Combinatorics of the Asymmetric Simple Exclusion Process

by

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Abstract

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The Asymmetric Simple Exclusion Process (ASEP) is a process from statistical physics that describes the dynamics of interacting particles hopping right and left on a one-dimensional finite lattice with open boundaries. The ASEP is a Markov chain on 2^n states denoted by words of length n in particles and holes with three hopping parameters α, β, and q. Particles may enter at the left with rate α, they may exit at the right with rate β, and in the bulk particles can hop to an empty location to the right with rate 1 and to the left with rate q.

A main goal in the study of the ASEP is to discover concrete formulae that compute its steady state probabilities. One can compute these probabilities as sums over combinatorial objects such as the *alternative tableaux* of Figure 1.4 (a). In Chapter 2, we give a determinantal formula for the weight generating function of these tableaux at $q = 0$, and thus explicitly compute the steady state probabilities for the ASEP at $q = 0$.

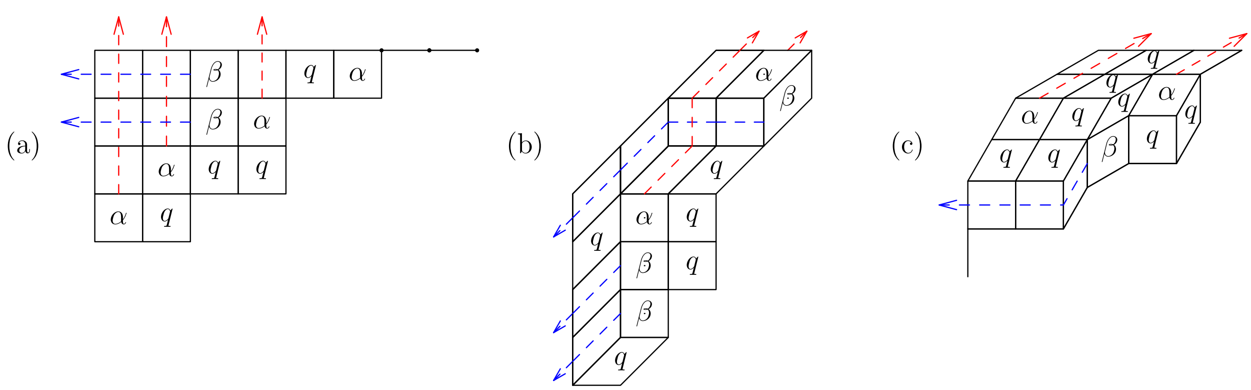

Figure 0.1: (a) an alternative tableau, (b) a rhombic alternative tableau, and (c) a 3-rhombic alternative tableau.

The two-species ASEP is a generalization in which there are two species of particles, *heavy* and *light*. Only the *heavy* particles are able to enter and exit at the left and right

of the lattice and with rates α and β, respectively. If particles of two different species are adjacent, they can swap with rate 1 if the heavier particle is on the left, and rate q if it is on the right. In Chapter 3, we give a combinatorial formula for the steady state probabilities of the two-species ASEP at by introducing the *rhombic alternative tableaux* of Figure 0.1 (b). We show that the weight generating function of these tableaux gives a formula for the steady state probabilities of the two-species ASEP. We give a second proof of this tableaux formula by constructing a Markov Chain on the rhombic alternative tableaux that projects to the two-species ASEP.

In Chapter 4, we introduce a k-species ASEP that generalizes the two-species ASEP. We prove a Matrix Ansatz that expresses the steady state probabilities of states of this k-species ASEP as a certain matrix product, which generalizes an analogous result for the two-species ASEP. In this k-species ASEP, there are k species of particles of varying heaviness. As with the two-species ASEP, only the heaviest particle is allowed to enter and exit at the boundaries of the lattice, with the same respective rates α and β. Moreover, adjacent particles of different species can swap with rate 1 if the heavier particle is on the left, and rate q if it is on the right. Using the generalized Matrix Ansatz, we introduce tableaux called the *k-rhombic tableaux* of Figure 0.1 (c), which give a combinatorial formula for the probabilities of the k-species ASEP.

Dedicated to:
my mother Svetlana,
my father Vladimir,
my sister Yelena,
and my brother Andrei.

Contents

Chapter 1

Introduction

The asymmetric simple exclusion process (ASEP) is a model from statistical physics introduced in the 1960's independently by biologists and mathematicians. It describes the dynamics of particles hopping left and right on a one-dimensional lattice with open boundaries. At the boundaries of the lattice, particles can enter on the left with rate α and exit on the right with rate β. The lattice has n sites, with at most one particle per site. Moreover, at most one particle can hop at a time: a particle at location i can hop to the right with rate 1 if location $i+1$ is empty, and to the left with rate q if location $i-1$ is empty. An empty location can also be denoted by a *hole*, and in this case we describe hopping as a *swap* between adjacent particles and holes.

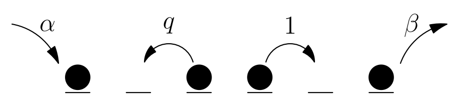

Figure 1.1: ASEP parameters.

A state of the ASEP of size n is denoted by a word of length n in 0's and 1's, or equivalently in ○'s and ●'s, where a 1 or ● represents a particle and a 0 or ○ represents a hole (or absence of a particle). For the remainder of this section we will alternate between denoting states by $X \in \{0,1\}^n$ and $X \in \{\circ, \bullet\}^n$.

The ASEP is a Markov chain on 2^n states denoted by words of length n in particles and holes. A discrete Markov chain is a stochastic model with a set of states and a set of transition probabilities between the states. Let X and Y be words in $\{\bullet, \circ\}$. Then the transitions of this process are:

$$X \bullet \circ Y \overset{1}{\underset{q}{\rightleftharpoons}} X \circ \bullet Y$$

$$\circ X \overset{\alpha}{\rightharpoonup} \bullet X$$

$$X \bullet \overset{\beta}{\rightharpoonup} X \circ$$

where by $X \stackrel{u}{\rightharpoonup} Y$ we mean that the transition from X to Y has probability $\frac{u}{n+1}$, n being the length of X (and also Y). Figure 1.1 shows the parameters of the ASEP, with α, β, and q denoting the rates of the hopping particles. Observe that the ASEP has a certain particle-hole symmetry: if we were to exchange the roles of the particles and the holes, we would obtain an equivalent process, but one where movement is directed from right to left. In this equivalent process, the holes are "entering on the left" with rate β and "exiting on the right" with rate α. Holes can swap with adjacent particles to their left with rate 1 and they can swap with adjacent particles to their right with rate q. Thus exchanging the roles of the particles and the holes is equivalent to exchanging α and β, which results in a symmetry between α and β.

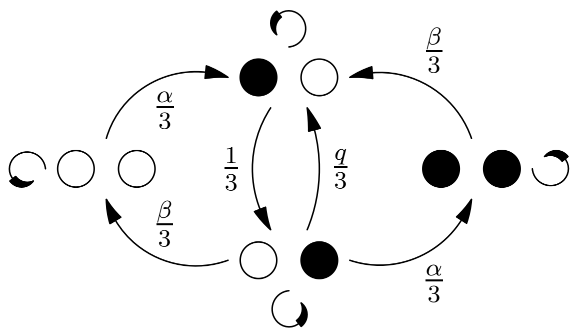

Figure 1.2: The transitions for an ASEP of size $n = 2$.

The ASEP is a non-equilibrium process that exhibits boundary-induced phase transitions (as seen in Figure 1.3). Typically such processes are very complex, but the ASEP is notable due to the existence of exact solutions for its stationary distribution, which makes it a canonical example of non-equilibrium processes in statistical mechanics. In recent years, the ASEP and related processes have attracted quite a lot of interest. On the practical side, the ASEP arises in a variety of contexts, for instance as a model for traffic flow, translation in protein synthesis, a one-dimensional gas, and more. The popularity of the ASEP is furthermore attributed to its surprising and rich algebraic and combinatorial structure. There arise numerous connections of the ASEP with a wide range of areas of mathematics: orthogonal polynomials, the XXZ model, the formation of shocks, total positivity on the Grassmanian, and random matrix theory.

A main goal of much work on the ASEP is to understand the stationary distribution of the ASEP. The steady state probability of a state of a Markov process in general terms is the probability of encountering that state at time "infinity", and in our case is given by the unique left eigenvector of the transition matrix with eigenvalue 1. For example, for the

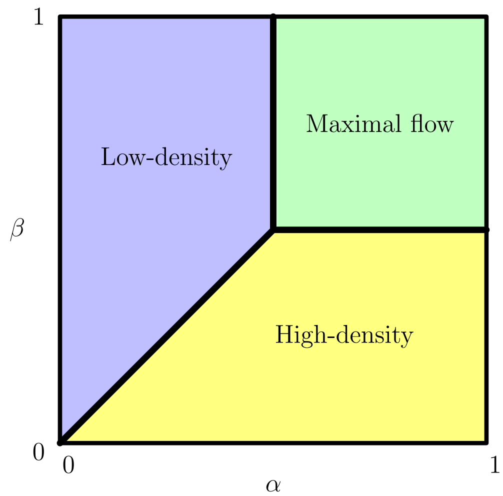

Figure 1.3: The phase diagram that represents three different boundary-induced phases of the ASEP. At $\alpha < \min(\beta, \frac{1}{2})$ the low-density phase occurs, at $\beta < \min(\alpha, \frac{1}{2})$ the high-density phase occurs, and at $\alpha, \beta > \frac{1}{2}$, the phase of maximal flow occurs.

ASEP of size $n = 2$ whose states and transitions are shown in Figure 1.2, the transition matrix is

$$
\begin{array}{c}
\\
\bullet\,\bullet \\
\circ\,\bullet \\
\bullet\,\circ \\
\circ\,\circ
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\quad\bullet\,\bullet\quad & \quad\circ\,\bullet\quad & \quad\bullet\,\circ\quad & \quad\circ\,\circ\quad
\end{array} \\
\begin{pmatrix}
1-\frac{\beta}{3} & 0 & \frac{\beta}{3} & 0 \\
\frac{\alpha}{3} & 1-\frac{\alpha+\beta+q}{3} & \frac{q}{3} & \frac{\beta}{3} \\
0 & \frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & \frac{\alpha}{3} & 1-\frac{\alpha}{3}
\end{pmatrix}
\end{array},
$$

and the steady state probabilities are the following:

$$
\mathrm{Prob}(\bullet\,\bullet) = \frac{1}{Z_2}\alpha^2 \qquad\qquad \mathrm{Prob}(\circ\,\bullet) = \frac{1}{Z_2}\alpha\beta
$$

$$
\mathrm{Prob}(\bullet\,\circ) = \frac{1}{Z_2}\alpha\beta(\alpha+\beta+q) \qquad\qquad \mathrm{Prob}(\circ\,\circ) = \frac{1}{Z_2}\beta^2
$$

where $Z_2 = \alpha^2 + \beta^2 + \alpha\beta(\alpha+\beta+q+1)$.

Surprisingly, the ASEP has rich combinatorial structure, and one can compute the steady state probabilities for the ASEP as sums over combinatorial objects. Combinatorial approaches to understanding the ASEP have been studied by many. In 2004, Duchi and Schaeffer [9] were the first to give a combinatorial formula for the stationary distribution of TASEP (the specialization of the ASEP at $q = 0$). In 2006, Corteel and Williams [7] described the steady state of ASEP in terms of *permutation tableaux*, which are certain fillings of Young diagrams with 1's and 0's (such tableaux are in bijection with permutations). In 2008, X. Viennot [22] improved upon the result of Corteel and Williams by reformulating their theorem in terms of *alternative tableaux*, which are certain fillings of Young diagrams with α's, β's, and q's (intended to correspond to the α, β, and q parameters of the ASEP). The alternative tableaux are in simple bijection with the permutation tableaux, but have symmetries that are consistent with the particle-hole symmetry of the ASEP. Finally in 2009, Corteel and Williams [5] generalized the alternative tableaux to *staircase tableaux*, which give a combinatorial formula for probabilities of a more general 5-parameter ASEP, the discussion of which we omit in this thesis.

Another reason why the ASEP has attracted significant attention is its strong connection to orthogonal polynomials, in particular the Askey-Wilson polynomials [5, 20]. The Askey-Wilson polynomials are important because they are at the top of the hierarchy of orthogonal polynomials in one variable, specializing to many other well-known classical orthogonal polynomials (Hermite, Laguerre, Jacobi, etc.). In 2011, Corteel and Williams found that the moments of the Askey-Wilson polynomials can be expressed using the partition function for the 5-parameter ASEP, and thus the staircase tableaux mentioned above give a combinatorial formula for these moments [5]. The Koornwinder polynomials (also known as Macdonald polynomials of type BC) are a multi-variate generalization of the Askey-Wilson polynomials that specialize or limit to many important multi-variate orthogonal polynomials, of which the Macdonald polynomials (playing an important role in algebraic geometry and representation theory) are a notable example. In recent work, Corteel and Williams found a surprising close connection between the Koornwinder polynomials and the two-species ASEP [6]. This result sparked the original interest of the author in studying the combinatorics of two-species ASEP, since such combinatorial results would also provide an interpretation for the moments of Koornwinder polynomials.

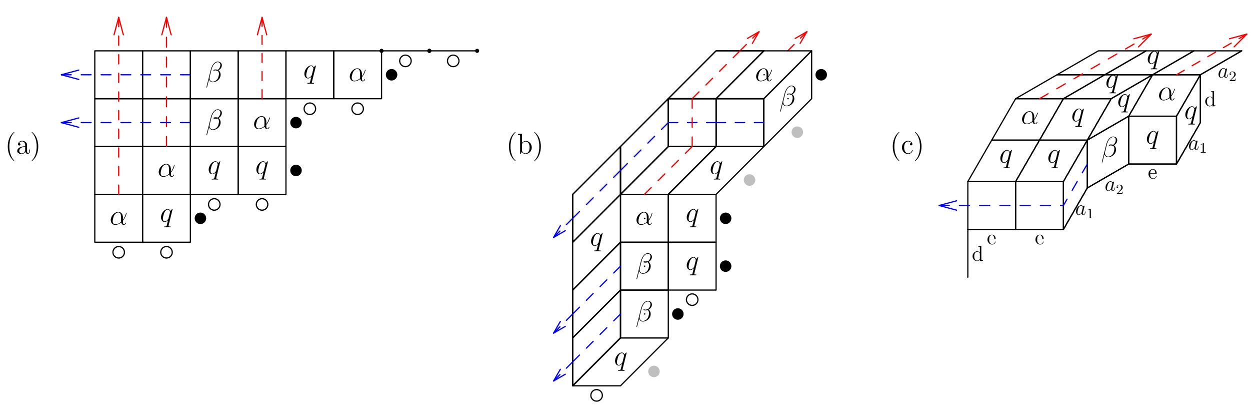

Figure 1.4: (a) an alternative tableau of type ○ ○ ● ○ ○ ● ● ○ ○ ● ○ ○, (b) a rhombic alternative tableau of type ● ● ● ● ● ○ ● ● ○, and (c) a 3-rhombic alternative tableau of type $a_2da_1ea_2a_1eed$.

In Chapter 2, we provide a determinantal formula that explicitly enumerates the alternative tableaux corresponding to states of the ASEP at $q = 0$. An example of an alternative tableau is shown in Figure 1.4 (a), and they are described in detail in Chapter 2 Section 2.1. The *weight* of such a tableau is proportional to the product of the symbols in its filling. The beautiful result of Corteel and Williams, which is central to the work presented in this thesis, expresses the steady state probabilities of the ASEP as sums of the weights of such tableaux. Thus our result gives an explicit determinantal formula for the steady state probabilities of the ASEP at $q = 0$.

In Chapter 3 we follow the line of research of Corteel and Williams by introducing certain *rhombic alternative tableaux* that generalize the alternative tableaux, as in Figure 1.4 (b). We show these tableaux provide an interpretation for the steady state probabilities for a certain two-species ASEP, as an analogue to the role of the alternative tableaux with respect to the usual ASEP. In Chapter 4 we introduce an even more general ASEP with k species of particles and corresponding tableaux called the *k-rhombic alternative tableaux*, such as in Figure 1.4 (c). We summarize these results below.

Chapter 2: A determinantal formula for TASEP probabilities

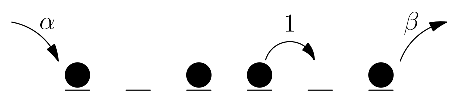

Figure 1.5: Parameters of the TASEP (ASEP at $q = 0$).

The totally asymmetric simple exclusion process (TASEP) is the specialization of the ASEP where $q = 0$, meaning that particles can only hop to the right, with parameters shown in Figure 2.1. Despite its simplicity, the TASEP exhibits boundary induced phase transitions, and so is still a rather interesting problem.

Our main result for the TASEP is an explicit determinantal formula for the steady state probabilities of the process. Such an explicit formula is particularly useful for computations, since determinants are efficient to compute. The strategy for this result was to use the Lingström-Gessel-Viennot determinant by constructing a weight-preserving bijection between alternative tableaux with $q = 0$ and non-crossing weighted lattice paths. The following theorem states the main result (with all necessary definitions provided in Chapter 2).

Theorem 1.0.1 (M. [11]). *Let X be a word in $\{\bullet, \circ\}^n$ with k $\bullet$'s, representing a state of the TASEP of length n with exactly k particles. Let $\lambda(X) = (\lambda_1, \ldots, \lambda_k)$ be the partition associated with the shape of the tableau of type X. Let $A^{\alpha,\beta}_{\lambda(X)} = (A_{ij})_{1\le i,j\le k}$ with*

$$A_{ij} = \left(\binom{\lambda_{j+1}}{j-i+1} + \frac{1}{\beta}\binom{\lambda_{j+1}}{j-i}\right) + \sum_{p=1}^{\lambda_j-\lambda_{j+1}} \left(\frac{1}{\alpha}\right)^p \left(\binom{\lambda_{j+1}+p-1}{j-i} + \frac{1}{\beta}\binom{\lambda_{j+1}+p-1}{j-i-1}\right).$$

The stationary probability of state X is proportional to

$$\mathrm{Prob}(X) = \alpha^{k+\lambda_1}\beta^n \det A^{\alpha,\beta}_{\lambda(X)}.$$

Chapter 3: Combinatorics of the two-species ASEP

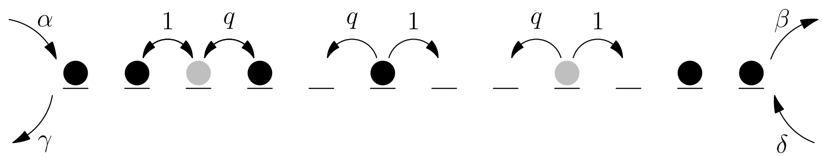

Figure 1.6: Two-species ASEP parameters.

The two-species ASEP is a generalization of the ASEP with two species of particles, one *heavy* and one *light*. (We treat the *hole* as a third type of particle of weight 0.) In this model, the heavy particle can enter and exit the lattice with rates α and β as shown in Figure 4.7. Moreover, the heavy particle can swap places with both the hole and the light particle when they are adjacent, and the light particle can swap places with the hole when they are adjacent. Each of these possible swaps occur at rate 1 when the heavier particle is to the left of the lighter one, and at rate q when the heavier particle is to the right. This process has also been studied by many for its combinatorial structure [2, 9, 19]. As mentioned above, recent interest in studying this process was sparked by a surprising connection to Koornwinder polynomials (Macdonald polynomials of type BC).

Our goal for the two-species ASEP was to find combinatorial results for the two-species process analogous to the combinatorial formulas for the usual one-species ASEP. This work was based on a Matrix Ansatz of Uchiyama [19] (further discussed in Chapter 3 Section 3.2). In [13], we obtained a combinatorial formulas in terms of certain tableaux for probabilities of the two-species ASEP for the case $q = 0$. In subsequent joint work with X. Viennot, we improved this result with a tableaux formula for general q, using certain tableaux objects called *rhombic alternative tableaux*. Specifically, in [15] we obtained the following theorem, which is the main result of Chapter 3.

Theorem 1.0.2 (M., Viennot, [15])**.** *Let X be a state of the two-species ASEP. Then*

$$\mathrm{Prob}(X) = \sum_T \mathrm{wt}(T)$$

is the unnormalized stationary probability of state X, where the sum is over all rhombic alternative tableaux T of type X.

A second proof of Theorem 1.0.2 is obtained by constructing a Markov chain on the rhombic alternative tableaux that projects to the two-species ASEP, from [12]. This result is contained in Chapter 3, Section 3.4.

Theorem 1.0.3 (M., [12])**.** *There is a Markov chain on the rhombic alternative tableaux that projects to the two-species ASEP. This implies the tableaux formula of Theorem 1.0.2.*

Chapter 4: Combinatorics of the k-species ASEP

A natural extension of the two-species ASEP is a more general k-species ASEP, where instead of two species of particles, there are now k species of particles of varying heaviness. As before, the particles are hopping left and right on a one-dimensional lattice on n sites with open boundaries. Again, only the heaviest particle can enter and exit at the left and right of the lattice respectively, and just as in the two-species process, a heavier particle can swap places with an adjacent lighter particle with rates 1 and q if the heavier particle is on the left or right, respectively.

The Matrix Ansatz is an important algebraic tool for solving for the stationary distribution of systems of interacting particles, and it has been used extensively in studies of the original ASEP. For the k-species ASEP, we proved a generalization of the Matrix Ansatz of Derrida, Evans, Hakim, and Pasquier given in Theorem 2.1.1. This k-species Matrix Ansatz gives a formula in terms of a certain matrix product to compute all steady state probabilities of the k-species ASEP [12]. In the case that $k = 2$, our theorem specializes to a theorem of Uchiyama [19].

Using the k-species Matrix Ansatz, we defined the *k-rhombic alternative tableaux* that generalize the rhombic alternative tableaux, and provide a combinatorial interpretation for the probabilities of the k-species ASEP (see Figure 1.4 (c)). The following theorem states the second main result of Chapter 4.

Theorem 1.0.4 (M., [12]). *Let X be a state of the k-species ASEP. Then*

$$\mathrm{Prob}(X) = \sum_{T} \mathrm{wt}(T)$$

is the unnormalized stationary probability of state X, where the sum is over all k-rhombic alternative tableaux T of type X.

Chapter 2

Determinantal formula for the TASEP

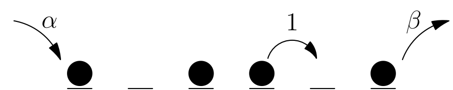

Figure 2.1: Parameters of the TASEP (ASEP at $q = 0$).

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Our main result for the TASEP is an explicit determinantal formula for the steady state probabilities of the process. Such an explicit formula is particularly useful for computations, since determinants are efficient to compute. The strategy for this result was to use the Lingström-Gessel-Viennot determinant by constructing a weight-preserving bijection between alternative tableaux with $q = 0$ and non-crossing weighted lattice paths. The following Theorem states the main result (with all necessary definitions provided in Chapter 1).

Theorem 2.0.1 (M. [11]). *Let X be a word in $\{\bullet, \circ\}^n$ with k $\bullet$'s representing a state of the TASEP of length n with exactly k particles. Let $\lambda(X) = (\lambda_1, \ldots, \lambda_k)$ be the partition associated with the shape of the tableau of type X. Let $A^{\alpha,\beta}_{\lambda(X)} = (A_{ij})_{1 \le i,j \le k}$ with*

$$A_{ij} = \left(\binom{\lambda_{j+1}}{j-i+1} + \frac{1}{\beta}\binom{\lambda_{j+1}}{j-i} \right) + \sum_{p=1}^{\lambda_j - \lambda_{j+1}} \left(\frac{1}{\alpha}\right)^p \left(\binom{\lambda_{j+1}+p-1}{j-i} + \frac{1}{\beta}\binom{\lambda_{j+1}+p-1}{j-i-1} \right).$$

The stationary probability of state X is proportional to

$$\mathrm{Prob}(X) = \alpha^{k+\lambda_1}\beta^n \det A^{\alpha,\beta}_{\lambda(X)}.$$

Acknowledgements. I gratefully acknowledge Lauren Williams for suggesting the problem to me, and for numerous helpful conversations. I also acknowledge Xavier Viennot for enlightening conversations that inspired this work. Some proofs were improved after some fruitful conversations with Benjamin Young and Adrien Boussicault. Finally, I would also like to thank the anonymous referees who gave some very detailed and useful comments during the submission of this work. I was supported by the NSF grant DMS-1049513.

2.1 Introduction

The TASEP (totally asymmetric exclusion process) is a special case of the ASEP in which $q = 0$, meaning that particles only hop to the right. One could think of the TASEP as a primitive traffic model describing cars on a one-lane street, entering the street with some rate α and exiting with some rate β, and moving forward whenever there's an empty space ahead. Even in this very simple case of the TASEP, there are boundary induced phase transitions, which indicate it is still quite an interesting and complex problem.

Derrida, Evans, Hakim, and Pasquier [8] provided a Matrix Ansatz solution for the stationary distribution of the ASEP, given in Theorem 2.1.1. The Matrix Ansatz is a theorem that expresses the steady state probabilities of a process in terms of a certain matrix product.

Theorem 2.1.1 (Derrida et. al. [8]). *Let $X = \{X_1, \dots, X_n\}$ with $X_i \in \{\bullet, \circ\}$ for $1 \leq i \leq n$ represent a state of the two-species ASEP of length n. Suppose there are matrices D and E and vectors $\langle w|$ and $|v\rangle$ which satisfy the following conditions:*

$$\begin{aligned} DE &= D + E + qED \\ \langle w|E &= \frac{1}{\alpha}\langle w| \\ D|v\rangle &= \frac{1}{\beta}|v\rangle. \end{aligned}$$

If $Z_{n,r} = \langle w|(D+E)^n|v\rangle$, then the steady state probability of state X is

$$\operatorname{Prob}(X) = \frac{1}{Z_n}\langle w| \prod_{i=1}^{n} D\, \mathbb{1}_{(X_i=\bullet)} + E\, \mathbb{1}_{(X_i=\circ)}\, |v\rangle. \tag{2.1}$$

Note that in Equation (2.1), the matrix product that computes the steady state probability for state X is a product of matrices D and E in order corresponding to X where D is in the place of each $\bullet$ and E is in the place of each $\circ$.

Example. For $X = \circ \bullet \circ \bullet \circ \circ$,

$$\operatorname{Prob}(X) = \frac{1}{Z_n}\langle w|EDEDEE|v\rangle.$$

The Matrix Ansatz does not imply existence or uniqueness of matrices D and E and vectors $\langle w|$ and $|v\rangle$. Derrida et. al. provided matrices corresponding to the ASEP with the parameters α, β, and q in the form of infinite matrices whose entries are polynomials in α, β, and q. Such matrices are not unique. Furthermore, a very similar Matrix Ansatz holds even for a more general case of the ASEP with parameters $\alpha, \beta, \delta, \gamma$, and q where δ and γ denote the rates of particles *entering from the right* and *exiting from the right*, respectively. However, the matrices D and E that satisfy the conditions of this more general Matrix Ansatz are extremely complicated.

Even though the Matrix Ansatz does give an exact solution for the probabilities of the ASEP, this solution is not considered combinatorial. To explore the combinatorics of the ASEP, we introduce the alternative tableaux, which arose from the work of X. Viennot building upon the work of Corteel and Williams. The alternative tableaux are a vital object for this thesis since the rhombic alternative tableaux described in Chapter 3 build upon them.

First we give a preliminary definition of a Young diagram.

Definition 2.1.2. A *Young diagram* is a collection of boxes arranged in left-justified rows, with the row lengths weakly decreasing. The shape of a Young diagram is identified with a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ where row i has λ_i boxes for each i.

Our convention is to have a Young diagram of k rows and shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ be contained in the top left corner of a box of size $m \times k$, where $m \geq \lambda_1$. We identify the southeast boundary of the Young diagram with the lattice path that coincides with this boundary from the top-right corner to the bottom-left corner of the $m \times k$ box.

Finally we define the alternative tableaux.

Definition 2.1.3. Let $X \in \{\bullet, \circ\}^n$ be a word denoting a state of the ASEP of size n with k $\bullet$'s. We associate to X a Young diagram $Y(X)$ contained in a box of size $n - k \times k$. An *alternative tableau* of *type* X is a filling of $Y(X)$ with α's, β's, and q's according to the following rules:

i. Every box above and in the same column as an α must be empty.

ii. Every box left and in the same row as a β must be empty.

iii. Every box without an α below it or a β to its right must contain an α, β, or q.

Definition 2.1.4. The *type* of an alternative tableau T is the word X in $\{\bullet, \circ\}$ that corresponds to the shape of T, and is denoted by $\text{type}(T)$. The notation $\text{shape}(T)$ denotes the partition λ that describes the shape of the Young diagram associated to T. If X has length n and k $\bullet$'s, we say the *size* of T is (n, k), denoted by $\text{size}(T)$. In some cases, we say simply $\text{size}(T) = n$.

Note that a tableau of size (n, k) is contained within a box of size $n - k \times k$, and so it has a total of k rows (some of which may contain 0 boxes).

Definition 2.1.5. The *weight* of an alternative tableau T of size (n, k) is the product of the symbols contained in its filling times $\alpha^k \beta^{n-k}$. The weight is denoted by $\mathrm{wt}(T)$.

Note that for a tableau of size (n, k), the factor $\alpha^k \beta^{n-k}$ is considered the weight of the boundary. Figure 2.2 shows an example of an alternative tableau of size $(12, 4)$ of type ○ ○ ● ○ ○ ● ● ○ ○ ● ○ ○ and weight $(\alpha^4\beta^8)\alpha^4\beta^2q^4$.

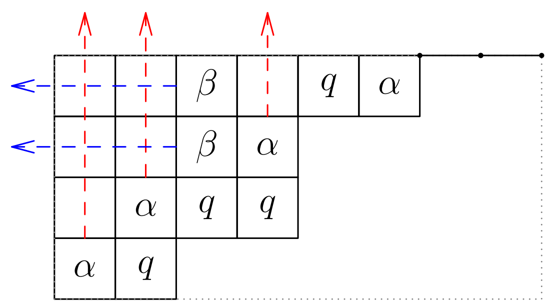


Figure 2.2: An alternative tableau of type ○ ○ ● ○ ○ ● ● ○ ○ ● ○ ○, size $(12, 4)$, and weight $(\alpha^4\beta^8)\alpha^4\beta^2q^4$. The red arrows denote boxes that are forced to be empty by an α below, and the blue arrows denote boxes that are forced to be empty by a β to the right. The dotted lines indicate the dimension of the 8×4 box that contains this tableau.

The Theorem below states the beautiful result of Corteel and Williams allows us to interpret the probabilities of the ASEP in terms of the weight generating function of the alternative tableaux.

Theorem 2.1.6 (Corteel, Williams [7]). *Let X be a state of the ASEP of size n. Let*

$$Z_n = \sum_{T:\ \mathrm{size}(T)=n} \mathrm{wt}(T)$$

be the sum of the weights of all tableaux of size n. The steady state probability of state X is

$$\mathrm{Prob}(X) = \frac{1}{Z_n} \sum_{T:\ \mathrm{type}(T)=X} \mathrm{wt}(T).$$

In this chapter, we work with a specialization of the alternative tableaux where $q = 0$, that correspond to the TASEP. Such alternative tableaux have nonzero weight if and only if they contain 0 q's. These tableaux are sometimes called *Catalan tableaux*. This is because there are C_{n+1} such tableaux corresponding to states of the TASEP of size n, where $C_n = \binom{2n}{n}\frac{1}{n+1}$ denotes the n'th Catalan number due to Steingrimsson and Williams [17].

