Strings on AdS2 and the High-Energy Limit of Noncritical M-Theory*

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Abstract: Noncritical M-theory in $2+1$ dimensions has been defined as a double-scaling limit of a nonrelativistic Fermi liquid on a flat two-dimensional plane. Here we study this noncritical M-theory in the limit of high energies, analogous to the $\alpha' \to \infty$ limit of string theory. In the related case of two-dimensional Type 0A strings, it has been argued that the conformal $\alpha' \to \infty$ limit leads to AdS_2 with a propagating fermion whose mass is set by the value of the RR flux. Here we provide evidence that in the high-energy limit, the natural ground state of noncritical M-theory similarly describes the $AdS_2 \times S^1$ spacetime, with a massless propagating fermion. We argue that the spacetime effective theory in this background is captured by a topological higher-spin extension of conformal Chern-Simons gravity in $2+1$ dimensions, consistently coupled to a massless Dirac field. Intriguingly, the two-dimensional plane populated by the original nonrelativistic fermions is essentially the twistor space associated with the symmetry group of the $AdS_2 \times S^1$ spacetime; thus, at least in the high-energy limit, noncritical M-theory can be nonperturbatively described as a "Fermi liquid on twistor space."

Contents

1. Introduction

Noncritical string theories in $1+1$ dimensions (see [1–5] for reviews) have long been a useful playground for studying stringy physics. Unlike their ten-dimensional cousins, two-dimensional string theories are exactly solvable; thus, questions which are difficult to study in full string theory prove themselves more approachable in the theater of noncritical strings.

In this paper, we use the setting of noncritical theories to examine some of the mysteries of M-theory. In the full critical string case, we know very little about full M-theory beyond the use of either nonperturbative dualities, or the low-energy limit as described by eleven-dimensional supergravity. Following the resurgence of interest in two-dimensional Type 0A and 0B string theories [6–8], in [9,10] we proposed a nonperturbative definition of *noncritical M-theory* in $2+1$ dimensions, related to Type 0A and 0B strings in two dimensions by a string/M-theory duality. The definition of noncritical M-theory as given in [9] is in terms of a double-scaling limit of a nonrelativistic Fermi liquid on a rigid two-dimensional plane. In the double scaling limit, the number of fermions N goes to infinity and the potential felt by the fermions becomes that of an an inverted harmonic oscillator. Various ways of filling a Fermi sea correspond to various classical solutions of the theory. This is also how two-dimensional Type 0A and 0B string vacua with a linear dilaton and RR flux are reproduced as solutions of noncritical M-theory: we recover their nonperturbative description as particular Fermi liquids of matrix-model eigenvalues. In this correspondence, the role of the "extra" dimension of M-theory is played by the angular coordinate on the plane populated by the fermions: the Type 0A D0-brane charge is identified with the KK momentum along the extra dimension (*i.e.*, the angular momentum on the plane), mimicking the well-known correspondence from the critical case.[1]

In addition to the two-dimensional string vacua, noncritical M-theory also contains a natural ground state, which we term the $|M\rangle$ state [9]. This state is the noncritical analog of the eleven-dimensional M-theory vacuum solution. As was shown in [9, 10], the $|M\rangle$ state exhibits many features expected of a $2+1$ dimensional spacetime solution. However, its description in terms of an effective gravity theory in a dynamical $2+1$-dimensional spacetime remains unknown. In the related case of two-dimensional strings, the relation between the physical spacetime and the space populated by the fermions is quite subtle [2]. The time dimension is the same between the two pictures. However, the spatial Liouville dimension x of the linear dilaton background is related to the spatial eigenvalue dimension λ by an intricate integral transform, which can be viewed as an early form of string duality. In noncritical M-theory, the situation is worse: so far, only the fermionic description has been developed, and how it maps to a physical spacetime picture is not yet understood. It is the purpose of this paper to remedy this situation, and provide further evidence that the ground state of noncritical M-theory does indeed correspond to spacetime physics in $2+1$ dimensions, with an effective gravity description. We will address this problem in the conformal limit of the theory, where our analysis will be facilitated by the larger symmetries of the system.

In string theory, the conformal limit [12–14] corresponds to sending $\alpha' \to \infty$, which can also be interpreted as a high-energy limit [15–17] (see also [18–20]). This fact provides another motivation for this paper: taking the high-energy limit of a mysterious theory in order to learn more about its underlying degrees of freedom is a classic strategy, pursued in string theory since its early days. It has been widely speculated that in the high-energy limit, the theory might reveal an "unbroken phase" in which the massive string modes become massless [17,21]. Alternatively, one could probe the underlying degrees of freedom by heating the system to high temperature [22]. We applied this latter strategy to noncritical M-theory in [10], and found

[1] The full Type 0A theory in two dimensions has two separate RR fluxes [7]. Making both nonzero simultaneously requires the presence of a nonzero number of long strings [11]. Only one of the RR-fluxes – identified with a D0-charge – plays a role in our definition of noncritical M-theory. Whether or not the noncritical M-theory framework can be extended to incorporate the long strings and both RR fluxes is an interesting open question.

a surprising connection between thermal noncritical M-theory and the topological strings of the A-model on the resolved conifold. In this correspondence, the radius of the Euclidean time circle (*i.e.*, the inverse temperature) on the M-theory side plays the role of the A-model string coupling, a relation expected of topological M-theory [23]. In the present paper, we complement the analysis of [10] and begin to probe the ground state of noncritical M-theory in another extreme regime, of high energies.

This paper is organized as follows. After providing a quick review of noncritical M-theory in Section 2, and of the Type 0A conformal limit in Section 3, we will explore the same limit in the M-theory case in Section 4, and argue that it describes an $AdS_2 \times S^1$ spacetime. The spectrum of propagating modes corresponds to the quanta of a single massless Dirac fermion on this background. In Section 5, we address the question of an effective description of this system on the spacetime side. First we embed the $AdS_2 \times S^1$ as a vacuum solution to conformal $SO(3,2)$ Chern-Simons gravity in 2+1 dimensions. Then we extend the theory to a higher-spin Chern-Simons gauge theory, which not only incorporates the infinite symmetry of noncritical M-theory, but also allows a coupling to the propagating fermionic matter using the "unfolded formalism" of Vasiliev *et al.* [24–26] (see also [27] for a review). Finally, we present our conclusions in Section 6, together with the amusing observation that from the spacetime point of view, the underlying Fermi liquid system can be viewed as living on twistor space associated with the conformal group $SO(3,2)$ of the $2+1$-dimensional dynamical spacetime.

2. Review of Noncritical M-Theory

2.1 Definition as a Fermi Liquid

Following [9], we define noncritical M-theory by starting with a regulated inverted harmonic oscillator potential on a two-dimensional plane $\mathbf{R}^2$, with coordinates λ_i, $i = 1, 2$, filled with N fermions. The two-dimensional plane carries a fixed flat metric

$$ds^2 = d\lambda_1^2 + d\lambda_2^2. \tag{2.1}$$

This metric is not dynamical. The dynamical spacetime with a fluctuating metric field emerges as an effective structure associated with a particular solution of the theory. This is exactly parallel to the case of two-dimensional string theory, wherein the eigenvalues of the matrix model live on a rigid space related to the spacetime Liouville dimension by an integral transform [2].

Then we take a double-scaling limit, simultaneously reducing the potential to an inverted harmonic oscillator while taking the number of fermions to infinity. When the result of this process is written in the second-quantized language, with $\Psi(\lambda_i, t)$ a spinless fermion field, the appropriate action takes the nonrelativistic form

$$S = \int dt d^2\lambda \left(i\Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{1}{2} \sum_{i=1,2} \frac{\partial \Psi^\dagger}{\partial \lambda_i} \frac{\partial \Psi}{\partial \lambda_i} + \frac{1}{2}\omega_0^2 \sum_{i=1,2} \lambda_i^2 \Psi^\dagger \Psi + \ldots \right), \tag{2.2}$$

where ω_0 is the fundamental frequency of the theory, and "..." stand for nonuniversal regulating terms in the potential that are scaled away in the double-scaling limit.[2] In the string-theory solutions of noncritical M-theory [9], ω_0 is related to α' by

$$\omega_0 = \frac{1}{\sqrt{2\alpha'}}. \tag{2.3}$$

In order to simplify our terminology, we will frequently refer to $1/(2\omega_0^2)$ as α' in the case of the M-theory vacuum as well.

We will later also use the first quantized action, which is simply given by

$$S = \frac{1}{2}\int dt \sum_{i=1,2} \left(\dot{\lambda}_i^2 + \omega_0^2 \lambda_i^2\right). \tag{2.4}$$

As explored in [9], the richness of noncritical M-theory comes from the freedom to pick any N states to fill with fermions. However, there is still a most natural second quantized ground state. We construct this state by filling every fermion with individual energy below $-\mu$, while everything with higher energy is kept empty. This is the natural M-theory vacuum solution, and we will call it $|M\rangle$. Its properties depend on the (double-scaled) value of the Fermi energy μ, which plays the role of a coupling constant in the M-theory vacuum [9, 10].

2.2 Embedding of the Type 0A String

In addition to the M-theory state, the vacua of two-dimensional Type 0A and 0B theories string theories are also solutions of noncritical M-theory as defined via the Fermi liquid system. Here we will concentrate on the embedding of the Type 0A linear dilaton vacuum with RR flux (see [9, 10] for 0B). In order to find the Type 0A state in noncritical M-theory, we first change variables from λ_i to the polar coordinates λ and θ (with λ the radial coordinate). Elementary separation of variables allows us to label the fermion quanta by their angular momentum q, and then discuss only the dependence on the radial coordinate, λ. Thus, we are left with a set of one-dimensional fermions, labelled by q, each living in the following potential:

$$V(\lambda) = -\frac{1}{2}\omega_0^2\lambda^2 + \frac{M}{2\lambda^2}, \tag{2.5}$$

where $M = q^2 - 1/4$. The ground state describing the Type 0A vacuum with q_0 units of RR flux simply corresponds to placing all N fermionic quanta in the lowest N states with $q = q_0$ and taking the double scaling limit. (In the Type 0A language, the individual fermion corresponds to the open-string mode of a D0-$\overline{\text{D0}}$ pair, and q_0 is the excess DO-brane charge.) Since this solution of noncritical M-theory has been prepared such that all single-particle states with $q \neq q_0$ are kept empty in this ground state, all excitations in sectors with $q \neq q_0$

[2] The coordinates λ_i before and after the double-scaling limit differ by an overall rescaling factor; in order to avoid notational clutter, we keep this factor implicit, and refer the reader to [9] for the exact technical details of the double-scaling limit.

are infinitely energetic with respect to this ground state and therefore decouple, leaving precisely the excitations of the two-dimensional Type 0A vacuum at q_0 units of RR flux.

This embedding of the Type 0A vacua as solutions in noncritical M-theory sheds new light on the M-theory ground state solution $|M\rangle$. Indeed, we can think of $|M\rangle$ as a coherent sum of Type 0A vacua for all values of RR flux. All single-particle (or hole) excitations are now finitely energetic with respect to the Fermi surface of $|M\rangle$, and represent physical excitations of the $|M\rangle$ state.

