta, \epsilon)$, the section C is contained in $\mathbb{R} \times Y_{m,H,\tilde{\epsilon}}$;*
4. *Finally, if such a section does not have any end over $x = 0$ or $x = 1$, then it is completely contained in $\mathbb{R} \times (Y_D - Y_H)$ or $\mathbb{R} \times (Y_H - Y_D)$.*

The proof is similar to that of Lemma 2.3.11, but it is worthwhile to write down the details.

Proof. For any $\epsilon_1 \in (\delta, \epsilon)$, let us denote the homology classes of $C \cap F_{\pm\epsilon_1}$ by $(p^\pm, q^\pm)$ (the choice of ϵ_1 does not matter here). Since all wrapping numbers of C vanish, we conclude that $p^\pm = 0$.

To prove the first bullet point, suppose C is not contained in either region. Without loss of generality we could assume there is some ϵ_1 such that $C \cap F_{\pm\epsilon_1} \neq \emptyset$. Now Lemma 2.4.2 tells us that

$$m\epsilon_1 q^+ > 0 > m\epsilon_1 q^-. \tag{2.46}$$

So $q^+ \geq 1$ and $q^- \leq -1$. Notice that for punctures of C that are contained in $F_{[-\epsilon_1,\epsilon_1]}$, the homology class is $(0,1)$. Let us assume there are $d_\infty \in \{0,1\}$ (resp. $d_{-\infty}$) many of such positive (resp. negative) punctures, and we have:

$$d_{-\infty}(0,1) + (0,q^-) = d_\infty(0,1) + (0,q^+). \tag{2.47}$$

But this is not possible, because otherwise

$$2 \leq q^+ - q^- = d_{-\infty} - d_\infty \leq 1. \tag{2.48}$$

To prove the second bullet point, it suffices to show that for any $\epsilon_1 \in (\delta,\epsilon)$, we have $C \cap F_{\pm\epsilon_1} = \emptyset$ and $C \cap F_{1\pm\epsilon_1} = \emptyset$. Without loss of generality suppose $C \cap F_{\epsilon_1} \neq \emptyset$ or $C \cap F_{-\epsilon_1} \neq \emptyset$. By the same argument as in the previous paragraph, we have $q^+ - q^- \geq 1$, but now we have $d_\infty = 1$, so

$$1 \leq q^+ - q^- = d_{-\infty} - d_\infty \leq 0, \tag{2.49}$$

a contradiction.

To prove the third bullet point, simply notice that otherwise such a section is completely contained in $\mathbb{R} \times Y_D$ by the first bullet point. Now observe that Reeb orbits in Y_D that are over different values of x have different homology classes, it follows that both ends of C are over $x = 0$ or $x = 1$. Now the second bullet point shows that such a section is contained in $\mathbb{R} \times Y_{m,H,\tilde{\epsilon}}$ as well.

Finally, to prove the last bullet point, observe that (in the same notation as before) $d_{\pm\infty} = 0$ forces that $q^\pm = 0$, hence $C \cap F_{\pm\epsilon_1} = \emptyset$. Similarly $C \cap F_{1\pm\epsilon_1} = \emptyset$ for any ϵ_1. □

More about J-holomorphic sections in the twist region

To prove Theorem 2.4.1, the final ingredient we need is a more detailed understanding of J-holomorphic sections that are contained in the twist region X_D. Let us recall that, for J-holomorphic sections of $X_{m,n}$ that are contained in the twist region $X_D = B_0 \times (-\epsilon, 1+\epsilon)_x \times S^1_y$, the asymptotic Reeb orbits can occur over:

1. $x = \frac{i}{m}$ $(i \in \{0,1,\cdots,m\})$ for the first positive end;
2. $x = \frac{j}{n}$ $(j \in \{0,1,\cdots,n\})$ for the second positive end;
3. $x = \frac{k}{m+n}$ $(k \in \{0,1,\cdots,m+n\})$ for the negative end.

The next lemma tells us that for J-holomorphic sections of $X_{m,n}$ that are contained in the twist region, the three ends must in fact lie over the same x-coordinate.

Lemma 2.4.4. *Let J be a fibration-compatible almost complex structure on $X_{m,n}$, and C be a J-holomorphic section that is completely contained in $X_D = B_0 \times (-\epsilon, 1+\epsilon)_x \times S^1_y$. Then the x coordinate of the three cylindrical ends of C asymptote to the same value. Furthermore, C itself is completely contained in the δ-neighborhood of the slice F_x (the subscript x denotes the x value to which the ends of C asymptote.)*

Proof. The Reeb vector field near the first positive end is $\partial_t - mx\partial_y$, so the homology class of any Reeb orbit over $x = \frac{i}{m}$ is $[S^1_{t_1}] - i[S^1_y] \in H_1(X_D)$. Similarly, for Reeb orbits over the second positive end with the x-coordinate $\frac{j}{n}$, the homology class is $[S^1_{t_2}] - j[S^1_y] \in H_1(X_D)$; the homology class for Reeb orbits over the negative end with the x-coordinate $\frac{k}{m+n}$ is $[S^1_{t_1}] + [S^1_{t_2}] - k[S^1_y] \in H_1(X_D)$.

It follows that for a J-holomorphic section that is completely contained in X_D, we have $k = i+j$ for homological reasons. Now if the three ends do not share the same x-coordinates, without loss of generality we can assume that $\frac{i}{m} < \frac{i+j}{m+n} < \frac{j}{n}$. Pick some $x_0 \in (\frac{i+j}{m+n}, \frac{j}{n})$ such that C intersects the slice F_{x_0} transversely, then the homology class of $C \bigcap F_{x_0}$ in $H_1(X_D)$ is $(-j, 0, 1)$. Using Lemma 2.3.2, we have

$$-j + x_0(m \cdot 0 + n \cdot 1) \geq 0 \tag{2.50}$$

which implies that $x_0 \geq \frac{j}{n}$, a contradiction.

Now suppose C is not contained in the δ-neighborhood of the slice F_x, we can choose some $\tilde{\epsilon}$ slightly bigger than δ such that C intersects $F_{x\pm\tilde{\epsilon}}$ transversely and the intersect is nonempty. But notice that $C \cap F_{x\pm\tilde{\epsilon}}$ are both null-homologous, so this is a violation of Lemma 2.3.2. □

Remark 2.4.5. The second part of the above Lemma also holds for J-holomorphic sections of $X^{m,n}$ that are completely contained in the twist region. Namely, if all three ends of such a section share the same x-coordinate, then the entire section is contained in the δ-neighborhood of the slice F_x.

Proof of "no crossing" for general J (Theorem 2.4.1)

Now we are ready to prove the main result of this section.

Proof of Theorem 2.4.1. Fix some $\tilde{\epsilon} \in (\delta, \epsilon)$. Suppose that the statement of Theorem 2.4.1 fails, by the SFT compactness theorem, we can find a subsequence of $\{C_k\}$, still denoted by $\{C_k\}$, such that :

1. For every k, C_k is not contained in X_H or X_D, and

2. $\{C_k\}$ converges to a J-holomorphic building $\mathcal{B}$.

Let us first observe that by our assumptions, $\pi_2(X_{m,n})$, $\pi_2(X^{m,n})$, $\pi_2(Y_{\phi^m})$ are all trivial, so bubbling off of J- holomorphic spheres can not occur in any level of $\mathcal{B}$. Notice also that it is not possible for any component of any level of $\mathcal{B}$ to have only positive or only negative punctures, simply by homological considerations. The above two observations imply that

1. $\mathcal{B}$ has no nodes;
2. The main level of $\mathcal{B}$ is a J-holomorphic section of X, and
3. Every other level of $\mathcal{B}$ is a (resp. pair of) holomorphic cylinder in the symplectization of $Y_{\phi^{m+n}}$ (resp. $Y_{\phi^m} \coprod Y_{\phi^n}$).

We note that all levels of $\mathcal{B}$ must have vanishing wrapping numbers. The reason is that the wrapping number is homological, so the sum of the wrapping number from all different level is equal to zero. By Remark 2.3.10, all wrapping numbers are non-negative, so they have to vanish in each level as well.

If the main level of $\mathcal{B}$ is contained in $X_{H,\tilde{\epsilon}}$, we can use Lemma 2.4.3 and induction to show that all other levels of $\mathcal{B}$ are contained in $\mathbb{R} \times Y_{m+n,H,\tilde{\epsilon}}$ (or $\mathbb{R} \times (Y_{m,H,\tilde{\epsilon}} \coprod Y_{n,H,\tilde{\epsilon}})$ respectively). For example, the first level above the main level consists of one J_+-holomorphic cylinder in $\mathbb{R} \times Y_{\phi^{m+n}}$ (if $X = X^{m,n}$) or a pair of J_+-holomorphic cylinders in $\mathbb{R} \times (Y_{\phi^m} \coprod Y_{\phi^n})$ (if $X = X_{m,n}$). In either case, those J_+-holomorphic cylinders have negative ends which are either over $x = 0, 1$ inside the twist regions, or outside the twist region. Now the third and forth bullets points of Lemma 2.4.3 tell us that these cylinders are entirely contained in $\mathbb{R} \times Y_{m+n,H,\tilde{\epsilon}}$ or $\mathbb{R} \times (Y_{m,H,\tilde{\epsilon}} \coprod Y_{n,H,\tilde{\epsilon}})$. We conclude, using induction, that all levels above the main level are contained in the same region. Now let us consider the first level under the main level. For any such J_--holomorphic cylinder, if the positive end is over $x = 0, 1$, then by the second bullet point of Lemma 2.4.3, they are completely contained in $F_{(-\tilde{\epsilon},\tilde{\epsilon})}$; if the positive end is contained outside of the twist regions, then the third and forth bullet points of Lemma 2.4.3 imply that the cylinders are contained in $\mathbb{R} \times Y_{m+n,H,\tilde{\epsilon}}$ or $\mathbb{R} \times (Y_{m,H,\tilde{\epsilon}} \coprod Y_{n,H,\tilde{\epsilon}})$. Again, we can repeat the above analysis to find that all levels under the main level are contained in $\mathbb{R} \times Y_{m+n,H,\tilde{\epsilon}}$ or $\mathbb{R} \times (Y_{m,H,\tilde{\epsilon}} \coprod Y_{n,H,\tilde{\epsilon}})$. In summary, the entire building is contained in the (slightly shrunk) non-twist region, which implies that for sufficiently large k, the section C_k is contained in X_H as well, a contradiction.

If the main level of $\mathcal{B}$ is contained in $X_{D,\tilde{\epsilon}} - X_{H,\tilde{\epsilon}}$, then again we can use the fourth bullet point of Lemma 2.4.3 and induction to deduce that all other levels of $\mathcal{B}$ are contained in $\mathbb{R} \times Y_{m+n,D,\tilde{\epsilon}}$ or $\mathbb{R} \times (Y_{m,D,\tilde{\epsilon}} \coprod Y_{n,D,\tilde{\epsilon}})$. It follows that for sufficiently large k, the section C_k is completely contained in X_D, a contradiction.

□

2.5 The product

In this section, we use the no-crossing results to calculate the pair-of-pants product defined in Section 2.1:

$$\mathrm{HF}_*(\phi^m) \otimes \mathrm{HF}_*(\phi^n) \longrightarrow \mathrm{HF}_*(\phi^{m+n}) \tag{2.51}$$

where ϕ is the (Hamiltonian perturbed) positive Dehn twist along a homologically nontrivial simple closed curve $\gamma \subset \Sigma$, with the extra conditions stated in Theorem 2.2.1.

As reviewed in Section 2.1, we fix the cobordism $X = X_{m,n}$ and a generic Hamiltonian perturbation. We always assume that the almost complex structure J is C^∞ close to a fibration-compatible one, as in Section 2.3. Furthermore, we require that near each boundary component of $\partial\Sigma$ with local coordinates (x_i, y_i), the almost complex structure J is fibration-compatible, and is induced from the almost complex structure on Ver that sends ∂_{x_i} to ∂_{y_i}. Lemma 2.3.8 tells us that J-holomorphic sections cannot approach $\partial\Sigma$ by the maximum principle.

Several remarks on monotonicity

To define fixed point Floer homology without using Novikov rings, we need a monotonicity condition. In what follows, we will use a slightly stronger version of "weak monotonicity" introduced in [Cot09].

Definition 2.5.1 ([Cot09] Condition 2.5)**.** *Let ψ be a symplectomorphism of (Σ, ω_0). Let ω_ψ denote the 2-form on the mapping torus Y_ψ induced by ω_0, and* Ver *the vertical distribution of $Y_\psi \to S^1$. We say ψ is weakly monotone if $[\omega_\psi]$ vanishes on the kernel of*

$$c_1(\mathrm{Ver}) : H_2(Y_\psi) \longrightarrow \mathbb{R}. \tag{2.52}$$

We have the following:

Lemma 2.5.2. *For the positive Dehn twist $\phi : (\Sigma, \omega_0) \to (\Sigma, \omega_0)$, the map ϕ^m is weakly monotone for any positive integer m.*

Proof. The proof is almost verbatim to that of [HS05] Lemma 5.1, and the only difference is that in our setting $\langle [\Sigma], c_1(\mathrm{Ver})\rangle = 2 - 2g(\Sigma)$ if $\partial\Sigma = \emptyset$. □

The above lemma tells us that the count in (1.3) is finite, so the fixed point Floer homology $\mathrm{HF}_*(\phi^m)$ is well-defined without using the Novikov rings. Similarly, we need a weak monotonicity condition for the count (1.8) to be finite.

Definition 2.5.3. *Let $\pi : (E, \omega) \to B$ be a symplectic fiber bundle, and* $\mathrm{Ver} := \mathrm{Ker}(d\pi)$ *be the vertical distribution. We say the symplectic bundle is weakly monotone if $[\omega]$ vanishes on the kernel of*

$$c_1(\mathrm{Ver}) : H_2(E) \longrightarrow \mathbb{R}. \tag{2.53}$$

Similarly, we have the following lemma, which tells us that the count (1.8) is finite (see the discussions following the proof of Lemma 2.5.4), and hence the product and coproduct structures induced by $X_{m,n}$ and $X^{m,n}$ are well-defined without use of Novikov rings.