The main result is an explicit determinantal formula for the steady state probabilities of the states of the TASEP, which we state in the following Theorem.

Theorem 2.1.7 (M. [11])**.** *Let X be a word in $\{\bullet,\circ\}^n$ with k $\bullet$'s representing a state of the TASEP of length n with exactly k particles. Let $\lambda(X) = (\lambda_1, \ldots, \lambda_k)$ be the partition associated with the shape of the tableau of type X. Let $A^{\alpha,\beta}_{\lambda(X)} = (A_{ij})_{1\le i,j\le k}$ with*

$$A_{ij} = \left(\binom{\lambda_{j+1}}{j-i+1} + \frac{1}{\beta}\binom{\lambda_{j+1}}{j-i}\right) + \sum_{p=1}^{\lambda_j-\lambda_{j+1}} \left(\frac{1}{\alpha}\right)^p \left(\binom{\lambda_{j+1}+p-1}{j-i} + \frac{1}{\beta}\binom{\lambda_{j+1}+p-1}{j-i-1}\right). \tag{2.2}$$

Then the stationary probability of state X is proportional to

$$\mathrm{Prob}(X) = \alpha^{k+\lambda_1}\beta^n \det A^{\alpha,\beta}_{\lambda(X)}.$$

In this chapter, we present a bijective proof for Formula (2.2) of Theorem 2.1.7 using the Lindström-Gessel-Viennot Lemma.

In Section 2.2 of this chapter, we define the bijection from Catalan tableaux to weighted paths which is central to our main results. In Section 2.3 we describe a bijection from weighted paths on a Young diagram to disjoint weighted paths, which gives the desired determinantal formula in terms of α, β when combined with the Lindström-Gessel-Viennot Lemma. Finally, Section 2.4 contains a formula for the number of Catalan tableaux of size (n, k) for fixed n and k, and the related corollaries.

We obtain the following definition by setting $q = 0$ in Definition 2.1.3.

Definition 2.1.8. Let $X \in \{\bullet,\circ\}^n$ be a word denoting a state of the TASEP of size n with k $\bullet$'s. We associate to X a Young diagram $Y(X)$ contained in a box of size $n-k \times k$. A *Catalan tableau* of *type* X is a filling of $Y(X)$ with α's and β's according to the following rules:

i. Every box above and in the same column as an α must be empty.

ii. Every box left and in the same row as a β must be empty.

iii. Every box without an α below it or a β to its right must contain an α or a β.

Note that item (iii.) is the only difference from Definition 2.1.3.

Definition 2.1.9. Let T have size (n, k). We associate to T a lattice path $L = L(T)$ with steps south and west, which starts at the northeast corner of the $n-k\times k$ rectangle containing T and ends at the southwest corner, and follows the southeast border of shape λ.

The definitions of size, weight, type, and shape pertaining to a Catalan tableau T are the same as for the alternative tableaux. Note that the type of T can also be obtained by reading L from northeast to southwest and assigning a $\bullet$ to a south-step and a $\circ$ to a west-step.

Definition 2.1.10. A row of a Catalan tableau is called *β-free* if it contains no β's in the filling of its boxes (or if it contains no boxes). A column of a Catalan tableau is called *α-free* if it contains no α's in the filling of its boxes (or if it contains no boxes). We can also simply call such rows and columns *free rows* and *free columns*. Conversely, if a row contains a β, this row is called β-*indexed*, and if a column contains an α, this column is called α-*indexed*.

Lemma 2.1.11. *The weight of a Catalan tableau T of size (n, k) is*

$$\mathrm{wt}(T) = (\alpha\beta)^n \left(\frac{1}{\alpha}\right)^c \left(\frac{1}{\beta}\right)^r \tag{2.3}$$

with r the number of β-free rows and c the number of α-free columns in the filling of T.

Proof. According to Definition 2.1.5 $\mathrm{wt}(T) = \alpha^{k+j}\beta^{n-k+\ell}$ where j is the number of α's and ℓ is the number of β's in the filling of T. Since each row of T can contain at most one β in its filling and there are a total of k rows, we have $k - \ell$ is the number of β-free rows. Similarly, each column of T can contain at most one α in its filling and there is a total of $n - k$ columns, so $n - k - j$ is the number of α-free columns. Consequently, Equation (2.3) gives an equivalent definition for the weight of a tableau T as given in Definition 2.1.5. ☐

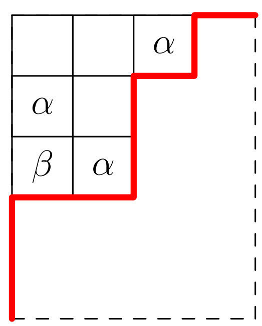
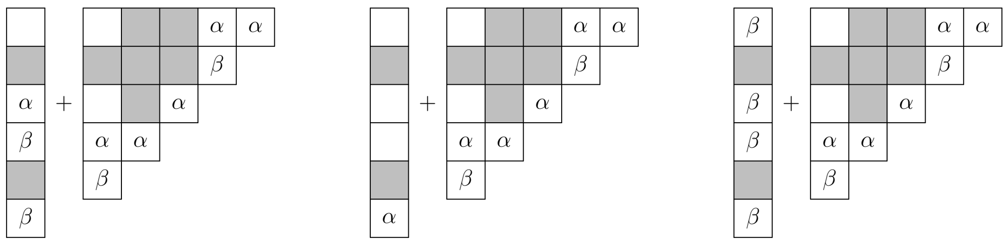


Figure 2.3: A Catalan tableau of type ○ ● ○ ● ● ○ ○ ● ●. The Catalan tableau has size $(9, 5)$, shape $\mathrm{shape}(T) = (3, 2, 2, 0, 0)$, and weight $\mathrm{wt}(T) = \alpha^8\beta^5$. The path outlined in bold on the Catalan tableau is the lattice path $L(T)$.

We give some intuition for the structure of Catalan tableaux. One way to increase the size of a Catalan tableau T from size n to size $n + 1$ is to add a new edge to the southwest corner of $L(T)$. Suppose T is contained in a $n - k \times k$ rectangle. If the new edge is a south edge, then one free row containing 0 boxes is added to the bottom of T, and there is no change to the filling of T. The size of T becomes $(n + 1, k + 1)$ and the size of the rectangle containing T increases to $n - k \times k + 1$. Figure 2.4 (a) shows the addition of a new free row to a Catalan tableau.

If the new edge is a west edge, then one column of length k is added to the left of T. The size of T becomes $(n + 1, k)$ and the size of the rectangle containing T increases to $n - k + 1 \times k$. Suppose T has r free rows. Due to (iii.) of Definition 2.1.8, the only allowed empty boxes in the new column are precisely those that lie above an α, left of a β, or both. Hence this new column must be, starting from the bottom, a (possibly empty) sequence of β's followed by an α, or just a sequence of β's, such that every free row is occupied by a β until the α is reached. Figure 2.4 (b) shows two cases for the allowed fillings of a new column added to a Catalan tableau.



(a) (b) (c)

Figure 2.4: (a) A south edge is added to the southwest corner of $L(T)$, which results in the addition of a new free row to T. (b) A west edge is added to the southwest corner of $L(T)$, which results in the addition of a new column to T. The new column we add can contain in its free rows either a (possibly empty) sequence of β's followed by an α, or a β in every free row.

To connect back to the TASEP, let X be a word of length n in the letters $\{\bullet,\circ\}$ representing a state of the TASEP. We draw a lattice path L with steps south and west by reading X from left to right, and by drawing a step south for a $\bullet$ and a step west for a $\circ$. We obtain a Young diagram Y of shape λ whose southeast border coincides with L. The size of the rectangle containing Y is $n-k \times k$, where k is the number of $\bullet$'s in X. More precisely, $\lambda = (\lambda_1, \ldots, \lambda_k)$, where λ_i the number of $\circ$'s to the right of the i'th $\bullet$. Then any filling with α's and β's of Y according to Definition 2.1.8 yields a Catalan tableau of type X, and the steady state probability $\text{Prob}(X)$ is proportional to $\sum \text{wt}(T)$ where the sum is over all Catalan tableaux T of type X. We can also refer to L, Y, and λ by $L(X)$, $Y(X)$, and $\lambda(X)$.

Remark 2.1.12. Note that when j_1 of the λ_i's of the Catalan tableau T of type X are equal to 0, this means that X ends with a a string of j_1 $\bullet$'s. Furthermore, when $(n-k)-\lambda_1 = j_2$, this means that X begins with a string of j_2 $\circ$'s. Thus keeping track of the size of the rectangle containing the Young diagram associated to T is important for preserving the weight of the Catalan tableau. We can see an example of this in Figure 2.3, where $j_1 = 2$ and $j_2 = 1$.

Remark 2.1.13. Catalan tableaux are essentially the alternative tableaux studied by Viennot in [22]. See also [23] for a closely related object. Viennot [23] states a further characterization of the steady state probabilities that is given by the enumeration of certain weighted lattice paths, which we call Catalan paths and define in the following section. A specialization of this result for the case $\alpha = \beta = 1$ is presented in [16].

2.2 From Catalan tableaux to weighted Catalan paths

In this section, we present a canonical bijection from a filling of the Catalan tableau with associated Young diagram Y to a lattice path on a Young diagram of the same shape. X. Viennot describes an analogous bijection from Catalan permutation tableaux (which are in bijection to the Catalan tableaux) to weighted lattice paths in [23]. We reformulate this bijection for the Catalan tableaux and assign the weights to the resulting lattice path in a particular way.

Weighted Catalan path

Let Y be a Young diagram contained within a $n-k \times k$ rectangle.

Definition 2.2.1. A *lattice path constrained by* Y is a path that begins in the northeast corner and ends at the southwest corner of rectangle, and takes the steps south and west in such a way that it never crosses the southeast boundary of Y.

Definition 2.2.2. A *Catalan path* C of size (n, k) with associated Young diagram Y is a lattice path constrained by Y with the following weights on its edges:

- A south edge that coincides with the east border of the rectangle receives a $\frac{1}{\beta}$.
- A south edge that does not coincide with the east border of the rectangle receives a 1.
- A west edge that coincides with the south boundary of Y receives a $\frac{1}{\alpha}$.
- A west edge that does not coincide with the south boundary of Y receives a 1.

Definition 2.2.3. The **path weight** $\mathrm{p\,wt}(C)$ of the Catalan path C is the product of the weights on its edges. We call the total weight of the Catalan path $\mathrm{wt}(C)$, with $\mathrm{wt}(C) = (\alpha\beta)^n \,\mathrm{p\,wt}(C)$.

The following Lemma describes a natural correspondence between the Catalan tableaux and the Catalan paths.

Lemma 2.2.4. *There is a weight-preserving bijection between the set of Catalan paths of size (n, k) constrained by the Young diagram Y to the set of Catalan tableaux of size (n, k) of type X such that $\lambda(X)$ is the same partition that describes Y.*

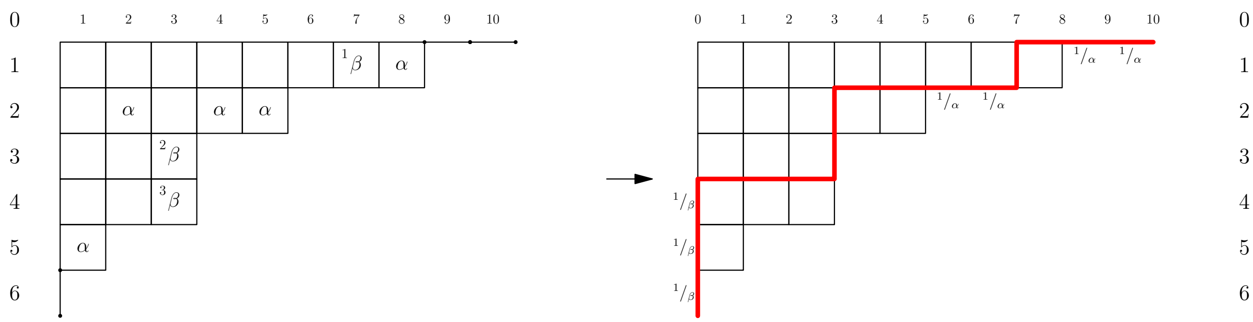


Figure 2.5: A Catalan tableau T and its corresponding weighted Catalan path C on a tableau of shape $\lambda = (8, 5, 3, 3, 1, 0)$ and weight $\mathrm{wt}(T) = \alpha^{12}\beta^{13}$. On the left, the β's are labeled such as to generate the partition $(7, 3, 3, 0, 0, 0)$ where the column containing the ith beta is the length of the ith row of the partition. This partition is precisely the shape of the path in the figure on the right. The path weight of C is $\mathrm{p\,wt}(C) = \left(\frac{1}{\beta}\right)^3 \left(\frac{1}{\alpha}\right)^4$, and so $\mathrm{wt}(C) = (\alpha\beta)^{16} \left(\frac{1}{\beta}\right)^3 \left(\frac{1}{\alpha}\right)^4 = \mathrm{wt}(T)$.

Proof. Let a Catalan path C of size (n, k) constrained by a Young diagram Y of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ be described by the partition $(C_1, \ldots, C_k)$ that is weakly smaller than λ. In other words, $C_1 \geq C_2 \cdots \geq C_k$ and $0 \leq C_i \leq \lambda_i$, where C_i is the position of the south step of the lattice path that occurs in row i of the $n - k \times k$ rectangle.

We map $(C_1, \ldots, C_k)$ to a Catalan tableau T as follows. First we label the columns of the $n - k \times k$ rectangle with 1 through $n - k$ from left to right. Then, for i in $\{1, \ldots, k\}$, if $C_i > 0$, we place a β in column C_i of Y such that it is the south-most position possible with the condition that there is at most one β per row. We now place an α in the lowest possible β-free row of every column. (Consequently, a column does not receive an α if and only if it has zero β-free rows.) It is easy to check that this construction results in a valid Catalan tableau.

Conversely, to map a Catalan tableau T to the partition $(C_1, \ldots, C_k)$, we label the β's in the filling of Y from left to right and top to bottom with $1, \ldots, \ell$ where ℓ is the number of β's, and we let C_i be the label of the column containing the i'th beta. We let $C_{\ell+1} = \cdots = C_k = 0$. In this construction, the labels on the β's decrease as the labels on the columns decrease, as in the left image of Figure 2.5, so $C_i \geq C_{i+1}$. The partition $(C_1, \ldots, C_k)$ is then directly mapped to the Catalan path P.

Now we show the weight $\mathrm{wt}(C)$ of the Catalan path C is the same as the weight $\mathrm{wt}(T)$ of the Catalan tableau T. Let $\{C_{i_1}, \ldots, C_{i_m}\}$ be the subset of $\{C_1, \ldots, C_k\}$ that represents the south steps that touch the south boundary of Y. Then the contribution of the $\left(\frac{1}{\alpha}\right)$ to the weight of the path is $\prod_{j=1}^{m} \left(\frac{1}{\alpha}\right)^{C_{i_j} - \lambda_{i_j+1}}$. This is because, for each j, if C_{i_j} touches the south boundary of Y, we know that there are zero β-free rows in the column i_j. In particular, no column of the Catalan tableau between λ_{i_j+1} and i_j can contain an α, so every west-edge of the path in those columns carries a weight of $\frac{1}{\alpha}$. It follows that both the Catalan tableau and the Catalan path have the same power of $\frac{1}{\alpha}$ contributed to their weight.

As for the factor of $\frac{1}{\beta}$, by the construction of the path, it must be $\left(\frac{1}{\beta}\right)^t$, where t is the number of C_j that equal 0. But we already know that if $C_j = 0$, it means that row j of the Catalan tableau is β-free, and so contributes a $\frac{1}{\beta}$ to the weight of the tableau. Thus $\mathrm{wt}(C) = \mathrm{wt}(T) = (\alpha\beta)^n \left(\frac{1}{\beta}\right)^t \prod_{j=1}^m \left(\frac{1}{\alpha}\right)^{C_{i_j}-\lambda_{i_j+1}}$ where $t, i_1, \ldots, i_m$ were defined in the above paragraphs. □

2.3 Weighted lattice path bijection

In this section we present a bijection from a weighted lattice path on a Young diagram of k rows to k disjoint weighted paths on a related shape.

Let D be a digraph where we assume finitely many paths between any two vertices. Let $\mathbf{e} = (e_1, \ldots, e_k)$ and $\mathbf{v} = (v_1, \ldots, v_k)$ be k-tuples of vertices of D. Let every edge of D be assigned a weight.

Definition 2.3.1. A **k-path** from $\mathbf{e}$ to $\mathbf{v}$ is a k-tuple of paths $\mathbf{P}(\mathbf{e}, \mathbf{v}) = (P_1, \ldots, P_k)$ where for some fixed $\pi \in S_k$, P_i is a path from e_i to $v_{\pi(i)}$. The k-path $\mathbf{P}$ is **disjoint** if the paths P_i are all vertex disjoint.

Definition 2.3.2. The **weight** $\mathrm{wt}(P_i)$ of a path P_i is the product of the weights on its edges. The weight $\mathrm{wt}(\mathbf{P})$ of the k-path $\mathbf{P} = (P_1, \ldots, P_k)$ is the sum of the weights of its components, in other words $\mathrm{wt}(\mathbf{P}) = \sum_{i=1}^k \mathrm{wt}(P_i)$.

Following the notation from these definitions, we provide the following well-known result of [10] (see also [18]).

Theorem 2.3.3 (Lindström, Gessel-Viennot)**.** *Let D be a digraph, and let $\boldsymbol{u} = (u_1, \ldots, u_k)$ and $\boldsymbol{y} = (y_1, \ldots, y_k)$ be k-tuples of vertices of D. Let $\mathcal{P}_{ij}$ be the set of paths from u_i to y_j. Define $w_{ij} = \sum_{p \in \mathcal{P}_{ij}} \mathrm{wt}(p)$. Then*

$$\sum_{\pi \in S_k} \sum_{\boldsymbol{P}} \mathrm{sgn}(\pi)\, \mathrm{wt}(\boldsymbol{P}) = \det\left(w_{ij}\right)_{1 \le i,j \le k}.$$

where $\boldsymbol{P}$ ranges over all disjoint k-paths $\boldsymbol{P}(\boldsymbol{u}, \pi(\boldsymbol{y}))$.

In this section, we describe a bijection from a Catalan path on a Young diagram Y to a disjoint k-path on a corresponding digraph with appropriately assigned weights on the edges. Ignoring the weights, we obtain the canonical bijection from lattice paths constrained by a Young diagram to disjoint k-paths.[1] This bijection allows us to enumerate the Catalan paths as an application of the Lindström-Gessel-Viennot Lemma.

Let C be a Catalan path of size (n, k) with associated Young diagram Y of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$. We label the vertical lines in the $n - k \times k$ rectangle from left to right with $\{0, 1, \ldots n - k\}$. Let C be described by the partition $(C_1, \ldots, C_k)$ where C_i is the label of the south-step of C in row i. Since C consists of only south- and west- steps, we necessarily have $C_1 \geq \cdots \geq C_k \geq 0$.

Now we define a twisted tableau $\tilde{Y}$ from Y as follows: for $1 \leq i \leq k$, draw a row of λ_i parallelograms consisting of east and southeast edges, and left-justify the rows as in the middle image of Figure 2.6. In each row, we label the southeast edges of the parallelograms with $0, 1, 2, \ldots$ from left to right. We put weights on the edges of the parallelograms in the following way:

- the edges with label 0 receive a $\frac{1}{\beta}$,
- otherwise if an edge in row i has label t and $t > \lambda_{i+1}$, the edge receives a $\left(\frac{1}{\alpha}\right)^{t-\lambda_{i+1}}$.

Every other edge receives a weight of 1.

We mark the left-most vertices of each row of parallelograms as the k special points $e_1, \ldots, e_k$ from top to bottom. We also mark the right-most vertices of each row of parallelograms as the k special points $v_1, \ldots, v_k$. Finally, we convert $\tilde{Y}$ into a digraph by directing all its edges from northwest to southeast. We denote by $\mathcal{P}_{ij}$ the set of weighted paths from e_i to v_j.

We map the partition $(C_1, \ldots, C_k)$ on Y to a k-path $\mathbf{P}(C) = \mathbf{P}(\mathbf{e}, \mathbf{v})$ on $\tilde{Y}$ in the following way. We write $\mathbf{P}(C) = (p_{11}, \ldots, p_{kk})$ where $p_{ii} \in \mathcal{P}_{ii}$. For each i in $\{1, \ldots, k\}$, we define p_{ii} as follows: let the single diagonal step in p_{ii} be the southeast edge in row i with label C_i. The rest of the edges in p_{ii} must necessarily be the horizontal edges that connect that diagonal step from e_i to v_i. From Figure 2.6, it is easy to see this is a one to one correspondence.

Remark 2.3.4. It is important to note that the segment of C that lies in the columns $\{\lambda_1 + 1, \ldots, n - k\}$ is ignored in the construction of $\mathbf{P}(C)$. This is permissible since any Catalan path constrained by λ must necessarily have the same such segment. Thus it suffices to simply adjust the weight of $\mathbf{P}(C)$ by the weight contribution of that segment, which is $\left(\frac{1}{\alpha}\right)^{n-k-\lambda_1}$.

Lemma 2.3.5. *Based on the construction of the k-path $\boldsymbol{P}(C)$ above, we claim that (i.) $\boldsymbol{P}(C)$ is disjoint if and only if $C_1 \geq \cdots \geq C_k$ and (ii.)* $\operatorname{p\,wt}(C) = \left(\frac{1}{\alpha}\right)^{n-k-\lambda_1} \operatorname{wt}(\boldsymbol{P}(C))$.

[1] We can treat the Catalan path and the Young diagram that contains it simply as nested lattice paths. The duality of nested lattice paths with disjoint k-paths is known in the literature and is described as the Kreweras-Narayana determinant. In particular, this duality is described in slides by Viennot [24], and the case for $\alpha = \beta = 1$ of our problem is solved therein.

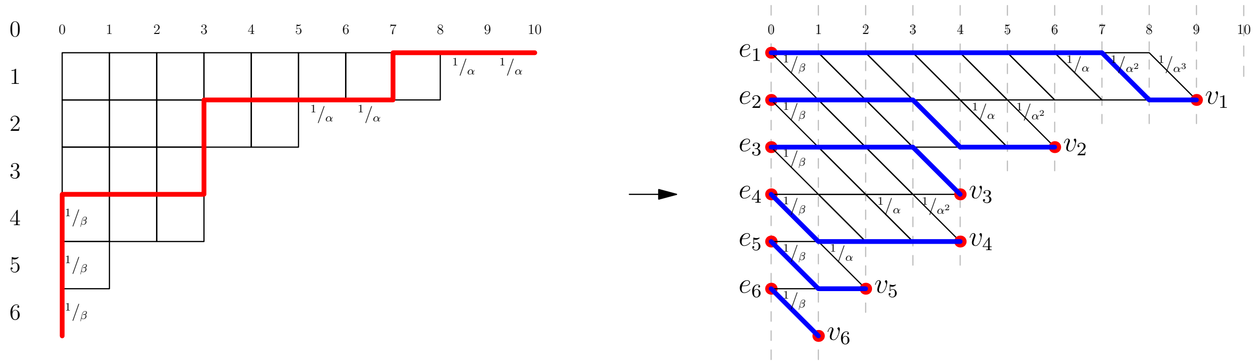


Figure 2.6: A Catalan path represented by partition $(7, 3, 3, 0, 0, 0)$ on a Young Diagram with rows $(C_1, \ldots, C_6) = (1, \ldots, 6)$ and the corresponding set of paths $\{p_{ii}\}_{1 \le i \le 6}$ where $p_{ii} \in \mathcal{P}_{ii}$ has a single diagonal step at edge labeled C_i. This Catalan path is the same one as in Figure 2.5.

Proof. [*i.*] It is easy to see from the construction that $C_i \ge C_{i+1}$ if and only if the diagonal edge in row i is strictly to the right of the diagonal edge in row $i+1$. That implies p_{ii} is strictly to the northeast of $p_{i+1\ i+1}$. Since the p_{ii}'s are nested paths, this implies $\mathbf{P}(C)$ is disjoint.

[*ii.*] We prove the equality by comparing $\mathrm{wt}(p_{ii})$ to the weight contribution of the segment of C that is in row i (including the south border of the row), and showing they are equal for each $1 \le i \le k$.

- First, if $C_i = 0$, then $\mathrm{wt}(p_{ii}) = \frac{1}{\beta}$, and also the weight contribution of row i in C is $\frac{1}{\beta}$. See rows 3-6 in the example in Figure 2.6.

- When $C_i > 0$, there is no contribution of $\frac{1}{\beta}$ to the segment of C in row i or to p_{ii}, so we consider only the contribution of $\frac{1}{\alpha}$. If $0 < C_i \le \lambda_{i+1}$, the south-step of C in row i does not touch the south boundary of Y, so there is no contribution of $\frac{1}{\alpha}$ from that segment of the path, and hence the total weight contribution is 1. Similarly, p_{ii} does not contain any edges with non-unit weight and so $\mathrm{wt}(p_{ii}) = 1$. See rows 2-3 in the example in Figure 2.6.

- If $C_i > \lambda_{i+1}$, the south-step of C in row i touches the south boundary of Y, so that segment of the path has $C_i - \lambda_{i+1}$ west-edges that coincide with the south boundary of Y and thus carry the weight $\frac{1}{\alpha}$. Thus the total contribution to the weight of the segment of C in row i is $\left(\frac{1}{\alpha}\right)^{C_i - \lambda_{i+1}}$. By the construction, p_{ii} has weight $\left(\frac{1}{\alpha}\right)^{C_i - \lambda_{i+1}}$ on its diagonal edge, and that also equals $\mathrm{wt}(p_{ii})$. See row 1 in the example in Figure 2.6.

From the above, for each i, the contribution of the weight of the segment of C in row i equals $\mathrm{wt}(p_{ii})$. By Remark 2.3.4, we have excluded from $\mathbf{P}(C)$ the contribution of the weight of the segment of C that lies to the northeast of Y. Consequently, we have $\mathrm{p\,wt}(C) = \left(\frac{1}{\alpha}\right)^{n-k-\lambda_i} \mathrm{wt}(\mathbf{P}(C))$ as desired. □

Proof of Theorem 2.1.7

We make the simple observation that a k-path $(P_i, \ldots, P_k)$ from the $\mathbf{e}$ to $\mathbf{v}$ is disjoint if and only if each path P_i is from e_i to v_i. As before, let $w_{ij} = \sum_{p \in \mathcal{P}_{ij}} \mathrm{wt}(p)$ for $\mathcal{P}_{ij}$ the collection of paths from e_i to v_j. Then from the bijection above and from Theorem 2.3.3, we obtain

$$\sum_C \mathrm{p\,wt}(C) = \left(\frac{1}{\alpha}\right)^{n-k-\lambda_i} \sum_{\mathbf{P}} \mathrm{wt}(\mathbf{P}) = \left(\frac{1}{\alpha}\right)^{n-k-\lambda_i} \det\left(w_{ij}\right)_{1 \le i,j \le k},$$

where C ranges over the Catalan tableaux constrained by Y, and $\mathbf{P}$ ranges over the disjoint k-paths from $\mathbf{e}$ to $\mathbf{v}$ on $\tilde{Y}$.

It is not difficult to check that w_{ij} for $i, j > 0$ equals precisely the entry A_{ij} from Theorem 2.1.7. We describe the calculations below.

Consider the paths from e_i to v_j that have weight generating function w_{ij}. First, if $i > j+1$, there are zero such paths since all paths can only take east and southeast steps. Next, if $i = j+1$, there is exactly one path, namely the one that takes only horizontal steps from e_i, and so the weight on that path is 1, and thus $w_{i,i-1} = 1$. Finally, assume $i \le j$. Then any path in $\mathcal{P}_{ij}$ takes $j - i + 1$ southeast steps, of which at most one step could have a weight of $\frac{1}{\beta}$, and at most one other step could have a weight of $\left(\frac{1}{\alpha}\right)^{\ell}$ for some $\ell > 0$. Thus we count four cases for paths in $\mathcal{P}_{ij}$:

1. *A path has all its steps of weight 1.* The path necessarily takes the first step east and goes to the right-most vertex of parallelogram number λ_{i+1} in the ith row. This can happen in $\binom{\lambda_{i+1}}{j-i+1}$ ways, and every such path has weight 1.

2. *A path has one step of weight $\frac{1}{\beta}$ and the rest of weight 1.* The path necessarily takes the first step southeast and goes to the right-most vertex of parallelogram number λ_{i+1} in the ith row. This can happen in $\binom{\lambda_{i+1}}{j-i+1}$ ways, and every such path has weight $\frac{1}{\beta}$.

3. *A path has one step of weight $\left(\frac{1}{\alpha}\right)^{\ell}$ and the rest of weight 1.* The path necessarily takes the first step east and goes to the right-most vertex of parallelogram number $\lambda_{i+1} + \ell - 1$ in row $i - 1$. This can happen in $\binom{\lambda_{i+1}+\ell}{j-i}$ ways, and every such path has weight $\left(\frac{1}{\alpha}\right)^{\ell}$, where $1 \le \ell \le \lambda_i - \lambda_{i+1}$.

4. *A path has one step of weight $\frac{1}{\beta}$, one step of weight $\left(\frac{1}{\alpha}\right)^{\ell}$, and the rest of weight 1.* The path necessarily takes the first step southeast and goes to the right-most vertex of parallelogram number $\lambda_{i+1} + \ell - 1$ in row $i - 1$. This can happen in $\binom{\lambda_{i+1}+\ell}{j-i-1}$ ways, and every such path has weight $\frac{1}{\beta}\left(\frac{1}{\alpha}\right)^{\ell}$, where $1 \le \ell \le \lambda_i - \lambda_{i+1}$.

We combine the above to obtain $A_\lambda = (w_{ij})_{1\le i,j\le k}$ as desired.

Finally, if C is the Catalan path corresponding to the Catalan tableau T, since $\mathrm{wt}(T) = \mathrm{wt}(C) = (\alpha\beta)^n \,\mathrm{p\,wt}(C) = \beta^n\alpha^{k+\lambda_1}\,\mathrm{wt}(\mathbf{P}(C))$, we obtain the desired formula.

Corollary 2.3.6. *The un-normalized steady state probability that the TASEP with n sites has particles in precisely the locations $1 \le x_1 < \cdots < x_k \le n$ is*

$$\mathcal{P}\left[\{x_1,\dots,x_k\}\right] = \det A_\lambda^{\alpha,\beta},$$

where $A_\lambda^{\alpha,\beta}$ is given by

$$A_{ij} = \beta^{j-i}\alpha^{i-(j+1)+x_{j+1}-x_i}\left(\binom{n-k+j+1-x_{j+1}}{j-i} + \beta\binom{n-k+j+1-x_{j+1}}{j-i+1}\right)$$
$$+\,\beta^{j-i}\alpha^{i-j+x_j-x_i}\sum_{\ell=0}^{x_{j+1}-x_j-1}\alpha^\ell\left(\binom{n-k+j-x_j-\ell-1}{j-i-1} + \beta\binom{n-k+j-x_j-\ell-1}{j-i}\right).$$

Proof. We refer to Theorem 2.1.6 to connect back to the TASEP from the Catalan tableaux. A TASEP state of length n with k particles in locations $\{x_1,\dots,x_k\}$ corresponds to a word W in $\{\bullet,\circ\}^n$ with the ith $\bullet$ in location x_i. From Definition 2.1.8, this state corresponds to Catalan tableaux of shape $\lambda(\tau) = (n-k+1-x_1, n-k+2-x_2,\dots,n-k+k-x_k)$. Equivalently, $\lambda(W) = (\lambda_1,\dots,\lambda_k)$ where λ_j is the number of holes to the right of particle j, meaning $\lambda_j = n-k+j-x_j$. Thus Theorem 2.1.7 implies the desired formula. □

2.4 Enumeration of Catalan tableaux of size (n, k)

In this section, we provide an explicit combinatorial formula for the weight generating function for Catalan tableaux of size (n,k). Let $m = n-k$, and define $N_{m,k}(\alpha,\beta)$ to be the weight generating function for Catalan tableaux of size $(m+k,k)$. In other words, the Young diagrams associated to these tableaux are contained in an $m\times k$ rectangle.

