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# Higher Comparison Maps for the Spectrum of a Tensor Triangulated Category

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Beren James Sanders

2014

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# ABSTRACT OF THE DISSERTATION

Higher Comparison Maps for the Spectrum of a Tensor Triangulated Category

by

Beren James Sanders

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Paul Balmer, Chair

For each object in a tensor triangulated category, we construct a natural continuous map from the object's support—a closed subset of the category's triangular spectrum—to the Zariski spectrum of a certain commutative ring of endomorphisms. When applied to the unit object this recovers a construction of P. Balmer. These maps provide an iterative approach for understanding the spectrum of a tensor triangulated category by starting with the comparison map for the unit object and iteratively analyzing the fibers of this map via “higher” comparison maps. We illustrate this approach for the stable homotopy category of finite spectra. In fact, the same underlying construction produces a whole collection of new comparison maps, including maps associated to (and defined on) each closed subset of the triangular spectrum. These latter maps provide an alternative strategy for analyzing the spectrum by iteratively building a filtration of closed subsets by pulling back filtrations of affine schemes.

The dissertation of Beren James Sanders is approved.

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2014

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# CHAPTER 1

## Introduction

Triangulated categories have their origins in Verdier’s work on the homological foundations of Grothendieck’s algebraic geometry and, independently, in Puppe’s work on the foundations of stable homotopy theory. In the intervening half century, the notion of a triangulated category has proved to be a predestined structure that is found throughout the algebraic and topological branches of modern pure mathematics. Indeed, examples of triangulated categories arise in a truly diverse range of mathematical disciplines from algebraic geometry and homological algebra to stable homotopy theory, noncommutative topology and modular representation theory. The abstract theory is motivated by a rich variety of concrete examples and provides theorems that are applicable to a wide range of subjects. At the same time, triangulated categories provide a platform for transferring ideas and techniques between these disparate subjects. For example, one can take an idea from algebraic geometry (such as gluing sheaves), generalize the idea to the abstract world of triangulated categories, and then see what it says about “gluing representations” in modular representation theory. It is a truly interdisciplinary subject.

Many triangulated categories arising in nature come equipped with natural  $\otimes$ -product structures—that is, they are *tensor* triangulated categories—and in recent years there has been a growing appreciation for the significance of these  $\otimes$ -structures. For example, a (nice) scheme can be recovered from the tensor triangulated structure of its derived category of

perfect complexes, but not from the triangulated structure alone (see [Bal10a, Remark 64], for example). Using the  $\otimes$ -structure, Balmer [Bal05] has introduced the *spectrum* of a tensor triangulated category. Just as the spectrum of a commutative ring provides a geometric approach to commutative algebra, the spectrum of a tensor triangulated category provides a geometric approach to the study of tensor triangulated categories—an approach referred to as *tensor triangular geometry* by its originators. This dissertation makes a contribution to tensor triangular geometry and the antenatal reader is referred to [Bal10b] for an introduction to this relatively new field and for additional background that leads to the present work.

Determining the spectrum of a given tensor triangulated category is a highly non-trivial problem, which is essentially equivalent to classifying the thick triangulated  $\otimes$ -ideals in the category—in other words, classifying the objects of the category up to the naturally available structure:  $\otimes$ -products,  $\oplus$ -sums,  $\oplus$ -summands, suspensions, and cofibers. Major classification theorems in algebraic geometry, modular representation theory and stable homotopy theory give complete descriptions of the spectrum in several important examples, but one of the goals of tensor triangular geometry is to go the other way—to develop techniques for determining the spectrum (and thereby solve the classification problem), or to at least say something interesting about the spectrum when a full determination proves to be too ambitious.

In any tensor triangulated category, the endomorphism ring of the unit is commutative, and the first step towards saying something about the spectrum of a general tensor triangulated category was taken in [Bal10a] where continuous maps

$$\rho : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(\mathrm{End}_{\mathcal{K}}(\mathbb{1})) \quad \text{and} \quad \rho^{\bullet} : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}^{\mathrm{h}}(\mathrm{End}_{\mathcal{K}}^{\bullet}(\mathbb{1}))$$

were defined going from the triangular spectrum to the (homogeneous) spectrum of the

(graded) endomorphism ring of the unit. These “comparison maps” are often surjective and so attention focuses on understanding conditions under which they are injective and more generally on understanding their fibers. If  $\mathcal{K} = \mathrm{D}_{\mathrm{perf}}(A)$  is the derived category of perfect complexes of a commutative ring  $A$ , then  $\mathrm{End}_{\mathcal{K}}(\mathbb{1})$  is isomorphic to  $A$  and  $\rho$  turns out to be a homeomorphism. This can be proved directly and provides an alternative proof of the affine case of the Hopkins-Neeman-Thomason theorem. On the other hand, if  $G$  is a finite group,  $k$  is a field, and  $\mathcal{K} = \mathrm{D}^b(kG\text{-mod})$  with  $\otimes = \otimes_k$ , then  $\mathrm{End}_{\mathcal{K}}^{\bullet}(\mathbb{1})$  is group cohomology  $H^{\bullet}(G, k)$  and it is known using the classification theorem of Benson-Carlson-Rickard that the map  $\rho^{\bullet}$  is a homeomorphism. A more direct proof of the injectivity of  $\rho^{\bullet}$  in this example would provide a new proof of the Benson-Carlson-Rickard theorem. In general, however, one cannot expect the (graded) endomorphisms of the unit to determine the global structure of the whole category and we are left with the important general problem of understanding the fibers of these comparison maps.

In this dissertation, we will construct new comparison maps which generalize those mentioned above. More specifically, for each object  $X$  in a tensor triangulated category  $\mathcal{K}$  we will define maps

$$\rho_X : \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_X) \quad \text{and} \quad \rho_X^{\bullet} : \mathrm{supp}(X) \rightarrow \mathrm{Spec}^h(R_X^{\bullet})$$

from the support of  $X$  (a closed subset of the triangular spectrum) to the (homogeneous) spectrum of a certain (graded-)commutative ring of (graded) endomorphisms of  $X$ , which recover the original comparison maps when  $X = \mathbb{1}$ . The author’s initial interest in these new comparison maps stems from the fact that they provide a method for studying the fibers of the original maps. This in turn leads to an iterative strategy for studying the spectrum based on a repeated analysis of the fibers of a sequence of generalized comparison maps. The idea runs as follows. Given an arbitrary tensor triangulated category  $\mathcal{K}$ , we can take

the unit object and consider the comparison map  $\rho_1 : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(R_1)$ . Understanding the fibers of this map reduces by a localization technique to the case when  $R_1$  is a local ring. If the unique closed point  $\mathfrak{m} = \langle f_1, \dots, f_n \rangle$  is finitely generated then it is straightforward to show that the fiber  $\rho_1^{-1}(\{\mathfrak{m}\})$  is equal to the support of the object  $X_1 := \mathrm{cone}(f_1) \otimes \dots \otimes \mathrm{cone}(f_n)$ . This fiber can then be examined more closely by considering the “higher” comparison map  $\rho_{X_1} : \mathrm{supp}(X_1) \rightarrow \mathrm{Spec}(R_{X_1})$  associated with the object  $X_1$ . The same procedure can then be used to study the fibers of  $\rho_{X_1}$  and the process repeats itself. Following any particular thread in this process produces a linear filtration

$$\mathrm{Spc}(\mathcal{K}) \supset \mathrm{supp}(X_1) \supset \mathrm{supp}(X_2) \supset \dots \supset \mathrm{supp}(X_n)$$

which can be extended for however long the rings involved possess finitely generated primes.

One of the difficulties with this method is that to understand the fiber over a non-closed point we must first apply a localization procedure. The reason is that for a finitely generated prime  $\mathfrak{p} = \langle f_1, \dots, f_n \rangle$ , the support of  $\mathrm{cone}(f_1) \otimes \dots \otimes \mathrm{cone}(f_n)$  is actually the preimage of the closure  $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$  rather than the fiber over  $\mathfrak{p}$ . More generally,

$$\rho_X^{-1}(V(I)) = \mathrm{supp}(\mathrm{cone}(f_1) \otimes \dots \otimes \mathrm{cone}(f_n))$$

for any finitely generated ideal  $I = \langle f_1, \dots, f_n \rangle$ . Thus, rather than examining the fibers of a comparison map  $\rho_X$ , an alternative strategy is to take a look at the preimages of all of the Thomason closed subsets  $V(I) \subset \mathrm{Spec}(R_X)$ .<sup>1</sup> Choosing generators of the ideal  $I$  provides us with an object of  $\mathcal{K}$  whose support is the closed subset  $\rho_X^{-1}(V(I))$  and we can examine this subset further via the comparison map associated with this “generator” object.<sup>2</sup>

---

<sup>1</sup>Recall that a Thomason closed subset is the same thing as a closed subset whose complement is quasi-compact. In the case of an affine scheme  $\mathrm{Spec}(A)$  this is a closed set of the form  $V(I)$  for a finitely generated ideal  $I \subset A$ , while in the case of  $\mathrm{Spc}(\mathcal{K})$  this is a closed subset of the form  $\mathrm{supp}(a)$  for an object  $a \in \mathcal{K}$ . These notions will be reviewed in Section 2.4.

<sup>2</sup>A “generator” of a closed subset  $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$  is an object  $a \in \mathcal{K}$  with  $\mathrm{supp}(a) = \mathcal{Z}$ .

Both of these strategies suffer from the fact that (a) they only deal with finitely generated primes and Thomason closed subsets (which may be an undesirable limitation in non-noetherian situations) and (b) they involve non-canonical choices of generators. The fundamental idea in both approaches is to examine a Thomason closed subset  $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$  by the comparison map associated with an object which generates  $\mathcal{Z}$ , but this comparison map depends on the choice of generator. Such considerations lead to the desire for a “generator-independent” comparison map which only depends on the Thomason closed subset on which it is defined, and more generally for a comparison map associated to *every* closed subset of the spectrum.

Indeed, the map  $\rho_X$  is just one of a host of new comparison maps introduced in this work. The most general construction associates a natural, continuous map

$$\rho_\Phi : \bigcap_{X \in \Phi} \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_\Phi)$$

to each set of objects  $\Phi \subset \mathcal{K}$  that is closed under the  $\otimes$ -product. Taking  $\Phi = \{X^{\otimes n} \mid n \geq 1\}$  gives the map  $\rho_X$  above, while taking  $\Phi = \{a \in \mathcal{K} \mid \mathrm{supp}(a) \supset \mathcal{Z}\}$  gives a map  $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathrm{Spec}(R_{\mathcal{Z}})$  associated to (and defined on) an arbitrary closed subset of the spectrum. Following on from the previous discussion, the latter “closed set” comparison maps  $\rho_{\mathcal{Z}}$  afford perhaps the most robust strategy for studying the spectrum. The idea is to iteratively build a filtration of closed subsets by pulling back filtrations of the affine schemes  $\mathrm{Spec}(R_{\mathcal{Z}})$ . This idea has the advantage that it utilizes all closed subsets (not just Thomason ones) and is purely deterministic: no choices are involved. The hope is that ultimately the filtration will become fine enough to completely determine the spectrum. Although certain difficulties prevent these strategies from working out in the full generality that one might hope, the author nevertheless considers them to be the primary justification for the theory developed in this dissertation.

In any case, one example where none of the difficulties arise is the stable homotopy category of finite spectra  $\mathrm{SH}^{\mathrm{fin}}$ . This is an elusive example for tensor triangular geometry. Although the structure of the space  $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$  is known via the work of Devinatz, Hopkins and Smith (cf. Section 6.5), the unit comparison map

$$\begin{array}{ccccccc}
 & \mathcal{C}_{2,\infty} & \mathcal{C}_{3,\infty} & \cdots & \mathcal{C}_{p,\infty} & \cdots \\
 & \vdots & \vdots & & \vdots & \\
 \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) = & \mathcal{C}_{2,n+1} & \mathcal{C}_{3,n+1} & \cdots & \mathcal{C}_{p,n+1} & \cdots \\
 & \vdots & \vdots & & \vdots & \\
 & \mathcal{C}_{2,n} & \mathcal{C}_{3,n} & \cdots & \mathcal{C}_{p,n} & \cdots \\
 & \vdots & \vdots & & \vdots & \\
 & \mathcal{C}_{2,2} & \mathcal{C}_{3,2} & \cdots & \mathcal{C}_{p,2} & \cdots \\
 & & \searrow & & \swarrow & \\
 & & \mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}} & & & \\
 \downarrow \rho_1 & & & & & \\
 \mathrm{Spec}(\mathbb{Z}) = & 2\mathbb{Z} & 3\mathbb{Z} & \cdots & p\mathbb{Z} & \cdots \\
 & & \searrow & & \swarrow & \\
 & & (0) & & & 
 \end{array}$$

is far from injective and understanding the fibers (which are given by the collection of Morava  $K$ -theories) is related to the important problem of understanding residue fields in tensor triangular geometry. In any case, the iterative procedure we have mentioned above works out very nicely in this example, and it provides one illustration of how the higher comparison maps can work out in practice. However, determining the comparison maps in this example—in particular, determining the structure of the rings  $R_X^\bullet$ —requires the full strength of the results in [HS98] on nilpotence and periodicity in stable homotopy theory. In particular, it presupposes knowledge of the classification of thick subcategories in  $\mathrm{SH}^{\mathrm{fin}}$ . Nevertheless, these results allow us to show that the new comparison maps refine the view of  $\mathrm{SH}^{\mathrm{fin}}$  provided by Balmer’s original comparison map  $\rho_1$ . This is an important test for our theory as other generalizations of the original maps have failed to provide ad-

ditional insight into this example. Stated differently, although our application of the theory to  $\mathrm{SH}^{\mathrm{fin}}$  ultimately depends on the classification of thick subcategories, our work nevertheless demonstrates that the spectrum of this topological category is completely seen by purely algebro-geometric invariants. The original algebraic invariant  $\mathrm{Spec}^h(\mathrm{End}_{\mathcal{K}}^\bullet(\mathbb{1}))$  sees very little, but our “higher” algebraic invariants see everything.

## Outline of the dissertation

The original results of the dissertation are confined to the last two chapters. The first four chapters are preliminary in nature. In detail:

Chapter 2 contains miscellaneous material on idempotent completion, monoidal categories, graded rings and spectral spaces. It could arguably be put as an appendix but since it contains remarks that we would like to bring to the reader’s attention we have opted to include it as part of the narrative.

Chapter 3 covers the theory of triangulated categories from the basic definitions to Brown representability and Bousfield localization. In particular, we define the notion of a *tensor* triangulated category in Section 3.3 with a careful treatment of the compatibility between the tensor structure and the triangulated structure. Examples are considered in Section 3.8. Finally, in the last section we set the stage for the next chapter by briefly discussing the thick subcategory classification theorems of Hopkins-Neeman-Thomason and Benson-Carlson-Rickard.

