Title
PROGRESS REPORT ON THE SOLUTION OF THE GENERAL MAGNETOSTATIC EQUATION

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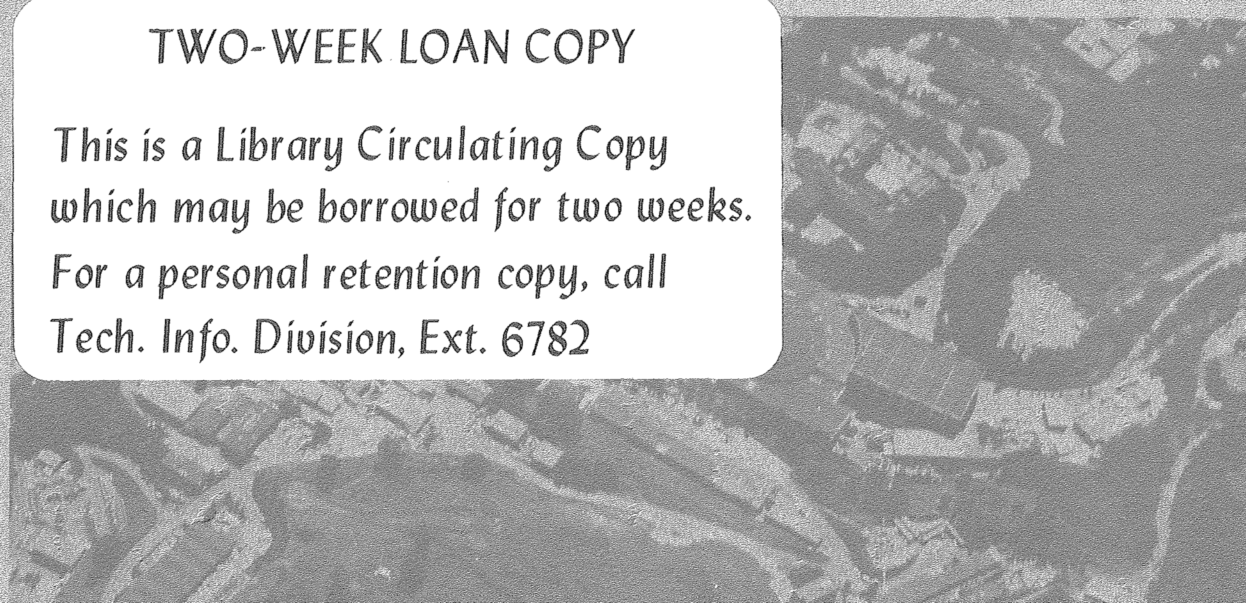

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I. Introduction

This report summarizes our efforts and findings in connection with the formulation and solution of the non-linear "magnetostatic equation", and some of our efforts towards the design and implementation of a computer program to solve this equation efficiently.

II. The Magnetostatic Equation

A large class of problems in physics and engineering is formulated by the non-linear Poisson's equation which, with modern digital computers and advances in numerial methods, has been solved for a variety of geometrical configurations of extreme complexity and various boundary conditions.

In magnetostatics, this equation has been solved by both integral and differential operators, and many well established computer programs exist producing solutions in two dimensions usually in terms of the single component vector potential. This differential operator approach produces accurate results, however, it imposes undesirable characteristics such as: artificial boundaries which can have significant effect on the results, and the necessity to differentiate the potential function to obtain the field quantities, a process which can become very difficult especially near the surface of a discontinuity.

These difficulties can be overcome, (at the expense of computational cost) by solving the integral form of the equations in terms of field components directly. An added advantage of this approach is that only regions containing material media (e.g. iron) are discretized. One has to be careful in the choice of the order of discretization, since above a moderate level, computing time escalates rapidly because of the fully populated matrix associated with integral operators. In this respect the use of the differential operator is preferred.

Such integral schemes are very useful in extending to three dimensions where the need to have a mesh of elements connecting many different regions of complex shape is considered as a limitation of the differential approach.

The above brief description of the differential and integral approach, leads to the conclusion that for the formulation of an efficient three dimensional program it is prefered to have a formulation that couples the differential and integral equation, or, more exactly:

1) Use the differential operator for non-linear regions. Methodology has already been developed for this, offering the greatest economy (sparse system).