This formal decomposition of the ground state of noncritical M-theory into Type 0A sectors is useful technically, for example in the evaluation of the vacuum energy of the solution [9, 10]. It also leads to a crucial "correspondence principle": because the finite-energy excitations of every individual Type 0A sector carry finite energy in the $|M\rangle$ state, the effective 0A physics of each sector must be reproduced by the properties of the $|M\rangle$ state as well. As one application of this correspondence principle, one can argue that since the vacua of Type 0A string theory can be described by an effective action containing two-dimensional gravity, the $|M\rangle$ state should have a gravitational description also. We will use this correspondence as one of our guiding principles throughout this paper.

2.3 Symmetries of the $|M\rangle$ and $|0A\rangle$ States

Now that we have reviewed the basic noncritical M-theory setup, let us discuss a few implications of its symmetries. The theory has an underlying infinite-dimensional symmetry algebra $\mathcal{W}$, first discussed in Section 8.2 of [9]. This algebra is the M-theory analog of the w_∞ symmetry algebras known from two-dimensional string theories [2]. $\mathcal{W}$ arises from four basic conserved charges, given in the classical limit by

$$a_i = \frac{1}{\sqrt{2}}(p_i + \omega_0\lambda_i)e^{-\omega_0 t}, \qquad b_i = \frac{1}{\sqrt{2}}(p_i - \omega_0\lambda_i)e^{\omega_0 t}. \tag{2.6}$$

Here λ_i with $i = 1, 2$ are again the Cartesian coordinates on $\mathbf{R}^2$, and p_i are the conjugate momenta. a_i and b_j satisfy commutation relations implied by the canonical Poisson brackets between the momenta p_i and coordinates λ_j. The four charges (a_i, b_j) form a natural coordinate system on the phase space $\mathcal{T} = \mathbf{R}^4$ of the fermions.

The full algebra $\mathcal{W}$ has a basis consisting of the Weyl-ordered products of an arbitrary finite number of a_i and b_j. We can assign a "degree" to the elements of this basis, simply defined as the total degree of the corresponding monomial in a_i and b_j. Linear combinations of the ten independent charges of degree two,

$$a_1^2,\ a_2^2,\ b_1^2,\ b_2^2,\ a_1a_2,\ b_1b_2,\ a_1b_2,\ a_2b_1,\ \frac{1}{2}(a_1b_1 + b_1a_1),\ \frac{1}{2}(a_2b_2 + b_2a_2), \tag{2.7}$$

form a finite-dimensional subalgebra in the full infinite symmetry algebra $\mathcal{W}$. This algebra of quadratic charges is isomorphic to the Lie algebra of $SO(3, 2)$ or, equivalently, of the noncompact version $Sp(4, \mathbf{R})$ of the symplectic group. Taking appropriate linear combinations of these charges, one can show that the full algebra $\mathcal{W}$ (and, in particular, the $SO(3, 2)$ subalgebra of quadratic charges) is maintained in the $\alpha' \to \infty$ limit.

Even though the Fermi liquid theory exhibits this large symmetry $\mathcal{W}$, any given *solution* will typically break some of $\mathcal{W}$. In particular, those solutions that are described by a semiclassical Fermi surface will generally break $\mathcal{W}$ to the subalgebra that preserves the Fermi surface. As an example, consider again the Type 0A string theory background with RR flux q_0. For this solution of noncritical M-theory, the relevant symmetries in $\mathcal{W}$ are those that commute with the angular momentum generator

$$J = \frac{1}{2\omega_0}\left(a_1 b_2 - a_2 b_1\right) \tag{2.8}$$

on the two-dimensional plane. This is dictated by the fact that the Type 0A solution of M-theory corresponds to filling all available states of $J = q_0$ up to μ, while keeping states with $J \neq q_0$ empty. Out of the ten quadratic generators in $\mathcal{W}$, four survive; J itself, plus the three diagonal combinations

$$a_1^2 + a_2^2, \qquad b_1^2 + b_2^2, \qquad \frac{1}{2}(a_1 b_1 + b_1 a_1 + a_2 b_2 + b_2 a_2). \tag{2.9}$$

The four quadratic charges that commute with J form an $SL(2, \mathbf{R}) \times U(1)$ subalgebra in the $SO(3, 2)$ algebra of quadratic charges in $\mathcal{W}$. From the perspective of Type 0A string theory, the $SL(2, \mathbf{R})$ factor of the surviving symmetry algebra corresponds precisely to the generators of the ground ring. Of course, the $SL(2, \mathbf{R})$ symmetry may be further broken by the level μ of the Fermi sea in the Type 0A vacuum. This embedding of the Type 0A symmetries into $\mathcal{W}$ will be important below.

3. Review of The Conformal Limit of the 0A Matrix Model

The conformal limit of two-dimensional Type 0A string vacua vith RR flux has been studied in [12–14]. Taking α' to infinity in the linear dilaton spacetime with RR flux q leads to the AdS_2 geometry. This limit can also be viewed as a near-horizon limit of an extremally charged two-dimensional black hole [28–30]. In [14], Aharony and Patir provide further evidence for this behavior, by analyzing the spectrum of the model in this limit.

By examining the potential in (2.5), we see that the limit of $\omega_0 \to 0$ allows us to ignore the λ^2 term. We can view this limit alternatively as probing small λ. Either way, the quantum mechanics of the individual eigenvalues reduces in this limit to

$$S = \frac{1}{2}\int dt \left(\dot{\lambda}^2 - \frac{M}{\lambda^2}\right). \tag{3.1}$$

This is a conformal field theory in 0+1 dimensions, studied a long time ago in [31]. The ground state of the second-quantized Fermi liquid consists of N eigenvalues occupying all available states up to Fermi energy $\mu = 0$, which ensures that conformal invariance is maintained. (Conformal invariance would also result from completely emptying or completely filling the entire Fermi sea.)

Two results are of importance here: first, the action (3.1) is invariant under the conformal symmetry $SO(2,1) \sim SL(2,\mathbf{R})$, which of course is equivalent to the isometries of an AdS_2 spacetime. The $SL(2,\mathbf{R})$ generators are

$$\begin{aligned} H &= \frac{1}{2}\left(\dot{\lambda}^2 + \frac{M}{\lambda^2}\right) \\ D &= -\frac{1}{4}\left(\lambda\dot{\lambda} + \dot{\lambda}\lambda\right) + tH \\ K &= \frac{1}{2}\lambda^2 + 2tD - t^2 H. \end{aligned} \tag{3.2}$$

H is the Hamiltonian following from the action (3.1), and it of course exhibits a continuous spectrum. The proposal of [12] is to interpret this Hamiltonian on the AdS_2 dual, as the evolution operator in the Poincaré time. The duality to AdS_2 suggests that we should consider the evolution with respect to the compact operator[3]

$$\widetilde{H} = \frac{1}{2}\left(\frac{1}{\mathcal{R}}K + \mathcal{R}H\right), \tag{3.3}$$

where $\mathcal{R}$ is an arbitrary constant scale, to be identified with the curvature radius of the AdS_2. $\widetilde{H}$ represents the time evolution with respect to the *global time* on AdS_2 [12–14]. $\widetilde{H}$ has a discrete spectrum, with the n-th level eigenvalue given by

$$h_n = \frac{1}{2} + n + \frac{|q|}{2}. \tag{3.4}$$

On the AdS_2 side, this should be interpreted as the spectrum of propagating matter in global time. Aharony and Patir have shown that this spectrum exactly matches the spectrum of a free Dirac fermion on AdS_2 with mass

$$m = \frac{|q|}{2\mathcal{R}}. \tag{3.5}$$

In this sense, the conformal limit of the type $0A$ matrix model is dual to a theory on AdS_2 whose matter excitations are precisely those of a spinor field of mass given in (3.5). In particular, the bosonic degrees of freedom of the tachyon do not survive in the conformal limit. Indeed, from the point of view of the compact generator $\widetilde{H}$, the ground state of the system is empty, and there is no macroscopic Fermi sea and thus no collective bosonic modes. In the near-horizon interpretation of this limit, this means that the propagating excitations of the Fermi sea do not make it to the near-horizon region of the black hole, leaving just the individual fermionic eigenvalues as the only propagating excitations in that regime.

4. Large α' Limit of Noncritical M-Theory

As we have discussed in Section 3, the ground state of noncritical M-theory can be viewed in polar coordinates as a coherent collection of ground states of Type 0A string theory, all

[3] In calling this generator $\widetilde{H}$, we differ slightly from the commonly accepted convention in the literature.

filled to the common Fermi level μ. We now take the conformal limit of this $|M\rangle$ state. Thus, we set $\mu = 0$ in order to maintain conformal invariance of the vacuum. This leads to manifest $SL(2,\mathbf{R})$ invariance of the vacuum. On the dual spacetime side, it is thus natural to expect that AdS_2 will be a part of the spacetime geometry. The rest of the geometry can be inferred as follows. Invoking our "correspondence principle," one can predict the spectrum of excitations of the $|M\rangle$ state in the conformal limit, from the knowledge of the Type 0A spectrum as reviewed in Section 4. As $\alpha' \to \infty$, each individual Type 0A sector with fixed q will contribute one copy of a massive fermion, with mass m given by (3.5). In the M-theory vacuum, we thus get an infinite collection of Dirac fermions on AdS_2, with masses

$$m = \frac{|q|}{2\mathcal{R}}, \qquad q \in \mathbf{Z}. \tag{4.1}$$

This spectrum of masses represents a Kaluza-Klein tower, obtained from the reduction of a massess fermion in $2+1$ dimensions on S^1 of radius $2\mathcal{R}$. Thus, we expect that noncritical M-theory in the conformal limit corresponds to spacetime that is $AdS_2 \times S^1$. The matching of the spectra of propagating modes will be one of the tests of this conjecture. In the rest of the paper, we will subject this conjecture to several additional tests.