Chapter 4 provides a bare-bones account of the spectrum of a tensor triangulated category as introduced by Paul Balmer. We only include those notions and definitions which are required for understanding the rest of the dissertation.

Chapter 5 is devoted to our main contribution: the theory of higher comparison maps. In

this chapter we construct the new comparison maps and lay the foundations of their basic theory. For example, we establish their naturality and universality, show that passing to the idempotent completion does not change anything, and develop a technique for localizing with respect to primes in  $R_\Phi$  (which has been alluded to in the discussion above and generalizes the “central localization” of [Bal10a]). Other results include establishing that the object comparison maps  $\rho_X$  are invariant under a number of natural operations that can be performed on the object  $X$  such as taking suspensions, or duals, or  $\otimes$ -powers, etc. In addition, we establish some connections of a topological nature between the target affine scheme  $\mathrm{Spec}(R_\Phi)$  and the domain of  $\rho_\Phi$ ; for example, we show that the domain is connected if and only if  $\mathrm{Spec}(R_\Phi)$  is connected. Other results of that nature include establishing that the image of  $\rho_\Phi$  is always dense in  $\mathrm{Spec}(R_\Phi)$ .

Chapter 6 is devoted to stable homotopy theory. The purpose of this chapter is to illustrate how the iterative method for analyzing the spectrum via higher comparison maps works out in the example of the stable homotopy category of finite spectra. This is accomplished in the final section of the dissertation: Section 6.6. The first five sections provide background from chromatic homotopy theory and are included for the benefit of the reader.

The results of this dissertation have been published in [San13]. Besides some more detailed proofs and more extensive preliminary discussion, all the main results can be found in that paper.

Finally, it is worth mentioning that [DS14] has also defined generalizations of the original comparison maps from [Bal10a]. However, that work focuses on invertible objects in the category and goes in quite a different direction than our theory.

**A remark about foundations:** In our exposition of the theory of triangulated categories, we are careful to mention places where set-theoretic difficulties arise, but we are

nevertheless loose (and somewhat schizophrenic) with our language. For example, we say that a category “need not exist in our universe” synonymously with “need not be locally small.” Which statement is the accurate one depends on which choice of foundations you choose. In any case, the theory of higher comparison maps—and tensor triangular geometry in general—applies to essentially small tensor triangulated categories, for which such issues do not arise.

Finally, recall that a subcategory is said to be *replete* if it is closed under isomorphisms.

## CHAPTER 2

### Preliminaries

This chapter contains miscellaneous material on idempotent completion, monoidal categories, graded rings, and spectral spaces. The theory of triangulated categories will be covered in the next chapter. The reader might be perplexed by the amount of detail we provide in the section on monoidal categories. We have included this material in order to be precise about the axiomatics of tensor triangulated categories (and also to make some remarks missing from the standard references) but the quizzical (and well-informed) reader can fitfully skim that section. The material on idempotent completion and spectral spaces is perhaps less standard.

#### 2.1 Idempotent completion

Morally speaking, an additive category is idempotent complete if it has all the direct summands it “should have.” Every category can be embedded in an idempotent complete category and it is a theorem of Balmer and Schlichting [BS01] that the idempotent completion of a triangulated category remains triangulated. Idempotent completion makes several prominent appearances in modern mathematics—for example, in Grothendieck’s construction of categories of pure motives (cf. [And04, Chapter 4]) and in Thomason’s work on localization in algebraic  $K$ -theory (cf. [TT90] and Remark 3.7.17). In the theory of triangulated categories, the significance of *thick subcategories* rather than mere triangulated subcategories

hints at the relevance of idempotent completion in that context. As we shall see, idempotent completion is a very mild construction and we should not feel guilty about performing it whenever convenient. In particular, it doesn't affect the spectrum (see Proposition 4.4.3) and in Section 5.5 we shall show that a tensor triangulated category and its idempotent completion have “the same” theory of higher comparison maps.

**Definition 2.1.1.** An idempotent endomorphism  $e : X \rightarrow X$  is a *split idempotent* if there exists a factorization

$$\begin{array}{ccc} X & \xrightarrow{e} & X \\ & \searrow f & \nearrow g \\ & Y & \end{array}$$

with the property that  $f \circ g = \text{id}_Y$ . A category is *idempotent complete* if every idempotent endomorphism splits.

*Remark 2.1.2.* In an additive category, the idempotent  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : A \oplus B \rightarrow A \oplus B$  is a split idempotent. Conversely, if  $e : X \rightarrow X$  is an idempotent such that both  $e$  and  $1 - e$  are split then  $X \simeq \text{im } e \oplus \text{ker } e$  in such a way that  $e$  becomes the idempotent  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Thus, in an idempotent complete additive category, every idempotent arises from a direct-sum decomposition.

**Definition 2.1.3.** Let  $\mathcal{C}$  be a category. The *idempotent completion* of  $\mathcal{C}$  is a category  $\mathcal{C}^\sharp$  defined as follows. The objects are pairs  $(X, e)$  where  $X$  is an object of  $\mathcal{C}$  and  $e : X \rightarrow X$  is an idempotent endomorphism, and the maps are defined by

$$\text{Hom}_{\mathcal{C}^\sharp}((X, e), (Y, f)) := \{\phi \in \text{Hom}_{\mathcal{C}}(X, Y) \mid f \circ \phi = \phi = \phi \circ e\}$$

with composition induced by composition in  $\mathcal{C}$ . The functor  $\mathcal{C} \rightarrow \mathcal{C}^\sharp$  given by  $X \mapsto (X, \text{id}_X)$  is fully faithful and embeds  $\mathcal{C}$  as a full replete subcategory of  $\mathcal{C}^\sharp$ .

*Remark 2.1.4.* The category  $\mathcal{C}^\sharp$  is idempotent complete. Indeed, an idempotent  $(X, e) \xrightarrow{\phi} (X, e)$

splits as

$$\begin{array}{ccc} (X, e) & \xrightarrow{\phi} & (X, e) \\ & \searrow \phi \quad \nearrow \phi & \\ & (X, \phi) & \end{array}$$

since  $\phi^2 = \phi$  is the identity morphism of  $(X, \phi)$  in  $\mathcal{C}^\sharp$ . In fact, it is easy to show that the canonical functor  $i : \mathcal{C} \rightarrow \mathcal{C}^\sharp$  is the universal functor from  $\mathcal{C}$  to an idempotent complete category. More precisely, if  $F : \mathcal{C} \rightarrow \mathcal{D}$  is any functor to an idempotent complete category  $\mathcal{D}$  then there exists a functor  $\tilde{F} : \mathcal{C}^\sharp \rightarrow \mathcal{D}$  such that  $\tilde{F} \circ i = F$  and, moreover, any two such functors  $\tilde{F}$  are naturally isomorphic. This property characterizes  $\mathcal{C}^\sharp$  up to equivalence of categories.

*Remark 2.1.5.* A category is idempotent complete iff the canonical functor  $\mathcal{C} \rightarrow \mathcal{C}^\sharp$  is an equivalence of categories. In fact, this is the same as saying that  $\mathcal{C} \rightarrow \mathcal{C}^\sharp$  is an *isomorphism* of categories since the image of  $\mathcal{C}$  is a *replete* subcategory of  $\mathcal{C}^\sharp$ .

*Remark 2.1.6.* If  $\mathcal{A}$  is an additive category then  $\mathcal{A}^\sharp$  is also additive and the canonical functor  $\mathcal{A} \hookrightarrow \mathcal{A}^\sharp$  embeds  $\mathcal{A}$  as a full additive subcategory of  $\mathcal{A}^\sharp$ . We'll see in Section 4.4 that the idempotent completion of a (tensor) triangulated category is again a (tensor) triangulated category.

**Proposition 2.1.7** (Freyd). *Let  $\mathcal{A}$  be an additive category which has countable coproducts and which has the property that an idempotent  $e$  splits if and only if  $1 - e$  splits. Then every idempotent in  $\mathcal{A}$  splits. In other words,  $\mathcal{A}$  is idempotent complete.*

*Proof.* The following proof is taken from [Fre66]. Consider an idempotent  $e : A \rightarrow A$  in  $\mathcal{A}$ .

Let  $B := \coprod_{n \in \mathbb{N}} A$  and define maps  $f : B \rightarrow B$  and  $g : B \rightarrow B$  by

$$B \xrightarrow{f = \begin{pmatrix} 1-e & & & \\ e & 1-e & & \\ & e & 1-e & \\ & & & \ddots \end{pmatrix}} B \quad \text{and} \quad B \xrightarrow{g = \begin{pmatrix} 1-e & e & & \\ & 1-e & e & \\ & & 1-e & \\ & & & \ddots \end{pmatrix}} B.$$

Then  $g \circ f = \text{id}_B$  so that  $f \circ g : B \rightarrow B$  is a split idempotent. By our hypotheses,  $1 - f \circ g$  must

also split: there exists an object  $C$  and maps  $B \xrightarrow{\beta} C$  and  $C \xrightarrow{\gamma} B$  such that  $1 - f \circ g = \gamma \circ \beta$  and  $\beta \circ \gamma = \text{id}_C$ :

$$\begin{array}{ccc} B & \xrightarrow{1-f \circ g} & B \\ & \searrow \beta & \nearrow \gamma \\ & C & \end{array}$$

Note that  $\beta \circ f \circ g = \beta \circ (1 - \gamma \circ \beta) = \beta - \beta \circ \gamma \circ \beta = 0$  and hence  $\beta \circ f = 0$  since  $g$  is (split) epi.

Similarly,  $f \circ g \circ \gamma = (1 - \gamma \circ \beta) \circ \gamma = \gamma - \gamma \circ \beta \circ \gamma = 0$  so that  $g \circ \gamma = 0$ . Now observe that

$$f \circ g = \begin{pmatrix} 1-e & & & \\ e & 1-e & & \\ & e & 1-e & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1-e & e & & \\ & 1-e & e & \\ & & 1-e & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} 1-e & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

and hence

$$\gamma \circ \beta = 1 - f \circ g = \begin{pmatrix} e & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}.$$

In other words,  $\gamma \circ \beta = B \xrightarrow{p_1} A \xrightarrow{e} A \xrightarrow{i_1} B$  where  $i_1$  and  $p_1$  denote the canonical maps for the coproduct. On the other hand,

$$0 = \beta \circ f = (\beta \circ i_1 \quad \beta \circ i_2 \quad \beta \circ i_3 \quad \dots) \begin{pmatrix} 1-e & & & \\ e & 1-e & & \\ & e & 1-e & \\ & & & \ddots \end{pmatrix}$$

implies that  $\beta \circ i_n \circ (1-e) + \beta \circ i_{n+1} \circ e = 0$  for  $n \geq 1$ . Hence  $(\beta \circ i_n \circ (1-e) + \beta \circ i_{n+1} \circ e) \circ (1-e) = \beta \circ i_n \circ (1-e) = 0$  for  $n \geq 1$  and similarly  $(\beta \circ i_n \circ (1-e) + \beta \circ i_{n+1} \circ e) \circ e = \beta \circ i_{n+1} \circ e = 0$  for  $n \geq 1$ .

It follows that  $\beta \circ i_n = \beta \circ i_n \circ ((1-e) + e) = 0$  for  $n \geq 2$ . Hence  $B \xrightarrow{\beta} C = B \xrightarrow{p_1} A \xrightarrow{i_1} B \xrightarrow{\beta} C$ . Now define maps  $s: A \rightarrow C$  and  $t: C \rightarrow A$  by  $s = A \xrightarrow{i_1} B \xrightarrow{\beta} C$  and  $t = C \xrightarrow{\gamma} B \xrightarrow{p_1} A$  and observe that  $t \circ s = p_1 \circ \gamma \circ \beta \circ i_1 = p_1 \circ i_1 \circ e \circ p_1 \circ i_1 = e$  and  $s \circ t = \beta \circ i_1 \circ p_1 \circ \gamma = \beta \circ \gamma = \text{id}_C$ . Hence we have a splitting

$$\begin{array}{ccc} A & \xrightarrow{e} & A \\ & \searrow s & \nearrow t \\ & C & \end{array}$$

which proves the claim. □

*Remark 2.1.8.* We'll see in the next chapter that every triangulated category has the property that an idempotent  $e$  splits iff  $1 - e$  splits; see Lemma 3.1.21. It will therefore follow that any triangulated category which has countable coproducts is idempotent complete.

## 2.2 Monoidal categories

Monoidal categories form a basic piece of category theory. The standard reference is [Mac98] but [EK66] contains additional material.

**Definition 2.2.1.** A *monoidal category* consists of a category  $\mathcal{C}$ , a functor  $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and an object  $\mathbb{1} \in \mathcal{C}$  called the unit, together with natural isomorphisms  $l_a : \mathbb{1} \otimes a \xrightarrow{\sim} a$ ,  $r_a : a \otimes \mathbb{1} \xrightarrow{\sim} a$ , and  $\alpha_{a,b,c} : a \otimes (b \otimes c) \xrightarrow{\sim} (a \otimes b) \otimes c$ , called the left unitor, right unitor, and associator, respectively, such that the diagrams

$$\begin{array}{ccc} a \otimes (b \otimes (c \otimes d)) & \xrightarrow{\alpha} & (a \otimes b) \otimes (c \otimes d) \xrightarrow{\alpha} ((a \otimes b) \otimes c) \otimes d \\ \downarrow 1 \otimes \alpha & & \uparrow \alpha \otimes 1 \\ a \otimes ((b \otimes c) \otimes d) & \xrightarrow{\alpha} & (a \otimes (b \otimes c)) \otimes d \end{array} \quad (2.2.2)$$

and

$$\begin{array}{ccc} a \otimes (\mathbb{1} \otimes b) & \xrightarrow{\alpha} & (a \otimes \mathbb{1}) \otimes b \\ a \otimes l_b \downarrow & & \downarrow r_a \otimes b \\ a \otimes b & \xlongequal{\quad} & a \otimes b \end{array} \quad (2.2.3)$$

commute for any  $a, b, c, d \in \mathcal{C}$ .

*Remark 2.2.4.* The equality  $l_{\mathbb{1}} = r_{\mathbb{1}}$  is included as one of the axioms in [Mac98] but it was shown in [Kel64] that it follows from (2.2.2) and (2.2.3). Thus most authors do not include it as part of the definition (e.g. [Bor94, JS93, Kel82]). It is significant for us because it implies that endomorphisms of the unit object commute. In order to prove this fact, we require the following lemma.

