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Witten Genera on Generalized Spin Structures

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ABSTRACT OF THE DISSERTATION

Witten Genera on Generalized Spin Structures

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We focus on generalizations of the Witten genus on so-called spin^k manifolds (that is, oriented manifolds embeddable into spin manifolds with codimension k), and applications of these generalized genera to vanishing theorems of the Witten genus on related spin manifolds.

We utilize two Dirac operators constructible on a spin^k manifold M, one previously constructed by Mayer and one not before considered, to construct two generalized Witten genera on such M. We show that these genera are rigid with respect to particular circle actions on M. We then show that these two Witten genera are equal to the standard Witten genus on certain related spin manifolds N and $\widetilde{M}$ constructed geometrically from M: N a codimension k submanifold of M and $\widetilde{M}$ a branched cover of M. We finally show that the rigidity of our generalized Witten genera on M implies vanishing theorems for said genera in certain cases where M admits a group action, which thus gives vanishing theorems for the standard Witten genera on N and $\widetilde{M}$.

These vanishing theorems for the standard Witten genus are qualitiatively unlike any of those already known in the literature, in the sense that they don't require the spin manifolds to either be equipped themselves with a Lie group action or to be complete intersections. We show this also provides new evidence for the Stolz conjecture, particularly in the case of certain complete intersections in weighted projective space.

The dissertation of Kyle Gettig is approved.

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CHAPTER 1

Introduction

The Witten genus of a manifold M was initially constructed in [Wit87] as a partition function in the context of superstring theory with target manifold M and elaborated on in [Wit88] as the equivariant index of the Dirac operator on the loop space LM. Since then it's been found to be fundamental in the still rather mysterious field of elliptic cohomology, in particular as the complex orientation of this generalized cohomology theory [AHS01].

One direction of exploration has involved generalizing the Witten genus, whose natural home is on spin (or, more specifically, string) manifolds, to more general manifold structures; for instance, [Liu96] for almost complex manifolds, [Des99] for manifolds equipped with a complex vector bundle, [CHZ11] for spin^c manifolds, [HM22] for non-compact manifolds with particular Lie group actions, and [GWL24] for dimension $4n-1$ spin manifolds. The commonality in these generalizations is the modularity of the genus over $SL(2,\mathbb{Z})$ and the rigidity of the genus with respect to certain S^1-actions on the manifold.

Another common direction has been to expand the family of manifolds on which it's known that the Witten genus vanishes. This is particularly interesting in view of the Stolz conjecture [Sto96], which predicts that the Witten genus vanishes on any string manifold which admits a Riemannian metric with positive Ricci curvature. Results in this direction include the vanishing of Witten genera on string complete intersections in complex projective space (see [HBJ92], pp. 87-88), string complete intersections in products of complex spaces [CH08], in products of Grassmanians [ZZ14], in flag manifolds [Zhu16], and recently in the most generality in spin^c manifolds with particular Lie group actions [Wie24b]. In a slightly different direction, there are also vanishing results for the Witten genus on certain manifolds with torus actions; see [Wie17], [Wie24a], or for some string complete intersections of toric manifolds, [Xia17].

In this dissertation, we begin in the former direction, constructing two generalized Witten genera on spin^k manifolds; that is, oriented manifolds with an additional real, oriented rank k vector bundle whose second Steifel-Whitney class agrees with that of the tangent bundle. We then connect this to the latter direction by showing that the rigidity properties of our generalized Witten genera imply the vanishing of standard Witten genera on related spin manifolds.

It's our hope that the further generality of our Witten genus vanishing theorems (in particular, the non-necessity of a complete intersection structure) may point in the direction of a proof that general string Fano manifolds have vanishing Witten genus. This is suggested by the Stolz conjecture and the fact that Fano manifolds have metrics of positive Ricci curvature; one bit of hope that our results might lead in this direction is that Fano manifolds can be embedded naturally in complex projective space.

Detailed chapter summaries are given before each chapter, so we'll be extra brief here.

Chapter 2 presents preliminary necessities, including standard constructions in Chern-Weil theory, elliptic operators, and the Atiyah-Singer index theorem, as well as introduces basic constructions and results on spin^k manifolds as they were first presented in [AM21].

Chapter 3 defines our two Dirac operators on a spin^k manifold M as well as presents our two geometric constructions of N and $\widetilde{M}$ from M, before showing how index theory relates these two geometric constructions with the two Dirac operators.

Chapter 4 constructs the generalized Witten genera on M, proves rigidity of these genera with respect to particular circle actions, and demonstrates how this rigidity coupled with certain Lie group action conditions on M provide new vanishing theorems for the standard Witten genus on N and $\widetilde{M}$.

The appendix covers in detail constructions and results on equivariant cohomology and equivariant characteristic classes which are needed in Chapter 4.

CHAPTER 2

Preliminaries

This chapter will cover preliminary material that will be necessary to begin presenting our main results. The first three sections of this chapter, on Chern-Weil Theory, manifolds with G-structures, Dirac operators, and Atiyah-Singer, are standard material in the field. The final section, on so-called spin^k manifolds, is a brief overview of the definitions and results of [AM21], and is necessary reading even for those familiar with the previous topics.

The standard reference for characteristic classes is [MS74], which also covers the Chern-Weil construction in its Appendix C. G-structures on manifolds are covered in detail in [Ste65]. [KN63] is a classic reference covering both more general principal bundles as well as Chern-Weil theory. The Atiyah-Singer index theorem was introduced in the classic series of papers [AS68b, AS68a, AS68c]. Dirac operators and their index theory is expounded upon in the classic [LM89].

2.1 Chern-Weil Theory and Characteristic Classes

In this section, we review the construction of the Chern and Pontryagin classes via the curvature 2-form, as well as the axiomatic definition of the Steifel-Whitney class.

Let E be a complex vector bundle over a closed base manifold M.

Definition 2.1.1. Let ∇ be a linear map on sections $\nabla : \Gamma(E) \to \Gamma(E \otimes_{\mathbb{R}} T^*M)$, and for each vector field $X \in \Gamma(TM)$ on M, define $\nabla_X : \Gamma(E) \to \Gamma(E)$ to be ∇ composed with contraction along X. Then ∇ is a *covariant derivative* if:

1. The map $\Gamma(TM) \to \mathrm{Hom}(\Gamma(E), \Gamma(E))$ given by $X \mapsto \nabla_X$ is linear.

2. Each ∇_X satisfies the product rule $\nabla_X(fs) = f\nabla_X(s) + (Xf)s$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

In a slight abuse of terminology, we also refer to a *covariant derivative* as a *connection.*

If E additionally comes with a Riemannian metric g, we can generally choose a connection ∇ compatible with the metric in the following sense: for all $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$,

$$Xg(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2).$$

Furthermore, if $E = TM$, we can also impose the torsion-free condition on ∇: $[X,Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \Gamma(TM)$. The two conditions of metric compatibility and being torsion-free together uniquely determine ∇, which is then called the *Levi-Civita connection* on M. We will generally assume, unless given reason to otherwise, that a connection on a Riemannian bundle E is metric-compatible and that a connection on a Riemannian manifold M is the Levi-Civita connection.

For any connection ∇ on E, we can *locally* write

$$\nabla = d + \omega,$$

where d is the standard external derivative extended to functions of E and ω is a $k \times k$ matrix of 1-forms acting on $\Gamma(E)$ by matrix multiplication, where $k = \mathrm{rk}(E)$. We interpret this as being locally a section of $T^*M \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the adjoint bundle of the $GL(k,\mathbb{C})$-bundle associated to E. We refer to ω as the *connection form* or *connection 1-form* associated to ∇. If $g \in C^\infty(U, GL(k,\mathbb{C}))$ represents a change of basis in local coordinates of E for a neighborhood $U \subseteq M$, then ω transforms via the gauge transformation

$$\omega \mapsto g\omega g^{-1} + gd(g^{-1}).$$

This transformation law indicates that ω isn't *globally* well-defined as an element of $\Gamma(T^*M \otimes \mathfrak{g})$. However, one can define the following:

Definition 2.1.2. The *curvature form* or *curvature 2-form* Ω of a vector bundle E with connection form ω is defined as $\Omega = d\omega + \omega \wedge \omega$.

Here $\omega \wedge \omega$ denotes the matrix-valued 2-form given by $(\omega \wedge \omega)(X, Y) = [\omega(X), \omega(Y)]$ for any $X, Y \in \Gamma(TM)$. In particular, $\omega \wedge \omega \neq 0$ generally, despite ω's components being 1-forms.

One can directly compute the gauge transformation of Ω from that previously given of ω under a local change of coordinates g:

$$\Omega \mapsto g\Omega g^{-1}.$$

In particular, Ω *is* a globally defined element of $\Gamma(\wedge^2 T^*M \otimes \mathfrak{g})$. Furthermore, as local coordinate changes of Ω only amount to matrix conjugation, any expression of Ω which is conjugation-invariant, such as the trace, will be globally well-defined. This is the basis for the Chern-Weil construction of the Chern class.

Definition 2.1.3. The *(total) Chern class* $c(E, \nabla)$ of a complex vector bundle E of rank k is defined as

$$c(E, \nabla) = \det\left(\frac{i}{2\pi}\Omega + I\right) \in H^*(M, \mathbb{Z}),$$

where I is the $k \times k$ identity matrix. The *ith Chern class* $c_i(E, \nabla) \in H^{2i}(M, \mathbb{Z})$ is the homogeneous part of $c(E, \nabla)$ of degree $2i$.

It's a rather non-trivial fact (which may be proven with the splitting principle, which we present later) that the Chern classes take values in cohomology with *integer* coefficients, rather than complex (or real) coefficients.

Moreover, one can show that the cohomology class of $c(E, \nabla)$ is actually independent of the choice of connection ∇, and so we generally simply write $c(E)$ as a cohomology element unless there's reason to do otherwise. The Chern class satisfies the following properties:

- $c_i(E) = 0$ if $i > \mathrm{rk}(E)$ (and also, of course, if $i > n$).

- $c_i(\bar{E}) = (-1)^i c_i(E)$, where $\bar{E}$ is the conjugate bundle of E.

- (Naturality) If E is a complex vector bundle over N and $f : M \to N$ is a smooth map of manifolds, then $c(f^*E) = f^*c(E)$.

- (Whitney Sum Formula) If E, F are complex vector bundles over M, then $c(E \oplus F) = c(E)c(F)$.

When dealing with Chern classes computationally, it's particularly convenient to use Chern roots:

Definition 2.1.4. The *Chern roots* of E are formal 2-forms $x_1, \cdots, x_k$ defined via the formal factorization

$$c(E) = (1 + x_1)\cdots(1 + x_k).$$

We emphasize that the Chern roots are *not* necessarily well-defined 2-forms; rather, only their *symmetric sums* are well-defined cohomology elements (specifically, the Chern classes). This allows us to conveniently write certain cohomology classes as expressions in the Chern roots, and as long as these expressions are symmetric polynomials, they represent well-defined cohomological forms. For example:

Definition 2.1.5. The *Chern character* of a complex vector bundle E is $\mathrm{ch}(E) = \sum_{i=1}^{k} e^{x_i} \in H^*(M;\mathbb{Q})$, where $\{x_i\}_{i=1}^k$ are the Chern roots of E. Equivalently, we can write $\mathrm{ch}(E) = \mathrm{tr}\left(\exp\left(\frac{i}{2\pi}\Omega\right)\right)$ as a cohomology class.

The Chern character has the nice properties that $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F)$ for any complex vector bundles E, F.

Recall that the K-theory of a manifold M is the Grothendieck completion of the monoid of complex vector bundles on E with addition given by the Whitney sum. In other words, the abelian group $K(M)$ consists of pairs (E, F) of complex vector bundles on M quotiented by the identification $(E_1, F_1) \sim (E_2, F_2)$ if $E_1 \oplus F_2 \cong E_2 \oplus F_1$. Formally we imagine (E, F) as representing a "virtual bundle" $E - F$ over M. The tensor product $\otimes$ on vector bundles gives $K(M)$ a ring structure.

The relevance of this is that the Whitney sum formula $c(E \oplus F) = c(E)c(F)$ now allows us to extend Chern classes to not just proper vector bundles but to all of $K(M)$; in particular, we define for virtual bundles $c(E - F) = c(E)/c(F)$, which is viable since $c(F)$ has scalar term 1. This is consistent with the Whitney sum property of the Chern class on true vector bundles. Formally, the Chern class is now a map from $K(M)$ to $H^*(M;\mathbb{Q})$.

We can extend the Chern character in the same way as a map $\mathrm{ch} : K(M) \to H^*(M;\mathbb{Q})$. This map is now in fact a ring homomorphism by the previously stated Chern character properties.

We're now prepared to present the Splitting Principle. We give the form as it's presented in [Ati67] (as Corollary 2.7.11).

Theorem 2.1.6 (Splitting Principle)**.** Let E be a complex vector bundle over M. Then there exists a space F and a map $\pi : F \to M$ such that:

- $\pi^* : K(M) \to K(F)$ is injective.
- $\pi^* E$ is a sum of complex line bundles.

As suggested by the notation, we can take F to be the total space of the flag bundle of E and $\pi : F \to M$ to be the projection, though the particulars of the model won't usually matter. The convenience of this theorem is as follows: suppose a particular K-theoretic or characteristic class identity satisfies the following properties:

- It holds for any complex vector bundle E which is a sum of complex line bundles.
- Both sides are functorial in E.

Then for a general complex vector bundle E over M, we can consider $\pi^* E$ over F, for which the identity holds since $\pi^* E$ is a sum of line bundles. Then since both sides are functiorial, we get that π^* of the left hand side equals π^* of the right hand side, and we can finally remove the π^* since it's an injective function. Hence the splitting principal lets us prove K-theoretic identities in E by immediately reducing to the case that E is a sum of line bundles.

We move on to the case of characteristic classes on real vector bundles. Suppose now that E is a *real* vector bundle over M. We then have a complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ which is a complex vector bundle of the same rank; in the future, we denote this as just $E_{\mathbb{C}}$.

Definition 2.1.7. The *Pontryagin classes* of a real vector bundle E over a manifold M are defined as $p_i(E) = (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(M;\mathbb{Z})$. The *total Pontryagin class* is defined as $p(E) = 1 + p_1(E) + p_2(E) + \cdots \in H^*(M;\mathbb{Z})$.

Remark that as $E_{\mathbb{C}}$ is invariant under conjugation, $c_{2i+1}(E_{\mathbb{C}})$ is 2-torsion. The Pontryagin class inherits some of the nice properties of the Chern class, such as:

- $p_i(E) = 0$ if $i > \frac{1}{2}\mathrm{rk}(E)$.
- (Naturality) If E is a real vector bundle over N and $f : M \to N$ is a smooth map of manifolds, then $p(f^*E) = f^*p(E)$.
- (Whitney Sum Formula) If E, F are real vector bundles over M, then $p(E \oplus F) = p(E)p(F) + 2$-torsion.

Finally, note that again as $E_{\mathbb{C}}$ is invariant under conjugation, the Chern roots of $E_{\mathbb{C}}$ are invariant as a set under negation, and so may be written as $\{\pm x_i\}$, possibly with some number of zeros. The Pontryagin classes of E are then the symmetric sums of the x_i^2; in particular, for example, $p_1(E) = \sum_i x_i^2$.

The Pontryagin class isn't the only standard characteristic class on real vector bundles; we also have the Steifel-Whitney class, which has $\mathbb{Z}_2$-coefficients[1]. As the Steifel-Whitney class doesn't have such a direct, computationally convenient construction, we instead present it axiomatically.

Theorem 2.1.8. For real vector bundles E over M, there exists a unique characteristic class $w(E) = w_0(E) + w_1(E) + \cdots \in H^*(M; \mathbb{Z}_2)$ (where $w_i(E) \in H^i(M; \mathbb{Z}_2)$) called the *Steifel-Whitney class* satisfying all of the following properties:

- If E is the Möbius real line bundle over S^1, then $w_1(E)$ is the unique non-trivial element of $H^1(S^1; \mathbb{Z}_2)$.
- $w_0(E) = 1$, and $w_i(E) = 0$ for $i > \mathrm{rk}(E)$.
- (Naturality) If E is a real vector bundle over N and $f : M \to N$ is a smooth map of manifolds, then $w(f^*E) = f^*w(E)$.

[1] Here and in the future we use $\mathbb{Z}_2$ to denote the field with 2 elements, with apologies to the algebraists who consider it the 2-adic integers.

- (Whitney Sum Formula) If E, F are real vector bundles over M, then $w(E \oplus F) = w(E)w(F)$.

2.2 G-Structures

In general, for a closed manifold M of dimension n, the structure bundle is a principal $GL(n, \mathbb{R})$-bundle, as that's the group in which the transition functions between local trivializations of the tangent bundle lie. In many cases, however, we can change this transition function codomain group.

Definition 2.2.1. Given a group G and a homomorphism $\phi : G \to GL(n, \mathbb{R})$, a *$G$-structure* on a manifold M is a principal G-bundle $B \to M$ and a bundle map $f : B \to F$, where $F \to M$ is the principal $GL(n, \mathbb{R})$ frame bundle of M which is compatible with the group actions of the respective bundles in the following way:

$$f(b \cdot g) = f(b) \cdot \phi(g)$$

for any $b \in B$ and $g \in G$, where the left hand $\cdot$ is the free action of G on B and the right hand $\cdot$ is the free action of $GL(n, \mathbb{R})$ on F.

One may also call a G-structure a *reduction of the structure group* to G. I find this terminology a bit misleading; in this definition, G isn't required to be "larger" or "smaller" in any sense than the original structure group. Often, the exact homomorphism $G \to GL(n, \mathbb{R})$ will be clear from context. A particular manifold may or may not be equippable with a G-structure for a particular G; cohomological obstructions are a common way of determining whether a G-structure exists.

Any manifold M is equippable with a Riemannian metric g on its tangent bundle. With such a metric, we can locally orthogonalize the bases of TM on each coordinate chart, making sure the transition functions lie in $O(n) \subseteq GL(n, \mathbb{R})$. Hence every manifold has a $O(n)$-structure. In fact, the isomorphism class of this principal $O(n)$-bundle is independent of the choice of metric, and so we generally refer to this unique $O(n)$-bundle as *the* structure bundle of M.

Definition 2.2.2. A manifold M is *orientable* if its structure group reduces from $O(n)$ to $SO(n)$.

This is of course equivalent to any standard definition of orientability, as we can ensure the transition functions in $O(n)$ are also always of positive determinant, and hence of unit determinant. In fact, this gives us our first cohomological obstruction.

Theorem 2.2.3. A manifold M is orientable if and only if $w_1(TM) = 0 \in H^1(M; \mathbb{Z}_2)$, where $w_1(TM)$ is the first Steifel-Whitney class of TM.