4.1 Symmetry Generators

Let us begin our discussion of the large α', or small ω_0, limit of noncritical M-theory by considering the symmetry generators on the fermion side. As in the 0A case above, we find that the action (2.4) simplifies in the small ω_0 limit:

$$S = \frac{1}{2}\int dt \left(\dot{\lambda}_1^2 + \dot{\lambda}_2^2\right) \tag{4.2}$$

Now, we would like to consider the $|M\rangle$ state with μ set exactly to zero. If we think of the $|M\rangle$ state as a coherent collection of Type 0A ground states of all possible values of RR flux q, each of these Type 0A sectors will be filled up to Fermi energy $\mu = 0$. In the conformal limit, the entire $SL(2,\mathbf{R})$ respects the Type 0A vacuum, and the same will be true of the M-theory ground state. In the $\alpha' \to \infty$ limit of noncritical M-theory, the generators of this $SL(2,\mathbf{R})$ subalgebra of $\mathcal{W}$ are

$$\begin{aligned}
H &= \frac{1}{4}(a_1+b_1)^2 + \frac{1}{4}(a_2+b_2)^2 = \frac{1}{2}\left(\dot{\lambda}_1^2 + \dot{\lambda}_2^2\right) \\
K &= \frac{1}{4\omega_0^2}(a_1-b_1)^2 + \frac{1}{4\omega_0^2}(a_2-b_2)^2 = \frac{1}{2}\left(\lambda_1 - \dot{\lambda}_1 t\right)^2 + \frac{1}{2}\left(\lambda_2 - \dot{\lambda}_2 t\right)^2 \\
D &= \frac{1}{4\omega_0}\left(a_1^2 + a_2^2 - b_1^2 - b_2^2\right) = \frac{1}{4}\left(\lambda_1\dot{\lambda}_1 + \dot{\lambda}_1\lambda_1 + \lambda_2\dot{\lambda}_2 + \dot{\lambda}_2\lambda_2 - 2\dot{\lambda}_1^2 t - 2\dot{\lambda}_2^2 t\right).
\end{aligned} \tag{4.3}$$

They all commute with the angular momentum generator J,

$$J = \frac{1}{2\omega_0}(a_1 b_2 - a_2 b_1) = \frac{1}{2}\left(\lambda_1\dot{\lambda}_2 - \lambda_2\dot{\lambda}_1\right). \tag{4.4}$$

It turns out that, from the point of view of the first-quantized formulation, these $SL(2,\mathbf{R})\times U(1)$ generators enjoy a special status among all quadratic charges: they can be realized geometrically by a change of coordinates on the space of (t,λ_1,λ_2) which preserves the foliation of this space by constant time slices. The action (4.2) is indeed symmetric under the following sets of transformations, generated by (4.3) and (4.4):

$$\begin{array}{llll} H: & t'=t-\omega, & \lambda_1'(t')=\lambda_1(t), & \lambda_2'(t')=\lambda_2(t)\\ D: & t'=e^{-\omega}t, & \lambda_1'(t')=e^{-\omega/2}\lambda_1(t), & \lambda_2'(t')=e^{-\omega/2}\lambda_2(t)\\ K: & t'=\frac{t}{\omega t+1}, & \lambda_1'(t')=(1+\omega t)^{-1}\lambda_1(t), & \lambda_2'(t')=(1+\omega t)^{-1}\lambda_2(t)\\ J: & t'=t, & \lambda_1'(t')=(\cos\omega)\lambda_1-(\sin\omega)\lambda_2, & \lambda_2'(t')=(\cos\omega)\lambda_2+(\sin\omega)\lambda_1 \end{array} \tag{4.5}$$

These four generators have one algebraic relation:

$$\frac{1}{2}(HK+KH)-D^2-J^2=\frac{1}{4}. \tag{4.6}$$

The quadratic charges in $SO(3,2)$ that are not in this $SL(2,\mathbf{R})\times U(1)$ are not realized geometrically on (t,λ_i). This does not mean that they cannot survive as "hidden" symmetries, but we do not expect them to be realized by Killing symmetries of the gravitational background. We shall see in Section 5 that this picture is indeed correct.

As an example, let us consider what happens to a particular generator of $SO(3,2)$ that does not belong to the $SL(2,\mathbf{R})\times U(1)$ subalgebra:

$$H'=\dot\lambda_1^2-\dot\lambda_2^2. \tag{4.7}$$

This generator acts on $(t,\lambda_i(t))$ via

$$H':\ t=t',\qquad \lambda_1'(t')=\lambda_1(t-\omega),\qquad \lambda_2'(t')=\lambda_2(t+\omega). \tag{4.8}$$

There is no coordinate transformation on the λ_i and t which can represent this symmetry. It is a symmetry of the action, but not one which can be represented geometrically. One can easily check that the same holds true for the other generators of $SO(3,2)$ not in the $SL(2,\mathbf{R})\times U(1)$ subalgebra. As such, we do not expect any of these generators to be associated with an isometry in the dual spacetime picture. $SL(2,\mathbf{R})\times U(1)$ of course matches the isometries of $AdS_2\times S^1$.

4.2 Spectrum Matching: Fermions on $AdS_2\times S^1$

As in the Type 0A case, the spectrum of H in (4.3) is continuous, and corresponds to the Hamiltonian evolution in the Poincaré time on $AdS_2\times S^1$. We again switch to the global time Hamiltonian $\widetilde{H}$, defined by (3.3), now in terms of the M-theory operators H and K of (4.3). It is straightforward but reassuring to see that the spectrum of $\widetilde{H}$ matches that of a massless fermion on $AdS_2\times S^1$, as expected from our "correspondence principle". If we transform our action from (4.2) via

$$\widetilde{\lambda}_i(\tau)=\frac{\sqrt{\mathcal{R}}}{\sqrt{\mathcal{R}^2+t^2}}\lambda(t),\qquad \tau=\arctan\,(t/\mathcal{R}), \tag{4.9}$$

we find

$$S = \frac{1}{2} \int d\tau \left[(\partial_\tau \widetilde{\lambda}_1)^2 + (\partial_\tau \widetilde{\lambda}_2)^2 - \widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2 \right]. \tag{4.10}$$

Here τ corresponds to the global time, and the individual eigenvalues see a rightside-up planar harmonic oscillator potential. $\widetilde{H}$ generates translations along τ. Its spectrum is

$$h_{n,m} = \frac{1}{2}(n + m + 1), \tag{4.11}$$

where n and m both are non-negative integers. This is equivalent to the spectrum for the $1+1$ dimensional $\widetilde{H}$ as given in (3.4), if one allows q to range over all integers. The second-quantized ground state of $\widetilde{H}$ with the rightside-up harmonic oscillator potential is again empty of all fermions, and there is no macroscopic Fermi surface with propagating bosonic excitations.

Now, let us consider the spectrum of a free fermion, as calculated in global coordinates, on an S^1 fibered over AdS_2, as suggested from the symmetry arguments above. As shown in Appendix A, the only fibering which produces the same spectrum is the direct product spacetime, that is $AdS_2 \times S^1$. Moreover, the spectrum matching requires the radius of the S^1 factor to equal $2\mathcal{R}$, where $\mathcal{R}$ is the curvature radius of AdS_2.

Thus, we conjecture that *in the $\alpha' \to \infty$ limit, the natural ground state $|M\rangle$ of noncritical M-theory describes a theory on a dynamical $AdS_2 \times S^1$ spacetime, with propagating matter described by a single free massless Dirac fermion.*

This conjecture leads to two remarkable phenomena: (*i*) We get a relativistic dual out of a nonrelativistic Fermi liquid,[4] and (*ii*) the angular dimension on the flat plane populated by the nonrelativistic fermions corresponds under this duality to a fixed-radius circle fibered trivially over the AdS_2 base spacetime of string theory; this of course is the traditional behavior of the extra dimension in the simplest forms of string/M-theory duality.

4.3 A Family of Solutions with $SL(2, \mathbf{R})$ Symmetry

In passing, we wish to point out that the conformal $AdS_2 \times S^1$ vacuum belongs to an interesting multi-parameter family of solutions of noncritical M-theory, all of which share the $SL(2, \mathbf{R})$ symmetry of the Type 0A conformal vacua.

Consider the ground state of the global-time Hamiltonian $\widetilde{H}$ of (3.3) in the conformal limit of Type 0A theory. As reviewed above, in the Fermi liquid picture this ground state is empty, *i.e.*, all the single-particle states are unoccupied by the fermions. Another way of preparing a conformally invariant ground state would be to keep all single-particle states occupied, and treat the holes as elementary excitations. Of course, in Type 0A theory these two solutions are isomorphic by the particle-hole duality, and we gain nothing by switching from one description to the other.

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[4]Of course, this phenomenon is already present in two-dimensional string theory, in the duality between its Fermi liquid and spacetime descriptions, and therefore does not represent a novelty of noncritical M-theory. What is perhaps new is that such a duality extends to a dimension higher than $1+1$.

In $2+1$ dimensions, the situation is more interesting. The simplest ground state of the global-time Hamiltonian $\widetilde{H}$ is empty, leading to the $AdS_2 \times S^1$ solution which is the main focus of the present paper. Using particle-hole duality, it is again possible to switch between filled and empty states. Doing so simultaneously for all values q of the angular momentum would result in an equivalent solution. However, unlike in $1+1$ dimensions, we now have the additional freedom of deciding separately for each value of q whether all states are empty or full, without losing the $SL(2,\mathbf{R})$ symmetry of the state. This leads to an interesting multi-parameter family of solutions, parametrized as follows. We start with the ground state of $\widetilde{H}$ with all states empty, choose a sequence

$$q_1 < \ldots < q_n, \tag{4.12}$$

and prepare a new state such that all available states with angular momentum q in the range $q_{2k+1} \leq q < q_{2k}$ for $k = 0, 1, \ldots$ are occupied while those in sectors with $q_{2k} \leq q < q_{2k+1}$ remain empty.

These solutions have n sheets of the Fermi surface, located at $J = q_i$, $i = 1, \ldots n$. All of them inherit the $SL(2,\mathbf{R})$ symmetry from the Type 0A decomposition. It would be very interesting to investigate the spacetime interpretation of this family of solutions. When the individual Fermi surfaces are far separated, the physics of their collective excitations is almost decoupled. However, since there is one common $SL(2,\mathbf{R})$ symmetry shared by the n sheets of the Fermi surface, we expect the solution to have the structure of a multi-sheeted version of AdS_2, sharing the same boundary with a common conformal dual description. This structure is very reminiscent of the multi-sheeted AdS_5 with a common CFT dual, studied in [32, 33].

5. Spacetime Effective Action in the $AdS_2 \times S^1$ Background

An observer in the $AdS_2 \times S^1$ spacetime is not likely to describe the system in terms of the two-dimensional nonrelativistic Fermi liquid on the $\mathbf{R}^2$ flat plane. Instead, the physics of excitations that are sufficiently close to the ground state should be encoded in a spacetime effective action. This action should contain spacetime gravity, it should have $AdS_2 \times S^1$ as a solution, and should reproduce the symmetries and spectrum of the ground state. It is the goal of this Section to propose a natural effective action that satisfies such constraints.

We are working in a coordinate system (x^μ), $\mu = 0, 1, 2$, given by global coordinates $(x^0, x^1) = (t, \rho)$ on AdS_2, and a periodic coordinate $x^2 = y$ of periodicity 2π on S^1. The metric takes the following form,

$$ds^2 = -\mathcal{R}^2 \cosh^2 \rho \, dt^2 + \mathcal{R}^2 d\rho^2 + 4\mathcal{R}^2 dy^2. \tag{5.1}$$

The only nonzero component of the Einstein tensor in this spacetime is

$$R_{yy} - \frac{1}{2} R g_{yy} = 4. \tag{5.2}$$

Thus, unlike an AdS_3 spacetime, $AdS_2 \times S^1$ will not solve the vacuum Einstein equations for any value of the cosmological constant.