**Lemma 2.2.5.** *Let  $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$  be a monoidal category. The diagrams*

$$\begin{array}{ccc} \mathbb{1} \otimes (a \otimes b) & \xrightarrow{\alpha} & (\mathbb{1} \otimes a) \otimes b \\ & \searrow l_{a \otimes b} & \swarrow l_{a \otimes b} \\ & a \otimes b & \end{array} \quad \text{and} \quad \begin{array}{ccc} a \otimes (b \otimes \mathbb{1}) & \xrightarrow{\alpha} & (a \otimes b) \otimes \mathbb{1} \\ & \searrow a \otimes r_b & \swarrow r_{a \otimes b} \\ & a \otimes b & \end{array}$$

*commute for any  $a, b \in \mathcal{C}$ .*

*Proof.* We'll only demonstrate the commutativity of the first diagram since the second diagram is disposed of similarly. Note that  $\mathbb{1} \otimes - : \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence—indeed, the left unitor provides a natural isomorphism between  $\mathbb{1} \otimes -$  and the identity functor—so in order to show that the diagram commutes, it suffices to show that it commutes after applying  $\mathbb{1} \otimes -$ .

The diagram that results is the left triangle of the following diagram:

$$\begin{array}{ccccc} & \xrightarrow{1 \otimes \alpha} & \mathbb{1} \otimes ((\mathbb{1} \otimes a) \otimes b) & \xrightarrow{\alpha} & (\mathbb{1} \otimes (\mathbb{1} \otimes a)) \otimes b & \xrightarrow{\alpha \otimes 1} & ((\mathbb{1} \otimes \mathbb{1}) \otimes a) \otimes b \\ & \searrow 1 \otimes l_{a \otimes b} & \downarrow 1 \otimes (l_a \otimes 1) & & \downarrow (1 \otimes l_a) \otimes 1 & \swarrow (r_1 \otimes 1) \otimes 1 & \\ \mathbb{1} \otimes (\mathbb{1} \otimes (a \otimes b)) & & \mathbb{1} \otimes (a \otimes b) & \xrightarrow{\alpha} & (\mathbb{1} \otimes a) \otimes b & & \end{array}$$

The center square commutes by naturality and the right triangle is  $- \otimes b$  applied to (2.2.3). In order to show that the left triangle commutes, it suffices to show that the outline of the diagram commutes since all the maps are isomorphisms. This is demonstrated by the following diagram:

$$\begin{array}{ccccc} & \xrightarrow{1 \otimes \alpha} & \mathbb{1} \otimes ((\mathbb{1} \otimes a) \otimes b) & \xrightarrow{\alpha} & (\mathbb{1} \otimes (\mathbb{1} \otimes a)) \otimes b & \xrightarrow{\alpha \otimes 1} & ((\mathbb{1} \otimes \mathbb{1}) \otimes a) \otimes b \\ & \searrow 1 \otimes l_{a \otimes b} & \downarrow 1 \otimes (l_a \otimes 1) & & \downarrow (1 \otimes l_a) \otimes 1 & \swarrow (r_1 \otimes 1) \otimes 1 & \\ \mathbb{1} \otimes (\mathbb{1} \otimes (a \otimes b)) & \xrightarrow{\alpha} & (\mathbb{1} \otimes \mathbb{1}) \otimes (a \otimes b) & \xrightarrow{\alpha} & ((\mathbb{1} \otimes \mathbb{1}) \otimes a) \otimes b & & \\ & \searrow 1 \otimes l_{a \otimes b} & \downarrow 1 \otimes (l_a \otimes 1) & & \downarrow (1 \otimes l_a) \otimes 1 & \swarrow (r_1 \otimes 1) \otimes 1 & \\ & \mathbb{1} \otimes (a \otimes b) & \xrightarrow{\alpha} & (\mathbb{1} \otimes a) \otimes b & & \end{array}$$

The top is (2.2.2), the bottom-left is (2.2.3), and the bottom-right commutes by naturality.  $\square$

**Lemma 2.2.6.** *Let  $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$  be a monoidal category. Then  $l_{\mathbb{1}} = r_{\mathbb{1}}$ .*

*Proof.* The diagram

$$\begin{array}{ccc} 1 \otimes (1 \otimes 1) & \xrightarrow{l_{1 \otimes 1}} & 1 \otimes 1 \\ \downarrow 1 \otimes l_1 & & \downarrow l_1 \\ 1 \otimes 1 & \xrightarrow{l_1} & 1 \end{array}$$

commutes by naturality. It follows that  $l_{1 \otimes 1} = 1 \otimes l_1$  since  $l_1$  is an isomorphism. Lemma 2.2.5 then implies that the diagram

$$\begin{array}{ccc} 1 \otimes (1 \otimes 1) & \xrightarrow{\alpha} & (1 \otimes 1) \otimes 1 \\ & \searrow 1 \otimes l_1 = l_{1 \otimes 1} & \swarrow l_1 \otimes 1 \\ & 1 \otimes 1 & \end{array}$$

commutes, while axiom (2.2.3) implies that the diagram

$$\begin{array}{ccc} 1 \otimes (1 \otimes 1) & \xrightarrow{\alpha} & (1 \otimes 1) \otimes 1 \\ & \searrow 1 \otimes l_1 & \swarrow r_1 \otimes 1 \\ & 1 \otimes 1 & \end{array}$$

commutes. Precomposing by  $\alpha^{-1}$  we conclude that  $l_1 \otimes 1 = r_1 \otimes 1$  and hence that  $l_1 = r_1$  since  $- \otimes 1: \mathcal{C} \rightarrow \mathcal{C}$  is an equivalence.  $\square$

**Lemma 2.2.7.** *Let  $(\mathcal{C}, \otimes, 1, \alpha, l, r)$  be a monoidal category. Any two endomorphisms of the unit object commute.*

*Proof.* Using the fact that  $l_1 = r_1$ , the commutativity of

$$\begin{array}{ccccc} & & f & & \\ & \swarrow r_1 & & \searrow r_1 & \\ 1 & & 1 \otimes 1 & \xrightarrow{f \otimes 1} & 1 \otimes 1 \\ & \swarrow l_1 & & \searrow l_1 & \\ & & 1 \otimes 1 & \xrightarrow{f \otimes 1} & 1 \otimes 1 \\ & \swarrow r_1 & & \searrow r_1 & \\ & & 1 & & 1 \end{array} \begin{array}{c} \\ \\ g \\ \\ \\ \\ g \end{array}$$

shows that any two endomorphisms  $f, g \in \text{End}_{\mathcal{C}}(1)$  commute.  $\square$

*Remark 2.2.8.* The significance of the peculiar choice of axioms in Definition 2.2.1 lies in Mac Lane’s coherence theorem [Mac63]. The basic idea can be explained as follows. Given a sequence of objects  $x_1, \dots, x_n$  in a monoidal category, there are various ways we can form their  $\otimes$ -product depending on how we choose to bracket the expression  $x_1 \otimes \dots \otimes x_n$ . For any two such choices, we can construct an isomorphism from one to the other by iteratively applying the associator. However, there will typically be several ways to construct such an isomorphism and we would like all such isomorphisms to coincide. Axiom (2.2.2) asserts that two particular such isomorphisms

$$a \otimes (b \otimes (c \otimes d)) \xrightarrow{\sim} ((a \otimes b) \otimes c) \otimes d$$

coincide, and it is a non-trivial theorem—due to Saunders Mac Lane—that this one “coherence axiom” for the tensor product of four objects implies all “higher coherence axioms” for the tensor product of an arbitrary number of objects. The inclusion of axiom (2.2.3) enables one to prove a similar coherence result which takes the unitors and unit object into account.

*Remark 2.2.9.* The coherence theorem is sometimes expressed casually as the statement that “all diagrams built from  $\alpha$ ,  $l$  and  $r$  commute” but this is not correct. A more accurate statement is that all “formal” diagrams built from  $\alpha$ ,  $l$  and  $r$  commute (see [Mac98, §VII.2] for a precise statement). The problem with the more general false statement is the possibility that two “formally different” bracketings might coincide in any particular monoidal category. For example, it is possible that there might be an equality  $x_1 \otimes (x_2 \otimes x_3) = (x_1 \otimes x_2) \otimes x_3$  without  $\alpha : x_1 \otimes (x_2 \otimes x_3) \xrightarrow{\sim} (x_1 \otimes x_2) \otimes x_3$  being the identity, in which case

$$\begin{array}{ccc} x_1 \otimes (x_2 \otimes x_3) & \xrightarrow{\alpha} & (x_1 \otimes x_2) \otimes x_3 \\ \alpha \downarrow & & \uparrow \alpha \\ (x_1 \otimes x_2) \otimes x_3 & = & x_1 \otimes (x_2 \otimes x_3) \end{array}$$

is a diagram built from  $\alpha$  which does not commute. An explicit example of this phenomenon is provided by the small skeleton of the category of sets with monoidal structure induced

from the Cartesian product. In this monoidal category the countable set  $\mathbb{N}$  has the property that  $\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) = (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$  but the associator is not the identity (see [Mac98, page 164]).

**Remark 2.2.10.** Despite these caveats, one form of Mac Lane’s theorem states that any monoidal category is monoidally equivalent (see Definition 2.2.40 below) to a *strict* monoidal category—one in which  $\alpha$ ,  $l$  and  $r$  are identities. By invoking this theorem, some authors make the blanket assumption that their monoidal categories are strict and thereby drop  $\alpha$ ,  $l$  and  $r$  from their consideration and notation. In the present work, we will typically omit associators from our diagrams but in doing so we do not intend to replace our category by a strict monoidal category; rather, our suppression of associators is just a notational convenience (or a notational abuse) used to keep our diagrams to a digestible size. All the diagrams can be inflated and reworked to include the missing associators.

**Notation 2.2.11.** While on the topic of notational abuses, let us agree to write  $(\mathcal{C}, \otimes, \mathbb{1})$  for a monoidal category rather than the exhausting  $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ .

**Definition 2.2.12.** A *symmetric monoidal category* is a monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  equipped with a natural isomorphism  $\tau_{a,b} : a \otimes b \xrightarrow{\sim} b \otimes a$  called the symmetry such that

$$\begin{array}{ccc} a \otimes \mathbb{1} & \xrightarrow{\tau_{a,\mathbb{1}}} & \mathbb{1} \otimes a \\ & \searrow r_a \quad \swarrow l_a & \\ & a & \end{array} \quad (2.2.13)$$

$$\begin{array}{ccc} a \otimes b & \xrightarrow{\text{id}} & a \otimes b \\ & \searrow \tau_{a,b} \quad \swarrow \tau_{b,a} & \\ & b \otimes a & \end{array} \quad (2.2.14)$$

$$\begin{array}{ccccc} (a \otimes b) \otimes c & \xrightarrow{\alpha^{-1}} & a \otimes (b \otimes c) & \xrightarrow{\alpha \otimes \tau_{b,c}} & a \otimes (c \otimes b) \\ \downarrow \tau_{a \otimes b, c} & & & & \downarrow \alpha \\ c \otimes (a \otimes b) & \xrightarrow{\alpha} & (c \otimes a) \otimes b & \xrightarrow{\tau_{c,a} \otimes b} & (a \otimes c) \otimes b \end{array} \quad (2.2.15)$$

commute for any  $a, b, c \in \mathcal{C}$ .

*Remark 2.2.16.* Symmetric monoidal categories also admit a coherence theorem (see [Mac98, §XI.1] for a precise statement) but in this context it is particularly important to appreciate the “formal” nature of the result. For example, there is no reason to expect that a diagram like

$$\begin{array}{ccc} x \otimes (x \otimes x) & \xrightarrow{1 \otimes \tau} & x \otimes (x \otimes x) \\ \downarrow \alpha & & \downarrow \alpha \\ (x \otimes x) \otimes x & \xrightarrow{\tau \otimes 1} & (x \otimes x) \otimes x \end{array}$$

should commute. This diagram is not “formal” because it does not make sense if the three objects are “formally distinct” (cf. Remark 2.2.9) and can only be formed because certain equalities serendipitously happen to hold. For example, compare the above diagram with the following one:

$$\begin{array}{ccc} x_1 \otimes (x_2 \otimes x_3) & \xrightarrow{1 \otimes \tau} & x_1 \otimes (x_3 \otimes x_2) \\ \downarrow \alpha & & \downarrow ? \\ (x_1 \otimes x_2) \otimes x_3 & \xrightarrow{\tau \otimes 1} & (x_2 \otimes x_1) \otimes x_3. \end{array}$$

Nevertheless, the coherence theorem for symmetric monoidal categories ensures that given a permutation  $\sigma \in S_n$  and a collection of objects  $x_1, \dots, x_n$  there is a unique isomorphism  $x_1 \otimes \dots \otimes x_n \xrightarrow{\sim} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$  constructed using the symmetry (i.e., constructed using “transpositions”) which effects the permutation  $\sigma$  on the  $\otimes$ -factors. (Note that we have dropped associators in accordance with Remark 2.2.10.) This basic fact about symmetric monoidal categories is used implicitly in the work of Chapter 5. As a consequence, it is not immediately clear how much of the theory of higher comparison maps (if any) might work in the more general setting of braided monoidal categories [JS93]; the author has not given this point serious consideration. (In a braided monoidal category it is the braid group  $B_n$  which acts on an  $n$ -fold tensor product rather than the symmetric group  $S_n$ .)

*Remark 2.2.17.* If  $(\mathcal{C}, \otimes, \mathbb{1})$  is a monoidal category then the opposite category  $\mathcal{C}^{\text{op}}$  inherits a monoidal structure  $(\mathcal{C}^{\text{op}}, \hat{\otimes}, \mathbb{1})$  where the tensor product is taking in the opposite order:

$X \hat{\otimes} Y := Y \otimes X$ . If  $\mathcal{C}$  is symmetric then we can just take  $X \hat{\otimes} Y := X \otimes Y$ .

**Definition 2.2.18.** A *monoid*  $(A, \mu, \eta)$  in a monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  is an object  $A \in \mathcal{C}$  equipped with maps  $\mu : A \otimes A \rightarrow A$  and  $\eta : \mathbb{1} \rightarrow A$  such that

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\ \downarrow 1 \otimes \mu & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} A \otimes \mathbb{1} & \xrightarrow{1 \otimes \eta} & A \otimes A & \xleftarrow{\eta \otimes 1} & \mathbb{1} \otimes A \\ & \searrow r_A & \downarrow \mu & \swarrow l_A & \\ & & A & & \end{array}$$

commute. In other words,  $\mu$  is associative and  $\eta$  is a two-sided identity for  $\mu$ . If the monoidal category is symmetric then a monoid is said to be *commutative* if  $\mu = \mu \circ \tau_{A,A}$ .

*Example 2.2.19.* If  $R$  is a commutative ring then the category of  $R$ -modules forms a symmetric monoidal category  $(R\text{-Mod}, \otimes_R, R)$ . The (commutative) monoids in this category are the (commutative)  $R$ -algebras.