Let $N'_{m',k'}(\alpha,\beta)$ be the weight generating function for Catalan tableaux whose Young diagrams have first row equal to m' and which have precisely k' rows. In other words the Young diagram can be described by the partition $\lambda' = (\lambda'_1,\dots,\lambda'_{k'})$ where $1 \le \lambda'_{k'} \le \cdots \le \lambda'_1 = m'$. The following gives the relation between $N_{m,k}(\alpha,\beta)$ and $N'_{m',k'}(\alpha,\beta)$:

$$N_{m,k}(\alpha,\beta) = \alpha^k\beta^m\sum_{m'=0}^{m}\sum_{k'=0}^{k}\frac{1}{\alpha^{k'}\beta^{m'}}N'_{m',k'}(\alpha,\beta). \tag{2.4}$$

Here we multiplied by a factor of $\alpha^k\beta^m$ to account for the weight of the lattice path $L(T)$ that is associated with a Catalan tableau T of size $(k,k+m)$.

Enumerating all the Catalan tableaux of size $(m+k,k)$ whose Young diagrams have k nonzero rows and first row of length m is equivalent to taking the sum

$$N'_{m,k}(\alpha,\beta) = \alpha^k\beta^m\sum_{1\le\lambda_k\le\cdots\le\lambda_2\le m}\det\ A_{\{m,\lambda_2,\lambda_3,\dots,\lambda_k\}}.$$

The above gives rise to the following Lemma.

Lemma 2.4.1. *The weight generating function* $N'_{m,k}(\alpha,\beta)$ *equals*

$$\alpha^k\beta^m\sum_{\ell=0}^{k}\sum_{j=0}^{m}\alpha^j\beta^\ell\left(\binom{m+\ell-2+\delta_{jm}}{m-1}\binom{k+j-2+\delta_{\ell k}}{k-1}-\binom{m+\ell-2+\delta_{jm}}{m}\binom{k+j-2+\delta_{\ell k}}{k}\right) \tag{2.5}$$

where δ_{rs} *is the Kronecker* δ.

Summation of Formula (2.5) of Lemma 2.4.1 according to (2.4) yields the proof of the following Theorem:

Theorem 2.4.2. *The weight generating function for Catalan tableaux of size* (n,k) *with* $n=m+k$ *is*

$$N_{m,k}(\alpha,\beta)=\alpha^k\beta^m\sum_{j=0}^{m}\sum_{\ell=0}^{k}\alpha^j\beta^\ell\left(\binom{k+j-1}{j}\binom{m+\ell-1}{\ell}-\binom{k+j-1}{j-1}\binom{m+\ell-1}{\ell-1}\right). \tag{2.6}$$

Proof of Lemma 2.4.1. We prove Formula (2.5) by induction on m and k. As seen in Figure 2.7, a Young diagram with k nonzero rows and with first row of length m can be formed by the addition of a k–m hook with a row of length m and column of length k to the top and left edges of a Catalan tableau contained in a $m-1\times k-1$ rectangle.

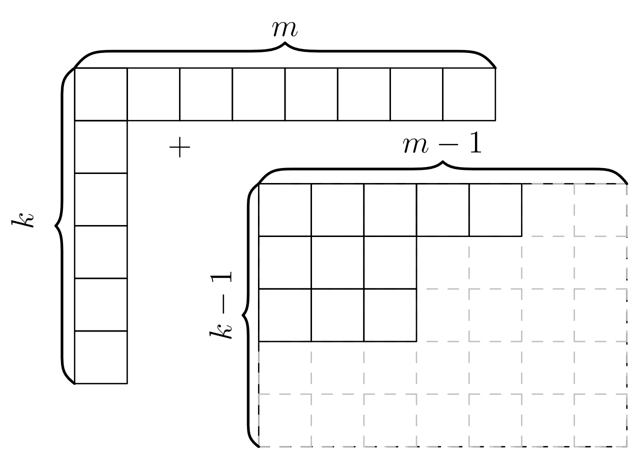
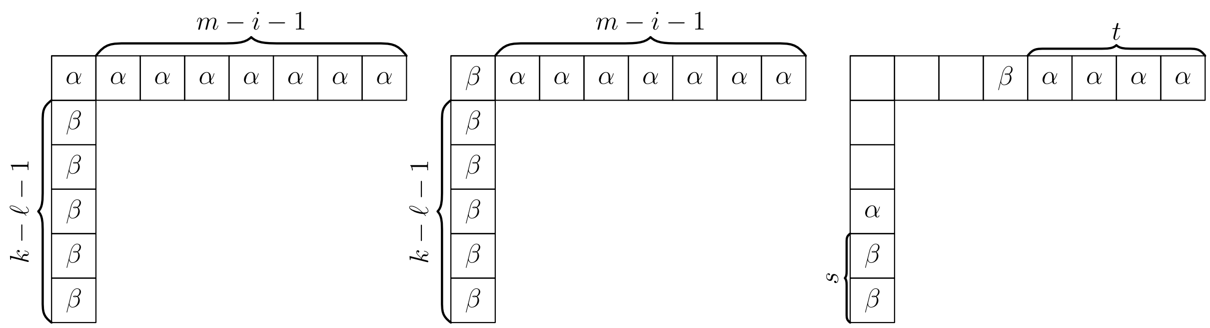


Figure 2.7: Constructing a Catalan tableau with k nonzero rows and first row of length m by adding a k–m hook to a tableau of size $(m+k-2,k-1)$.

Let $H^{m,k}_{p,q}$ be the sum of the weights of the possible fillings of the k–m hook, when the inside tableau has p rows that are α-indexed and q columns that are β-indexed. If the inside tableau has weight $\alpha^j\beta^\ell$, then it must contain ℓ β's, and so there are $k-1-\ell$ rows that are

α-indexed since there is always at most one β per row. By a similar argument, the inside tableau contains j α's, and hence then there must be $m-1-j$ columns that are β-indexed, since there is always at most one α per column. Figure 2.8 shows the cases that result in the following expression:

$$H^{m,k}_{k-1-\ell,\ m-1-j} = \alpha^{m-j} \sum_{s=0}^{k-\ell-1} \beta^s + \beta^{k-\ell} \sum_{t=0}^{m-j-1} \alpha^t + \sum_{t=1}^{m-j-1} \sum_{s=1}^{k-\ell-1} \alpha^t \beta^s. \tag{2.7}$$



(a) weight $\alpha^{m-i}\beta^{k-\ell-1}$ (b) weight $\alpha^{m-i-1}\beta^{k-\ell}$ (c) weight $\alpha^{t+1}\beta^{s+1}$ for $s = 0, 1, \ldots$ and $t = 0, 1, \ldots$

Figure 2.8: The weights for the three cases for fillings of a k–m hook with $k-1-\ell$ free rows and $m-1-j$ free columns added to a Catalan tableau of size $(m+j-2, k-1)$ and with ℓ rows that are β-indexed and j columns that are α-indexed.

Recall that if $f(\alpha, \beta)$ is a polynomial in α and β, then $[\alpha^j\beta^\ell]f(\alpha,\beta)$ denotes the coefficient of $\alpha^j\beta^\ell$ in $f(\alpha,\beta)$.

Hence for $m, k \geq 2$ we obtain the following recursion:

$$N'_{m,k}(\alpha,\beta) = \alpha^k\beta^m \sum_{j=0}^{m-1}\sum_{\ell=0}^{k-1} H^{m,k}_{k-1-\ell,\ m-1-j}\alpha^j\beta^\ell \ \left[\alpha^j\beta^\ell\right] \frac{1}{\alpha^{k-1}\beta^{m-1}} N_{m-1,k-1}(\alpha,\beta). \tag{2.8}$$

Note that the coefficient of $\alpha^j\beta^\ell$ in $\frac{1}{\alpha^{k-1}\beta^{m-1}}N_{m-1,k-1}(\alpha,\beta)$ gives the number of tableaux contained in an $m-1 \times k-1$ rectangle with j α-indexed columns and ℓ β-indexed rows. By the induction hypothesis and from (2.4) we know that to be

$$\binom{k+j-2}{j}\binom{m+\ell-2}{\ell} - \binom{k+j-2}{j-1}\binom{m+\ell-2}{\ell-1}.$$

The recursion is now straightforward to verify. On the right hand side of (2.8), we have

$$\alpha^k\beta^m \Bigg[\alpha^m \sum_{l=0}^{k-1} \beta^l \left(\binom{m+l-1}{m-1}\binom{k+m-2}{k-1} - \binom{m+l-1}{m}\binom{k+m-2}{k}\right)$$
$$+ \beta^k \sum_{j=0}^{m-1} \alpha^j \left(\binom{m+k-2}{m-1}\binom{k+j-1}{k-1} - \binom{m+k-2}{m}\binom{k+j-1}{k}\right)$$
$$+ \sum_{l=1}^{k-1}\sum_{j=1}^{m-1} \alpha^j\beta^l \left(\binom{m+l-2}{m-1}\binom{k+j-2}{k-1} - \binom{m+l-2}{m}\binom{k+j-2}{k}\right)\Bigg],$$

where we have used that $\sum_{i=0}^{a}\binom{b+i}{c} = \binom{b+a+1}{c+1} - \binom{b}{c+1}$.

This formula equals (2.5), which is the left hand side of (2.8) that we desire.

It remains to check the base cases for $N'_{m,k}(\alpha,\beta)$ when $m = 1$ or $k = 1$. If we plug $m = 1$ into (2.5), we obtain

$$N'_{1,k}(\alpha,\beta) = \alpha^k\beta\left(\beta^k + \alpha\sum_{\ell=0}^{k-1}\beta^\ell\right),$$

which is the sum of the weights of Catalan tableaux of the shape $\lambda = (1,\dots,1)$ of k rows. Similarly, plugging $k = 1$ into (2.5) yields

$$N'_{m,1}(\alpha,\beta) = \alpha\beta^m\left(\alpha^m + \beta\sum_{i=0}^{m-1}\alpha^i\right),$$

which is the sum of the weights of Catalan tableaux of the shape $\lambda = (m)$, and so the proof is complete. □

Bijective proof of Theorem 2.4.2

We can also prove Theorem 2.4.2 with a nice bijection. Our bijection is a combination of the bijection of A. Boussicault [1] from binary trees to polyomino parallelograms and the bijection of X. Viennot from Catalan tableaux to binary trees [23].

To get a binary tree on $n+1$ vertices from a Catalan tableau of size n, we do the following:

1. Add an extra row to the top border of the Young shape and put a vertex in every box whose column does not contain an α. Add an extra column to the left border of the Young shape and put a vertex in every box whose row does not contain a β. In the box in the top left corner of the resulting shape, put a vertex. This will be the root of the tree.

2. Place a vertex in each box inside the Young shape that contains α or β.

3. Connect all pairs of vertices that are in the same row with horizontal lines and all pairs of vertices in the same column with vertical lines.

The resulting object is a binary tree for the following reasons:

- due the structure of the Catalan tableau, the configuration is avoided. This is precisely the property that every vertex has at most one parent
- the vertices placed in the extra row and column above and left of the Young shape ensure that each non-root vertex has either some vertex to its left in the same row or some vertex above in the same column, and hence that each vertex has a parent.
- the grid structure is the property that every vertex can have a child to its right, a child below, neither, or both.

To get a binary tree on $n+1$ vertices from a polyomino parallelogram of semi-perimeter $n+1$, we place a vertex in every box that has a west edge or a north edge on the boundary of the polyomino. Now we connect every pair of vertices in the same row with a horizontal line, and every pair of vertices in the same column with a vertical line.

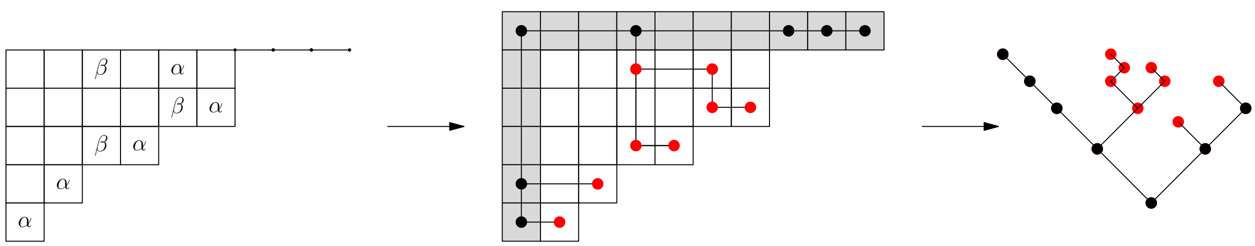


Figure 2.9: The bijection from Catalan tableaux to binary trees illustrated by an example. The red vertices are the ones that correspond to the α's and β's.

The statistics on the polyomino of semi-perimeter $n+1$ that we associate with the Catalan tableau of size n with k rows, $m = n-k$ columns, and a weight of $\alpha^j\beta^l$ in the interior are the following:

- The number of rows in the polyomino is $k+1$,
- The number of columns in the polyomino is $m+1$,
- The length of the first horizontal segment on the N border of the polyomino is $k-j$,
- The length of the first vertical segment on the W border of the polyomino is $m-l$.

Therefore, a Catalan tableau with k rows and m columns and weight $\alpha^j\beta^l$ in its filling is precisely a polyomino with $k+1$ rows and $m+1$ columns whose first horizontal segment on the north border has length $k-j$ and first vertical segment on the west border has length $m-l$. Such a polyomino is defined by two non-crossing lattice paths that start at the end

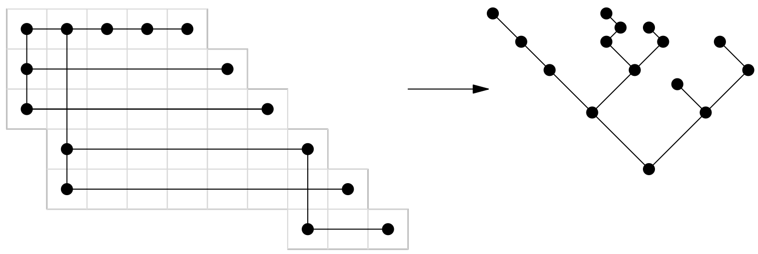


Figure 2.10: The bijection from polyomino parallelograms to binary trees illustrated by an example.

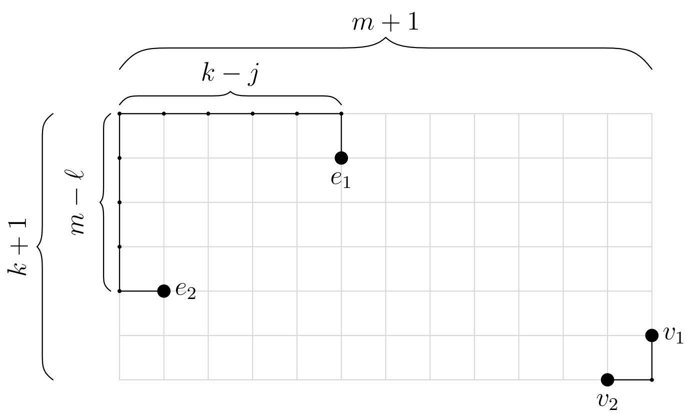


Figure 2.11: A polyomino parallelogram with the desired statistics corresponds to a pair of non crossing lattice paths from e_1 to v_1 and from e_2 to v_2.

of those fixed borders and end at the junction with the last box. In Figure 2.11, those are the lattice paths joining the points (e_1, v_1) and the points (e_2, v_2).

Let $p_{(e_i \to v_j)}$ be the total number of lattice paths from e_i to v_j. Then the number of desired pairs of non-crossing paths is given by the Lindström-Gessel-Viennot formula, which is the determinant of the matrix

$$\begin{pmatrix} p_{(e_1\to v_1)} & p_{(e_1\to v_2)} \\ p_{(e_2\to v_1)} & p_{(e_2\to v_2)} \end{pmatrix}.$$

We have:

$$p_{(e_1\to v_1)} = \binom{k+j-1}{j+1}, \qquad p_{(e_1\to v_2)} = \binom{k+j-1}{j},$$
$$p_{(e_2\to v_1)} = \binom{m+l-1}{l}, \qquad p_{(e_2\to v_2)} = \binom{m+l-1}{l+1}.$$

Combining all of the above results in the following weight generating function for the Catalan tableaux:

$$N_{m,k}(\alpha,\beta)=\alpha^k\beta^m\sum_{j=0}^{m}\sum_{\ell=0}^{k}\alpha^j\beta^\ell\left(\binom{k+j-1}{j}\binom{m+\ell-1}{\ell}-\binom{k+j-1}{j-1}\binom{m+\ell-1}{\ell-1}\right).$$

Enumerative consequences

Definition 2.4.3. Let $Z_n(\alpha,\beta)=\sum_{k=0}^{n}N_{n-k,k}(\alpha,\beta)$ be the weight generating function for the Catalan tableaux of size n, or equivalently, all Catalan tableaux that fit in a rectangle of semi-perimeter n.

Remark 2.4.4. Derrida provides the following formula in [8]:

$$Z_n(\alpha,\beta)=\alpha^n\beta^n\sum_{p=1}^{n}\frac{p}{2n-p}\binom{2n-p}{n}\frac{\alpha^{-p-1}-\beta^{-p-1}}{\alpha^{-1}-\beta^{-1}}. \tag{2.9}$$

This expression normalizes the previously derived stationary probabilities of the TASEP, as we see below in Corollary 2.4.5.

Derrida's formula can be derived from (2.4) as follows:

$$\begin{aligned}
[\alpha^{n-t}\beta^{n+t-s}]\sum_{k=0}^{n}N_{n-k,k} &= \sum_{k=0}^{n}[\alpha^{n-t}\beta^{n+t-s}]\sum_{j=0}^{n-k}\sum_{\ell=0}^{k}\alpha^{j+k}\beta^{\ell+n-k}\left(\binom{k+j-1}{k-1}\binom{n-k+\ell-1}{n-k-1}\right.\\
&\qquad\left.-\binom{k+j-1}{k}\binom{n-k+\ell-1}{n-k}\right)\\
&= \sum_{k=0}^{n}\left(\binom{k+(n-k-t)-1}{k-1}\binom{n-k+(n+t-s-n+k)-1}{n-k-1}\right.\\
&\qquad\left.-\binom{k+(n-k-t)-1}{k}\binom{n-k+(n+t-s-n+k)-1}{n-k}\right)\\
&= \frac{s}{2n-s}\binom{2n-s}{n}.
\end{aligned} \tag{2.10}$$

where in the third step the Vandermonde convolution is used.

Since (2.10) is independent of t, we obtain (2.9) by summing over k.

$$\begin{aligned}
Z_n(\alpha,\beta)=\sum_{k=0}^{n}N_{n-k,k} &= \sum_{s=1}^{n}\frac{s}{2n-s}\binom{2n-s}{n}\sum_{t=0}^{s}\alpha^{n-t}\beta^{n+t-s}\\
&= \sum_{s=1}^{n}\frac{s}{2n-s}\binom{2n-s}{n}\alpha^n\beta^n\frac{\alpha^{-s-1}-\beta^{-s-1}}{\alpha^{-1}-\beta^{-1}},
\end{aligned}$$

which matches (2.9), as desired.

Corollary 2.4.5. *The stationary probability of a TASEP of length n and containing exactly k particles is $N_{n-k,k}(\alpha,\beta)$ of (2.4), normalized by $Z_n(\alpha,\beta)$ from (2.9).*

Chapter 3

Combinatorics of the 2-species ASEP

The ASEP has been generalized to allow multiple "species" of interacting particles. In these processes, some priority rules permit adjacent particles of different species to swap with each other. For some of these multi-species processes, interesting combinatorial structures have been discovered. In this chapter, we consider a simple two-species ASEP with three parameters α, β and q, which are inherited from the ordinary ASEP (see [19, 2, 9]), with parameters shown in Figure 3.1.

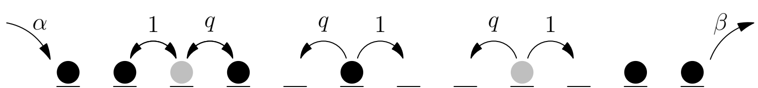


Figure 3.1: The parameters α, β, and q of the two-species ASEP. "Heavy" particles are denoted by ● and "light" particles are denoted by ●.

The two-species ASEP we study has two species of particles, one *heavy* and one *light*, hopping right and left on a one-dimensional lattice of length n with open boundaries. We consider the hole to be a third type of "particle" of weight 0. Then the hopping of particles to adjacent locations is equivalent to swapping two adjacent particles of different species. We denote the heavy particle by ●, the light particle by ●, and the hole by ○. The heavy particle can enter the lattice on the left with rate α,and exit the lattice on the right with rate β. Moreover, the heavy particle can swap places with both the hole and the light particle when they are adjacent, and similarly the light particle can swap places with the hole when they are adjacent. Each of these possible swaps occur at rate 1 when the heavier particle is to the left of the lighter one, and at rate q when the heavier particle is to the right. Since only the heavy particle can enter or exit the lattice, the number of light particles must stay fixed. Let r be the parameter representing the number of light particles. Note that when $r = 0$, we recover the original ASEP.

More precisely, the two-species ASEP of size n with r light particles is a Markov chain on $2^{n-r}\binom{n}{r}$ states, which are words in $\{\bullet, {\color{gray}\bullet}, \circ\}$ of length n and with exactly r ${\color{gray}\bullet}$'s. Let X, Y be some words in $\{\bullet, {\color{gray}\bullet}, \circ\}$. The transitions of the two-species ASEP are:

$$X{\color{gray}\bullet}\circ Y \underset{q}{\overset{1}{\rightleftharpoons}} X\circ{\color{gray}\bullet} Y \qquad X\bullet\circ Y \underset{q}{\overset{1}{\rightleftharpoons}} X\circ\bullet Y \qquad X\bullet{\color{gray}\bullet} Y \underset{q}{\overset{1}{\rightleftharpoons}} X{\color{gray}\bullet}\bullet Y$$

$$\circ X \overset{\alpha}{\rightharpoonup} \bullet X \qquad\qquad X\bullet \overset{\beta}{\rightharpoonup} X\circ$$

where by $X \overset{u}{\rightharpoonup} Y$ we mean that the transition from X to Y has probability $\frac{u}{n+1}$, n being the length of X (and also Y).

Uchiyama provided an extended Matrix Ansatz to express the stationary probabilities of the two-species ASEP as certain matrix products. Furthermore, Uchiyama provided matrices that satisfy the conditions of the Ansatz, thus giving a formula to compute the steady state probabilities.

Theorem 3.0.1 (Uchiyama [19]). *Let $W = \{W_1, \dots, W_n\}$ with $W_i \in \{\bullet, {\color{gray}\bullet}, \circ\}$ for $1 \le i \le n$ represent a state of the two-species ASEP of length n with r ${\color{gray}\bullet}$'s. Suppose there are matrices D, E, and A and vectors $\langle w|$ and $|v\rangle$ which satisfy the following conditions:*

$$DE = D + E + qED \qquad DA = A + qAD \qquad AE = A + qEA$$

$$\langle w|E = \frac{1}{\alpha}\langle w| \qquad D|v\rangle = \frac{1}{\beta}|v\rangle.$$

Then

$$\mathrm{Prob}(W) = \frac{1}{Z_{n,r}}\langle w|\prod_{i=1}^{n} D\,\mathbb{1}_{(W_i=\bullet)} + A\,\mathbb{1}_{(W_i={\color{gray}\bullet})} + E\,\mathbb{1}_{(W_i=\circ)}\,|v\rangle$$

where $Z_{n,r}$ is the coefficient of y^r in $\frac{\langle w|(D+yA+E)^n|v\rangle}{\langle w|A^r|v\rangle}$.

Theorem 3.0.1 specializes to Theorem 2.1.1 at $r = 0$.

Inspired by Uchiyama's Matrix Ansatz, the author of this thesis studied the case of the two-species ASEP for $q = 0$ in [13], and introduced an object called the "multi-Catalan tableaux" that gives an interpretation for the steady state probabilities of the two-species ASEP at $q = 0$. In this chapter, which is based on joint work with X. Viennot, the result is generalized for all q with a new object called the *rhombic alternative tableaux* (RAT). These tableaux are defined in Section 3.1, but we state our main theorem below.

Theorem 3.0.2. *Let W be a state of the two-species ASEP of size n with exactly r light particles. Then the stationary probability of state W is*

$$\mathrm{Prob}(W) = \frac{1}{\mathcal{Z}_{n,r}}\sum_{T}\mathrm{wt}(T)$$

where T ranges over the rhombic alternative tableaux corresponding to W, $\mathrm{wt}(T)$ is the weight of such a tableau, and $\mathcal{Z}_{n,r}$ is the weight generating function for the set of rhombic alternative tableaux corresponding to the state space of W.

In Section 3.1 of this chapter, we introduce the rhombic alternative tableaux, and in Section 3.2 we prove Theorem 3.0.2. In Section 3.3 we provide some enumerative results for the two-species ASEP. Finally, in Section 3.4, we describe a Markov chain on the rhombic alternative tableaux that projects to the two-species ASEP, which gives an alternate proof of Theorem 3.0.2.

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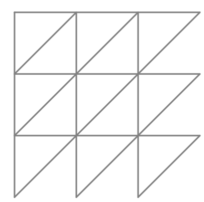
3.1 Rhombic alternative tableaux

The rhombic alternative tableaux (RAT) are an analog on a "triangular lattice" of the alternative tableaux [22] that correspond to the ordinary ASEP. By triangular lattice, we mean one which has as its vertices the integer points (i, j), and the possible edges are the south edges with vertices $\{(i, j), (i, j-1)\}$, west edges with vertices $\{(i, j), (i-1, j)\}$, and southwest edges with vertices $\{(i, j), (i-1, j-1)\}$ for integers i, j, as in the figure on the right.



Definition of the RAT

Definition 3.1.1. Let W be a word in the letters $\{\bullet, \textcolor{gray}{\bullet}, \circ\}$ with k $\bullet$'s, ℓ $\circ$'s, and r $\textcolor{gray}{\bullet}$'s of total length $n := k + \ell + r$. Define P_1 to be the path obtained by reading W from left to right and drawing a south edge for a $\bullet$, a west edge for an $\circ$, and a southwest edge for an $\textcolor{gray}{\bullet}$. From here on, we call any south edge a D-edge, any west edge an E-edge, and any southwest edge an A-edge. Define P_2 to be the path obtained by drawing ℓ west edges followed by r southwest edges, followed by k south edges. A *rhombic diagram* $\Gamma(W)$ of *type* W is a closed shape on the triangular lattice that is identified with the region obtained by joining the northeast and southwest endpoints of the paths P_1 and P_2 as in Figure 3.2.

Definition 3.1.2. A *tiling* $\mathcal{T}$ of a rhombic diagram is a collection of open regions of the following three parallelogram shapes as seen in Figure 3.3, the closure of which covers the diagram:

- A parallelogram with south and west edges which we call a DE tile.
- A parallelogram with southwest and west edges which we call an AE tile.
- A parallelogram with south and southwest edges which we call a DA tile.

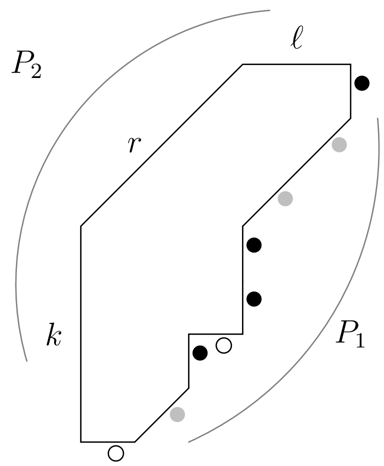


Figure 3.2: $\Gamma(W)$ and the two paths P_1 and P_2 for $W = $ ● ◍ ◍ ● ● ○ ● ◍ ○ with $\ell = 2$, $r = 3$, and $k = 4$.

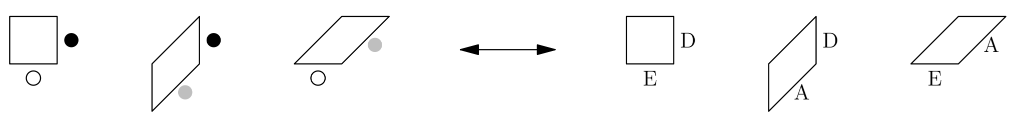


Figure 3.3: The tiles DE, DA, and AE.

We define the *area* of a tiling to be the total number of tiles it contains.

Lemma 3.1.3. *For each word W in* {●, ◍, ○}*, there exists a tiling of* $\Gamma(W)$.

Proof. We prove the above by induction on the area of $\Gamma(W)$. Let W be a word with k ●'s, ℓ ○'s, and r ◍'s of length $n = k + \ell + r$. First, if W contains no instances of a consecutive pair ● ○, ● ◍, or ◍ ○, then $W = $ ○$^\ell$◍^{r}●^{k}. Then the southeast boundary P_1 of $\Gamma(W)$ is identical to its northwest boundary P_2, so the area of the convex region is 0. Thus a tiling trivially exists.

Now suppose $\Gamma(W)$ has nonzero area m, and we make the hypothesis that any triangular region with area at most $m - 1$ has a tiling. By the above, W necessarily contains some instance of ● ○, ● ◍, or ◍ ○. Let X and Y be the {●, ◍, ○} subwords of W that occur respectively before and after that instance. In other words, $W = X *_1 *_2 Y$ for $*_1 *_2$ equal to ● ○, ● ◍, or ◍ ○. For each of these cases, we perform the following operation:

Let $W = X *_1 *_2 Y$. Then we place a $*_1 *_2$ tile adjacent to the $*_1$–$*_2$ edges of $\Gamma(W)$. Since the numbers of ●'s, ◍'s, and E's in the word $W' = X *_2 *_1 Y$ is equal to those of W, the northwest boundaries $P_2(W)$ and $P_2(W')$ of $\Gamma(W)$ and $\Gamma(W')$ are equal. Thus the region remaining after placing the tile $*_1 *_2$ is equivalent to the rhombic diagram $\Gamma(X *_2 *_1 Y)$. The area of $\Gamma(X *_2 *_1 Y)$ is $m - 1$, and therefore has a tiling by the inductive hypothesis.

Thus there exists a tiling for any $\Gamma(W)$. □

By convention, we label the E-edges of the southeast boundary of the rhombic diagram with 1 through ℓ from right to left, and the D-edges with 1 through k from top to bottom.

Definition 3.1.4. A *north-strip* on a rhombic diagram with a tiling is a maximal strip of adjacent tiles of types DE or AE, where the edge of adjacency is always an E-edge. A *west-strip* is a maximal strip of adjacent tiles of types DE or DA, where the edge of adjacency is always a D-edge. The i'th north-strip is the north-strip whose bottom-most edge is the i'th (from right to left) E-edge on the boundary of the rhombic diagram. The j'th west-strip is the west-strip whose right-most edge is the j'th (from top to bottom) D-edge on the boundary of the rhombic diagram. Figure 3.4 shows an example of the west- and north-strips.