With the hindsight afforded by AdS/CFT correspondence, it might be tempting to postulate the existence of a propagating $U(1)$ gauge field, whose flux through AdS_2 (and the dual flux through S^1) could provide just the right energy-momentum tensor to turn $AdS_2 \times S^1$ into a solution of the coupled Einstein-Maxwell equations.[5] This is indeed possible, and leads to a solution of the coupled system

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G_N T_{\mu\nu}, \tag{5.3}$$

where

$$T_{\mu\nu} = F_{\mu\lambda} F_\nu^\lambda - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \tag{5.4}$$

is the conventional energy-momentum tensor of the Maxwell Lagrangian.

Let us consider $U(1)$ flux F along the AdS_2 factor of the geometry, with F proportional to the area two-form on AdS_2:

$$F_{t\rho} = -F_{\rho t} = f_0 \cosh\rho \tag{5.5}$$

The energy-momentum tensor is

$$T_{\mu\nu} = \frac{f_0^2}{2\mathcal{R}^2} \begin{pmatrix} \cosh^2\rho & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.6}$$

The Einstein equations are now satisfied by $AdS_2 \times S^1$ if we pick

$$\Lambda = -\frac{4}{5\mathcal{R}^2}, \qquad f_0 = \frac{\mathcal{R}}{\sqrt{5\pi G_N}}. \tag{5.7}$$

Of course, the two form field strength F is exact, with the gauge field given by

$$A = \frac{\mathcal{R}\sinh\rho}{\sqrt{5\pi G_N}} dt; \tag{5.8}$$

the Maxwell equations for A are trivially satisfied.

The fact that there there is an electric flux through AdS_2 implies that the S^1 carries a dual, magnetic flux. In $2+1$ dimensions, the dual to a $U(1)$ one-form gauge field A is a scalar ϕ, related to the field strength F of A by $*F = d\phi$. By tracking this duality for our background (5.7), one can easily see that ϕ has a flux through the S^1 factor of $AdS_2 \times S^1$.

However, even though $AdS_2 \times S^1$ is a solution to the coupled Einstein-Maxwell system with negative cosmological constant in $2+1$ dimensions, this theory cannot be a good approximation to the effective theory describing the conformal limit of noncritical M-theory, for a simple reason. The analysis of the spectrum in Section 4 has revealed the existence of a single propagating matter field, a massless Dirac fermion in $AdS_2 \times S^1$. On the other

[5] AdS/CFT correspondence for $AdS_n \times S^1$ spaces has been previously encountered in [34] in the connection with higher-dimensional noncritical superstrings.

hand, the spectrum of low-energy excitations of the Einstein-Maxwell theory would contain a propagating photon, an excitation of which there is no evidence on the noncritical M-theory side. Hence, we must look for another effective theory that has $AdS_2 \times S^1$ as a solution, but with fewer propagating degrees of freedom.

It turns out that the correct starting point is the Chern-Simons theory with $SO(3,2)$ gauge symmetry, *i.e.*, conformal gravity in $2+1$ dimensions.

5.1 $AdS_2 \times S^1$ in $SO(3,2)$ Chern-Simons Gravity

In [35], Horne and Witten extend the work of [36] and show that conformal gravity in $2+1$ dimensions can be rewritten as an $SO(3,2)$ Chern-Simons gauge theory of the conformal group. We will now show that our $AdS_2 \times S^1$ spacetime is a solution of this theory.

Recall that conformal gravity on a $2+1$-dimensional manifold $\mathcal{M}$ (with coordinates x^μ) can be described by

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{5.9}$$

where A is an $SO(3,2)$ Lie algebra-valued one-form gauge field.[6] We write A in components as[7]

$$A_\mu = e_\mu{}^a \mathcal{P}_a + \omega_\mu{}^a \mathcal{J}_a + \zeta_\mu{}^a \mathcal{K}_a + \phi_\mu \mathcal{D}. \tag{5.10}$$

Here a runs over $0,1,2$ for each of $\mathcal{P}_a$, $\mathcal{J}_a$, and $\mathcal{K}_a$. In the interpretation of this theory as conformal gravity, $e_\mu{}^a$ are the components of the vielbein, while ω^a is the corresponding spin connection, Hodge-dualized in its internal Lorentz indices using the ϵ^{abc} tensor associated with $\eta_{ab} = \mathrm{diag}(-1,1,1)$. The commutation relations are

$$\begin{aligned}
&[\mathcal{J}_a, \mathcal{J}_b] = \epsilon_{abc}\mathcal{J}^c, \qquad &&[\mathcal{P}_a, \mathcal{P}_b] = [\mathcal{K}_a, \mathcal{K}_b] = [\mathcal{J}_a, \mathcal{D}] = 0,\\
&[\mathcal{P}_a, \mathcal{K}_b] = \eta_{ab}\mathcal{D} - \epsilon_{abc}\mathcal{J}^c, && \\
&[\mathcal{P}_a, \mathcal{J}_b] = \epsilon_{abc}\mathcal{P}^c, &&[\mathcal{K}_a, \mathcal{J}_b] = \epsilon_{abc}\mathcal{K}^c,\\
&[\mathcal{P}_a, \mathcal{D}] = \mathcal{P}_a, &&[\mathcal{K}_a, \mathcal{D}] = -\mathcal{K}_a.
\end{aligned} \tag{5.11}$$

The equations of motion of (5.10) are just the flatness conditions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{5.12}$$

An interesting class of solutions to (5.12) can be constructed as follows. First we assume that our vielbein is invertible, and then we pick a gauge in which $\phi_\mu = 0$. In such

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[6] Here "Tr" is the trace defined via the unique quadratic invariant on the simple group $SO(3,2)$. The coupling k is quantized because $\pi_3(SO(3,2)) = \pi_3(SO(3))$ is nontrivial. The precise quantization condition for k will depend on the exact choice of the gauge group, *i.e.*, on whether we choose the $SO(3,2)$ group itself or one of its covers. We shall briefly return to this point in Section 5.4 below.

[7] We use essentially the same notation as [35], with the only exception that we refer to the gauge fields associated to the special conformal transformations $\mathcal{K}_a$ as ζ^a_μ, while [35] used λ^a_μ.

circumstances, the equations of motion (5.12) reduce to

$$de^a - e_b \wedge \omega^{ab} = 0 \tag{5.13}$$

$$d\zeta^a - \zeta_b \wedge \omega^{ab} = 0 \tag{5.14}$$

$$e^a \wedge \zeta_a = 0 \tag{5.15}$$

$$-d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b + e^a \wedge \zeta^b - e^b \wedge \zeta^a = 0. \tag{5.16}$$

Before we discuss the embedding of our $AdS_2 \times S^1$ spacetime into this framework, let us discuss how AdS_3, dS_3, and the Minkowski space can be interpreted as solutions of this theory. First, we note that Eqn. (5.13) is simply the torsion-free condition. Setting $\zeta_\mu = \zeta e_\mu$ where ζ is a constant, we find that Eqn. (5.14) reduces to the torsion-free condition as well. Also, Eqn. (5.15) is trivially satisfied. Using $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, Eqn. (5.16) becomes

$$R^a{}_b - 2\zeta e^a \wedge e_b = 0. \tag{5.17}$$

Consequently, when (5.17) is satisfied, the Einstein tensor reduces to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -2\zeta g_{\mu\nu}. \tag{5.18}$$

In this way, the vacuum with cosmological constant $\Lambda = 2\zeta$ is embedded as a solution to conformal Chern-Simons gravity in $2+1$ dimensions. This solution is described by the following gauge field,

$$A_\mu = e_\mu{}^a \left(\mathcal{P}_a + \frac{\Lambda}{2} \mathcal{K}_a \right) + \omega_\mu{}^a \mathcal{J}_a. \tag{5.19}$$

The explanation of the existence of such solutions is very simple. The solution (5.19) only excites gauge field components of a certain subalgebra of $SO(3,2)$. When $\Lambda < 0$, the nonzero components in (5.19) belong to $SO(2,2) \subset SO(3,2)$, for $\Lambda > 0$ they span $SO(3,1) \subset SO(3,2)$, and if $\Lambda = 0$ we get $ISO(2,1) \subset SO(3,2)$. In all cases, the flatness of the $SO(3,2)$ connection reduces to the flatness in the corresponding subalgebra.

Now, let us return to consider the embedding of our $AdS_2 \times S^1$ spacetime. Our vielbein and spin connection components are

$$\begin{aligned} e^0 &= \mathcal{R} \cosh \rho dt \\ e^1 &= \mathcal{R}\, d\rho \\ e^2 &= 2\mathcal{R}\, dy \\ \omega^2 &= -\sinh \rho\, dt. \end{aligned} \tag{5.20}$$

For the ζ^a, we choose

$$\zeta^0 = -\frac{1}{2\mathcal{R}^2} e^0, \qquad \zeta^1 = -\frac{1}{2\mathcal{R}^2} e^1, \qquad \zeta^2 = \frac{1}{2\mathcal{R}^2} e^2. \tag{5.21}$$

Thus, the $SO(3,2)$ gauge field that desribes $AdS_2 \times S^1$ can finally be written as

$$A_\mu = e_\mu{}^0 \left(\mathcal{P}_0 - \frac{1}{2\mathcal{R}^2} \mathcal{K}_0 \right) + e_\mu{}^1 \left(\mathcal{P}_1 - \frac{1}{2\mathcal{R}^2} \mathcal{K}_1 \right) + e_\mu{}^2 \left(\mathcal{P}_2 + \frac{1}{2\mathcal{R}^2} \mathcal{K}_2 \right) + \omega_\mu{}^2 J_2. \tag{5.22}$$

Since we have again chosen $\phi_\mu = 0$, the flatness conditions $F_{\mu\nu} = 0$ reduce to Eqns. (5.13) – (5.16). Simple algebra will show that these equations are satisfied. Thus, the background A_μ of (5.22) is a solution of $SO(3,2)$ Chern-Simons gauge theory, and consequently of conformal gravity in $2+1$ dimensions.

The explanation for the existence of such a solution is again simple. (5.22) corresponds to the embedding of $SL(2,\mathbf{R})\times U(1)$, the isometry of $AdS_2\times S^1$, into $SO(3,2)$. In particular, the $U(1)$ factor is generated by

$$\mathcal{P}_2 + \frac{1}{2\mathcal{R}^2}\mathcal{K}_2, \tag{5.23}$$

which indeed commutes with the three generators of $SL(2,\mathbf{R})$ also excited by the background gauge field (5.22). In more generality, for any embedding of a direct product $G_1 \times G_2$ into the Chern-Simons gauge group G, the flatness conditions for G factorize into the flatness in the G_1 and G_2 factors if the gauge field belongs to $G_1 \times G_2$.

The $SO(3,2)$ Chern-Simons gauge theory has the following good features, which make it a suitable starting point for constructing our effective action: (i) $AdS_2 \times S^1$ is a solution of this Chern-Simons theory, and the gauge group $SO(3,2)$ naturally coincides with the group of all the quadratic charges of the symmetry algebra $\mathcal{W}$. (ii) The isometry $SL(2,\mathbf{R})\times U(1)$ of $AdS_2\times S^1$ is embedded in $SO(3,2)$ in precisely the manner expected from the Type 0A string theory and noncritical M-theory arguments of Section 4. (iii) Unlike the Einstein-Maxwell system considered at the beginning of Section 5, the $SO(3,2)$ Chern-Simons gravity has no propagating bosonic degree of freedom, just as noncritical M-theory in the conformal limit.