Lemma 2.5.4. *If $\partial\Sigma \neq \emptyset$ or Σ is closed with genus at least 2, then both $X_{m,n}$ and $X^{m,n}$ are weakly monotone.*

Proof. Take a closed surface $C \subset X$ such that $[C]$ lies in the kernel of $c_1(\mathrm{Ver})$. Using the same notation as in Lemma 2.3.2, let (p, q_1, q_2) denote the homology class of $[C \cap F_0] = [C \cap F_1] \in H_1(X_D)$ (isotope C slightly to make the two intersections transverse).

Let us start with the situation where γ is non-separating and Σ is closed. It is not difficult to see that:

$$\langle [C], c_1(\mathrm{Ver})\rangle = (2 - 2g(\Sigma))\eta(C) \tag{2.54}$$

where η is the wrapping number. We conclude that the wrapping number of C is zero. As explained in the proof of Lemma 2.3.11, we have

$$\eta(C) = p = p + mq_1 + nq_2 = 0. \tag{2.55}$$

Now if γ is separating and Σ is closed, let Σ_1, Σ_2 denote the two components of $\Sigma - [0,1]_x \times S^1_y$. By our assumption: $g(\Sigma_1), g(\Sigma_2) \geq 1$. Similar to the above, we have:

$$\langle [C], c_1(\mathrm{Ver})\rangle = (1 - 2g(\Sigma_1))\eta_1(C) + (1 - 2g_2(\Sigma_2))\eta_2(C). \tag{2.56}$$

So $[C] \in \mathrm{Ker}(c_1(\mathrm{Ver}))$ implies that both wrapping numbers of C vanish, hence we have $p = p + mq_1 + nq_2 = 0$ again.

It is not difficult to calculate, using the explicit expression of ω_X, that $\int_C \omega_X$ is a linear combination of p and $mq_1 + nq_2$. So $[\omega_X]$ indeed vanishes on $[C]$.

Finally, if $\partial\Sigma \neq \emptyset$, then the wrapping number of C is automatically zero if γ is non-separating. If γ is separating, then:

1. If both components of $\Sigma - [0,1]_x \times S^1_y$ contains at least one component of ∂C, then both wrapping numbers of C automatically vanishes;

2. If only one of the components of $\Sigma - [0,1]_x \times S^1_y$, say Σ_2, contains components of $\partial\Sigma$ (so η_2 vanishes), then we have

$$\langle [C], c_1(\mathrm{Ver})\rangle = (1 - 2g(\Sigma_1))\eta_1(C). \tag{2.57}$$

   So $[C] \in ker(c_1(\mathrm{Ver})$ implies that η_1 vanishes as well.

The conclusion is that $p = p + mq_1 + nq_2 = 0$ regardless. Using the exact same argument as above, we conclude that $[\omega_X]$ vanishes on $[C]$ as well. □

Lemma 2.5.2 tells us that for any positive integer m, $\mathrm{HF}_*(\phi^m)$ is well defined without use of Novikov coefficients. In fact, it is well-known (see for example [Sei96; HS05]) that

$$\mathrm{HF}_*(\phi^m) \cong H_*(\Sigma_0; \mathbb{Z}_2) \oplus (\oplus_{i=1}^{m-1} H_*(S^1)) \tag{2.58}$$

where the i-th component of $\oplus_{i=1}^{m-1} H_*(S^1)$ comes from the Reeb orbits inside the Dehn twist region over $x = \frac{i}{m}$.

Before proving Theorem 2.2.1, some remarks on the *Hofer energy* of the J-holomorphic sections are needed. In the following, we view the cobordism $X_{m,n}$ as the completion of the compact cobordism $K_{m,n}$ with the same symplectic form Ω_X.

Definition 2.5.5 (See for example [Hof93; Wen])**.** *Let J be a tame almost complex structue on $X_{m,n}$. The Hofer energy of a J-holomorphic section u is defined as*

$$E^{\text{Hofer}}(u) = \sup_{f \in \mathcal{T}} \int_{B_0} u^* \omega_f$$

Where $\mathcal{T} = \{f \in C^\infty(\mathbb{R}, (-\epsilon, \epsilon)) | f' > 0$ and $f(x) = x$ near $[-\delta, \delta]\}$ (for sufficiently small ϵ and δ), and

$$\omega_f = \begin{cases} \Omega_X & \text{on } K_{m,n} \\ d(f(s_i)dt_i) + \omega_X & \text{near the three punctures} \end{cases}$$

The discussions about the monotonicity conditions imply that when the J-holomorphic sections have the same indices and the same asymptotes, they share the same integral $\int_{B_0} \omega_X$. Hence we have a uniform bound on the Hofer energy for sections in $\mathcal{M}^J_{x,y;z}$, so the SFT compactness theorem implies that (1.8) (as well as the corresponding count for the coproduct structure) is a finite count.

All sections have vanishing wrapping numbers

In this subsection, we explain that in our setting, all J-holomorphic sections of $X_{m,n} \to B_0$ with Fredholm index 0 have vanishing wrapping numbers. This observation will allow us to use the no crossing results from Section 2.3 and 2.4.

Theorem 2.5.6. *Let J be an almost complex structure on $X_{m,n}$ that is close to a fibration-compatible one. Suppose*

- *If γ is non-separating, then $\partial\Sigma \neq \emptyset$ or Σ is closed with genus at least 2;*
- *If γ is separating, then each component of $\Sigma - \gamma$ either contains a component of $\partial\Sigma$ or has genus at least 2.*

Then for any Fredholm index zero J-holomorphic section C with cylindrical ends, C has vanishing wrapping numbers.

To prove Theorem 2.5.6, let us recall the Fredholm index formula. Let C be a J-holomorphic section in $X_{m,n}$ with positive asymptotes α_i and negative asymptote β, represented by a map $u : B_0 \to X_{m,n}$. Fix a trivialization τ of the vertical distribution along each Reeb orbit, and denote by $\langle c_1^\tau(TX_{m,n}), [C]\rangle$ the first Chern number of the complex vector bundle $u^*TX_{m,n}$ over B_0 with respect to the trivialization τ and the natural splitting $TX_{m,n}|_\gamma \cong \text{Ver} \bigoplus \mathbb{R}\langle R, \partial_s\rangle$ over the ends. Here $\mathbb{R}\langle R, \partial_s\rangle$ denotes the distribution spanned by the Reeb vector field and the symplectization direction. For each asymptotic orbit, let CZ_τ be the Conley-Zehnder index with respect to τ. We have the Fredholm index formula:

$$\text{ind}(C) = 1 + 2\langle c_1^\tau(TX_{m,n}), [C]\rangle + \sum \text{CZ}_\tau(\alpha_i) - \text{CZ}_\tau(\beta). \tag{2.59}$$

Notice that in our setting, the map u is a section of the fibration $X_{m,n} \to B_0$, so $u^*TX_{m,n}$ naturally splits as $u^*TX_{m,n} \cong TB_0 \bigoplus u^* \mathrm{Ver}$. In light of this splitting, we have:

$$\langle c_1^{\tau}(TX_{m,n}), [C] \rangle = -1 + \langle c_1^{\tau}(\mathrm{Ver}), [C] \rangle. \tag{2.60}$$

So the index formula can be rewritten as:

$$\mathrm{ind}(C) = -1 + 2\langle c_1^{\tau}(\mathrm{Ver}), [C] \rangle + \sum \mathrm{CZ}_{\tau}(\alpha_i) - \mathrm{CZ}_{\tau}(\beta). \tag{2.61}$$

Now recall that the Reeb orbits can be divided into two types: those coming from critical points of mH_0, nH_0 or $(m+n)H_0$ outside of N and those lying inside the twist region. There is a natural choice of the trivialization τ of the distribution Ver over these Reeb orbits: for the critical points of H, the distribution Ver can be identified with the tangent space $T\Sigma$ at the point; and over the Dehn twist region N, we can identify Ver with TN. We will always choose τ as above, and the Conley-Zehnder index CZ_{τ} with respect to such a trivialization is:

1. -1, if the orbit comes from a local minimum of H, or is an elliptic orbit inside the Dehn twist region;
2. 0, if the orbit comes from a saddle point of H, or is a hyperbolic orbit inside the Dehn twist region;
3. 1, if the orbit comes from a local maximum of H.

Following [HS05], we now demonstrate a lemma relating the relative first Chern number $\langle c_1^{\tau}(\mathrm{Ver}), [C] \rangle$ to the wrapping number $\eta(C)$ (similar ideas were applied in the proof of Lemma 2.5.4; we present a proof of the generalization of equation (2.54) here):

Lemma 2.5.7. *If Σ is a closed surface with genus g, the loop γ is non-separating, and C is a J-holomorphic section, then*

$$\langle c_1^{\tau}(\mathrm{Ver}), [C] \rangle = (2-2g)\eta(C). \tag{2.62}$$

Proof. Choose a generic point outside of the twist region, which we denote by $pt \in \Sigma_0$, such that C intersects $B_0 \times \{pt\}$ transversely. Recall that $\eta(C)$ is by definition the algebraic intersection number $\#C \bigcap (B_0 \times \{pt\})$.

Choose a section ψ of Ver over $X_{m,n}$, with the following property:

1. When restricted to the Reeb orbits, ψ is constant with respect to τ;
2. There are l points $p_1, p_2, \cdots, p_l$ concentrated in an arbitrarily small neighborhood of pt, such that on each fiber of $X_{m,n} \to B_0$, the section ψ has transverse zeroes at precisely $p_1, p_2, \cdots, p_l$, with total degree $2-2g$.

We can also arrange ψ so that C intersects each $B_0 \times \{p_i\}$ transversely. Now by definition, $\langle c_1^\tau(\mathrm{Ver}), [C]\rangle$ is the algebraic count of zeroes of $u^*\psi$. Observe that the zeroes of $\psi|_C$ occurs at precisely $C \bigcap (B_0 \times \{p_1, p_2, \cdots, p_l\})$, and the algebraic count of these zeroes is $(2-2g)\eta(C)$. □

Now we are ready to prove Theorem 2.5.6.

Proof of Theorem 2.5.6. Let us start with the case where γ is non-separating. As remarked before, if $\partial\Sigma \neq \emptyset$, then $\eta(C)$ is automatically zero. If Σ is closed, by equation (2.54), we have the following:

$$0 = \mathrm{ind}(C) = -1 + 2(2-2g)\eta(C) + \sum \mathrm{CZ}_\tau(\alpha_i) - \mathrm{CZ}_\tau(\beta). \tag{2.63}$$

But since $\mathrm{CZ}_\tau \in \{-1, 0, 1\}$, we have

$$\mathrm{ind}(C) \leq 2 + (4-4g)\eta(C). \tag{2.64}$$

This, together with the fact that $\eta(C) \geq 0$ and the assumption $g \geq 2$, forces that $\eta(C) = 0$.

Now let us deal with the case where γ is separating. As before, let us denote by Σ_1 and Σ_2 the two components of $\Sigma - N$. If Σ is closed, then similar to Lemma 2.5.7, we have:

$$\langle c_1^\tau(\mathrm{Ver}), [C]\rangle = (1-2g(\Sigma_1))\eta_1(C) + (1-2g(\Sigma_2))\eta_2(C). \tag{2.65}$$

So we have:

$$\begin{aligned} 0 &= \mathrm{ind}(C) \\ &= -1 + 2(1-2g(\Sigma_1))\eta_1(C) + 2(1-2g(\Sigma_2))\eta_2(C) + \sum \mathrm{CZ}_\tau(\alpha_i) - \mathrm{CZ}_\tau(\beta) \\ &\leq 2 + 2(1-2g(\Sigma_1))\eta_1(C) + 2(1-2g(\Sigma_2))\eta_2(C). \end{aligned} \tag{2.66}$$

By our assumption, $g(\Sigma_1), g(\Sigma_2) \geq 2$. Combined with the fact that $\eta_1, \eta_2 \geq 0$, we have

$$\eta_1(C) = \eta_2(C) = 0, \tag{2.67}$$

as desired.

The case where both Σ_1 and Σ_2 contain a component of $\partial\Sigma$ is easy: we only need to observe as before that if Σ_i contains a component of $\partial\Sigma$, then η_i is automatically zero. The only remaining situation is the following: only one of the two components of $\Sigma - N$ contains a component of $\partial\Sigma$, and the other one has genus at least 2. Without loss of generality let us assume Σ_2 is the one containing $\partial\Sigma$ (so η_2 vanishes automatically). Similar to what we saw in Lemma 2.5.7, we have the following:

$$\langle c_1^\tau(\mathrm{Ver}), [C]\rangle = (1-2g(\Sigma_1))\eta_1(C) \tag{2.68}$$

which implies that

$$\begin{aligned} 0 &= \operatorname{ind}(C) \\ &= -1 + 2(1-2g(\Sigma_1))\eta_1(C) + \sum \mathrm{CZ}_\tau(\alpha_i) - \mathrm{CZ}_\tau(\beta) \\ &\leq 2 + 2(1-2g(\Sigma_1))\eta_1(C). \end{aligned} \tag{2.69}$$

Again, since $g(\Sigma_1) \geq 2$ and $\eta_1 \geq 0$, we conclude that $\eta_1(C)$ has to vanish as well.

□

Computation of the product (proof of Theorem 2.2.1)

We are now ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. We choose a generic tame almost complex structure J on $X_{m,n}$ such that all moduli spaces of Fredholm index zero sections are cut out transversely. We further assume that J is C^∞ close to a fibration-compatible almost complex structure so that Theorem 2.4.1 applies. Theorem 2.4.1 and Theorem 2.5.6 tell us that all the J-holomorphic sections are either contained in X_H or X_D.