Note that the number of tiles in the i'th north-strip is the total number of ●'s and ●'s in the word W preceding the i'th ○. Similarly, the number of tiles in the j'th west-strip is the total number of ○'s and ●'s in the word W following the j'th ●.

Example. For the tableau of type ● ● ● ● ● ○ ● ● ○ in Figure 3.4, the ●'s (from top to bottom) have 5, 3, 3, and 2 ○'s and ●'s to their right, which corresponds to the west-strips having lengths 5, 3, 3, and 2 from top to bottom. Similarly, the ○'s (from right to left) have 5 and 7 ●'s and ●'s to their right, which corresponds to the north-strips having lengths 5 and 7 from right to left.

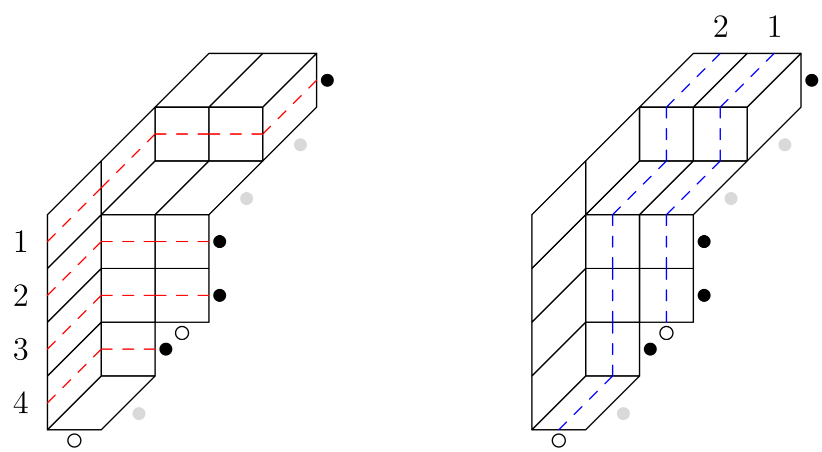


Figure 3.4: *(Left)* west-strips and *(right)* north-strips.

Finally we define the rhombic alternative tableaux, with an example of one shown in Figure 3.5.

Definition 3.1.5. A *rhombic alternative tableau* (RAT) of *type W* is a rhombic diagram $\Gamma(W)$ and an arbitrary tiling $\mathcal{T}$ with DE, DA, and AE tiles, and a filling F of $\mathcal{T}$ with α's and β's under the following conditions:

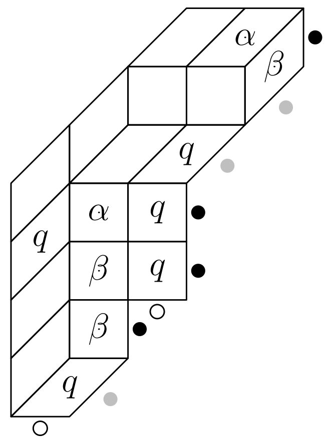


Figure 3.5: An example of a RAT of size $(9, 3, 4)$ with type ● ◍ ◍ ◍ ● ● ○ ● ◍ ○ and weight $\alpha^6\beta^5q^4$.

i. A DE tile is empty or contains an α or a β.

ii. A DA tile is empty or contains a β.

iii. An AE tile is empty or contains an α.

iv. Any tile above and in the same north-strip as an α must be empty.

v. Any tile to the left and in the same west-strip as a β must be empty.

We define $\mathrm{fi}(W, \mathcal{T})$ to be the set of fillings of tiling $\mathcal{T}$ of the rhombic diagram $\Gamma(W)$. In other words, $F \in \mathrm{fi}(W, \mathcal{T})$ means F is a filling of type W of the tiling $\mathcal{T}$.

Definition 3.1.6. A *north line* is a line drawn through each north-strip containing an α, starting at the tile directly above that α. A *west line* is a line drawn through each west-strip containing a β, starting at the tile directly left of that β. An example of the RAT with the north- and west lines is shown in Figure 3.6.

In terms of the north- and west lines, we rewrite the conditions (iv) and (v) of Definition 3.1.5 by (equivalently) requiring that any tile that contains a north line or a west line must be empty.

Definition 3.1.7. The *size* of a RAT of type W is (n, r, k), where k is the number of ●'s in W, r is the number of ◍'s in W, and n is the total number of letters in W. We can also call this the size of a filling F of type W. We can also refer to the size of a tableau as simply (n, r), where we do not keep track of the number of ●'s.

Definition 3.1.8. To compute the *weight* $\mathrm{wt}(F)$ of a filling F, first a q is placed in every empty tile that does not contain a north line or a west line. Next, $\mathrm{wt}(F)$ is the product of all the symbols inside F times $\alpha^k\beta^\ell$, for F a filling of size $(k + \ell + r, r, k)$.

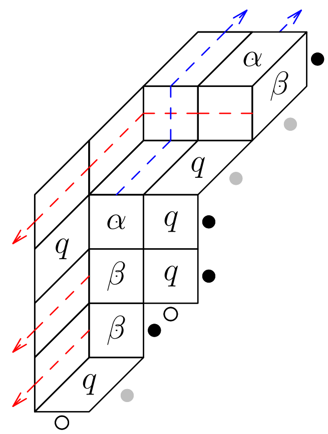


Figure 3.6: A complete representation of a RAT that is equivalent to the example on the left.

We will prove in Proposition 3.1.9 that the sum of the weights of all fillings of $\Gamma(W)$ does not depend on the tiling $\mathcal{T}$.

Independence of tilings and definition of weight(W)

Proposition 3.1.9. *Let W be a word in $\{\bullet, {\color{gray}\bullet}, \circ\}$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ represent two different tilings of a rhombic diagram $\Gamma(W)$ with DE, DA, and AE tiles. Then*

$$\sum_{F \in \mathrm{fi}(W, \mathcal{T}_1)} \mathrm{wt}(F) = \sum_{F' \in \mathrm{fi}(W, \mathcal{T}_2)} \mathrm{wt}(F').$$

Definition 3.1.10. Consider a hexagon with vertices $\{(i,j), (i,j-1), (i-1,j-2), (i-2,j-2), (i-2,j-1), (i-1,j)\}$ for some integers i, j that is tiled with a DE-, a DA-, and an AE tile. A *maximal hexagon* is when the tiles within the hexagon have the configuration of Figure 3.7 (left), and a *minimal hexagon* is when the tiles within the hexagon have the configuration of Figure 3.7 (right).

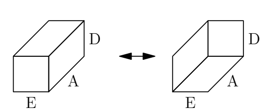


Figure 3.7: A flip from a maximal (left) to a minimal hexagon (right).

Definition 3.1.11. Let W be a word in $\{\bullet, {\color{gray}\bullet}, \circ\}$. We define the *minimal tiling* of $\Gamma(W)$ to be the tiling that that does not contain an instance of a maximal hexagon, such as the example in Figure 3.8. We refer to such a tiling by $\mathcal{T}_{min}$. (In the remark following the proof of Lemma 3.1.13, we show that $\mathcal{T}_{min}$ is the unique minimal tiling.) One can construct $\mathcal{T}_{min}$ by placing tiles from P_1 inwards, and always placing an AE tile whenever possible. In other words, all the west strips of $\mathcal{T}_{min}$ are, from right to left, a strip of adjacent DE boxes followed by a strip of adjacent DA boxes, as in Figure 3.8 (left).

Similarly, a *maximal tiling* is one that that does not contain an instance of a minimal hexagon, and is referred to by $\mathcal{T}_{max}$. The maximal tiling can be constructed by placing tiles from P_1 inwards, and always placing a DA tile whenever possible. In other words, all the north strips of $\mathcal{T}_{max}$ are, from bottom to top, a strip of adjacent DE boxes followed by a strip of adjacent AE boxes, as in Figure 3.8 (right).

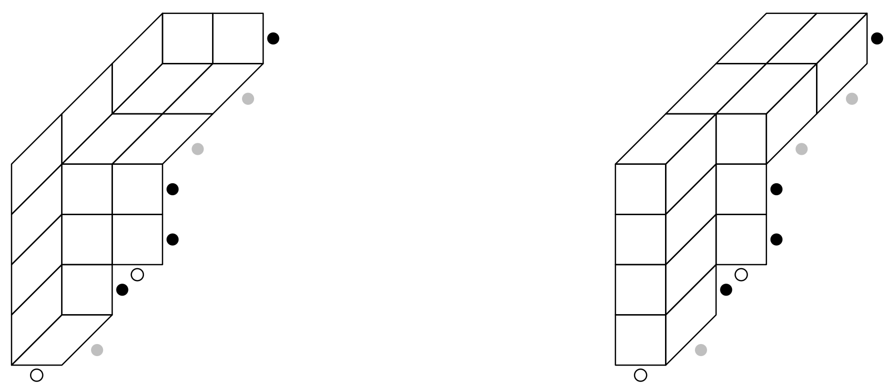


Figure 3.8: We see the minimal tiling $\mathcal{T}_{min}$ (left) and the maximal tiling $\mathcal{T}_{max}$ (right) of the rhombic diagram $\Gamma(X)$ for $X = $ ● ● ● ○ ○ ● ○ ● ○ of Figure 3.2.

Definition 3.1.12. A *flip* is an involution that switches between a maximal hexagon and a minimal hexagon, and is the particular rotation of tiles that is shown in Figure 3.7.

The lemma below contains a generally known result, notably in the case of a *plane partition*.

Lemma 3.1.13. *Let $\Gamma(W)$ be a rhombic diagram of type W. For any two tilings $\mathcal{T}$ and $\mathcal{S}$ of $\Gamma(W)$, $\mathcal{T}$ can be obtained from $\mathcal{S}$ by some series of flips.*

Remark 3.1.14. To make the paper self-contained, we will give here a proof that tilings are in bijection with configurations of non-crossing paths. In particular, our proof defines a classical construction in the case of a plane partition, where the bijection will give a configuration of paths related to a binomial determinant (by the Lindström-Gessel-Viennot Lemma). This determinant can be expressed by a simple formula giving the well-known MacMahon formula for plane partitions (or 3D Ferrers diagrams) within a box of size (a, b, c). We note that the case of a plane partition within a box of size (k, r, ℓ) is equivalent to the tilings of $\Gamma(W)$ where $W = ●^k ●^r ○^\ell$.

Definition 3.1.15. A *non-crossing configuration* $\{\tau\} = \{\tau_1, \tau_2, \ldots\}$ is a collection of lattice paths where no two paths τ_i, τ_j for $i \neq j$ share a vertex (and hence are non-crossing).

Proof of Lemma 3.1.13. We prove the lemma by constructing a bijection from certain non-crossing configurations on $\Gamma(W)$ to tilings of $\Gamma(W)$. We define a height function height$(\{\tau\})$ on the non-crossing configurations. We define the *minimal* non-crossing configuration $\{\tau\}_{min}$ to be the one with height 0. We also define a local move called a *slide* on $\{\tau\}$ which we show corresponds to a flip from a maximal to a minimal hexagon on $\mathcal{T}$. The reverse operation is called a *reverse slide*, and it corresponds to a flip from a minimal to a maximal hexagon. We show that a slide diminishes the height of a non-crossing configuration by 1, and that any non-crossing configuration with height greater than 0 admits a slide. Thus it is possible to obtain $\{\tau\}_{min}$ from any $\{\tau\}$ by some set of slides, and so it is possible to obtain any $\{\sigma\}$ from any $\{\tau\}$ by some set of slides and reverse slides. This translates to the desired result due to the bijection.

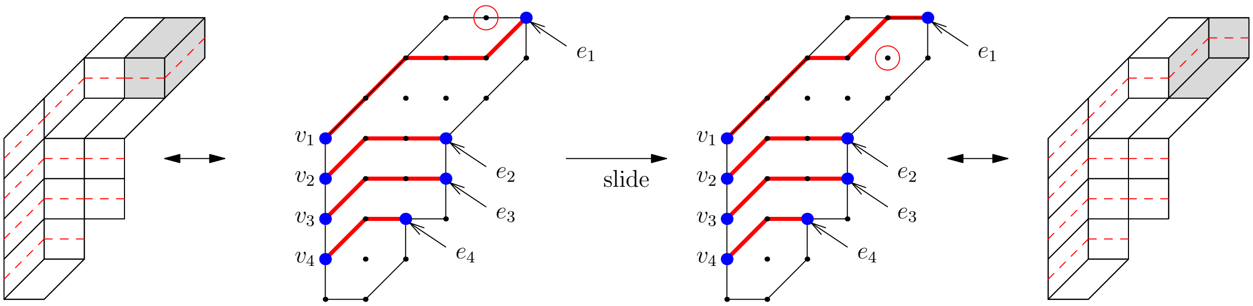


Figure 3.9: *(Left)* a tiling $\mathcal{T}$ with west-strips indicated, bijection to the non-crossing configuration $\{\tau\}$ with height$(\{\tau\}) = 2$. *(Right)* a slide performed at the indicated free lattice point to obtain the non-crossing configuration $\{\tau'\}$ with height$(\{\tau'\}) = 1$, which is in bijection with the tiling $\mathcal{T}'$ (which is obtained from $\mathcal{T}$ by performing a flip at the hexagon that is shaded grey).

Bijection from tilings on $\Gamma(W)$ *to non-crossing configurations on* $\Gamma(W)$. Recall that $\Gamma(W)$ is on a lattice with integer points $\{(i,j)\}$, and a tiling on $\Gamma(W)$ consists of tiles DE, DA, and AE whose corners are on those lattice points. We define a bijection from a tiling $\mathcal{T}$ to a non-crossing configuration $\{\tau\}$ on the lattice contained within $\Gamma(W)$. Let $\{\tau\} = \{\tau_i, \ldots, \tau_k\}$, where each τ_i is a lattice path consisting of west and southwest steps $(j,\ell) \to (j-1,\ell)$ and $(j,\ell) \to (j-1,\ell-1)$ respectively, that starts at e_i and ends at v_i for some sets $\{e_i\}$ and $\{v_i\}$. We let e_i be the north endpoint of the i'th D-edge on the southeast boundary of $\Gamma(W)$ (from top to bottom), and let v_i be the north endpoints of the i'th D-edge on the northwest boundary of $\Gamma(W)$ (from top to bottom).

To obtain $\{\tau\}$ from $\mathcal{T}$, let τ_i be the path that coincides with the northwest boundary of the i'th west-strip (from top to bottom). Since the west-strips do not cross each other, it is clear that $\{\tau\}$ is well defined in this way (see Figure 3.9).

Define a *free lattice point* to be a lattice point that is not part of any path in $\{\tau\}$, and does not lie on the southeast boundary of $\Gamma(W)$. To obtain $\mathcal{T}$ from $\{\tau\}$, we do the following: for each τ_i, place a DE tile directly below and adjacent to each west step of τ_i, and place a DA tile directly below and adjacent to each southwest step, so that the northwest boundaries of the DE and DA tiles coincide with the steps of τ_i. Thus τ_i corresponds to a west-strip. Now, for every free lattice point that is not on the southeast boundary of $\Gamma(W)$, place an AE tile so that its northwest corner coincides with that lattice point. We claim that we obtain in this way a valid tiling of $\Gamma(W)$. To check this, we must simply verify that the construction above results in no tiles overlapping, and a complete covering of the shape. This is easily verified inductively on the number of tiles in $\Gamma(W)$ by combining Lemma 3.1.3 with this bijection.

Height of $\{\tau\}$*.* For each i, define the *minimal path* m_i to be the one starting at e_i and taking a maximal possible number of west steps followed by a maximal possible number of southwest steps to v_i. Define the *height* of $\tau_i \in \{\tau\}$ (i.e. height(τ_i)) to be the area between τ_i and m_i (i.e. the number of free lattice points strictly northwest of τ_i and weakly southeast of m_i). We define the height of $\{\tau\}$ to be

$$\text{height}(\{\tau\}) = \sum_{\tau_i \in \{\tau\}} \text{height}(\tau_i).$$

It is clear that there is a unique $\{\tau\}$ of height 0, by letting each τ_i be m_i. We call this $\{\tau_{min}\}$.

A slide on $\{\tau\}$*.* Let p be a free lattice point (i,j) such that $(i+1,j)$, $(i,j-1)$, and $(i-1,j-1)$ are all not free lattice points. Then, those lattice points must necessarily belong to the same path in the non-crossing configuration $\{\tau\}$, say τ_i. A *slide* on $\{\tau\}$ at the location of p means exchanging the steps southwest and west for the steps west and southwest in τ_i, and thereby passing through p and creating a new free lattice point below τ_i, as in Figure 3.9. More precisely, the steps $(i+1,j) \to (i,j-1) \to (i-1,j-1)$ in τ_i are exchanged for $(i+1,j) \to (i,j) \to (i-1,j-1)$ to make τ_i'. Clearly τ_i' does not cross $\{\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots\}$, and so the new collection of paths $\{\tau'\}$ formed by replacing τ_i with τ_i' is also a non-crossing configuration. Furthermore, since height$(\tau_i') =$ height$(\tau_i) - 1$, we have height$(\{\tau'\}) =$ height$(\{\tau\}) - 1$.

We have already established that a southwest step of τ_i in $\{\tau\}$ corresponds to a DA box in the west-strip i, and a west step of τ_i corresponds to a DE box. A free lattice point necessarily corresponds to an AE tile, since all DA and DE tiles must be part of some west-strips. From Figure 3.9, it is easy to see how a slide corresponds to a flip from a maximal hexagon to a minimal hexagon, and a reverse slide (the inverse operation) corresponds to the reverse flip.

Notice that if no such free lattice point p exists such that its three neighbors east, south, and southwest are all not free lattice points, this implies $\tau_i = m_i$ for each i. Then $\{\tau\}$ is the minimal non-crossing configuration, as in Figure 3.10. Thus $\{\tau\}$ admits no slides if and only if it equals $\{\tau_{min}\}$.

Now we complete the proof. Let $\{\tau\}$ be some non-crossing configuration with height$(\{\tau\}) = k > 0$. Then by the above there is at least one free lattice point that admits a slide. After performing a slide at that location, we obtain a new non-crossing configuration $\{\tau'\}$ with height$(\{\tau'\}) =$ height$(\{\tau\}) - 1$. Recursively, this implies that by applying some series of slides, we can get from $\{\tau\}$ to a non-crossing configuration with height 0. However, $\{\tau_{min}\}$ is the unique such non-crossing configuration, so we have shown that we can get from $\{\tau\}$ to $\{\tau_{min}\}$ with some set of slides. We can now equivalently define height$(\{\tau\})$ as the minimal number of slides required to get from $\{\tau\}$ to $\{\tau_{min}\}$.

Let $\{\sigma\}$ be a different set of non-crossing configurations. There is similarly a set of slides to get from $\{\sigma\}$ to $\{\tau_{min}\}$. Thus we can get from $\{\tau\}$ to $\{\sigma\}$ by a series of slides, by first applying the slides to get from $\{\tau\}$ to $\{\tau_{min}\}$, and then by applying slides in reverse to get from $\{\tau_{min}\}$ to $\{\sigma\}$. Let the tiling $\mathcal{T}$ correspond to the non-crossing configuration $\{\tau\}$, and the tiling $\mathcal{S}$ to the non-crossing configuration $\{\sigma\}$. Since the slides on the paths correspond to flips on the tilings, we obtain that one can get from $\mathcal{T}$ to $\mathcal{S}$ with a series of flips, as desired. □

Remark 3.1.16. By the above lemma, since the minimal tiling $\mathcal{T}_{min}$ of $\Gamma(W)$ is a tiling that admits zero flips from a maximal hexagon to a minimal hexagon, we see that $\mathcal{T}_{min}$ corresponds to $\{\tau_{min}\}$, which is the non-crossing configuration admitting zero slides and having height 0, as in Figure 3.10. Since $\{\tau_{min}\}$ is the unique non-crossing configuration of height 0, $\mathcal{T}_{min}$ must be the unique minimal tiling according to our definition. (The maximal tiling $\mathcal{T}_{max}$ is also unique by a similar argument.) Furthermore, the minimal number of flips required to get from $\mathcal{T}$ to $\mathcal{T}_{min}$ is commonly referred to as the height of a tiling $\mathcal{T}$.

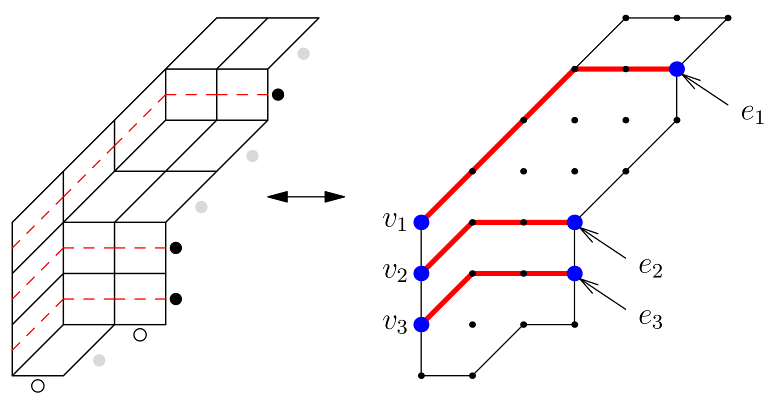


Figure 3.10: $\mathcal{T}_{min}$ and $\{\tau_{min}\}$.

For the proof of Proposition 3.1.9, we introduce a more explicit set of tiles, where tiles can now contain α, β, q, a north line, a west line, or both a north line and a west line, as in

Figure 3.11. We can now describe a *complete* covering of S by compatible tiles according to the following definition.

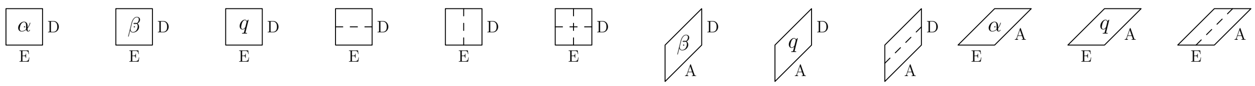


Figure 3.11: A more explicit set of tiles that is in simple correspondence with the RAT fillings.

Definition 3.1.17. Two adjacent tiles are *compatible* if:

1. a tile has a north line through it if and only if its south E-edge is adjacent to a tile containing an α or a north line, and
2. a tile has a west line through it if and only if its east D-edge is adjacent to a tile containing a β or a west line.

Proof of Proposition 3.1.9. For any rhombic diagram $\Gamma(W)$, any two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ can be obtained from each other by some series of flips by Lemma 3.1.13. Thus it is sufficient to show that if $\mathcal{T}_1$ and $\mathcal{T}_2$ differ by a single flip, then there is a weight-preserving bijection between $\mathrm{fi}(W, \mathcal{T}_1)$ and $\mathrm{fi}(W, \mathcal{T}_2)$. Let this flip occur at a certain "special hexagon" ($\mathfrak{h}_1$ in $\mathcal{T}_1$ and $\mathfrak{h}_2$ in $\mathcal{T}_2$). Without loss of generality, let $\mathfrak{h}_1$ be of minimal type as on the left of Figure 3.7, and let $\mathfrak{h}_2$ be of maximal type as on the right of Figure 3.7. The rest of the tiles are identical in $\mathcal{T}_1$ and $\mathcal{T}_2$.

We define the bijection from $F \in \mathrm{fi}(W, \mathcal{T}_1)$ to some $F' \in \mathrm{fi}(W, \mathcal{T}_2)$ with an involution ϕ. To begin, ϕ sends every tile including its contents in $\mathcal{T}_1 \backslash \mathfrak{h}_1$ to its identical copy in $\mathcal{T}_2 \backslash \mathfrak{h}_2$. Then, ϕ sends the tiles and contents of $\mathfrak{h}_1$ to a rearrangement of those tiles according to the cases shown in Figure 3.12. It is easy to see that this map preserves the weights of the fillings, since the quantities of β's, α's, and q's are preserved for each case.

We claim that the map ϕ also preserves the compatibility of the tiles, as defined in Definition 3.1.17. We confirm that in each possible case of $\mathfrak{h}_1$, the tile adjacent to the south E-edge contains an α or a north line if and only if the tile adjacent to the south E-edge of $\phi(\mathfrak{h}_1)$ contains an α or a north line. Similarly, the tile adjacent to the east D-edge of $\mathfrak{h}_1$ contains a β or a west line if and only if the tile adjacent to the east D-edge of $\phi(\mathfrak{h}_1)$ contains a β or a west line.

Thus ϕ indeed gives a weight-preserving bijection from $\mathrm{fi}(W, \mathcal{T}_1)$ to $\mathrm{fi}(W, \mathcal{T}_2)$, and so the proposition follows. $\square$

Thus we are able to make the following definition.

Definition 3.1.18. Let W be a word in $\{\bullet, \textcolor{gray}{\bullet}, \circ\}$, and let $\mathcal{T}$ be an arbitrary tiling of $\Gamma(W)$. Then the *weight* of a word W is

$$\mathrm{weight}(W) = \sum_{F \in \mathrm{fi}(W,\mathcal{T})} \mathrm{wt}(F).$$

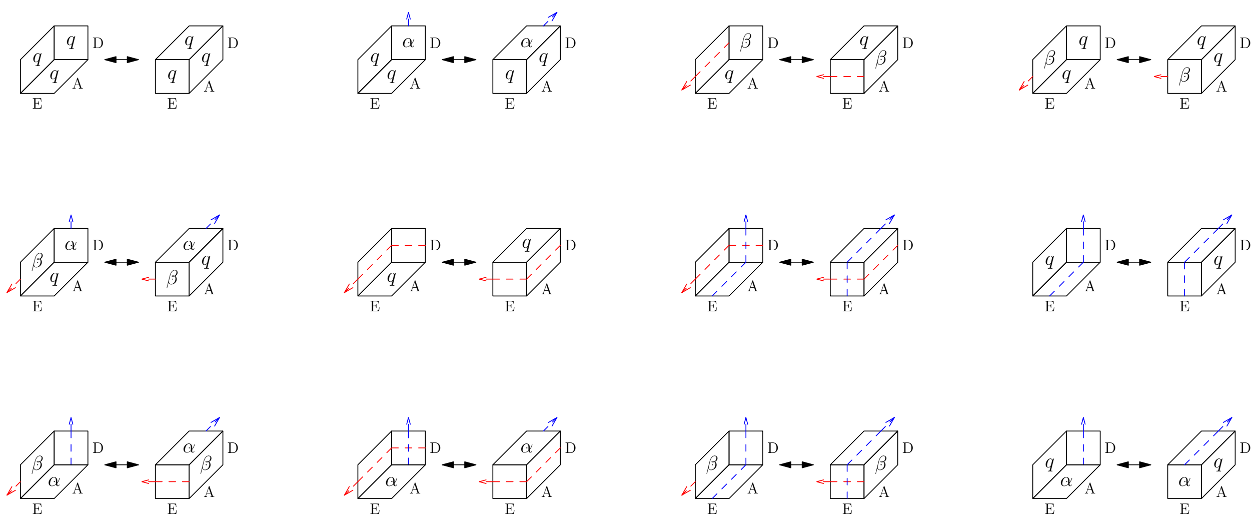


Figure 3.12: The involution ϕ from each possible filling of a minimal hexagon (left) to a maximal hexagon (right). The dashed arrows imply compatibility requirements.

In fact, we can define equivalence classes of rhombic alternative tableaux with the following definitions.

Definition 3.1.19. A *weight-preserving flip* on a RAT with tiling $\mathcal{T}$ is the transformation ϕ (or the inverse of the ϕ) given by Figure 3.12 on some hexagon $\mathfrak{h}$ of $\mathcal{T}$ and the symbols contained in it, while preserving the filling of $\mathcal{T} \setminus \mathfrak{h}$.

Definition 3.1.20. Let W be a word in $\{\bullet, \textcolor{gray}{\bullet}, \circ\}$, and let $T_1 \in \mathrm{fi}(W, \mathcal{T}_1)$ and $T_2 \in \mathrm{fi}(W, \mathcal{T}_2)$ be rhombic alternative tableaux of type W for arbitrary tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\Gamma(W)$. Then T_1 and T_2 are *equivalent* if and only if T_2 can be obtained from T_1 by a series of weight-preserving flips.

3.2 Steady state probabilities of the two-species ASEP

The main theorem of this chapter is the following.

Theorem 3.2.1. *Let W be a word in $\{$●, ◍, ○$\}^n$ that represents a state of the two-species ASEP with exactly r ◍'s. The stationary probability of state W is*

$$\Pr(W) = \frac{1}{\mathcal{Z}_{n,r}} \operatorname{weight}(W), \tag{3.1}$$

where weight(W) *is defined in Definition 3.1.18.*

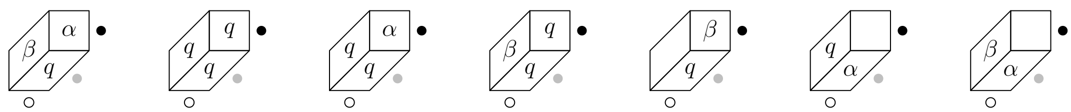


Figure 3.13: All fillings for the minimal tiling of a rhombic diagram of type ● ◍ ○.

Example. All seven fillings of the minimal tiling of a rhombic diagram corresponding to the state ● ◍ ○ are shown in Figure 3.13. From the sum of the weights of these fillings, we obtain

$$\Pr(\text{● ◍ ○}) = \frac{1}{\mathcal{Z}_{3,1}} \left(q^3 + \alpha q^2 + \alpha q + \beta q^2 + \beta q + \alpha\beta + \alpha\beta q\right).$$

To facilitate our proof, we provide a more flexible Matrix Ansatz that generalizes Theorem 3.0.1 with the same argument as in an analogous proof for the ordinary ASEP of Corteel and Williams [5, Theorem 5.2]. For a word $W \in \{$●, ◍, ○$\}^n$ with r ◍'s, we define unnormalized weights $f(W)$ which satisfy

$$\Pr(W) = f(W)/Z_{n,r}$$

for $Z_{n,r} = \sum_{W'} f(W')$ where the sum is over all words W' of length n and with r ◍'s.

Theorem 3.2.2. *Let λ be a constant. Let $\langle w|$ and $|v\rangle$ be row and column vectors with $\langle w||v\rangle = 1$. Let D, E, and A be matrices such that for any words X and Y in $\{D, A, E\}$ representing products of the matrices in the corresponding order, the following conditions are satisfied:*

(I) $\langle w|X(DE - qED)Y|v\rangle = \lambda\langle w|X(D+E)Y|v\rangle$,

(II) $\langle w|X(DA - qAD)Y|v\rangle = \lambda\langle w|XAY|v\rangle$,

(III) $\langle w|X(AE - qEA)Y|v\rangle = \lambda\langle w|XAY|v\rangle$,

(IV) $\beta\langle w|XD|v\rangle = \lambda\langle w|X|v\rangle$,

(V) $\alpha\langle w|EY|v\rangle = \lambda\langle w|Y|v\rangle$.