Now, recall that the group $SO(n)$ has fundamental group $\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}_2$ for $n \geq 3$, and so there's a unique connected double cover of $SO(n)$ which we denote by $\mathrm{Spin}(n)$. For $n = 1$, $SO(1)$ is a single point, and we define $\mathrm{Spin}(1)$ to be two disjoint points, so that $\mathrm{Spin}(1) \to SO(1)$ is still a double cover. The topological space $\mathrm{Spin}(n)$ inherits a group structure from that of $SO(n)$.

Definition 2.2.4. An n-dimensional manifold M is *spin* if it can be equipped with a $\mathrm{Spin}(n)$-structure along the projection $\mathrm{Spin}(n) \to SO(n)$.

In general use it can be ambiguous whether referring to a "spin" manifold means it actually has a particular $\mathrm{Spin}(n)$-structure equipped or simply has the ability to have one equipped. Some authors have combatted this ambiguity by using the term "spin" for the former and "spinnable" for the latter, in analogy with "oriented" and "orientable." We won't be doing this; rather, "spin" for us will mean that a $\mathrm{Spin}(n)$-structure simply exists, and if we need one assigned, we'll make it explicit which one.

Theorem 2.2.5. An n-dimensional manifold has a $\mathrm{Spin}(n)$-structure, i.e. is spin, if and only if $w_1(TM) = 0 \in H^1(M; \mathbb{Z}_2)$ and $w_2(TM) = 0 \in H^2(M; \mathbb{Z}_2)$.

That is, while the first Steifel-Whitney class is a complete obstruction to orientability, the first two Steifel-Whitney classes together form a complete obstruction to spinnability.

The main topic of this thesis will be manifolds with $\mathrm{Spin}^k(n)$ structures, where $\mathrm{Spin}^k(n) = (\mathrm{Spin}(n) \times \mathrm{Spin}(k))/\mathbb{Z}_2$. We'll elaborate on this construction in later sections.

We conclude this section by noting that essentially everything in it, both definitions and theorems, can be extended from the tangent bundle TM to a general rank k real vector bundle E over M and considering the reduction of its structure group $GL(k, \mathbb{R})$.

2.3 Dirac Operators and the Atiyah-Singer Index Theorem

We now proceed to a more analytic definition.

Definition 2.3.1. Given complex bundles E and F over a manifold M, a *differential operator* over M is a linear map $D : \Gamma(E) \to \Gamma(F)$ which locally looks like

$$D(s(x)) = \sum_{|\alpha| \le k} A_\alpha(x)(\partial^\alpha s)(x),$$

where the sum is over multi-indices α and each A_α is a locally-defined smooth matrix-valued function. The minimum value of k that we can take for a particular D is the *order* of D.

General differential operators can be tricky to get a handle on, but for our purposes, we need only look at a more well-behaved class of such operators called elliptic differential operators. Let $\pi : T^*M \to M$ be the standard projection of the cotangent bundle.

Definition 2.3.2. With a differential operator D being given locally as above, the *total symbol* of D is a bundle map $\sigma(D) : \pi^*E \to \pi^*F$ given locally by

$$\sigma(D)(x, \xi) = \sum_{|\alpha| \le k} A_\alpha(x)(i\xi)^\alpha,$$

where (x, ξ) are the local coordinates of T^*M and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$ is a complex scalar factor. The *principal symbol* of D is a bundle map $\hat{\sigma}(D) : \pi^*E \to \pi^*F$ given locally by

$$\hat{\sigma}(D)(x, \xi) = \sum_{|\alpha| = k} A_\alpha(x)(i\xi)^\alpha.$$

One also sees the symbol interpreted as a family of bundle maps $\sigma_\xi(D) : E \to F$ indexed by covectors ξ on M. With this, we're ready to define elliptic differential operators.

Definition 2.3.3. A differential opeartor $D : \Gamma(E) \to \Gamma(F)$ is *elliptic* if the principal symbol $\hat{\sigma}(D)(x, \xi)$ is a fiber-wise linear isomorphism at all $\xi \neq 0$.

This is, as one might expect, a fairly strong condition. Clearly if $D : \Gamma(E) \to \Gamma(F)$ is elliptic, then E and F must have the same rank. Another important, quite non-trivial consequence of ellipticity is as follows.

Theorem 2.3.4. If $D : \Gamma(E) \to \Gamma(F)$ is elliptic, then the vector spaces $\ker(D) \subseteq \Gamma(E)$ and $\operatorname{coker}(D) = \Gamma(F)/\operatorname{Im}(D)$ are finite-dimensional.

In particular, the index of an elliptic differential operator, defined as follows, is finite.

Definition 2.3.5. The *index* of an elliptic differential operator D is defined as

$$\operatorname{Ind}(D) = \dim\ker(D) - \dim\operatorname{coker}(D).$$

Computing the index of elliptic differential operators will be of great interest to us, as we'll see later. For now, we move on to constructing a particularly useful elliptic differential operator: the Dirac operator.

Recall that, given a vector space V equipped with a metric g, one can define the Clifford algebra $\mathrm{Cl}(V)$ as the quotient of the Fock space $\bigoplus_{n=0}^{\infty} V^{\otimes n}$ by the ideal generated by $\{v \otimes v + g(v,v)1 \mid v \in V\}$; in other words, one adds a free multiplication to V subject to the condition that $v^2 = -g(v,v)$. This construction can be adapted to smooth bundles over a manifold:

Definition 2.3.6. The *real Clifford bundle* over a manifold M is the real bundle given fiberwise by $\mathrm{Cl}(TM)$. The *complex Clifford bundle* is its complexification $\mathrm{Cl}_{\mathbb{C}}(TM) = \mathrm{Cl}(TM) \otimes_{\mathbb{R}} \mathbb{C}$.

We'll primarily be interested in the complex case. Note that the bundle TM naturally embeds as a subbundle of $\mathrm{Cl}_{\mathbb{C}}(TM)$.

Definition 2.3.7. A complex vector bundle E is a *spinor bundle* if it's equipped with a smooth injective algebra bundle homomorphism $f : \mathrm{Cl}_{\mathbb{C}}(TM) \to \operatorname{End}(E)$; in other words, the complex Clifford bundle has a fiber-wise action on E.

Spinor bundles are the natural setting for Dirac operators, as the following construction shows. Suppose E comes equipped with a covariant derivative ∇.

Definition 2.3.8. The *Dirac operator* on the complex spinor bundle E with Clifford bundle action given by f as above is defined by $D : \Gamma(E) \to \Gamma(E)$ satisfying

$$D\sigma = \sum_{i=1}^{n} f(e_i)(\nabla_{e_i}\sigma)$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of TM.

The independence of choice of orthonormal basis is clear. A standard computation then gives that the principal symbol of D is given fiberwise by $\sigma_\xi = if(\xi^\#)$, where $\xi^\#$ is the vector dual to ξ with respect to the metric. In particular, since f is injective, we get that

Theorem 2.3.9. Dirac operators are elliptic differential operators.

There's a standard spinor bundle and hence Dirac operator on an n-dimensional manifold with a Spin(n)-structure. Specifically, the complex spinor representation of Spin(n) with the natural Clifford algebra action[2] makes the associated bundle Δ to the principal Spin(n)-bundle a spinor bundle. Moreover, if n is even, this spinor bundle splits into chiral subbundles $\Delta \cong \Delta^+ \oplus \Delta^-$, with the Dirac operator flipping chiralities. We thus generally consider the chiral Dirac operator $D : \Gamma(\Delta^+) \to \Gamma(\Delta^-)$.

Let $\{\pm x_i\}_{i=1}^n$ denote the Chern roots of $TM \otimes_{\mathbb{R}} \mathbb{C}$. We define the $\hat{A}$-form of M to be

$$\hat{A}(TM) = \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)},$$

and the $\hat{A}$-genus of M to be $\int_M \hat{A}(TM)$, where the integral is of the top degree form.

Theorem 2.3.10. If D is the (chiral) Dirac operator on a spin manifold M, then

$$\operatorname{Ind}(D) = \int_M \hat{A}(TM).$$

One can twist the Dirac operator D by any other complex vector bundle E to get $D \otimes E : \Gamma(\Delta^+ \otimes E) \to \Gamma(\Delta^- \otimes E)$.

[2] Recall that the spinor representation of the spin group is by definition the restriction of a Clifford algebra representation.

Theorem 2.3.11 (Atiyah-Singer for Spin Manifolds)**.** If D is the Dirac operator on a spin manifold M and E is any complex vector bundle, then

$$\mathrm{Ind}(D \otimes E) = \int_M \hat{A}(TM)\mathrm{ch}(E).$$

In the case of spin manifolds, this form of Atiyah-Singer is actually fully general, since any elliptic operator over a spin manifold is isomorphic to some twisted Dirac operator. The following is another convenient computational form of Atiyah-Singer.

Theorem 2.3.12 (Atiyah-Singer Index Theorem)**.** Let M be a $2n$-dimensional closed manifold and $D : \Gamma(E) \to \Gamma(F)$ an elliptic differential operator over M on complex vector bundles E and F. Suppose the Euler class of M is non-vanishing; that is, $e(TM) \neq 0$. Then

$$\mathrm{Ind}(D) = (-1)^n \int_M \frac{\mathrm{ch}(E) - \mathrm{ch}(F)}{e(TM)} \hat{A}(TM)^2.$$

This condition that $e(TM) \neq 0$ looks alarming, as it seems that there's a whole class of manifolds on which we can just never use this convenient formulation. However, in practice, it's often the case that when one expands $\mathrm{ch}(E) - \mathrm{ch}(F)$ in terms of the Chern roots $\{\pm x_i\}_{i=1}^n$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$, one can then formally cancel a factor of $e(TM) = x_1 \cdots x_n$ and arrive at a correct index formula anyway. Specifically, suppose the following conditions are satisfied:

- The manifold M has a G-structure for some $G \to SO(2n)$.
- The bundles E and F are vector bundles associated to the principal G-bundle over M with respect to some complex representations.
- The maximal torus of G has no fixed non-zero vector when acting on $\mathbb{R}^{2n}$ via the map $G \to SO(2n)$.

If these conditions are satisfied, the above formal cancellation of $e(TM)$ can be justified[3].

There is, of course, a fully general index theorem for all elliptic operators over all closed even-dimensional manifolds, given by Theorem 2.12 in [AS68c]. As this takes a bit more

[3]This is essentially the content of Proposition 2.17 in [AS68c], with the understanding that in this set-up the computation of the Chern character of a complex representation is essentially the same as the computation of the Chern character of the vector bundle associated to such a representation.

overhead to present and we won't need its full generality, we leave its investigation to the curious reader.

2.4 Spin^k Manifolds

We're now ready to introduce our main object of study. For integers $n > 0, k \geq 0$, define the group

$$\mathrm{Spin}^k(n) := (\mathrm{Spin}(n) \times \mathrm{Spin}(k))/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ is generated by $(-1,-1) \in \mathrm{Spin}(n) \times \mathrm{Spin}(k)$. There's a clear natural map $\mathrm{Spin}^k(n) \to SO(n)$, and so we can talk about n-dimensional manifolds with $\mathrm{Spin}^k(n)$-structures, which we refer to as spin^k manifolds.

This general structure has been studied periodically in the past from the perspective of elliptic operators, index theory, and integrality results, for instance in [May65], [Bar93], [Bar99]. However, it wasn't until recently in [AM21] that this sort of structure was formalized and given a ground-up treatment. It's from this latter paper that we take the notation $\mathrm{Spin}^k(n)$, as well as most of the results in this section.

Note that small k yield well-known special cases.

$$\begin{aligned} \mathrm{Spin}^1(n) &\cong \mathrm{Spin}(n) \\ \mathrm{Spin}^2(n) &\cong \mathrm{Spin}^c(n) \\ \mathrm{Spin}^3(n) &\cong \mathrm{Spin}^h(n). \end{aligned}$$

Spin and spin^c structures on manifolds are of course classical; see, for example, the classic text [LM89]. Spin^h structures, while known for some time, have attracted increasing interest in recent years; see, for instance, the recent short review [Law23].

Note, as in [AM21], that the group $\mathrm{Spin}^k(n)$ is the pullback of the natural maps $SO(n) \times SO(k) \to SO(n+k)$ and $\mathrm{Spin}(n+k) \to SO(n+k)$, so that there's a commutative diagram

$$\begin{tikzcd} \mathrm{Spin}^k(n) \arrow[r] \arrow[d] & \mathrm{Spin}(n+k) \arrow[d] \\ SO(n) \times SO(k) \arrow[r] & SO(n+k). \end{tikzcd}$$

As an alternative interpretation, $\mathrm{Spin}^k(n)$ is the subgroup of $\mathrm{Spin}(n+k)$ lying over $SO(n) \times SO(k) \subseteq SO(n+k)$. The $\mathbb{Z}_2$ quotient comes from the fact that the standard map $\mathrm{Spin}(n) \times \mathrm{Spin}(k) \to \mathrm{Spin}(n+k)$ isn't injective but rather has kernel exactly this $\mathbb{Z}_2$ group by which we quotient.

The natural map $\mathrm{Spin}^k(n) \to SO(k)$ yields a k-dimensional real representation of $\mathrm{Spin}^k(n)$, which thus associates to any spin^k manifold M a real oriented rank k vector bundle E, which is called the *canonical bundle*.

Theorem 2.4.1 (Proposition 3.2 in [AM21])**.** The following are equivalent for a smooth manifold M of dimension n.

- M is spin^k, i.e. M can be given a $\mathrm{Spin}^k(n)$-structure.
- There is a real orientable vector bundle E of rank k such that $TM \oplus E$ is spin, i.e. $w_2(TM) = w_2(E)$.
- M can be immersed in a spin manifold with codimension k.
- M can be embedded in a spin manifold with codimension k.

Moreover, if M is closed, we can take the ambient spin manifold in the immersion/embedding to also be closed.

Note that a choice of spin^k structure on M is equivalent up to isomorphism to a choice of oriented canonical bundle E of rank k satisfying $w_2(E) = w_2(TM)$ along with a choice of spin structure on $TM \oplus E$.

We'll therefore, in a slight abuse of terminology, refer to the pair (M, E) as a spin^k structure when M is a closed oriented manifold and E is a real oriented rank k vector bundle over M satisfying $w_2(TM) = w_2(E)$, with the understanding that there's also a choice of spin structure on $TM \oplus E$.

It's instructive to see some basic examples.

- If M has a spin structure, we can take E to be the real rank 0 bundle, and write $(M, 0)$ as the associated spin^0 structure on M.

- For any closed, oriented dimension n manifold M, (M, TM) is a spin^n structure. Actually, more accurately, we would need to choose a spin structure on the bundle $TM \oplus TM$ first, as there isn't generally a canonical such choice, but in general we might refer to (M, TM) as the universal spin^n structure.

- If (M, E) is a spin^k structure, then $(M, E \oplus \mathbb{R}^l)$ is a spin^{k+l} structure.

- If (M, E) is a spin^k structure, then $(M, E^{\oplus(2n+1)})$ is a spin$^{(2n+1)k}$ structure for any $n \geq 0$.

- If (M, E) and (N, F) are spin^k and spin^l structures, then $(M \times N, \pi_M^* E \oplus \pi_N^* F)$ is a spin^{k+l} structure, where π_M, π_N are the projection maps on $M \times N$.

- If the tangent bundle TM of an oriented manifold M splits into $TM \cong E \oplus F$ where E is spin, then (TM, F) is a spin^k structure, where $k = \mathrm{rk}(F)$.

- As a specific case of the previous example, let M be the total space of a fiber bundle of manifolds over a spin base space B, and suppose the fibers have dimension k. If V is some choice of vertical bundle over M, then (M, V) is a spin^k structure.

CHAPTER 3

Index Theory on Spin^k Manifolds

In this section, we'll construct two differential operators over a given spin^k manifold; the first was orginally constructed by Mayer [May65], while the second, as far as we know, hasn't been explicitly presented in the literature. We'll then present two geometric constructions, each assigning to a (bordism class of a) spin^k manifold a (bordism class of a) spin manifold. Finally, we'll make the connection between the two pairs of constructions, showing that the geometric constructions leave the indices of the corresponding Dirac operators invariant up to a fixed factor.

3.1 Elliptic Operators on Spin^k Manifolds

We now present what will be our main analytic feature of even-dimensional spin^k manifolds: the existence of two natural Dirac operators if k is even, only one if k is odd. We denote these by D_+ and D_-, with the latter being the one requiring evenness of k. Our D_+ operator originated with Mayer in [May65] and we follow his construction[1]; our D_- operator, as far as we know, has not been explicitly presented in the literature. We begin with Mayer's construction of D_+.

Let (M, E) be a spin^k structure on a $2n$-dimensional manifold M. By definition, this gives a G-bundle $\mathcal{G}$ over M for $G = \mathrm{Spin}^k(2n) = \mathrm{Spin}(2n) \times \mathrm{Spin}(k)/\mathbb{Z}_2$. Let Δ_{2n} denote the complex spinor representation of $\mathrm{Spin}(2n)$; since $2n$ is even, this splits into chiral representations $\Delta^{\pm}_{2n}$. We thus have that $\Delta^{\pm}_{2n} \otimes \Delta_k$ are representations for $\mathrm{Spin}(2n) \times \mathrm{Spin}(k)$ which

[1] In Mayer's original paper, what we call D_+ is called instead D^+. Mayer also writes D^- for the formal adjoint of D^+, and this is not analogous to our D_-. Mayer also defines his own D_+ and D_- which, to reiterate, are different from ours. We trust that this is perfectly clear.

descend to representations for G since $(-1,-1) \in \mathrm{Spin}(2n) \times \mathrm{Spin}(k)$ acts as the identity; denote these $(n+\lfloor k/2 \rfloor)$-dimensional complex representations briefly as $\rho_\pm$. We can therefore construct the associated vector bundles

$$\sigma^\pm = \mathcal{G} \times_{\rho_\pm} \mathbb{C}^{n+\lfloor k/2 \rfloor}.$$

If $\mathcal{C}(TM)$ denotes the Clifford bundle of TM, we have that $\sigma^\pm$ are spinor bundles via the map $f : \mathcal{C}(TM) \to \mathrm{End}(\sigma^\pm)$ given by $f([m,a]) = [m, \Phi_{2n}(a) \otimes \mathrm{Id.}]$. Here a is an element of the complex Clifford algebra of $TM|_m$ and Φ_{2n} is the (unique) irreducible complex representation of this Clifford algebra. Therefore, since $\sigma^\pm$ are spinor bundles, we can define

Definition 3.1.1 ([May65])**.** The elliptic differential operator $D_+ : \Gamma\sigma^+ \to \Gamma\sigma^-$ is (locally) defined by $D\phi = \sum_i f(e_i)(\nabla_{e_i}\phi)$, where $\{e_i\}$ is any local orthonormal basis of $\mathcal{C}(TM)$.