*Example 2.2.20.* If  $A$  is any ring then the category of  $A$ - $A$ -bimodules provides an example of a monoidal category  $(A\text{-Mod-}A, \otimes_A, A)$  which is not symmetric.

*Example 2.2.21.* If  $R$  is a commutative ring then the chain complexes of  $R$ -modules form a monoidal category  $(\text{Ch}(R), \otimes, R)$  with the usual tensor product of complexes:  $(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes_R Y_j$  with differential  $d(x \otimes y) := dx \otimes y + (-1)^{|x|} x \otimes dy$ . The unit is  $R$  regarded as a complex concentrated in degree zero. A monoid in  $\text{Ch}(R)$  is the same thing as a differential graded  $R$ -algebra. Similarly, the category of graded abelian groups  $\text{Ab}^{\mathbb{Z}}$  is a monoidal category whose monoids are the same thing as graded rings.

*Example 2.2.22.* The Cartesian product  $\times$  provides the category of sets with the structure of a monoidal category  $(\text{Sets}, \times, *)$  with a one-point set  $*$  serving as unit object. A monoid in this monoidal category is just a monoid in the ordinary sense: a set equipped with an associative, unital multiplication.

*Example 2.2.23.* Let  $G$  be a finite group and let  $k$  be a field. If  $M$  and  $N$  are two  $kG$ -modules then the  $k$ -linear tensor product  $M \otimes_k N$  inherits the structure of a  $kG$ -module by letting  $G$

act diagonally:  $g.(m \otimes n) = (g.m \otimes g.n)$ . With this tensor product the category of  $kG$ -modules forms a symmetric monoidal category  $(kG\text{-Mod}, \otimes_k, k)$ . A monoid in this category is the same thing as a  $k$ -algebra equipped with an action of  $G$  via algebra automorphisms.

*Remark 2.2.24.* A monoidal category can admit several symmetries turning it into a symmetric monoidal category. For example, the category of graded abelian groups  $\text{Ab}^{\mathbb{Z}}$  has two natural symmetries:  $x \otimes y \mapsto y \otimes x$  and  $x \otimes y \mapsto (-1)^{|x| \cdot |y|} y \otimes x$ . With respect to the second symmetry, the commutative monoids are the graded rings which are “graded-commutative” while the commutative monoids with respect to the first symmetry are the graded rings which are genuinely commutative. We’ll discuss graded-commutative rings in more detail in Section 2.3.

*Remark 2.2.25.* Monoids in *additive* monoidal categories behave like rings. In such circumstances it is common to use the term “ring” or “algebra” instead of monoid. For example, monoids in the stable homotopy category of spectra (see Chapter 6) are called “ring spectra.”

**Definition 2.2.26.** Let  $(A, \mu, \eta)$  be a monoid in a monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$ . A *left  $A$ -module* is an object  $M \in \mathcal{C}$  equipped with a map  $\rho : A \otimes M \rightarrow M$  such that

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{1 \otimes \rho} & A \otimes M \\ \downarrow \mu \otimes 1 & & \downarrow \rho \\ A \otimes M & \xrightarrow{\rho} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{1} \otimes M & \xrightarrow{\eta \otimes 1} & A \otimes M \\ & \searrow l_M & \downarrow \rho \\ & & M \end{array}$$

commute. These are the natural analogues of the usual module axioms:  $(a_1 a_2).x = a_1.(a_2.x)$  and  $1.x = x$ . Every object  $X \in \mathcal{C}$  gives rise to a so-called “free  $A$ -module”  $A \otimes X$  with action  $\rho : A \otimes A \otimes X \xrightarrow{\mu \otimes 1} A \otimes X$ .

**Definition 2.2.27.** A *lax monoidal functor*  $(\mathcal{C}, \otimes, \mathbb{1}) \rightarrow (\mathcal{D}, \otimes, \mathbb{1})$  between monoidal categories consists of a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  together with a morphism  $\varphi_0 : \mathbb{1}_{\mathcal{D}} \rightarrow F(\mathbb{1}_{\mathcal{C}})$  and a natural

transformation  $\varphi_{a,b} : Fa \otimes Fb \rightarrow F(a \otimes b)$  such that

$$\begin{array}{ccc} Fa \otimes (Fb \otimes Fc) & \xrightarrow{\alpha} & (Fa \otimes Fb) \otimes Fc \xrightarrow{\varphi_{a,b} \otimes 1} F(a \otimes b) \otimes Fc \\ \downarrow 1 \otimes \varphi_{b,c} & & \downarrow \varphi_{a \otimes b, c} \\ Fa \otimes F(b \otimes c) & \xrightarrow{\varphi_{a,b \otimes c}} & F(a \otimes (b \otimes c)) \xrightarrow{F(\alpha)} F((a \otimes b) \otimes c) \end{array} \quad (2.2.28)$$

$$\begin{array}{ccc} Fa \otimes \mathbb{1} & \xrightarrow{1 \otimes \varphi_0} & Fa \otimes F\mathbb{1} \\ \downarrow r_{Fa} & & \downarrow \varphi_{a, \mathbb{1}} \\ Fa & \xleftarrow{F(r_a)} & F(a \otimes \mathbb{1}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{1} \otimes Fa & \xrightarrow{\varphi_0 \otimes 1} & F\mathbb{1} \otimes Fa \\ \downarrow l_{Fa} & & \downarrow \varphi_{\mathbb{1}, a} \\ Fa & \xleftarrow{F(l_a)} & F(\mathbb{1} \otimes a) \end{array} \quad (2.2.29)$$

commute for any  $a, b, c \in \mathcal{C}$ . A *strong monoidal functor* is a lax monoidal functor with the property that  $\varphi_{a,b}$  and  $\varphi_0$  are isomorphisms. A (lax or strong) *symmetric monoidal functor* is a (lax or strong) monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between symmetric monoidal categories such that

$$\begin{array}{ccc} Fa \otimes Fb & \xrightarrow{\varphi_{a,b}} & F(a \otimes b) \\ \downarrow \tau_{Fa, Fb} & & \downarrow F(\tau_{a,b}) \\ Fb \otimes Fa & \xrightarrow{\varphi_{b,a}} & F(b \otimes a) \end{array} \quad (2.2.30)$$

commutes for each  $a, b \in \mathcal{C}$ .

*Remark 2.2.31.* A lax (symmetric) monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves (commutative) monoids. Indeed, if  $(A, \mu, \eta)$  is a (commutative) monoid in  $\mathcal{C}$  then the maps

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\varphi_0} F\mathbb{1}_{\mathcal{C}} \xrightarrow{F(\eta)} FA \quad \text{and} \quad FA \otimes FA \xrightarrow{\varphi_{A,A}} F(A \otimes A) \xrightarrow{F(\mu)} FA$$

give  $FA$  the structure of a (commutative) monoid in  $\mathcal{D}$ . Similarly, if  $(X, \rho)$  is a left  $A$ -module then the map

$$FA \otimes FX \xrightarrow{\varphi_{A,X}} F(A \otimes X) \xrightarrow{F(\rho)} FX$$

gives  $FX$  the structure of a left  $FA$ -module in  $\mathcal{D}$ .

**Definition 2.2.32.** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two lax monoidal functors. A *monoidal natural transformation*  $\theta : F \rightarrow G$  is a natural transformation such that

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\varphi_0} & F\mathbb{1} \\ & \searrow \varphi_0 & \downarrow \theta \\ & & G\mathbb{1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Fa \otimes Fb & \xrightarrow{\theta_a \otimes \theta_b} & Ga \otimes Gb \\ \downarrow \varphi & & \downarrow \varphi \\ F(a \otimes b) & \xrightarrow{\theta_{a \otimes b}} & G(a \otimes b) \end{array}$$

commute for all  $a, b \in \mathcal{C}$ .

*Remark 2.2.33.* Monoidal categories, lax monoidal functors, and monoidal natural transformations form a 2-category  $\text{MonCat}$ . An “equivalence of monoidal categories” is best defined to be an equivalence in this 2-category but the next couple of results establish that this is the same thing as a strong monoidal functor that is an equivalence of the underlying categories (see Definition 2.2.40 below).

**Lemma 2.2.34.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a strong monoidal functor which admits a right adjoint. The right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$  inherits the structure of a lax monoidal functor such that the unit  $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$  and counit  $\epsilon : FG \rightarrow \text{id}_{\mathcal{D}}$  are monoidal natural transformations.*

*Proof.* Define  $Ga \otimes Gb \rightarrow G(a \otimes b)$  to be the natural map adjoint to

$$F(Ga \otimes Gb) \xrightarrow{\varphi^{-1}} FGa \otimes FGb \xrightarrow{\epsilon \otimes \epsilon} a \otimes b$$

and define  $\mathbb{1} \rightarrow G\mathbb{1}$  to be the map adjoint to  $F\mathbb{1} \xrightarrow{\varphi_0^{-1}} \mathbb{1}$ . It is a routine verification that these maps give  $G$  the structure of a lax monoidal functor in such a way that the unit and counit are monoidal natural transformations. This is a well-known fact although it is hard to find a proof in the literature. In any case, it follows from the more general results of [Kel74].  $\square$

**Corollary 2.2.35.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a strong monoidal functor that is an equivalence of categories. Then  $F$  is an equivalence in the 2-category  $\text{MonCat}$ .*

*Remark 2.2.36.* This tells us that a strong monoidal functor is an equivalence of monoidal categories (in the sense of being an equivalence in  $\text{MonCat}$ ) if and only if it is an equivalence of the underlying categories. However, since the 1-morphisms in  $\text{MonCat}$  are the lax monoidal functors, we need the following lemma to complete the picture:

**Lemma 2.2.37.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a lax monoidal functor that is an equivalence in the 2-category  $\text{MonCat}$ . Then  $F$  is a strong monoidal functor.*

*Proof.* According to our hypotheses there exists a lax monoidal functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  together with monoidal natural isomorphisms  $\eta : \text{id}_{\mathcal{C}} \xrightarrow{\sim} GF$  and  $\epsilon : FG \xrightarrow{\sim} \text{id}_{\mathcal{D}}$ . The fact that these natural isomorphisms are monoidal (cf. Definition 2.2.32) implies that

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\varphi_0} F\mathbb{1} & \xrightarrow{F\varphi_0} FG\mathbb{1} \\ & \searrow \text{id} & \downarrow \epsilon_{\mathbb{1}} \\ & & \mathbb{1} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\varphi_0} G\mathbb{1} & \xrightarrow{G\varphi_0} GF\mathbb{1} \\ \downarrow \text{id} & \nearrow \eta_{\mathbb{1}} & \\ \mathbb{1} & & \end{array} \quad (2.2.38)$$

commute. The first diagram provides a left inverse for  $\varphi_0 : \mathbb{1} \rightarrow F\mathbb{1}$  and the following diagram demonstrates that it is also a right inverse:

$$\begin{array}{ccccc} F\mathbb{1} & \xrightarrow{F\varphi_0} & FG\mathbb{1} & \xrightarrow{\epsilon_{\mathbb{1}}} & \mathbb{1} \xrightarrow{\varphi_0} F\mathbb{1} \\ & \searrow F\eta_{\mathbb{1}} & \downarrow FG\varphi_0 & \nearrow \epsilon_{F\mathbb{1}} & \\ & & FGF\mathbb{1} & & \end{array}$$

Here the left triangle is  $F$  applied to the second diagram in (2.2.38) and the right triangle commutes by the naturality of  $\epsilon$ . It remains to check that  $\varphi : Fa \otimes Fb \rightarrow F(a \otimes b)$  is an isomorphism. Since  $G$  is an ordinary equivalence of categories, it suffices to check that  $G\varphi : G(Fa \otimes Fb) \rightarrow GF(a \otimes b)$  is an isomorphism. The fact that  $\eta$  and  $\epsilon$  are monoidal (cf. Definition 2.2.32) implies that

$$\begin{array}{ccc} & GF(a \otimes b) & \\ \eta_{a \otimes b} \nearrow & \uparrow G\varphi & \\ a \otimes b & G(Fa \otimes Fb) & \\ \eta_a \otimes \eta_b \searrow & \uparrow \varphi & \\ & GFa \otimes GFb & \end{array} \quad \text{and} \quad \begin{array}{ccc} FG(a \otimes b) & & \\ F\varphi \uparrow & \epsilon_{a \otimes b} \searrow & \\ F(Ga \otimes Gb) & & a \otimes b \\ \varphi \uparrow & \nearrow \epsilon_a \otimes \epsilon_b & \\ FGa \otimes FGb & & \end{array} \quad (2.2.39)$$

commute. The left diagram shows that

$$GF(a \otimes b) \xrightarrow{\eta_{a \otimes b}^{-1}} a \otimes b \xrightarrow{\eta_a \otimes \eta_b} GFa \otimes GFb \xrightarrow{\varphi} G(Fa \otimes Fb)$$

is a right inverse for  $G\varphi : G(Fa \otimes Fb) \rightarrow GF(a \otimes b)$  and it remains to show that it is also a left inverse. Since  $F$  is an equivalence we need only show this after applying  $F$ . To this end,

consider the following commutative diagram:

$$\begin{array}{ccccccc}
 FG(Fa \otimes Fb) & \xrightarrow{FG\varphi} & FGF(a \otimes b) & \xrightarrow{F\eta^{-1}} & F(a \otimes b) & \xrightarrow{F(\eta \otimes \eta)} & F(GFa \otimes GFb) \xrightarrow{F\varphi} FG(Fa \otimes Fb) \\
 \uparrow F\varphi & & \searrow \epsilon & & \parallel & & \parallel \\
 & & & & F(a \otimes b) & \xrightarrow{F(\eta \otimes \eta)} & F(GFa \otimes GFb) \\
 & & \nearrow \varphi & & & & \uparrow F\varphi \\
 F(GFa \otimes GFb) & & Fa \otimes Fb & & & & \\
 \uparrow \varphi & \nearrow \epsilon \otimes \epsilon & & \searrow F\eta \otimes F\eta & & & \uparrow \varphi \\
 FGFa \otimes FGFb & \xrightarrow{id} & & & FGFa \otimes FGFb & & 
 \end{array}$$

The top line is the morphism that we claim is the identity. This follows from the commutativity of the diagram since the morphism

$$FGFa \otimes FGFb \xrightarrow{\varphi} F(GFa \otimes GFb) \xrightarrow{F\varphi} FG(Fa \otimes Fb)$$

which forms the left and right sides of the diagram is an isomorphism by (2.2.39).  $\square$

**Definition 2.2.40.** An *equivalence of monoidal categories* is a strong monoidal functor that is an equivalence of the underlying categories. By Corollary 2.2.35 and Lemma 2.2.37 this is the same thing as an equivalence in the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations. In fact, these results also show that it is the same thing as an equivalence in the 2-category of monoidal categories, strong monoidal functors, and monoidal natural transformations.