Now the count of J-holomorphic sections contained in X_H precisely corresponds to the intersection product of $H_*(\Sigma_0;\mathbb{Z}_2) \subset \mathrm{HF}_*(\phi^m)$ and $H_*(\Sigma_0;\mathbb{Z}_2) \subset \mathrm{HF}_*(\phi^n)$ in the sense of the decomposition (2.58). By the classical results from [PSS96; FS07; Lan16], the cobordism map of the pair-of-pants product in this case can be identified with the intersection pairing (notice that in our case $\pi_2(\Sigma_0) = 0$, so no Novikov rings are needed here):

$$H_*(\Sigma_0;\mathbb{Z}_2) \otimes H_*(\Sigma_0;\mathbb{Z}_2) \xrightarrow{\cap} H_*(\Sigma_0;\mathbb{Z}_2). \tag{2.70}$$

To finish the proof, we only need to show that the count of sections contained in the twist region contributes to zero in the cobordism map. By Lemma 2.4.4, any such J-holomorphic section must be contained in the δ-neighborhood of some slice F_x, where $x = \frac{i}{d}$ for some $i \in \{1, 2 \cdots, d-1\}$, here $d := \gcd(m,n)$ (the reason is that we need mx, nx and $(m+n)x$ to be integers simultaneously for the three Reeb orbits to exist inside a δ-neighborhood of F_x). Recall that, near $x = \frac{i}{d}$, the symplectic fiber bundle is given by the trivial product $X_i := B_0 \times (\frac{i}{d} - \delta, \frac{i}{d} + \delta)_x \times S^1_y$ with the fiberwise symplectic closed 2-form ω_X, which is a small Hamiltonian perturbation of $\omega_{X,0} = dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n})$. Now we use a change-of-coordinate trick to show that the above symplectic fiber bundle is equivalent to another one, which calculates the pair-of-pants product of the Hamiltonian Floer homology of *small* Hamiltonians on $(\frac{i}{d}-\delta, \frac{i}{d}+\delta)_x \times S^1_y$. To do this, let X_0 be the trivial bundle $B_0 \times (-\delta, \delta)_{x'} \times S^1_{y'}$ together with the fiberwise symplectic form $dx' \wedge dy' + d(\frac{1}{2}x'^2\beta_{m,n})$. Define a diffeomorphism $\mu : X_i \to X_0$ by:

$$\begin{cases} x' = x - \frac{i}{d} \\ y' = y + i \cdot g'_{m,n}(p) \end{cases} \tag{2.71}$$

where p denotes the coordinate on B_0. It's easy to see that μ preserves the fibers, and that μ pulls $dx' \wedge dy' + d(\frac{1}{2}x'^2\beta_{m,n})$ back to $\omega_{X,0}$, because (recall that $d \cdot g'_{m,n} = g_{m,n}$):

$$\begin{aligned}\mu^*(dx' \wedge dy' + d(\frac{1}{2}x'^2\beta_{m,n})) &= dx \wedge (dy + i \cdot g'_{m,n}) + d(\frac{1}{2}(x - \frac{i}{d})^2\beta_{m,n}) \\ &= dx \wedge dy + \frac{i}{d}dx \wedge \beta_{m,n} + (x - \frac{i}{d})dx \wedge \beta_{m,n} \\ &= dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n}) \\ &= \omega_{X,0}.\end{aligned} \tag{2.72}$$

For any almost complex structure J on X_i, we have the following one-to-one correspondence:

$$\{J\text{-holomorphic sections of } X_i\} \overset{1:1}{\longleftrightarrow} \{\mu_*(J)\text{-holomorphic sections of } X_0\}.$$

Now, similar to the J-holomorphic sections that are contained in X_H, it is clear that sections contained in X_0 computes the pair-of-pants product of (a small perturbation of) the fixed points of time-1 maps of $m \cdot \frac{1}{2}x'^2$ and $n \cdot \frac{1}{2}x'^2$. This corresponds to the intersection product of $H_*((-\delta, \delta)_{x'} \times S^1_{y'})$, which is identically zero. This concludes the proof of Theorem 2.2.1. □

2.6 The coproduct

In this section, we generalize the methods used in Section 2.5 further to compute the pair-of-pants coproduct of fixed point Floer homology of Dehn twists:

$$\Delta : \mathrm{HF}_*(\phi^{m+n}) \to \mathrm{HF}_*(\phi^m) \otimes \mathrm{HF}_*(\phi^n).$$

Note that Lemma 2.5.4 tells us that the cobordism map is well-defined even without the use of Novikov rings. The goal of this section is to prove Theorem 2.2.2. To begin with, similar to Theorem 2.5.6, we have the following:

Theorem 2.6.1. *Let J be an almost complex structure on $X^{m,n}$ that is close to a fibration-compatible one. Suppose*

- *If γ is non-separating, then $\partial\Sigma \neq \emptyset$ or Σ is closed with genus at least 2;*
- *If γ is separating, then each component of $\Sigma - \gamma$ either contains a component of $\partial\Sigma$ or has genus at least 2.*

Then for any index zero J-holomorphic section C with cylindrical ends, C has vanishing wrapping numbers.

And the proof is almost the same as that of Theorem 2.5.6 (we only need to slightly modify the Conley-Zehnder index term). Combined with Theorem 2.4.1, it tells us the following:

Corollary 2.6.2. *Suppose the almost complex structure J, and the loop γ satisfy the same condition as in Theorem 2.6.1, then all J-holomorphic sections of $X^{m,n} \to B_0$ that have Fredholm index zero must be contained in X_H or X_D.*

What is different from Section 2.5 is that the count of sections contained in the twist region does not contribute to zero in the cobordism map. In the following two subsections, we first give a detailed understanding of the moduli space of all J-holomorphic sections in the Morse-Bott (unperturbed) setting, then explain what will happen if we perturb the form $\omega_{X,0}$ to break the Morse-Bott degeneracy.

J-holomorphic sections inside the twist region

In this subsection, we analyze possible J-holomorphic sections inside X_D. Unless otherwise specified, the almost complex structure J is assumed to be fibration-compatible, and when restricted to Ver, it sends ∂_x to ∂_y (so J is completely determined by the fiberwise symplectic 2-form ω). We start with the fiberwise symplectic 2-form $\omega_{X,0} = dx \wedge dy + d(\frac{1}{2}x^2\beta_{m,n})$ and view $(X_D, \omega_{X,0})$ as part of $\bar{X}_D := B_0 \times \mathbb{R}_x \times S^1_y$ with the same fiberwise symplectic 2-form. Note that by extending the cobordism, we are not introducing new curves: Lemma 2.3.2 ensures that if the x-coordinates of the asymptotic Reeb vector fields of a given J-holomorphic section are contained in $(-\epsilon, 1+\epsilon)$, then the entire J-holomorphic section of $\bar{X}_D$ is in fact entirely contained in X_D.

Notice that with this Morse-Bott setting, the Reeb orbits at the ends come in S^1-families. The possible x-coordinates of such families are:

- $x = \frac{k_\infty}{m+n}$ at the positive end,
- $x = \frac{k_1}{m}$ at the first negative end, and
- $x = \frac{k_2}{n}$ at the second negative end.

Observe that if a J-holomorphic section has cylindrical ends at $x = \frac{k_\infty}{m+n}, \frac{k_1}{m}$, and $\frac{k_2}{n}$, then $k_\infty = k_1 + k_2$.

We make the following two basic observations about J-holomorphic sections of $\bar{X}_D$ with cylindrical ends. In what follows, for any fiberwise symplectic 2-form ω on $\bar{X}_D$ that coincides with $\omega_{X,0}$ outside of some compact subset $K \subset \bar{X}_D$, let $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ denote the moduli space of J-holomorphic sections of $(\bar{X}_D, \omega)$ whose ends have x-coordinates asymptoting to $\frac{k_\infty}{m+n}, \frac{k_1}{m}$, and $\frac{k_2}{n}$, where $k_\infty = k_1 + k_2$.

Remark 2.6.3. We make an observation about $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ that is already implicit in its definition. Note here the Reeb orbits come in S^1 families, and in defining $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ we allow the ends of its elements to land on any Reeb orbit on a given S^1 family. In other words, the ends of a J-holomorphic section are "free". We can also require ends of J-holomorphic sections to land on a specific Reeb orbit in a S^1 family, in which case the ends are "fixed". This distinction will be important to us when we pass from Morse-Bott case to the perturbed, non-degenerate case.

Lemma 2.6.4. *There is a one-to-one correspondence between $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$ and $\mathcal{M}_{\omega_{X,0}}(k_\infty + m + n; k_1 + m, k_2 + n)$.*

Proof of Lemma 2.6.4. Define a diffeomorphism $\mu : \bar{X}_D \to \bar{X}_D$, $(p, x, y) \mapsto (p, x', y')$ by

$$\begin{cases} x' = x - 1 \\ y' = y + g^{m,n}(p). \end{cases} \tag{2.73}$$

Observe that μ preserves Ver, and a simple calculation shows $\mu^*\omega_{X,0} = \omega_{X,0}$. So $\mu^*J = J$, and hence if u is a J-holomorphic section contained in $\mathcal{M}_{\omega_{X,0}}(k_\infty + m + n; k_1 + m, k_2 + n)$, then $\mu \circ u$ is a J-holomorphic section contained in $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$, and vice versa. □

Lemma 2.6.5. *For any $u \in \mathcal{M}_\omega(k_\infty; k_1, k_2)$, $\mathrm{ind}(u) = 1$ and u is cut out transversely.*

Proof of Lemma 2.6.5. Similar to what we did in Section 2.5, choose the trivialization τ : Ver $\to T(\mathbb{R}_x \times S^1_y) \cong \mathbb{R}^2$. Let α and β_1, β_2 denote the three ends of u. The index formula is:

$$\begin{aligned} \mathrm{ind}(u) &= 1 + 2\langle c_1^\tau(TX), [C]\rangle + \mathrm{CZ}_\tau^+(\alpha) - \sum \mathrm{CZ}_\tau^-(\beta_i) \\ &= -1 + \mathrm{CZ}_\tau^+(\alpha) - \sum \mathrm{CZ}_\tau^-(\beta_i) \\ &= -1 + 0 - (-1) - (-1) = 1. \end{aligned} \tag{2.74}$$

Now the automatic transversality theorem (see, for example, [Wen10] Theorem 1) applies here, because

$$1 = \mathrm{ind}(u) > c_N(u) + Z(du) = 0 + 0. \tag{2.75}$$

□

Notice that Lemma 2.6.5 does not require ω to be $\omega_{X,0}$. Let us consider a 1-parameter family of closed fiberwise symplectic 2-forms ω_λ for $\lambda \in [0, 1]$, where:

1. $\omega_0 = \omega_{X,0}$,
2. When restricted to each fiber of $\bar{X}_D \to B_0$, we have $\omega_\lambda|_{\text{fiber}} = dx \wedge dy$,
3. There is a compact subset of $\bar{X}_D$ outside of which all ω_λ, with $\lambda \in [0, 1]$, agree.

Observe that any closed fiberwise symplectic 2-form ω that agrees with $\omega_{X,0}$ outside of a compact set and restricts to $dx \wedge dy$ on each fiber can be connected to $\omega_{X,0}$ using a family ω_λ described above: one can simply put $\omega_\lambda = \lambda\omega + (1-\lambda)\omega_{X,0}$. For each λ, let J_λ denote the fibration compatible almost complex structure on $\bar{X}_D$ determined by ω_λ. As before, let $\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)$ denote the moduli space of J_λ-holomorphic sections of $(\bar{X}_D, \omega_\lambda)$ whose ends have x-coordinates $\frac{k_\infty}{m+n}$, $\frac{k_1}{m}$, $\frac{k_2}{n}$, where $k_\infty = k_1 + k_2$.

Let $\{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]}$ denote the parametrized moduli space:

$$\{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]} := \{(u,\lambda)|u \in \mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2), \lambda \in [0,1])\} \tag{2.76}$$

We now describe the specific type of deformed fiberwise symplectic 2-form that we will use. Fix a compact subset $K_1 \subset B_0$ such that the complement of K_1 is contained in the cylindrical ends. For any $R > 0$, denote $[-R, R]_x \times S^1_y \subset \mathbb{R}_x \times S^1_y$ by Q_R.

Definition 2.6.6. *A 1-form $\sigma \in \Omega^1(B_0, C^\infty(\mathbb{R} \times S^1))$ is called* admissible *if there exists some compact subset $K_2 \subset B_0$ containing K_1, and some $R > 0$, such that $\sigma = \beta^{m,n} \cdot \frac{1}{2}x^2$ outside of $K_2 \times Q_R$.*

For any admissible 1-form σ, we define the corresponding closed fiberwise symplectic 2-form ω to be $dx \wedge dy + d\sigma$. The following simple observation asserts that if $\omega = dx \wedge dy + d\sigma$ and σ is admissible, then all pseudo-holomorphic sections in $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ have a uniform upper bound on the vertical energy (see definition below) and the range of its x component is bounded.

Definition 2.6.7. *Fix an admissible 1-form σ and the corresponding almost complex structure J. Let g_J denote the metric determined by $\omega = d\sigma + dx \wedge dy$ and J. For any smooth map $u : B_0 \to \bar{X}_D$, define the vertical energy $E(u)$ to be*

$$E(u) = \frac{1}{2}\int_{B_0} |\partial_s u - \partial_s^\#|^2_{g_J} + |\partial_t u - \partial_t^\#|^2_{g_J} ds \wedge dt \tag{2.77}$$

where (s,t) is a local conformal coordinate on B_0, and $\partial_s^\#$ and $\partial_t^\#$ are the horizontal lifts of the vector fields ∂_s and ∂_t, respectively.