Let $W = \{W_1, \ldots, W_n\}$ for $W_i \in \{\bullet, \textcolor{gray}{\bullet}, \circ\}$ for $1 \leq i \leq n$. Then for any state W of the two-species ASEP of length n with r light particles,

$$f(W) = \frac{1}{\langle w|A^r|v\rangle} \langle w| \prod_{i=1}^{n} D\, \mathbb{1}_{(W_i=\bullet)} + A\, \mathbb{1}_{(W_i=\textcolor{gray}{\bullet})} + E\, \mathbb{1}_{(W_i=\circ)}\, |v\rangle.$$

Proof. The proof of Theorem 4.3.1 follows exactly that of [5, Theorem 5.2]. Note that the above implies that

$$Z_{n,r} = [y^r] \frac{\langle w|(D + yA + E)^n|v\rangle}{\langle w|A^r|v\rangle}.$$

□

Proof of the main theorem

Proof of 3.2.1. The Matrix Ansatz of Theorem 4.3.1 implies that the steady state probabilities for the two-species ASEP satisfy certain recurrences (that in turn determine all probabilities). The strategy of our proof is to show that the weight generating function for RAT of fixed type satisfies the same recurrences. Specifically, we use these recurrences with the constant $\lambda = \alpha\beta$, to show by induction that for W a word in $\{\bullet, \textcolor{gray}{\bullet}, \circ\}^n$ with r $\textcolor{gray}{\bullet}$'s,

$$f(W) = \text{weight}(W). \tag{3.2}$$

We do the induction on the number of tiles M in our tableaux. For the rest of this proof, we call the number of tiles in a rhombic diagram its *area*.

Definition 3.2.3. Let $x(W)$ be a word in $\{D, A, E\}$ representing a matrix product of the matrices D, A, and E, corresponding to the two-species ASEP word W in the letters $\{\bullet, \textcolor{gray}{\bullet}, \circ\}$, where D, A, and E correspond to $\bullet$, $\textcolor{gray}{\bullet}$, and $\circ$ respectively.

Example. For $W = \bullet \circ \textcolor{gray}{\bullet} \textcolor{gray}{\bullet} \circ$, we have $x(W) = DEAAE$.

According to this definition, for $W \in \{\bullet, \textcolor{gray}{\bullet}, \circ\}^n$ a word with r $\textcolor{gray}{\bullet}$'s we have

$$f(W) = \frac{1}{\langle w|A^r|v\rangle} \langle w|x(W)|v\rangle$$

by Theorem 4.3.1.

To start, if the area of $\Gamma(W)$ is 0, then necessarily $W = \circ^\ell \textcolor{gray}{\bullet}^r \bullet^k$ for some ℓ and k. For every such case, there is a single tiling of $\Gamma(W)$ and a single RAT on that tiling, and both of these are trivial (i.e. there are zero tiles to be filled.) We obtain weight$(\circ^\ell \textcolor{gray}{\bullet}^r \bullet^k) = \beta^\ell \alpha^k$. From Theorem 4.3.1, it is clear that the base case indeed satisfies Equation (3.1).

Now suppose that any word W' such that $\Gamma(W')$ has area $M < m$ satisfies Equation (3.1). Let W be a word of length n with r $\textcolor{gray}{\bullet}$'s, such that $\Gamma(W)$ has area m. Outside of the base case, we assume that at least one of the following must occur:

i. W contains an instance of ● ○.

ii. W contains an instance of ● $\color{gray}{\bullet}$.

iii. W contains an instance of $\color{gray}{\bullet}$ ○.

Based on the occurrence of one of the above, we will express weight(W) in terms of the weight of some other words whose rhombic diagrams have areas smaller than m. Throughout the following, we let X and Y represent some arbitrary words in {●, $\color{gray}{\bullet}$, ○}, and we let T represent a RAT of type W.

(i.) W contains an instance of ● ○. We write $W = X$● ○ Y. We can choose an arbitrary tiling $\mathcal{T}$ of $\Gamma(W)$, since any such tiling will contain a DE tile adjacent to the chosen ● ○ edges. We call this DE corner tile the *chosen corner*. Let $T \in \text{fi}(W, \mathcal{T})$. The chosen corner of T must contain either an α, a β, or a q, so we can decompose the possible fillings of T into three cases.

If the chosen corner contains an α, then all the tiles above it in the same north-strip are empty, and so its entire north-strip has no effect on the rest of the tableau. Thus such T can be mapped to a filling of a smaller RAT on tiling $\mathcal{T}'$ with that north-strip removed, which would have type X● Y (similar to the example in Figure 3.14 (a)). It is easy to check that this operation results in a valid tableau, since any two symbols in the same west-strip of T remain in the same relative position in a west-strip of T'. (And similarly for the north-strips, save for the one that was removed). This map gives a bijection between tableaux of type X● ○ Y on tiling $\mathcal{T}$ with an α in the chosen DE corner and tableaux of type X● Y on tiling $\mathcal{T}'$. The removed column with the α in its bottom-most box has total weight $\alpha\beta$.[1]

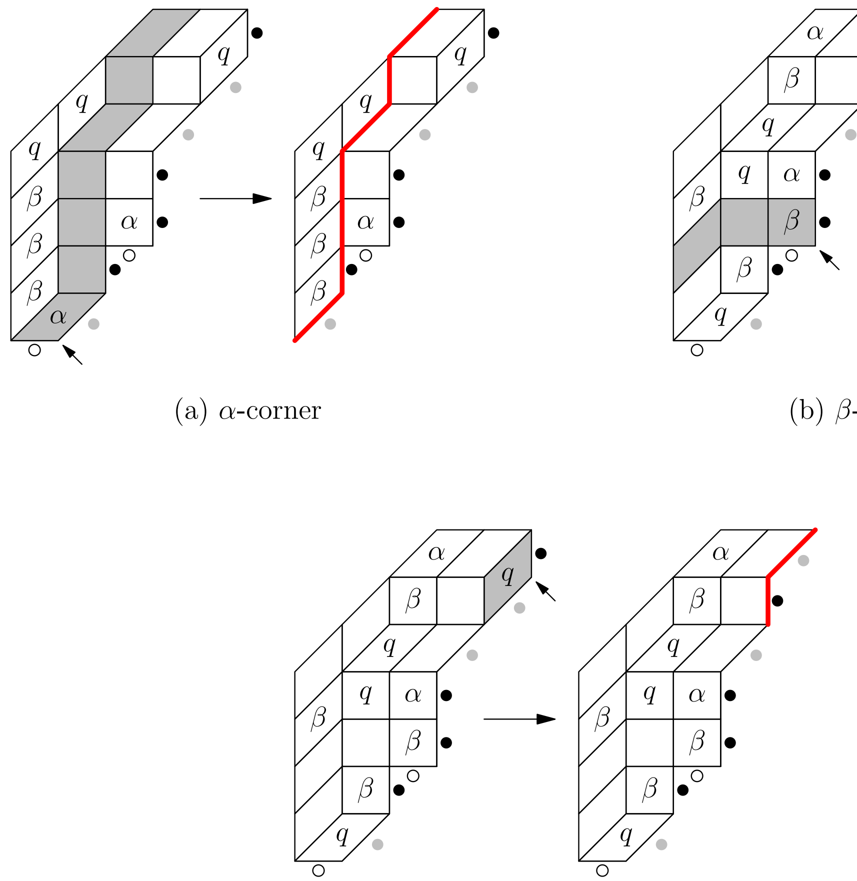
Similarly, if the chosen corner contains a β, then the tiles to its left in the same west-strip must be empty, and so its entire west-strip has no effect on the rest of the tableau. Hence such T can be mapped to a smaller RAT on tiling $\mathcal{T}''$ with that west-strip removed, which would have type X○ Y as in Figure 3.14 (b). This map gives a bijection between tableaux of type X● ○ Y on tiling $\mathcal{T}$ with a β in the chosen DE corner and tableaux of type X○ Y on tiling $\mathcal{T}''$. The removed west-strip with the β in its right-most tile also has total weight $\alpha\beta$.

Finally, if the chosen corner contains a q, then this tile has no effect on the rest of the tableau. Hence such T can be mapped to a RAT of area $m-1$ on tiling $\mathcal{T}'''$ with that DE corner tile removed, which would have type X○ ● Y (similar to the example in Figure 3.14 (c)). This map gives a bijection between tableaux of type X● ○ Y on tiling $\mathcal{T}$ with a q in the chosen DE corner and tableaux of type X○ ● Y on tiling $\mathcal{T}'''$. The removed tile with the q has total weight q.

Consequently, we have the sum of the weights of the fillings:

$$\text{weight}(X{\bullet}\circ Y) = \text{weight}(X{\bullet} Y)\cdot \alpha\beta + \text{weight}(X{\circ} Y)\cdot\alpha\beta + q\,\text{weight}(X{\circ}{\bullet} Y).$$

[1] In the total weight of a column, we include the weight of the bottom-most edge, which is a component of the southeast boundary of T. When the column removed is an E-column, the weight of the boundary component is β, so the total weight of the column with an α at the bottom is $\alpha\beta$. Similar reasoning is used in the other cases.



(a) α-corner

(b) β-corner

(c) q-corner

Figure 3.14: (a) $X{\color{gray}\bullet}\circ Y \mapsto X{\color{gray}\bullet} Y$, (b) $X\bullet\circ Y \mapsto X\bullet Y$, and (c) $X\bullet{\color{gray}\bullet} Y \mapsto qX{\color{gray}\bullet}\bullet Y$.

By the induction hypothesis, since the areas of $\Gamma(X\bullet Y)$, $\Gamma(X\circ Y)$, and $\Gamma(X\circ\bullet Y)$ are all strictly smaller than m, we have weight$(X\bullet Y) = f(X\bullet Y)$, weight$(X\circ Y) = f(X\circ Y)$, and weight$(X\circ\bullet Y) = f(X\circ\bullet Y)$. Thus we obtain

$$\begin{aligned} f(X\bullet\circ Y) &= (\alpha\beta)\langle w|x(X)\ (D+E)\ x(Y)|v\rangle + q\langle w|x(X)\ ED\ x(Y)|v\rangle \\ &= \langle w|x(X)\ DE\ x(Y)\ |v\rangle \\ &= \langle w|x(W)|v\rangle. \end{aligned}$$

Hence by Theorem 4.3.1 with $\lambda = \alpha\beta$, it follows that W satisfies Equation (3.2).

(ii.) W contains an instance of ${\color{gray}\bullet}\circ$. We write $W = X{\color{gray}\bullet}\circ Y$. We choose tiling $\mathcal{T}$ of $\Gamma(W)$ such that there is an AE tile adjacent to the chosen ${\color{gray}\bullet}\circ$ edges (and we allow the rest of

the tiling to be arbitrary). We call this AE corner tile the *chosen corner*. Let $T \in \mathrm{fi}(W, \mathcal{T})$. The chosen corner of T must contain either an α or a q.

If the chosen corner contains an α, then the tiles above it in the same north-strip must be empty, and so its entire north-strip has no effect on the rest of the tableau. Hence such T can be mapped to a smaller RAT on tiling $\mathcal{T}'$ with that north-strip removed, which would have type $X{\color{gray}\bullet}\, Y$, as in Figure 3.14 (a). This map gives a bijection between tableaux of type $X{\color{gray}\bullet} \circ Y$ on tiling $\mathcal{T}$ with an α in the chosen AE corner and tableaux of type $X{\color{gray}\bullet}\, Y$ on tiling $\mathcal{T}'$. The removed north-strip with the α in its bottom-most tile has total weight $\alpha\beta$.

On the other hand, if the chosen corner contains a q, then this tile has no effect on the rest of the tableau. Hence such T can be mapped to a RAT of area $m-1$ on tiling $\mathcal{T}''$ with that AE corner tile removed, which would have type $X\circ {\color{gray}\bullet}\, Y$ (similar to the example in Figure 3.14 (c)). This map gives a bijection between tableaux of type $X{\color{gray}\bullet} \circ Y$ on tiling $\mathcal{T}$ with a q in the chosen AE corner and tableaux of type $X\circ {\color{gray}\bullet}\, Y$ on tiling $\mathcal{T}''$. The removed tile with the q has total weight q. Thus we obtain the sum of the weights of the fillings:

$$\mathrm{weight}(X{\color{gray}\bullet} \circ Y) = \mathrm{weight}(X{\color{gray}\bullet}\, Y) \cdot \alpha\beta + q\,\mathrm{weight}(X\circ {\color{gray}\bullet}\, Y).$$

Similar reasoning to the DE case completes the argument.

(iii.) W contains an instance of $\bullet\, {\color{gray}\bullet}$. We write $W = X\bullet {\color{gray}\bullet}\, Y$. We choose tiling $\mathcal{T}$ of $\Gamma(W)$ such that there is a DA tile adjacent to the chosen $\bullet\,{\color{gray}\bullet}$ edges (and we allow the rest of the tiling to be arbitrary). We call this DA corner tile the *chosen corner*. Let $T \in \mathrm{fi}(W, \mathcal{T})$. The chosen corner of T must contain either a β or a q.

If the chosen corner contains a β, then the tiles to its left in the same west-strip must be empty, and so its entire west-strip has no effect on the rest of the tableau. Hence such T can be mapped to a smaller RAT on tiling $\mathcal{T}'$ with that west-strip removed, which would have type $X{\color{gray}\bullet}\, Y$ (similar to the example in Figure 3.14 (b)). This map gives a bijection between tableaux of type $X\bullet {\color{gray}\bullet}\, Y$ on tiling $\mathcal{T}$ with a β in the chosen DA corner and tableaux of type $X{\color{gray}\bullet}\, Y$ on tiling $\mathcal{T}'$. The removed west-strip with the β in its right-most tile has total weight $\alpha\beta$.

On the other hand, if the chosen corner contains a q, then this tile has no effect on the rest of the tableau. Hence such T can be mapped to a RAT of area $m-1$ on tiling $\mathcal{T}''$ with that DA corner tile removed, which would have type $X{\color{gray}\bullet} \bullet Y$, as in Figure 3.14 (c). This map gives a bijection between tableaux of type $X\bullet {\color{gray}\bullet}\, Y$ on tiling $\mathcal{T}$ with a q in the chosen DA corner and tableaux of type $X{\color{gray}\bullet} \bullet Y$ on tiling $\mathcal{T}''$. The removed tile with the q has total weight q. Thus we obtain the sum of the weights of the fillings:

$$\mathrm{weight}(X\bullet {\color{gray}\bullet}\, Y) = \mathrm{weight}(X{\color{gray}\bullet}\, Y) \cdot \alpha\beta + q\,\mathrm{weight}(X{\color{gray}\bullet} \bullet Y).$$

Similar reasoning to the DE case completes the argument.

From the above cases, we obtain that for any M, any word W with $\Gamma(W)$ of area M satisfies Equation (3.2), which is the desired result. $\square$

Section 3.4 features an independent proof of Theorem 3.2.1 obtained by constructing a Markov chain on the rhombic alternative tableaux that projects to the two-species ASEP.

Complements: the cellular ansatz.

It is possible to give another proof of Theorem 3.2.1 in the spirit of the general theory called *cellular ansatz*, introduced and developed by X. Viennot in [21].

In the simple case $r = 0$ of the ASEP, the Matrix Ansatz defines an algebra with generators D and E, with relation $DE = qED + E + D$. In this algebra, any word W with letters D and E can be written in a unique way as a sum of monomials $q^tE^iD^j$.

The proof relies on a *planarization* of the *rewriting rules* $DE \mapsto qED$, $DE \mapsto E$, and $DE \mapsto D$ (see an example on slides 25-50 of Chapter 3a of [21]). In this context, alternative tableaux appear naturally. We obtain an identity expressing the word W as $\sum_T \mathrm{wt}(T)$, where the sum is over alternative tableaux T, and $\mathrm{wt}(T)$ is a certain monomial of the form $q^tE^iD^j$, where i, j, and t are defined from the tableau T (see slide 59 of Chapter 3a of [21]). By applying the Matrix Ansatz for the ASEP, we get immediately the interpretation of the stationary probabilities in terms of alternative tableaux (slide 60 of Chapter 3a of [21]).

The general theory of the cellular ansatz works with some family of quadratic algebras $\mathcal{Q}$, having two families of generators, with some commutations relations. Any word W in those generators can be expressed as a sum of monomials over generalized tableaux called *complete $\mathcal{Q}$-tableaux*, in bijection with *$\mathcal{Q}$-tableaux* (see Chapter 6a, slides 41-46, 54-56 of [21]).

The Matrix Ansatz for the two-species ASEP defines an algebra with three generators D, E, A and three commutation relations $DE = qED + E + D$, $DA = qAD + A$, and $AE = qEA + A$. This quadratic algebra does not quite fit in the general cellular ansatz theory of Chapter 6a of [21], but the theory can be extended to such an algebra, by replacing the quadratic lattice by a tiling $\mathcal{T}$ of the diagram $\Gamma(W)$. The corresponding *$\mathcal{Q}$-tableaux* are the RAT, and in a similar way, one can prove that any word W in letters $\{D, A, E\}$ can be expressed in a unique way as a sum of monomials $q^tE^iA^mD^j$. Here D, A, and E are identified with ●, ●, and ○ respectively.

More precisely we have the identity

$$W = \sum_{F \in \mathrm{fi}(W,\mathcal{T})} q^tE^iA^mD^j,$$

where i is the number of north-strips of F not containing an α, j is the number of west-strips of F not containing a β, and t is the number of cells weighted q as in the definition of $\mathrm{wt}(F)$ in Section 3.1. Note that the weight $\mathrm{wt}(F)$ defined in Definition 3.1.8 is equal to the monomial $q^t\alpha^{n-r-i}\beta^{n-r-j}$ where n is the length of W and r is the number of ●'s it contains.

Applying to the above the two-species Matrix Ansatz, we obtain immediately Theorem 3.2.1.

3.3 Enumeration of the rhombic alternative tableaux

In this section, we compute the partition function at $q = 1$ for the rhombic alternative tableaux, and provide some more refined enumeration for the case $q = 0$.

Definition 3.3.1. Define the *partition function*

$$\mathcal{Z}_{n,r}(\alpha, \beta, q) = \sum_{W} \text{weight}(W)$$

for W ranging over all words in $\{\bullet, {\color{gray}\bullet}, \circ\}^n$ with r ${\color{gray}\bullet}$'s. By convention, let $\mathcal{Z}_{n,n}(\alpha, \beta, q) = 1$.

Proposition 3.1.9 allows us to make the following definition.

Definition 3.3.2. Let W be a state of the two-species ASEP and $\mathcal{T}$ and $\mathcal{T}'$ be some tilings of $\Gamma(W)$. A RAT $F \in \text{fi}(W, \mathcal{T})$ is *equivalent* to a RAT $F' \in \text{fi}(W, \mathcal{T}')$ if F can be obtained from F' by some series of weight-preserving flips.

Let Ω_r^n be the set of states of the two-species ASEP of size n with exactly r light particles. Let $\Psi_{(n,r)}$ be the set of equivalence classes of RAT whose type belongs to Ω_r^n. More precisely, $\psi \in \Psi_{(n,r)}$ is some set of RAT of a single type such that for any $F, F' \in \psi$, F and F' are equivalent. Moreover, if F and F' are equivalent and $F \in \psi$ and $F' \in \psi'$, then $\psi = \psi'$.

From [15], we also have the following theorem.

Theorem 3.3.3 ([15] Theorem 2.19)**.**

$$\mathcal{Z}_{n,r}(\alpha, \beta, 1) = \binom{n}{r} \prod_{i=r}^{n-1} (\alpha + \beta + i\alpha\beta). \tag{3.3}$$

This implies the following corollary.

Corollary 3.3.4.

$$|\Psi_{(n,r)}| = \binom{n}{r} \frac{(n+1)!}{(r+1)!}.$$

Proof of Theorem 3.3.3. Let $Z_{n,r,k}(\alpha, \beta, q)$ be the weight generating function for the RAT (with the maximal tiling) with exactly k west-strips that do not contain a β.

We also define

$$Z_{n,r}(x) = \sum_{k \geq 0} Z_{n,r,k}(\alpha, \beta, 1) x^k \tag{3.4}$$

with $\mathcal{Z}_{n,r}(\alpha, \beta, 1) = Z_{n,r}(1)$. We claim that

$$Z_{n,r}(x) = \binom{n}{r} \prod_{i=r}^{n-1} (x\alpha + \beta + i\alpha\beta). \tag{3.5}$$

We will prove Equation (3.5) and hence Equation (3.3) by induction on n in terms of the more refined $Z_{n,r,k}$'s.

First, when $n = 1$, $Z_{1,0}(x) = x\alpha + \beta$ and $Z_{1,1} = 1$.

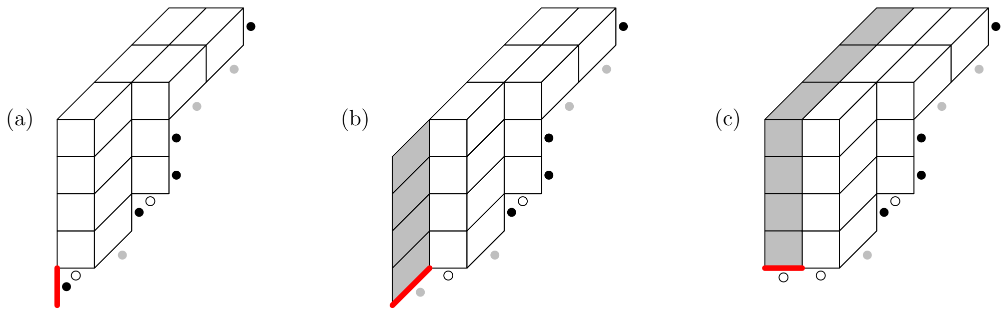


Figure 3.15: Adding a (a) ●, (b) ◍, or (c) ○ to the end of W.

Now, we suppose that Equation (3.5) holds for any $N \leq n$ and any $r \leq n-1$. (Again, by convention $Z_{n,n} = 1$ for all n.) We will show that the formula holds as well for $N = n+1$, for all $r \leq n$.

We begin by observing that Equation (3.5) satisfies

$$Z_{n+1,r}(x) = (x\alpha + \beta + r\alpha\beta)Z_{n,r}(x+\beta) + Z_{n,r-1}(x+\beta). \tag{3.6}$$

We now construct a recursion for $Z_{n+1,r,k}$ in terms of the functions $\{Z_{n,r',k'}\}$ by keeping track of the terms after the addition of a ●, ◍, or ○ to the end of a word $W \in \{$●, ◍, ○$\}^n$. For the following, we denote by T' a tableau in $\mathrm{fi}(W, \mathcal{T}_{max}(W))$ (for $\mathcal{T}_{max}(W)$ the maximal tiling of $\Gamma(W)$), and by T a tableau in $\mathrm{fi}(Wx, \mathcal{T}_{max}(Wx))$ for $x \in \{$●, ◍, ○$\}$. We consider all possible cases for W and corresponding T' such that the resulting $T \in \mathrm{fi}(Wx, \mathcal{T}_{max}(Wx))$ has size $(n+1, r)$ and exactly k west-strips that do not contain a β.

1. *We add a ● to the end of ASEP word W of length n with r ◍'s.* On the tableau level, this corresponds simply to the addition of a D-edge to the southwest end of each $T' \in \mathrm{fi}(W, \mathcal{T}_{max}(W))$ as in Figure 4.1 (a). Since the filling of T is the same as that of T',
$$Z_{n+1,r,k} = \alpha Z_{n,r,k-1}. \tag{3.7}$$
This contributes to $\sum_{T \in \mathrm{fi}(W●, \mathcal{T}_{max}(W●))} \mathrm{wt}(T)|_{(q=1)}$.

2. *We add a ◍ to the end of a ASEP word W of length n with $r-1$ ◍'s.* On the tableau level, this corresponds to the addition of a vertical column of some DA tiles to the left boundary of T' to form a tableau with the maximal tiling of $\Gamma(W$◍$)$ as in Figure 4.1 (b). Suppose T' has $\ell \geq k$ west-strips that do not contain a β. Then to obtain T with exactly k west-strips that do not contain β, the $\ell - k$ DA tiles that *do* contain a β can be chosen in $\binom{\ell}{k}$ ways, with the other k DA tiles containing q. Therefore, we obtain
$$Z_{n+1,r,k}(\alpha, \beta, 1) = \sum_{\ell \geq k} \binom{\ell}{k} \beta^{\ell-k} Z_{n,r-1,\ell}. \tag{3.8}$$
This contributes to $\sum_{T \in \mathrm{fi}(W◍, \mathcal{T}_{max}(W◍))} \mathrm{wt}(T)|_{(q=1)}$.

3. *We add a ○ to the end of a ASEP word W of length n with r ●'s.* On the tableau level, this corresponds to the addition of a vertical column of some DE tiles followed by a strip of r AE tiles to the left boundary of T' to form a tableau with the maximal tiling of $\Gamma(W○)$ as in Figure 4.1 (c). Suppose T' has $\ell \geq k$ west-strips that do not contain a β. We have two cases.

   (1) For the first case, there is no α in the newly added DE tiles. Then to obtain T with exactly k west-strips that do not contain β, the $\ell - k$ DE tiles that *do* contain a β can be chosen in $\binom{\ell}{k}$ ways, with the other k DE tiles containing q. Following this, the AE tiles can either contain all q's, or some consecutive string of q's followed by an α.

   (2) For the second case, there is an α in the newly added DE tiles, with some $\ell - k \leq u \leq \ell - 1$ free DE tiles below it. (Recall that a DE tile is free if there is no β to its right in the same west-strip, and no α below it in the same north-strip.) Then, to obtain T with exactly k west-strips that do not contain β, the $\ell - k$ DE tiles that *do* contain a β can be chosen in $\binom{u}{\ell-k}$ ways, with the other $u - (\ell - k)$ DE tiles that lie below the α containing q, and the rest of the tiles empty.

   Combining the above two cases, we obtain

$$Z_{n+1,r,k}(\alpha,\beta,1) = \beta \sum_{\ell \geq k} \binom{\ell}{k} \beta^{\ell-k}(r\alpha+1) Z_{n,r,\ell} + \sum_{u=\ell-k}^{\ell-1} \binom{u}{\ell-k} \alpha\beta^{\ell-k} Z_{n,r,\ell}. \tag{3.9}$$

   This contributes to $\sum_{T \in \text{fi}(W○, \mathcal{T}_{max}(W○))} \text{wt}(T)|_{(q=1)}$.

Combining Equations (4.12), (4.13), and (4.14) and summing over k, we obtain

$$\begin{aligned} Z_{n+1,r}(x) = \sum_{k\geq 0} \Bigg(& x\alpha Z_{n,r,k-1} + \sum_{\ell \geq k}\binom{\ell}{k}\beta^{\ell-k}x^k Z_{n,r-1,\ell} + \beta \sum_{\ell\geq k}\binom{\ell}{k}\beta^{\ell-k}x^k Z_{n,r,\ell}(r\alpha+1) \\ & +\beta \sum_{\ell\geq k}\sum_{u=\ell-k}^{\ell-1}\binom{u}{\ell-k}\alpha\beta^{\ell-k}x^k Z_{n,r,\ell}\Bigg) \\ = {} & x\alpha Z_{n,r}(x) + Z_{n,r-1}(x+\beta) + \beta(r\alpha+1)Z_{n,r}(x+\beta) \\ & + \alpha\beta\sum_{k\geq 0}\sum_{\ell\geq k}\binom{\ell}{k-1}\beta^{\ell-k}x^k Z_{n,r,\ell} \\ = {} & x\alpha Z_{n,r}(x) + Z_{n,r-1}(x+\beta) + (r\alpha\beta+\beta+x\alpha)Z_{n,r}(x+\beta) - x\alpha\sum_{k\geq 0} x^{k-1} Z_{n,r,k-1}. \end{aligned}$$

which simplifies to Equation (3.6). Since $Z_{n+1,r}$ satisfies the desired recursion, we thus obtain that Equation (3.5) indeed holds for $N = n+1$, and so our proof is complete. □

Remark 3.3.5. A recent work by the author of this thesis and X. Viennot features a bijective proof for Theorem 3.3.3 [14]. The rhombic alternative tableaux are enumerated by the Lah numbers, which also enumerate certain *assemblées of permutations*. In [14] we describe a bijection between the rhombic alternative tableaux and these assemblées, and provide an insertion algorithm that gives a weight generating function for the assembées. Combining these results, we obtain a bijective proof for the weight generating function for the rhombic alternative tableaux of Equation (3.3).

Enumeration of rhombic alternative tableau with $q = 0$

Finally, for RAT with $q = 0$, there are some more refined enumerative results from [13] and also [2, 9].

Theorem 3.3.6. *The weight generating function for RAT at $q = 0$ of size n and whose type has r ●'s is*

$$\mathcal{Z}^0_{n,r}(\alpha, \beta, 0) = (\alpha\beta)^{n-r} \sum_{p=1}^{n-r} \frac{2r+p}{2n-p} \binom{2n-p}{n+r} \frac{\alpha^{-p-1} - \beta^{-p-1}}{\alpha^{-1} - \beta^{-1}}.$$

Theorem 3.3.7. *The number of RAT at $q = 0$ of size n and whose type has r ●'s is*

$$\mathcal{Z}^0_{n,r}(1, 1, 0) = \frac{2(r+1)}{n+r+2} \binom{2n+1}{n-r}.$$

Theorem 3.3.8. *Let $n := r + k + \ell$. The number of RAT at $q = 0$ of size n and whose type has r ●'s and k ●'s is*

$$\frac{r+1}{n+1} \binom{n+1}{k} \binom{n+1}{\ell}.$$

3.4 A Markov chain on the RAT

We restate here the definition of a Markov chain that projects to another, and describe the RAT as a Markov chain that projects to the two-species ASEP. Such results exist for the alternative tableaux which project to the regular ASEP. Those results were originally described in terms of permutation tableaux (which are in simple bijection with the alternative tableaux) in [4]. Our Markov chain has the same flavor as the existing Markov chain defined by Corteel and Williams. The following definition is from [4, Definition 3.20].

Definition 3.4.1. Let M and N be Markov chains on finite sets X and Y, and let f be a surjective map from X to Y. We say that M *projects* to N if the following properties hold:

- If $x_1, x_2 \in X$ with $Prob_M(x_1 \to x_2) > 0$, then $Prob_M(x_1 \to x_2) = Prob_N(f(x_1) \to f(x_2))$.

- If y_1 and y_2 are in Y and $Prob_N(y_1 \to y_2) > 0$, then for each $x_1 \in X$ such that $f(x_1) = y_1$, there is a unique $x_2 \in X$ such that $f(x_2) = y_2$ and $Prob_M(x_1 \to x_2) > 0$; moreover, $Prob_M(x_1 \to x_2) = Prob_N(y_1 \to y_2)$.

Furthermore, we have Proposition 3.4.2 below, which implies Corollary 3.4.3.

Let $\mathrm{Prob}_m(x_0 \to x; t)$ denote the probability that if we start at state x_0 at time 0, then we are in state x at time t. From the following proposition of [4], we obtain that if M projects to N, then a walk on the state diagram of M is indistinguishable from a walk on the state diagram of N.

Proposition 3.4.2. *Suppose that M projects to N. Let $x_0 \in X$ and $y_0, y_1 \in Y$ such that $f(x_0) = y_0$. Then*

$$\mathrm{Prob}_N(y_0 \to y_1) = \sum_{x_1 \ s.t. \ f(x_1)=y_1} \mathrm{Prob}_M(x_0 \to x_1)$$

Corollary 3.4.3. *Suppose M projects to N via the map f. Let $y \in Y$ and let*

$$X' = \{x \in X \mid f(x) = y\}.$$

Then the steady state probability that N is in state y is equal to the steady state probabilities that M is in any of the states $x \in X'$.

In our case, N is the two-species ASEP (which we call the ASEP chain), and M is the Markov chain on the RAT (which we call the RAT chain).

Recall that Ω_r^n denotes the states of the two-species ASEP of size n with exactly r light particles. We specify the states of the RAT chain to be $\Psi_{(n,r)}$, the set of the RAT equivalence classes of size (n, r), based on the fact that different tilings can be chosen to yield equivalent tableaux, as mentioned in Remark 3.3.2.