2) Use the integral operator for linear regions. Here, we need a mesh or elements on surfaces only.

These objectives were also defined in [1] (Simkin & Trowbridge, 1979) and in [2] (Mcdonald and Wexler, 1978). Wexler's work also suggests mutually constrained partial differential and integral equation field formulations. The technique utilized involves the enclosure of the region containing iron within an artificial boundary, e.g. a picture frame; then the differential formulation is used inside; and the constraints, in the form of integral equation representing the outside region, are used on the boundary. This technique introduces new degrees of freedom corresponding to the Lagrangian multiplier, and the resulting matrix is not positive definite. A closely related question is what potential to use inside: scalar or vector? In [1] the numerical experiments concerning this question are discussed in some details.

III. Mathematical Formulation

Our starting point was the integral equation formulation. The objectives to be reached were:

a. The mathematical investigation of the integral equation. Four papers were submitted for publication, dealing with this investigation. Part I [3] considers the solution of eq. (1):

$$\bar{H}(\bar{M}(x)) + \frac{1}{4\pi}\, \text{grad div} \int_{\Omega} \bar{M}(y) \cdot \text{grad}_y \frac{1}{r}\, dy = \bar{B}_A(x),$$

and making use of the monotone operator method, proves the existence and uniqueness of the solution, and also justifies the application of the Galerkin and Ritz methods in solving eq. (1).

In Part II [4] the task of establishing the Tucker stability of the Galerkin procedure is undertaken so that one may obtain perturbation estimates which are very useful in engineering applications.

Part III [5] continues the theoretical analysis of the nonlinear integral equation by presenting more analysis of the spectrum of the singular integral operator which is very important for the derivation and investigation of the mixed integral differential formulations.

The fourth paper [6] discusses the finite element approximation of an integral equation of the second kind deduced from a linear magnetostatic problem.

b. The derivation of the mixed formulations satisfying the two conditions mentioned in section II earlier.

We tried several different approaches, some of them are shown in detail in [7],[8]. Our approach was direct in the sense that: a) we did not introduce new boundaries (differential formulation is only inside the magnetic material, producing a sparce matrix, while the integral equation is used on the iron boundary); b) the formulation is not a constrained one, i.e. we did not introduce Lagrangian multipliers and; c) the particular case $\mu = \infty$ is obtained from the general formulation by taking the corresponding limits. Our methodology assumes that the magnetic field resulting from the current sources in free space is computed separately so that existing efficient programs can be used for its computation.

Perturbation estimates detailed in [4] provide useful information relating to the numerical solution of the original integral equation. For instance, in [9], we present a simple example which clearly indicates the advantages of formulating the problem in terms of $\bar{B}$ rather than in terms of $\bar{H}$.

However, in general case, the choice between $\bar{B}$ or $\bar{H}$ is more involved and in [4] a mathematical formalism is discussed which provides the appropriate analysis.

We start with the integral equation formulation in terms of $\bar{B}$

$$(1) \qquad (R\bar{B})(x) \equiv \bar{H}(\bar{B}(x)) + \frac{1}{4\pi}\int_\Omega \bar{M}(\bar{B}(y))\cdot \mathrm{grad}_y \frac{1}{r}\, dy \equiv \bar{H}(\bar{B}(x))$$
$$- \frac{1}{4\pi}\,\mathrm{grad\,div}\int_\Omega \frac{\bar{B}(y) - \bar{H}(\bar{B}(y))}{r}\, dy = B_a(x), \quad \text{in } \Omega,$$
$$x = (x_1, x_2, x_3) \in \Omega, \quad dy = dy_1 dy_2 dy_3,$$

where Ω is a domain with the boundary Γ in 3-D space which we imagine to be filled with magnetic material. $\bar{B}_a(x)$ is an applied field, produced by currents in free space, a known function. In an operator form (1) is written as

$$(2) \quad R\bar{B} = \bar{H}(\bar{B}) + A(\bar{B} - \bar{H}(\bar{B}))$$

We choose to solve (1) by the Galerkin method in the space of functions $\bar{B}$ satisfying $\bar{B} = \nabla \times \bar{A}$:

$$(3) \quad \int_\Omega R\bar{B}\cdot\bar{L}\, dx = \int_\Omega \bar{B}_a\cdot\bar{L}\, dx, \qquad \bar{L} = \nabla\times\bar{\varphi} \text{ is arbitrary.}$$