Lemma 2.6.8. *Let $\sigma_\lambda|_{\lambda\in[0,1]}$ be a family of admissible 1-form as in Definition 2.6.6 (where R is assumed to be sufficiently large compared to k_∞, k_1 and k_2), and $\omega_\lambda = dx \wedge dy + d\sigma_\lambda$ be the corresponding closed fiberwise symplectic 2-form on $\bar{X}_D$. Let J_{ω_λ} denote the fibration-compatible almost complex structure determined by ω_λ. Then for any $u \in \mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)$,*

1. *u is contained in $\{-2R \le x \le 2R\}$, and*

2. *The vertical energy $E(u)$ has a uniform bound.*

Proof of Lemma 2.6.8. Assume that there is some $u \in \mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)$ not contained in $\{-2R \leq x \leq 2R\}$. It's easy to show that such a section cannot be horizontal (otherwise, inside the region $\{R < |x| < 2R\}$ the function x would be locally constant, a contradiction). Outside of $\{-2R \leq x \leq 2R\}$ the 2-form ω coincides with $\omega_{X,0}$, so after picking some x_0 such that $|x_0| > 2R$ and u intersects $\{x = x_0\}$ transversely in a non-empty way, the homology class of the intersection $u \bigcap \{x = x_0\}$ satisfies the local energy inequality

$$p + x_0(mq_1 + nq_2) > 0, \tag{2.78}$$

which contradicts the fact that $u \bigcap \{x = x_0\}$ is null-homologous. For the second bullet point, we first observe that for a sequence of subdomains D_k exhausting B_0, we have

$$\begin{aligned} \int_{B_0} u^*\omega_\lambda &= \lim_{k\to\infty} \int_{D_k} u^*\omega_\lambda \\ &= \lim_{k\to\infty} \int_{\partial D_k} u^*(xdy + \sigma_\lambda) \\ &= \lim_{k\to\infty} \int_{\partial D_k} u^*(xdy + \beta^{m,n} \cdot \frac{1}{2}x^2). \end{aligned} \tag{2.79}$$

And the last limit does not depend on λ and u. We also observe that given the fact $u \in \mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)$, this term is finite.

Next, similar to what we did in Lemma 2.3.5, we calculate $E(u)$ for a J-holomorphic section u (again using local conformal coordinates (s,t)):

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{B_0} |\partial_s u - \partial_s^{\#}|^2_{g_J} + |\partial_t u - \partial_t^{\#}|^2_{g_J} ds \wedge dt \\ &= \int_{B_0} \omega_\lambda(\partial_s u - \partial_s^{\#}, \partial_t u - \partial_t^{\#}) ds \wedge dt \\ &= \int_{B_0} \omega_\lambda(\partial_s u, \partial_t u) - \omega_\lambda(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt \\ &= \int_{B_0} u^*\omega_\lambda - \int_{B_0} \omega_\lambda(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt \end{aligned} \tag{2.80}$$

So it suffices to estimate the term $\int_{B_0} \omega_\lambda(\partial_s^{\#}, \partial_t^{\#}) ds \wedge dt$. Write σ_λ as $F^\lambda(s,t,x,y)ds + G^\lambda(s,t,x,y)dt$, and we have:

$$\omega_\lambda(\partial_s^{\#}, \partial_t^{\#}) = \frac{\partial G^\lambda}{\partial s} - \frac{\partial F^\lambda}{\partial t} + \frac{\partial G^\lambda}{\partial x}\frac{\partial F^\lambda}{\partial y} - \frac{\partial G^\lambda}{\partial y}\frac{\partial F^\lambda}{\partial x} \tag{2.81}$$

Now we simply observe that on the union of cylindrical ends Z where $\sigma_\lambda = \frac{1}{2}x^2\beta^{m,n}$, the term F^λ is identically zero and G^λ is independent of s. Consequently, $\omega_\lambda(\partial_s^{\#}, \partial_t^{\#})$ is compactly supported on

$$(s,t,x,y,\lambda) \in (B_0 - Z) \times [-2R, 2R] \times S^1 \times [0,1] \tag{2.82}$$

and hence has a uniform bound for all $u \in \mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)$.

□

Lemma 2.6.8, together with the usual SFT compactness argument developed in [Bou+03] (see also [Wen] for a nice account), tells us that for any closed fiberwise symplectic 2-form $\omega = dx \wedge dy + d\sigma$ where σ is admissible, the moduli space $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ is compact. To see that no SFT-type breaking can occur for a sequence of sections in $\mathcal{M}_\omega(k_\infty; k_1, k_2)$, we observe that levels in the symplectizations are necessarily cylinders, and such cylinders asymptote to orbits in the same Morse-Bott family for homological reasons. Now such cylinders have zero vertical energy, hence are trivial cylinders. We also observe no bubbles appear, as π_2 of the bundle is trivial.

More generally, for such ω, if we define the 1-parameter family $\omega_\lambda := \lambda\omega + (1-\lambda)\omega_{X,0}$, then the parametric moduli space $\{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]}$ is compact as well. Lemma 2.6.5 tells us that for such ω, both $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ and $\{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]}$ are transversely cut out, so $\{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]}$ is a compact cobordism between two closed 1-dimensional manifolds. Observe also that for each fixed $\lambda' \in [0, 1]$, the slice

$$\mathcal{M}_{\omega_{\lambda'}}(k_\infty; k_1, k_2) \subset \{\mathcal{M}_{\omega_\lambda}(k_\infty; k_1, k_2)\}|_{\lambda\in[0,1]}$$

is a closed 1-manifold, so we get the following:

Corollary 2.6.9. *For any $\omega = dx \wedge dy + d\sigma$ where σ is admissible, $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ is diffeomorphic to $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$.*

The next observation (Corollary 2.6.13) asserts that $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$ has at most one component. To get started, let us observe that there is an S^1 symmetry of $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$. In the following, we will view J-holomorphic sections u of $(\bar{X}_D, \omega)$ as maps $\bar{u} : B_0 \to \mathbb{R} \times S^1$, so it is handy to establish the following Lemma:

Lemma 2.6.10. *Choose a local conformal coordinate (s, t) of B_0 and suppose locally*

$$\sigma = \frac{1}{2}x^2(F(s,t)ds + G(s,t)dt)$$

$$\omega_{X,0} = dx \wedge dy + d\sigma.$$

A map $u : B_0 \to (\bar{X}_D, \omega)$ is a J-holomorphic section if and only if the corresponding map $\bar{u} = (x(s,t), y(s,t)) : B_0 \to \mathbb{R} \times S^1$ in our coordinate system solves the following PDE:

$$\begin{cases} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial s} + xF = 0 \\ \frac{\partial y}{\partial t} - \frac{\partial x}{\partial s} + xG = 0. \end{cases} \tag{2.83}$$

Proof of Lemma 2.6.10. Let $v^\#$ denote the horizontal lift (with respect to ω) of any vector $v \in TB_0$. A simple calculation shows that:

$$\partial_s^\# = \partial_s - xF\partial_y, \quad \partial_t^\# = \partial_t - xG\partial_y. \tag{2.84}$$

So by definition, $J(\partial_s - xF\partial_y) = \partial_t - xG\partial_y$. Recall that we required that J always sends ∂_x to ∂_y, so this shows:

$$J(\partial_s) = \partial_t - xF\partial_x - xG\partial_y. \tag{2.85}$$

Now suppose $u : B_0 \to \bar{X}_D$, $(s,t) \mapsto (s,t,x,y)$ is J-holomorphic, i.e.

$$J(\partial_s + \frac{\partial x}{\partial s}\partial_x + \frac{\partial y}{\partial s}\partial_y) = \partial_t + \frac{\partial x}{\partial t}\partial_x + \frac{\partial y}{\partial t}\partial_y. \tag{2.86}$$

Combine the above equations and collect the coefficients of ∂_x and ∂_y, and we get the desired equations. □

The following corollary is immediate:

Corollary 2.6.11. *If $u : B_0 \to \bar{X}_D$ is a map given by $(s,t) \mapsto (s,t,x(s,t),y(s,t))$, and u is an element of $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$, then for any $y_0 \in S^1$, $(s,t) \mapsto (s,t,x(s,t),y(s,t)+y_0)$ is also an element of $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$.*

In other words, if $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2) \neq \emptyset$, then S^1 acts freely on $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$ by translating the y-coordinate. We now show that this S^1 action is also transitive.

Lemma 2.6.12. *Suppose u_1 and u_2 are two different J-holomorphic sections in $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$, and let $\bar{u}_i : B_0 \to \mathbb{R} \times S^1$ denote the corresponding maps to the fiber, which we write as $(s,t) \mapsto (x_i, y_i)$. Then $x_1 = x_2$, and there is some $y_0 \in S^1$ such that $y_2(s,t) = y_1(s,t) + y_0$.*

Proof of Lemma 2.6.12. Consider $w := \bar{u}_1 - \bar{u}_2 : B_0 \to \mathbb{R} \times S^1$, $(s,t) \mapsto (l(s,t), m(s,t))$. By Lemma 2.6.10, locally the following PDEs are satisfied:

$$\begin{cases} \frac{\partial l}{\partial t} + \frac{\partial m}{\partial s} + lF = 0 \\ \frac{\partial m}{\partial t} - \frac{\partial l}{\partial s} + lG = 0. \end{cases} \tag{2.87}$$

Observe that by our assumption, u_1 and u_2 have the same asymptotics on their cylindrical ends, so it follows that w induces the trivial map on π_1, hence can be lifted to $\tilde{w} : B_0 \to \mathbb{R} \times \mathbb{R}$, $(s,t) \mapsto (l(s,t), \tilde{m}(s,t))$, where $(l, \tilde{m})$ solves the same PDE. Let us denote the covering map $\mathbb{R} \times \mathbb{R} \to \mathbb{R} \times S^1$ by π.

Now for any $\zeta \in \mathbb{R}$, the map $\bar{u}_2 + \pi(\zeta \cdot \tilde{w})$ solves the same PDE, hence gives an element in $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$. This implies that $l(s,t)$ is identically zero and hence $\tilde{m}(s,t)$ is constant, because otherwise,

$$\bar{u}_2 + \pi(\zeta \cdot \tilde{w} + \xi \cdot (0,1)) \tag{2.88}$$

solves the same PDE for any $\zeta, \xi \in \mathbb{R}$, giving us a two dimensional family of solutions. This contradicts the fact that u_2 is cut out transversely and $\mathrm{ind}(u_2) = 1$. □

A corollary of the above discussion is the following:

Corollary 2.6.13. *For any admissible σ, the moduli space $\mathcal{M}_{dx\wedge dy + d\sigma}(k_\infty; k_1, k_2)$ is either empty, or diffeomorphic to S^1.*

We now proceed to show that $\mathcal{M}_{dx\wedge dy+d\sigma}(k_\infty; k_1, k_2)$ is not empty. By Lemma 2.6.4, we can assume that k_∞, k_1 and k_2 are all positive. In light of Corollary 2.6.9, it suffices to find *one* special admissible σ, and show that $\mathcal{M}_\omega(k_\infty; k_1, k_2)$ is not empty for this special σ, where $\omega = dx \wedge dy + d\sigma$. The rest of this subsection describes how one can construct an admissible σ with a nonempty moduli space $\mathcal{M}_{dx\wedge dy+d\sigma}(k_\infty; k_1, k_2)$.

To this end, we make the following observation. Fix cylindrical ends Z_∞, Z_1 and Z_2 of B_0 outside of K_1, each with conformal coordinates $(s_i, t_i) \in [N, \infty) \times S^1)$ (or $(-\infty, -N] \times S^1)$, and choose cutoff functions $\chi_i : Z_i \to [0, 1]$ such that $1-\chi_i$ are compactly supported. Suppose $v : B_0 \to \mathbb{R} \times S^1$ is a smooth map, such that (notice the resemblance to the formulations in [Sch95]):

1. The restriction of $v = (x, y)$ to the cylindrical end Z_∞ solves the equation:

$$\begin{cases} \frac{\partial x}{\partial t_\infty} + \frac{\partial y}{\partial s_\infty} = 0 \\ \frac{\partial y}{\partial t_\infty} - \frac{\partial x}{\partial s_\infty} + (m+n)\chi_\infty(s_\infty, t_\infty)x = 0 \end{cases} \tag{2.89}$$

2. The restriction of $v = (x, y)$ to the cylindrical end Z_1 solves the equation:

$$\begin{cases} \frac{\partial x}{\partial t_1} + \frac{\partial y}{\partial s_1} = 0 \\ \frac{\partial y}{\partial t_1} - \frac{\partial x}{\partial s_1} + m\chi_1(s_1, t_1)x = 0 \end{cases} \tag{2.90}$$

3. The restriction of $v = (x, y)$ to the cylindrical end Z_2 solves the equation:

$$\begin{cases} \frac{\partial x}{\partial t_2} + \frac{\partial y}{\partial s_2} = 0 \\ \frac{\partial y}{\partial t_2} - \frac{\partial x}{\partial s_2} + n\chi_2(s_2, t_2)x = 0 \end{cases} \tag{2.91}$$

4. The restriction of $v = (x, y)$ to the complement of $Z_1 \bigcup Z_2 \bigcup Z_\infty$ is holomorphic:

$$\frac{\partial x}{\partial t} + \frac{\partial y}{\partial s} = \frac{\partial y}{\partial t} - \frac{\partial x}{\partial s} = 0 \tag{2.92}$$

5. The map v approaches the projections (to the fiber $S^1 \times \mathbb{R}$) of three Reeb orbits at $x_1 = \frac{k_1}{m}, x_2 = \frac{k_2}{n}, x_\infty = \frac{k_\infty}{m+n}$ at its corresponding cylindrical ends. Note the Reeb orbits whose projections the map v approaches all live in S^1 families. We do not care which orbits in these S^1 families v approaches.