Now, we define the transitions on $\Psi_{(n,r)}$ in the RAT chain that correspond to transitions on Ω_r^n in the ASEP chain. We introduce the following terminology, as in Figure 3.16.

Definition 3.4.4. A *corner* is a pair of consecutive D and E, D and A, or A and E-edges on the boundary of a RAT. If there is a DE tile, a DA tile, or an AE tile (respectively) adjacent to the corresponding edges of the boundary, we call that tile a *corner tile*.

An *inner corner* is a pair of consecutive E and D, A and D, or E and A-edges on the boundary of a RAT.

An *empty E-strip* corresponds to an E-edge on the boundary of the RAT that coincides with its top-most boundary.

An *empty D-strip* corresponds to a D-edge on the boundary of the RAT that coincides with its left-most boundary.

Lemma 3.4.5. *Let $\psi \in \Psi_{(n,r)}$ be a RAT equivalence class and let $F \in \psi$. If F has a corner of type DE, DA, or AE, then there exists an equivalent $F' \in \psi$ that has, respectively, a DE tile, a DA tile, or an AE tile at that corner.*

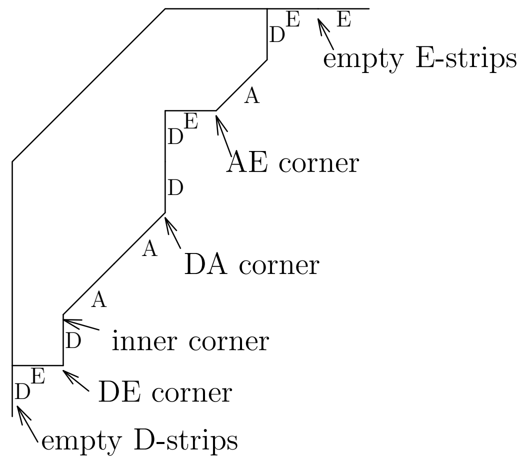


Figure 3.16: The features of a tableau.

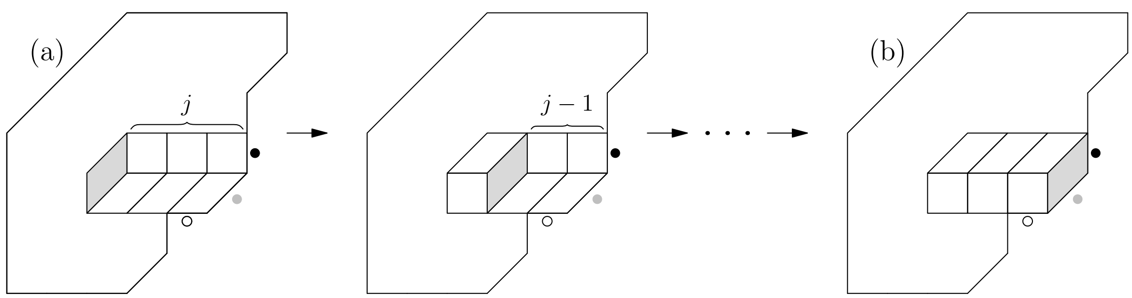


Figure 3.17: If the boundary of the tableau contains consecutively a D, A, and E-edge, and there is no DA tile adjacent to the DA corner, then a "stack of boxes" as in (a) must occur in the tiling, for some value of j. After performing j flips, the configuration in (b) is obtained, with a DA tile adjacent to the DA corner, as desired.

Proof. First, it is clear that any tiling of a rhombic diagram with a DE corner must have a DE tile at that corner, so for the DE case the lemma is obvious.

Now, for the DA and the AE cases, it suffices to prove the lemma for only one of them, since by taking the transpose of a tableau and swapping the roles of α and β, we end up exchanging the D-edges with the E-edges's (and consequently the DA corners with the AE corners), and so by symmetry, these cases will have the same properties. Thus we will prove the DA case.

First, if the DA corner already has a DA tile adjacent to it, we are done. Thus we assume there is not a DA tile, which means the tiling of the rhombic diagram must contain the tiles shown in Figure 3.17 (a). More precisely, as seen in the figure, the tiles must be a row of

$j \geq 1$ DE tiles on top of j AE tiles, with one adjacent DA tile on the left. Now it is easy to check that with j flips, we end up with the configuration in Figure 3.17 (b), and moreover, there will a β in the corner DA tile in the tiling (b) if and only if there is a β in the right-most DE tile in the tiling (a) (and otherwise there will be a q). Thus with j flips, we obtain an equivalent tableau with a DA tile in the DA corner, as desired. □

Based on the above lemma, we make the following definition:

Definition 3.4.6. Let F be a tableau with a corner. We call that corner a *q-corner* (or an *α-corner*, or a *β-corner*) if a tableau T contains a q in the tile adjacent to that corner (or respectively, an α, or a β) for some T that is equivalent to F and has a corner tile adjacent to the corner.

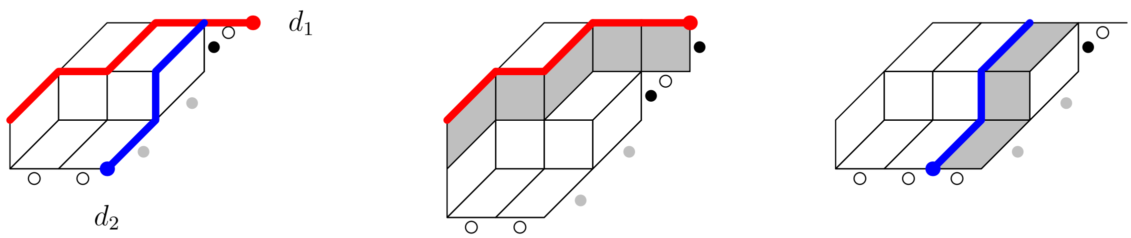


Figure 3.18: Let d_1 and d_2 be the indicated D- and E-paths on $\mathcal{T}$. Then $\mathcal{T}$ is the compression of $\mathrm{bl}(\mathcal{T}, d_1)$ and $\mathrm{bl}(\mathcal{T}, d_2)$ at the highlighted D- and E-strips, respectively.

Definition 3.4.7. Let $\mathcal{T}$ be a tiling of a rhombic diagram F. A *D-path* on $\mathcal{T}$ is a path from some point on $P_1(F)$ to some point on $P_2(F)$ consisting of A- and E-edges. An *E-path* on $\mathcal{T}$ is a path from some point on $P_1(F)$ to some point on $P_2(F)$ consisting of D- and A-edges. We introduce the operation of *compressing* a D-strip in $\mathcal{T}$ to obtain a new tiling $\mathcal{T}'$ with a D-path in place of the D-strip (respectively, E-strip and E-path). We also introduce the inverse operation of *blowing up* a D-path in $\mathcal{T}'$ to obtain a new tiling $\mathcal{T}''$ with a D-strip in place of the D-path (respectively, E-path and E-strip).

Compressing a D-strip means selecting its northern border to be the D-path, and then gluing together the north and south E-edges and A-edges of each tile in the D-strip, thereby replacing the D-strip by the D-path. Similarly, compressing an E-strip means selecting its western border to be the E-path, and then gluing together the west and east D-edges and A-edges of each tile in the E-strip, thereby replacing the E-strip with the E-path. If $\mathfrak{s}$ is a D- or E-strip of $\mathcal{T}$, then we denote by $\mathrm{com}(\mathcal{T}, \mathfrak{s})$ the new tiling $\mathcal{T}'$ that results from compressing at $\mathfrak{s}$.

For the inverse, blowing up a D-path means replacing each E-edge of the path with a DE tile, and each A-edge with a DA tile, to obtain a D-strip from the new tiles. Similarly, blowing up an E-path means replacing each D-edge of the path with a DE tile, and each A-edge with an AE tile, to obtain an E-strip from the new tiles. If $\mathfrak{p}$ is a D- or E-path of $\mathcal{T}$, then we denote by $\mathrm{bl}(\mathcal{T}, \mathfrak{p})$ the new tiling $\mathcal{T}''$ that results from blowing up at $\mathfrak{p}$. Figure 3.18 illustrates these definitions.

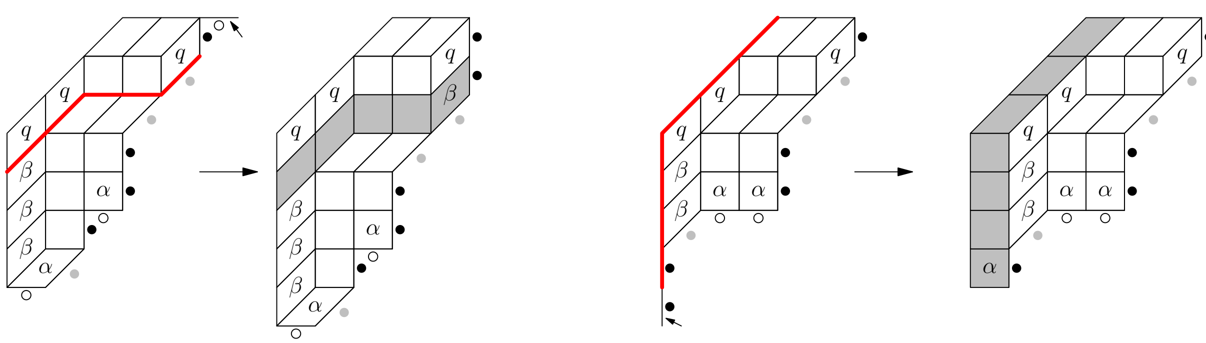
By convention, if $\mathfrak{p}$ is a path of length 0, then blowing up $\mathfrak{p}$ results in replacing it by an empty E-strip or an empty D-strip (depending on whether $\mathfrak{p}$ coincides with the west boundary or the north boundary of the rhombic diagram, respectively). Conversely, compression of an empty E-strip or an empty D-strip results in replacing those strips with a single point.

It is easy to see that compressing is the inverse of blowing up.

Let F be a RAT of size (n, r, k) with tiling $\mathcal{T}$, and let $\psi(F) \in \Psi_{(n,r)}$ denote the equivalence class that F belongs to. Below we describe the RAT chain transitions on F, which are also transitions on $\psi(F)$.

A ● enters from the left.

If F has an empty E-strip $\mathfrak{e}$, then there is a transition in the RAT chain from F that corresponds to a heavy particle entering from the left in the ASEP. Let the type of F be eW.



(a) heavy particle hops on at the left

(b) heavy particle hops off at the right

Figure 3.19: For both examples, let the left tableau have type W and tiling $\mathcal{T}$, and denote the indicated empty (E- or D-) strip by $\mathfrak{e}$ and the marked (E- or D-) path by $\mathfrak{p}$. Then (a) shows the transition $\circ\, W \to \bullet\, W$ and (b) shows the transition $W\bullet\ \to W\circ$. We obtain a new diagram with tiling $\mathrm{bl}(\mathrm{com}(\mathcal{T}, \mathfrak{e}), \mathfrak{p})$, and in (a) a β is placed in the resulting D-strip and in (b) an α is placed in the resulting E-strip.

We define a new RAT T as follows. Let p be the south-most point on $P_1(F)$ (the southeast boundary of F) such that there are exactly $n - k - 1$ E- and A- edges on $P_1(F)$ southwest of p. Let $\mathfrak{p}$ be any D-path originating at p. Let $\mathcal{T}' = \mathrm{bl}(\mathrm{com}(\mathcal{T}, \mathfrak{e}), \mathfrak{p})$. It is easy to check that $\mathcal{T}'$ is a valid tiling of $\Gamma(\bullet\, X)$ which has size $(n, r, k + 1)$.

If $n-k-1>0$, the new D-strip of $\mathcal{T}'$ is non-empty, so we place a β in its right-most tile, which is valid since that tile must be either a DE tile or a DA tile. Furthermore, p was chosen to be the south-most point such that there are $n-k-1$ E- and A-edges southwest of it, so the right-most tile of the new D-strip is also the bottom-most tile of the A- or E-strip it lies in, and thus does not interfere with the rest of the filling of the tableau. We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{\alpha}{N+1}$. The weight of F with the exception of $\mathfrak{e}$ equals the weight of T with the exception of the newly added D-strip. The weight of the new D-strip of T is $\alpha\beta$, and the weight of $\mathfrak{e}$ is β. Therefore, $\mathrm{wt}(T) = \frac{\alpha\beta}{\beta}\,\mathrm{wt}(F)$, and so

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\mathrm{wt}(T)}{N+1}.$$

For the exceptional case, if $n-k-1=0$, then the newly added D-strip of T is empty, and thus has total weight α. In this case, the ASEP state corresponding to F is of the form ed^{n-1}, and the ASEP state corresponding to T is d^n. Then $\mathrm{wt}(F) = \beta\alpha^{n-1}$, $\mathrm{wt}(T) = \alpha^{n-1}$, and so in this case we have

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\beta\,\mathrm{wt}(T)}{N+1}.$$

A $\bullet$ exits from the right.

If F has an empty D-strip $\mathfrak{e}$, then there is a transition in the RAT chain from F that corresponds to a heavy particle exiting from the right in the ASEP. Let the type of F be $W\bullet$.

We define a new RAT T as follows. Let p be the east-most point on $P_1(F)$ such that there are exactly $r+k-1$ D- and A- edges on $P_1(F)$ northeast of p. Let $\mathfrak{p}$ be any E-path originating at p. Let $\mathcal{T}' = \mathrm{bl}(\mathrm{com}(\mathcal{T},\mathfrak{e}),\mathfrak{p})$. It is easy to check that $\mathcal{T}'$ is a valid tiling of $\Gamma(We)$ which has size $(n,r,k-1)$.

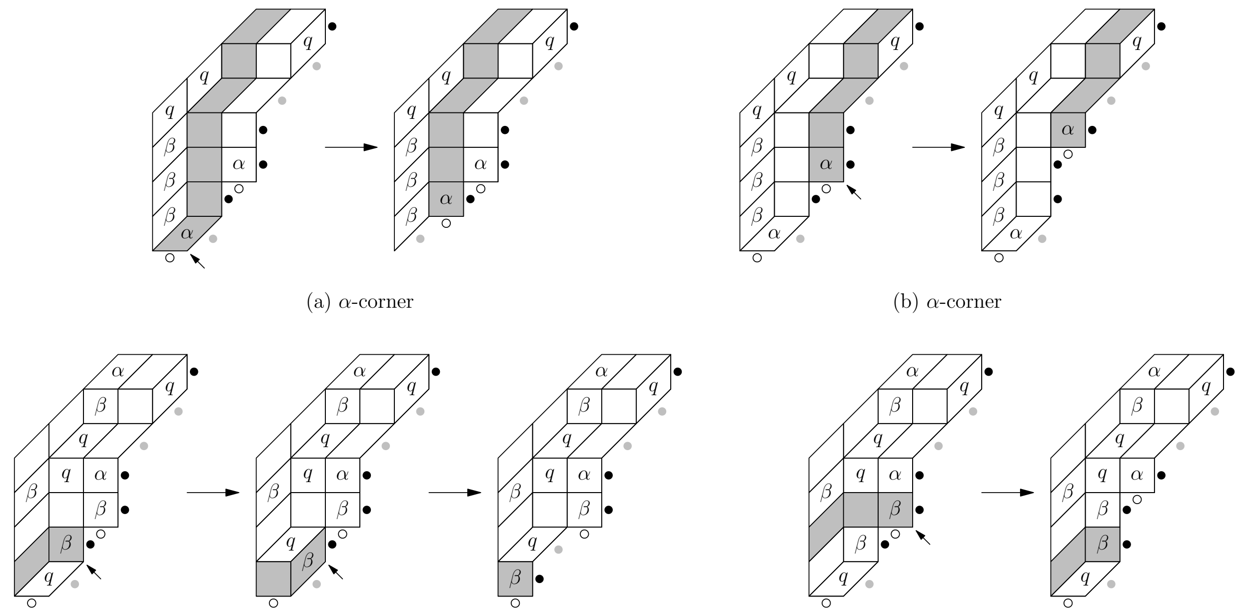
If $r+k-1>0$, the new E-strip of $\mathcal{T}'$ is non-empty, so we place an α in its bottom-most tile, which is valid since that tile must be either a DE tile or an AE tile. Furthermore, p was chosen to be the east-most point such that there are $r+k-1$ D- and A-edges northeast of it, so the bottom-most tile of the new E-strip is also the right-most tile of the A- or D-strip it lies in, and thus does not interfere with the rest of the filling of the tableau. We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{\beta}{N+1}$. The weight of F with the exception of $\mathfrak{e}$ equals the weight of T with the exception of the newly added D-strip. The weight of the new D-strip of T is $\alpha\beta$, and the weight of $\mathfrak{e}$ is α. Therefore, $\mathrm{wt}(T) = \frac{\alpha\beta}{\alpha}\,\mathrm{wt}(F)$, and so

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\mathrm{wt}(T)}{N+1}.$$

For the exceptional case, if $r+k-1=0$, then the newly added E-strip of T is empty, and thus has total weight β. In this case, the ASEP state corresponding to F is of the form

$\bigcirc^{n-1}\bullet$, and the ASEP state corresponding to T is $\bigcirc^n$. Then $\mathrm{wt}(F) = \beta\alpha^{n-1}$, $\mathrm{wt}(T) = \alpha^{n-1}$, and so in this case we have

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\alpha\,\mathrm{wt}(T)}{N+1}.$$



(a) α-corner (b) α-corner

(c) β-corner (d) β-corner

Figure 3.20: (a) A ● ○ → ○ ● transition and a (b) ● ○ → ○ ● transition at an α-corner, and (c) a ● ● → ● ● transition and (d) a ● ○ → ○ ● transition at a β-corner.

A ● exchanges with a ○.

If F has a DE corner, then there is a transition in the RAT chain from F that corresponds to a ● swapping places with a ○ in the ASEP. Let the type of F be W● ○ Y, and suppose it has tiling $\mathcal{T}$. The DE corner necessarily corresponds to a DE tile. This tile contains an α, a β, or a q. We describe these three cases below.

The DE corner tile contains a β.

We define a new RAT T as follows. Let the D-strip containing the DE corner tile have length λ. Let p be the south-most point on $P_1(F)$ such that there are exactly $\lambda - 1$ E- and A- edges on $P_1(F)$ southwest of p. Let $\mathfrak{p}$ be any D-path originating at p. Let $\mathcal{T}' = \mathrm{bl}(\mathrm{com}(\mathcal{T}, \mathfrak{e}), \mathfrak{p})$. It is easy to check that $\mathcal{T}'$ is a valid tiling of $\Gamma(W$○ ● $Y)$, as in Figure 3.20 (d).

If $\lambda - 1 > 0$, then we place a β in the right-most box of the newly inserted D-strip $\mathfrak{s}$. Such a filling is valid since the right-most box (containing the new β) is necessarily the bottom-most box of the E- (or A-) strip that contains it, and so $\mathfrak{s}$ does not interfere with any of the other tiles in T. We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{1}{N+1}$. The weight of T equals the weight of F. Therefore,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\mathrm{wt}(T)}{N+1}.$$

If $\lambda - 1 = 0$, then necessarily F corresponds to a ASEP state $W\bullet\circ\bullet^j$ for some j, and T corresponds to the state $W\circ\bullet^{j+1}$. The newly added D-strip is empty, and so $\mathrm{wt}(F) = \beta\,\mathrm{wt}(T)$. Therefore,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\beta\,\mathrm{wt}(T)}{N+1}.$$

The DE corner tile contains an α.

We define a new RAT T as follows. Let the E-strip containing the DE corner tile have length λ. Let p be the east-most point on $P_1(F)$ such that there are exactly $\lambda - 1$ D- and A- edges on $P_1(F)$ northeast of p. Let $\mathfrak{p}$ be any E-path originating at p. Let $\mathcal{T}' = \mathrm{bl}(\mathrm{com}(\mathcal{T}, \mathfrak{e}), \mathfrak{p})$. It is easy to check that $\mathcal{T}'$ is a valid tiling of $\Gamma(W\circ\bullet Y)$, as in Figure 3.20 (b).

If $\lambda - 1 > 0$, then we place an α in the bottom-most box of the newly inserted E-strip $\mathfrak{s}$. Such a filling is valid since the bottom-most box (containing the new α) is necessarily the right-most box of the D- (or A-) strip that contains it, and so $\mathfrak{s}$ does not interfere with any of the other tiles in T. We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{1}{N+1}$. The weight of T equals the weight of F. Therefore,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\mathrm{wt}(T)}{N+1}.$$

If $\lambda - 1 = 0$, then necessarily F corresponds to a ASEP state $\circ^j\bullet\circ Y$ for some j, and T corresponds to the state $\circ^{j+1}\bullet Y$. The newly added D-strip is empty, and so $\mathrm{wt}(F) = \alpha\,\mathrm{wt}(T)$. Therefore,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\alpha\,\mathrm{wt}(T)}{N+1}.$$

The DE corner tile contains a q.

We define a new RAT T by simply removing the DE corner tile from F. We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{1}{N+1}$. Since a single tile of weight q was removed, $\mathrm{wt}(F) = q\,\mathrm{wt}(T)$. Therefore,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{q\,\mathrm{wt}(T)}{N+1}.$$

A ● exchanges with a ●.

If F has a DA corner, then there is a transition in the RAT chain from F that corresponds to a ● swapping places with a ● in the ASEP. Let the type of F be W● ● Y. By Lemma 3.4.5, we can assume that F has a DA tile at the DA corner. This tile contains a β or a q. We describe these two cases below.

The DA corner tile contains a β.

We perform exactly the same operation as for the DE case containing a β. Once again, we define $\text{Prob}_{RAT}(F \to T) = \frac{1}{N+1}$. In all but the exceptional case, the weight of T equals the weight of F. Therefore,

$$\text{wt}(F)\,\text{Prob}_{RAT}(F \to T) = \frac{\text{wt}(T)}{N+1}.$$

In the special case, if F corresponds to a ASEP state W● ● ●^{j} for some j, and T corresponds to the state W● ●$^{j+1}$, then we have $\text{wt}(F) = \beta\,\text{wt}(T)$. Therefore,

$$\text{wt}(F)\,\text{Prob}_{RAT}(F \to T) = \frac{\beta\,\text{wt}(T)}{N+1}.$$

The DA corner tile contains a q.

We perform exactly the same operation as for the DE case containing a q. Again,

$$\text{wt}(F)\,\text{Prob}_{RAT}(F \to T) = \frac{q\,\text{wt}(T)}{N+1}.$$

A ● exchanges with a ○.

If F has an AE corner, then there is a transition in the RAT chain from F that corresponds to a ● swapping places with a ○ in the ASEP. Let the type of F be W● ○ Y. By Lemma 3.4.5, we can assume that F has an AE tile at the DA corner. This tile contains an α or a q. We describe these two cases below.

The AE corner tile contains an α.

We perform exactly the same operation as for the DE case containing an α. Once again, we define $\text{Prob}_{RAT}(F \to T) = \frac{1}{N+1}$. In all but the exceptional case, the weight of T equals the weight of F. Therefore,

$$\text{wt}(F)\,\text{Prob}_{RAT}(F \to T) = \frac{\text{wt}(T)}{N+1}.$$

In the special case, if F corresponds to a ASEP state ○^{j}● ○Y for some j, and T corresponds to the state ○$^{j+1}$● Y, then we have $\text{wt}(F) = \alpha\,\text{wt}(T)$. Therefore,

$$\text{wt}(F)\,\text{Prob}_{RAT}(F \to T) = \frac{\alpha\,\text{wt}(T)}{N+1}.$$

The AE corner tile contains a q.

We perform exactly the same operation as for the DE case containing a q. Again,

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{q\,\mathrm{wt}(T)}{N+1}.$$

A lighter particle type exchanges with a heavier particle type.

We describe only the $W\circ\bullet Y \to W\bullet\circ Y$ transition, but the same holds true for $W{\color{gray}\bullet}\bullet Y \to W\bullet{\color{gray}\bullet} Y$ and $W\circ{\color{gray}\bullet} Y \to W{\color{gray}\bullet}\circ Y$ if the corresponding letters are used. If F has an inner ED corner, then there is a transition in the RAT chain from F that corresponds to a $\circ$ swapping places with a $\bullet$ in the ASEP. Let the type of F be $W\circ\bullet Y$. Then to form the tableau T, we simply append a DE tile to the outside of F, adjacent to the ED inner corner. We place a q inside the tile, and thus obtain a valid filling T of type $W\bullet\circ Y$ with a q in its DE corner.

We define $\mathrm{Prob}_{RAT}(F \to T) = \frac{q}{N+1}$. Therefore, since $q\,\mathrm{wt}(F) = \mathrm{wt}(T)$, we have

$$\mathrm{wt}(F)\,\mathrm{Prob}_{RAT}(F \to T) = \frac{\mathrm{wt}(T)}{N+1}.$$

The operator PR is clearly a surjective map from the set $\Psi_{(n,r)}$ to Ω_r^n. It is easy to see by our description of the transitions on the RAT chain that it indeed projects to the ASEP chain. Figure 3.21 shows a few transitions on some states of RAT of size $(4,1)$.

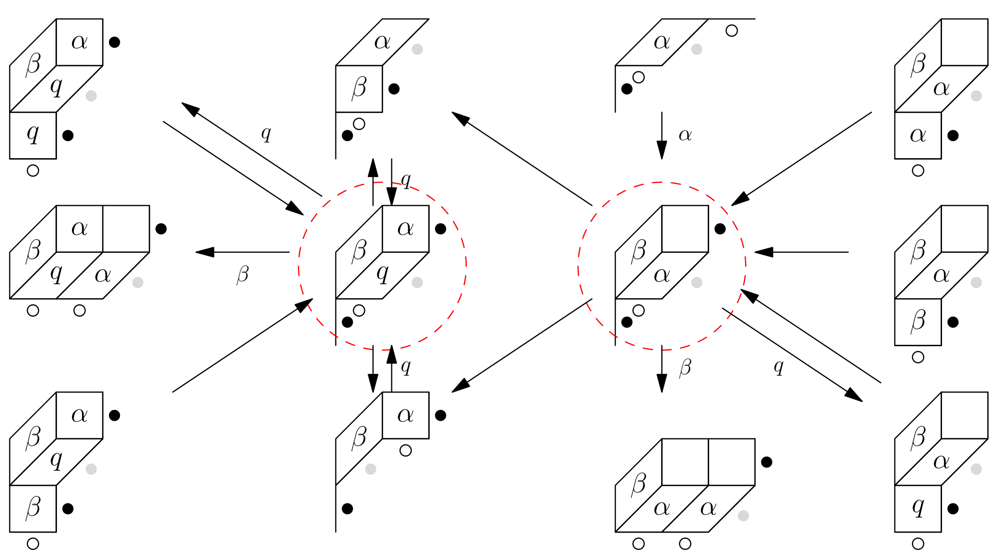


Figure 3.21: Some of the transitions on some of the states in Ω_1^4. All the transitions involving the circled tableaux are included.

Stationary probabilities of the RAT chain

We carefully summarize the transitions out of a RAT F (and consequently from the equivalence class of F), depending on the chosen corner at which the transition occurs. We will be referring to these cases further on. First we make the following definitions. Let F have size (n, r, k) and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be the partition given by the lengths of the D-strips from top to bottom. Assume that λ has at least one non-zero part.

Definition 3.4.8. We define λ_R be the indicator that equals 1 if F has an empty E-strip, and 0 otherwise. We define λ_L be the indicator that equals 1 if F has n empty D-strip, and 0 otherwise.

Definition 3.4.9. We call a *q-corner* a corner that contains a q. (Refer to Definition 3.4.6 for the precise definition.) We call a *top-most corner* an α- or β-corner such that the length of the D-strip containing it equals λ_1. (If the corner in the top-most position contains a q, we do not call it a top-most corner). We define the indicator δ^R_β which equals 1 if the top-most corner contains a β, and 0 if it contains an α. Analogously, we call a *bottom-most corner* an α- or β-corner such that the length of the row containing it equals the length of the smallest non-zero row of λ. (If the corner in the bottom-most position contains a q, we do not call it a bottom-most corner). We define the indicator δ^L_α which equals 1 if the bottom-most corner contains an α, and 0 if it contains a β. We call a *middle corner* an α- or β-corner that is neither a top-most corner or a bottom-most corner (and not a q-corner).

Summary of transitions $F \to T$

Denote by $\pi(F \to T)$ the rate of transition from tableau F to T (where by rate we mean the unnormalized probability). We obtain the following cases for the transitions from F to T.

1. For a transition at a middle corner, a top-most corner with $\delta^R_\beta = 1$, or a bottom-most corner with $\delta^L_\alpha = 1$, we have $\mathrm{wt}(T) = \mathrm{wt}(F)$, and $\pi(F \to F) = 1$.

2. For a transition at a top-most corner with $\delta^R_\beta = 0$ such that the length of the E-strip containing it is greater than 1, we have $\mathrm{wt}(T) = \mathrm{wt}(F)$ and $\pi(F \to T) = 1$. Then the top-most corner of T will be an α-corner.

3. For a transition at a bottom-most corner with $\delta^L_\alpha = 0$ such that the length of the row containing it is greater than 1, we have $\mathrm{wt}(T) = \mathrm{wt}(F)$ and $\pi(F \to T) = 1$. Then the bottom-most corner of T will be a β-corner.

4. For a transition at a top-most corner with $\delta^R_\beta = 0$ such that the length of the E-strip containing it is 1, we have $\mathrm{wt}(T) = \frac{1}{\alpha}\,\mathrm{wt}(F)$ and $\pi(F \to T) = 1$.

5. For a transition at a bottom-most corner with $\delta^L_\alpha = 0$ such that the length of the D-strip containing it is 1, we have $\mathrm{wt}(T) = \frac{1}{\beta}\,\mathrm{wt}(F)$ and $\pi(F \to T) = 1$.

6. For a transition at an empty E-strip, we have $\mathrm{wt}(T) = \alpha\,\mathrm{wt}(F)$ and $\pi(F \to T) = \alpha$. T will not have an empty E-strip, and it will have a top-most corner that contains a β.

7. For a transition at an empty D-strip, we have $\mathrm{wt}(T) = \beta\,\mathrm{wt}(F)$ and $\pi(F \to T) = \beta$. T will not have an empty D-strip, and it will have a bottom-most corner that contains an α.

8. For a transition at an inner corner, we have $\mathrm{wt}(T) = q\,\mathrm{wt}(F)$ and $\pi(F \to T) = q$.

9. For a transition at a q-corner, we have $\mathrm{wt}(T) = \frac{1}{q}\,\mathrm{wt}(F)$ and $\pi(F \to T) = 1$.

Our main theorem is the following.

Theorem 3.4.10. *Consider the RAT chain on $\Psi_{(n,r)}$, the RAT equivalence classes of size (n, r). Fix a RAT F and its equivalence class ψ. Then the steady state probability of state ψ is proportional to* $\mathrm{wt}(F)$.