Eq. (3) is not in a form appropriate for numerical solution since its subsequent dicretization would lead to a fully populated matrix, therefore, we proceed to integrate (3) by parts to obtain the following two equivalent variational formulations.

$$(4)\quad (R\bar{B}, \nabla\times\bar{\varphi}) \equiv \int_\Omega \bar{H}(\bar{B})\cdot\nabla\times\bar{\varphi}\,dx + \int_\Gamma \bar{n}\cdot\left(\frac{1}{4\pi}\,\mathrm{grad\,div}\int_\Omega \frac{\bar{B}(y)-\bar{H}(\bar{B}(y))}{r}\,dy\right)$$

$$\times\ \bar{\varphi}(x)\,d\gamma_x = \int_\Omega \bar{B}_A\cdot\nabla\times\bar{\varphi}\,dx.$$

$$(5)\quad (R\bar{B},\nabla\times\bar{\varphi}) \equiv \int_\Omega \bar{H}(\bar{B})\cdot\nabla\times\bar{\varphi}\,dx + \int_\Gamma \left(\frac{1}{4\pi}\int_\Gamma \frac{\bar{B}(y)\cdot\bar{n}_y}{r}\,d\gamma_y\right)(\nabla\times\bar{\varphi}(x))\cdot\bar{n}_x\,d\gamma_x$$

$$+\int_\Gamma \bar{n}\times\left[\bar{H}(\bar{B}(x)) + \frac{1}{4\pi}\nabla\times\int_\Gamma \frac{\bar{n}_y\times\bar{H}(\bar{B}(y))}{r}\,d\gamma_y\right]\Big|_{int\,\Gamma}\cdot\bar{\varphi}(x)\,d\gamma_x$$

$$=\int_\Omega \bar{B}_A\cdot\nabla\times\bar{\varphi}\,dx$$

The third equivalent formulation, which is not variational is obtained directly from (1):

$$(6)\quad \begin{cases} \mathrm{div}\,\bar{B} = 0 & \text{in } \Omega \\ \nabla\times\bar{H}(\bar{B}) = 0 & \text{in } \Omega \\ \bar{n}\times\left[\frac{1}{4\pi}\nabla\int_\Gamma \frac{\bar{B}(y)\cdot\bar{n}_y}{r}\,d\gamma_y - \frac{1}{4\pi}\nabla\times\int_\Gamma \frac{\bar{n}_y\times\bar{H}(\bar{B}(y))}{r}\,d\gamma_y - B_A(x)\right]\Big|_{int\,\Gamma} = 0 \end{cases}$$

All three formulations lead to sparse matrices. The 2-D cases are obtained by replacing $\frac{1}{4\pi}$ by $\frac{1}{2\pi}$ and $\frac{1}{r}$ by $\log\frac{1}{r}$ in (4) - (6). Further in the 2-D case and with μ = constant, (4) leads to

$$(4')\quad (R\bar{B},\bar{L}) \equiv \frac{1}{\mu}\int_\Omega \bar{B}\cdot\bar{L}\,dx + \frac{\mu-1}{2\pi\mu}\int_\Gamma\left(\int_\Gamma \bar{B}(y)\cdot\bar{n}_y\,\ln\frac{1}{r}\,d\gamma_y\right)\bar{L}(x)\cdot\bar{n}_x\,d\gamma_x = \int_\Omega \bar{B}_A\cdot\bar{L}\,dx,$$

and (5) leads to

$$(5') \quad (R\bar{B}, \bar{L}) \equiv \frac{1}{\mu}\int_{\Omega} B\cdot\nabla\times\varphi\, dx + \frac{1}{2\pi}\int_{\Gamma}\Big(\int_{\Gamma}\bar{B}(y)\cdot\bar{n}_y \ln\frac{1}{r}\, d\gamma_y\Big)\Big(-\frac{\partial\varphi(x)}{\partial t}\Big)\, d\gamma_x$$

$$+\int_{\Gamma}\Big[-\frac{1}{2\mu}B_t(x) + \frac{1}{2\pi\mu}\int_{\Gamma}B_t(y)\frac{\partial}{\partial n_x}\ln\frac{1}{r}\, d\gamma_y\Big]\varphi(x)\, d\gamma_x = \int_{\Omega}\bar{B}_A\cdot\nabla\times\bar{\varphi}\, dx$$