Mayer then computes the index of D_+ via a direct application of Proposition 2.17 in [AS68c].

Proposition 3.1.2 ([May65])**.** The index of D_+ is given by

$$\mathrm{Ind}(D_+) = (-1)^n 2^s \int_M \hat{A}(TM) \prod_{i=1}^{s} \cosh\left(\frac{y_i}{2}\right),$$

where $k = 2s$ or $k = 2s+1$ and $\{\pm y_i\}_{i=1}^s$ are the Chern roots of $E \otimes_{\mathbb{R}} \mathbb{C}$.

We now give our definition of D_-. Suppose M is $2n$-dimensional and E is of rank $k = 2s$ (so we in particular now assume that E has even rank). By our assumption of a spin^k structure, $TM \oplus E$ has a spin structure, and therefore induces two complex bundles $\Delta^\pm(TM \oplus E)$. Now, the bundle TM is naturally included as a subbundle of $TM \oplus E$, meaning there's a natural inclusion of Clifford bundles $\mathcal{C}(TM) \hookrightarrow \mathcal{C}(TM \oplus E)$. The usual action of $\mathcal{C}(TM \oplus E)$ on $\Delta^\pm(TM \oplus E)$ therefore induces an action of $\mathcal{C}(TM)$ on the same, making these bundles spinor bundles.

Definition 3.1.3. The elliptic differential operator $D_- : \Gamma(\Delta^+(TM \oplus E)) \to \Gamma(\Delta^-(TM \oplus E))$ is the Dirac operator on these spinor bundles with respect to the above Clifford bundle action.

We can easily compute the index of this operator.

Proposition 3.1.4. The index of D_- is given by

$$\mathrm{Ind}(D_-) = (-1)^n 2^s \int_M \hat{A}(TM) \prod_{i=1}^{s} \sinh\left(\frac{y_i}{2}\right),$$

where $\{\pm y_i\}_{i=1}^s$ are the Chern roots of $E \otimes_{\mathbb{R}} \mathbb{C}$.

Proof. Let $\{\pm x_i\}_{i=1}^n$ be the Chern roots of $TM \otimes_{\mathbb{R}} \mathbb{C}$. Then we have

$$\mathrm{ch}\left(\Delta^+(TM \oplus E) - \Delta^-(TM \oplus E)\right) = \prod_{i=1}^{n}(e^{x_i/2} - e^{-x_i/2}) \prod_{j=1}^{s}(e^{y_j/2} - e^{-y_j/2}).$$

Therefore,

$$\begin{aligned}
\mathrm{Ind}(D_-) &= (-1)^n \int_M \prod_{i=1}^{n}(e^{x_i/2} - e^{-x_i/2}) \prod_{j=1}^{s}(e^{y_j/2} - e^{-y_j/2}) \left(\prod_{i=1}^{n} x_i\right)^{-1} \left(\prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}\right)^2 \\
&= (-1)^n \int_M \prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \prod_{j=1}^{s}(e^{y_j/2} - e^{-y_j/2}) \\
&= (-1)^n 2^s \int_M \hat{A}(TM) \prod_{j=1}^{s} \sinh\left(\frac{y_j}{2}\right).
\end{aligned}$$

Note that, based on the comment in the previous section and the fact that no non-zero vector of $SO(2n)$ is fixed by the maximal torus of $\mathrm{Spin}^k(2n)$ in the representation of $\mathrm{Spin}^k(2n)$ induced by the map $\mathrm{Spin}^k(2n) \to SO(2n)$, our formal division by the Euler class $\prod_{i=1}^{n} x_i$ is justified. $\square$

A brief remark on sign ambiguity: if $\{\pm y_j\}$ are the Chern roots of $E \otimes_{\mathbb{R}} \mathbb{C}$, and we write $\prod_{j=1}^{s} f(y_j)$ for an odd function f, this has a sign ambiguity: instead of taking a given y_j, we could instead take $-y_j$ and get the opposite sign. This is resolved by writing

$$\prod_{j=1}^{s} f(y_j) = \left(\prod_{j=1}^{s} \frac{f(y_j)}{y_j}\right) e(E),$$

where $e(E)$ is the Euler class of E. Here $\frac{f(y_j)}{y_j}$ is now even and hence is invariant under flipping signs of y_j, while the sign of $e(E)$ is well-defined (and depends on the orientation of M). In other words, we fix a choice of sign of $\prod_{j=1}^{s} y_j$ to correspond with the Euler class. In the future, whenever we write a product of forms in this way, this will be the assumed sign convention.

We make some basic observations based on the above index formulas. If n is odd (i.e. if $\dim M$ isn't divisible by 4), then $\mathrm{Ind}(D_+) = 0$ by dimensionality. Also by dimensionality, if $2n \not\equiv k \pmod 4$, then $\mathrm{Ind}(D_-) = 0$. Also, if $k > n$, then $\mathrm{Ind}(D_-) = 0$ as well.

It's interesting to check what these indices reduce to in the basic spin^k structure cases. If M is spin and $E = 0$, it's easy to see that both $\mathrm{Ind}(D_+)$ and $\mathrm{Ind}(D_-)$ reduce to just the $\hat{A}$-genus on M; in fact, both D_+ and D_- are then just the standard Dirac operator on M.

If $E = TM$, then $\mathrm{Ind}(D_-) = (-1)^n \chi(M)$ is up to a sign the Euler characteristic of M. Furthermore in this case, $\mathrm{Ind}(D_+) = \sigma(M)$ is the signature of M (note the $(-1)^n$ sign wouldn't matter, as the signature vanishes when n is odd anyway).

3.2 D_- and Geometry

We'll show in this section that the index of D_- on a spin^k manifold is equal to the index of the standard Dirac operator on a related spin manifold.

Let (M, E) be a spin^k manifold, with M having dimension $2n$ and E having rank $k = 2s$. Let $s : M \to E$ be a generic smooth section of E; that is, a section transverse to the zero section. Define $i : N \subseteq M$ to be the inclusion of the zero locus of s; that is, $N = s^{-1}(0)$. By transversality, N is a smooth submanifold of M.

It's a standard result (see, e.g. §6 in [BT82]) that i^*E is the normal bundle to N in M, and that N is Poincaré dual to $e(E) \in H^*(M, \mathbb{Z})$; in particular, N is also naturally oriented. From $i^*TM \cong TN \oplus i^*E$, we thus have that $w_2(TN) = w_2(i^*TM) + w_2(i^*E) = i^*w_2(TM \oplus E) = 0$ by (M, E) being a spin^k structure. Therefore, N is a spin manifold.

Remark that the manifold N in this construction isn't generally unique; it will depend on the choice of generic section s of E. However, the oriented bordism class of N will be well-defined independently of this choice, and so one can talk about the genera of N[2]

Proposition 3.2.1. With the construction as above, let D_- be the spin^k Dirac operator on (M, E) with M $2n$-dimensional and let D be the standard Dirac operator on N. Then $\mathrm{Ind}(D_-) = (-1)^n \mathrm{Ind}(D)$.

[2]Recall that a genus is a ring homomorphism from some bordism ring (such as oriented bordism, spin bordism, etc.) to some target manifold R (often $\mathbb{Z}$ or $\mathbb{Q}$). Genera are therefore bordism-invariants with respect to the relevant bordism theory.

Proof. From the isomorphism $i^*TM \cong TN \oplus i^*E$, we get that $\hat{A}(TN) = i^*\left(\frac{\hat{A}(TM)}{\hat{A}(E)}\right)$. Therefore,

$$\begin{aligned}
\mathrm{Ind}(D) &= \int_N \hat{A}(TN) \\
&= \int_N i^*\left(\frac{\hat{A}(TM)}{\hat{A}(E)}\right) \\
&= \int_M \frac{\hat{A}(TM)}{\hat{A}(E)} e(E) \\
&= 2^l \int_M \hat{A}(TM) \prod_{i=1}^{l} \sinh\left(\frac{y_i}{2}\right) \\
&= (-1)^n \mathrm{Ind}(D_-).
\end{aligned}$$

□

From this construction, we have a geometric interpretation of the previous vanishing conditions of $\mathrm{Ind}(D_-)$. If $2n \not\equiv k \,(\mathrm{mod}\, 4)$, then the dimension of N (which is $2n - k$) isn't divisible by 4, and so the $\hat{A}$-genus of N vanishes. If $k > n$, then the zero locus of a generic section of E is empty, and one can imagine N as an empty manifold with vanishing $\hat{A}$-genus.

This construction was simple enough, but it motivates to consider whether there's a similar geometric construction that relates the index of D_+ on a spin^k manifold with the index of the standard Dirac operator on some related spin manifold.

3.3 D_+ and Geometry

We provide a partial positive answer to this question; specifically in the case that E splits into a sum of oriented subbundles of rank at most 2. This condition is equivalent to the structure group of E being reducible to an abelian structure group. We'll start with the spin^c case; that is, $E = L$ for L a complex line bundle.

We begin with a construction originally due to Hirzebruch[3] [Hir69].

[3]In Hirzebruch's original paper, the item of interest wasn't this construction per se, but its general implication of the existence of branched double covers. Extensions of that paper seem to largely go in the direction of knot theory and low dimensions, i.e. as in [Nag00], which is a precursor to this work.

Let M be a (closed, oriented) manifold with an associated complex line bundle L. Let $s : M \to L \otimes L$ be a generic section (i.e. a section transverse to the zero section). Define $\rho : L \to L \otimes L$ to be the smooth bundle map induced by the vector space map $v \mapsto v \otimes v$.

Proposition 3.3.1. *ρ is transverse to $s(M) \subseteq L \otimes L$.*

Proof. This is fairly easy to see; the differential $d\rho$ is given locally by

$$d\rho(v, l_1)|_{(m,l_2)} = (v, 2l_1 \otimes l_2)|_{(m,l_2 \otimes l_2)},$$

where (m, l_2) and $(m, l_2 \otimes l_2)$ are points in the total bundles of L and $L \otimes L$ respectively, and (v, l_1) and $(v, 2l_1 \otimes l_2)$ are vectors at those points split into horizontal and vertical parts. This differential is evidently surjective away from the zero section of L (i.e. when $l_2 \neq 0$); at the zero section of L, it's only surjective onto the horizontal bundle of $L \otimes L$, but then ds is surjective onto the vertical bundle by the assumed transversality of s to the zero section of $L \otimes L$. □

By this transversality, then, the preimage $\rho^{-1}(s(M)) \subseteq L$ is an oriented manifold of the same dimension as M, which we denote as $\widetilde{M}$.

Definition 3.3.2. We refer to $\widetilde{M}$ as the *Hirzebruch branched cover* of M with the associated complex line bundle L.

We remark briefly that $\widetilde{M}$ is evidently not unique, as it will generally depend on the choice of section $s : M \to L \otimes L$. It is, however, unique up to oriented bordism, a fact which we'll abuse for now before elaborating on later.

As suggested by our chosen name, $\widetilde{M}$ is indeed a ramified double cover of M with branching locus the zero locus of the section s; alternatively, the branching locus is Poincaré dual to $e(L \otimes L) = 2c_1(L)$. If L is the trivial bundle $\mathbb{C}$-bundle, $\widetilde{M}$ is simply two copies of M (and so is unramified). The projection map $\pi : \widetilde{M} \to M$ is given by $\pi = s^{-1} \circ \rho$.

Before proceeding, we present a more-or-less concrete example.

Proposition 3.3.3. *Let $M = \mathbb{C}P^{2n}$ with canonical complex line bundle $L = \mathcal{O}(1)$ the dual to the tautological line bundle. Then $\widetilde{M}$ is oriented bordant to the zero locus of a generic section of $\mathcal{O}(2)$ over $\mathbb{C}P^{2n+1}$; that is, a quadric hypersurface in $\mathbb{C}P^{2n+1}$.*

Proof. Let E denote the total space of the tautological complex line bundle $\mathcal{O}(1)$ over $N = \mathbb{C}P^{2n+1}$. Let S_1 and S_2 denote generic smooth sections of $\mathcal{O}(1)$ and $\mathcal{O}(2)$, respectively, over N. We construct the subset W of $E \times [0,1]$ given by

$$W = \{(n, v, t) \,|\, S_2(n) = t(v \otimes v), tS_1(n) + (1-t)v = 0\},$$

where n is a point in N, v is a coordinate on $\mathcal{O}(1)$, and $t \in [0,1]$.

That W is properly an oriented manifold is a standard transversality argument no more complex than the proof of Proposition 3.3.1. Hence W defines an oriented bordism between

$$W_0 = \{(n, v) \,|\, S_2(n) = 0, v = 0\}$$

and

$$W_1 = \{(n, v) \,|\, S_2(n) = v \otimes v, S_1(n) = 0\}.$$

W_0 is immediately the zero locus of a generic section of $\mathcal{O}(2)$ over $N = \mathbb{C}P^{2n+1}$. To interpret W_1, recall that the zero set of a generic section of $\mathcal{O}(1)$ over $\mathbb{C}P^{2n+1}$ is just (bordant to) $\mathbb{C}P^{2n}$, and so W_1 is the restriction of the Hirzebruch branched cover of $\mathbb{C}P^{2n+1}$ with associated bundle $\mathcal{O}(1)$ to $\mathbb{C}P^{2n}$, which is easily seen to be the Hirzebruch branched cover of $\mathbb{C}P^{2n}$ with associated bundle $\mathcal{O}(1)$. $\square$

Between this proposition and the general construction, it's clear to see that the Hirzebruch double cover of $\mathbb{C}P^n$ with associated line bundle $\mathcal{O}(d)$ is the hyperplane $z_{n+1}^2 = f(z_0, \cdots, z_n)$ in (complex) $(n+1)$-dimensional weighted projective space $\mathbb{C}P(d, 1, \cdots, 1)$, where f is a non-singular homogeneous polynomial of degree $2d$. We'll revisit this construction in more generality later.

Back to generalities, the following lemma will be useful.

Lemma 3.3.4. Let $\pi : \widetilde{M} \to M$ be the Hirzebruch branched cover with associated complex line bundle L. Then we have an isomorphism of topological vector bundles $\pi^*(TM \oplus L) \cong T\widetilde{M} \oplus \pi^*(L \otimes L)$[4].

[4]We're being slightly sloppy here in failing to distinguish real vs. complex vector bundles. The tensor product $L \otimes L$ is a complex tensor product (and so evaulates to a complex line bundle), and after that all complex bundles are then considered as real.

Proof. As by construction $\widetilde{M}$ is naturally embedded into the total space of the line bundle L over M, it's easy to see that the pullback of the tangent bundle of this total space to $\widetilde{M}$ is $\pi^*(TM \oplus L)$. It therefor suffices to show that $\pi^*(L \otimes L)$ is isomorphic to the normal bundle of $\widetilde{M}$; however, by the transversality from Proposition 3.3.1, the normal bundle to $\widetilde{M}$ is isomorphic to the pullback of the normal bundle to $s(M)$ in $L \otimes L$, which altogether is $\pi^*(L \otimes L)$ as desired. □

This lemma then quickly gives us the following result.

Proposition 3.3.5. Let M be a spin^c manifold whose spin^c structure has canonical complex line bundle L. Then the Hirzebruch branched cover $\widetilde{M}$ is a spin manifold, with spin strucutre naturally inherited from the spin^c structure on M.

Proof. Taking the total Steifel-Whitney class of the isomorphism in Lemma 3.3.4, we get

$$\pi^*(w(TM \oplus L)) = w(T\widetilde{M}) \cdot \pi^*(w(L \otimes L)).$$

As everything is oriented, restricting to the second Steifel-Whitney class gives $\pi^*(w_2(TM \oplus L)) = w_2(T\widetilde{M}) + \pi^*(w_2(L \otimes L))$. Now $L \otimes L$ is a spin vector bundle, since its first Chern class is $2c_1(L)$ and hence even, so $w_2(L \otimes L) = 0$. Moreover, since (M, L) is spin^c, $TM \oplus L$ is a spin vector bundle as well, so $w_2(TM \oplus L) = 0$. Hence $w_2(T\widetilde{M}) = 0$ and $\widetilde{M}$ is spin.

To show that $\widetilde{M}$ also has a naturally inherited spin structure, firstly recall that once the canonical bundle L for the spin^c structure on M is chosen, the choice of spin^c structure is equivalent to a choice of spin structure on $TM \oplus L$, so $\pi^*(TM \oplus L)$ has a naturally inherited spin structure. Moreover, a choice of spin structure on $\pi^*(L \otimes L)$ is equivalent to a choice of square root of this bundle, of which we certainly have a natural one, $\pi^* L$. Therefore, as $\pi^*(TM \oplus L)$ and $\pi^*(L \otimes L)$ both have natural spin structures, by the two-out-of-three lemma on the short exact sequence

$$0 \to T\widetilde{M} \to \pi^*(TM \oplus L) \to \pi^*(L \otimes L) \to 0,$$

the bundle $T\widetilde{M}$ inherits a natural spin structure. □

It's shown in [LWY16] that this argument also essentially works in the complex category, provided one can find generic holomorphic sections. Specifically, if M is a complex Fano manifold with anticanonical bundle K_M^*, then the Hirzebruch branched cover of (M, K_M^*) is Calabi-Yau.

We now return to the issue of uniqueness up to bordism via the following theorem, which shows that considering the Hirzebruch branched cover construction as a map on bordism groups rather than on manifolds may be the natural interpretation.

Theorem 3.3.6. If M is an n-dimensional spin^c manifold with associated complex line bundle L, the construction from (M, L) to $\widetilde{M}$ induces a natural morphism on bordism groups $\psi : \Omega_n^{\mathrm{spin}^c} \to \Omega_n^{\mathrm{spin}}$.

Proof. We must show that the final bordism class in Ω_n^{spin} is independent firstly of the choice of representative M in a given class of $\Omega_n^{\mathrm{spin}^c}$ and secondly of the choice of generic section of $L \otimes L$ over M.

Suppose firstly that we have a fixed M and a fixed L, but two choices $s_0, s_1 : M \to L$ of generic sections. The fact that the zero loci of s_0 and s_1 are bordant is a standard result; if we take $M \times [0,1]$ with projection π onto M, we can construct the interpolated section $s(m,t) = (1-t)s_0(m) + ts_1(m)$ of $\pi^* L$. This is transverse to the zero locus at $t = 0$ and $t = 1$, and we can perturb its interior so that it's generic everywhere, so the full zero locus of the perturbed section as a submanifold of $M \times [0,1]$ gives a bordism between the zero loci of s_1 and s_2. In fact, this shows the stronger result that the zero loci are bordant relative to their inclusions into M.