5.2 Extension to Higher-Spin Gauge Theory

Despite its good properties, the $SO(3,2)$ Chern-Simons gauge theory cannot be the whole story, for two reasons: (1) $SO(3,2)$ is only a subalgebra of the infinite symmetry algebra of noncritical M-theory, and (2) we have seen evidence that in the conformal limit, the matter content of the noncritical M-theory vacuum is that of a propagating massless Dirac fermion. In the conventional approach to Chern-Simons gravity, it is unknown how to couple second-quantized matter to the gravity sector as described by the gauge connection. We shall now show that once the correct infinite symmetries of Problem (1) are properly taken into account, Problem (2) will acquire a natural solution as well.

In order to resolve Problem (1), we are in need of a Chern-Simons gravity theory based on an infinite-dimensional extension of $SO(3,2)$. Remarkably, this theory is already available. It is the bosonic version of a supersymmetric Chern-Simons theory of an infinite hierarchy of conformal higher-spin fields constructed in [26].

Higher-spin gauge theories have a rich history, going back to the original work of Fradkin and Vasiliev [37] (see [27] for a review). In $2+1$ dimensions, higher-spin gauge theories as Chern-Simons gauge theories were first written down by Blencowe [38]. The higher-spin version of conformal Chern-Simons gravity in $2+1$ dimensions appeared first in the work of Pope and Townsend [39], and Fradkin and Linetsky [40]. We shall follow most closely the detailed construction by Shaynkman and Vasiliev [26]; see also [41].

The first thing we need is a convenient parametrization of the higher-spin symmetry algebra. In order to construct this algebra, Shaynkman and Vasiliev [26] first define operators[8] $\hat{a}_\alpha$ and $\hat{a}^{+\alpha}$, where the index $\alpha = 1, 2$ parametrizes the spinor representation of the Lorentz group in $2+1$ dimensions, and subject them to the commutation relations

$$[\hat{a}_\alpha, \hat{a}^{+\beta}] = \delta_\alpha{}^\beta, \qquad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^{+\alpha}, \hat{a}^{+\beta}] = 0. \tag{5.24}$$

Rather than use an operator realization, it is convenient to use techniques of noncommutative geometry on $\mathbf{R}^4$ parametrized by commuting coordinates a_α and $a^{+\alpha}$ and endowed with the star product:

$$(f \star g)(a, a^+) = f(a, a^+) \exp\left\{ \frac{1}{2} \left(\frac{\overleftarrow{\partial}}{\partial a_\alpha} \frac{\overrightarrow{\partial}}{\partial a^{+\alpha}} - \frac{\overleftarrow{\partial}}{\partial a^{+\alpha}} \frac{\overrightarrow{\partial}}{\partial a_\alpha} \right) \right\} g(a, a^+). \tag{5.25}$$

This definition results in the following $\star$ commutators:

$$[a_\alpha, a^{+\beta}]_\star = a_\alpha \star a^{+\beta} - a^{+\beta} \star a_\alpha = \delta_\alpha^\beta, \qquad [a_\alpha, a_\beta]_\star = [a^{+\alpha}, a^{+\beta}]_\star = 0. \tag{5.26}$$

The associative algebra of the $\star$-product defined by generators consisting of all powers of a and a^+ is called A_2 in [26]. On A_2, one can define the structure of a Lie superalgebra, by first assigning even (or odd) grading to the generators given by even-degree (or odd-degree) monomials in a_α and $a^{+\alpha}$, and then defining the (anti)commutation relations via the $\star$-product (anti)commutator. In the rest of the paper, we shall call this higher-spin superalgebra $\widetilde{\mathcal{W}}$.

The subalgebra of quadratic charges in $\widetilde{\mathcal{W}}$ is $Sp(4, \mathbf{R})$, which is isomorphic to the $SO(3, 2)$ algebra. It has a $\star$-product realization by

$$\begin{aligned} \mathcal{P}_a &= \frac{1}{2} \sigma_a{}^{\alpha\beta} a_\alpha a_\beta, & \mathcal{J}_a &= \frac{1}{2} \sigma_a{}^\alpha{}_\beta a_\alpha a^{+\beta}, \\ \mathcal{K}_a &= -\frac{1}{4} \sigma_{a\alpha\beta} a^{+\alpha} a^{+\beta}, & \mathcal{D} &= \frac{1}{4} \left(a_\alpha a^{+\alpha} + a^{+\alpha} a_\alpha \right), \end{aligned}$$

where $\sigma_a{}^{\alpha\beta}$ are symmetric matrices given by

$$\sigma_0{}^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1{}^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2{}^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.27}$$

and the spinor indices are raised and lowered as $c^\alpha = \epsilon^{\alpha\beta} c_\beta$, $c_\beta = \epsilon_{\alpha\beta} c^\alpha$, with $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = \epsilon^{12} = 1$.

Clearly, this construction of $\widetilde{\mathcal{W}}$ is very closely related to the construction of the $\mathcal{W}$ symmetry algebra in noncritical M-theory which we reviewed briefly in Section 4. More precisely, the infinite subalgebra $\mathcal{W}_0$ of all even-degree charges in $\mathcal{W}$ coincides with the maximal bosonic subalgebra in $\widetilde{\mathcal{W}}$. Our generators a_i, b_j are related to $a_\alpha, a^{+\beta}$ of [26] by a linear transformation that preserves the commutation relations, *i.e.*, by an $Sp(4, \mathbf{R})$ symplectomorphism of the phase space $\mathcal{T}$ of the Fermi liquid.

[8] Despite appearances, $\hat{a}$ and $\hat{a}^+$ are not Hermitian conjugates of each other [26].

The conformal higher-spin theory that will be relevant for the conformal limit of noncritical M-theory is based on the bosonic higher-spin Lie algebra $\mathcal{W}_0$. This algebra contains generators that correspond to all integer spins; there is no evidence, in the noncritical M-theory vacuum that we consider here, of the half-integer fermionic spins. We shall comment on a possible supersymmetric extension in Section 5.4.

Thus, we consider the bosonic higher-spin theory, described again by the Chern-Simons action,

$$S_{HCS} = \frac{k}{4\pi}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right). \tag{5.28}$$

with the gauge field one-form $\mathcal{A}$ now taking values in the infinite-dimensional higher-spin Lie algebra $\mathcal{W}_0$ of even-degree bosonic charges.[9] In general, $\mathcal{A}$ can be expanded in components,

$$\mathcal{A}(x|a,a^+) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{2\ell}\frac{1}{m!(2\ell-m)!}\mathcal{A}_{\alpha_1\ldots\alpha_m}{}^{\alpha_{m+1}\ldots\alpha_{2\ell}}(x)a^{+\alpha_1}\ldots a^{+\alpha_m}a_{\alpha_{m+1}}\ldots a_{\alpha_{2\ell}}. \tag{5.29}$$

Each component $\mathcal{A}_{\alpha_1\ldots\alpha_m}{}^{\alpha_{m+1}\ldots\alpha_{2\ell}}(x)$ is a one-form on $\mathcal{M}$.

The equations of motion of (5.28) yield the flatness condition,

$$d\mathcal{A} + \mathcal{A}\star\wedge\mathcal{A} = 0. \tag{5.30}$$

The gauge field A of (5.22), which describes the $AdS_2\times S^1$ solution of $SO(3,2)$ Chern-Simons gravity, can be embedded into the higher-spin theory by setting the components of $\mathcal{A}$ in the $SO(3,2)$ subalgebra equal to A, and all others to zero. In the $\star$-product language, our $AdS_2\times S^1$ background is described by

$$\mathcal{A} = \frac{1}{2}\left[e^a_\mu\sigma_a^{\alpha\beta}\left(a_\alpha a_\beta + \frac{1}{4\mathcal{R}^2}a^+_\alpha a^+_\beta\right) - e^2_\mu\sigma_2^{\alpha\beta}(\frac{1}{2\mathcal{R}^2}a^+_\alpha a^+_\beta) + \omega^2_\mu{\sigma_2}^\alpha{}_\beta(a_\alpha a^{+\beta})\right]dx^\mu. \tag{5.31}$$

Thus, $AdS_2\times S^1$ is a solution of (5.30).

The theory (5.28) is invariant under the full set of higher spin conformal transformations given by

$$\delta\mathcal{A} = d\varepsilon + [\mathcal{A},\varepsilon]_\star \tag{5.32}$$

with ε a scalar function with values in the infinite-dimensional higher-spin Lie algebra. For any given solution $\mathcal{B}$ of the equations of motion (5.30), we are interested in its global symmetries, *i.e.*, gauge transformations ε that preserve $\mathcal{B}$:

$$d\varepsilon + [\mathcal{B},\varepsilon]_\star = 0. \tag{5.33}$$

[9] The "Tr" in (5.28) is defined as the bosonic restriction to $\mathcal{W}_0$ of the natural supertrace defined on $\widetilde{\mathcal{W}}$ (see [42] and also Section 3 of [43]). Conversely, one could try to keep the odd-degree generators as bosonic symmetries, *i.e.*, replace their anticommutation relations with commutation relations, as defined again via the $\star$-product algebra. However, the hypothetical gauge theory of $\mathcal{W}$ would contain bosonic gauge field components of half-integer spins, leading to many conceptual difficulties; consequently, we will not consider this option in this paper.

On a topologically trivial spacetime $\mathcal{M}$, one global symmetry can be constructed for each element of the symmetry algebra, as follows. Consider a solution $\mathcal{B}$ of (5.30). Since $\mathcal{B}$ is flat, it can be written as

$$\mathcal{B} = g^{-1}(x) \star dg(x) \tag{5.34}$$

for some function $g(x)$ with values in the Lie group $\mathcal{W}_0$. Then, for any fixed, constant element ξ from the Lie algebra of $\mathcal{W}_0$,

$$\varepsilon = g^{-1}(x) \star \xi \star g(x) \tag{5.35}$$

is a symmetry of the background $\mathcal{B}$, *i.e.*, a solution of (5.33).