*Remark 2.2.41.* Similarly, an *equivalence of symmetric monoidal categories* is a strong symmetric monoidal functor that is an equivalence of the underlying categories. As above this is the same thing as an equivalence in the analogous 2-categories of symmetric monoidal categories (with lax or strong symmetric monoidal functors and monoidal natural transformations). The only point to check is that if the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  in Lemma 2.2.34 is a strong *symmetric* monoidal functor then the induced lax monoidal structure on its right adjoint is also symmetric monoidal.

**Proposition 2.2.42.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an equivalence of categories and suppose that  $\mathcal{C}$  has the structure of a (symmetric) monoidal category. Then  $\mathcal{D}$  admits the structure of a (symmetric) monoidal category such that  $F$  is an equivalence of (symmetric) monoidal categories.*

*Proof.* Choose a quasi-inverse  $G : \mathcal{D} \rightarrow \mathcal{C}$  with natural isomorphisms  $\eta : \text{id}_{\mathcal{C}} \xrightarrow{\sim} GF$  and  $\epsilon : FG \xrightarrow{\sim} \text{id}_{\mathcal{D}}$ . Define the bifunctor  $- \wedge - : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$  to be the composite

$$\mathcal{D} \times \mathcal{D} \xrightarrow{G \times G} \mathcal{C} \times \mathcal{C} \xrightarrow{- \otimes -} \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

In other words,  $x \wedge y := F(Gx \otimes Gy)$ . Let  $\mathbb{1}_{\mathcal{D}} := F\mathbb{1}_{\mathcal{C}}$  and define natural “unitor” maps

$$\mathbb{1}_{\mathcal{D}} \wedge x = F(GF\mathbb{1}_{\mathcal{C}} \otimes Gx) \xrightarrow{F(\eta_1^{-1} \otimes 1)} F(\mathbb{1} \otimes Gx) \xrightarrow{F(l_{Gx})} FGx \xrightarrow{\epsilon_x} x$$

and

$$x \wedge \mathbb{1}_{\mathcal{D}} = F(Gx \otimes GF\mathbb{1}_{\mathcal{C}}) \xrightarrow{F(1 \otimes \eta_1^{-1})} F(Gx \otimes \mathbb{1}) \xrightarrow{F(r_{Gx})} FGx \xrightarrow{\epsilon_x} x.$$

Furthermore, define a natural “associator” map  $x \wedge (y \wedge z) \rightarrow (x \wedge y) \wedge z$  by

$$\begin{aligned} F(Gx \otimes GF(Gy \otimes Gz)) &\xrightarrow{F(1 \otimes \eta^{-1})} F(Gx \otimes (Gy \otimes Gz)) \\ &\downarrow F(\alpha) \\ F((Gx \otimes Gy) \otimes Gz) &\xrightarrow{F(\eta \otimes 1)} F(GF(Gx \otimes Gy) \otimes Gz). \end{aligned}$$

Finally, if  $\mathcal{C}$  is symmetric then define a natural “symmetry” map

$$x \wedge y = F(Gx \otimes Gy) \xrightarrow{F(\tau_{Gx, Gy})} F(Gy \otimes Gx) = y \wedge x.$$

The commutativity of the unit axiom (2.2.3) can be checked by a brute force expansion of the definitions—but the diagram is too large to be typeset on the page. Alternatively, we can observe the following. Note that the diagram we need to check is  $F$  applied to a diagram of the following form

$$\begin{array}{ccc} Ga \otimes GF(-) & \xrightarrow{\quad - \quad} & GF(-) \otimes Gb. \\ & \searrow 1 \otimes - \quad \quad \quad \swarrow - \otimes 1 & \\ & Ga \otimes Gb & \end{array}$$

By precomposing this diagram with the appropriate unit morphism  $\eta$  we obtain a diagram

$$\begin{array}{ccc}
 Ga \otimes - & \xrightarrow{\quad - \quad} & - \otimes Gb \\
 \searrow 1 \otimes \eta & & \swarrow \eta \otimes 1 \\
 Ga \otimes GF(-) & \xrightarrow{\quad} & GF(-) \otimes Gb \\
 \searrow 1 \otimes - & \swarrow - \otimes 1 & \\
 & Ga \otimes Gb &
 \end{array}$$

and since the unit morphism is an isomorphism it suffices to check the commutativity of this “larger” diagram. However, one readily sees from the definitions that this larger diagram is merely

$$\begin{array}{ccc}
 Ga \otimes (GF1 \otimes Gb) & \xrightarrow{\quad \alpha \quad} & (Ga \otimes GF1) \otimes Gb \\
 \downarrow 1 \otimes (\eta^{-1} \otimes 1) & & \downarrow (1 \otimes \eta^{-1}) \otimes 1 \\
 Ga \otimes (1 \otimes Gb) & \xrightarrow{\quad \alpha \quad} & (Ga \otimes 1) \otimes Gb \\
 \searrow 1 \otimes l & \swarrow r \otimes 1 & \\
 & Ga \otimes Gb &
 \end{array}$$

which evidently commutes. A similar angle of approach can be used to dispose of the hexagon axiom (2.2.2). By peeling away the outer layer of tensor products, observe that the diagram we need to check is  $F$  applied to a diagram like the inner rectangle of the following diagram:

$$\begin{array}{ccccccc}
 & & id & \nearrow & Ga \otimes - & \longrightarrow & (Ga \otimes Gb) \otimes (Gc \otimes Gd) & \longrightarrow & - \otimes Gd & \xleftarrow{id} & - \otimes Gd & & \\
 & & & & \uparrow 1 \otimes \eta^{-1} & & \downarrow \eta \otimes \eta & & \downarrow \eta \otimes 1 & & \uparrow \eta \otimes 1 & & \\
 Ga \otimes - & \xrightarrow{1 \otimes \eta} & Ga \otimes GF(-) & \longrightarrow & GF(-) \otimes GF(-) & \longrightarrow & GF(-) \otimes Gd & \xleftarrow{\eta \otimes 1} & - \otimes Gd & & & & \\
 \downarrow 1 \otimes - & & \downarrow 1 \otimes GF(-) & & & & \uparrow GF(-) \otimes 1 & & \uparrow \eta \otimes 1 & & \downarrow - \otimes 1 & & \\
 Ga \otimes - & \xrightarrow{1 \otimes \eta} & Ga \otimes GF(-) & \longrightarrow & GF(-) \otimes Gd & \xleftarrow{\eta \otimes 1} & - \otimes Gd & & & & & & \\
 & & \downarrow 1 \otimes \eta^{-1} & & & & \uparrow \eta \otimes 1 & & \uparrow id & & & & \\
 & & id & \searrow & Ga \otimes - & \longrightarrow & - \otimes Gd & & & & & &
 \end{array}$$

As above it suffices to check that the larger diagram commutes. By looking at the definitions

one sees fairly immediately that this “larger” diagram is actually just the following

$$\begin{array}{ccccc}
Ga \otimes (Gb \otimes (Gc \otimes Gd)) & \xrightarrow{\alpha} & (Ga \otimes Gb) \otimes (Gc \otimes Gd) & \xrightarrow{\alpha} & ((Ga \otimes Gb) \otimes Gc) \otimes Gd \\
\downarrow 1 \otimes \alpha & & & & \uparrow \alpha \otimes 1 \\
Ga \otimes ((Gb \otimes Gc) \otimes Gd) & \xrightarrow{\alpha} & (Ga \otimes (Gb \otimes Gc)) \otimes Gd. & & 
\end{array}$$

Finally, suppose that  $\mathcal{C}$  is symmetric. The commutativity of (2.2.13) is shown by the following

$$\begin{array}{ccc}
F(Ga \otimes GF\mathbb{1}_{\mathcal{C}}) & \xrightarrow{F(\tau)} & F(GF\mathbb{1}_{\mathcal{C}} \otimes Ga) \\
\downarrow F(1 \otimes \eta^{-1}) & & \downarrow F(\eta^{-1} \otimes 1) \\
F(Ga \otimes \mathbb{1}_{\mathcal{C}}) & \xrightarrow{F(\tau)} & F(\mathbb{1}_{\mathcal{C}} \otimes Ga) \\
\searrow F(r_{Ga}) & & \swarrow F(l_{Ga}) \\
& FGa & \\
\downarrow \epsilon_a & & \\
a & & 
\end{array}$$

while the commutativity of (2.2.14) is immediate. Finally, observe that diagram (2.2.15) is

$F$  applied to a diagram like the inner rectangle of the following diagram

$$\begin{array}{ccccc}
- \otimes Gc & \xleftarrow{-} & Ga \otimes - & \xrightarrow{1 \otimes -} & Ga \otimes - \\
\downarrow \eta \otimes 1 & & \downarrow 1 \otimes \eta & & \downarrow \eta \otimes 1 \\
GF(-) \otimes Gc & \xleftarrow{-} & Ga \otimes GF(-) & \xrightarrow{1 \otimes GF(-)} & Ga \otimes GF(-) \\
\downarrow \tau & & & & \downarrow - \\
Ga \otimes GF(-) & \xrightarrow{-} & GF(-) \otimes Gb & \xrightarrow{GF(-) \otimes 1} & GF(-) \otimes Gb \\
\uparrow 1 \otimes \eta & & \uparrow \eta \otimes 1 & & \uparrow 1 \otimes \eta \\
Ga \otimes - & \xrightarrow{-} & - \otimes Gb & \xrightarrow{- \otimes 1} & - \otimes Gb
\end{array}$$

and it suffices to check that the outer diagram commutes. We see from the definitions that

the outer diagram is precisely

$$\begin{array}{ccccc}
(Ga \otimes Gb) \otimes Gc & \xleftarrow{\alpha} & Ga \otimes (Gb \otimes Gc) & \xrightarrow{1 \otimes \tau} & Ga \otimes (Gc \otimes Gb) \\
\downarrow \tau & & & & \downarrow \alpha \\
Gc \otimes (Ga \otimes Gb) & \xrightarrow{\alpha} & (Gc \otimes Ga) \otimes Gb & \xrightarrow{\tau \otimes 1} & (Ga \otimes Gc) \otimes Gb.
\end{array}$$

This establishes that  $\mathcal{D}$  has the structure of a (symmetric) monoidal category. Next we wish

to show that  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a strong (symmetric) monoidal functor (cf. Definition 2.2.27). To

this end, define  $\varphi_0 : \mathbb{1}_{\mathcal{D}} \rightarrow F\mathbb{1}_{\mathcal{C}}$  to simply be the identity  $\text{id}_{F\mathbb{1}_{\mathcal{C}}}$  and define a natural isomorphism  $\varphi : Fx \wedge Fy \xrightarrow{\sim} F(x \otimes y)$  by

$$F(GFx \otimes GFy) \xrightarrow{F(\eta_x^{-1} \otimes \eta_y^{-1})} F(x \otimes y).$$

The commutativity of the left diagram in (2.2.29) is given by the following diagram

$$\begin{array}{ccc} F(GFa \otimes GF\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\text{id}} & F(GFa \otimes GF\mathbb{1}_{\mathcal{C}}) \\ \downarrow F(1 \otimes \eta^{-1}) & & \downarrow F(\eta^{-1} \otimes \eta^{-1}) \\ F(GFa \otimes \mathbb{1}_{\mathcal{C}}) & \xrightarrow{F(\eta^{-1} \otimes 1)} & F(a \otimes \mathbb{1}_{\mathcal{C}}) \\ \downarrow F(r) & & \downarrow F(r) \\ FGFa & \xrightarrow{\epsilon_{Fa} = F(\eta_a)^{-1}} & Fa \end{array}$$

and the commutativity of the right diagram in (2.2.29) is similar. Next we check the commutativity of (2.2.28). Using the same technique as earlier, we see that the diagram is  $F$  applied to the middle rectangle of

$$\begin{array}{ccccc} GFa \otimes (-) & \xrightarrow{(-)} & - \otimes GFc & \xrightarrow{(-) \otimes 1} & - \otimes GFc \\ \downarrow 1 \otimes \eta & & \downarrow \eta \otimes 1 & & \downarrow \eta \otimes 1 \\ GFa \otimes GF(-) & \xrightarrow{(-)} & GF(-) \otimes GFc & \xrightarrow{GF(-) \otimes 1} & GF(-) \otimes GFc \\ \downarrow 1 \otimes GF(-) & & \downarrow 1 \otimes GF(-) & & \downarrow (-) \\ GFa \otimes GF(-) & \xrightarrow{(-)} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \\ \uparrow 1 \otimes \eta & & \parallel & & \parallel \\ GFa \otimes (-) & \xrightarrow{(-)} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \end{array}$$

and from the definitions we see that the outline of the diagram is

$$\begin{array}{ccccc} GFa \otimes (GFb \otimes GFc) & \xrightarrow{\alpha} & (GFa \otimes GFb) \otimes GFc & \xrightarrow{(\eta^{-1} \otimes \eta^{-1}) \otimes 1} & (a \otimes b) \otimes GFc \\ \downarrow 1 \otimes (\eta^{-1} \otimes \eta^{-1}) & & & & \downarrow 1 \otimes \eta^{-1} \\ GFa \otimes (b \otimes c) & \xrightarrow{\eta^{-1} \otimes 1} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \end{array}$$

which commutes by the naturality of  $\alpha$ . Finally, in the symmetric case the commutativity of (2.2.30) is immediate from the definitions. This completes the proof. The author has proved this proposition partly to exercise his muscles—but it does actually turn up; e.g., see Remark 3.7.6. □

**Definition 2.2.43.** A *closed symmetric monoidal category* is a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  equipped with a functor  $F(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , called “internal hom,” and an isomorphism

$$\text{Hom}_{\mathcal{C}}(a \otimes b, c) \simeq \text{Hom}_{\mathcal{C}}(a, F(b, c))$$

natural in all three variables  $a, b, c \in \mathcal{C}$ .