Once we have a generic section s' of $\pi^* L$ on $M \times [0,1]$ that restricts to s_0 and s_1 on $t = 0$ and $t = 1$, respectively, we can simply perform the Hirzebruch branched cover construction on $M \times [0,1]$ with associated bundle $\pi^* L$ and section s, which clearly results in a bordism between the corresponding construction on the two boundary pieces.

Now suppose we choose two different representatives M_1 and M_2 of the same bordism class in $\Omega_n^{\mathrm{spin}^c}$, with a bordism W connecting them so that the spin^c structure on W restricts to the correct spin^c structure on its boundaries. The Hirzebruch branched cover construction on the entirety of W with complex line bundle corresponding to its spin^c structure will then, like before, give a spin manifold $\widetilde{W}$ with boundary $\widetilde{M_1} \sqcup \widetilde{M_2}$, as desired. □

Our capstone result for this chapter is now the following; note that by showing that this construction is a well-defined function on bordism groups, this shows that the Dirac operator indices are well-defined.

Theorem 3.3.7. Let M be a $4n$-dimensional spin^c manifold with complex line bundle L associated to the spin^c structure, and let $\widetilde{M}$ be the Hirzebruch branched cover of M associated with L. Let D and D^c denote the spin and spin^c Dirac operators on $\widetilde{M}$ and M, respectively. Then $\operatorname{Ind}(D) = 2\operatorname{Ind}(D^c)$.

Proof. Our starting point will be the isomorphism of bundles in Proposition 3.3.4:

$$\pi^*(TM \oplus L) \cong T\widetilde{M} \oplus \pi^*(L \otimes L).$$

Taking the A-hat genus of both sides,

$$\hat{A}(\pi^* TM)\hat{A}(\pi^* L) = \hat{A}(T\widetilde{M})\hat{A}(\pi^*(L \otimes L)) + \text{torsion}^5.$$

Let $c_1(L) = u$, so that $\hat{A}(\pi^* L) = \frac{(\pi^* u)/2}{\sinh((\pi^* u)/2)}$ and $\hat{A}(\pi^* L \otimes \pi^* L) = \frac{\pi^* u}{\sinh(\pi^* u)}$. Then

$$\begin{aligned}
\hat{A}(\pi^* TM)\frac{(\pi^* u)/2}{\sinh((\pi^* u)/2)} &= \hat{A}(T\widetilde{M})\frac{\pi^* u}{\sinh(\pi^* u)} + \text{torsion} \\
\implies \pi^*(\hat{A}(TM))\frac{\sinh(\pi^* u)}{2\sinh((\pi^* u)/2)} &= \hat{A}(T\widetilde{M}) + \text{torsion} \\
\implies \pi^*(\hat{A}(TM))\cosh((\pi^* u)/2) &= \hat{A}(T\widetilde{M}) + \text{torsion},
\end{aligned}$$

using the identity $\sinh(2x) = 2\sinh(x)\cosh(x)$. Now, integrating the top degree of both sides over $\widetilde{M}$ (at which point, as described in the footnote, the torsion contribution vanishes),

$$\begin{aligned}
\int_{\widetilde{M}} \hat{A}(T\widetilde{M}) &= \int_{\widetilde{M}} \pi^*\left(\hat{A}(TM)\cosh(u/2)\right) \\
&= 2\int_M \hat{A}(TM)\cosh(u/2) \\
&= 2\int_M \hat{A}(TM)e^{u/2},
\end{aligned}$$

[5] The existence of a possible torsion discrepancy here is a subtle but important point. The previous isomorphism of vector bundles, as noted, was purely topological, and not necessarily respected by any metrics or connections. It's a famous result of Novikov [Nov65] that the *rational* Pontryagin classes are topological invariants, but Milnor showed the same isn't necessarily true of the *integral* Pontryagin classes, from which the A-hat genus is constructed; there may be some torsion difference. Ultimately this is fine for us, as we'll see, since we're only integrating top degree forms, and the top degree cohomology group is free, so torsion contributions to the operator indices will vanish.

where the second inequality is since $\pi : \widetilde{M} \to M$ is a true double cover except on a set of measure zero, and the third is since M is $4n$-dimensional, only the even terms of $e^{u/2}$ contribute to the top degree cohomology, so it can be symmetrized to $\cosh(u/2)$ without changing the value of the integral. These are now the standard Atiyah-Singer integral formulas for the corresponding Dirac operators. □

We remark that this is an alternate proof of a more general genus computation that appeared in [Hat71].

Theorem 3.3.7 gives a convenient way of computing ψ in dimension 4. Recall that $\Omega_4^{\mathrm{spin}^c} \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the standard spin^c structures induced from the complex structures, while $\Omega_4^{\mathrm{spin}} \cong \mathbb{Z}$ is generated by a K3 surface (see, for instance, [BC22]). The Todd genus of the former two generators is 1, while the $\hat{A}$-genus of the K3 surface is 2, so we immediately get that $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ as a map in dimension 4 has $\psi(0,1) = \psi(1,0) = 1$. This is consistent with the K3 surface being a branched double cover of $\mathbb{C}P^2$ branched over a sextic curve.

(In general, ψ won't be surjective, even on the free parts. As in the above reference, Ω_8^{spin} is generated by Bott space $\mathbb{B}$ and hyperbolic projective space $\mathbb{H}P^2$; the former has $\hat{A}$-genus 1 and the latter 0, while any representative in the image of ψ has even $\hat{A}$-genus.)

The result of Theorem 3.3.7 can be rewritten in the following somewhat suggestive way. There are obvious identifications $K^{-4n}(\mathrm{pt}) \cong \mathbb{Z} \cong KO^{-4n}(\mathrm{pt})$ given simply by taking ranks. The natural forgetful map $K^{-4n}(\mathrm{pt}) \to KO^{-4n}(\mathrm{pt})$ is then multiplication by 2 with respect to this identification. Moreover, we have the ABS maps $\alpha : \Omega_{4n}^{\mathrm{spin}}(\mathrm{pt}) \to KO^{-4n}(\mathrm{pt})$ and $\alpha^c : \Omega_{4n}^{\mathrm{spin}^c}(\mathrm{pt}) \to K^{-4n}(\mathrm{pt})$; see [LM16], §II.7 (where it's called the Atiyah-Milnor-Singer invariant). These can be explicitly described as follows: identify as before the codomain of these maps with $\mathbb{Z}$. Then for $M \in \Omega_{2k}^{\mathrm{spin}^c}(\mathrm{pt})$, $\alpha^c(M)$ is the index of the spin^c Dirac operator on M. For $M \in \Omega_{8k}^{\mathrm{spin}}(\mathrm{pt})$, $\alpha(M)$ is the index of the standard Dirac operator on M, while for $M \in \Omega_{8k+4}^{\mathrm{spin}}(\mathrm{pt})$, $\alpha(M)$ is $\frac{1}{2}$ this index.

Define now $\hat{\psi} : \Omega_{4n}^{\mathrm{spin}^c}(\mathrm{pt}) \to \Omega_{4n}^{\mathrm{spin}}(\mathrm{pt})$ by $\hat{\psi} = \psi$ when n is even and $\hat{\psi} = 2\psi$ when n is odd. Then our previous theorem says that

$$\begin{tikzcd} \Omega_{4n}^{\mathrm{spin}^c}(\mathrm{pt}) \arrow[r, "\hat{\psi}"] \arrow[d, "\alpha^c"'] & \Omega_{4n}^{\mathrm{spin}}(\mathrm{pt}) \arrow[d, "\alpha"] \\ K^{-4n}(\mathrm{pt}) \arrow[r] & KO^{-4n}(\mathrm{pt}) \end{tikzcd}$$

is commutative, where again the bottom arrow is the forgetful map from complex to real bundles.

It's interesting to compare this to the more standard commutative diagram (now with M a general target manifold)

$$\begin{tikzcd} \Omega_{4n}^{\mathrm{spin}^c}(M) \arrow[d, "\alpha^c"'] & \Omega_{4n}^{\mathrm{spin}}(M) \arrow[l] \arrow[d, "\alpha"] \\ K^{-4n}(M) & KO^{-4n}(M) \arrow[l] \end{tikzcd}$$

where here the *top* arrow is the forgetful map, now from spin to spin^c structures, and the bottom arrow is complexification.

It would be interesting to see whether this diagram remains commutative when pt is replaced by a general target manifold M. One imagines the first step of such a proof would be to check whether $\hat{\psi}$ is induced from a map on spectra $\mathrm{MSpin}^c \to \mathrm{MSpin}$[6].

We can naturally extend this construction as follows. Suppose instead of M being a spin^c manifold with canonical line bundle L, we have that (M, E) is a spin^k manifold, and that E splits into a sum of complex line bundles and real trivial bundles. Any trivial real bundles will be irrelevant for what follows, so we simply assume that $E \cong L_1 \oplus \cdots \oplus L_l$ is a sum of l complex line bundles, $k = 2l$. We now define

$$\rho : E \to \Psi^2 E,$$

induced by $\rho(v_1, \cdots, v_l) = (v_1 \otimes v_1, \cdots, v_l \otimes v_l)$, where $\Psi^2 E = (L_1 \otimes L_1) \oplus \cdots \oplus (L_l \otimes L_l)$. Exactly the same sort of construction as previously works.

[6]This isn't true in what would be the "obvious" way, which is such a map being induced by a homomorphism $\mathrm{Spin}^c(n) \to \mathrm{Spin}(2n)$; no such natural homomorphism exists.

Definition 3.3.8. If (M, E) is a spin^k vector bundle such that E splits into a sum of complex line bundles and real trivial bundles, let E' denote just the sum of the complex line bundles. Then we define the *Hirzebruch branched cover* $\widetilde{M}$ *of* M *with respect to* E to be $\rho^{-1}(s(M)) \subseteq E'$, where $s : M \to \Psi^2 E'$ is a generic section.

As before, the diffeomorphism class of $\widetilde{M}$ is generally not well-defined as it depends on the choice of section, but the bordism class is unique. Also exactly as before, we get the topological isomorphism

$$\pi^*(TM \oplus E') \cong T\widetilde{M} \oplus \pi^* \Psi^2 E',$$

where $\pi = s^{-1} \circ \rho$.

One can see that in this construction, $\widetilde{M}$ is still a "branched cover" in the following sense: if l is the number of complex line bundles in the splitting of E, then $\pi : \widetilde{M} \to M$ is a degree 2^l cover away from a codimension 2 submanifold dual to $c_1(\Psi^2 E') = 2c_1(E')$. Then, over this codimension 2 submanifold, π is a degree 2^{l-1} cover away from a further codimension 2 submanifold dual to $c_2(\Psi^2 E') = 4c_2(E')$ with respect to M. This pattern continues until we either run out of dimension or we end at a degree 1 covering of a submanifold dual to $c_{2l}(\Psi^2 E') = 2^l e(E')$. So it's not technically accurate to call $\widetilde{M}$ a "branched cover" in the standard sense; it's more of a "stratified branch cover," with each stratification halving the degree of the cover. In any case, we continue to use the slightly abusive name "Hirzebruch branched cover" out of convenience.

We hope the following will give a little more intuition.

Proposition 3.3.9. The Hirzebruch branched cover of a complete intersection in complex projective space is a complete intersection in some weighted projective space.

Proof. Suppose M is the zero locus of a generic section of $\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_m)$ in $\mathbb{C}P^N$ for $N = n + m$; in other words, it's the solution set $\{[z_0, \cdots, z_N] \mid 0 = f_i(z_0, \cdots, z_N)\ \forall\, 1 \le i \le m\}$, where f_i is a non-degenerate homogeneous polynomial of degree d_i. Let $E = L_1 \oplus \cdots \oplus L_l$ be a sum of complex line bundles over M. By the Lefschetz hyperplane theorem, each L_i is the pullback of some $\mathcal{O}(e_i)$ on $\mathbb{C}P^N$ for some $e_i \ge 0$ (possibly by reversing the natural orientation of $\mathcal{O}(-e_i)$). It's then easy to see that the Hirzebruch branched cover of M with respect to E is topologically the set

$$\{[z_0, \cdots, z_{N+l}] \,|\, 0 = f_i(z_0, \cdots, z_N),\ z_{N+j}^2 = g_j(z_0, \cdots, z_N)\ \forall\, 1 \le i \le m, 1 \le j \le l\},$$

in the relevant weighted projective space, where g_j is a non-degenerate homogeneous polynomial of degree $2e_j$. In particular, this is a complete intersection inside the weighted projective space $\mathbb{C}P(2, \cdots, 2, 2e_1, \cdots, 2e_l)$, which is diffeomorphic to $\mathbb{C}P(1, \cdots, 1, e_1, \cdots, e_l)$. □

Back to analogies for the spin^c case, we get just as before,

Proposition 3.3.10. If (M, E) is a spin^k structure such that E splits into oriented subbundles of rank at most 2, then the Hirzebruch double cover $\widetilde{M}$ of M with respect to E is spin, with spin structure induced from the spin^k structure on (M, E).

Theorem 3.3.11. In this case, $\mathrm{Ind}(D) = 2^l \mathrm{Ind}(D_+)$, where l is the number of rank 2 bundles in the splitting of E, D is the standard Dirac operator on $\widetilde{M}$, and D_+ is the spin^k Mayer-Dirac operator on (M, E).

CHAPTER 4

Rigidity, Witten Genera, and Vanishing Theorems

In this chapter, we consider now spin^k manifolds with a smooth S^1-action. Firstly we discuss the concept of rigidity of a differential operator, and demonstrate a condition under which the operatords D_+ and D_- are rigid. Then we use D_+ and D_- to give generalizations of the Witten genus, which we show are the indices of rigid operators under a certain equivariant cohomological condition. This rigidity is used to show vanishing theorems for these generalized Witten genera. Finally, using the geometric constructions of the previous chapter, we show that vanishing theorems for our generalized Witten genera actually imply new vanishing theorems for the standard Witten genus on the corresponding spin manifold constructions.

4.1 Operator Rigidity

We consider now manifolds with a smooth S^1-action. Let M be such an S^1-manifold, and suppose $D : \Gamma(E) \to \Gamma(F)$ is an elliptic differential operator on complex vector bundles E and F such that:

1. The S^1 action on M lifts to an action on E and F, in the sense that it lifts to a smooth action on the total spaces of E and F which is a linear map on the fibers and which fills out the obvious commutative diagram with the projection map and the S^1 action on M.

   (Note that this in particular implies that S^1 also acts on the vector spaces of smooth sections $\Gamma(E)$ and $\Gamma(F)$, or in other words, that these spaces form S^1-representations.)

2. The operator D commutes with this lifted action; that is, if $g \in S^1$ and $s \in \Gamma(E)$, then $g \cdot Ds = D(g \cdot s)$ in $\Gamma(F)$.

Remark that if S^1 acts on $V = \mathbb{C}^n$, then we can split V into 1-dimensional invariant subspaces $V \cong V_1 \oplus \cdots \oplus V_n$ such that $g \in S^1$ acts on $V_k \cong \mathbb{C}$ as multiplication by g^{n_k} for some power n_k; the $\{n_k\}_{k=1}^n$ are the *weights* of this representation.

With the above assumptions, we have that S^1 also acts on $\ker D$ and $\operatorname{coker} D$, which are finite-dimensional complex vector spaces since D is elliptic. Hence we can split $\ker D \cong \bigoplus_{n \in \mathbb{Z}} V_n$ and $\operatorname{coker} D \cong \bigoplus_{n \in \mathbb{Z}} W_n$ as above, where V_n and W_n have weight n.

Definition 4.1.1 (Equivariant Index)**.** Let M be a manifold with a smooth S^1 action and $D : \Gamma(E) \to \Gamma(F)$ an elliptic differential operator on complex vector bundles E and F such that the above conditions 1 and 2 are satisfied. Then the *equivariant index* of D with respect to the S^1 action is

$$\operatorname{Ind}_g(D) = \sum_{n \in \mathbb{Z}} (\dim(V_n) - \dim(W_n)) g^n,$$

where V_n and W_n are the invariant factors of $\ker D$ and $\operatorname{coker} D$, respectively, with weight n, as above.

We re-emphasize that the sum in the above definition is finite since D is elliptic, or in other words, $\operatorname{Ind}_g(D)$ is a Laurent polynomial in g. Furthermore, at $g = 1$, the equivariant index becomes just the standard index, so the equivariant index is a refinement of the standard index.

Definition 4.1.2 (Rigidity)**.** With the same setup as Definition 4.1.1, the elliptic differential operator D is *rigid* with respect to the S^1 action if $\operatorname{Ind}_g(D)$ is a constant independent of g.

This constant is of course necessarily $\operatorname{Ind}(D)$. So the index of a rigid operator D is "localized" at sections where S^1 acts as the identity.

One fundamental property of rigid genera, as shown by Ochanine in [Och06], is that a genus which is rigid with respect to all S^1-actions is also multiplicative over all fiber bundles with compact connected Lie structure groups. However, as we'll see, our Witten genera won't have this rigidity property over *all* S^1-actions.

The primary tool in computing equivariant indices, and hence in determining whether an operator is rigid, is the Atiyah-Bott-Singer fixed point formula. If S^1 acts on a manifold M, let $i : M^g \hookrightarrow M$ be the inclusion of a fixed point submanifold. Note that the normal bundle to M^g in M splits into a sum of complex line bundles which are invariant with respect to the S^1-action; moreover, for any complex vector bundle E on M to which the S^1 action lifts, the pullback i^*E also splits into a sum of invariant complex line bundles, possibly plus an extra real bundle factor on which S^1 acts as the identity.

We define the *equivariant Chern character* for an S^1-action lifting to a complex vector bundle E as follows. Suppose $E \cong E_0 \oplus L_1 \oplus \cdots \oplus L_k$ is the splitting of E into S^1-invariant bundles, with $g \in S^1$ acting on E_0 as the identity and on L_i as multiplication by g^{n_i}. The equivariant Chern character of E with respect to this action is then.

$$\mathrm{ch}_g(E) = \mathrm{ch}(E_0) + \sum_{i=1}^{k} e^{c_1(L_i)} g^{n_i}.$$

We now have the tools to state the fixed point formula.

Theorem 4.1.3 (Atiyah-Bott-Singer Fixed Point Formula (3.9 in [AS68c]))**.** Let M be a closed $2n$-dimensional manifold equipped with an S^1-action with fixed point submanifold $i : M^g \hookrightarrow M$ of dimension $2k$. Let $\{\pm x_j\}_{j=1}^{k}$ and $\{\pm y_j\}_{j=1}^{n-k}$ be Chern roots of $TM^g \otimes \mathbb{C}$ and the complexified normal bundle $N^g \otimes \mathbb{C}$ of M^g in M, respectively. Suppose $D : E \to F$ is an elliptic differential operator on M such that the S^1 action on M lifts to E and F and commutes with D.