Then we can construct an admissible σ such that the moduli space

$$\mathcal{M}_{dx\wedge dy+d\sigma}(k_\infty; k_1, k_2)$$

is not empty. The reason is that $\tilde{v} : (s, t) \mapsto (s, t, v(s, t))$ is a smooth section of $\bar{X}_D \to B_0$ with the desired asymptotes, and the image of v is contained in Q_R for some large R. If we define an admissible 1-form σ such that

$$\sigma = \begin{cases} \frac{m+n}{2}x^2\chi_\infty(s_\infty, t_\infty)dt_\infty, & \text{in } Z_\infty \times Q_R \\ \frac{m}{2}x^2\chi_1(s_1, t_1)dt_1, & \text{in } Z_1 \times Q_R \\ \frac{n}{2}x^2\chi_2(s_2, t_2)dt_2, & \text{in } Z_2 \times Q_R \\ 0, & \text{in } (B_0 - Z_1 \bigcup Z_2 \bigcup Z_\infty) \times Q_R \end{cases} \tag{2.93}$$

Then Lemma 2.6.10 tells us that $\tilde{v} \in \mathcal{M}_{dx\wedge dy + d\sigma}(k_\infty; k_1, k_2)$.

We now construct v as described above. In the following, inside the cylindrical ends Z_i, we will always assume that χ_i's are t_i-independent and monotone, and that:

$$y = \begin{cases} -k_\infty t_\infty, & \text{in } Z_\infty \\ -k_1 t_1, & \text{in } Z_1 \\ -k_2 t_2, & \text{in } Z_2 \end{cases} \tag{2.94}$$

So inside Z_i, x is t_i-independent, and the equations simplify to ODE's

$$\frac{\partial x}{\partial s_\infty} + k_\infty - (m+n)\chi_\infty(s_\infty)x = 0 \tag{2.95}$$

$$\frac{\partial x}{\partial s_1} + k_\infty - m\chi_1(s_1)x = 0 \tag{2.96}$$

$$\frac{\partial x}{\partial s_2} + k_\infty - n\chi_2(s_2)x = 0. \tag{2.97}$$

If we let

$$a_\infty(s_\infty) := \int_N^{s_\infty} -(m+n)\chi_\infty(\rho)d\rho \tag{2.98}$$

$$a_1(s_1) := \int_{-N}^{s_1} -m\chi_1(\rho)d\rho \tag{2.99}$$

and

$$a_2(s_2) := \int_{-N}^{s_2} -n\chi_2(\rho)d\rho, \tag{2.100}$$

then the solutions to the above ODEs can be explicitly written down:

$$x(s_\infty) = e^{-a_\infty}(c_\infty - k_\infty \int_N^{s_\infty} e^{a_\infty(\rho)}d\rho) \tag{2.101}$$

$$x(s_1) = e^{-a_1}(c_1 - k_1 \int_{-N}^{s_1} e^{a_1(\rho)}d\rho) \tag{2.102}$$

$$x(s_2) = e^{-a_2}(c_2 - k_2 \int_{-N}^{s_2} e^{a_2(\rho)}d\rho). \tag{2.103}$$

Based on our assumptions, it follows from direct computations that for the first expression, there is a unique choice of $c_\infty \in \mathbb{R}$ such that $x(s_\infty)$ stays at $\frac{k_\infty}{m+n}$ when s_∞ is large enough. On the other hand, whatever choice of c_1 or c_2 is, the second(resp. the third expression) will converge to $\frac{k_1}{m}$ (resp. $\frac{k_2}{n}$) when s_i close enough to $-\infty$. Notice also that near the boundaries of Z_i (i.e. when s_i is close to $\pm N$) the functions $x(s_i)$ are linear with slopes $-k_i$.

For a suitable choice of $c_\infty < c_1 = c_2$, it is not hard to construct a holomorphic k_∞-fold branched cover $v = (x, y) : B_0 - Z_1 \bigcup Z_2 \bigcup Z_\infty \to [c_\infty, c_1] \times S^1$ such that near ∂Z_∞ (resp. ∂Z_1, ∂Z_2), $(x, y) = (c_\infty - k_\infty(s_\infty - N), -k_\infty t_\infty)$ (resp. $(c_1 - k_1(s_1 + N), -k_1 t_1)$ and $(c_2 - k_2(s_2 + N), -k_2 t_2)$). So for such choices of $c_\infty < c_1 = c_2$, the map v glues smoothly with the three solutions on Z_i's. The above discussion shows the following:

Proposition 2.6.14. *For any admissible σ, $\mathcal{M}_{dx\wedge dy+d\sigma}(k_\infty; k_1, k_2)$ is diffeomorphic to S^1.*

Morse-Bott theory and enumeration of sections after the perturbation

In this subsection we use Morse-Bott theory to enumerate sections after we replace $\omega_{X,0}$ with ω_X by adding a small Hamiltonian perturbation that breaks the Morse-Bott degeneracy (see Section 2.1). We know from Theorem 2.6.1 and Theorem 2.4.1 that no holomorphic sections cross from X_H to X_D or vice versa, so we only need to consider what happens to J-holomorphic sections that stay in X_D as we break the Morse-Bott degeneracy.

For simplicity let us work instead in $\bar{X}_D$. By abuse of notation, we will use the same notation ω_X to denote the 2-form on the completion $\bar{X}_D$, and likewise for $\omega_{X,0}$. We first recall some conventions for J-holomorphic sections whose ends land on Morse-Bott submanifolds. We fix J, the fibration-compatible almost complex structure (we will have a bit more to say about the choice of J after we describe cascades). In our case all Reeb orbits come in S^1 families, corresponding to $x = \frac{k_1}{m}, \frac{k_2}{m}, \frac{k_1+k_2}{m+n}$ in $\bar{X}_D$. Hence we have tori that are foliated by Reeb orbits (we will call such tori "Morse-Bott tori").

Recall for J-holomorphic sections ending on Morse-Bott tori, we can consider moduli spaces of sections with "fixed" end points and moduli spaces of sections with "free" end points. These are conditions we impose on the given cylindrical ends of the holomorphic section. "Fixed" end points condition means the given end of the section must land on a specific Reeb orbit in the Morse-Bott torus, whereas the "free" end condition means that the given end of a section in the moduli space is allowed to freely move around on the Morse-Bott torus. The dimension of the moduli space of course depends on how many ends are specified as fixed or free. For instance the moduli space we constructed in the previous section, $\mathcal{M}_{dx\wedge dy+d\sigma}(k_\infty; k_1, k_2)$ has all ends free by this convention (we implicitly assumed this in our previous construction).

We shall invoke some standard Morse-Bott theory (which is easily adapted to cobordisms) to figure out how to count our J-holomorphic sections. However, the statement of the entire theory is rather cumbersome, instead we will just state the small parts we need. We refer the reader to [Yao22] for a more detailed account.

We recall that in breaking the Morse-Bott degeneracy via perturbations, each torus of Morse Bott orbits breaks into an hyperbolic orbit h_k at maximum of $h(y)$ and an elliptic orbit e_k at minimum of $h(y)$. These orbits form the generators of the fixed point Floer homology.

For our purposes, J-holomorphic sections of degree one and Fredholm index zero in $(\bar{X}_D, \omega_X)$ which we need to count correspond to cascades in $(\bar{X}_D, \omega_{X,0})$. The cascades, generally speaking, take the following form:

- There is a main level $u_0 : B_0 \to (\bar{X}_D, \omega_{X,0})$;
- There are upper levels indexed by $i \in \mathbb{Z}_{>0}$. We write them as $u_{i>0} : S^1 \times \mathbb{R} \to (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}, \omega_{X,0})$;
- There are lower levels indexed by $j \in \mathbb{Z}_{<0}$ labelling the level number, and $k \in \{1, 2\}$ labeling which negative puncture it corresponds to. We write them as: $u^k_{j<0} : S^1 \times \mathbb{R} \to (S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}, \omega_{X,0})$.
- For each map u_i let π denote the projection to the base in the codomain, then $\pi \circ u_i$ is the identity; likewise for u^k_j;
- Let $ev^{\pm}(u_i)$ denote the Reeb orbit u_i approaches as $s \to \pm\infty$, let ϕ_T denote the gradient flow of $h(y)$ for time T, then there exists $T_i \in (0, \infty)$ so that $\phi_{T_i}(ev^-(u_i)) = ev^+(u_{i-1})$;
- For the main level we have numbers $T_1, T^0_k \in (0, \infty)$ for $k \in \{1, 2\}$ so that:

$$\phi_{T_1}(ev^-(u_1)) = ev^+(u_0)$$

$$\phi_{T^k_0}(ev^-(u_0)) = ev^+(u^k_{-1}).$$

- For the lower levels, there are numbers $T^k_j \in (0, \infty)$ for $k \in \{1, 2\}$ so that $\phi_{T^k_j}(ev^-(u^k_j)) = ev^+(u^k_{j-1})$.

Generally speaking there are more conditions we can achieve for cascades by choosing generic J, however for our case we immediately observe for homological reasons $u_{i\neq 0}$ are cylinders, in fact they must all be trivial cylinders by energy considerations, so there is only the main level, which for ease of notation we denote by u. Since our entire cascade only has one level, in order to count u, it must live in a moduli space of dimension zero.

Thus we arrive at the following description of the cascades we must count:

Proposition 2.6.15. *The (1-level) cascades that we need to count takes the following form:*

- $u : B_0 \to (\bar{X}_D, \omega_{X,0})$ *is a J-holomorphic section.*

- *Let the ends of B_0 be labelled $\{1, 2, \infty\}$. Then one of the ends in $\{1, 2, \infty\}$ is fixed, the other 2 are free. Hence u belongs in a (transversely cut out) moduli space of index zero. (Fixing one end reduces the virtual dimension by one).*

- *All free ends avoid critical points of $h(y)$ (this can be achieved for generic J). If the ends labelled $1, 2$ are fixed, then they land on the maximum of $h(y)$. If the ∞ end is fixed, it lands on the minimum of $h(y)$.*

That the cascades above are the ones we need to count to compute the co-product map is supplied by the following correspondence theorem:

Proposition 2.6.16. *Given a perfect Morse function $h(y)$, we make the Morse perturbation smaller by rescaling it to be $\bar{\delta}h(y)$, where $\bar{\delta} > 0$ is a small positive real number. For small enough $\bar{\delta} > 0$, there is a 1-1 correspondence between J-holomorphic sections in the nondegenerate case and J holomorphic cascades. Given a cascade u of the form described in the previous proposition, for the positive end of u, if it is a free end we assign the generator h_k, and if it a fixed end we assign it the generator e_k. We reverse the assignments for the two negative ends $1, 2$. Then each cascade gives rise bijectively to one J-holomorphic section beginning and ending at the generators we assigned to the ends of cascade.*

Proof. See [Yao22] for a proof of this statement and how the correspondence works in the general case of multiple level cascades. However since we are only working in the simple case of 1-level cascades, the correspondence theorem that we need is also established in the Appendix of [Col+] cowritten with Yao. □

Remark 2.6.17. Note by the above correspondence theorem, the only requirement we need to impose on J in the Morse-Bott case is that all moduli spaces of index zero sections listed above are transversely cut out, and that the free ends avoid critical points of $h(y)$. It will be apparent from the paragraph following this remark that the fibration compatible J we chose in Section 2.6, with the help of automatic transversality, suffices for the purpose of showing the index zero sections are transversely cut out. To ensure the free ends avoid critical points of h, instead of further perturbing J, we shall instead choose generic h. This kind of strategy was also undertaken when Morse-Bott techniques were employed in [HS06].

We also remark that in the correspondence between cascades and J holomorphic sections we need to perturb the almost complex structure from J in the Morse-Bott case in $\bar{X}_D$ to a generic $J_{\bar{\delta}}$ in the non-degenerate case to establish a correspondence between cascades and holomorphic sections, see [Yao22] for more details. The difference between $J_{\bar{\delta}}$ and J can be taken to be C^∞ small. Hence for small enough $\bar{\delta} > 0$ we do not need to worry about this change in almost complex structure since we already established in Section 2.4 that for C^∞ small perturbations of J the no-crossing results continues to hold.

Recall that Proposition 2.6.14 tells us the moduli space of sections with all three ends free in the Morse-Bott case come in S^1 families, and this S^1 family is precisely given by rotation around the ∂_y direction, i.e. rotation along the Morse-Bott tori. To obtain a cascade of

Fredholm index zero in the form specified above, it is readily apparent we restrict one of the three ends fixed and the rest two free. All of our cascades we need arise this way.

Instead of perturbing J further to ensure that when we have fixed an end, all the remaining free ends avoid critical points of h, we choose generic Morse perturbations h. To be specific, we choose perfect Morse functions $\{h_l^i(y)\}$, where $l \in \{1, 2, \infty\}$ labels which cylindrical end of the base we are referring to, and $i \in \mathbb{Z}$ refers to the specific Morse-Bott torus in that end. (For example when $l = 1$ and $i = 1$, this refers to the Morse Bott torus at $x = 1$ for the negative end labelled by $k = 1$). We then use these functions to break the Morse-Bott degeneracy.

Given some large integer N, for a generic choice of $\{h_l^i(y)\}$ we can arrange that for tuples (k_1, k_2, k_∞) satisfying $0 \leq k_1, k_2, k_\infty \leq N$ and $k_\infty = k_1 + k_2$, if we consider the moduli space $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$ (here all ends are free) which is diffeomorphic to S^1, if any element in this S^1 family has one end landing on a critical point of any of the Morse functions in $\{h_l^i(y)\}$, then all the other ends avoid critical points of elements in $\{h_l^i(y)\}$. We can arrange for this to happen by picking generic collection $\{h_l^i(y)\}$ because we only need to consider finite number of moduli spaces.

It is then apparent if a section in the S^1 family $\mathcal{M}_{\omega_{X,0}}(k_\infty; k_1, k_2)$ has one end hitting a critical point, it is the only section in this S^1 family with an end on that critical point. This, combined with the cascade and holomorphic section correspondence (Proposition 2.6.16) thus produces the computation required for the coproduct structure restricted to $\bar{X}_D$.

Remark 2.6.18. Even though all our previous proofs (e.g. no-crossing) assumed we used a single Morse function $h(y)$ to break the Morse-Bott degeneracy, one can check that using a collection of Morse functions $\{h_l^i(y)\}$ to break the degeneracy does not make a difference to our previous results.