*Example 2.2.44.* Let  $R$  be a commutative ring. The set of  $R$ -linear maps  $\text{Hom}_R(X, Y)$  between any two  $R$ -modules is itself an  $R$ -module and provides an internal hom for the symmetric monoidal category  $(R\text{-Mod}, \otimes_R, R)$ . A more interesting example is provided by the category of chain complexes of  $R$ -modules  $\text{Ch}(R)$ . The internal hom is given by a complex  $F(X, Y)$  whose component in degree  $n$  is  $F(X, Y)_n = \prod_{i \in \mathbb{Z}} \text{Hom}_R(X_i, Y_{i+n})$  with differential defined by  $(df)(x) = d(fx) - (-1)^{|f|} f(dx)$ . The zero cycles of this complex give the ordinary maps of chain complexes, while the zeroth homology gives the homotopy classes of maps of chain complexes:

$$\text{Hom}_{\text{Ch}(R)}(X, Y) = Z_0(F(X, Y)) \quad \text{and} \quad \text{Hom}_{K(R)}(X, Y) = H_0(F(X, Y)).$$

*Remark 2.2.45.* A closed symmetric monoidal category is essentially the same thing as a symmetric monoidal category with the property that  $- \otimes b$  has a right adjoint for each  $b \in \mathcal{C}$ . Indeed, if we denote the right adjoint of  $- \otimes b$  by  $F(b, -) : \mathcal{C} \rightarrow \mathcal{C}$  then we have isomorphisms  $\text{Hom}_{\mathcal{C}}(a \otimes b, c) \simeq \text{Hom}_{\mathcal{C}}(a, F(b, c))$  which are natural in  $a$  and  $c$ . However, it is a basic fact from category theory (see [Mac98, §IV.7, Thm. 3]) that the functors  $F(b, -)$  extend uniquely to a bifunctor  $F(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$  such that the above isomorphisms are also natural in  $b$ . Thus, if all the  $- \otimes b$  have right adjoints then we can produce the required internal hom. Any two such choices of internal hom will be canonically isomorphic so there is very little difference between these two points of view.

**Notation 2.2.46.** The unit and counit of the adjunction  $- \otimes b \dashv F(b, -)$  will be denoted by

$$\text{coev} : a \rightarrow F(b, a \otimes b) \quad \text{and} \quad \text{ev} : F(b, a) \otimes b \rightarrow a$$

and called “coevaluation” and “evaluation,” respectively.

**Definition 2.2.47.** An object  $X$  in a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  is said to be *dualizable* if there exists an object  $DX$  and morphisms  $\eta : \mathbb{1} \rightarrow DX \otimes X$  and  $\epsilon : X \otimes DX \rightarrow \mathbb{1}$  such that

$$X \simeq X \otimes \mathbb{1} \xrightarrow{1 \otimes \eta} X \otimes DX \otimes X \xrightarrow{\epsilon \otimes 1} \mathbb{1} \otimes X \simeq X \quad (2.2.48)$$

and

$$DX \simeq \mathbb{1} \otimes DX \xrightarrow{\eta \otimes 1} DX \otimes X \otimes DX \xrightarrow{1 \otimes \epsilon} DX \otimes \mathbb{1} \simeq DX \quad (2.2.49)$$

are the identity morphisms. This is equivalent to saying that there is an object  $DX$  such that  $DX \otimes -$  is right adjoint to  $- \otimes X$ .

*Remark 2.2.50.* Dualizability is a kind of finiteness condition. For example, the dualizable objects in the category of vector spaces over a field  $k$  are precisely the finite-dimensional vector spaces. More generally, the dualizable objects in  $R\text{-Mod}$  for a commutative ring  $R$  are the finitely generated projective modules [DP80, Example 1.4] and a complex of  $R$ -modules is dualizable in  $\text{Ch}(R)$  if and only if it is a bounded complex of finitely generated projective modules [DP80, Example 1.5].

**Lemma 2.2.51.** *If  $(D_1X, \eta_1, \epsilon_1)$  and  $(D_2X, \eta_2, \epsilon_2)$  are two duals of an object  $X$  then the map  $\theta : D_1X \rightarrow D_2X$  defined by*

$$D_1X \simeq \mathbb{1} \otimes D_1X \xrightarrow{\eta_2 \otimes 1} D_2X \otimes X \otimes D_1X \xrightarrow{1 \otimes \epsilon_1} D_2X \otimes \mathbb{1} \simeq D_2X$$

*is an isomorphism with inverse  $\theta^{-1} : D_2X \rightarrow D_1X$  given by*

$$D_2X \simeq \mathbb{1} \otimes D_2X \xrightarrow{\eta_1 \otimes 1} D_1X \otimes X \otimes D_2X \xrightarrow{1 \otimes \epsilon_2} D_1X \otimes \mathbb{1} \simeq D_1X.$$

*Proof.* The following diagram demonstrates that  $\theta^{-1} \circ \theta = \text{id}_{D_1X}$ :

$$\begin{array}{ccccc}
D_1X & \xrightarrow{\eta_2 \otimes 1} & D_2X \otimes X \otimes D_1X & \xrightarrow{1 \otimes \epsilon_1} & D_2X \\
\downarrow \eta_1 \otimes 1 & & \downarrow \eta_1 \otimes 1 & & \downarrow \eta_1 \otimes 1 \\
D_1X \otimes X \otimes D_1X & \xrightarrow{1 \otimes 1 \otimes \eta_2 \otimes 1} & D_1X \otimes X \otimes D_2X \otimes X \otimes D_1X & \xrightarrow{1 \otimes 1 \otimes 1 \otimes \epsilon_1} & D_1X \otimes X \otimes D_2X \\
& \searrow \text{id} & \downarrow 1 \otimes \epsilon_2 \otimes 1 \otimes 1 & & \downarrow 1 \otimes \epsilon_2 \\
& & D_1X \otimes X \otimes D_1X & \xrightarrow{1 \otimes \epsilon_1} & D_1X.
\end{array}$$

A similar diagram establishes that  $\theta \circ \theta^{-1} = \text{id}_{D_2X}$ . □

*Remark 2.2.52.* Any two duals of  $X$  are thus *canonically* isomorphic *provided* we remember the structure maps.

*Remark 2.2.53.* The uniqueness of duals can of course be interpreted in terms of  $D_1X \otimes -$  and  $D_2X \otimes -$  both being right adjoints of  $- \otimes X$ .

*Remark 2.2.54.* If  $X$  is dualizable then its dual  $DX$  is itself dualizable; in fact,  $X$  is a dual of  $DX$ . Indeed, the maps

$$1 \xrightarrow{\eta_X} DX \otimes X \simeq X \otimes DX \quad \text{and} \quad DX \otimes X \simeq X \otimes DX \xrightarrow{\epsilon_X} 1$$

provide the relevant structure maps. It follows from Lemma 2.2.51 that if  $D^2X$  denotes any dual of  $DX$  then there is an isomorphism  $\theta_X : X \xrightarrow{\sim} D^2X$  given by

$$X \simeq 1 \otimes X \xrightarrow{\eta_{DX}} D^2X \otimes DX \otimes X \xrightarrow{1 \otimes \tau} D^2X \otimes X \otimes DX \xrightarrow{1 \otimes \epsilon_X} D^2X \otimes 1 \simeq D^2X \quad (2.2.55)$$

with inverse  $\theta_X^{-1} : D^2X \xrightarrow{\sim} X$  given by

$$D^2X \simeq 1 \otimes D^2X \xrightarrow{\eta_X \otimes 1} DX \otimes X \otimes D^2X \xrightarrow{\tau \otimes 1} X \otimes DX \otimes D^2X \xrightarrow{1 \otimes \epsilon_{DX}} X \otimes 1 \simeq X. \quad (2.2.56)$$

This will be used in the proof of the following proposition.

**Proposition 2.2.57.** *Let  $(\mathcal{C}, \otimes, 1)$  be a symmetric monoidal category. Every object is dualizable if and only if there exists a functor  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  and an isomorphism*

$$\text{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \text{Hom}_{\mathcal{C}}(X, DY \otimes Z) \quad (2.2.58)$$

natural in  $X$ ,  $Y$  and  $Z$ . The functor  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  is unique up to isomorphism and is an equivalence of categories. It is also a strong monoidal functor when  $\mathcal{C}^{\text{op}}$  is given the induced monoidal structure. In other words,  $D(X \otimes Y) \simeq DX \otimes DY$  and  $D\mathbb{1} \simeq \mathbb{1}$ .

*Proof.* If every object is dualizable then we can construct a functor  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  by choosing a dual  $DX$  for every object  $X \in \mathcal{C}$  and sending a morphism  $f : X \rightarrow Y$  to the morphism  $Df : DY \rightarrow DX$  given by

$$DY \simeq \mathbb{1} \otimes DY \xrightarrow{\eta_X \otimes 1} DX \otimes X \otimes DY \xrightarrow{1 \otimes f \otimes 1} DX \otimes Y \otimes DY \xrightarrow{1 \otimes \epsilon_Y} DX \otimes \mathbb{1} \simeq DX.$$

For example,  $D(\text{id}_X)$  is exactly the composite (2.2.49) which is  $\text{id}_{DX}$  by assumption—so  $D$  preserves identity morphisms. To show that  $D$  preserves composition, let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be a pair of composable morphisms and consider the following diagram:

$$\begin{array}{ccccccc}
DZ & \xrightarrow{\eta \otimes 1} & DY \otimes Y \otimes DZ & \xrightarrow{1 \otimes g \otimes 1} & DY \otimes Z \otimes DZ & \xrightarrow{1 \otimes \epsilon} & DY \\
\downarrow \eta \otimes 1 & & \downarrow \eta \otimes 1 \otimes 1 & & \downarrow \eta \otimes 1 & & \downarrow \eta_Z \otimes 1 \\
DX \otimes X \otimes DZ & \xrightarrow{1 \otimes \eta \otimes 1} & DX \otimes X \otimes DY \otimes Y \otimes DZ & \xrightarrow{1 \otimes g \otimes 1} & DX \otimes X \otimes DY \otimes Z \otimes DZ & \xrightarrow{1 \otimes \epsilon_Z} & DX \otimes X \otimes DY \\
\downarrow 1 \otimes f \otimes 1 & & \downarrow 1 \otimes f \otimes 1 & & \downarrow 1 \otimes f \otimes 1 & & \downarrow 1 \otimes f \otimes 1 \\
DX \otimes Y \otimes DZ & \xrightarrow{1 \otimes \eta \otimes 1} & DX \otimes Y \otimes DY \otimes Y \otimes DZ & \xrightarrow{1 \otimes g \otimes 1} & DX \otimes Y \otimes DY \otimes Z \otimes DZ & \xrightarrow{1 \otimes \epsilon_Z} & DX \otimes Y \otimes DY \\
& \searrow \text{id} & \downarrow 1 \otimes \epsilon \otimes 1 & & \downarrow 1 \otimes \epsilon_Y \otimes 1 & & \downarrow 1 \otimes \epsilon \\
& & DX \otimes Y \otimes DZ & \xrightarrow{1 \otimes g \otimes 1} & DX \otimes Z \otimes DZ & \xrightarrow{1 \otimes \epsilon} & DX
\end{array}$$

Going along the top and down the right-hand side we get  $D(f) \circ D(g)$ , while going down and along the bottom we get  $D(g \circ f)$ . This establishes the functoriality of our construction.

Next we wish to show that our functor  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  has the adjunction property (2.2.58). Given a map  $X \otimes Y \xrightarrow{f} Z$ , we can construct a map  $X \rightarrow DY \otimes Z$  by

$$X \simeq \mathbb{1} \otimes X \xrightarrow{\eta_Y \otimes 1} DY \otimes Y \otimes X \simeq DY \otimes X \otimes Y \xrightarrow{1 \otimes f} DY \otimes Z$$

and given a map  $X \xrightarrow{g} DY \otimes Z$  we can construct a map  $X \otimes Y \rightarrow Z$  by

$$X \otimes Y \simeq Y \otimes X \xrightarrow{1 \otimes g} Y \otimes DY \otimes Z \xrightarrow{\epsilon_Y \otimes 1} \mathbb{1} \otimes Z \simeq Z.$$

One readily checks that these constructions are inverse to each other. Moreover, checking that the resulting isomorphisms  $\text{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \text{Hom}_{\mathcal{C}}(X, DY \otimes Z)$  are natural in  $X, Y$  and  $Z$  is a straightforward expansion of the definitions. Note that the construction of  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  depends on our initial choice of duals  $DX$  for each object  $X \in \mathcal{C}$ . Nevertheless, any two functors  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  satisfying the adjunction property (2.2.58) are canonically isomorphic by formal nonsense.

Next we wish to show that our functor  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  is an equivalence of categories. By Remark 2.2.54, given any  $X \in \mathcal{C}$  there is a canonical isomorphism  $\theta_X : X \xrightarrow{\sim} DX$  which commutes with the structure maps. We just need to check that  $\theta_X : X \xrightarrow{\sim} D^2X$  is natural in  $X$ . From the definitions (2.2.55) and (2.2.56), one can check that the following diagram commutes

$$\begin{array}{ccccc}
1 & \xrightarrow{\eta_X} & DX \otimes X & \xrightarrow{1 \otimes \theta_X} & DX \otimes D^2X \\
\downarrow \eta_{DX} & & \downarrow \tau & & \downarrow \epsilon_{DX} \\
D^2X \otimes DX & \xleftarrow{\theta_X \otimes 1} & X \otimes DX & \xrightarrow{\epsilon_X} & 1
\end{array}$$

and this can be used to check the commutativity of the following monstrous diagram:

$$\begin{array}{ccccccc}
D^2X & \xrightarrow{D^2f} & & & & & D^2Y \\
\parallel & & & & & & \parallel \\
D^2X & \xrightarrow{\eta \otimes 1} & D^2Y \otimes DY \otimes D^2X & \xrightarrow{1 \otimes Df \otimes 1} & D^2Y \otimes DX \otimes D^2X & \xrightarrow{1 \otimes \epsilon} & D^2Y \\
\uparrow & & \uparrow \theta \otimes 1 \otimes \theta & & \uparrow \theta \otimes 1 \otimes \theta & & \uparrow \theta \\
X \otimes DY \otimes X & \xrightarrow{1 \otimes Df \otimes 1} & Y \otimes DX \otimes X & & & & \\
\downarrow \sim & & \downarrow \sim & & & & \\
X \otimes DY \otimes Y & \xrightarrow{1 \otimes Df \otimes 1} & X \otimes DX \otimes Y & & & & \\
\downarrow 1 \otimes \eta \otimes 1 \otimes 1 & & \downarrow 1 \otimes \epsilon \otimes 1 & & & & \\
X \otimes DX \otimes X \otimes DY \otimes Y & \xrightarrow{1 \otimes 1 \otimes f \otimes 1 \otimes 1} & X \otimes DX \otimes Y \otimes DY \otimes Y & & & & \\
\downarrow \epsilon \otimes 1 \otimes 1 \otimes 1 & & \downarrow \epsilon \otimes 1 \otimes 1 \otimes 1 & & & & \\
X \otimes DY \otimes Y & \xrightarrow{f \otimes 1 \otimes 1} & Y \otimes DY \otimes Y & & & & \\
\downarrow & & \downarrow & & & & \\
X & \xrightarrow{f} & Y & & & & 
\end{array}$$

Finally, one can check directly that  $DX \otimes DY$  is a dual of  $X \otimes Y$  and we obtain a canonical isomorphism  $D(X \otimes Y) \simeq DX \otimes DY$ . It is also trivial to check that  $D\mathbb{1} \simeq \mathbb{1}$ . The verification that these isomorphisms provide  $D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  with the structure of a strong monoidal functor is left to the reader.  $\square$

*Remark 2.2.59.* If  $\mathcal{C}$  is a *closed* symmetric monoidal category then there is a natural map

$$v : F(X, Y) \otimes Z \rightarrow F(X, Y \otimes Z) \quad (2.2.60)$$

defined for any three objects  $X, Y, Z \in \mathcal{C}$  which is adjoint to the map

$$F(X, Y) \otimes Z \otimes X \simeq F(X, Y) \otimes X \otimes Z \xrightarrow{\text{ev} \otimes 1} Y \otimes Z.$$

An object  $X$  is dualizable in the sense of Definition 2.2.47 if and only if  $v$  is an isomorphism for all  $Y$  and  $Z$ . In fact, it turns out that  $X$  is dualizable if and only if the single map  $F(X, \mathbb{1}) \otimes X \rightarrow F(X, X)$  is an isomorphism. These claims are discussed in [LMS86, §III.1] but we will provide a proof in Lemma 2.2.61 below. An object that is dualizable in this sense (that is, in the sense of Definition 2.2.47) is sometimes called “strongly dualizable” in the literature in order to differentiate it from other weaker notions. For example, in [DP80] an object  $X$  is said to have a “weak dual” if the functor  $\text{Hom}_{\mathcal{C}}(- \otimes X, \mathbb{1})$  is representable. In a closed symmetric monoidal category, every object has a weak dual—namely  $F(X, \mathbb{1})$ —but this object need not be a dual in the sense of Definition 2.2.47.