Proof. To prove the theorem, it suffices to show that for each RAT F, the following detailed balance condition holds. Let $\mathcal{R}$ be the set of RAT such that there exists a transition from F to $T \in \mathcal{R}$. Let $\mathcal{S}$ be the set of *equivalence classes* of RAT such that for each $\psi \in \mathcal{S}$, there exists some $S \in \psi$ such that there is a transition from S to F. Though we actually work with the equivalence classes, we write for simplicity $S \in \mathcal{S}$.

$$\mathrm{wt}(F) \sum_{T \in \mathcal{R}} \pi(F \to T) = \sum_{S \in \mathcal{S}} \mathrm{wt}(S)\pi(S \to F). \tag{3.10}$$

Let the RAT F have type W. First we treat the transitions going out of F to $T \in \mathcal{R}$. By the construction of the RAT chain, it is clear that there is a transition with probability 1 for every corner (including the top-most-, bottom-most-, middle-, and q-corners), a transition with probability α for an empty E-strip, a transition with probability β for an empty D-strip, and a transition with probability q for every inner corner. These transitions directly correspond to all the possible transitions out of the two-species ASEP state W. Suppose F has C_0 q-corners, C α- or β-corners, and I inner corners. Thus we obtain

$$\sum_{T \in \mathcal{R}} \pi(F \to T) = C + C_0 + qI + \alpha\delta_L + \beta\delta_R. \tag{3.11}$$

For the transitions going into F from some $S \in \mathcal{S}$, we observe that any transition from one tableau to another ends with a q-corner or an α- or β-corner, an empty E-strip, an empty D-strip, or an inner corner. Thus it is sufficient to examine all such properties of F to enumerate all the possibilities for $S \in \mathcal{S}$. We examine the pre-image of the cases for the possible transitions going into F to obtain the following cases for S.

1. For a middle corner, a top-most corner with $\delta_\beta^R = 0$, or a bottom-most corner with $\delta_\alpha^L = 0$, we have $\mathrm{wt}(S) = \mathrm{wt}(F)$ and $\pi(S \to F) = 1$. This is the inverse of Case 1 of

Section 3.4. This gives a contribution of $\mathrm{wt}(F)(C-2+(1-\delta_\beta^R)+(1-\delta_\alpha^L))$ to the right hand side (RHS) of the detailed balance equation.[2]

2. For a top-most corner with $\delta_\beta^R=1$ and $\delta_R=0$, we have a transition involving an empty E-strip of S, so $\mathrm{wt}(S)=\frac{1}{\alpha}\mathrm{wt}(F)$ and $\pi(S\to F)=\alpha$. This is the inverse of Case 2 of Section 3.4. This gives a contribution of $\alpha\frac{1}{\alpha}\mathrm{wt}(F)\delta_\beta^R(1-\delta_R)$ to the RHS of the detailed balance equation.

3. For a bottom-most corner with $\delta_\alpha^L=1$ and $\delta_L=0$, we have a transition involving an empty D-strip of S, so $\mathrm{wt}(S)=\frac{1}{\beta}\mathrm{wt}(F)$ and $\pi(S\to F)=\beta$. This is the inverse of Case 3 of Section 3.4. This gives a contribution of $\beta\frac{1}{\beta}\mathrm{wt}(F)\delta_\alpha^L(1-\delta_L)$ to the RHS of the detailed balance equation.

4. For a top-most corner with $\delta_\beta^R=1$ and $\delta_R=1$, there are two possibilities. For the first, S could fall into Case 2 of Section 3.4, meaning that the top-most corner of S is a β-corner, which results in the usual transition with $\mathrm{wt}(S)=\mathrm{wt}(F)$. For the second possibility, S could fall into Case 4 of Section 3.4, meaning that the top-most corner of S is an α-corner and the column containing it has length 1. In that case, $\mathrm{wt}(S)=\alpha\,\mathrm{wt}(F)$. In both situations, $\pi(S\to F)=1$. We obtain a contribution of $\mathrm{wt}(F)\delta_\beta^R\,(\delta_R+\alpha(1-\delta_R))$ to the RHS of the detailed balance equation.

5. For a bottom-most corner with $\delta_\alpha^L=1$ and $\delta_L=1$, there are two possibilities. For the first, S could fall into Case 3 of Section 3.4, meaning that the bottom-most corner of S is an α-corner, which is the usual transition with $\mathrm{wt}(S)=\mathrm{wt}(F)$. For the second possibility, S could fall into Case 5 of Section 3.4, meaning that S has a bottom-most corner containing a β and the row containing it has length 1. In that case, $\mathrm{wt}(S)=\beta\,\mathrm{wt}(F)$. In both situations, $\pi(S\to F)=1$. We obtain a contribution of $\mathrm{wt}(F)\delta_\alpha^L\,(\delta_L+\beta(1-\delta_L))$ to the RHS of the detailed balance equation.

6. For a q-corner, we have $\mathrm{wt}(S)=\frac{1}{q}\mathrm{wt}(F)$ and $\pi(S\to F)=q$. This is the inverse of Case 9 of Section 3.4. We obtain a contribution of $\mathrm{wt}(F)$ to the RHS of the detailed balance equation.

7. For an inner corner, we have $\mathrm{wt}(S)=q\,\mathrm{wt}(F)$ and $\pi(S\to F)=1$. This is the inverse of Case 8 of Section 3.4. We obtain a contribution of $q\,\mathrm{wt}(F)$ to the RHS of the detailed balance equation.

We sum up the contributions to the RHS of the detailed balance equation to obtain

$$\sum_{S\in\mathcal{S}}\mathrm{wt}(S)\pi(S\to F)=\mathrm{wt}(F)(C+C_0+qI-\delta_\beta^R-\delta_\alpha^L+\delta_\beta^R(1-\delta_R)+\delta_\alpha^L(1-\delta_L)$$
$$+\delta_\beta^R(\delta_R+\alpha(1-\delta_R))+\delta_\alpha^L(\delta_L+\beta(1-\delta_L))). \quad (3.12)$$

[2]Note that if $C<2$, the formulas we give have some degeneracies. However, it is easy to verify that these do not cause any problems due to cancellation of all the degenerate terms.

We see that after simplification, Equation 3.12 equals Equation 3.11, so indeed the desired Equation 4.3 holds for "most" F, save for the easily-verified degenerate cases.

$\square$

The proof above circumvents the use of the Matrix Ansatz, and is another way to prove our main result of Theorem 3.2.

Chapter 4

Combinatorics of the k-species ASEP

Following the study of the two-species ASEP, it is natural to study generalizations with k species of particles. We describe one such process in this chapter. In Section 4.1 we describe a generalization of the two-species ASEP to a k-species ASEP. In Section 4.2, we generalize the two-species Matrix Ansatz of Uchiyama to a k-species Matrix Ansatz. In Section 4.3, we provide a proof for Theorem 3.2.1 of the previous chapter by explicitly defining the matrices that both provide the weight generating function of the rhombic alternative tableaux, and also satisfy the Matrix Ansatz hypothesis. Our proofs are analogous to the proofs for the two-species case. In Section 4.4, we define the k-rhombic alternative tableaux, which provide an interpretation for the stationary probabilities of the k-species ASEP, and show that these tableaux satisfy the k-species Matrix Ansatz.

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4.1 The k-species ASEP

We now describe a generalization of the two-species ASEP to a k-species ASEP of a similar flavor. In our new model, we consider k particle species of varying heaviness on a one-dimensional lattice of size n. We call the heaviest particle a d particle, followed by $a_1 > a_2 > \cdots > a_{k-1}$. For easier notation, we also introduce another particle which we call an e particle to represent a hole, and we allow this to be the lightest particle in our set of species. Thus, in our model, every location on the lattice contains exactly one particle out of the set of species $\{d, a_1, \ldots, a_{k-1}, e\}$. Moreover, the d particle is allowed to "enter" on the left at location 1 by replacing an e particle at that location (with rate α), and it is allowed to "exit" on the right at location n by being replaced with an e particle at that location (with rate β). The particles of type a_i are not allowed to enter or exit, so we fix the numbers of particles of those species to be r_i for $i = 1, \ldots, k-1$.

For two particle types A and B, we write $A > B$ (respectively, $A < B$ or $A = B$) to mean that A is a heavier particle type than B (respectively, A is lighter than B, or they are equal). The dynamics in the bulk are the following: a heavier particle of species A can swap with an adjacent lighter particle of species B with rate 1 if A is to the left of B, and with rate $0 \leq q_{AB} \leq 1$ if A is to the right of B. This means that heavier particles have a tendency to move to the right of the lattice. Our notation is shown in the table below:

A	B		q_{AB}
d	a_i	$1 \leq i \leq k-1$	q_{0i}
d	e		$q_{0\infty}$
a_i	a_j	$1 \leq j < i < k-1$	q_{ij}
a_i	e	$1 \leq i \leq k-1$	$q_{i\infty}$.

More precisely, our process is a Markov chain with states represented by words of length n in the letters $\{d, a_1, \ldots, a_{k-1}, e\}$. The transitions in the Markov chain are the following, with X and Y representing arbitrary words in these letters.

$$Xa_ieY \underset{q_{i\infty}}{\overset{1}{\rightleftharpoons}} Xea_iY \qquad XdeY \underset{q_{0\infty}}{\overset{1}{\rightleftharpoons}} XedY \qquad Xda_iY \underset{q_{0i}}{\overset{1}{\rightleftharpoons}} Xa_idY \qquad Xa_ia_jY \underset{q_{ij}}{\overset{1}{\rightleftharpoons}} Xa_ja_iY$$

$$eX \overset{\alpha}{\rightharpoonup} dX \qquad\qquad Xd \overset{\beta}{\rightharpoonup} Xe$$

for $1 \leq i \leq k-1$ and $1 \leq j < i$.

where by $X \overset{u}{\rightharpoonup} Y$ we mean that the transition from X to Y has probability $\frac{u}{n+1}$, n being the length of X (and also Y).

Definition 4.1.1. For a given k-species ASEP, we fix n to be the size of the lattice and r_i to be the number of particles of species a_i for $1 \leq i \leq k-1$. We define $\Omega^n_{r_1,\ldots,r_{k-1}}$ to be the set of words of length n in the letters $\{d, a_1, \ldots, a_{k-1}, e\}$ with r_i instances of the letter a_i for each i. We also define

$$\Omega^n = \bigcup_{r_1,\ldots,r_{k-1}} \Omega^n_{r_1,\ldots,r_{k-1}}.$$

Remark 4.1.2. In Section 4.2, we will provide a Matrix Ansatz solution for the model with different parameters q_i for every type of transition. However, so far we only have nice combinatorics when all the q_i's are set to equal a single constant q. Furthermore, it is easy to see that if $k = 2$, we recover the two-species ASEP that we described in the previous section, and if $k = 1$, we recover the original ASEP.

4.2 The Matrix Ansatz for the k-species ASEP

Building on a Matrix Ansatz solution for the usual ASEP by Derrida at. al. [8] and a more general solution for the two-species ASEP by Uchiyama in [19], we have the following generalization for the k-species process.

Theorem 4.2.1. *Let $W = W_1 \dots W_n$ with $W_i \in \{d, a_1, \dots, a_{k-1}, e\}$ for $1 \le i \le n$ represent a state of the k-species ASEP in $\Omega^n_{r_1,\dots,r_{k-1}}$. Suppose there are matrices D, $A_1, \dots, A_{k-1}$, and E and a row vector $\langle w|$ and a column vector $|v\rangle$ (with $\langle w||v\rangle = 1$) which satisfy the following conditions*

$$DE - q_{0\infty}ED = D + E, \quad DA_i - q_{0i}A_iD = A_i, \quad A_iE - q_{i\infty}EA_i = A_i, \quad A_iA_j - q_{ij}A_jA_i = 0, \tag{4.1}$$

$$\langle w|E = \frac{1}{\alpha}\langle w|, \qquad D|v\rangle = \frac{1}{\beta}|v\rangle, \tag{4.2}$$

then

$$\mathrm{Prob}(W) = \frac{1}{Z_{n,r_1,\dots,r_{k-1}}}\langle w|\prod_{i=1}^{n} D\,\mathbb{1}_{(W_i=d)} + E\,\mathbb{1}_{(W_i=e)} + \sum_{i=1}^{k-1} A_i\,\mathbb{1}_{(W_i=a_i)}\,|v\rangle$$

where $Z_{n,r_1,\dots,r_{k-1}}$ is the coefficient of $y_1^{r_1}\dots y_{k-1}^{r_{k-1}}$ in

$$\frac{\langle w|(D + y_1A_1 + \dots + y_{k-1}A_{k-1} + E)^n|v\rangle}{\langle w|A_{k-1}^{r_{k-1}}\cdots A_1^{r_1}|v\rangle}.$$

Proof. For W a word of length n, we define the weight

$$f_n(W) = \langle w|\prod_{i=1}^{n} D\,\mathbb{1}_{(W_i=d)} + E\,\mathbb{1}_{(W_i=e)} + \sum_{i=1}^{k-1} A_i\,\mathbb{1}_{(W_i=a_i)}\,|v\rangle.$$

We show that $f_n(W)$ satisfies the detailed balance conditions

$$f_n(W)\sum_{W\to V}\mathrm{Pr}(W\to V) = \sum_{X\to W} f_n(X)\,\mathrm{Pr}(X\to W) \tag{4.3}$$

for each $W \in \Omega^n$, where by $\mathrm{Pr}(W \to V)$ and $\mathrm{Pr}(X \to W)$ we denote the probabilities of the transitions $W \to V$ and $X \to W$ respectively. This would imply that the stationary probability of state W is proportional to $f_n(W)$, which would complete the proof.

We observe that for fixed W, the only terms $f_n(X)\,\mathrm{Pr}(X \to V)$ for some $X, V \in \Omega^n$ appearing in (4.3), are precisely the terms:

i. $f_n(eW_2 \dots W_n)\alpha$,

ii. $f_n(W_1 \dots W_{n-1}d)\beta$,

iii. and $\{f_n(W_1 \dots W_{i-1}BCW_{i+2} \dots W_n)\cdot 1, -f_n(W_1 \dots W_{i-1}CBW_{i+2} \dots W_n)\cdot q_{BC}\}$ where $W_iW_{i+1} = BC$ for $B > C$ over $1 \le i \le n-1$.

This is because these terms are precisely the terms out of which possible transitions can occur to go into or out of W. Moreover, whether the terms of (iii.) appear on the left hand side of Equation (4.3) or the right hand side is determined by whether $W_iW_{i+1} = BC$

or $W_iW_{i+1} = CB$ for $B > C$. In other words, the terms in the bulk are given a sign of $(-1)^{\mathbb{1}_{(W_{i+1}>W_i)}}$ for each i, and the boundary terms of (i.) and (ii.) are given a sign of $(-1)^{\mathbb{1}_{(W_1=d)}}$ and $(-1)^{\mathbb{1}_{(W_n=e)}}$ for the left and right boundaries, respectively.

Thus Equation (4.3) can be rewritten as the following:

$$\begin{aligned}\mathbb{1}_{(W_1=d \text{ or } e)}(-1)^{\mathbb{1}_{(W_1=d)}}\alpha f_n(eW_2\dots W_n)& \\ +\mathbb{1}_{(W_n=d \text{ or } e)}(-1)^{\mathbb{1}_{(W_n=e)}}\beta f_n(W_1\dots W_{n-1}d)& \\ +\sum_{i=1}^{n-1}\mathbb{1}_{(W_i\neq W_{i+1})}(-1)^{\mathbb{1}_{(W_{i+1}>W_i)}}\Big(f_n(W_1\dots W_{i-1}B_iC_iW_{i+2}\dots W_n)& \\ -q_{B_iC_i}f_n(W_1\dots W_{i-1}C_iB_iW_{i+2}\dots W_n)\Big)&\end{aligned} \tag{4.4}$$

where in the above we use $B_i := \max(W_i, W_{i+1})$ and $C_i := \min(W_i, W_{i+1})$.

The reduction rules of Equation (4.1) or (4.2) apply whenever $W_1 = d$ or e, or $W_n = d$ or e, or whenever $W_i \neq W_{i+1}$ for $1 \le i < n$. We obtain the following.

$$f_n(W'deW'') - q_{0\infty}f_n(W'edW'') = f_{n-1}(W'dW'') + f_{n-1}(W'eW''), \tag{4.5}$$
$$f_n(W'da_iW'') - q_{0i}f_n(W'a_idW'') = f_{n-1}(W'a_iW''), \tag{4.6}$$
$$f_n(W'a_ieW'') - q_{i\infty}f_n(W'ea_iW'') = f_{n-1}(W'a_iW''), \tag{4.7}$$
$$f_n(W'deW'') - q_{ij}f_n(W'edW'') = 0, \tag{4.8}$$
$$\alpha f_n(eW'') = f_{n-1}(W''), \tag{4.9}$$
$$\beta f_n(W'd) = f_{n-1}(W'). \tag{4.10}$$

For $W = W_1\dots W_n$, we introduce the notation $f^i_{n-1}(W) = f_{n-1}(W_1\dots\hat{W}_i\dots W_n)$ to be the weight of the word W with the letter W_i cut out. With this notation, using the reduction rules of Equation (4.5), Equation (4.4) becomes the sum $a_0 + a_1 + \dots + a_{n-1} + a_n$, where

$$a_0 = \begin{cases} f^1_{n-1}(W) & W_1 = e \\ -f^1_{n-1}(W) & W_1 = d \end{cases}, \qquad a_n = \begin{cases} f^n_{n-1}(W) & W_n = d \\ -f^n_{n-1}(W) & W_n = e \end{cases},$$

$$\text{and} \quad a_i = \begin{cases} f^i_{n-1}(W) + f^{i+1}_{n-1}(W) & \text{if } W_iW_{i+1} = de \text{ or } ed \\ f^i_{n-1}(W) & \text{if } W_iW_{i+1} = da_i \\ -f^{i+1}_{n-1}(W) & \text{if } W_iW_{i+1} = a_id \\ f^{i+1}_{n-1}(W) & \text{if } W_iW_{i+1} = a_ie \\ -f^i_{n-1}(W) & \text{if } W_iW_{i+1} = ea_i \end{cases} \quad \text{for } 1 \le i \le n-1. \tag{4.11}$$

Notice that for all $i > j$, the terms $f_n(W'a_ia_jW'') - q_{i,j}f_n(W'a_ja_iW'') = 0$.

Suppose there are a total of s transitions in the bulk. For $j = 1, \dots, s$, label the location i where the j'th transition occurs (i.e. the j'th i for which $W_i \neq W_{i+1}$) by W_{t_j}. The strategy of our proof is to show that all the f_{n-1} terms that arise from the transitions at the locations $\{t_j\}_{1\leq j\leq s}$ cancel with other terms Equation (4.11) with an opposite sign. We describe these cancellations in the cases that follow.

(a.) $W_{t_j}W_{t_j+1} = de$, so the contribution of terms from this transition is $f_{n-1}^{t_j}(W) + f_{n-1}^{t_j+1}(W)$. Then $W_{t_{j+1}}W_{t_{j+1}+1}$ is necessarily either ed or ea_t for some t, in which case it contributes the term $-f_{n-1}^{t_{j+1}}(W)$. Similarly, $W_{t_{j-1}}W_{t_{j-1}+1}$ is necessarily either de or a_ue for some u, in which case it contributes the term $-f_{n-1}^{t_{j-1}+1}(W)$. However, the former of these cancels with the term $f_{n-1}^{t_j}(W)$, and the latter cancels with $f_{n-1}^{t_j+1}(W)$, as desired.

There are two exceptions to the above. First, if $j = 1$, then there is no t_{j-1} term. However, in this case, W necessarily begins with a d, and so the $f_{n-1}^{t_j}(W)$ term cancels with the left boundary term $-f_{n-1}^{1}(W)$. Second, if $j = n$, then there is no t_{j+1} term. However, in this case, W necessarily ends with an e, and so the $f_{n-1}^{t_j+1}(W)$ term cancels with the right boundary term $-f_{n-1}^{n}(W)$.

(b.) $W_{t_j}W_{t_j+1} = ed$, so the contribution of terms from this transition is $-f_{n-1}^{t_j}(W) - f_{n-1}^{t_j+1}(W)$. Then $W_{t_{j-1}}W_{t_{j-1}+1}$ is necessarily either de or a_te for some t, in which case it contributes the term $f_{n-1}^{t_{j-1}+1}(W)$. Similarly, $W_{t_{j+1}}W_{t_{j+1}+1}$ is necessarily either de or da_u for some u, in which case it contributes the term $f_{n-1}^{t_{j+1}}(W)$. However, the former of these cancels with the term $-f_{n-1}^{t_j}(W)$, and the latter cancels with $-f_{n-1}^{t_j+1}(W)$, as desired.

There are two exceptions to the above. First, if $j = 1$, then there is no t_{j-1} term. However, in this case, W necessarily begins with an a, and so the $-f_{n-1}^{t_j}(W)$ term cancels with the left boundary term $f_{n-1}^{1}(W)$. Second, if $j = n$, then there is no t_{j+1} term. However, in this case, W necessarily ends with a d, and so the $-f_{n-1}^{t_j+1}(W)$ term cancels with the right boundary term $f_{n-1}^{n}(W)$.

The rest of the cases are similar. Below, we describe the cancellations that occur for each transition location.

(c.) $W_{t_j}W_{t_j+1} = da_t$, so the contribution of terms from this transition is $f_{n-1}^{t_j}(W)$. This term cancels with the term $-f_{n-1}^{t_{j-1}+1}(W)$ since $W_{t_{j-1}}W_{t_{j-1}+1}$ must equal ed or a_ud for some u.

(d.) $W_{t_j}W_{t_j+1} = a_td$, so the contribution of terms from this transition is $-f_{n-1}^{t_j+1}(W)$. This term cancels with the term $f_{n-1}^{t_{j+1}}(W)$ since $W_{t_{j+1}}W_{t_{j+1}+1}$ must equal de or da_u for some u.

(e.) $W_{t_j}W_{t_j+1} = a_te$, so the contribution of terms from this transition is $f_{n-1}^{t_j+1}(W)$. This term cancels with the term $-f_{n-1}^{t_{j+1}}(W)$ since $W_{t_{j+1}}W_{t_{j+1}+1}$ must equal ed or ea_u for some u.

(f.) $W_{t_j}W_{t_j+1} = ea_t$, so the contribution of terms from this transition is $-f_{n-1}^{t_j}(W)$. This term cancels with the term $f_{n-1}^{t_{j-1}+1}(W)$ since $W_{t_{j-1}}W_{t_{j-1}+1}$ must equal de or $a_u e$ for some u.

The cancellations of the boundary terms are treated as the exceptions in cases of (a) and (b).

It is easy to check from the above that every term cancels with another term in Equation (4.4), so indeed, it equals zero. Thus the function f_n satisfies the detailed balance in Equation (4.3), as desired. □

4.3 Matrix Ansatz proof of Theorem 3.2

In this section we return to the two-species ASEP and give a new proof of Theorem 3.2, the main result of Chapter 3, by explicitly defining matrices D, A, and E and row vector $\langle v|$ and column vector $|w\rangle$ that satisfy the hypotheses of a slightly more general Matrix Ansatz, and also have a combinatorial interpretation in terms of the rhombic alternative tableaux. This construction will serve as a warm-up for the proof of the more general analogue of the theorem for the k-species ASEP, which we provide in Section 4.4.

Definition of our matrices

Our matrices are infinite and indexed by a pair of non-negative integers in both row and column, so $D = [D_{(i,j)(u,v)}]_{i,j,u,v\geq 0}$, $A = [A_{(i,j)(u,v)}]_{i,j,u,v\geq 0}$, and $E = [E_{(i,j)(u,v)}]_{i,j,u,v\geq 0}$. Our vectors are also indexed by a pair of integers, so $\langle v| = [v_{(i,j)}]_{i,j\geq 0}$ and $|w\rangle = [w_{(u,v)}]^T_{u,v\geq 0}$.

We define $v_{(i,j)} = 1$ for $i = 0, j = 0$, and 0 for all other indices. We define $w_{(u,v)} = 1$ for all indices. Also, let

$$D_{(i,j)(i+1,j)} = \frac{1}{\beta} \tag{4.12}$$

and $D_{(i,j)(k,\ell)} = 0$ for all other indices i, j, k, ℓ. Let

$$A_{(i,j)(u,j+1)} = \binom{i}{u} q^u \beta^{i-u} \tag{4.13}$$

for $0 \leq u \leq i$ and $A_{(i,j)(k,\ell)} = 0$ for all other indices i, j, k, ℓ. Finally, let

$$E_{(i,j)(u,j)} = \frac{\beta^{i-u}}{\alpha}\left[\binom{i}{u} q^u (q^j + \alpha[j]_q) + \alpha \sum_{w=0}^{u-1} \binom{i-u+w}{i-u} q^w\right] \tag{4.14}$$

for $0 \leq u \leq i$ and $E_{(i,j)(k,\ell)} = 0$ for all other indices i, j, k, ℓ. (Here $[j]_q = q^{j-1} + \ldots + 1$.)

Since (i, j) specify the row of the matrices, and (u, v) specify the columns, multiplication is defined as

$$(MN)_{(i,j),(k,\ell)} = \sum_{u,v} M_{(i,j),(u,v)} N_{(u,v),(k,\ell)}.$$

Note that in the case of the matrices D, A, and E of Equations (4.12), (4.13), and (4.14), all products are given by finite sums, since the matrix entries are 0 for $u \geq i+1$ or $v \geq j+1$.

To facilitate our proof, we provide a more flexible Matrix Ansatz that generalizes Theorem 3.0.1 with the same argument as in an analogous proof for the ordinary ASEP of Corteel and Williams [5, Theorem 5.2]. For consistency with the k-species ASEP notation, in this section we denote ● by a d particle, ● by an a particle, and ○ by an e particle. For a word $W \in \{d,a,e\}^n$ with r a's, as before we define unnormalized weights $f(W)$ which satisfy

$$\Pr(W) = f(W)/Z_{n,r}$$

where $Z_{n,r} = \sum_{W'} f(W')$ where the sum is over all words $W' \in \{d,a,e\}$ of length n and with r a's.

Theorem 4.3.1. *Let λ be a constant. Let $\langle w|$ and $|v\rangle$ be row and column vectors with $\langle w||v\rangle = 1$. Let $\tilde{D}$, $\tilde{E}$, and $\tilde{A}$ be matrices such that for any words X and Y in $\{\tilde{D}, \tilde{A}, \tilde{E}\}$ representing a product of those matrices in the corresponding order, the following conditions are satisfied:*

I. $\langle w|X(\tilde{D}\tilde{E} - q\tilde{E}\tilde{D})Y|v\rangle = \lambda\langle w|X(\tilde{D}+\tilde{E})Y|v\rangle$,

II. $\langle w|X(\tilde{D}\tilde{A} - q\tilde{A}\tilde{D})Y|v\rangle = \lambda\langle w|X\tilde{A}Y|v\rangle$,

III. $\langle w|X(\tilde{A}\tilde{E} - q\tilde{E}\tilde{A})Y|v\rangle = \lambda\langle w|X\tilde{A}Y|v\rangle$,

IV. $\beta\langle w|X\tilde{D}|v\rangle = \lambda\langle w|X|v\rangle$,

V. $\alpha\langle w|\tilde{E}Y|v\rangle = \lambda\langle w|Y|v\rangle$.

Let $W = W_1 \dots W_n$ with $W_i \in \{d,a,e\}$ for $1 \leq i \leq n$ represent a state of the two-species ASEP of length n with r a's. Then

$$f(W) = \frac{1}{\langle w|\tilde{A}^r|v\rangle}\langle w|\prod_{i=1}^{n} \tilde{D}\,\mathbb{1}_{(W_i=d)} + \tilde{A}\,\mathbb{1}_{(W_i=a)} + \tilde{E}\,\mathbb{1}_{(W_i=e)}\,|v\rangle.$$

Proof. The proof of Theorem 4.3.1 follows exactly that of [5, Theorem 5.2]. Note that the above implies that

$$Z_{n,r} = [y^r]\frac{\langle w|(\tilde{D}+y\tilde{A}+\tilde{E})^n|v\rangle}{\langle w|\tilde{A}^r|v\rangle}.$$

□

Combinatorial interpretation of the matrices in terms of tableaux

Let W be an arbitrary word in $\{d, a, e\}$ with rhombic diagram $\Gamma(W)$ with the maximal tiling $\mathcal{T}_{max}$, and let weight(W) be the weight generating function for fi$(W, \mathcal{T}_{max})$.

Definition 4.3.2. We call a *D-strip* the section of a west-strip to the left of one of the D-edges in the strip. A **free D-strip** is a D-strip that does not contain a β.

Similarly, we call an *E-strip* the section of a north-strip to above one of the E-edges in the strip. A **free E-strip** is an E-strip that does not contain an α.

A *free DA tile* is one that is contained in a free D-strip. A *free AE tile* is one that is contained in a free E-strip. A **free DE tile** is one that is contained in a free D-strip and a free E-strip.

For a word $W \in \{d, a, e\}$, we fix the maximal tiling $\mathcal{T}_{\max}$ of $\Gamma(W)$. We will show that the matrices D, A, and E of Equations (4.12), (4.13), and (4.14) represent the addition of a D-edge, an A-edge, and an E-edge to the bottom of $\Gamma(W)$ to form the rhombic diagram $\Gamma(Wd)$, $\Gamma(Wa)$, and $\Gamma(We)$ respectively. Recall that these matrices have rows indexed by the pair (i, j) and columns indexed by the pair (u, v). We let i represent the number of free D-strips in a tableau $F \in$ fi$(W, \mathcal{T}_{max}(W))$, and j the number of a's in W. For the columns, we let u represent the number of free D-strips in a tableau $F' \in$ fi$(Wd, \mathcal{T}_{max}(Wd))$ (and respectively, $\Gamma(Wa)$ and $\Gamma(We)$), and v the number of a's in Wd (and respectively, Wa and We).

Recall from Definition 3.2.3 that $x(W)$ is a word in $\{D, A, E\}$ representing a matrix product corresponding to the two-species ASEP word W in the letters $\{d, a, e\}$, where D, A, and E correspond to d, a, and e respectively.

Theorem 4.3.3. *Let W be a word in $\{d, a, e\}$, and let $X = x(W)$. Then:*

- $X_{(i,j)(u,v)}$ *is the generating function for all ways of adding $|W|$ new edges of type W to the southwest boundary of a rhombic alternative tableau with i free D-strips and j A-strips, to obtain a new rhombic alternative tableau with u free D-strips and v A-strips.*

- $(\langle w|X)_{(u,v)}$ *is the generating function for rhombic alternative tableaux of type W, which have u free D-strips and v A-strips.*

- $\langle w|X|v\rangle$ *is the generating function for all rhombic alternative tableaux of type W.*

We prove Theorem 4.3.3 with the following lemma, which says that the matrices D, A, and E of Equations (4.12), (4.13), and (4.14) are "transfer matrices" for building rhombic alternative tableaux with the maximal tiling.

Lemma 4.3.4. *For the matrices D, A, and E of Equations* (4.12), (4.13), *and* (4.14),

- $D_{(i,j)(u,v)}$ *is the generating function that represents the addition of a D-edge,*

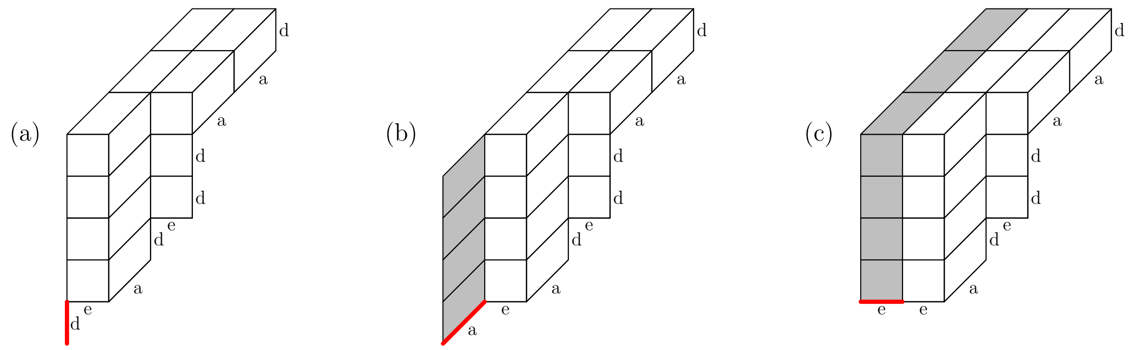

Figure 4.1: Adding a (a) d, (b) a, or (c) e to the end of W.