Now we are ready to compute the coproduct map induced by $X^{m,n}$. Let $h_{k_\infty}^{m+n}$ (resp. $e_{k_\infty}^{m+n}$) be the hyperbolic (resp. elliptic) orbit over $x = \frac{k_\infty}{m+n}$ at the positive end, $h_{k_i}^m$ (resp. $e_{k_i}^m$) be the hyperbolic (resp. elliptic) orbit over $x = \frac{k_i}{m}$ at the first negative end, and $h_{k_i}^n$ (resp. $e_{k_i}^n$) be the hyperbolic (resp. elliptic) orbit over $x = \frac{k_i}{n}$ at the second negative end. Let Δ be the coproduct map induced by $(X^{m,n}, \omega_X)$. The above discussion shows the following:

Corollary 2.6.19.

$$\Delta([e_{k_\infty}^{m+n}]) = \sum_{k_i \in \{0,1,\cdots,m\},\ k_\infty - k_i \in \{0,1,\cdots,n\}} [e_{k_i}^m] \otimes [e_{k_\infty - k_i}^n] \tag{2.104}$$

$$\Delta([h_{k_\infty}^{m+n}]) = \sum_{k_i \in \{0,1,\cdots,m\},\ k_\infty - k_i \in \{0,1,\cdots,n\}} [e_{k_i}^m] \otimes [h_{k_\infty - k_i}^n] + [h_i^m] \otimes [e_{k_\infty - k_i}^n] \tag{2.105}$$

For all $k_\infty \in \{0, 1, \cdots, m+n\}$.

Remark 2.6.20. Notice that in equations (2.104) and (2.105), the homology classes $[e_0^{m+n}]$, $[h_0^{m+n}]$, $[e_{m+n}^{m+n}]$, $[h_{m+n}^{m+n}]$, $[e_0^m]$, $[h_0^m]$, $[e_m^m]$, $[h_m^m]$, $[e_0^n]$, $[h_0^n]$, $[e_n^n]$, $[h_n^n]$ refer to homology classes in $H_*(\Sigma_0;\mathbb{Z}_2)$ in the sense of the decomposition (2.58). It's also not difficult to see that the computations for $\Delta([e_0^{m+n}]$, $\Delta([h_0^{m+n}])$, $\Delta([e_{m+n}^{m+n}])$ and $\Delta([h_{m+n}^{m+n}])$ coincide with that of the coproduct structure Δ_0 on $H_*(\Sigma_0;\mathbb{Z}_2)$.

Finally, We are able to complete the proof of Theorem 2.2.2.

Proof of Theorem 2.2.2. Let J be an almost complex structure on $X^{m,n}$ sufficiently close to a fibration-compatible one in the sense of Theorem 2.4.1. By Corollary 2.6.2, all J-holomorphic sections that are counted in the cobordism map are either contained in the twist region X_D or X_H. Similar to what we observed in the proof of Theorem 2.2.1, the count of J-holomorphic sections contained in X_H precisely corresponds to the coproduct

$$\mathrm{HF}_*(\phi^{m+n}) \supset H_*(\Sigma_0;\mathbb{Z}_2) \xrightarrow{\Delta_0} H_*(\Sigma_0;\mathbb{Z}_2) \otimes H_*(\Sigma_0;\mathbb{Z}_2) \longrightarrow \mathrm{HF}_*(\phi^m) \otimes \mathrm{HF}_*(\phi^n) \quad (2.106)$$

in the sense of the decomposition (2.58). The remaining parts of Theorem 2.2.2 readily follows from Corollary 2.6.19.

□

Chapter 3

Symplectic cohomology of a nodal curve

In this chapter, we define the notion of the symplectic cohomology of a nodal curve[1], and use the results from chapter 2 to give an explicit computation. We then briefly explain how this computation gives a direct proof of closed-string mirror symmetry for nodal curves of genus at least two.

3.1 Setup and summary of results from chapter 2

Throughout this chapter, we fix a closed symplectic surface (Σ_g, ω) of genus $g \geq 2$. For each positive integer d, we consider the iterated positive (perturbed) Dehn twist

$$\phi^d : \Sigma_g \to \Sigma_g$$

along a non-separating simple closed curve γ, as described in section 2.1. Recall that there are two types of fixed points of ϕ^d: those corresponding to the critical points of the Hamiltonian dH_0 over $\Sigma_0 = \Sigma - N'$, and those coming from the perturbation of the Morse-Bott S^1 family of fixed points inside the twist region N. For the elliptic (resp. hyperbolic) fixed point over $x = i/d$ inside the twist region N, we will denote it by e_i^d (resp. h_i^d). Notice that the four fixed points e_0^d, h_0^d, e_d^d and h_d^d can be viewed both as critical points of dH_0 and as fixed points over the twist region N. For simplicity of notations, throughout this chapter, we will make the additional assumption that e_0^1 and e_1^1 are the only two local minima of H_0.

We now define the fixed point Floer cohomology of ϕ^d to be the cohomology of the cochain complex $\mathrm{Hom}(\mathrm{CF}_*(\phi^d), \mathbb{C})$ and denote it by $\mathrm{HF}^*(\phi^d)$. Notice that in this chapter, we are changing the coefficient[2] of both fixed point Floer homology and cohomology of ϕ^d to $\mathbb{C}$ (instead of $\mathbb{Z}/2$ in the previous two chapters). We shall remark that for all of our purposes

[1] In this chapter, we sometimes refer to (smooth or nodal) symplectic surfaces as "curves", as they can be viewed as algebraic curves.

[2] The reason for this more complicated setup is to match the convention of the "B-model" invariants.

in this chapter, the results we need from chapter 2 still hold. See [JYZ23] Remark 14 for a more detailed account.

In what follows, it is necessary to consider the $\mathbb{Z}/2$-grading on the fixed point Floer cohomology, an ingredient that is missing from the previous discussions. Recall that for each fixed point x of ϕ^d (or equivalently, each Reeb orbit γ_x of the mapping torus M_{ϕ^d}), there is a Conley-Zehnder index $\mathrm{CZ}_\tau(\gamma_x)$. The Conley-Zehnder index depends on the specific choice of the trivialization τ, but the parity does not, which we will denote by $|x|$. The $\mathbb{Z}/2$ grading on $\mathrm{CF}^*(\phi^d)$ is defined on generators as $gr(x) := |x| + 1$, which descends to a $\mathbb{Z}/2$ grading on $\mathrm{HF}^*(\phi^d)$. In what follows, we will use the notation $\mathrm{HF}^k(\phi^d)$ to denote the grading-k part of the cohomology.

With the above point of view, for any pair of positive integers m, n, we can define the product of fixed point Floer cohomology of iterations of a single Dehn twist

$$\cdot : \mathrm{HF}^*(\phi^m) \otimes \mathrm{HF}^*(\phi^n) \to \mathrm{HF}^*(\phi^{m+n}) \tag{3.1}$$

as the dual of the corresponding coproduct on the fixed point Floer homologies. To be more specific, we first identify generators of CF_* with generators of CF^* in the obvious way[3]. For any generators $\alpha \in \mathrm{CF}_*(\phi^m)$, $\beta \in \mathrm{CF}_*(\phi^n)$ and $\gamma \in \mathrm{CF}_*(\phi^{m+n})$, we define

$$\langle \alpha \cdot \beta, \gamma \rangle := \langle \Delta(\gamma), \alpha \otimes \beta \rangle. \tag{3.2}$$

It is not difficult to see, from the Fredholm index formula, that the product structure preserves the $\mathbb{Z}/2$ grading.

As mentioned in chapter 2, equation (2.8), for each positive integer d, the fixed point Floer homology $\mathrm{HF}_*(\phi^d)$ is isomorphic to

$$H_*(\Sigma_0; \mathbb{C}) \oplus (\oplus_{i=1}^{d-1} H_*(S^1; \mathbb{C})). \tag{3.3}$$

Similarly, the fixed point Floer cohomology $\mathrm{HF}^*(\phi^d)$ is isomorphic to

$$H^*(\Sigma_0; \mathbb{C}) \oplus (\oplus_{i=1}^{d-1} H^*(S^1; \mathbb{C})). \tag{3.4}$$

We remark that for each $i = 1, 2, \cdots, d-1$, the i-th component of the second summand from the above decomposition is generated by the classes $[e_i^d]$ and $[h_i^d]$, with gradings 0 and 1 respectively. On the other hand, $e_0^d + e_d^d$ and $h_0^d + h_d^d$ both represent cohomology classes in $H^*(\Sigma_0, \mathbb{C})$ (with $\mathbb{Z}/2$-gradings 0 and 1 respectively), which we will denote by $[f^d]$ and $[g^d]$, respectively.

We are now ready to state the following lemma, which is a rephrase of Theorem 2.2.2 in terms of fixed point Floer cohomologies:

[3] There are, however, choices of signs. We will choose the signs in such a way that the product formulas from the previous chapters hold verbatim. See [JYZ23] Remark 14 for a brief explanation.

Lemma 3.1.1. *For each pair of positive integers m, n, and indices $i \in \{1, 2, \cdots, m-1\}$, $j \in \{1, 2, \cdots, n-1\}$, we have the following relations:*

$$[e_i^m] \cdot [e_j^n] = [e_{i+j}^{m+n}], \quad [h_i^m] \cdot [e_j^n] = [h_{i+j}^{m+n}], \quad [h_i^m] \cdot [h_j^n] = 0,$$

$$[f^m] \cdot [e_j^n] = [e_j^{m+n}] + [e_{j+m}^{m+n}], \quad [g^m] \cdot [e_j^n] = [h_j^{m+n}] + [h_{j+m}^{m+n}],$$

$$[f^m] \cdot [f^n] = [e_m^{m+n}] + [e_n^{m+n}] + [f^{m+n}],$$

$$[g^m] \cdot [f^n] = [h_m^{m+n}] + [h_n^{m+n}] + [g^{m+n}],$$

$$[g^m] \cdot [g^n] = 0.$$

Proof. This is a direct consequence of Theorem 2.2.2 and equation (3.2). □

3.2 The Seidel class and the symplectic cohomology of a nodal curve

In this section, we briefly explain the concept of the Seidel class (see [JYZ23] section 4.2 and section 6 for a more detailed account). We then define the symplectic cohomology $\mathrm{SH}^*(\Sigma_g^0)$ of a nodal curve Σ_g^0 in terms of the fixed point Floer cohomologies and the Seidel class.

The Seidel class

A key ingredient of the definition of symplectic cohomology of a nodal curve is the Seidel class of the positive Dehn twist ϕ. We briefly review the construction of the Seidel class of ϕ in this subsection, and refer to [Sei03; Sei08; JYZ23] for more details.

Associated to ϕ, Seidel in [Sei03] describes a standard Lefschetz fibration $\pi : E \to \mathbb{C}$. Roughly speaking, (E, Ω_E) is a symplectic manifold, and π is a symplectic fiber bundle except over $0 \in \mathbb{C}$. For each smooth fiber $F_z := \pi^{-1}(z)$ with $z \neq 0$, $(F_z, \Omega_E|_{F_z})$ is symplectomorphic to (Σ_g, ω). With the above identification, the counterclockwise parallel transformation along the unit circle gives the symplectomorphism ϕ, and $\pi|_{\{|z|\geq 1\}}$ is isomorphic to the symplectization of the mapping torus M_ϕ.

Fix a generic Floer perturbation data (J, K) as described in [Sei08]. For each fixed point x of ϕ, we may consider the moduli space $\mathcal{M}(E, J, K, x)$ that consists of Fredholm index 0, K-perturbed J-holomorphic sections of $\pi : E \to \mathbb{C}$, that are asymptotic to the Reeb orbit γ_x when $|z| \to \infty$ (after identifying $\pi|_{\{|z|\geq 1\}}$ with the symplectization of the mapping torus M_ϕ).

For a generic choice of (J, K), the moduli space $\mathcal{M}(E, J, K, x)$ is a compact, zero-dimensional manifold, and we denote by $\#\mathcal{M}(E, J, K, x)$ the (signed) count of points in the moduli space. We define the Seidel element $S \in \mathrm{CF}^0(\phi)$ to be

$$S := \sum_x \#\mathcal{M}(E, J, K, x) \cdot x, \tag{3.5}$$

where the right hand side sums over all fixed points of ϕ. The Seidel element is a cocycle, and we define the Seidel class $[S] \in \mathrm{HF}^0(\phi)$ to be the corresponding cohomology class.

For our purpose of an explicit computation, it is important to know the exact form of the Seidel class of a single Dehn twist ϕ:

Theorem 3.2.1. *The Seidel class $[S]$ of the positive Dehn twist ϕ is either $[f^1]$ or $-[f^1]$.*

Proof. See [JYZ23] Theorem 6.1. □

We shall remark that, to determine the exact sign of the Seidel class, we will have to go through the setups of the orientations of the corresponding moduli spaces. However, for the purpose of our computations, different signs will give the same result, as will be clear from the proofs of Theorem 3.3.6 and Theorem 3.3.9. In this sense, throughout the rest of this chapter, we will assume that the Seidel class $[S]$ is equal to $[f^1]$.

The symplectic cohomology $\mathrm{SH}^*(\Sigma_g^0)$

In this subsection, we explain the notion of the symplectic cohomology of a nodal curve Σ_g^0. For our purposes, a *nodal curve* Σ_g^0 of genus $g \geq 2$ can be viewed as the quotient space Σ_g/γ. In other words, Σ_g^0 can be thought of as the smooth, genus g Riemann surface Σ_g pinched along a (non-separating) simple closed curve γ.