**Lemma 2.2.61.** *Let  $X$  be an object in a closed symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$ . The following are equivalent:*

- (1)  *$X$  is dualizable in the sense of Definition 2.2.47.*
- (2)  *$v : F(X, Y) \otimes Z \rightarrow F(X, Y \otimes Z)$  is an isomorphism for all  $Y, Z \in \mathcal{C}$ .*
- (3)  *$v : F(X, \mathbb{1}) \otimes X \rightarrow F(X, X)$  is an isomorphism.*

*Proof.* If  $X$  is dualizable then both  $DX \otimes -$  and  $F(X, -)$  are right adjoints of  $- \otimes X$ . Hence there is a natural isomorphism  $\alpha_W : DX \otimes W \xrightarrow{\sim} F(X, W)$  determined by the units and counits of the two adjunctions. One can check from the definitions that

$$\begin{array}{ccc} F(X, Y) \otimes Z & \xrightarrow{\nu} & F(X, Y \otimes Z) \\ \alpha_Y \otimes Z \uparrow & \nearrow \alpha_{Y \otimes Z} & \\ DX \otimes Y \otimes Z & & \end{array} \quad (2.2.62)$$

commutes and hence conclude that  $\nu$  is a natural isomorphism. Let us sketch the proof of this claim. Expanding the definitions of  $\alpha_{Y \otimes Z}$  and  $\nu$  we obtain the following diagram:

$$\begin{array}{ccccc} DX \otimes Y \otimes Z & \xrightarrow{\text{coev}} & F(X, DX \otimes Y \otimes Z \otimes X) & \xrightarrow{F(1, \tau)} & F(X, Y \otimes Z \otimes X \otimes DX) \\ \downarrow \alpha_Y \otimes 1 & & \downarrow F(1, \alpha_Y \otimes 1 \otimes 1) & & \downarrow F(1, 1 \otimes 1 \otimes \epsilon) \\ F(X, Y) \otimes Z & \xrightarrow{\text{coev}} & F(X, F(X, Y) \otimes Z \otimes X) & & F(X, Y \otimes Z \otimes \mathbb{1}) \\ & & \downarrow F(1, 1 \otimes \tau) & & \downarrow \sim \\ & & F(X, F(X, Y) \otimes X \otimes Z) & \xrightarrow{F(1, \text{ev} \otimes 1)} & F(X, Y \otimes Z). \end{array}$$

The commutativity of (2.2.62) thus reduces to the commutativity of

$$\begin{array}{ccc} DX \otimes Y \otimes Z \otimes X & \xrightarrow{\tau} & Y \otimes Z \otimes X \otimes DX \\ \downarrow \alpha_Y \otimes 1 \otimes 1 & & \downarrow 1 \otimes 1 \otimes \epsilon \\ F(X, Y) \otimes Z \otimes X & & Y \otimes Z \otimes \mathbb{1} \\ \downarrow 1 \otimes \tau & & \downarrow \sim \\ F(X, Y) \otimes X \otimes Z & \xrightarrow{\text{ev} \otimes 1} & Y \otimes Z \end{array}$$

and this can be verified using

$$\begin{array}{ccc} DX \otimes Y \otimes X & \xrightarrow{\text{id}} & DX \otimes Y \otimes X \\ \downarrow \text{coev} \otimes 1 & & \downarrow \tau \\ F(X, DX \otimes Y \otimes X) \otimes X & \xrightarrow{\text{ev}} & DX \otimes Y \otimes X \\ \downarrow F(1, \tau) \otimes 1 \otimes 1 & & \downarrow 1 \otimes \epsilon \\ F(X, Y \otimes X \otimes DX) \otimes X & & Y \otimes X \otimes DX \\ \downarrow F(1, 1 \otimes \epsilon) \otimes 1 & & \downarrow 1 \otimes \epsilon \\ F(X, Y \otimes \mathbb{1}) \otimes X & & Y \otimes \mathbb{1} \otimes Z \\ \downarrow \sim & & \downarrow \sim \\ F(X, Y) \otimes X & \xrightarrow{\text{ev}} & Y. \end{array}$$

$\alpha_Y \otimes 1$  (curved arrow from  $DX \otimes Y \otimes X$  to  $F(X, Y) \otimes X$ )

Modulo the minor details left to the reader, this establishes that (2.2.62) commutes and hence  $v : F(X, Y) \otimes Z \rightarrow F(X, Y \otimes Z)$  is an isomorphism. On the other hand, if  $F(X, \mathbb{1}) \otimes X \rightarrow F(X, X)$  is an isomorphism then we can use the inverse to define a map

$$\eta : \mathbb{1} \xrightarrow{\text{coev}} F(X, X) \xrightarrow{v^{-1}} F(X, \mathbb{1}) \otimes X$$

and one can check that together with the map

$$\epsilon : X \otimes F(X, \mathbb{1}) \simeq F(X, \mathbb{1}) \otimes X \xrightarrow{\text{ev}} \mathbb{1}$$

the pair  $(X, F(X, \mathbb{1}))$  satisfies Definition 2.2.47 so that  $X$  is dualizable with dual  $F(X, \mathbb{1})$ . For example, showing that

$$X \simeq X \otimes \mathbb{1} \rightarrow X \otimes F(X, X) \xrightarrow{1 \otimes v^{-1}} X \otimes F(X, \mathbb{1}) \otimes X \simeq F(X, \mathbb{1}) \otimes X \otimes X \xrightarrow{\text{ev} \otimes 1} \mathbb{1} \otimes X \simeq X$$

is the identity boils down to showing that

$$\begin{array}{ccc} F(X, \mathbb{1}) \otimes X \otimes X & \xrightarrow{1 \otimes \tau} & F(X, \mathbb{1}) \otimes X \otimes X \\ \downarrow v \otimes 1 & & \downarrow \text{ev} \otimes 1 \\ & & \mathbb{1} \otimes X \\ & & \downarrow \sim \\ F(X, X) \otimes X & \xrightarrow{\text{ev}} & X \end{array}$$

commutes. This can be shown using the definition of  $v$  and the naturality of evaluation:

$$\begin{array}{ccc} F(X, \mathbb{1}) \otimes X \otimes X & \xrightarrow{\text{id}} & F(X, \mathbb{1}) \otimes X \otimes X \\ \downarrow \text{coev} \otimes 1 & & \downarrow 1 \otimes \tau \\ F(X, F(X, \mathbb{1}) \otimes X \otimes X) \otimes X & \xrightarrow{\text{ev}} & F(X, \mathbb{1}) \otimes X \otimes X \\ \downarrow F(1, 1 \otimes \tau) \otimes 1 & & \downarrow \text{ev} \otimes 1 \\ F(X, F(X, \mathbb{1}) \otimes X \otimes X) \otimes X & \xrightarrow{\text{ev}} & F(X, \mathbb{1}) \otimes X \otimes X \\ \downarrow F(1, \text{ev} \otimes 1) \otimes 1 & & \downarrow \sim \\ F(X, \mathbb{1} \otimes X) \otimes X & \xrightarrow{\text{ev}} & \mathbb{1} \otimes X \\ \downarrow \sim & & \downarrow \sim \\ F(X, X) \otimes X & \xrightarrow{\text{ev}} & X. \end{array}$$

A curved arrow labeled  $v \otimes 1$  points from the top-left node  $F(X, \mathbb{1}) \otimes X \otimes X$  to the bottom-left node  $F(X, X) \otimes X$ .

Similar techniques can be used to establish that  $F(X, \mathbb{1}) \xrightarrow{\eta \otimes 1} F(X, \mathbb{1}) \otimes X \otimes F(X, \mathbb{1}) \xrightarrow{1 \otimes \epsilon} F(X, \mathbb{1})$  is the identity.  $\square$

*Remark 2.2.63.* If  $X$  is a dualizable object in a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  then the maps  $\mathbb{1} \xrightarrow{\eta} DX \otimes X$  and  $DX \otimes X \otimes DX \otimes X \xrightarrow{1 \otimes \epsilon \otimes 1} DX \otimes X$  provide  $DX \otimes X$  with the structure of a monoid in  $\mathcal{C}$ . If  $\mathcal{C}$  is closed symmetric monoidal then this monoid structure on  $DX \otimes X \simeq F(X, \mathbb{1}) \otimes X \simeq F(X, X)$  corresponds to the “endomorphism monoid” structure on  $F(X, X)$ .

## 2.3 Graded rings

All graded rings in this dissertation will be  $\mathbb{Z}$ -graded. Recall that the category of graded abelian groups  $\text{Ab}^{\mathbb{Z}}$  is a monoidal category and that graded rings are precisely the ring objects in this category (cf. Remark 2.2.25). Note that we take the “external” point of view in which a graded ring consists of a collection of abelian groups  $A_i$  equipped with multiplication maps  $A_i \otimes A_j \rightarrow A_{i+j}$ ; we never speak of the direct sum  $\bigoplus_{i \in \mathbb{Z}} A_i$  and only ever consider homogeneous elements.

*Remark 2.3.1.* There are two symmetries that turn the monoidal category  $\text{Ab}^{\mathbb{Z}}$  into a symmetric monoidal category. A commutative ring object with respect to one of the symmetries is a commutative graded ring while a commutative ring object with respect to the other symmetry is a “graded-commutative” graded ring: a graded ring satisfying  $x \cdot y = (-1)^{|x| \cdot |y|} y \cdot x$ . Much of commutative ring theory extends to (graded-)commutative graded rings, including

- (1) the theory of localization; and
- (2) the theory of the (homogeneous) spectrum.

Few references discuss these constructions for graded-commutative rings. It is true that one can get away with only commutative graded rings by the trick of squaring elements,

but it is conceptually cleaner and aesthetically nicer to just deal directly with the graded-commutative rings that arise in practice. One reference which explicitly discusses the theory for graded-commutative rings is [DS13, Section 2].

**Definition 2.3.2.** Let  $S$  be a multiplicative set ( $1 \in S$ ,  $S \cdot S \subset S$ ) of central homogeneous elements in a (graded-)commutative graded ring  $R^\bullet$ . The *localization*  $S^{-1}R^\bullet$  is defined by

$$(S^{-1}R^\bullet)^i := \left\{ \frac{r}{s} \mid \deg(r) - \deg(s) = i \right\}.$$

It is a (graded-)commutative graded ring in an obvious manner and the map  $R^\bullet \rightarrow S^{-1}R^\bullet$  which sends  $f$  to  $f/1$  is the universal graded ring homomorphism out of  $R^\bullet$  which inverts the elements of  $S$ .

*Remark 2.3.3.* Every element of even degree in a graded-commutative graded ring is central. If  $T$  is a multiplicative subset of (not necessarily central) homogeneous elements then  $S := \{\pm t^2 \mid t \in T\}$  is a multiplicative set of central homogeneous elements and the localization map  $R^\bullet \rightarrow S^{-1}R^\bullet$  is the universal graded ring homomorphism out of  $R^\bullet$  which inverts the elements of  $T$ . Therefore the assumption in Definition 2.3.2 that  $S$  consists of *central* homogeneous elements results in no real loss of generality.

**Definition 2.3.4.** Let  $R^\bullet$  be a graded ring. The *graded-center* of  $R^\bullet$  is defined by

$$\text{Center}_{\text{gr}}(R^\bullet)^i := \left\{ f \in R^i \mid f \cdot g = (-1)^{ij} g \cdot f \text{ for each } g \in R^j \text{ and } j \in \mathbb{Z} \right\}.$$

It is a graded-commutative graded subring of  $R^\bullet$ . Moreover,  $R^\bullet$  is graded-commutative if and only if  $R^\bullet = \text{Center}_{\text{gr}}(R^\bullet)$ .

*Remark 2.3.5.* Every one-sided homogeneous ideal in a graded-commutative graded ring is automatically two-sided. This is the fundamental observation which enables one to define the homogeneous spectrum of a graded-commutative graded ring.

**Definition 2.3.6.** The *homogeneous spectrum*  $\text{Spec}^h(R^\bullet)$  of a (graded-)commutative graded ring  $R^\bullet$  is the set of homogeneous prime ideals equipped with the Zariski topology. The closed sets are the sets  $V(I^\bullet) := \{\mathfrak{p}^\bullet \in \text{Spec}^h(R^\bullet) \mid \mathfrak{p}^\bullet \supset I^\bullet\}$  for each homogeneous ideal  $I^\bullet \subset R^\bullet$ . This construction is functorial with respect to the (graded-)commutative graded ring  $R^\bullet$ .