Let $N^g \cong J_1 \oplus \cdots \oplus J_k$ be a decomposition into S^1-invariant complex line bundles so that $c_1(J_m) = y_m$ and $g \in S^1$ acts on J_m as multiplication by g^{n_m}. Then

$$\mathrm{Ind}_g(D) = \int_{M^g} \left(\mathrm{ch}_g(i^*(E - F)) \prod_m \frac{1}{(e^{y_m/2} g^{n_m/2} - e^{-y_m/2} g^{-n_m/2})^2} \prod_j \frac{x_j}{(e^{x_j/2} - e^{-x_j/2})^2} \right).[1]$$

(There's a bit of sloppiness in the way we've written this formula; the right-most product isn't well defined, since it has a pole at the x_j. But these extra factors of x_j generally cancel with factors in the equivariant Chern character term, so it works in practice.)

[1]If the fixed point submanifold M^g has multiple connected components, possibly of different dimensions, we should consider this integral as a sum of the corresponding integrals over all connected components.

In spirit, if $g = e^{i\theta}$, one can imagine transforming the non-equivariant Atiyah-Singer formula into the above equivariant formula by localizing to the fixed point set M^g and replacing the Chern roots $\{\pm z_j\}_{j=1}^{2n}$ of $i^*TM \otimes \mathbb{C}$ with the "equivariant Chern roots" $\{\pm(z_j + i\theta n_j)\}_{j=1}^{2n}$ for the corresponding weights n_j, with the exception of a factor of $e(N^g)$.

It's interesting to ask whether the operators D_+ and D_- are rigid.

Proposition 4.1.4. Let (M, E) be a $2n$-dimensional spin^{2k} manifold with an S^1-action. Suppose that the S^1 action lifts to the domain and codomain bundles of D_-, resp. D_+. Suppose further that E is a (topological) subbundle of TM. Then D_-, resp. D_+, is rigid with respect to this S^1 action.

Proof. We'll show this in detail for D_-, and D_+ is essentially the same. Let $i : M^g \hookrightarrow M$ be the inclusion of a $2l$-dimensional fixed point submanifold M^g with normal bundle N^g. Then

$$i^*\Big(\Delta^+(TM \oplus E) - \Delta^-(TM \oplus E)\Big) \cong \Big(\Delta^+(TM^g) - \Delta^-(TM^g)\Big) \otimes \Big(\Delta^+(N^g) - \Delta^-(N^g)\Big) \otimes \Big(\Delta^+(E) - \Delta^-(E)\Big).$$

and we easily see that if $\{\pm x_j\}_{j=1}^{l}$, $\{\pm y_r\}_{r=1}^{n-l}$, and $\{\pm z_s\}_{s=1}^{k}$ are Chern roots of $TM^g \otimes \mathbb{C}$, $N^g \otimes \mathbb{C}$, and $E \otimes \mathbb{C}$, respectively, and if $\{m_r\}_{r=1}^{n-l}$ and $\{n_s\}_{s=1}^{k}$ are the weights of the S^1 action on N^g and E, respectively, then

$$\mathrm{ch}_g\Big(i^*\Big(\Delta^+(TM \oplus E) - \Delta^-(TM \oplus E)\Big)\Big) = \prod_j \left(e^{x_j/2} - e^{-x_j/2}\right) \times \prod_r \left(e^{y_r/2} g^{m_r/2} - e^{-y_r/2} g^{-m_r/2}\right) \times \prod_s \left(e^{z_s/2} g^{n_s/2} - e^{-z_s/2} g^{-n_s/2}\right).$$

Substituting this into the fixed point formula yields

$$\mathrm{Ind}_g(D_-) = \int_{M^g} \left(\prod_{j,r,s} \left(\frac{x_j}{e^{x_j/2} - e^{-x_j/2}}\right)\left(\frac{1}{e^{y_r/2} g^{m_r/2} - e^{-y_r/2} g^{-m_r/2}}\right)\left(e^{z_s/2} g^{n_s/2} - e^{-z_s/2} g^{-n_s/2}\right)\right).$$

The critical condition that E be a subbundle of TM now implies that all of the $e^{z_s/2} g^{n_s/2} - e^{-z_s/2} g^{-n_s/2}$ factors cancel with corresponding factors in the denominator, leaving us with the form

$$\mathrm{Ind}_g(D_-) = \int_{M^g} \left(\prod_{j,t} x_j \frac{1}{P_t(g)}\right),$$

where $P_t(g) = \alpha g^{s_t} - \alpha^{-1} g^{-s_t}$ for $\alpha \in H^*(M^g, \mathbb{Q})$ and $s_t \in \frac{1}{2}\mathbb{Z}$. If $s_t = 0$ for all t (which can only happen if the N^g terms cancel completely), then $\mathrm{Ind}_g(D_-)$ is constant in g and we're immediately done. Otherwise, consider the analytic continuation of $\mathrm{Ind}_g(D_-)$ to the complex plane, and see easily that

$$\lim_{g\to 0} \mathrm{Ind}_g(D_-) = \lim_{g\to\infty} \mathrm{Ind}_g(D_-) = 0.$$

On the other hand, from Definition 4.1.1, $\mathrm{Ind}_g(D_-)$ is a Laurent polynomial in g, and so its analytic continuation to the complex plane can only have poles at 0 or ∞; as we've just shown there are no poles at these points, $\mathrm{Ind}_g(D_-)$ must analytically continue to the entire Riemann sphere, and hence must be constant in g (actually, must vanish, by the above limit values). Hence in both cases, $\mathrm{Ind}_g(D_-)$ is constant in g, and so D_- is rigid.

The case of D_+ is essentially identical, mutatis mutandis. □

This proof also serves as an exposition for how to work with general fixed point sets; from now on, for simplification, we'll assume the fixed point sets of our S^1 actions are discrete points, with the understanding that extending to general S^1 actions offers no non-notational complications. A straightforward unwrapping of the fixed point formula shows that if M^g is a discrete collection of points,

$$\mathrm{Ind}_g(D) = \sum_{\text{fixed pts.}} \left(\left(\sum_j g^{e_j} - \sum_j g^{f_j} \right) \prod_m \frac{1}{(g^{n_m/2} - g^{-n_m/2})^2} \right),$$

where $\{e_j\}, \{f_j\}$, and $\{n_m\}$ are the weights of the S^1 action on E, F, and TM, respectively, localized at each fixed point.

Finally, we note that the idea of rigidity can be extended to operators twisted by general K-theory bundles by linearity; define $\mathrm{Ind}_g(D \otimes (V_1 - V_2)) = \mathrm{Ind}_g(D \otimes V_1) - \mathrm{Ind}_g(D \otimes V_2)$, and rigidity of $D \otimes (V_1 - V_2)$ to again be the constancy of this character-valued index.

4.2 Computational Preliminaries

In this section, we'll introduce several standard constructions that will help us construct and compute with Witten genera in the next section. The first half of this section will be on symmetric and anti-symmetric tensor products of vector bundles, and the second half will be on Jacobi theta functions.

Let E be a complex vector bundle over a base manifold M. For each integer $k \geq 0$, define the vector bundles

$$S^k E = E^{\otimes k} / \sim_{\text{sym}}$$
$$\Lambda^k E = E^{\otimes k} / \sim_{\text{asym}},$$

where $\sim_{\text{sym}}$ is an equivalence relation identifying permutations of the factors in $E^{\otimes k}$, while $\sim_{\text{asym}}$ is the same but including an additional factor of the sign of the permutation. Then, for a formal variable q, define

$$S_q E = 1 + qS^1 E + q^2 S^2 E + \cdots \in K(m)[[q]]$$
$$\Lambda_q E = 1 + q\Lambda^1 E + q^2 \Lambda^2 E + \cdots \in K(m)[[q]].$$

Here 1 is the trivial line bundle. Note that $\Lambda_q E$ is a finite sum when E is a proper vector bundle since $\Lambda^k E = 0$ for $k > \text{rk}(E)$. A key feature of these constructions (which is straightforward to show) is that

$$S_q(E \oplus F) \cong S_q E \otimes S_q F$$
$$\Lambda_q(E \oplus F) \cong \Lambda_q E \otimes \Lambda_q F.$$

With these additive rules, we can extend the operators S_q and Λ_q to all $E \in K(M)$.

If L is a complex line bundle, then $S_q L = 1 + qL + qL^{\otimes 2} + \cdots$ while $\Lambda_q L = 1 + qL$, and so we get that $S_q L \otimes \Lambda_{-q} L = 1$. This immediately extends to when L is a sum of line bundles, then to when L is a general K-theory element by the splitting principle, and so

$$S_q E \otimes \Lambda_{-q} E = 1.$$

We can use this to determine the Chern characters of $S_q E$ and $\Lambda_q E$. The latter is easier: if $E = L_1 \oplus \cdots \oplus L_k$, then

$$\Lambda_q(E) \cong \Lambda_q(L_1) \otimes \cdots \otimes \Lambda_q(L_k) \cong (1 + qL_1) \otimes \cdots \otimes (1 + qL_k),$$

so

$$\mathrm{ch}(\Lambda_q(E)) = \prod_j (1 + qe^{u_j}),$$

where $\{u_j\}_{j=1}^k$ are the Chern roots of E. Again by the splitting principle we can extend this same formula to all K-theory bundles E. We now immediately have

$$\mathrm{ch}(S_q(E)) = \frac{1}{\mathrm{ch}(\Lambda_{-q}(E))} = \frac{1}{\prod_j (1 - qe^{u_j})}.$$

This is the extent to which we'll need these constructions, so now we move on to defining Jacobi theta functions.

The mathematical content of these definitions is absolutely standard; the exact conventions, unfortunately, are not. We use the definitional/notational conventions of [Cha85], which is the same as those of [CHZ11], whose presentation we essentially copy.

The four Jacobi theta functions are defined as

$$\begin{aligned}
\theta(z,\tau) &= 2q^{1/4}\sin(\pi z)\prod_{j=1}^{\infty}\left[(1-q^{2j})(1-e^{2\pi i z}q^{2j})(1-e^{-2\pi i z}q^{2j})\right],\\
\theta_1(z,\tau) &= 2q^{1/4}\cos(\pi z)\prod_{j=1}^{\infty}\left[(1-q^{2j})(1+e^{2\pi i z}q^{2j})(1+e^{-2\pi i z}q^{2j})\right],\\
\theta_2(z,\tau) &= \prod_{j=1}^{\infty}\left[(1-q^{2j})(1-e^{2\pi i z}q^{2j-1})(1-e^{-2\pi i z}q^{2j-1})\right],\\
\theta_3(z,\tau) &= \prod_{j=1}^{\infty}\left[(1-q^{2j})(1+e^{2\pi i z}q^{2j-1})(1+e^{-2\pi i z}q^{2j-1})\right],
\end{aligned}$$

where $q = e^{\pi i \tau}$. These functions have nice transformational properties under modular transformations of τ:

$$\theta(z,\tau+1) = e^{\pi i/4}\theta(z,\tau)$$
$$\theta(z,-1/\tau) = e^{\pi i/4}\tau^{1/2}e^{\pi i\tau z^2}\theta(\tau z,\tau)$$

$$\theta_1(z,\tau+1) = e^{\pi i/4}\theta_1(z,\tau)$$
$$\theta_1(z,-1/\tau) = e^{3\pi i/4}\tau^{1/2}e^{\pi i\tau z^2}\theta_2(\tau z,\tau)$$

$$\theta_2(z,\tau+1) = \theta_3(z,\tau)$$
$$\theta_2(z,-1/\tau) = e^{3\pi i/4}\tau^{1/2}e^{\pi i\tau z^2}\theta_1(\tau z,\tau)$$

$$\theta_3(z,\tau+1) = \theta_2(z,\tau)$$
$$\theta_3(z,-1/\tau) = e^{3\pi i/4}\tau^{1/2}e^{\pi i\tau z^2}\theta_3(\tau z,\tau).$$

We also have the following transformations under translations of z.

$$\theta(z+1,\tau) = -\theta(z,\tau)$$
$$\theta(z+\tau,\tau) = -q^{-1}e^{-2\pi iz}\theta(z,\tau)$$

$$\theta_1(z+1,\tau) = -\theta_1(z,\tau)$$
$$\theta_1(z+\tau,\tau) = q^{-1}e^{-2\pi iz}\theta(z,\tau)$$

$$\theta_2(z+1,\tau) = \theta_2(z,\tau)$$
$$\theta_2(z+\tau,\tau) = -q^{-1}e^{-2\pi iz}\theta_2(z,\tau)$$

$$\theta_3(z+1,\tau) = \theta_3(z,\tau)$$
$$\theta_3(z+\tau,\tau) = q^{-1}e^{-2\pi iz}\theta_3(z,\tau).$$

Finally, two further equations will prove very useful. The first is Jacobi's formula:

$$\theta'(0,\tau) = \pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau),$$

where the derivative is taken with respect to z. Lastly, we have the double angle formula for θ:

$$\theta(2z,\tau) = 2\frac{\theta(z,\tau)\theta_1(z,\tau)\theta_2(z,\tau)\theta_3(z,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}.$$

4.3 Generalized Witten Genera

We immediately recall the definition of the Witten genus of a (closed, oriented) $4n$-dimensional manifold M. Let $TM_{\mathbb{C}}$ denote the complexification of TM, and let $S_q(V) = \mathbb{C} + qV + q^2S^2(V) + q^3S^3(V) + \cdots$ for a complex vector bundle V, where $S^n(V)$ is the nth symmetrized tensor product of V with itself. Then the Witten genus of M is defined as

$$\begin{aligned} W(M) &= \int_M \hat{A}(TM)\mathrm{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^{2n}}(\widetilde{TM_{\mathbb{C}}})\right) \in \mathbb{Q}[[q]] \\ &= \int_M x_i \frac{\theta'(0,\tau)}{\theta(x_i,\tau)}, \end{aligned}$$

where $q = e^{\pi i\tau}$, $\{\pm 2\pi i x_j\}$ are the Chern roots of the complexified tangent bundle $TM_{\mathbb{C}}$, and $\widetilde{TM_{\mathbb{C}}} = TM_{\mathbb{C}} - \mathbb{C}^{4n}$ is the K-theoretic reduction.

The idea of the "Witten genus" of a manifold M arises in several distinct but interrelated constructions. In [Wit87] it's constructed (up to an anomalous factor, and without reducing $TM_{\mathbb{C}}$) as a refinement of the trace of $(-1)^{F_R}$ in a Type II closed superstring theory with target manifold M, where left-movers and right-movers are given Neveu-Schwarz and Ramond boundary conditions, respectively, and F_R is the fermion number operator for these right-movers. In [Wit88], it's constructed (again up to an anomalous factor and without reducing $TM_{\mathbb{C}}$) as the equivariant index of a Dirac operator on the loop space LM with respect to the standard S^1-action using a formal application of the the above Atiyah-Bott-Singer formula.

The connection between these two interpretations is as follows. Consider a $0+1$-dimensional non-linear sigma model with target space M a (real) n-dimensional spin Kähler manifold. One can then express the Dirac operator on M (up to a suitable isomorphism) as

$D = \sqrt{2}(\partial + \bar{\partial})$, acting on the space of anti-holomorphic forms on M twisted by the square root of the canonical bundle $K_M^{1/2}$ (which exists because M is spin). A good reference for this business is [Fri00]. Using Hodge theory, one can then show that

$$\mathrm{Ind}(D) = \sum_{i=0}^{n} (-1)^i \dim H^i(M; K_M^{1/2}),$$

where the $H^i(M; K_M^{1/2})$ are the sheaf cohomology groups (equivalently, the cohomology groups obtained from the chain complex of anti-holomorphic forms twisted by $K_M^{1/2}$). If one then interprets (the pullbacks of) twisted even anti-holomorphic forms as right-moving bosons and twisted odd anti-holomorphic forms as right-moving fermions[2], then we in fact get $\mathrm{Ind}(D) = \mathrm{tr}(-1)^F_R$.

This is well and good for the $0+1$-dimensional theory. If we add a compactified spacial dimension to get a $1+1$-dimensional theory with target space M, one can "reduce" this to the $0+1$-dimensional theory on the parameterized loop space LM by "transferring" the spacial degree of freedom of the worldsheet to the target space. In this way, by analogy with the previous $0+1$-dimensional analysis, one can consider the supertrace $\mathrm{tr}(-1)^F_R$ in the $1+1$-dimensional theory to be analogous with the index of the Dirac operator on the loop space; this is, up to a refinement, the Witten genus.

It was later understood that the Witten genus is (again up to a refinement) exactly the orientation map for elliptic cohomology; see [AHS01] for the initial presentation or [Lur09] for a more modern survey.

On a heuristic level, the Witten class on a manifold M is analagous to the $\hat{A}$-class on the loop space LM, which in turn is analagous to the inverse Euler class on double loop space LLM. More generally, we imagine elliptic cohomology on M to be somehow analagous to equivariant K-theory on LM, which in turn is somehow analagous to equivariant cohomology on LLM. To have a precise form of these analogies would represent a breakthrough in our understanding of the geometric form of elliptic cohomology; see the discussion at the end of [Liu95a].

[2] One specifically wants to use cohomology (that is, mod out by exact forms) by the analogy between the external derivative and the BRST operator.

Even in an interpretive vacuum, the Witten genus has several properties that make it appealing:

- If M is a spin manifold, then $W(M) \in \mathbb{Z}[[q]]$.

This is simply because $W(M)$ is in this case the index of a twisted Dirac operator:

$$W(M) = \mathrm{Ind}\left(D \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{TM_{\mathbb{C}}})\right).$$

- If $p_1(TM) = 0$ (or, more generally, $p_1(TM)$ is torsion), then $W(M)$ is a modular form with weight $\frac{1}{2}\dim(M)$.

This can be seen from a direct computation using the tools of the previous section.

The remainder of this chapter is devoted to presenting natural generalizations of the Witten genus to spin^k manifolds, using the above properties as a guide for what such generalizations should entail.

We'll also find, interestingly, that such generalized Witten genera are rigid with respect to S^1 actions satisfying certain conditions on the equivariant Pontryagin classes. This rigidity is consistent with previous results for the Witten or Witten-like genera; see Theorem 4 in [Liu95b], or Theorem 3.2 in [CHZ11], from which our results are generalized.