In the case that $\mu = \infty$ (4') and (5') coincide:

$$(7) \quad \frac{1}{2\pi}\int_{\Gamma}\Big(\int_{\Gamma}\bar{B}(y)\cdot\bar{n}_y \ln\frac{1}{r}\, d\gamma_y\Big)\bar{L}(x)\cdot\bar{n}_x\, d\gamma_x = \int_{\Omega}\bar{B}_A\cdot\bar{L}\, dx$$

The field outside the magnetic material is given by

$$(8) \quad \bar{B}(x) \equiv \bar{H}(x) = \bar{B}_A(x) + \frac{1}{4\pi}\,\mathrm{grad\,div}\int_{\Omega}\frac{\bar{B}(y) - \bar{H}(\bar{B}(y))}{r}\, dy$$

if we solved (4) and by

$$(9) \quad \bar{B}(x) \equiv \bar{H}(x) = \bar{B}_A(x) - \frac{1}{4\pi}\nabla\int_{\Gamma}\frac{\bar{B}(y)\cdot\bar{n}_y}{r}\, d\gamma_y + \frac{1}{4\pi}\nabla\times\int_{\Gamma}\frac{\bar{n}_y\times\bar{H}(\bar{B}(y))}{r}\, d\gamma_y$$

if we solved (5) or (6).

Let us now briefly reflect on our results. Formulation (4) leads to a symmetric matrix (since the operator A is self-adjoint); the formulations (5), (6) lead to the matrices which are close to symmetric matrices, but are not symmetric. Let us compare the asymptotic amount of work required to solve these equations in 3-D case, h being the mesh size, c being generic constant, and k being the number of iterations. To form the matrix of the original integral equation (1) (or (3)) we need $c_1 h^{-3}\cdot h^{-3} = c_1 h^{-6}$ operations. Then the operational count to solve the system is $c_1 h^{-6} + c_1' k_1 h^{-6}$.

To form the matrix of (4) we need $c_2 h^{-3}\cdot h^{-2} = c_2 h^{-5}$ operations, the total operational count then is $c_2 h^{-5} + c_2' k_2 h^{-4}$. Similarly, the operation count required to solve (5), or (6), is $c_3 h^{-4} + c_3' k_3 h^{-4}$.

The above formulations, as far as we know, are new and have not been used by other researchers in this field. Therefore, we have no quantitative knowledge relative to the numerical problems that one might encounter; we feel that adequate research is needed to investigate thoroughly their mathematical and computational behavior so that the proper choice can be made.

IV CONCLUSIONS

We began with the very ambitious task to formulate and derive new mixed integral - differential formulations for the solution of the general three-dimensional magnetostatic equation and we succeeded in deriving some formulations which show very promising results if one considers the mathematical developments mentioned earlier.

However, from here to developing a general purpose computer program, the road is hard and the requirements difficult. We feel that to develop an effective general three dimensional code, parallel mathematical investigations and computer experiments are required. The computer experiments should at first be concentrated at the 2-D level, with μ =constant , and H = H(B) (nonlinear case), in turn; later.

Some of the problems that need to be solved are:

1) Choice of finite elements (for (4),(5)) and the investigation of the order of the approximation for sources.
2) The questions regarding the relative orders of approximations needed for surface and source approximation.
3) Adequate catering of source singularities (due to corners, etc.).
4) Stability to perturbations due to errors in the computations and the original uncertainty in μ (since μ is obtained from experiments).
5) Choice of the adequate iterative techniques to solve the discretized problem.

In parallel to thinking of methods to solve the problems listed above, we began writing a pilot computer program to examine some of our assumptions as well as to test the methodology we developed. Presently, this program has produced results for the 2-D $\mu = \infty$ case, with results comparable to those obtained by more advanced programs such as TRIM. We hope to continue upgrading this program, with the next step being the μ = constant case.

Our recommendations for the future depend largely on the appropriations that this laboratory is willing to commit to such an effort. We have shown our efforts this summer have been both substantial and rewarding, providing a strong foundation which further work may depend upon.

We have also shown, that our theoretical considerations, novel as they might be, they are far from being conclusive and require considerably more work and additional funding which we hope this laboratory will provide.

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