The *symplectic cohomology* of Σ_g^0 is defined in terms of the smooth Riemann surface Σ_g and the positive Dehn twist ϕ along the pinching curve γ. To be more precise, we have the following:

Definition 3.2.2 ([JYZ23] Definition 1.1)**.** *The symplectic cohomology $\mathrm{SH}^*(\Sigma_g^0)$ is defined as*

$$\mathrm{SH}^*(\Sigma_g^0) := \varinjlim_d \mathrm{HF}^*(\phi^d), \tag{3.6}$$

where the direct limit is taken along multiplications by the Seidel class:

$$\mathrm{HF}^*(\phi^d) \xrightarrow{\cdot [S]} \mathrm{HF}^*(\phi^{d+1}). \tag{3.7}$$

The fixed point Floer cohomology $\oplus_{d\geq 1}\mathrm{HF}^*(\phi^d)$ comes with an ($\mathbb{Z}/2$-graded) algebra structure (we described the multiplication structure in the previous section), and the algebra structure descends to that of $\mathrm{SH}^*(\Sigma_g^0)$.

In the next section, we will briefly explain that our computations of the symplectic cohomology of the nodal curve Σ_g^0 gives the closed-string mirror symmetry of Σ_g^0. In the classical setup (see, for example, [GKR17]) where a smooth algebraic curve Σ_g is considered, closed-string mirror symmetry predicts that the symplectic cohomology of Σ_g, being the Hochschild cohomology of the Fukaya category (the "A-model" invariant) of Σ_g, is isomorphic to the Hoschild cohomology of the matrix factorization category (the "B-model" invariant) of the mirror. The above definition might seem strange at the first glance, but we see this as the appropriate "A-model" invariant for the nodal curve Σ_g^0 because of the following theorem:

Theorem 3.2.3 ([JYZ23] Theorem 4.1)**.** *The symplectic cohomology* $\mathrm{SH}^*(\Sigma_0^g)$ *is isomorphic to the Hochschild cohomology of the Fukaya category of* Σ_g^0 *defined in [Jef22]:*

$$\mathrm{SH}^*(\Sigma_g^0) \cong \mathrm{HH}^*(\mathcal{F}(\Sigma_g^0)). \tag{3.8}$$

3.3 Computations of the algebra $\mathrm{SH}^*(\Sigma_g^0)$

In this section, we will compute the graded algebra $\mathrm{SH}^*(\Sigma_g^0)$. The main ingredients that will be needed from the previous sections are the calculations of the product of the fixed point Floer cohomologies (Lemma 3.1.1), the description of the Seidel class (Theorem 3.2.1) and the definition of the symplectic cohomology of the nodal curve (Definition 3.2.2).

To simplify the proofs, we remark that the algebra $\oplus_{d\geq 1}\mathrm{HF}^*(\phi^d)$ has a $\mathbb{Z}$-grading (different from the $\mathbb{Z}/2$-grading mentioned before), simply by defining the grading of elements of $\mathrm{HF}^*(\phi^d)$ to be d.

The even part $\mathrm{SH}^0(\Sigma_g^0)$

We begin with the computations of $\mathrm{SH}^0(\Sigma_g^0)$. In the following, we first determine the algebra $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$, and then take into account the direct limit defined by the Seidel class. We begin with the following observations:

Lemma 3.3.1. *The algebra* $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$ *is generated by* $[f^1]$, $[e_1^2]$, *and* $[e_1^3]$.

Proof. Notice that $[f^2] = [f^1]\cdot[f^1] - [e_1^2] - [e_1^2]$ and $[e_2^3] = [f^1]\cdot[e_1^2] - [e_1^3]$. It follows that $\mathrm{HF}^0(\phi)$, $\mathrm{HF}^0(\phi^2)$, and $\mathrm{HF}^0(\phi^3)$ are all generated by the said elements. It suffices to show that for each $d \geq 4$, the classes $[e_i^d]$ (for $0 < i < d-1$) and $[f^d]$ are generated by elements of lower gradings, and the lemma follows by induction on d.

To begin with, we notice that if $1 < i < d-1$ and $d \geq 4$ then $[e_i^d] = [e_1^2]\cdot[e_{i-1}^{d-2}]$, so we only need to focus on $[f^d]$, $[e_1^d]$ and $[e_{d-1}^d]$.

Next we observe that for each $d \geq 4$, we have

$$[f^d] = [f^2]\cdot[f^{d-2}] - [e_2^d] - [e_{d-2}^d] = [f^2]\cdot[f^{d-2}] - [e_1^{d-2}]\cdot[e_1^2] - [e_{d-3}^{d-2}]\cdot[e_1^2], \tag{3.9}$$

so $[f^d]$ can be generated by elements of lower gradings.

Finally, we observe that for each $d \geq 4$,

$$[e_1^d] = [f^1]\cdot[e_1^{d-1}] - [e_2^d] = [f^1]\cdot[e_1^{d-1}] - [e_1^2]\cdot[e_1^{d-2}], \tag{3.10}$$

and

$$[e_{d-1}^d] = [f^1]\cdot[e_{d-2}^{d-1}] - [e_{d-2}^d] = [f^1]\cdot[e_{d-2}^{d-1}] - [e_1^2]\cdot[e_{d-3}^{d-2}]. \tag{3.11}$$

So $[e_1^d]$ and $[e_{d-1}^d]$ (with $d \geq 4$) are also generated by elements of lower degree. $\square$

Lemma 3.3.2. *The generators* $[f^1], [e_1^2], [e_1^3]$ *satisfy the relation*

$$[f^1] \cdot [e_1^2] \cdot [e_1^3] = [e_1^2]^3 + [e_1^3]^2 \tag{3.12}$$

Proof. This follows directly from Lemma 3.1.1. □

We next show that, in the sense of Lemma 3.3.1, equation (3.12) generates all of the relations in the algebra $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$. In other words, let

$$P(x, y, z) := xyz - y^3 - z^2, \tag{3.13}$$

then we have

Lemma 3.3.3. *Let* $L(x, y, z) \in \mathbb{C}[x, y, z]$ *be a nonzero polynomial with* $L([f], [e_1^2], [e_1^3]) = 0$, *then there is a polynomial* $Q(x, y, z) \in \mathbb{C}[x, y, z]$ *such that*

$$L(x, y, z) = Q(x, y, z) \cdot P(x, y, z). \tag{3.14}$$

To prove this, we need the following simple observation.

Lemma 3.3.4. $[e_1^2]$ *and* $[e_1^3]$ *are algebraically independent over* $\mathbb{C}$. *That is, if* $f \in \mathbb{C}[y, z]$ *satisfies* $f([e_1^2], [e_1^3]) = 0$ *then* $f = 0$.

Proof. f can be decomposed into homogeneous parts:

$$f = f_0 + f_1 + f_2 + \cdots \tag{3.15}$$

where the grading of the monomial $y^a z^b$ is given by $2a + 3b$. It is clear that $f_k([e_1^2], [e_1^3]) \in \mathrm{HF}^0(\phi^k)$, so we have $f_k([e_1^2], [e_1^3]) = 0$ for each k.

Suppose

$$f_k(y, z) = \sum_j \lambda_j y^{a_j} z^{b_j}, \tag{3.16}$$

where $2a_j + 3b_j = k$ for all indices j. By Lemma 3.1.1, we have

$$f_k([e_1^2], [e_1^3]) = \sum_j \lambda_j [e^k_{a_j+b_j}]. \tag{3.17}$$

Since the lower indices $a_j + b_j$ are distinct for different j, the classes $\{[e^k_{a_j+b_j}]\}$ are linearly independent over $\mathbb{C}$. So the coefficients λ_j are all zeroes. □

Proof of Lemma 3.3.3. We write $L(x, y, z) = g_k(y, z)x^k + g_{k-1}(y, z)x^{k-1} + \cdots + g_0(y, z)$. The assumption that L is nonzero implies $k > 0$, by Lemma 3.3.4.

We next apply the division algorithm in $\mathbb{C}(y, z)[x]$ to write

$$L(x, y, z) = \left(x - \frac{y^3 + z^2}{yz}\right)\left(r_{k-1}(y, z)x^{k-1} + ... + r_0(y, z)\right) + h_0(y, z) \tag{3.18}$$

where the coefficients $r_j(y,z)$ and $h_0(y,z)$ are elements of $\mathbb{C}(y,z)$.

It follows from induction that the coefficients are of the form

$$r_j(y,z) = \frac{\phi_j(y,z)}{(yz)^{m_j}} \tag{3.19}$$

for some $\phi_j(y,z) \in \mathbb{C}[y,z]$ and $m_j \in \mathbb{Z}_{\geq 0}$, and

$$h_0(y,z) = \frac{\rho_0(y,z)}{(yz)^M} \tag{3.20}$$

for some $\rho_0(y,z) \in \mathbb{C}[y,z]$, and $M \geq 0$.

In particular, this implies that there is a large enough N such that

$$(yz)^N L(x,y,z) = (xyz - y^3 - x^2)\left(\tilde{r}_{k-1}(y,z)x^{k-1} + \cdots + \tilde{r}_0(y,z)\right) + \tilde{h}_0(y,z) \tag{3.21}$$

where $\tilde{r}_j(y,z), \tilde{h}_0(y,z) \in \mathbb{C}[y,z]$. We next plug $x = [f^1]$, $y = [e_1^2]$ and $z = [e_1^3]$ into the above equation to get $\tilde{h}_0([e_1^2],[e_1^3]) = 0$. Again, since $[e_1^2]$ and $[e_1^3]$ are algebraically independent, we have

$$\tilde{h}_0(y,z) = 0. \tag{3.22}$$

Using the fact that $\mathbb{C}[x,y,z]$ is a UFD and that the polynomial $xyz - y^3 - z^2$ is irreducible, we conclude that $P(x,y,z) = xyz - y^3 - z^2$ divides L in $\mathbb{C}[x,y,z]$. This concludes the proof of the lemma. □

We are now ready to describe the graded algebra $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$:

Theorem 3.3.5. *There is an isomorphism of $\mathbb{Z}$-graded algebras between*

$$\bigoplus_{d\geq 1} \mathrm{HF}^0(\phi^d)$$

and

$$(\mathbb{C}[x,y,z]/(xyz - y^3 - z^2))_{\geq 1}, \tag{3.23}$$

the grading ≥ 1 part of the algebra $\mathbb{C}[x,y,z]/(xyz - y^3 - z^2)$, with the gradings given by $gr(x) = 1$, $gr(y) = 2$ and $gr(z) = 3$.

Proof. This follows directly from Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3; the isomorphism is defined by sending $[f]$ to x, $[e_1^2]$ to y and $[e_1^3]$ to z. It is also clear that the isomorphism sends $\mathrm{HF}^0(\phi^d)$ to

$$(\mathbb{C}[x,y,z]/(xyz - y^3 - z^2))_d, \tag{3.24}$$

the grading-d part of the algebra. □

Finally, we are ready to compute $\mathrm{SH}^0(\Sigma_g^0)$, the even part of the symplectic cohomology of the nodal curve Σ_g^0:

Theorem 3.3.6. *The algebra* $\mathrm{SH}^0(\Sigma_g^0)$ *is isomorphic to*

$$\mathbb{C}[Y,Z]/(YZ - Y^3 - Z^2). \tag{3.25}$$

Proof. Recall that

$$\mathrm{SH}^0(\Sigma_g^0) = \varinjlim_d \mathrm{HF}^0(\phi^d). \tag{3.26}$$

Under the isomorphism in Theorem 3.3.5, the direct system

$$\mathrm{HF}^0(\phi^d) \xrightarrow{\cdot[S]} \mathrm{HF}^0(\phi^{d+1}) \tag{3.27}$$

can be identified with

$$(\mathbb{C}[x,y,z]/(xyz - y^3 - z^2))_d \xrightarrow{\cdot x} (\mathbb{C}[x,y,z]/(xyz - y^3 - z^2))_{d+1}. \tag{3.28}$$

It is straightforward to see that the direct limit of the above system is isomorphic to the grading-0 part of the algebra

$$\mathbb{C}[x,y,z,x^{-1}]/(xyz - y^3 - z^2), \tag{3.29}$$

which is isomorphic to

$$\mathbb{C}[Y,Z]/(YZ - Y^3 - Z^2). \tag{3.30}$$

□

The odd part $\mathrm{SH}^1(\Sigma_g^0)$

We next determine $\mathrm{SH}^1(\Sigma_g^0)$, the odd part of the symplectic cohomology. We begin with the following elementary observation:

Lemma 3.3.7. *For any pair of positive integers* m *and* n*, the product*

$$\mathrm{HF}^1(\phi^m) \times \mathrm{HF}^1(\phi^n) \xrightarrow{\cdot} \mathrm{HF}^0(\phi^{m+n}) \tag{3.31}$$

is identically zero.

Proof. This follows directly from Lemma 3.1.1. □

Since $\mathrm{SH}^1(\Sigma_g^0) = \varinjlim_d \mathrm{HF}^1(\phi^d)$, we have the following

Corollary 3.3.8. *The product*

$$\mathrm{SH}^1(\Sigma_g^0) \times \mathrm{SH}^1(\Sigma_g^0) \xrightarrow{\cdot} \mathrm{SH}^0(\Sigma_g^0) \tag{3.32}$$

is identically zero.

As a consequence, to conclude the algebra structure of $\mathrm{SH}^*(\Sigma_g^0)$, it suffices to determine $\mathrm{SH}^1(\Sigma_g^0)$ as a $\mathrm{SH}^0(\Sigma_g^0)$ module. We first observe that, for each $d \geq 1$, the summand

$$H^1(\Sigma_0) \subset \mathrm{HF}^1(\phi^d) \tag{3.33}$$

is $2g-1$ dimensional. Aside from the class $[g^d]$, for each $d \geq 1$, we can choose $2g-2$ distinguished classes

$$[k_1^d], [k_2^d], \cdots, [k_{2g-2}^d] \in H^1(\Sigma_0) \subset \mathrm{HF}^1(\phi^d) \tag{3.34}$$

such that

1. $[g^d], [k_1^d], [k_2^d], \cdots, [k_{2g-2}^d]$ form a basis of $H^1(\Sigma_0) \subset \mathrm{HF}^1(\phi^d)$, and
2. $[k_j^d] \cdot [f^1] = [k_j^{d+1}]$ for each $j = 1, 2, \cdots, 2g-2$.