4.3.1 Witten Genus Constructed from D_-

Let M be a $2n$-dimensional closed Riemannian manifold with real orientable vector bundle E of rank $k = 2l$. Suppose $TM_{\mathbb{C}}$ has Chern roots $\{\pm 2\pi i x_j\}$ and $E_{\mathbb{C}}$ has Chern roots $\{\pm 2\pi i y_j\}$. We define the Witten form

$$\begin{aligned}
\mathcal{W}_-(M,E) &= (-1)^n 2^l \hat{A}(TM) \prod_{j=1}^{l} \sinh\left(\frac{y_j}{2}\right) \mathrm{ch}\left(\bigotimes_{n=1}^{\infty}\left[S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M}) \otimes \Lambda_{-q^{2n}}(\widetilde{E_{\mathbb{C}}})\right]\right) \\
&= (-1)^n 2^l \hat{A}(TM) \prod_{j=1}^{l} \sinh\left(\frac{y_j}{2}\right) \mathrm{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)\right) \\
&= (-1)^n (2\pi i)^l \prod_{i=1}^{n} x_i \frac{\theta'(0,\tau)}{\theta(x_i,\tau)} \prod_{j=1}^{l} \frac{\theta(y_j,\tau)}{\theta'(0,\tau)},
\end{aligned}$$

where as usual $q = e^{\pi i \tau}$. We then define the corresponding Witten genus

$$W_-(M,E) = \int_M \mathcal{W}_-(M,E) \in \mathbb{Q}[[q]].$$

This construction is similar to an old construction of Dessai for spin^c manifolds ([Des99], [Des00]); however, we consider here E to be a real bundle rather than complex. It also generalizes the $(4k+2)$-dimensional spin^c Witten genus constructed in [CHZ11].

We remark on a couple special cases: if $E = 0$ is the rank 0 bundle, $W_-(M,0) = W(M)$ is the standard Witten genus on M (note that the $(-1)^n$ sign is irrelevant, since the standard Witten genus vanishes when n is odd anyway). If $E = TM$, then $W_-(M,TM) = (-1)^n \chi(M)$ is up to a sign the Euler characteristic of the manifold.

The following integrality is now evident from what we know about the index of D_-.

Proposition 4.3.1. If (M,E) is a spin^k structure, that is, if $w_2(M) = w_2(E)$, then $W_-(M,E) \in \mathbb{Z}[[q]]$.

Proof. In this case, the Mayer-Dirac operator D_- exists, and

$$W_-(M,E) = \operatorname{Ind}\left(D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}} \left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}} \right) \right).$$

□

Furthermore, the above theta function formula for $\mathcal{W}_-(M,E)$ shows that it behaves well under Möbius transformations.

Proposition 4.3.2. Let M be a $2n$-dimensional closed Riemannian manifold and E a rank $k = 2l$ orientable real vector bundle over M. Suppose furthermore that $p_1(TM) = p_1(E)$, or more generally, that $p_1(TM) - p_1(E)$ is torsion. Then $W_-(M,E)$ is a modular form over $SL_2(\mathbb{Z})$ with weight $n - l$.

Proof. $W_-(M,E)$ is evidently invariant under the T transformation $\tau \mapsto \tau + 1$, as this only changes $\theta(z,\tau)$ by a phase of $e^{\pi i/4}$, which cancels by our normalization. For the S

transformation $\tau \mapsto -\frac{1}{\tau}$, check straightforwardly that $\theta'(0,\tau) \mapsto e^{\pi i/4}\tau^{3/2}\theta'(0,\tau)$. Therefore,

$$\begin{aligned}
\mathcal{W}_-(M,E) &\mapsto (-1)^n(2\pi i)^l \prod_{i=1}^n x_i \frac{e^{\pi i/4}\tau^{3/2}\theta'(0,\tau)}{e^{\pi i/4}\tau^{1/2}e^{\pi i\tau x_i^2}\theta(\tau x_i,\tau)} \prod_{j=1}^l \frac{e^{\pi i/4}\tau^{1/2}e^{\pi i\tau y_j^2}\theta(\tau y_j,\tau)}{e^{\pi i/4}\tau^{3/2}\theta'(0,\tau)} \\
&= (-1)^n(2\pi i)^l \exp\left(\pi i\tau\left(\sum_{j=1}^l y_j^2 - \sum_{i=1}^n x_i^2\right)\right)\prod_{i=1}^n(\tau x_i)\frac{\theta'(0,\tau)}{\theta(\tau x_i,\tau)}\prod_{j=1}^l \tau^{-1}\frac{\theta(\tau y_j,\tau)}{\theta'(0,\tau)} \\
&= \tau^{-l}(-1)^n(2\pi i)^l \sum_{i=1}^n(\tau x_i)\frac{\theta'(0,\tau)}{\theta(\tau x_i,\tau)}\prod_{j=1}^l \frac{\theta(\tau y_j,\tau)}{\theta'(0,\tau)},
\end{aligned}$$

where $\sum_{j=1}^l y_j^2 = \sum_{i=1}^n x_i^2$ by the assumption that $p_1(TM) = p_1(E)$. (If $p_1(TM) - p_1(E)$ is torsion, then its contribution will vanish once we integrate the above form.) Now, when we integrate $\mathcal{W}_-(M,E)$ over M to get $W_-(M,E)$, we're only looking at the top degree cohomology, which means terms homogeneous in the x_i and y_j of degree $2n$. As there's always a factor of τ attached to each of these Chern roots in the above expression, this contributes an extra factor of τ^n compared to $W_-(M,E)$ pre-S transformation. Therefore, under the S transformation $\tau \mapsto -\frac{1}{\tau}$,

$$W_-(M,E) \mapsto \tau^{n-l}W_-(M,E),$$

giving a modular form of weight $n-l$. □

A string structure on $TM \oplus E$ induces in the usual way (see [McL92]) a canonical cohomology class $\lambda_E \in H^4(M;\mathbb{Z})$ given by the image of a generator of $H^4(B\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}$, which satisfies $2\lambda_E = p_1(TM \oplus E)$. In analogy with the standard string condition $\lambda_0 = 0$ and the $2 \pmod 4$-dimensional string^c condition $\lambda_L = c_1(L)^2$ constructed in [CHZ11], we propose the *string^{k-} condition*

$$\lambda_E = p_1(E),$$

which is a strictly stronger condition in general than $p_1(TM) = p_1(E)$, but equivalent when $H^4(M;\mathbb{Z})$ has no 2-torsion.

It's immediate that if M is spin and $E = 0$, then the string^{k-} condition reduces to the standard string condition on M. The other obvious special case is slightly less immediate.

Proposition 4.3.3. If M is an n-dimensional closed Riemannian manifold, the universal spin^n structure (M,TM) satisfies the string $^{n-}$ condition.

Proof. We want to show that $\frac{1}{2}p_1(TM \oplus TM) = p_1(TM)$, where the left hand side is the usual first fractional Pontryagin class. Remark that $TM \oplus TM \cong TM \otimes_{\mathbb{R}} \mathbb{C}$ can be assigned a complex structure; then by Lemma 2.39 in [CN19],

$$\frac{1}{2}p_1(TM \oplus TM) = \frac{1}{2}p_1(TM \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(TM \otimes_{\mathbb{R}} \mathbb{C}) = p_1(TM),$$

as required. □

We'll now divert our attention to when M has an S^1-action, and look at the rigidity of $W_-(M,E)$ with respect to this action when (M,E) is spin^k. As before, we'll first compute the equivariant index using the Atiyah-Bott-Singer fixed point formula. As mentioned before, we'll now assume the S^1 action has only isolated fixed points, with the understanding that considering more general fixed point submanifolds gives no serious complication.

For what follows, we'll need the basics of equivariant cohomology and equivariant characteristic classes. We go into more detail constructing these things from the ground up in the Appendix, but to keep this somewhat self-contained, keeping the following as a black box is sufficient:

The equivariant Pontryagin class $p_1(V)_{S^1}$ for an S^1-equivariant real vector bundle V can be expanded as $p_1(V)_{S^1} = p_1(V) + \omega_2 u + \omega_0 u^2$ for a formal degree 2 variable u and differential forms ω_2, ω_4 of degrees 2 and 4, respectively. When one restricts $p_1(V)_{S^1}$ to a connected fixed point submanifold M^g with exponents $n_1, \cdots, n_k$, then $\omega_0|_{M^g} = n_1^2 + \cdots + n_k^2$.

The Appendix builds this up in more rigor, but this is all we really need to begin stating and proving our results.

Theorem 4.3.4. Let (M,E) be a spin^k structure with an S^1-action acting on M that lifts to said spin^k structure. Suppose that $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ for $m \in \mathbb{Z}$. If $m \geq 0$, then

$$\mathrm{Ind}_g\left(D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)\right)$$

is constant in g, and so this operator is rigid with respect to the given S^1 action. Moreover, if $m > 0$, then

$$\mathrm{Ind}_g\left(D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)\right) = 0$$

constantly.

Proof. The path of this proof essentially follows the ideas of [Liu96] and [Liu95a].

At a given fixed point, let $\{m_i\}_{i=1}^n$ and $\{n_j\}_{j=1}^l$ be the exponents of the action on TM and E, respectively, at that point. By the general principal discussed previously, in which the action exponents in the equivariant index are analagous to the Chern roots in the non-equivariant index, we have

$$\mathrm{ind}_g\left(D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)\right) = \sum_{\text{fixed pts.}} (-1)^n \prod_{i=1}^{n} \frac{\theta'(0,\tau)}{\theta(m_i t,\tau)} \prod_{j=1}^{l} \frac{\theta(n_j t,\tau)}{\theta'(0,\tau)} =: F(t,\tau),$$

where $g = e^{2\pi i t}$ and as usual $q = e^{\pi i \tau}$. We compute, similarly to before with the standard index $W_-(M,E)$, the transformation of $F(t,\tau)$ under $S : (t,\tau) \mapsto \left(\frac{t}{\tau}, -\frac{1}{\tau}\right)$ and $T : (t,\tau) \mapsto (t, \tau+1)$. The T transformation evidently leaves F invariant, so we focus on the S transformation. We have

$$\begin{aligned}
F\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) &= \sum_{\text{fixed pts.}} (-1)^n \prod_{i=1}^{n} \frac{\theta'(0,-1/\tau)}{\theta(m_i t/\tau, -1/\tau)} \prod_{j=1}^{l} \frac{\theta(n_j t/\tau, -1/\tau)}{\theta'(0,-1/\tau)} \\
&= \sum_{\text{fixed pts.}} (-1)^n \prod_{i=1}^{n} \frac{\tau^{3/2}\theta'(0,\tau)}{\tau^{1/2}\exp(\pi i \tau (m_i t/\tau)^2)\theta(m_i t,\tau)} \prod_{j=1}^{l} \frac{\tau^{1/2}\exp(\pi i \tau (n_j t/\tau)^2)\theta(n_j t,\tau)}{\tau^{3/2}\theta'(0,\tau)} \\
&= \sum_{\text{fixed pts.}} (-1)^n \tau^{n-l} \exp\left(\pi i \frac{t^2}{\tau}\left(\sum_{j=1}^{l} n_j^2 - \sum_{i=1}^{n} m_i^2\right)\right) \prod_{i=1}^{n} \frac{\theta'(0,\tau)}{\theta(m_i t,\tau)} \prod_{j=1}^{l} \frac{\theta(n_j t,\tau)}{\theta'(0,\tau)}.
\end{aligned}$$

Now, remark that the condition $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ implies that $\sum_{j=1}^{l} n_j^2 - \sum_{i=1}^{n} m_i^2 = -m$ at all fixed points. Therefore,

$$F\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) = \tau^{n-l} \exp\left(-m\pi i \frac{t^2}{\tau}\right) F(t,\tau).$$

This combined with invariance under T are exactly the transformation rules for a Jacobi form with weight $n-l$ and index $-\frac{m}{2}$; the only thing missing is showing that F is holomorphic on $\mathbb{C} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane. This will be the last main piece of the proof. The proof is essentially the same as in [Liu96], from which we borrow some of the notation: we show firstly that F is holomorphic in a neighborhood of $\mathbb{R} \times \mathbb{H}$, and then show that any potential pole outside that range can be transformed to a pole inside that range via a modular transformation, thus proving no poles can exist anywhere.

A priori, we defined g to be in S^1 so that $t \in \mathbb{R}$; we want to extend the definition of F meromorphically so that t can be any complex number. To denote this notationally, we'll replace g with z, so that $z = e^{2\pi i t} \in \mathbb{C}$. (As always, $\tau \in \mathbb{H}$, so that $|q| < 1$.) From the product expansion of θ, the potential poles of $F(t,\tau)$ occur either when $\sin(\pi m_i t) = 0$ or when $z^{m_i} = q^{\pm 2r}$ for some $1 \le i \le n, r \in \mathbb{Z}_{>0}$. In particular, as long as $|z| < |q|^{\pm 2/m_i}$ for all exponents m_i at all fixed points, the only poles occur when $\sin(\pi m_i t) = 0$, in particular, when $t \in \mathbb{R}$ and $z^{m_i} = 1$. We thus restrict our attention to the domain $\mathbb{D}_N := \left\{(t,\tau) \middle| |q|^{2/N} < |z| < |q|^{-2/N}, 0 < |q| < 1\right\}$, where N is larger than any exponent m_i at any fixed point.

In this domain, we can formally expand

$$F(t,\tau) = \sum_{n=0}^{\infty} b_n(z) q^n,$$

where each $b_n(z)$ is a meromorphic function with poles on $|z| = 1$; these b_n are themselves (when restricted to that unit circle) the equivariant index of the twisted D_- operator at the corresponding q mode in $D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)$. Therefore, each b_n has itself a Laurent series expansion $b_n(z) = \sum_{m \in \mathbb{Z}} a_{mn} z^m$, where the sum must be finite. All of these sums are formal; we make no claim yet about convergence.

On the other hand, as there are only finitely many possible poles for $F(t,\tau)$, namely where $z^{m_i} = 1$ for some exponent m_i at some fixed point, we can define $f(z) = \prod_{\text{fixed pts.}} \prod_{i=1}^{n} (z^{m_i} - 1)$, so that $f(z)F(t,\tau)$ is entire. We can thus express

$$f(z)F(t,\tau) = \sum_{n=0}^{\infty} c_n(z) q^n$$

for entire functions $c_n(z)$; this we claim does in fact converge on all of $\mathbb{C} \times \mathbb{H}$. Hence

$$\sum_{n=0}^{\infty} \frac{c_n(z)}{f(z)} q^n = F(t,\tau) = \sum_{n=0}^{\infty} b_n(z) q^n$$

on the domain $\mathbb{D}_N$. Therefore, $\frac{c_n(z)}{f(z)} = b_n(z)$; but on $\mathbb{D}_N$, the left hand side has poles only on $|z| = 1$, while the right hand side as a Laurent polynomial has no poles on $|z| = 1$, and so $\frac{c_n(z)}{f(z)}$ must in fact be holomorphic on $\mathbb{D}_N$. Hence, the expansion

$$F(t,\tau) = \sum_{n=0}^{\infty} \frac{c_n(z)}{f(z)} q^n$$

shows that F is indeed holomorphic on $\mathbb{D}_N$.

The next step is to show that F is in fact holomorphic on all of $\mathbb{C} \times \mathbb{H}$. Suppose for the sake of contradiction that F has a pole at $(t,\tau) = (t_0,\tau_0)$, so that $F(t_0,\tau_0) = \pm\infty$. Again from the product formula for θ, we can see that once $t_0 \in \mathbb{R}$ has been ruled out as a possible pole location, the only remaining possibilities are when $z^{\pm m}q^{2j} = 1$ for some $m, j \in \mathbb{Z}$; this is equivalent to $t_0 m + \tau_0 j = s$ for some $m, j, s \in \mathbb{Z}$ (absorbing the sign ambiguity into m). We can solve $t_0 = \frac{s - j\tau_0}{m} = \frac{d(s' - j'\tau_0)}{m}$, for $d = \gcd(s,j), s = ds', j = dj'$, so that $\gcd(s', j') = 1$. There thus exists $a, b \in \mathbb{Z}$ for which $aj' + bs' = -1$; we act the $SL_2(\mathbb{Z})$ element $\begin{bmatrix} a & b \\ s' & -j' \end{bmatrix}$ on (t, τ) using the Jacobi transformation law of F to get

$$F\left(\frac{t_0}{s'\tau_0 - j'}, \frac{a\tau_0 + b}{s'\tau_0 - j'}\right) = (s'\tau_0 - j')^{n-l} \exp\left(\frac{-\pi i m s' t_0^2}{s'\tau_0 - j'}\right) F(t_0, \tau_0) = \pm\infty.$$

Finally, note that $\frac{t_0}{s'\tau_0 - j'} = \frac{d}{m} \in \mathbb{R}$, so $\left(\frac{t_0}{s'\tau_0 - j'}, \frac{a\tau_0 + b}{s'\tau_0 - j'}\right) \in \mathbb{D}_N$, contradicting the fact that $\mathbb{D}_N$ contains no poles of F. We therefore conclude that $F(t, \tau)$ is holomorphic on the entirety of $\mathbb{C} \times \mathbb{H}$.

Therefore, we arrive at $F(t, \tau)$ being a proper holomorphic Jacobi form on $\mathbb{C} \times \mathbb{H}$ with weight $n - l$ and index $-\frac{m}{2}$. The second part of our theorem, when $m > 0$, now immediately follows from the non-existence of non-vanishing holomorphic Jacobi functions of negative index. On the other hand, if $m = 0$, note that the transformation law $\theta(t+\tau, \tau) = -\frac{1}{q}e^{-2\pi i t}\theta(t, \tau)$ can be iterated to get

$$\theta(t + a\tau, \tau) = (-1)^a e^{-2\pi i a t - \pi i a^2 \tau}\theta(t, \tau)$$

for $a \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
F(t + 2\tau, \tau) &= \sum_{\text{fixed pts.}} (-1)^n \prod_{i=1}^{n} \frac{\theta'(0, \tau)}{\theta(m_i t + 2m_i\tau, \tau)} \prod_{j=1}^{l} \frac{\theta(n_j t + 2n_j\tau, \tau)}{\theta'(0, \tau)} \\
&= \sum_{\text{fixed pts.}} (-1)^n \prod_{i=1}^{n} \frac{\theta'(0, \tau)}{\theta(m_i t, \tau)} \exp\left(4\pi i t m_i^2 + 4\pi i \tau m_i^2\right) \prod_{j=1}^{l} \frac{\theta(n_j t, \tau)}{\theta'(0, \tau)} \exp\left(-4\pi i t n_j^2 - 4\pi i \tau n_j^2\right) \\
&= \sum_{\text{fixed pts.}} (-1)^n \exp\left(4\pi i (t + \tau)\left(\sum_{i=1}^{n} m_i^2 - \sum_{j=1}^{l} n_j^2\right)\right) \prod_{i=1}^{n} \frac{\theta'(0, \tau)}{\theta(m_i t, \tau)} \prod_{j=1}^{l} \frac{\theta(n_j t, \tau)}{\theta'(0, \tau)} \\
&= F(t, \tau),
\end{aligned}$$

again using that $m = 0$ implies $p_1(TM)_{S^1} = p_1(E)_{S^1}$ which in turn implies $\sum_{i=1}^{n} m_i^2 = \sum_{j=1}^{l} n_j^2$.