In the case of topologically nontrivial $\mathcal{M}$, there could be obstructions against defining (5.35) globally over $\mathcal{M}$. Consequently, the actual symmetry can be reduced to a subalgebra. As an example, consider for simplicity the Minkowski space, described as a solution of Chern-Simons theory in (5.19), and focus on the quadratic charges belonging to $SO(3,2)$. Clearly, the spacetime-independent transformations

$$\varepsilon(\mathcal{P}) = \xi^a \mathcal{P}_a \tag{5.36}$$

solve (5.33) for any constant ξ^a; they represent the rigid translations of the Minkowski space. Less trivially, one can show that

$$\begin{aligned}
\varepsilon(\mathcal{J}) &= \xi^a \left(\mathcal{J}_a + e_\mu{}^b x^\mu \epsilon_{abc} \mathcal{P}^c \right), \\
\varepsilon(\mathcal{D}) &= \xi \left(\mathcal{D} - e_\mu{}^a x^\mu \mathcal{P}_a \right), \\
\varepsilon(\mathcal{K}) &= \xi^a \left(\mathcal{K}_a - e_\mu{}^b x^\mu \left(\epsilon_{abc} \mathcal{J}^c + \eta_{ab} \mathcal{D} \right) + \left(e_{\mu a} e_\nu{}^b x^\mu x^\nu - \frac{1}{2} \delta_a^b x_\mu x^\mu \right) \mathcal{P}_b \right)
\end{aligned} \tag{5.37}$$

are also solutions of (5.33), and thus represent global symmetries of Minkowski space viewed as a solution of $SO(3,2)$ Chern-Simons gauge theory.[10] Thus, on $\mathcal{M} = \mathbf{R}^3$, we see that the entire $SO(3,2)$ is a symmetry. However, if we compactify (say) x^2 on S^1, and require the global symmetries to be well-defined on S^1, only the linear combinations of (5.37) that are independent of the x^2 coordinate survive the compactification. The global symmetry group of the flat $\mathbf{R}^2 \times S^1$ background is reduced to $ISO(1,1) \times U(1)$.

On $AdS_2 \times S^1$ we are in a very similar situation. Consider again the algebra of quadratic charges $SO(3,2)$. If we sent the radius of S^1 to infinity, the global symmetry would correspond to the entire $SO(3,2)$. The main difference compared to the Minkowski example is that on $AdS_2 \times \mathbf{R}$, the solutions to (5.33) are in fact periodic along the coordinate x^2 on $\mathbf{R}$ with a fixed periodicity, set by the radius $\mathcal{R}$ of AdS_2. Which symmetries survive on $AdS_2 \times S^1$ is thus determined by the radius of S^1 in units of $\mathcal{R}$. For a generic radius of S^1, $\mathcal{W}_0$ is broken to the subalgebra of global charges that are independent of x^2.

The quadratic charges that are independent of x^2 form the $SL(2,\mathbf{R}) \times U(1)$ subalgebra of $SO(3,2)$. It turns out that the quadratic charges that do depend on x^2 are periodic on S^1 whose radius is $\mathcal{R}$ (or any integer multiple thereof). In order to check this, we can go back

[10] Note that only a smaller algebra corresponds to isometries of the background.

to the coordinates (t,ρ,y) of (5.1), and find six solutions to (5.33) that depend on y, such as for example

$$\begin{aligned}\varepsilon = \mathcal{R}\sinh\rho\,\cos(2y)\left(\mathcal{P}_2-\frac{1}{2\mathcal{R}^2}\mathcal{K}_2\right)-\mathcal{R}\cosh\rho\,\sin(2y)\left(\mathcal{P}_1+\frac{1}{2\mathcal{R}^2}\mathcal{K}_1\right)\\ -\cosh\rho\,\cos(2y)\,\mathcal{J}_0+\sinh\rho\,\sin(2y)\,\mathcal{D},\end{aligned} \tag{5.38}$$

Noting that y is periodic with periodicity 2π when the S^1 radius is $2\mathcal{R}$, we obtain the stated result. Thus, we see that if the radius of S^1 is an integer multiple of $\mathcal{R}$, all $SO(3,2)$ symmetries will be unbroken. Since in noncritical M-theory the radius of the S^1 factor is twice the radius of AdS_2, the entire $SO(3,2)$ symmetry survives the compactification, in accord with our expectations from the Fermi liquid side discussed in Section 4.1.

In fact, there is an interesting refinement of the story. The same $AdS_2\times S^1$ background would also be a solution of the supersymmetric extension of the theory, which would result from keeping both even- and odd-degree charges in the higher-spin algebra $\widetilde{\mathcal{W}}$. One can again ask what would be the periodicity of the odd-degree charges along x^2. It is intriguing that the odd-degree charges, and in particular the linear charges that correspond to the supercharges in the $OSp(1|4)$ supersymmetric extension of $SO(3,2)$, are all antiperiodic on S^1 of radius $\mathcal{R}$. This implies that the supercharges survive as global symmetries of $AdS_2\times S^1$ if the radius of S^1 is an *even* multiple of the AdS_2 radius. We see that the radius of S^1 in the conformal limit of noncritical M-theory is precisely given by the minimal value for which the $AdS_2\times S^1$ background would be supersymmetric, if embedded into the supersymmetrized version of the higher-spin theory. This suggests that our noncritical M-theory may be a simple $\mathbf{Z}_2$ orbifold of a supersymmetric theory, on which we comment further in Section 5.4.

5.3 Coupling the Fermion

Our analysis of the spectrum in Section 4 revealed that noncritical M-theory in the high-energy limit is not purely topological; the vacuum has at least one type of a propagating excitation, described by a second-quantized massless Dirac fermion field on the $AdS_2\times S^1$ background. In the full effective action, this matter sector should couple consistently to the topological Chern-Simons sector of the theory. The existence of such a coupling will represent another check of the proposed picture.

The standard lore of topological gravity is that the system cannot be coupled to propagating matter. This is sometimes avoided by representing matter in the first-quantized framework, essentially via a collection of Wilson lines coupled to the topological gauge field. The difficulty essentially stems from the fact that in order to write down the equations of motion for a second-quantized propagating field, we must invert the vielbein; however, this is an unnatural procedure in Chern-Simons theory where we interpret the vielbein as a part of the Chern-Simons gauge field.

Remarkably, this standard lore may no longer be valid once the gauge group becomes infinite dimensional. Vasiliev *et al.* have shown [26] (see [27] for a review) that propagating matter fields of low spins (in particular, a scalar and a spinor) can indeed be coupled the Chern-Simons higher spin gravity in $2+1$ dimensions.

Again following [26], we first introduce the Fock vacuum $|0\rangle$ defined to satisfy $a_\alpha|0\rangle = 0$. In the $\star$-product realization, this vacuum can be described by a projector

$$|0\rangle\langle 0| = 4\exp\left(-2a_\alpha a^{+\alpha}\right), \tag{5.39}$$

which satisfies

$$a_\alpha \star |0\rangle\langle 0| = |0\rangle\langle 0| \star a^{+\alpha} = 0, \qquad |0\rangle\langle 0| \star |0\rangle\langle 0| = |0\rangle\langle 0|. \tag{5.40}$$

In other words it is properly normalized, and annihilated on the left by a_α. The full set of Fock states on this vacuum can be created by action on the left with $a^{+\alpha}$. The matter fields on $\mathcal{M}$ will be represented by a section of the Fock bundle over $\mathcal{M}$,

$$|\Phi(x|a^+)\rangle = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} c_{\alpha_1\dots\alpha_\ell}(x) a^{+\alpha_1}\dots a^{+\alpha_\ell} \star |0\rangle\langle 0|. \tag{5.41}$$

For future reference, it will be natural to split $|\Phi\rangle$ into an even and odd part,

$$\begin{aligned} |\Phi_0\rangle &= \sum_{n=0}^{\infty} \frac{1}{2n!} c_{\alpha_1\dots\alpha_{2n}}(x) a^{+\alpha_1}\dots a^{+\alpha_{2n}} \star |0\rangle\langle 0|, \\ |\Phi_1\rangle &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{\alpha_1\dots\alpha_{2n+1}}(x) a^{+\alpha_1}\dots a^{+\alpha_{2n+1}} \star |0\rangle\langle 0|. \end{aligned} \tag{5.42}$$

Each component $c_{\alpha_1\dots\alpha_\ell}(x)$ is symmetric in all its indices. Moreover, it is natural to consider the component fields $c_{\alpha_1\dots\alpha_\ell}(x)$ as bosonic if ℓ is even and fermionic if ℓ is odd.

The dynamics of the matter fields in a Chern-Simons background $\mathcal{A}$ is encoded in the equations of motion,

$$d|\Phi\rangle + \mathcal{A} \star |\Phi\rangle = 0. \tag{5.43}$$

In terms of the components $c_{\alpha_1\dots\alpha_\ell}$, these equations become

$$\begin{aligned} 2\partial_\mu c_{\alpha_1\dots\alpha_\ell} &= c_{\alpha_1\dots\alpha_\ell\beta_1\beta_2} e_\mu{}^{\beta_1\beta_2} - \frac{\ell(\ell-1)}{4\mathcal{R}^2} c_{\alpha_1\dots\alpha_{\ell-2}} e_{\mu\alpha_{\ell-1}\alpha_\ell} \\ &\quad + \frac{\ell(\ell-1)}{2\mathcal{R}^2} c_{\alpha_1\dots\alpha_{\ell-2}} \sigma_{2\,\alpha_{\ell-1}\alpha_\ell} e_\mu{}^2 - (\ell+1)\omega_\mu{}^2 \sigma_2{}^\beta{}_{\alpha_1} c_{\alpha_2\dots\alpha_\ell\beta}, \end{aligned} \tag{5.44}$$

where $e_\mu{}^{\alpha\beta} = e^a_\mu \sigma_a{}^{\alpha\beta}$, and the symmetrization over $\alpha_1\dots\alpha_\ell$ is kept implicit on the right-hand side of (5.44). Assuming that the vielbein is invertible, the infinite chain of equations (5.44) can be used to solve algebraically for all the higher components $c_{\alpha_1\dots\alpha_\ell}$ in terms of only two independend fields, given by the two lowest components c and c_α. The full equation of motion (5.43) thus reduces to two dynamical equations for these remaining fields,

$$\left(g^{\mu\nu} D_\mu D_\nu - \frac{1}{4\mathcal{R}^2}\right) c(x) = 0, \tag{5.45}$$

$$e^\mu_a \sigma^{a\beta}{}_\alpha \left(\partial_\mu c_\beta + \omega_\mu{}^2 \sigma_2{}^\gamma{}_\beta c_\gamma\right) = 0. \tag{5.46}$$

Here $g^{\mu\nu}$ is the inverse metric and D_μ is the covariant derivative, built from the spin connection and the vielbein.

Some comments are now in order:

- Eqns. (5.45) and (5.46) are the Klein-Gordon and the Dirac equation for a massless spinor and scalar on $AdS_2 \times S^1$. All the components $c_{\alpha_1\dots\alpha_\ell}$ with $\ell \geq 2$ are formed from derivatives of c or c_α, and thus do not constitute separate degrees of freedom themselves.

- The equations of motion (5.44) decouple components $c_{\alpha_1\dots\alpha_\ell}$ with ℓ even from those with ℓ odd. Thus, in our theory with $\mathcal{W}_0$ gauge symmetry, the matter multiplet $|\Phi\rangle$ of (5.41) is reducible, and can be split into two irreducible components given by $|\Phi_0\rangle$ and $|\Phi_1\rangle$ of (5.42). From the perspective of the subalgebra of quadratic charges $SO(3,2) \sim Sp(4,\mathbf{R})$, $|\Phi_0\rangle$ and $|\Phi_1\rangle$ correspond essentially to the two irreducible metaplectic representations of $Sp(4,\mathbf{R})$, sometimes called Di and Rac in the representation theory of this algebra.