With the above preparation, we have the following

Theorem 3.3.9. *The* $\mathrm{SH}^0(\Sigma_g^0)$*-module* $\mathrm{SH}^1(\Sigma_g^0)$ *is isomorphic to*

$$\mathbb{C}[Y, Z]/(YZ - Y^3 - Z^2) \oplus \mathbb{C}^{2g-2}, \tag{3.35}$$

where $\mathrm{SH}^0(\Sigma_g^0) \cong \mathbb{C}[Y, Z]/(YZ - Y^3 - Z^2)$ *acts on the first factor by multiplication, and on the second factor by projection to* $\mathbb{C}$ *followed by diagonal multiplication.*

Proof. The proof follows that of Theorem 3.3.6 almost verbatim, so we only highlight the differences here. Throughout the proof, we identify $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$ with $(\mathbb{C}[x, y, z]/(xyz - y^3 - z^2))_{\geq 1}$ in the sense of Theorem 3.3.5, and $\mathrm{SH}^0(\Sigma_g^0)$ with $\mathbb{C}[Y, Z]/(YZ - Y^3 - Z^2)$ in the sense of Theorem 3.3.6.

As a $(\mathbb{C}[x, y, z]/(xyz - y^3 - z^2))_{\geq 1}$ module, $\oplus_{d\geq 1}\mathrm{HF}^1(\phi^d)$ is generated by $[g^1]$, $[h_1^2]$ and $[h_1^3]$, together with $[k_j^d]$ for all $d \geq 1$ and $1 \leq j \leq 2g-2$. The relations of these generators are generated by the following equations

$$y \cdot [g^1] = x \cdot [h_1^2],\ z \cdot [g^1] = x \cdot [h_1^3],\ z \cdot [h_1^2] = y \cdot [h_1^3],\ yz \cdot [g^1] = y^2 \cdot [h_1^2] + z \cdot [h_1^3] \tag{3.36}$$

together with

$$x \cdot [k_j^d] = [k_j^{d+1}]. \tag{3.37}$$

It follows that the $\mathbb{Z}$-graded $(\mathbb{C}[x, y, z]/(xyz - y^3 - z^2))_{\geq 1}$ module $\oplus_{d\geq 1}\mathrm{HF}^1(\phi^d)$ is isomorphic to

$$(\mathbb{C}[x, y, z]/(xyz - y^3 - z^2))_{\geq 1} \oplus ((\mathbb{C}[x])_{\geq 1})^{2g-2}. \tag{3.38}$$

The isomorphism is given by sending $[g^1]$ to x, $[h_1^2]$ to y, and $[h_1^3]$ to z in the first factor, and $[k_j^d]$ to x^d in the j-th summand of the second factor. The module structure is given by multiplication on the first factor, and by projection to $(\mathbb{C}[x])_{\geq 1}$ followed by diagonal multiplication on the second factor. Just as what we had for $\oplus_{d\geq 1}\mathrm{HF}^0(\phi^d)$, the above isomorphism is an isomorphism of $\mathbb{Z}$-graded modules, where the grading is given by $gr(x) = 1$, $gr(y) = 2$ and $gr(z) = 3$.

Finally, recall that $\mathrm{SH}^1(\Sigma_g^0)$ is defined by $\varinjlim_d \mathrm{HF}^1(\phi^d)$, and under the above isomorphism, the direct system is obtained by multiplying x. It is not difficult to see that the direct limit is isomorphic to the grading-0 part of the graded module

$$\mathbb{C}[x,y,z,x^{-1}]/(xyz - y^3 - z^2) \oplus (\mathbb{C}[x,x^{-1}])^{2g-2}, \tag{3.39}$$

which is isomorphic to

$$\mathbb{C}[Y,Z]/(YZ - Y^3 - Z^2) \oplus \mathbb{C}^{2g-2}. \tag{3.40}$$

□

Theorem 3.3.6, Corollary 3.3.8, and Theorem 3.3.9 completely determine the $\mathbb{Z}/2$-graded algebra $\mathrm{SH}^*(\Sigma_g^0)$.

3.4 Closed-string mirror symmetry for nodal curves

In this section, we explain the significance of Theorem 3.3.6 and Theorem 3.3.9. We begin with a brief review of closed-string mirror symmetry for smooth curves, and then explain the "B-model" invariants for nodal curves and its relation to the "A-model" invariants introduced in the previous sections. At the end, we will discuss how some of the results in this chapter can be generalized to a broader setup.

Smooth curves

Let (Σ_g, ω) be a smooth surface with genus $g \geq 2$. Homological mirror symmetry for such a smooth curve has been studied extensively, see for example, [Sei11; Efi12; Lee15; PS19; PS21; LP17; LP12]. Roughly speaking, homological mirror symmetry associates Σ_g to a Landau-Ginzburg model (X_g, W_g), where X_g is a three-dimensional algebraic variety and $W_g : X_g \to \mathbb{C}$ is a holomorphic function, whose critical locus is a trivalent configuration of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s [Efi12; AAK16], which we denote by Z_g. Homological mirror symmetry predicts that a Fukaya category [4] of Σ_g (the "A-model" invariant) is equivalent to the matrix factorization category $\mathrm{MF}(X_g, M_g)$ (the "B-model" invariant). After passing to the Hochschild cohomologies of both invariants, this implies the closed-string mirror symmetry [GKR17] in terms of the symplectic cohomology of Σ_g

$$\mathrm{SH}^*(\Sigma_g) \cong \mathrm{HH}(\mathrm{MF}(X_g, M_g)). \tag{3.41}$$

The Fukaya category and the symplectic cohomology of nodal curves

The goal of this section is to generalize equation (3.41) to include nodal curves as well. As before, we use Σ_g^0 to denote a genus $g \geq 2$ curve with a single node. It is expected that the

[4] Throughout this chapter, the term "Fukaya category" refers to the split closure of the A_∞ category of twisted complexes over the Fukaya category, wrapped in the sense of [GPS20] for punctured curves.

mirror (X_g^0, W_g^0) for the nodal curve Σ_g^0 has the property that the critical locus Z_g^0 of W_0 is Z_g with one smooth point removed. Figure 3.1 illustrates the case for $g = 2$.

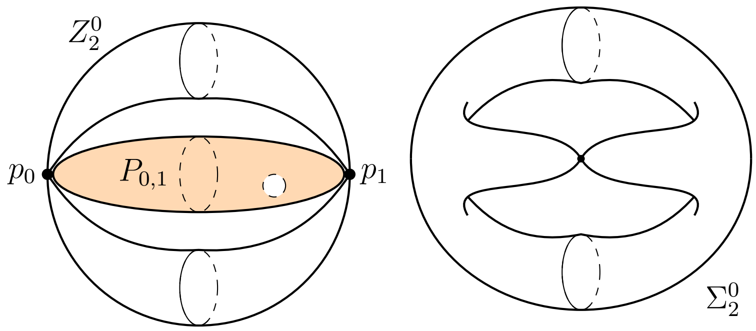

Figure 3.1: The mirror Z_2^0 of a nodal genus-2 curve Σ_2^0.

The A-model invariant that we consider for Σ_2^0 is $\mathcal{F}(\Sigma_g^0)$, the Fukaya category of the nodal curve defined in [Jef22]. The significance of Definition 3.2.2 is implied by the following

Theorem 3.4.1 ([JYZ23] Theorem 4.1)**.** *Suppose $f : X \to \mathbb{C}$ is a holomorphic function on a Stein manifold X with general fiber M and a single singular fiber M^0 over 0, and suppose ϕ is the counterclockwise monodromy of the fiber. Then there is an equivalence of graded algebras*

$$\mathrm{HH}^*(\mathcal{F}(M^0)) \cong \varinjlim_{d} \mathrm{HF}^*(\phi^d) \tag{3.42}$$

where the connecting map for the direct limit is defined by multiplication of the Seidel class.

In sum, what we call the symplectic cohomology of a nodal curve Σ_g^0, is indeed the Hochschild cohomology of the Fukaya category of Σ_g^0 in the sense of [Jef22], and hence is the analog of the left-hand side of equation (3.41) for nodal curves.

The B-model computations

To generalize equation (3.41) for nodal curves, it remains to compute the "B-model" invariants for the Landau-Ginzburg model (X_g^0, W_g^0) associated to the nodal curve Σ_g^0. To introduce the key results, we need the following

Definition 3.4.2 ([JYZ23] Definition 1.2)**.** *The sheaf $\tilde{T}_Z$ of balanced vector fields on a trivalent configuration Z of $\mathbb{A}^1$'s and $\mathbb{P}^1$'s, is the sheaf whose sections are vector fields on Z*

1. *that vanish at the nodes of Z, and*

2. *whose rotation numbers around every node sum up to zero.*

The following Hochschild-Kostant-Rosenberg theorem for matrix factorization factories is important in the computation of the B-model invariants. It states that the analog of the right-hand side of equation (3.41) for the nodal curve Σ_g^0 can be explicitly computed in terms of the critical locus Z_g^0 and the sheaf of balanced vector fields on it:

Theorem 3.4.3 ([JYZ23] Theorem 2.1)**.** *Let Z_g^0 be the critical locus of the Landau-Ginzburg model (X_g^0, W_g^0) mirror to Σ_g^0, then*

$$\mathrm{HH}^{\mathrm{even}}(\mathrm{MF}(X_g^0, W_g^0)) \cong H^0(\mathcal{O}_{Z_g^0}) \oplus H^1(\tilde{T}_{Z_g^0}), \tag{3.43}$$

$$\mathrm{HH}^{\mathrm{odd}}(\mathrm{MF}(X_g^0, W_g^0)) \cong H^1(\mathcal{O}_{Z_g^0}) \oplus H^0(\tilde{T}_{Z_g^0}). \tag{3.44}$$

The sheaf cohomologies of $\mathcal{O}_{Z_g^0}$ and $\tilde{T}_{Z_g^0}$ can be explicitly computed with a Zariski open cover of Z_g^0. Here is the result:

Theorem 3.4.4 ([JYZ23] Theorem 2.2)**.** *Let $A = \mathbb{C}[Y, Z]/(YZ - Y^3 - Z^2)$. We have*

$$H^0(\mathcal{O}_{Z_g^0}) \oplus H^1(\tilde{T}_{Z_g^0}) \cong A \tag{3.45}$$

as $\mathbb{C}$-algebras, and

$$H^1(\mathcal{O}_{Z_g^0}) \oplus H^0(\tilde{T}_{Z_g^0}) \cong A \oplus \mathbb{C}^{2g-2} \tag{3.46}$$

as A-modules.

The nodal curve Σ_g^0 and beyond

Theorem 3.3.6, Theorem 3.3.9, Theorem 3.4.3 and Theorem 3.4.4 imply the following closed-string mirror symmetry for the nodal curve Σ_g^0, which generalizes (3.41):

Theorem 3.4.5 ([JYZ23] Theorem 1.3)**.** *There is an equivalence of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras*

$$\mathrm{SH}^*(\Sigma_g^0) \cong \mathrm{HH}^*(\mathrm{MF}(X_g^0, W_g^0)) \tag{3.47}$$

Remark 3.4.6. Several remarks are needed here.

1. So far, we have only considered the case with a single node, which corresponds to positive Dehn twists along a single, non-separating closed curve γ. It is worth pointing out that the no-crossing results still hold if we replace γ with a non-separating collection $\mathcal{C}$ of simple closed curves, under mild topological conditions. As a consequence, the computations about the fixed-point cohomologies of iterations of positive Dehn twists along $\mathcal{C}$ can be carried over, which leads to an explicit computation of the symplectic cohomology of curves with multiple nodes, see [JYZ23] Theorem 7.8. Combined with the corresponding B-model computation ([JYZ23] Theorem 2.3), such a computation leads to the closed-string mirror symmetry for curves with multiple nodes.

2. We can also add punctures to Σ_g^0. In [JYZ23], we explicitly computes the invariants for $\Sigma_{g,1}^0$, the once punctured genus g nodal curve. The main difference in the setup is that we require ϕ to be fully wrapped near the puncture, and in computing the A-model invariants, the Seidel class includes one more fixed point near the puncture. For the B-model computations, the main difference is that the critical locus Z_g' of the Landau-Ginzburg model includes an $\mathbb{A}^1$-component. The explicit computations for both invariants are carried out in [JYZ23] Theorem 2.4 and Theorem 7.10.

3. Theorem 3.3.5 is treated as the first step in the computation of $\mathrm{SH}^*(\Sigma_g^0)$ here. In fact, by incorporating the $H^*(\Sigma_g)$-module structure of the fixed point Floer cohomologies (see, for example, [RT95; PSS96; Sei97]), Theorem 3.3.5 can be upgraded to a computation of the graded algebra $\bigoplus_{d\geq 0} \mathrm{HF}^*(\phi^d)$; see [JYZ23] Theorem 7.1 and Theorem 7.5 for a detailed account. Homological mirror symmetry predicts that the graded algebra

$$\bigoplus_{d\geq 0} \mathrm{HF}^*(\phi^d) \tag{3.48}$$

should be isomorphic to the B-model invariant

$$\mathrm{HH}^*(\mathrm{MF}(X_g, W_g), \mathcal{L}^{\otimes d}), \tag{3.49}$$

the Hochschild cohomology of the matrix factorization category of the Landau-Ginzburg model (X_g, W_g) with coefficients in a power of a line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ has degree 1 on one single $\mathbb{P}^1$ component of the critical locus Z_g, and is trivial over all the others. The corresponding version of the Hochschild-Kostant-Rosenberg theorem says that the above invariant is isomorphic to

$$\bigoplus_{d\geq 0} \bigoplus_{i+j\equiv *} H^i(\wedge^j \tilde{T}_{Z_g} \otimes \mathcal{L}^{\otimes d}). \tag{3.50}$$

An explicit computation ([JYZ23] Theorem 2.6) of the above sheaf cohomology shows that the two graded algebras (3.48) and (3.50) are isomorphic, proving the homological mirror symmetry for homogeneous coordinate rings ([JYZ23] Theorem 1.5).

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