That $F(t+2, \tau) = F(t, \tau)$ as well is immediate from the same being true for τ. Therefore, for a fixed non-real τ, F is a doubly periodic holomorphic function of t, and so is constant in

t, and the same immediately extends to real τ by continuity. We can thus finally conclude that $F(t,\tau)$ is independent of t, which says exactly that

$$D_- \otimes \bigotimes_{n=1}^{\infty} S_{q^{2n}}\left(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}\right)$$

is rigid with respect to the assigned S^1-action. □

We comment on this condition that $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ for $m \in \mathbb{Z}$, which we might call the equivariant string^{2k-} condition. In fact, one can loosen it slightly so that equality only has to hold modulo torsion. This condition of course implies the non-equivariant condition $p_1(TM) = p_1(E)$ (possibly up to torsion), and with certain natural conditions, it's actually equivalent.

Proposition 4.3.5 (Dessai, [Des99])**.** (a) Suppose an S^1-action on M lifting to E extends to a Pin(2)-action on M also lifting to E which has finite kernel and acts trivially on $H^*(M;\mathbb{Z})$. Then $p_1(TM) = p_1(E)$ modulo torsion implies $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ modulo torsion for some $m \in \mathbb{Z}$.

(b) Suppose M has a non-trivial action by a semisimple Lie group G which lifts to an action on E. Then there exists a non-trivial S^1-action on M lifting to E satisfying the conditions of part (a), and so $p_1(TM) = p_1(E)$ modulo torsion implies $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ modulo torsion for some $m \in \mathbb{Z}$.

This proposition won't, in general, give the sign of m, although in principle this can be checked from looking at the exponents of the S^1-action at any one fixed point submanifold. A simple situation also due to [Des99] is that if E splits into a sum of complex line bundles and the first Betti number $b_1(M) = 0$, then $m < 0$ in the above proposition.

We now move on to a geometric interpretation of this string^{2k-} Witten genus, analogous to our previous geometric construction for the index of the Mayer-Dirac operator D_-. The following is a straightforward computation.

Proposition 4.3.6. Let M be a $2n$-dimensional closed Riemannian manifold and (M,E) a spin^{2k} structure. Let $N \subseteq M$ be a codimension $2k$ submanifold which is the zero locus of a generic section of E. Then $W_-(M,E) = W(N)$, the standard Witten genus on N.

This naturally motivates the following.

Proposition 4.3.7. With (M, E) and N as above, if (M, E) is string^{2k-}, then N is string.

Proof. Let $i : N \hookrightarrow M$ be the inclusion map; we have the standard topological isomorphism of vector bundles $i^*TM \cong TN \oplus i^*E$, or when rewritten, $i^*(TM \oplus E) \cong TN \oplus i^*(E \oplus E)$ (it's interesting to note how similar this is to the corresponding formula for the Hirzebruch branched cover; just one $\otimes$ becomes a $\oplus$). As a quick lemma, we'll show that the first fractional Pontryagin class is additive over spin vector bundles. Suppose we have principal Spin(m) and Spin(n) bundles over a base manifold. Then we have the obvious maps

$$\begin{tikzcd}
 & H^4(B\mathrm{Spin}(m+n);\mathbb{Z}) \arrow[d, "B\oplus^*"'] & \\
 & H^4(B\mathrm{Spin}(m) \times B\mathrm{Spin}(n);\mathbb{Z}) & \\
H^4(B\mathrm{Spin}(m);\mathbb{Z}) \arrow[ur, "B\pi_1^*"'] & & H^4(B\mathrm{Spin}(n);\mathbb{Z}) \arrow[ul, "B\pi_2^*"]
\end{tikzcd}$$

where the three outer groups are isomorphic to $\mathbb{Z}$ and the inner group to $\mathbb{Z} \oplus \mathbb{Z}$. One can easily check that $B\oplus^*(1) = (1,1) = (1,0) + (0,1) = B\pi_1^*(1) + B\pi_2^*(1)$, so the image of $1 \in H^4(B\mathrm{Spin}(m+n);\mathbb{Z})$ in $H^4(M;\mathbb{Z})$ under the pullback of the classifying map is the sum of the images of $1 \in H^4(B\mathrm{Spin}(m);\mathbb{Z})$ and $1 \in H^4(B\mathrm{Spin}(n);\mathbb{Z})$.

With this additivity, since TN and $i^*(E \oplus E)$ are spin, we get that

$$i^*\left(\frac{1}{2}p_1(TM \oplus E)\right) = \frac{1}{2}p_1(TN) + i^*\left(\frac{1}{2}p_1(E \oplus E)\right) = \frac{1}{2}p_1(TN) + i^*p_1(E).$$

The string^{2k-} condition on (M, E) is that $\frac{1}{2}p_1(TM \oplus E) = p_1(E)$, so this would imply that $\frac{1}{2}p_1(TN) = 0$, as desired. $\square$

Combining our previous results, we get a vanishing theorem for the standard Witten genus.

Theorem 4.3.8. Let (M, E) be a spin^{2k} structure, and let there be an S^1-action on M which lifts to this structure. Suppose furthermore that $p_1(TM)_{S^1} - p_1(E)_{S^1} = mu^2$ for $m > 0$. Then if $N \subseteq M$ is the zero locus of a generic section of E, $W(N) = 0$.

As described in the introduction, this sort of theorem has several precursors in the literature ([CH08], [ZZ14], [Zhu16], [Wie24b]), all of which assume some sort of complete intersection structure on the manifold, which we don't. We hope that the trimming of this condition might move us in the direction of showing that string Fano manifolds all have vanishing Witten genus, as suggested by the Stolz conjecture [Sto96], noting that Fano manifolds can be embedded naturally into complex projective space, which itself has a semisimple Lie group action. It's interesting to note that the major condition for a smooth projective variety to be Fano and the major condition for the above theorem to hold are both positivity conditions.

We make one final remark before moving on. Revisiting the condition that E be a topological subbundle of TM, we can define the Ochanine-like genus

$$\operatorname{Ind}\left(D_{-} \otimes \Delta(TM - E) \otimes \bigotimes_{n=1}^{\infty} \left(S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}}) \otimes \Lambda_{-q^{2n}}(\widetilde{T_{\mathbb{C}}M} - \widetilde{E_{\mathbb{C}}})\right)\right).$$

Identical arguments to those already presented show that this is rigid with respect to any S^1-action which lifts to the spin^{2k} structure without any further condition on equivariant cohomology; hence, $D_{-} \otimes \Lambda(TM - E)$ is rigid, etc. This Ochanine-like genus is equal to the standard Ochanine genus on the submanifold N.

4.3.2 Witten Genus Constructed from D_+

As one might guess from the notation of the previous section, we can also construct a Witten-like genus on spin^k structures starting based on the D_+ operator rather than D_-. Much of the results will be analogous with essentially identical proofs, so we'll omit proofs or otherwise be brief when things work out the same.

As with the start of the previous section, let M be a $2n$-dimensional closed Riemannian manifold with real orientable vector bundle E of rank $k = 2l$ or $2l + 1$ (unlike previously, an odd rank vector bundle won't trivialize everything). Suppose $TM_{\mathbb{C}}$ has Chern roots $\{\pm 2\pi i x_j\}$ and $E_{\mathbb{C}}$ has Chern roots $\{\pm 2\pi i y_j\}$ (if k is odd, there'll be an unpaired 0 among these roots).

We then define the Witten form

$$\begin{aligned}\mathcal{W}_+(M,E) &= 2^l \hat{A}(TM)\prod_{j=1}^{l}\cosh\left(\frac{y_j}{2}\right)\mathrm{ch}\left(\bigotimes_{n=1}^{\infty}\left[S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M})\otimes\Lambda_{q^{2n}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{-q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\right]\right)\\ &= 2^l\prod_{i=1}^{n}x_i\frac{\theta'(0,\tau)}{\theta(x_i,\tau)}\prod_{j=1}^{l}\frac{\theta_1(y_j,\tau)\theta_2(y_j,\tau)\theta_3(y_j,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}\\ &= \prod_{i=1}^{n}x_i\frac{\theta'(0,\tau)}{\theta(x_i,\tau)}\prod_{j=1}^{l}\frac{\theta(2y_j,\tau)}{\theta(y_j,\tau)},\end{aligned}$$

where again $q = e^{\pi i\tau}$. Then we define the full genus

$$W_+(M,E) = \int_M \mathcal{W}_+(M,E) \in \mathbb{Q}[[q]].$$

There's a missing $(-1)^n$ compared to $W_-(M,E)$ since the top form and hence the integral vanishes when n is odd anyway. As with W_-, if $E = 0$, then $W_+(M,0) = W(M)$ is the standard Witten genus. If $E = TM$,

$$W_+(M,TM) = \int_M \prod_{i=1}^{n} x_i \frac{\theta'(0,\tau)\theta(2x_i,\tau)}{\theta(x_i,\tau)^2},$$

and we have no immediate interpretation of this. The following two propositions require nothing new to prove.

Proposition 4.3.9. If (M,E) is a spin^k structure, that is, if $w_2(M) = w_2(E)$, then $W_+(M,TM) \in \mathbb{Z}[[q]]$.

Proposition 4.3.10. Let M be a $2n$-dimensional closed Riemannian manifold and E a rank $k = 2l$ or $k = 2l+1$ orientable real vector bundle over M. Suppose furthermore that $p_1(TM) = 3p_1(E)$, or more generally, that $p_1(TM) - 3p_1(E)$ is torsion. Then $W_+(M,E)$ is a modular form over $SL_2(\mathbb{Z})$ with weight n.

As before, we have a canonical class $\lambda_E \in H^4(M;\mathbb{Z})$ constructed from the string structure on $TM \oplus E$ satisfying $2\lambda_E = p_1(TM \oplus E)$ (see [McL92]). We now propose the *string^{k+} condition*

$$\lambda_E = 2p_1(E).$$

As with the string^{k-} condition, having M spin and $E = 0$ reduces to the standard string condition on M. On the other hand, the string^{k+} condition is satisfied in the case of the universal spin^{2n} structure (M,TM) if and only if $p_1(TM) = 0$.

It's evident that if (M,E) is string^{k+}, then $(M, E\oplus E\oplus E)$ is string^{k-}.

The statement and proof of the rigidity theorem for $W_+(TM,E)$ is entirely analogous to the previous section. Recall that here $u\in H^2(ES^1\times_G M;\mathbb{Z})$ is the pullback of the generator of $H^*(BS^1;\mathbb{Z})$.

Theorem 4.3.11. Let (M,E) be a spin^k structure with an S^1 action acting on M that lifts to said spin^k structure. Suppose that $p_1(TM)_{S^1} - 3p_1(E)_{S^1} = mu^2$ for $m\in\mathbb{Z}$. If $m\geq 0$, then

$$\mathrm{Ind}_g\left(D_+\otimes\bigotimes_{n=1}^{\infty}\left[S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M})\otimes\Lambda_{q^{2n}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{-q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\right]\right)$$

is constant in g, and so this operator is rigid with respect to the given S^1 action. Moreover, if $m>0$, then

$$\mathrm{Ind}_g\left(D_+\otimes\bigotimes_{n=1}^{\infty}\left[S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M})\otimes\Lambda_{q^{2n}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{-q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\otimes\Lambda_{q^{2n-1}}(\widetilde{E_{\mathbb{C}}})\right]\right)=0$$

constantly.

As one can predict from the flow of this section, we'll now show that when E splits into a sum of trivial bundles and complex line bundles, $W_+(M,E)$ is exactly the standard Witten genus on the Hirzebruch branched cover of M. The proof is again a straightforward computation, which we omit.

Proposition 4.3.12. Suppose (M,E) is a spin^k structure, and suppose that E splits into a sum of orientable subbundles each of rank at most 2. Then $W_+(M,E) = W(\widetilde{M})$, the standard Witten genus of the Hirzebruch branched cover of M with respect to E.

Proposition 4.3.13. With (M,E) and N as above, if (M,E) is string^{k+}, then N is string.

We can combine our results to get:

Theorem 4.3.14. Let (M,E) be a spin^k structure, and let there be an S^1-action on M which lifts to this structure. Suppose furthermore that $p_1(TM)_{S^1} - 3p_1(E)_{S^1} = mu^2$ for $m>0$. Then if $\widetilde{M}$ is the Hirzebruch branched cover of M with respect to the bundle E, $W(\widetilde{M})=0$.

In fact, the S^1 action need only *stably* lift to the spin^{2k} structure, in the sense that it suffices to lift to some spin^j structure for $j \geq 2k$ trivially inherited from the spin^{2k} strucutre. This is because the branched cover construction and $W_+(M,E)$ are both stable in E; compare this to the corresponding situation for $W_-(M,E)$, where any trivial factor in E trivializes everything.

One nice thing about this theorem is that with the assumption that E splits, the condition that an S^1-action lifts to the spin^k structure can greatly simplify. By our stability, we can ignore trivial summands and assume E is a sum of l complex line bundles. In this case, the spin^{2l} structure is a torus bundle over the frame $SO(2n)$-bundle of M; let the total space of this bundle be Q. Of course, any S^1-action on M lifts to an S^1-action on Q. Now if $b_1(M) = 0$ (that is, the first Betti number of M vanishes), then $b_1(Q) = 0$ as well, and so any S^1-action on M lifts to any principal $U(1)$-bundle on Q; see, for instance, Proposition 3.3 in [Des99]. Once the S^1-action lifts to the torus bundle over Q, we can modify this lift as in (the proof of) Theorem 6.2 in [Pet72] so that it properly lifts to the whole spin^{2l} structure. The punchline, then, is that if assume $b_1(M) = 0$, then *every* S^1-action on M lifts to the spin^{2l} structure (M,E).

Moreover, [Des99] tells us more: if $b_1(M) = 0$, then any action of a semi-simple Lie group G on M induces an action of $\mathrm{Pin}_+(2)$ which lifts to all complex line bundles. At each fixed point set of the action, this lifting gives a complex 1-dimensional representation of $\mathrm{Pin}_+(2)$. Since $\mathrm{Pin}_+(2)$ is generated by reflection elements all of which square to 1, it's easy to see that any such representation is either trivial or a sign representation; in either case, the restriction of the representation to $S^1 \subseteq \mathrm{Pin}_+(2)$ is trivial. In particular, for this induced S^1 action on M (which, by our previous discussion, necessarily lifts to the spin^{2l} structure), provided this action has a fixed point, $p_1(E)_{S^1} = p_1(E)$, since all the exponents at any given fixed point vanish. This makes the condition $p_1(TM)_{S^1} - 3p_1(E)_{S^1} = mu^2$ for $m > 0$ simplify nicely for this induced S^1-action: it simply needs to hold non-equivariantly, i.e. $p_1(TM) - 3p_1(E) = 0$, and the action must be non-trivial and have at least one fixed point, so that the action on TM has non-zero exponents.

The final result of this whole discussion is the following.

Theorem 4.3.15. Let (M, E) be a string^{k+} structure such that E splits into a sum of orientable subbundles each of rank at most 2. Suppose also that $b_1(M) = 0$, and there is a non-trivial action of a semi-simple Lie group G on M. Then $W(\widetilde{M}) = 0$, where $\widetilde{M}$ is the Hirzebruch double cover of M constructed from E.

As usual, we can replace the condition of being string^{k+} with the condition that $p_1(TM) - 3p_1(E) \in H^4(M;\mathbb{Z})$ is torsion.

Corollary 4.3.16. Let $M = \mathbb{C}P^{2n}$ for $n \equiv 1 \pmod 3$, and let $d_1, \cdots, d_k$ be integers for which $\sum_{i=1}^{k} d_i^2 = \frac{2n+1}{3}$. Then $W(\widetilde{M}) = 0$, where $\widetilde{M}$ is the Hirzebruch branched cover of $\mathbb{C}P^{2n}$ associated to $E = \bigoplus_{i=1}^{k} \mathcal{O}(d_i)$.

Proof. This is a direct application of the theorem; note that the cohomology of $M = \mathbb{C}P^{2n}$ is concentrated in even degree so that $b_1(M) = 0$, and also the semi-simple group $SU(2n+1)$ acts naturally and non-trivially on M. It's practically immediate that the given conditions on d_i imply $p_1(TM) - 3p_1(E) = 0$ (of course, using that $p_1(\mathbb{C}P^{2n}) = (2n+1)x^2$ for $x \in H^*(\mathbb{C}P^{2n};\mathbb{Z})$ the generator). Notice finally that the given condition on the d_i implies $\sum_{i=1}^{k} d_i$ is odd, whence $TM \oplus E$ is spin, so all the conditions of the theorem are satisfied. □

Remark that, as shown in the previous chapter, such a $\widetilde{M}$ is a complete intersection of multidegree $(d_1, \cdots, d_k)$ in the generalized projective space $\mathbb{C}P(1, 1, \cdots, 1, d_1, \cdots, d_k)$, and hence is Fano, thus having a Kähler metric of positive Ricci curvature. This vanishing theorem therefore provides new evidence for the Stolz conjecture (note that none of the previously referenced Witten genus vanishing theorems for complete intersections permit an ambient space that isn't smooth).

One can also see that if $d_i = 1$ for all i, the corollary already follows from the classic Landweber-Strong theorem on the vanishing of the Witten genus on string complete intersections, noticing that in this case E would be the normal bundle to $\mathbb{C}P^{2n}$ inside $\mathbb{C}P^{2n} \subset \mathbb{C}P^{2n+1} \subset \cdots \subset \mathbb{C}P^{2n+k}$, and then $\widetilde{M}$ would be a complete intersection inside $\mathbb{C}P^{2n+k}$ with multidegree $(2, 2, \cdots, 2)$ (k times).

It's interesting to note that if $k = 1$, then $2n = 3d_1^2 - 1 \equiv 2 \pmod{12}$, which is both the expected weight of $W(\widetilde{M})$ (were it not to identically vanish) and the exact congruence condition on the weight that appears in the dimension formula for modular forms over $SL_2(\mathbb{Z})$. We offer no explanation for this.

4.4 Combining D_+ and D_-

We show in this section how in the general case that the associated bundle of a spin^k structure splits, one can create "mixed" Mayer-Dirac operators and their corresponding Witten genera. This construction is of course still based on Mayer's [May65].