In order to match our expected spectrum of noncritical M-theory in the conformal limit, as given in (3.4), we keep $|\Phi_1\rangle$ which contains the propagating massless Dirac fermion, but throw away $|\Phi_0\rangle$ which would contain the scalar.

Second-quantized propagating matter can thus be coupled to Chern-Simons theory at the level of the equations of motion. All matter interactions are mediated by the coupling to topological Chern-Simons theory, which itself does not propagate any physical degrees of freedom. It is unclear, however, how to formulate an action principle for this system of equations of motion. Interestingly, Vasiliev *et al.* [43] have suggested that such an action principle would have to be formulated not on $\mathcal{M}$ but on a space that also includes a_α.

5.4 Relation to a Supersymmetric Higher-Spin Theory

It is remarkable that the geometric properties of $AdS_2 \times S^1$ that are required to match the Fermi liquid spectrum in the conformal limit, are precisely such that the radius of S^1 equals the minimum possible value compatible with unbroken supersymmetry of the solution. This suggests that the effective theory that we propose for the description of the conformal limit of the noncritical M-theory vacuum is a simple $\mathbf{Z}_2$ orbifold of a supersymmetric theory, with our $AdS_2 \times S^1$ as a maximally supersymmetric solution.

Such a supersymmetic extension of the effective theory of higher-spin Chern-Simons plus propagating massless matter can be easily constructed. In fact, it has already been written down by Shaynkman and Vasiliev in [26].[11] The gauge symmetry of this supersymmetric theory is given by the higher-spin conformal superalgebra $\widetilde{\mathcal{W}}$. The Chern-Simons gauge field $\widetilde{\mathcal{A}}$ is now in the adjoint of $\widetilde{\mathcal{W}}$. It can be decomposed into components,

$$\widetilde{\mathcal{A}}(x|a,a^+) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{1}{m!(\ell-m)!}\widetilde{\mathcal{A}}_{\alpha_1\dots\alpha_m}{}^{\alpha_{m+1}\dots\alpha_\ell}(x)a^{+\alpha_1}\dots a^{+\alpha_m}a_{\alpha_{m+1}}\dots a_{\alpha_\ell}. \tag{5.47}$$

[11] In [26], the focus is on $\mathcal{N}=2$ supersymmetric theory; the $\mathcal{N}=1$ version can be obtained by setting their $\hat{k}$ to zero.

The component one-forms $\widetilde{\mathcal{A}}_{\alpha_1\ldots\alpha_m}{}^{\alpha_{m+1}\ldots\alpha_\ell}(x)$ are bosons if ℓ is even, and fermions for ℓ odd. The lowest fermionic component of the gauge field corresponds to terms linear in a_α and $a^{+\alpha}$, and they describe the massless spin-3/2 gravitino of $\mathcal{N}=1$ conformal supergravity in $2+1$ dimensions. The full theory is then a higher-spin extension of $OSp(1|4,\mathbf{R})$ Chern-Simons supergravity.

The Chern-Simons gauge sector is coupled to the massless matter supermultiplet, described by $|\Phi\rangle = |\Phi_0\rangle + |\Phi_1\rangle$ of Section 5.3. As we have seen there, in the $AdS_2\times S^1$ background $|\Phi_0\rangle$ gives rise to a propagating boson and $|\Phi_1\rangle$ describes a propagating fermion. Due to the periodicity properties of the generators of global symmetries, $AdS_2\times S^1$ is a supersymmetric solution of this theory if the radius of S^1 is an even multiple of the AdS_2 radius.

On this $\mathcal{N}=1$ supersymmetric theory, we can define the action of the orbifold group $\mathbf{Z}_2=\{1,\Omega\}$ via

$$\Omega: \quad x^\mu\to x^\mu, \quad a_\alpha\to -a_\alpha, \quad a^{+\alpha}\to -a^{+\alpha}, \tag{5.48}$$

and extend it to the action on the fields by

$$\Omega: \quad \mathcal{A}(x|a,a^+)\to\mathcal{A}(x|-a,-a^+), \tag{5.49}$$

$$|\Phi(x|a^+)\rangle \to -|\Phi(x|-a^+)\rangle. \tag{5.50}$$

The orbifold projection by Ω projects out the odd-degree part of the gauge field $\widetilde{\mathcal{A}}$ and the even-degree part $|\Phi_0\rangle$ of the matter multiplet. Thus, our effective theory describing the conformal limit of the noncritical M-theory vacuum is an orbifold of the supersymmetric theory under this $\mathbf{Z}_2$ action.

This fact can perhaps be seen as another check of our proposal, for the following reason. Just as in the critical spacetime dimension, two-dimensional Type 0A and 0B string theories are believed to be related to supersymmetric Type II cousins [44–47] by an orbifold construction. Naturally, such supersymmetric two-dimensional theories could also be tied together into a supersymmetric version of noncritical M-theory, much like Type 0A and 0B backgrounds were in [9]. If such a supersymmetric extension of noncritical M-theory exists, one would again expect it to be related to the nonsupersymmetric version by an orbifold. It is intriguing that in the conformal limit, such a simple and natural extension does exist, at least at the level of the effective spacetime description, and that $AdS_2\times S^1$ extends naturally to a supersymmetric solution of it.[12] It would certainly be interesting to investigate these issues further.

We conclude this Section with a few comments on the precise choice of the gauge group. In particular, the quadratic charges in $\mathcal{W}_0$ generate the Lie algebra of $SO(3,2)$, and the question is which of its covers should be chosen. Conformal gravity in $2+1$ dimensions arises naturally as the gauge theory of $SO(3,2)$. However, since our effective theory couples the

[12] A $\mathbf{Z}_2$-twisted version of the ground state of noncritical M-theory was studied in Section 9.4 of [9], where it was shown that its vacuum energy vanishes to all orders in the expansion in the coupling constant $1/\mu$. Whether this feature is explained by some form of hidden supersymmetry in this state is not known.

gravity sector to a propagating fermion, the natural gauge group should be at least as large as $Spin(3,2)$, the double-cover of $SO(3,2)$. The $Spin(3,2)$ group is isomorphic to $Sp(4,\mathbf{R})$. Since $Sp(4,\mathbf{R})$ also naturally arises on the Fermi-liquid side of noncritical M-theory, it might be tempting to propose it as the correct gauge group. However, $Sp(4,\mathbf{R})$ itself has a nontrivial topological structure, with $\pi_1(Sp(4,\mathbf{R}))=\mathbf{Z}$. As a result, it has a unique connected double cover, known as the "metaplectic group" $Mp(4,\mathbf{R})$. The metaplectic group is *not* a matrix group; its smallest faithful representation, known as the "metaplectic representation," is infinite-dimensional. In fact, this metaplectic representation is equivalent to the Fock space representation of the $\hat{a}_\alpha$ and $\hat{a}^{+\alpha}$ operator algebra! Thus, the metaplectic representation plays an essential role in the construction of our propagating matter multiplet. Since it is only a projective representation of $Sp(4,\mathbf{R})$, it is natural to expect that the correct gauge group is not $Sp(4,\mathbf{R})$ but its double cover $Mp(4,\mathbf{R})$. Perhaps, in noncritical M-theory in $2+1$ dimensions, "M" stands for "metaplectic"!

6. Conclusions

In this paper, we have presented evidence suggesting that the ground state of noncritical M-theory, in the conformal limit, describes $AdS_2\times S^1$ spacetime. The symmetries and spectrum of propagating modes in this limit are compatible with an effective theory given by the infinite higher-spin extension Chern-Simons gravity in $2+1$ dimensions, coupled to a propagating massless Dirac fermion. This correspondence represents the simplest M-theory analog of the duality relation between the Liouville spacetime dimension and the eigenvalue coordinate as known from two-dimensional string theory.

This correspondence leads to a nice matching of the "extra dimension" of noncritical M-theory in the $AdS_2\times S^1$ spacetime and in the Fermi liquid. In the Fermi liquid picture, the "extra dimension" corresponds to the orbits of the $U(1)$ rotations of the rigid plane populated by the eigenvalues. On the spacetime side, this $U(1)$ group becomes the group of translations along the S^1 factor of the $AdS_2\times S^1$ background, and the "extra dimension" acquires its traditional role as in critical string/M-theory. In particular, the radius of the S^1 as measured by the spacetime metric is constant everywhere along AdS_2. The fact that such a simple and intuitive picture emerges on the effective spacetime side lends further support to the original proposal of [9] that the extra dimension of M-theory indeed corresponds to the angular coordinate on the plane populated by the Fermi liquid.

The conformal limit of the theory involves sending $\alpha'\to\infty$, and one can think of it as a certain high energy limit of the theory. We have argued that in this limit, the effective spacetime theory is given by a higher-spin Chern-Simons gravity, coupled to a propagating fermionic degree of freedom. It is intriguing that such a connection to higher-spin gauge theories emerges in the high-energy limit of noncritical M-theory. Indeed, the possibility of a close relation between higher-spin theories and the high-energy limit of critical string theory has been suspected for a long time (see, *e.g.*, [27, 48–56] and references therein). We believe that the exactly solvable setting of noncritical M-theory in $2+1$ dimensions now provides an explicit testing ground for such ideas.[13]

Having found a dual interpretation of the ground-state solution of the Fermi liquid system in terms of a gravitational $AdS_2 \times S^1$ background, one can turn the relation around, and ask the following question: what is, from the perspective of an observer in $AdS_2 \times S^1$, the interpretation of the dual space on which the Fermi liquid resides? Since the double-scaling limit of the Fermi liquid involves taking a semiclassical limit, it is even more natural to look for an interpretation of the full phase space $\mathcal{T}$, as parametrized either by (λ_i, p_i) of Section 2 or equivalently by the conserved charges $(a_\alpha, a^{+\alpha})$ of Section 5. Amusingly, it turns out that this phase space is precisely the twistor space associated with the $2+1$ dimensional conformal group $SO(3,2)$. This relation can be made explicit by defining the twistor transform, which associates to a given element $(a_\alpha, a^{+\beta})$ of the twistor space a null vector p^μ at a spacetime point x^μ, via

$$p^\mu = \sigma^{\mu\,\alpha\beta} a_\alpha a_\beta, \qquad a^{+\alpha} = x^\mu \sigma_\mu{}^{\alpha\beta} a_\beta. \tag{6.1}$$

Perhaps the relation between the physical spacetime and the space of the Fermi liquid is simply related to such a twistor transform? In any case, whether or not the twistor transform proves useful in this context, it is certainly true that the Fermi liquid lives on the twistor space associated with the conformal group $SO(3,2)$ of the $2+1$-dimensional spacetime.[14]

This twistor perspective might shed some new light on another mysterious aspect of noncritical M-theory. As shown in [10], noncritical M-theory at finite temperature is closely related to the A-model topological strings on the resolved conifold, and plays essentially the role expected of topological M-theory. In turn, topological M-theory has been conjecturally described by an effective gauge theory action in seven dimensions [23]. It is puzzling why noncritical M-theory in $2+1$ dimensions should reproduce results expected of this seven-dimensional theory. We do not have answers to this question, but we at least wish to point out one reason why seven dimensions should be relevant for noncritical M-theory. As mentioned in Section 5.3, the equations of motion for the unfolded fermion (5.46) do not follow from any obvious action in three spacetime dimensions. Attempts by Vasiliev *et al.* [43] to resolve this problem suggest that an action might exist, but it would be naturally formulated on a bigger space that also includes the twistor coordinates. In particular, one can view the Chern-Simons gauge field $\mathcal{A}(x|a, a^+)$ of (5.29) as living on the seven-dimensional space $\mathcal{T} \times \mathcal{M}$, the total space of the double fibration $\mathcal{T} \leftarrow \mathcal{T} \times \mathcal{M} \rightarrow \mathcal{M}$ familiar from twistor theory.