- $A_{(i,j)(u,v)}$ *is the generating function that represents the addition of an A-edge, and*
- $E_{(i,j)(u,v)}$ *is the generating function that represents the addition of an E-edge*

to the southwest corner of a rhombic alternative tableau with the maximal tiling with i free D-strips and j A-strips, resulting in a rhombic alternative tableau with the maximal tiling with u free D-strips and j A-strips.

Proof. We describe the possible rhombic alternative tableaux that arise from the addition of a D-edge, an A-edge, and an E-edge respectively to the southwest corner of an existing RAT of shape W with the maximal tiling, and i free D-strips and j A-strips.

The addition of the D-edge to $\Gamma(W)$ does not affect the interior of the tableau, as in the example of Figure 3.5 (a), and the tiling of the new tableau is clearly still a maximal one. Thus for any $F \in \text{fi}(W, \mathcal{T}_{max}(W))$, we obtain $F' \in \text{fi}(Wd, \mathcal{T}_{max}(Wd))$ whose weight simply increases by α, the weight of the new D-edge. We have thus $\text{wt}(F') = \alpha \, \text{wt}(F)$. Moreover, the addition of the D-edge adds exactly one free D-strip to F. Recall

$$D_{(i,j)(i+1,j)} = \frac{1}{\beta}$$

and 0 for all other indices, so we obtain the desired entry in the matrix D.

The addition of the A-edge and a vertical strip of adjacent DA tiles to the left boundary of $\mathcal{T}_{max}(W)$ results in a maximal tiling of $\Gamma(Wa)$, as in the example of Figure 3.5 (b). Let us consider the entry $(i,j),(u,j+1)$ of A for $0 \leq u \leq i$. Each free DA tile contains either a q or a β with no restrictions on their positions, for a total of $i-u$ β's and u q's. Thus there are precisely $\binom{i}{u}$ ways to choose such a filling of the new tiles. Every such filling contributes a weight of $q^u\beta^{i-u}$. Wa now has $j+1$ a's, and it is clear that all other entries of A are zero. Recall

$$A_{(i,j)(u,j+1)} = \binom{i}{u} q^u \beta^{i-u}$$

for $0 \leq u \leq i$ and 0 for all other indices, so we obtain the desired entry in the matrix A.

The addition of the E-edge and a vertical strip of adjacent DE tiles followed by j adjacent AE tiles to the left boundary of $\mathcal{T}_{max}(W)$ results in a maximal tiling of $\Gamma(We)$, as in the example of Figure 3.5 (c). Let us call this strip of new tiles the new E-strip. There are three possible cases for this new E-strip. For the following, let us consider the entry $(i,j),(u,j)$ of E for $0 \leq u \leq i$.

Case 1: the new E-strip does not contain an α. Then each of the j AE tiles must contain a q, and each of the i free DE tiles contains either a q or a β, with no restrictions on their positions, with exactly $i-u$ β's and u q's. This gives a total weight contribution of $\binom{i}{u}\beta^{i-u}q^{u+j}$.

Case 2: the new E-strip contains an α in one of the AE tiles. Then each of the AE tiles below that α must contain a q, and each of the free i DE tiles contains either a q or a β, with no restrictions on their positions, with exactly $i-u$ β's and u q's. This gives a total weight contribution of $\binom{i}{u}\alpha\beta^{i-u}q^u[j]_q$.

Case 3: the new E-strip contains an α in one of the free DE tiles. Then exactly $i-u$ of the free DE tiles below the α must contain a β, and u of them contain a q. This gives a total weight contribution of $\beta^{i-u}\alpha\sum_{w=0}^{u-1}\binom{i-u+w}{i-u}q^w$.

Recall

$$E_{(i,j)(u,j)} = \frac{\beta^{i-u}}{\alpha}\left[\binom{i}{u}q^u(q^j+\alpha[j]_q) + \alpha\sum_{w=0}^{u-1}\binom{i-u+w}{i-u}q^w\right]$$

for $0 \leq u \leq i$ and 0 for all other indices, so we obtain the desired entry in the matrix E. □

Proof of 4.3.3. The first point is immediate from Lemma 4.3.4.

The second point is due to the following: $\langle w|$ is a row vector for which the entry with index $(0,0)$ is 1, and the rest are 0. By the first point, $(\langle w|X)_{(0,0),(u,v)}$ is, in particular, the generating function for adding $|W|$ new edges of type W to the southwest boundary of a trivial RAT of size 0, to result in a RAT of type W with the maximal tiling with u free D-strips and v A-strips.

The third point is due to the following: $|v\rangle$ is a column vector with every entry equal to 1. By the second point, the generating function for all possible RAT in $\mathrm{fi}(W,\mathcal{T}_{max}(W))$ is the sum of RAT of type W over all choices for the number of A-strips and free D-strips in the fillings. In other words, it is the sum over all (u,v) of $(\langle w|X)_{(0,0),(u,v)}$. It follows that $\langle w|X|v\rangle$ is the desired generating function. □

Combinatorial proof that our matrices satisfy the Matrix Ansatz

Using Theorem 4.3.3, we provide simple combinatorial proofs that our matrices satisfy the equations of Theorem 4.3.1. Let W be a word in $\{d, a, e\}$ with $\Gamma(W)$ its rhombic diagram. In this subsection, when we say "addition of a d (or a or e) to W", we mean adding a D-edge (or A- or E-edge) to the southwest point of $\Gamma(W)$, as described in the preceding subsection.

I. For D, E of Equations (4.12), (4.14), we have $DE - qED = \alpha\beta(D + E)$.

By our construction, consecutive addition of a d and a e to W results in a DE corner with a DE corner tile as the bottom-most tile of the E-strip that contains it (as well as the right-most tile of the D-strip that contains it). This DE corner tile contains an α, β, or q.

- If the DE corner tile contains an α, then the rest of the E-strip containing this tile must be empty. Thus the entire E-strip has weight $\alpha\beta$, and the rest of the tableau has the same weight as if the DE were replaced by a D-edge (with the same filling in the corresponding tiles).
- If the DE corner tile contains a β, then the rest of the D-strip containing this tile must be empty. Thus the entire D-strip has weight $\alpha\beta$, and the rest of the tableau has the same weight as if the DE were replaced by an E-edge (with the same filling in the corresponding tiles).
- If the DE corner tile contains a q, then this tile has no effect on the rest of the tableau which has the same weight as if the D- and E-edges were replaced by E- and D-edges (with the same tiling and filling), and the tile itself has weight q.

Combining the above cases, we obtain that $DE = qED + \alpha\beta(D + E)$, as desired.

II. For D, A of Equations (4.12), (4.13), we have $DA - qAD = \alpha\beta A$.

By our construction, consecutive addition of a D-edge and an A-edge results in a DA corner with a DA corner tile as the right-most tile of the D-strip that contains it. This DE corner tile contains a β or q.

- If the DA corner tile contains a β, then the rest of the D-strip containing this tile must be empty. Thus the entire D-strip has weight $\alpha\beta$, and the rest of the tableau has the same weight as if the DA were replaced by an A-edge (with the same filling in the corresponding tiles).
- If the DA corner tile contains a q, then this tile has no effect on the rest of the tableau which has the same weight as if the DA were replaced by an AD (with the same tiling and filling), and the tile itself has weight q.

Combining the above cases, we obtain that $DA = qAD + \alpha\beta A$, as desired.

III. For A, E of Equations (4.13), (4.14), we have $AE - qEA = \alpha\beta A$.

Definition 4.3.5. We call an **AE strip** the region of the rhombic diagram that corresponds to a maximal A-strip together with an adjacent maximal E-strip. (By maximal A- and E-strips, we mean A- and E-strips as they would appear in a maximal tiling of a rhombic diagram, i.e. a strip of adjacent DA tiles for the A-strip as in Figure 4.1 (b), and a vertical strip of adjacent DE tiles followed by a strip of adjacent AE tiles for the E-strip as in Figure 4.1 (c).) We allow any valid tiling for the AE strip, and we call an AE strip **maximal** if it has the maximal tiling, and we call it **minimal** if it has the minimal tiling. Note that a minimal AE strip has an AE corner tile in the AE corner.

By our construction, consecutive addition of an A-edge and an E-edge results in a maximal AE strip. For our proof, we consider the corresponding minimal AE strip. We apply a series of flips to convert the maximal AE strip to a minimal AE strip, and we consider the contents of its AE corner tile. This AE corner can contain an α or q. If the AE corner tile contains an α, then the rest of the E-strip containing this tile must be empty. Thus the entire E-strip has weight $\alpha\beta$, and the rest of the tableau has the same weight as if the E-strip were removed entirely. This operation is the same as if in the original tableau, the AE were replaced by an A-edge (with the same filling in the corresponding tiles).

For the other case, if the AE corner tile contains a q, then this tile has no effect on the rest of the tableau. Thus the weight of the tableau with the exception of the AE corner tile is the same as the weight of a tableau with the same tiling and filling with the AE replaced by an EA. Moreover, this new tableau (with the AE corner tile removed from the minimal AE strip) is in fact the maximal tableau that corresponds to replacing the AE by an EA. Thus we have as desired, $AE = qEA + \alpha\beta A$ from these two cases.

Remark 4.3.6. It is also possible to directly compute the $(i,j),(u,v)$ entry of each term of the equations of Theorem 4.3.1, and show that equality holds in each case.

4.4 k-rhombic alternative tableaux

In this section, we introduce a combinatorial object that generalizes the RAT to provide an interpretation for the probabilities of the k-species ASEP. This object, called the k-rhombic alternative tableau (or k-RAT) is of the same flavor as the RAT, and is similarly defined as follows.

Definition of the k-rhombic alternative tableaux

To a word $W \in \Omega^n_{r_1,\dots,r_{k-1}}$, we associate a k-rhombic diagram $\Gamma(W)$ as follows.

Definition 4.4.1. Let $W \in \Omega^n_{r_1,\dots,r_{k-1}}$, and let r_0 be the number of e's and r_k the number of d's in W. Let an E-edge be a unit edge oriented in the direction $-\pi$. Let a D-edge be a unit edge oriented in the direction $-\pi/2$. Let an A_i-edge be a unit edge oriented in the direction $-\frac{(k+i)\pi}{2k}$ (see Figure 4.2). Define $P_1(W)$ to be the lattice path composed of the

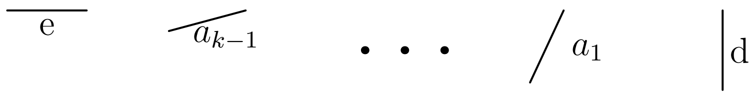

Figure 4.2: E-edge, A_{k-1}-edge, ..., A_1-edge, D-edge

E-, A_1-, ..., A_{k-1}-, and D-edges, placed end to end in the order the corresponding letters appear in the word W. Define $P_2(W)$ to be the path obtained by placing in the following order: r_0 E-edges, r_1 A_1-edges, r_2 A_2 edges, and so on, up to r_{k-1} A_{k-1}-edges, and then r_k D-edges. The k-**rhombic diagram** $\Gamma(W)$ is the closed shape that is identified with the region obtained by joining the northwest and southwest endpoints of $P_1(W)$ and $P_2(W)$ (see Figure 4.3).

Define a lattice path given by W to be composed of the edges in the order they appear in the word X, and let us associate this lattice path with the southeast boundary of our rhombic diagram. We complete the path to form the diagram by drawing in the following order: to connect the top-most corner of the lattice path to its bottom-most corner.

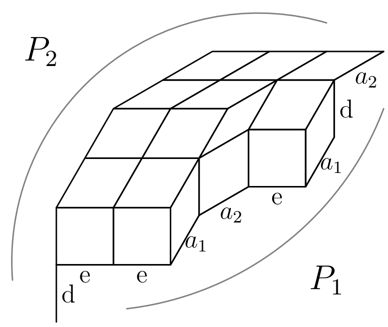

Figure 4.3: $\Gamma(a_2da_1ea_2a_1eed)$ defined by southeast boundary P_1 and northeast boundary P_2, with a maximal tiling.

Definition 4.4.2. A DE tile is a rhombus with D and E edges. A DA_i tile is a rhombus with D and A_i edges. An A_iE tile is a rhombus with A_i and E edges. An A_iA_j tile is a rhombus with A_i and A_j edges for $i > j$ (see Figure 4.4). We impose on the tiles the following partial ordering: $A_jX < A_iX' < DX''$, and $XE < XA_j < XA_i < XD$ for $i > j$ and for any edges X, X', X''. If tile C ¡ tile D according to our ordering, we say D is **heavier** than C.

Definition 4.4.3. A **maximal** tiling on a k-rhombic diagram is one in which tiles are always placed from southeast to northwest, and priority is always given to the "heaviest" tiles.

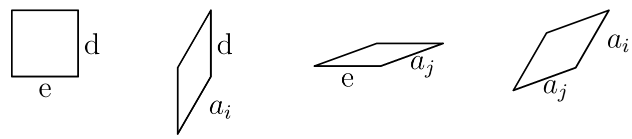

Figure 4.4: A DE-tile, DA_i-tile, A_iE-tile, and A_iA_j-tile (with $i < j$)

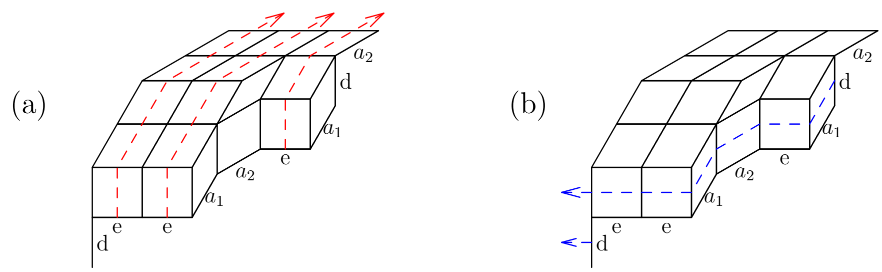

Figure 4.5: (a) E-strips and (b) D-strips.

Define a *maximal corner* to be a corner on $P_1(W)$ whose edges A and B are such that for any other corner on that diagram with edges C and D, $AB \geq CD$. The canonical way to tile the rhombic diagram with a maximal tiling would be to pick a maximal corner with some edges A and B, and place an AB tile adjacent to that corner. The rest of the surface would then itself be a rhombic diagram with the same P_2. We proceed to tile that surface in the same manner until the untiled region has area zero. It is easy to see that such a construction results in a maximal rhombic tiling of the k-rhombic diagram. Let us call this tiling $\mathcal{T}(W)$.

Definition 4.4.4. An **E-strip** is a maximal strip of adjacent tiles whose edge of adjacency is an E-edge, as in Figure 4.5 (a). A **D-strip** is a maximal strip of adjacent tiles whose edge of adjacency is a D-edge, as in Figure 4.5 (b). (This definition is the same for the k-RAT as it is for the RAT).

We now define a filling of $\mathcal{T}(W)$ with α's and β's as follows.

Definition 4.4.5. A filling of a k-rhombic alternative tableau (k-RAT) is defined by the following rules.

- A DE-tile is allowed to be empty or contain α or β.
- A DA_i tile is allowed to be empty or contain β, for each i.
- An A_iE tile is allowed to be empty or contain α, for each i.
- An A_iA_j tile must be empty, for each $i > j$.
- Any tile in the same E-strip and above an α must be empty.

- Any tile in the same D-strip and left of a β must be empty.

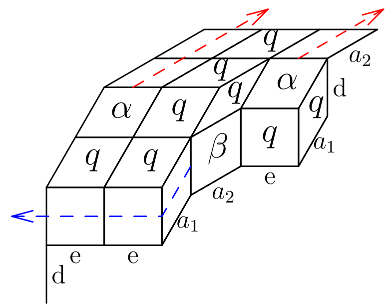

Figure 4.6: A 3-RAT of type $a_2da_1ea_2a_1eed$ of weight $\alpha^4\beta^4q^8$.

Denote the set of fillings of $\mathcal{T}(W)$ by $\text{fi}(W)$. We assign weights to a filling $F \in \text{fi}(W)$ from the rules above by placing a q in each tile that is not forced to be empty by some α below it in the same E-strip, or some β to the right in the same D-strip. Figure 4.6 shows an example of a 3-rhombic alternative tableau.[1]

Definition 4.4.6. Let $W \in \Omega^n$, and t be the number of d's and ℓ the number of e's in W. For $F \in \text{fi}(W)$, define the **weight** $\text{wt}(F)$ to be the product of the symbols in the filling of F times $\alpha^t\beta^\ell$.

Define

$$\mathcal{Z}_{n,r_1,\ldots,r_{k-1}} = \sum_W \sum_{F\in\text{fi}(W)} \text{wt}(F)$$

to be the sum of the weights over all k-RAT corresponding to states in $\Omega^n_{r_1,\ldots,r_{k-1}}$. Our main result for the k-RAT is the following, which we will prove in the next section.

Theorem 4.4.7. *Let $\text{fi}(W)$ denote the set of fillings of the rhombic diagram $\Gamma(W)$ with the maximal tiling, and let $\text{wt}(F)$ denote the weight of a filling in $\text{fi}(W)$.Then the stationary probability of state W of the k-species ASEP is*

$$\frac{1}{\mathcal{Z}_{n,r_1,\ldots,r_{k-1}}} \sum_{F\in\text{fi}(W)} \text{wt}(F).$$

Corollary 4.4.8. *Let $\mathcal{T}'$ be any tiling of the rhombic diagram $\Gamma(W)$ associated to a state W of the k-species ASEP. Let $\text{fi}(W,\mathcal{T}')$ denote the set of fillings of tiling $\mathcal{T}'$. Then the stationary probability of state W of the k-species ASEP is*

$$\frac{1}{\mathcal{Z}_{n,r_1,\ldots,r_{k-1}}} \sum_{F\in\text{fi}(W,\mathcal{T}')} \text{wt}(F).$$

[1]We allow the parameters q_{BC} that represent swapping rates between B-type and C-type particles to vary in Section 4.2. However, to keep the combinatorics "nice", we fix all these parameters to equal a single constant q.

Remark 4.4.9. For $k = 2$, the corollary follows from a special case of Proposition 3.1.9 of Chapter 3.

Proof of Corollary 4.4.8. We extend the proof of Proposition 3.1.9 for $k \geq 3$. Our proof is structured as follows. Let the notion of a flip on a k-RAT be precisely the same as for a 2-RAT, and let W be a word representing a state of the k-species ASEP. First we show that if two tilings $\mathcal{T}$ and $\mathcal{T}'$ of $\Gamma(W)$ differ by a single flip, then

$$\sum_{F \in \mathrm{fi}(W,\mathcal{T})} \mathrm{wt}(F) = \sum_{F \in \mathrm{fi}(W,\mathcal{T}')} \mathrm{wt}(F).$$

Next, we show that any tiling $\mathcal{T}$ can be obtained from $\mathcal{T}_{max}$ by some series of flips. The corollary follows from Theorem 4.4.7.

The proof of the first point is in fact precisely the same as the proof of Lemma 3.1.9. We define the transformation ϕ on some hexagon $\mathfrak{h}$ of $\mathcal{T}$ and the symbols contained in its filling. There are four cases for ϕ on a k-RAT, depending on the type of $\mathfrak{h}$ on which the flip occurs. These cases are:

i. $\mathfrak{h}$ is composed of a D-edge, an A_i-edge, and an E-edge.

ii. $\mathfrak{h}$ is composed of a D-edge, an A_i-edge, and an A_j-edge with $j > i$.

iii. $\mathfrak{h}$ is composed of an A_i-edge, an A_j-edge, and an E-edge with $j > i$.

iv. $\mathfrak{h}$ is composed of an A_i-edge, an A_j-edge, and an A_k-edge with $k > j > i$.

Case (i.) is a special case of a weight-preserving flip on a 2-RAT of Definition 3.1.19, whose construction is also given in Figure 3.12. (This applies to our case by replacing the A-edge in the 2-RAT by the A_i-edge.)

In Case (iv.), there is a single filling of $\mathfrak{h}$, where each tile must contain a q. The flip from $\mathfrak{h}$ to $\phi(\mathfrak{h})$ preserves this filling, and so is trivially weight-preserving.

Cases (ii.) and (iii.) are symmetric, so we only do the proof for Case (ii.). Since the A_iA_j tile must contain a q, there are four possible fillings for $\mathfrak{h}$: the D-strip is empty since there is a β in the same D-strip to the right, the D-strip contains a single β, the D-strip contains a q followed by a β, and the D-strip contains two q's. It is easy to check that in each of these four cases, the involution ϕ is indeed weight-preserving.

Thus for a single flip, ϕ indeed gives a weight-preserving involution on the fillings of a k-RAT.

Now we show that any tiling $\mathcal{T}$ can be obtained by some series of flips from $\mathcal{T}_{max}$. We obtain this by referring to the classical bijection of rhombic tilings of a convex shape with permutations of multi-words.

Therefore, for any tiling $\mathcal{T}$, the weight generating function of the fillings of $\mathcal{T}$ is well-defined as weight(W). $\square$

Matrix Ansatz proof for the k-RAT

We will prove Theorem 4.4.7 using the same strategy as in Section 4.3 for the RAT.

We provide matrices $D, E, A_1, \dots, A_{k-1}$ that correspond to the addition of a D-edge, E-edge, or a_i-edge for $1 \le i \le k-1$ to the bottom of the path corresponding to a word W of length n to form a new rhombic diagram with a maximal tiling of size $n+1$ that corresponds to the word Wd (or We, or Wa_i for $1 \le i \le k-1$ respectively). For $\lambda = \alpha\beta$, we show that these matrices satisfy the Matrix Ansatz relations

$$\begin{aligned} DE - qED &= \lambda(D+E), \\ DA_i - qA_iD &= \lambda A_i, \\ A_iE - qEA_i &= \lambda A_i, \\ A_iA_j &= qA_jA_i \text{ for } i > j. \end{aligned} \tag{4.15}$$

The k-species Matrix Ansatz of Theorem 4.2.1 would then imply that the steady state probability of k-species ASEP state W is proportional to a certain matrix product $\langle w|x(W)|v\rangle$ with the matrices $\{D, E, A_1, \dots, A_{k-1}\}$. (As in Section 4.2, we let $x(W)$ be the word in the matrices $\{D, E, A_1, \dots, A_{k-1}\}$ that corresponds to the word W in the letters $\{d, e, a_1, \dots, a_{k-1}\}$.)[2] Similarly to Section 4.3, we show that these matrices give a combinatorial interpretation to the construction of the k-RAT. Therefore, the fillings with α's, β's, and q's of the maximal tilings of the k-rhombic diagrams provide the steady state probabilities for the k-species ASEP.

In these matrices, the rows are indexed by the tuple $(i, j_1, \dots, j_{k-1})$ where i is the number of free D-strips in a tableau F of the maximal tiling of $\Gamma(W)$ and j_i is the number of a_i's in W. The columns of the matrices are indexed by the pair $(i', j'_1, \dots, j_{k-1})$, where k is the number of free D-strips in a tableau F' of the maximal tiling of $\Gamma(Wd)$ (and respectively, $\Gamma(We)$ and $\Gamma(Wa_s)$ for each s) and j'_i is the number of a_i's in Wd (and respectively, We and Wa_s for each s).

Analogously to the construction of the matrices in the two-species ASEP case, we have now

$$D_{(i,j_1,\dots,j_{k-1})(i+1,j_1,\dots,j_{k-1})} = \frac{1}{\beta}$$

and 0 for all other indices.

$$A_{(i,j_1,\dots,j_i,\dots,j_{k-1})(u,j_1,\dots,j_i+1,\dots,j_{k-1})} = \binom{i}{u} q^u \beta^{i-u} \prod_{s=i+1}^{k-1} q^{j_s}$$

for $0 \le u \le i$ and 0 for all other indices.

[2] In Equation (4.15), the constant $\lambda = \alpha\beta$ is used to slightly generalize the Matrix Ansatz of Theorem 4.2.1 in the same manner that Theorem 4.3.1 generalizes Theorem 3.0.1. The statement of the theorem and the proof are very similar to that of Theorem 4.3.1, so we do not provide them here.

$$E_{(i,j_1,\ldots,j_{k-1})(u,j_1,\ldots,j_{k-1})} = \frac{\beta^{i-u}}{\alpha}\left[\binom{i}{u}q^u(q^j+\alpha[j]_q)+\alpha\sum_{w=0}^{u-1}\binom{i-u+w}{i-u}q^w\right]$$

for $0 \le u \le i$ and 0 for all other indices, where we define $j = \sum_{s=1}^{k-1} j_s$, and $[j]_q = q^{j-1}+\ldots+1$.

The relations

$$\begin{aligned} DE - qED &= D + E \\ DA_i - qA_iD &= A_i \\ A_iE - qEA_i &= A_i \end{aligned}$$

are satisfied by the same arguments as in the two-species ASEP case, except with some additional powers of q in the equations. It remains to show that $A_tA_s = qA_sA_t$ for $t > s$.

First we compute the $(i, j_1, \ldots, j_s, \ldots, j_t, \ldots, j_{k-1})(u, j_1, \ldots, j_s + 1, \ldots, j_t + 1, \ldots, j_{k-1})$ entry of A_tA_s. (The entries of all other indices are automatically zero).

$$\begin{aligned}
&(A_tA_s)_{(i,j_1,\ldots,j_s,\ldots,j_t,\ldots,j_{k-1})(u,j_1,\ldots,j_s+1,\ldots,j_t+1,\ldots,j_{k-1})} \\
&= \sum_{w=u}^{i}(A_t)_{(i,j_1,\ldots,j_s,\ldots,j_t,\ldots,j_{k-1})(w,j_1,\ldots,j_s,\ldots,j_t+1,\ldots,j_{k-1})}(A_s)_{(w,j_1,\ldots,j_s,\ldots,j_t+1,\ldots,j_{k-1})(u,j_1,\ldots,j_s+1,\ldots,j_t+1,\ldots,j_{k-1})} \\
&= \sum_{w=u}^{i}\binom{i}{w}q^w\beta^{i-w}\prod_{r=t+1}^{k-1}q^{j_r}\cdot\binom{w}{u}q^u\beta^{w-u}\cdot q\prod_{r=s+1}^{k-1}q^{j_r} \\
&= q\sum_{w=u}^{i}\binom{i}{w}q^{w+u}\beta^{i-u}\prod_{r=t+1}^{k-1}q^{j_r}\cdot\prod_{r=s+1}^{k-1}q^{j_r} \qquad (4.16)
\end{aligned}$$

Similarly for A_sA_t,

$$\begin{aligned}
&(A_sA_t)_{(i,j_1,\ldots,j_s,\ldots,j_t,\ldots,j_{k-1})(u,j_1,\ldots,j_s+1,\ldots,j_t+1,\ldots,j_{k-1})} \\
&= \sum_{w=u}^{i}(A_s)_{(i,j_1,\ldots,j_s,\ldots,j_t,\ldots,j_{k-1})(w,j_1,\ldots,j_s+1,\ldots,j_t,\ldots,j_{k-1})}(A_t)_{(w,j_1,\ldots,j_s+1,\ldots,j_t,\ldots,j_{k-1})(u,j_1,\ldots,j_s+1,\ldots,j_t+1,\ldots,j_{k-1})} \\
&= \sum_{w=u}^{i}\binom{i}{w}q^w\beta^{i-w}\prod_{r=s+1}^{k-1}q^{j_r}\cdot\binom{w}{u}q^u\beta^{w-u}\cdot\prod_{r=t+1}^{k-1}q^{j_r} \\
&= \sum_{w=u}^{i}\binom{i}{w}q^{w+u}\beta^{i-u}\prod_{r=t+1}^{k-1}q^{j_r}\cdot\prod_{r=s+1}^{k-1}q^{j_r}. \qquad (4.17)
\end{aligned}$$

It is clear that $A_tA_s = qA_sA_t$, as desired.

Thus we obtain that the k-rhombic alternative tableaux indeed satisfy the k-species ASEP Matrix Ansatz of Theorem 4.2.1.

4.5 Additional results

We have other work that was done during graduate school that was not included in this thesis. We give a brief synopsis here.

There is a fascinating connection between the ASEP and orthogonal polynomials. Recall that the *partition function* of a Markov chain is the sum of the unnormalized probabilities over all the states. The *moments* of orthogonal polynomials are weight functions, which in one variable are generally integrals of x^n with respect to the measure. For the single-species ASEP with 5 parameters $\alpha, \beta, \gamma, \delta, q$, the partition function $Z_n(\alpha, \beta, \gamma, \delta, q)$ is closely connected to the moments of the Askey-Wilson polynomials.

One can generalize the two-species ASEP to a 5-parameter model with $\alpha, \beta, \gamma, \delta, q$ by allowing the heavy particles to enter and exit on both sides of the lattice, with parameters as shown in Figure 4.7. This model is significantly more difficult than the $\gamma = \delta = 0$ case – even for the single species process, solutions to the Matrix Ansatz for general γ, δ were not obtained until 20 years after the original Matrix Ansatz proof [20]. In the multi-variate case, the Koornwinder moments can be defined as integrals of the homogeneous symmetric polynomials with respect to the measure. For the two-species process, it turns out that the partition function $Z_{n,r}(\alpha, \beta, \gamma, \delta, q)$ (corresponding to a two-species 5 parameter ASEP of size n with exactly r *light* particles) corresponds precisely to these kinds of moments.

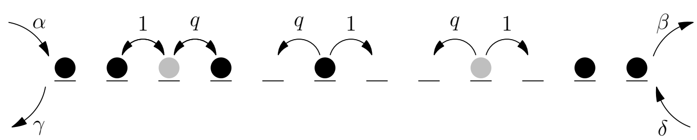

Figure 4.7: Two-species 5-parameter ASEP.

The connection of the single species ASEP to the Askey-Wilson polynomials, and of the two-species ASEP to Koornwinder polynomials, is only realized when all 5 parameters are general. This motivates the problem of finding tableaux formulae for probabilities of the two-species 5-parameter ASEP. In recent work with S. Corteel and L. Williams, we have defined “rhombic staircase tableaux”, which provides a combinatorial interpretation for the stationary probabilities of the two-species ASEP with the 5 parameters $\alpha, \beta, \gamma, \delta, q$. We show an example of these tableaux in Figure 4.8.

Theorem 4.5.1 (Corteel, M., Williams [3])**.** *Let X be a state of the two-species 5-parameter ASEP. Then the unnormalized stationary probability of state X is* $\mathrm{Prob}(X) = \sum_T \mathrm{wt}(T)$*, where the sum is over the rhombic staircase tableaux T.*

One can define more general Koornwinder moments K_λ as integrals of Schur polynomials. A long-term goal would be to find an explicit combinatorial formula for these K_λ.

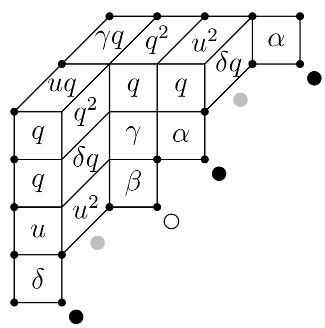

Figure 4.8: An example of a rhombic staircase tableau of type ● ● ● ○ ● ●.

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