Suppose that M is a spin^{2j+k} manifold with associated bundle $E \oplus F$, where E and F are rank $2j$ and k, respectively, oriented real bundles. Such a splitting gives a principal $G = (\mathrm{Spin}(2n+2j) \times \mathrm{Spin}(k))/\mathbb{Z}_2$ bundle. This structure group acts naturally on the product of spinor representations $\Delta_{2n+2j} \otimes \Delta_k$; let the corresponding associated vector bundle of this representation be denoted $\sigma = \sigma_{2n+2j} \otimes \sigma_k$. We can split this into $\sigma^+ = \sigma^+_{2n+2j} \otimes \sigma_k$ and $\sigma^- = \sigma^-_{2n+2j} \otimes \sigma_k$ in the usual way. We can then define a morphism $f : \mathrm{Cl}(TM) \to \mathrm{End}(\sigma)$ by $f([m, a]) = [m, \Phi(a) \otimes \mathrm{id}]$, where Φ is the standard representation of the $2n$-dimensional Clifford algebra Cl_{2n} on $\mathbb{C}^{2^{2n+2j}}$ passing through the representation of Cl_{2n+2j}. This turns σ into a Clifford representation bundle, from which we can construct a Dirac operator $D_{-+} : \sigma^+ \to \sigma^-$ in the usual way.

The standard method for computing the index from the representation of G applies here (Proposition 2.17 in [AS68c]), and we get (taking $k = 2l$ or $k = 2l+1$)

$$\mathrm{ind}(D_{-+}) = (-1)^n 2^{j+l} \int_M \hat{A}(TM) \prod_{m=1}^{j} \sinh\left(\frac{y_m}{2}\right) \prod_{n=1}^{l} \cosh\left(\frac{z_n}{2}\right),$$

where $\{\pm y_m\}$ and $\{\pm z_n\}$ are the Chern roots of the complexifications of E and F, respectively.

Essentially everything we do now is what we've done before with D_- and then D_+ with no surprises or hiccups; the second time around we were brief, this time we'll be completely exclusionary, presenting only the results.

Proposition 4.4.1. Let $(M, E \oplus F)$ be a spin^{2j+k} structure with E and F orientable; let $i : N \hookrightarrow M$ be the zero locus of a generic section of E. Then N naturally has a spin^k structure with canonical bundle i^*F and hence Mayer-Dirac operator $D_{N,+}$, and $\mathrm{Ind}(D_{-+}) = \mathrm{Ind}(D_{N,+})$.

We can then define the mixed Witten form

$$\mathcal{W}_{-+}(M,E,F) = (-1)^n 2^j (2\pi i)^l \prod_{p=1}^{n} x_p \frac{\theta'(0,\tau)}{\theta(x_p,\tau)} \prod_{q=1}^{j} \frac{\theta(y_q,\tau)}{\theta'(0,\tau)} \prod_{r=1}^{l} \frac{\theta_1(z_r,\tau)\theta_2(z_r,\tau)\theta_3(z_r,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}$$

and the mixed Witten genus

$$W_{-+}(M,E,F) = \int_M \mathcal{W}_{-+}(M,E,F),$$

where now $\{\pm 2\pi i y_q\}$ and $\{\pm 2\pi i z_r\}$ are the Chern roots of $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$, respectively.

Proposition 4.4.2. We have $W_{-+}(M,E,F) = W_+(N, i^*F)$, where $i : N \hookrightarrow M$ is the zero locus of a generic section of E.

Proposition 4.4.3. The Witten genus $W_{-+}(M,E,F)$ is the index of the operator D_{-+} twisted by a K-theory bundle one can easily write down, and so is in $\mathbb{Z}[[q]]$.

Proposition 4.4.4. Under the condition $p_1(TM) = p_1(E) + 3p_1(F)$, the Witten genus $W(M,E,F)$ is a modular form over $SL_2(\mathbb{Z})$ with weight $n - j$.

If $\lambda_{E\oplus F}$ is the fractional Pontryagin class associated to the spin structure on $TM \oplus E \oplus F$, we then define the *mixed string^{2j+k} condition*

$$\lambda_{E\oplus F} = p_1(E) + 2p_1(F).$$

Theorem 4.4.5. Let $(M, E \oplus F)$ be a spin^{2j+k} structure with E and F orientable of ranks $2j$ and k, respectively. Suppose S^1 acts on M and this action lifts to the spin^{2j+k} structure so that $p_1(TM)_{S^1} - p_1(E)_{S^1} - 3p_1(F)_{S^1} = mu^2$ for some $m \in \mathbb{Z}$. If $m \geq 0$, then the twisted D_{-+} operator from Proposition 4.4.3 is rigid; moreover, if $m > 0$, then its equivariant index identically vanishes.

Finally, in cases that F is a sum of complex line bundles, we'd like to get a manifold N for which $W_{-+}(M,E,F) = W(N)$ by performing a composite of our previous constructions. We might first take the zero section of a generic bundle of E and then take the branched cover of that with respect to the pullback of F, or we might first take the branched cover of M with respect to F and then restrict to the zero locus of a generic section of the pullback of E. It's nice to know that either order produces the same result, up to bordism. The proof is entirely straightforward, so in keeping with the theme, we omit it.

Proposition 4.4.6. Let $(M, E \oplus F)$ be a spin^{2j+k} structure, and suppose F splits into a sum of orientable subbundles of rank at most 2. Take $i : N \hookrightarrow M$ to be the zero locus of a generic section of E, and let $\widetilde{N}$ be the Hirzebruch branched cover of N with respect to i^*F. On the other hand, let $\pi : \widetilde{M} \to M$ be the Hirzebruch branched cover of M with respect to F. Then $\widetilde{N}$ is spin bordant to the zero locus of a generic section of π^*E in $\widetilde{M}$.

CHAPTER 5

Appendix

5.1 Appendix A: S^1-Equivariant Cohomology

In this appendix, we'll present all the necessary material on equivariant cohomology of S^1-manifolds, from the two models of its construction to computations with equivariant characteristic classes. All of this material is classical, and is presented in expanded form in [Tu20].

Let G be a Lie group. There exists a principal G-bundle $EG \to BG$ such that the total space EG is contractible[1]. Moreover, as EG is a principal G-bundle, G acts freely on it on the right. If M is a G-manifold, we can then construct the space

$$M_G = EG \times M / \sim,$$

where $\sim$ is the relation $(e, m) \sim (eg^{-1}, gm)$ for all $g \in G$. M_G is an M-bundle over BG.

Definition 5.1.1 (Borel Construction of Equivariant Cohomology)**.** For a coefficient ring R and a G-manifold M, the *equivariant cohomology ring* of M is $H^*_G(M; R) := H^*(M_G; R)$.

This is one possible construction of equivariant cohomology; its flavor is more geometric. When considering alternate constructions, a natural consideration is to take all standard cohomological forms on M and consider only those which are invariant under G in the following sense. Suppose $G = S^1$[2]. The infinitesimal action of S^1 induces a vector field X on

[1]In general, EG would only be weakly contractible, but when G is a Lie group we can give EG a CW-complex structure and thus have it be contractible by Whitehead's Theorem.

[2]This construction of course generalizes to all Lie groups, but is a bit more involved in general, so we restrict our attention to the simplest case, which happens to be the only one we need.

M. We then consider differential forms $\omega \in \Omega^*(M)$ satisfying $\mathcal{L}_X\omega = 0$ for all i, where $\mathcal{L}_X$ is the Lie derivative in the direction of X; the vector space of such forms is denoted $\Omega^*_{S^1}(M;R)$ if the coefficients are in a ring R.

The proper equivariant differential form on $\Omega^*_{S^1}(M;R)$ is then given by $d_{S^1} = d - i_X$. We follow some authors in introducing a formal degree 2 variable u and writing $d_{S^1} = d - ui_X$; this variable is only to keep terms homogeneous, and also to make the analogy between models explicit, as we'll see.

Definition 5.1.2 (Cartan Model of S^1-Equivariant Cohomology)**.** The *equivariant cohomology* $H^*_{S^1}(M;R)$ of an S^1-manifold M is the cohomology of the differential complex $\Omega^*_{S^1}(M;R)$ with differential d_{S^1}.

Remark that if $\omega \in \Omega^*_{S^1}(M;R)$, then

$$d_{S^1}(d_{S^1}\omega) = (d - ui_X)^2\omega = -u(di_X + i_Xd)\omega = -u\mathcal{L}_X\omega = 0$$

by the definition of $\Omega^*_{S^1}(M;R)$. Hence taking the cohomology makes sense.

We now make the connection between the two constructions in the case that $G = S^1$. In this case, $S^\infty \to \mathbb{C}P^\infty$ is a principal S^1-bundle with contractible total space, and so $ES^1 \simeq S^\infty$ and $BS^1 \simeq \mathbb{C}P^\infty$. We have by the Leray-Serre spectral sequence for this fiber bundle that $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[u]$, were u has degree 2. Moreover, the same sort of spectral sequence argument also gives that, since the cohomology of $\mathbb{C}P^\infty$ is concentrated in the even degrees,

$$H^*(M_{S^1};\mathbb{Z}) \cong H^*(M;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong H^*(M;\mathbb{Z})[u].$$

Hence any cohomology class on M_{S^1} has a representative element ω that can be written as

$$\omega = \omega_n + \omega_{n-2}u + \omega_{n-4}u^2 + \cdots,$$

where the ω_k are differential forms on M of degree k. It's convenient to write out S^1-equivariant differential forms in this expansion when doing computations. A straightforward local computation shows that if $d_{M_{S^1}}$ is the standard de Rham differential on M_{S^1}, then when forms are written as above, we have

$$d_{M_{S^1}} = d - ui_X,$$

the same as in the Cartan model. Finally, such an element ω must of course be a closed form on M_{S^1}. We can thus compute

$$0 = d_{M_{S^1}}\omega = (d - ui_X)(\omega_n + \omega_{n-2}u + \omega_{n-4}u^2 + \cdots) = d\omega_n + (d\omega_{n-2} - i_X\omega_n)u + \cdots$$

and see that if $\omega \in H^*_{S^1}(M)$, then $d\omega_n = 0$, $d\omega_{n-2} = i_X\omega_n$, etc. To make the final connection, note that this implies for each k that

$$\mathcal{L}_X\omega_k = di_X\omega_k + i_Xd\omega_k = d^2\omega_{k-2} + i_X^2\omega_{k+2} = 0$$

(where out-of-range indices give vanishing forms), so each ω_k that appears in ω is S^1-invariant, just as in the Cartan model. Hence the two given constructions for S^1-equivarint cohomology are equivalent.

We now consider equivariant characteristic classes. Let E be a vector bundle over a G-manifold M (with any base field). We say E can be given a G-vector bundle structure if the action of G on M lifts to a smooth action on the total space of E such that, for any $g \in G$ and $m \in M$, the induced map $g : E|_m \to E|_{g\cdot m}$ is a linear map of vector spaces. This is equivalent to the existence of a vector bundle E_G on M_G such that i^*E_G is isomorphic to E for any fiber inclusion $i : M \to M_G$.

A specific choice of G-vector bundle structure on E is a choice of lift of the G-action to the total space of E, or equivalently a choice of vector bundle E_G on M_G and a choice of smooth isomorphism $i^*E_G \cong E$ at each fiber of M_G. The following definition is then natural.

Definition 5.1.3 (Equivariant Characteristic Classes)**.** Given a G-vector bundle E on a G-manifold M, an *equivariant characteristic class* of E is the corresponding standard characteristic class of E_G as an element of $H^*_G(M)$ with the corresponding coefficient ring.

For instance, if E is a complex G-vector bundle, the first equivariant Chern class $c_1(E)_G$ is defined to be $c_1(E_G) \in H^2(M_G;\mathbb{Z})$.

It's immediate that for any fiber inclusion $i : M \to M_G$, the pullback i^* of an equivariant characteristic class is the corresponding non-equivariant characteristic class. In the case of $G = S^1$, this pullback can be written as

$$i^*(\omega_n + \omega_{n-2}u + \omega_{n-4}u^2 + \cdots) = \omega_n.$$

In particular, one can interpret equivariant characteristic classes as extensions of the non-equivariant classes.

For the rest of this appendix we'll continue restricting our attention strictly to $G = S^1$. Let $F \subseteq M$ be a connected fixed point submanifold of the S^1-action, and take any $m \in F$. Let g be a topological generator of S^1. Then g acts as the identity on $TF \subseteq TM|_m$, but will act non-trivially on the normal bundle $N|_m$ to F. In particular, $N|_m$ is a real representation of S^1, and since it can't have components on which S^1 acts trivially, it must be a direct sum of invariant dimension 2 representations, say $N|_m \cong N_1|_m \oplus \cdots \oplus N_k|_m$. S^1 then acts on $N_i|_m$ as a rotation with some integer winding number n_i.

Definition 5.1.4. For a given connected fixed point submanifold $F \subseteq M$, the *exponents* of the S^1-action at F are the above non-zero integers $\{n_1, \cdots, n_k\}$. Here $2k$ is the codimension of F in M.

Note that by continuity, the set of exponents is invariant under any choice of $m \in F$ as long as F is connected. Different connected components of the action may have different exponents, however.

We can perform the same construction for general S^1-vector bundles; in general, we may have exponents of 0.

Definition 5.1.5. For a given connected fixed point submanifold $F \subseteq M$ and an S^1-vector bundle E, the S^1-action on E splits $E|_m$ into invariant subspaces $E|_m \cong E_1|_m \oplus \cdots E_k|_m$ of dimension ≤ 2, on which S^1 acts either trivially or as a rotation. The *exponents* of the S^1-action on E at F are then the winding numbers $\{n_1, \cdots, n_k\}$ of these irreducible representations.

Note that the local splittings in these constructions do extend to global splittings; in particular, any S^1-vector bundle on M restricted to a fixed point submanifold must split over that submanifold into a direct sum of subbundles of rank at most 2 on which S^1 acts invariantly.

The punchline of this whole set-up that we're working towards can now be stated as follows: suppose E is an S^1-vector bundle on M, and let $F \subseteq M$ be a connected fixed point submanifold. Let $E|_F \cong E_1 \oplus \cdots \oplus E_k$ be a splitting of the restriction of E into invariant subbundles on which S^1 acts either trivially or as a rotation, and let n_i be the exponent of E_i.

Punchline 5.1.6. When performing practical computations with equivariant characteristic classes restricted from M_{S^1} to F_{S^1}, one can compute with the "equivariant Chern roots" $\{c_1(E_i)+n_i u\}_{i=1}^k$ using the same formulas as one would use to compute standard characteristic classes with standard Chern roots.

If E_i has real rank 1, we consider $c_1(E_i)$ and n_i to both be 0. We'll now give a formal derivation of the above punchline. It suffices to consider just when E is a complex line bundle, since the restriction of E_{S^1} to $F_{S^1} \subseteq M_{S^1}$ will split as a sum of line bundles anyway.

Let $i : F \hookrightarrow M$ be the inclusion, so that i^*E is the pullback of E to F. Through a slight abuse of notation, extend i to the inclusion $i : F_{S^1} \hookrightarrow M_{S^1}$, and we consider the pullback $i^*E_{S^1}$. We want to show that $c_1(i^*E_{S^1}) = c_1(i^*E) + nu$, where n is the weight of the S^1-action on i^*E.

Since S^1 acts trivially on F, F_{S^1} is a globally trivial product bundle over $BS^1 \simeq \mathbb{C}P^\infty$ with fiber F. If we choose points $x \in \mathbb{C}P^\infty$ and $f \in F$, it suffices to find the first Chern class of the pullbacks to $\{x\} \times F$ and $\mathbb{C}P^\infty \times \{f\}$.

The former is easy, as by construction the pullback of E_{S^1} to $F \subseteq M \subseteq M_{S^1}$ is just i^*E and hence has first Chern class $c_1(i^*E)$.

For the latter, locally coordinatize points in the total space $\mathbb{E}$ of i^*E over F by pairs (f, z) for $f \in F_{S^1}$ and $z \in \mathbb{C}$. Then the total space of $i^*E_{S^1}$ over F_{S^1} is given by

$$ES^1 \times \mathbb{E}/\sim,$$

where $\sim$ is given locally by $(x,(f,z)) \sim (xg^{-1}, g\cdot(f,z)) = (xg^{-1},(f,g^n z))$ for $g \in S^1$; note that g acts trivially on the f components.

But this is exactly the line bundle over BS^1 associated to the principal S^1-bundle $ES^1 \to BS^1$ and the S^1 representation $\rho(g) = g^n$. It's therefore the nth tensor power of the line bundle over BS^1 associated to the representation $\rho(g) = g$, which is exactly the line bundle associated to the S^1-bundle ES^1 itself. This is known to be the universal line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^\infty$, which has first Chern class just the generator u (note this depends on the choice of sign of this generator). Therefore, the pullback of $i^*E_{S^1}$ to $\mathbb{C}P^\infty \times \{f\}$ has first Chern class nu.

All together, since $F_{S^1} \cong \mathbb{C}P^\infty \times F$, we have that $c_1(i^*E_{S^1}) = c_1(i^*E) + nu \in H_G^*(F;\mathbb{Z}) \subseteq H_G^*(M;\mathbb{Z})$. This concludes the formal proof of the prior computational heuristic.

Now we'll give an example of an application of this now-proven heuristic to demonstrate a fact which was used in the main text.

Proposition 5.1.7. Let M be a manifold with an S^1-action, and let E_1, E_2 be two vector bundles over M to which this action extends. Let $F \subseteq M$ be a connected fixed point submanifold so that E_1 and E_2 have exponents $\{m_1, \cdots, m_j\}$ and $\{n_1, \cdots, n_k\}$ over F, respectively. Then the equality of equivariant Pontryagin classes $p_1(E_1)_{S^1} = p_1(E_2)_{S^1}$ implies $m_1^2 + \cdots + m_j^2 = n_1^2 + \cdots + n_k^2$.

Proof. Let the complexified bundles $E_1 \otimes_{\mathbb{R}} \mathbb{C}$ and $E_2 \otimes_{\mathbb{R}} \mathbb{C}$ have Chern roots $\{\pm c_1, \cdots, \pm c_j\}$ and $\{\pm d_1, \cdots, \pm d_k\}$, respectively. Recall that the first Pontryagin class is given by $p_1(E_1) = c_1^2 + \cdots + c_j^2$, and similarly for E_2. Then, by our heuristic, if $i : F \hookrightarrow M$ is the inclusion map,

$$\begin{aligned}(c_1 + m_1 u)^2 + \cdots + (c_j + m_j u)^2 &= i^* p_1(E_1)_{S^1} \\ &= i^* p_1(E_2)_{S^1} \\ &= (d_1 + n_1 u)^2 + \cdots + (d_k + n_k u)^2.\end{aligned}$$

Taking the u^2 coefficient (which is equivalent to pulling back via $H_G^*(F;\mathbb{Z}) \to H^*(\mathbb{C}P^\infty, \mathbb{Z})$) then gives the desired equality. □

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