[13] It also seems worth pointing out that our effective field theory for noncritical M-theory in $2+1$ dimensions is remarkably similar to the "holographic field theory" proposal of [57]. Indeed, the eleven-dimensional theory of [57] is a Chern-Simons gauge theory based on a higher-dimensional conformal superalgebra $itOSp(1|32) \times OSp(1|32)$, coupled to propagating (fermionic) matter. It thus appears that noncritical M-theory in $2+1$ dimensions might be a baby version of "holographic field theory" in the sense of [57].

[14] This reference may need a bit of explanation. In his after-dinner remarks at a Strings conference at (K)ITP Santa Barbara in the mid-1990's, Joe Polchinski proposed a "Fermi liquid on twistor space" as a half-joking answer to the question of "what is string theory?".

Various interesting open questions still remain. In particular, it would be interesting to understand how to restore finite α' and study the full dynamics of the theory away from the conformal limit. This should include the dynamics of other moduli, such as the ratio of the radii of $AdS_2 \times S^1$. It would also be nice to understand what is the role, if any, of the Type 0A backgrounds with long strings and both values of the RR flux in the context of noncritical M-theory. Answering this last question may require an understanding of the matrix model from which our Fermi liquid picture would follow. Our effective description of the conformal limit of the theory in terms of $AdS_2 \times S^1$ is a strong hint that a dual matrix model should exist, at least in this limit, in the form of a conformal quantum mechanics with a global $U(1)$ symmetry.

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A. Appendix A: Free Fermion Spectrum on S^1 Fibered over AdS_2

We wish to consider the spectrum of a free massless Dirac fermion on an S^1 fibration over AdS_2, assuming that the fibration preserves the $SL(2, \mathbf{R})$ symmetries of the base. We work in global coordinates on AdS_2, and will study the spectrum with respect to the evolution in the global time on AdS_2. The most general metric that preserves the $SL(2, \mathbf{R})$ symmetry of AdS_2 is

$$ds^2 = -\mathcal{R}^2 \cosh^2 \rho \, dt^2 + \mathcal{R}^2 d\rho^2 + \mathcal{R}^2(\gamma dy + \alpha \sinh \rho \, dt)^2. \tag{A.1}$$

The arbitrary constant α effectively measures the Kaluza-Klein flux of the off-diagonal metric components in the fibration. The other arbitrary constant γ parameterizes the ratio of the radius of S^1 and the curvature radius of AdS_2 and S^1 components; we take the S^1 coordinate y to run from 0 to 2π. Setting $\alpha = 0$ would reproduce the direct product metric on $AdS_2 \times S^1$. Alternatively, the AdS_3 Hopf fibration would be obtained by setting $\alpha = \gamma = 1$ and allowing y to run over $\mathbf{R}$.

We will keep α and γ arbitrary, in order to explore the full set of fibrations. Additionally we will not impose boundary conditions yet to preserve generality. The vielbein components

$$e^0 = \mathcal{R} \cosh \rho dt \tag{A.2}$$
$$e^1 = \mathcal{R} d\rho \tag{A.3}$$
$$e^2 = \mathcal{R}\gamma dy + \mathcal{R}\alpha \sinh \rho dt \tag{A.4}$$

imply the spin connection components

$$\begin{aligned}
\omega_{01} &= \tfrac{\alpha\gamma}{2}dy + \sinh\rho\left(\tfrac{\alpha^2}{2}-1\right)dt \\
\omega_{02} &= \tfrac{\alpha}{2}d\rho \\
\omega_{12} &= -\tfrac{\alpha\cosh\rho}{2}dt.
\end{aligned} \tag{A.5}$$

We will use the following γ matrices, which have no explicit factors of i:

$$\gamma^0 = i\sigma^2, \qquad \gamma^1 = \sigma^1, \qquad \gamma^2 = \sigma^3. \tag{A.6}$$

Next, we need to calculate $\Gamma_\mu = \frac{1}{8}\omega_\mu{}^{ba}\sigma^{ab}$:

$$\begin{aligned}
&\Gamma_y = -\tfrac{\alpha\gamma}{4}\sigma^3 \\
&\Gamma_\rho = \tfrac{\alpha}{4}\sigma^1 \\
&\Gamma_t = \tfrac{1}{2}\left[(1-\tfrac{\alpha^2}{2})\right]\sinh\rho\sigma^3 - \tfrac{\alpha}{2}\cosh\rho i\sigma^2
\end{aligned} \tag{A.7}$$

Finally, the massless Dirac equation

$$i\gamma^\mu\nabla_\mu\psi = (i\gamma^\mu\partial_\mu + i\gamma^\mu\Gamma_\mu)\psi = 0 \tag{A.8}$$

becomes

$$\begin{pmatrix} \cosh\rho\frac{1}{\gamma}\partial_y + \frac{\alpha}{4}\cosh\rho & \partial_t + \cosh\rho\partial_\rho - \frac{\alpha}{\gamma}\sinh\rho\partial_y - \frac{1}{2}\sinh\rho \\ -\partial_t + \cosh\rho\partial_\rho + \frac{\alpha}{\gamma}\sinh\rho\partial_y - \frac{1}{2}\sinh\rho & -\cosh\rho\frac{1}{\gamma}\partial_y + \frac{\alpha}{4}\cosh\rho \end{pmatrix}\psi = 0. \tag{A.9}$$

We will make the coordinate change $\cosh\rho = 1/\cos\theta$ additionally stipulating $\sinh\rho = -\tan\theta$. This gives

$$\begin{pmatrix} \sec\theta\frac{1}{\gamma}\partial_y + \frac{\alpha}{4}\sec\theta & \partial_t - \partial_\theta + \frac{\tan\theta}{2} + \frac{\alpha}{\gamma}\tan\theta\partial_y \\ -\partial_t - \partial_\theta + \frac{\tan\theta}{2} - \frac{\alpha}{\gamma}\tan\theta\partial_y & -\sec\theta\frac{1}{\gamma}\partial_y + \frac{\alpha}{4}\sec\theta \end{pmatrix}\psi = 0. \tag{A.10}$$

Now, let us choose ψ such that

$$\psi = e^{i\beta y}e^{-i\omega t}Y(\theta)\begin{pmatrix} e^{i\omega\theta}X_1(\theta)u(\theta) \\ e^{-i\omega\theta}X_2(\theta)v(\theta) \end{pmatrix}, \tag{A.11}$$

where

$$\partial_\theta Y = \frac{1}{2}\tan\theta Y, \quad \partial_\theta X_1 = -i\frac{\alpha\beta}{\gamma}\tan\theta X_1, \quad \partial_\theta X_2 = i\frac{\alpha\beta}{\gamma}\tan\theta X_2. \tag{A.12}$$

This choice of ψ reduces our Dirac equation to

$$\left(i\frac{\beta}{\gamma} + \frac{\alpha}{4}\right)\sec\theta e^{2i\omega\theta}\frac{X_1}{X_2}u - \partial_\theta v = 0, \tag{A.13}$$

$$-\frac{X_1}{X_2}e^{2i\omega\theta}\partial_\theta u + \left(-i\frac{\beta}{\gamma} + \frac{\alpha}{4}\right)\sec\theta v = 0. \tag{A.14}$$

These coupled equations reduce to the single second-degree equation

$$\left(\frac{\alpha^2}{16} + \left(\frac{\beta}{\gamma}\right)^2\right) u = \left(2i\omega\cos^2\theta - (1 + 2i\frac{\alpha\beta}{\gamma})\sin\theta\cos\theta\right)\partial_\theta u + \cos^2\theta\partial_\theta^2 u. \tag{A.15}$$

Following [14], we change variables to $z = (1 + \tan\theta)/2$ which gives us

$$\left(\frac{\alpha^2}{16} + \frac{\beta^2}{\gamma^2}\right) u + z(1-z)u'' + \left[\omega + \left(1 - 2i\alpha\frac{\beta}{\gamma}\right)\left(\frac{1}{2} - z\right)\right] = 0. \tag{A.16}$$

Note that the $\alpha = 0$ case reduces to Eqn. (A.17) in [14], provided we set $\frac{\beta}{\gamma} = m\mathcal{R}$. Although our Dirac norm contains instead a factor of $1/\cos^3\theta$, we still have the same requirement for u to vanish at $\theta = \pm\pi/2$, or $z \to \pm\infty$. Thus, for the particular case of $\alpha = 0$, we can use the result of [14] that

$$|\omega| = |\frac{\beta}{\gamma}| + \frac{1}{2} + n. \tag{A.17}$$

Now, even in the $\alpha = 0$ case, we must also consider the restrictions the boundary conditions for ψ in y put on β. Assuming periodicity of the fermions, we find

$$\beta = \frac{q}{2}, \tag{A.18}$$

which gives us exactly the spectrum (3.4) if we also set $\gamma = 2$.

We should also check that no other combination of α and γ produces the same spectrum. Let us proceed by comparing Equation (A.16) to the generic form for a hypergeometric equation

$$z(1-z)u'' + [c - (a+b+1)z]\,u' - abu = 0. \tag{A.19}$$

We can match Equation (A.16) to this equation by choosing

$$a = -i\alpha\frac{\beta}{\gamma} - s, \qquad b = -i\alpha\frac{\beta}{\gamma} + s, \qquad c = \omega + \frac{1}{2} - i\alpha\frac{\beta}{\gamma} \tag{A.20}$$

with

$$s = \sqrt{\frac{\alpha^2}{16} - \frac{\alpha^2\beta^2}{\gamma^2} + \frac{\beta^2}{\gamma^2}}. \tag{A.21}$$

A similar analysis to that done in [14] shows us the spectrum must be

$$|\omega| = n + \frac{1}{2} + s \tag{A.22}$$

for n a non-negative integer. Now, we would like to see if we can match s to the set of half integers $q/2$. For non-compact y, we find a continuous spectrum; for compact y, presuming $y \in [0, 2\pi]$, we find $\beta = q/2$, where q ranges over the integers. One can quickly check that only $\alpha = 0$ and $\gamma = 2$ will allow s to range over the set of half integers given by $q/2$. Thus, only the direct product spacetime with equal characteristic size for AdS_2 and S^1 will produce the desired spectrum.

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