**Lemma 2.3.7.** *Let  $R^\bullet$  be a (graded-)commutative graded ring. There is a natural continuous surjective map  $(-)^0 : \text{Spec}^h(R^\bullet) \rightarrow \text{Spec}(R^0)$  which sends a homogeneous prime ideal  $\mathfrak{p}^\bullet$  to  $\mathfrak{p}^0 := \mathfrak{p} \cap R^0$ .*

*Proof.* It is straightforward to check that  $(-)^0$  is well-defined and continuous. In order to show that it is surjective, fix a prime ideal  $\mathfrak{p} \in \text{Spec}(R^0)$  and let  $S := R^0 \setminus \mathfrak{p}$ . This is a multiplicative set of central elements in  $R^\bullet$  and we see that  $(S^{-1}R^\bullet)^0 = S^{-1}R^0$ . Since the diagram

$$\begin{array}{ccc} \text{Spec}^h(R^\bullet) & \xrightarrow{(-)^0} & \text{Spec}(R^0) \\ \uparrow & & \uparrow \\ \text{Spec}^h(S^{-1}R^\bullet) & \xrightarrow{(-)^0} & \text{Spec}(S^{-1}R^0) \end{array}$$

commutes, we are thus reduced to the case when  $R^0$  is a local ring with maximal ideal  $\mathfrak{p}$ . In this case, consider the homogeneous ideal  $R^\bullet \mathfrak{p}$  of  $R^\bullet$ . It is proper and hence contained in a homogeneous prime ideal  $\mathfrak{q}^\bullet$ . One then readily checks that  $R^0 \cap \mathfrak{q}^\bullet = \mathfrak{p}$ . Indeed  $\mathfrak{p} \subset R^0 \cap \mathfrak{q}^\bullet$  since  $\mathfrak{q}^\bullet$  contains  $\mathfrak{p}$ , while  $R^0 \cap \mathfrak{q}^\bullet \subset \mathfrak{p}$  since  $\mathfrak{p}$  is the maximal ideal of the local ring  $R^0$ .  $\square$

**Lemma 2.3.8.** *Let  $R^\bullet$  be a (graded-)commutative graded ring. The homogeneous spectrum  $\text{Spec}^h(R^\bullet)$  is connected iff  $\text{Spec}(R^0)$  is connected.*

*Proof.* A decomposition of  $\text{Spec}(R^0)$  into a disjoint union of non-empty closed sets induces a similar decomposition of  $\text{Spec}^h(R^\bullet)$  via the surjective continuous map  $(-)^0 : \text{Spec}^h(R^\bullet) \rightarrow \text{Spec}(R^0)$  of Lemma 2.3.7. On the other hand, suppose  $\text{Spec}^h(R^\bullet) = V(I^\bullet) \sqcup V(J^\bullet)$  is a disjoint union of non-empty closed subsets. Without loss of generality we may assume that the

graded nilradical of  $R^\bullet$  (i.e., the intersection of all homogeneous prime ideals) is zero. In this case,  $\text{Spec}^h(R^\bullet) = V(I^\bullet) \cup V(J^\bullet) = V(I^\bullet J^\bullet)$  implies that  $I^\bullet J^\bullet = 0$ . On the other hand,  $\emptyset = V(I^\bullet) \cap V(J^\bullet) = V(I^\bullet + J^\bullet)$  implies that  $I^\bullet + J^\bullet = R^\bullet$  because every proper homogeneous ideal is contained in a homogeneous prime ideal. In particular,  $R^0 = I^0 + J^0$  and  $I^0 J^0 = 0$ . Thus,  $1 = x + y$  for some  $x \in I^0$  and  $y \in J^0$  and using the fact that  $I^0 J^0 = 0$  we see that  $x^2 = x$  and  $y^2 = y$ . We conclude that  $R^0$  possesses a pair of orthogonal idempotents  $x$  and  $y$ . They are non-trivial since otherwise either  $I^\bullet = R^\bullet$  or  $J^\bullet = R^\bullet$  in which case  $V(I^\bullet)$  or  $V(J^\bullet)$  would be empty; hence  $\text{Spec}(R^0)$  is disconnected.  $\square$

**Definition 2.3.9.** A (graded-)commutative graded ring is said to be *graded-noetherian* if it satisfies the ascending chain condition on homogeneous ideals; equivalently, if every homogeneous ideal is finitely generated.

*Remark 2.3.10.* The closed sets of  $\text{Spec}^h(R^\bullet)$  which have quasi-compact complement are those sets of the form  $V(I^\bullet)$  for a finitely generated homogeneous ideal  $I^\bullet$ ; for example, see [BKS07, Lemma 2.2]. Thus  $\text{Spec}^h(R^\bullet)$  is a noetherian topological space precisely when  $R^\bullet$  is a graded-noetherian ring (mirroring the situation for ungraded commutative rings).

*Remark 2.3.11.* Every graded ring  $R^\bullet$  corresponds to another graded ring  $R_{\text{op}}^\bullet$  which has the “opposite” grading:  $R_{\text{op}}^i := R^{-i}$ . Evidently  $R_{\text{op}}^\bullet$  is graded-commutative iff  $R^\bullet$  is graded-commutative and in this case there is a canonical identification:  $\text{Spec}^h(R_{\text{op}}^\bullet) = \text{Spec}^h(R^\bullet)$ .

**Definition 2.3.12.** A graded ring is a *graded-division ring* if every nonzero homogeneous element is invertible. A commutative graded-division ring is called a *graded-field*. Examples include a field concentrated in degree zero and the ring of Laurent polynomials  $k[t, t^{-1}]$  over a field. (In fact, if we add “skew Laurent polynomial rings” to the list then we have described all graded-fields; see [NO82, Corollary I.4.3].)

*Remark 2.3.13.* Every graded module over a graded-field is graded-free in the sense that it

has a basis of homogeneous elements. One consequence is that  $M^\bullet \otimes N^\bullet = 0$  implies that either  $M^\bullet = 0$  or  $N^\bullet = 0$  for graded modules over a graded-field. This will have some significance in Chapter 6 when we consider the Morava  $K$ -theories—extraordinary cohomology theories whose coefficient rings (cf. Section 3.5) are graded-fields.

The final topic to discuss in this section is the notion of a graded-local ring. For the reader's convenience we recall the noncommutative notion of a local ring.

**Proposition 2.3.14.** *For a non-zero ring  $R$ , the following statements are equivalent:*

1.  *$R$  has a unique maximal left ideal;*
2.  *$R$  has a unique maximal right ideal;*
3.  *$R/\text{rad}(R)$  is a division ring;*
4. *the set of non-units in  $R$  forms an ideal;*
5. *the sum of two non-units is again a non-unit.*

*A non-zero ring satisfying these conditions is said to be a local ring.*

*Proof.* See [Lam01, Theorem 19.1]. □

The following graded-analogue of Proposition 2.3.14 is not as widely accessible in the literature.

**Proposition 2.3.15.** *For a non-zero graded ring  $R^\bullet$ , the following statements are equivalent:*

1.  *$R^\bullet$  has a unique homogeneous left ideal that is maximal among the proper homogeneous left ideals;*

2.  $R^\bullet$  has a unique homogeneous right ideal that is maximal among the proper homogeneous right ideals;
3.  $R^\bullet/\text{rad}^g(R^\bullet)$  is a graded-division ring;
4. the homogeneous non-units form a two-sided homogeneous ideal of  $R^\bullet$ ;
5. the sum of two homogeneous non-units of the same degree is again a non-unit;
6.  $R^0$  is a local ring.

A non-zero graded ring is said to be *graded-local* if these conditions hold.

*Proof.* The proofs of the equivalences of (1) through (5) follow the proofs of the non-graded case using familiar properties of the “graded Jacobson radical”  $\text{rad}^g(R)$ . See for example [NV79, II.8] for a discussion of the graded Jacobson radical. The only potentially surprising part is the relationship with  $R^0$ : that a graded ring  $R^\bullet$  is graded-local iff  $R^0$  is a local ring. For a proof of this claim see [Li12, Section 2]. □

## 2.4 Spectral spaces

**Definition 2.4.1.** A topological space is said to be *spectral* if it is  $T_0$  and quasi-compact; the quasi-compact open subsets are closed under finite intersection and form an open basis; and every non-empty irreducible closed subset has a generic point.

*Remark 2.4.2.* Hochster [Hoc69] showed that a topological space is spectral if and only if it is homeomorphic to the Zariski spectrum of a commutative ring. On the other hand, the spectrum of a tensor triangulated category is spectral (cf. Chapter 4) and it follows from the results of Hochster, Thomason, and Balmer that every spectral space arises in this way.

**Definition 2.4.3.** A subset  $\mathcal{Y} \subset \mathcal{X}$  of a spectral space is *Thomason* if it is a union of closed subsets each of which has quasi-compact complement.

*Remark 2.4.4.* Hochster showed that every spectral space admits a “dual” spectral topology whose open sets are precisely the Thomason subsets. The nomenclature comes from the prominent role these “dual-open” sets play in the work of Thomason [Tho97]; cf. Theorem 4.2.3 in Chapter 4.

*Example 2.4.5.* If  $X$  is a noetherian topological space then Thomason subsets are the same thing as unions of closed sets—the so-called “specialization-closed” subsets. For example, it is well-known that a closed subset of an affine scheme  $\text{Spec}(A)$  has quasi-compact complement iff it is of the form  $V(I)$  for a finitely generated ideal  $I \subset A$  (cf. Remark 2.3.10). Noetherian spectral spaces are those of the form  $\text{Spec}(A)$  for a noetherian ring  $A$ .

**Definition 2.4.6.** A *spectral map* between spectral spaces is a continuous map with the property that the preimage of any quasi-compact open subset is again quasi-compact. This is equivalent to being a continuous map that is also continuous with respect to the dual spectral topologies.

*Remark 2.4.7.* Any closed subset of a spectral space is again a spectral space and the inclusion is a spectral map. A cheap way to see this is to use Hochster’s theorem to reduce consideration to the case of an affine scheme  $\text{Spec}(A)$ . That a closed subset  $V(I) \subset \text{Spec}(A)$  is spectral is clear from the homeomorphism  $V(I) \simeq \text{Spec}(A/I)$  and the fact that the inclusion  $\varphi : \text{Spec}(A/I) \hookrightarrow \text{Spec}(A)$  is spectral is clear from the observation that  $\varphi^{-1}(V(a_1, \dots, a_n)) = V(\bar{a}_1, \dots, \bar{a}_n)$ .

*Remark 2.4.8.* The homogeneous spectrum of a (graded-)commutative graded ring is a spectral space (see [BKS07, Prop. 2.5] and [DS14, Prop. 2.43]); moreover, the continuous surjective map  $(-)^0 : \text{Spec}^h(R^\bullet) \rightarrow \text{Spec}(R^0)$  from Lemma 2.3.7 is a spectral map. Our comparison

maps will be spectral maps defined on closed subsets of the spectrum of a tensor triangulated category and mapping to the (homogeneous) spectrum of a (graded-)commutative (graded) ring.

The result of the following lemma is a minor technical point that will be needed.

**Proposition 2.4.9.** *Let  $f : X_1 \rightarrow X_2$  be a spectral map of spectral spaces and suppose that  $Z_1 \subset X_1$ ,  $Z_2 \subset X_2$  are two closed subsets such that  $f(Z_1) \subset Z_2$ . Then  $f|_{Z_1} : Z_1 \rightarrow Z_2$  is a spectral map.*

*Proof.* Let  $U \subset Z_2$  be quasi-compact and relatively open. From the quasi-compactness of  $U$  and the fact that the quasi-compact opens form an open basis of  $X_2$  it follows that there are quasi-compact opens  $U_1, U_2, \dots, U_n$  of  $X_2$  such that  $U = Z_2 \cap (U_1 \cup \dots \cup U_n)$ . Now  $Z_1 \cap f^{-1}(Z_2) = Z_1$  since  $f(Z_1) \subset Z_2$ , so  $Z_1 \cap f^{-1}(U) = Z_1 \cap f^{-1}(Z_2) \cap f^{-1}(U_1 \cup \dots \cup U_n) = Z_1 \cap f^{-1}(U_1 \cup \dots \cup U_n)$ . Since  $f$  is spectral, each  $f^{-1}(U_i)$  is quasi-compact open in  $X_1$ . Since a finite union of quasi-compact opens remains quasi-compact,  $f^{-1}(U_1 \cup \dots \cup U_n)$  is a quasi-compact open of  $X_1$ . Since the inclusion  $Z_1 \hookrightarrow X_1$  is spectral, it follows that  $Z_1 \cap f^{-1}(U_1 \cup \dots \cup U_n) = Z_1 \cap f^{-1}(U)$  is quasi-compact. □

## CHAPTER 3

### Triangulated categories

In this chapter we present the basic theory of triangulated categories. The most comprehensive textbook on the subject is [Nee01], although [Ver96], [Wei94, Chapter 10], and [GM03, Chapter 4] provide good introductions. The first two sections cover very standard material. A few proofs will be given to illustrate the kind of manipulations one performs with triangulated categories, but many results will be deferred to the standard references. In Section 3.3, we will discuss *tensor* triangulated categories with a careful treatment of the compatibility between the triangulated structure and the tensor structure. This material is less standard although still well-known. Throughout the chapter we are careful to mention how standard constructions from the theory of triangulated categories interact with the tensor structure. In Section 3.4, we discuss rigid tensor triangulated categories (those in which every object is dualizable) as well as rigidly-compactly generated categories. The latter serve as a convenient replacement for the notion of a compactly generated triangulated category in the setting of tensor triangulated categories. Our outline of the general theory is completed in Sections 3.5–3.7 with a discussion of generalized homology theories, Verdier quotients and Bousfield localization. Examples will be discussed in Section 3.8. Finally, the last section is devoted to thick subcategory classification theorems. This sets the stage for the topic of the next chapter: the theory of tensor triangular geometry.

### 3.1 Basic definitions

Let  $\mathcal{T}$  be an additive category equipped with a self-equivalence  $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ . A *triangle* in  $(\mathcal{T}, \Sigma)$  is a diagram of the form  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ . The terminology comes from the fact that if we use the notation  $Z \xrightarrow{\circ} X$  to denote a “degree one” morphism  $Z \longrightarrow \Sigma X$  then we can regard a triangle as a diagram of the form

$$\begin{array}{ccc} & Z & \\ \swarrow \circ & & \nwarrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

The author agrees with the reader that this isn’t a very good reason for using the word “triangle.” In any case, a *morphism* of triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

which we may abbreviate by  $(u, v, w)$ . It is an isomorphism if  $u, v$  and  $w$  are isomorphisms.

**Definition 3.1.1.** A *triangulated category* is an additive category  $\mathcal{T}$  equipped with a self-equivalence  $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$  and a distinguished class of “exact” triangles satisfying the following five axioms:

TR0. Every triangle isomorphic to an exact triangle is exact, and for each object  $X$  in  $\mathcal{T}$  the triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

is exact.

TR1. For each morphism  $f : X \rightarrow Y$  in  $\mathcal{T}$  there exists an exact triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

TR2. A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is exact if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is exact. This is called the “rotation axiom.”

TR3. A commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ u \downarrow & & \downarrow v & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

where the rows are exact triangles can always be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ u \downarrow & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'. \end{array}$$

TR4. Suppose we are given two composable morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  together with chosen exact triangles

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X,$$

$$Y \xrightarrow{g} Z \longrightarrow C_g \longrightarrow \Sigma Y,$$

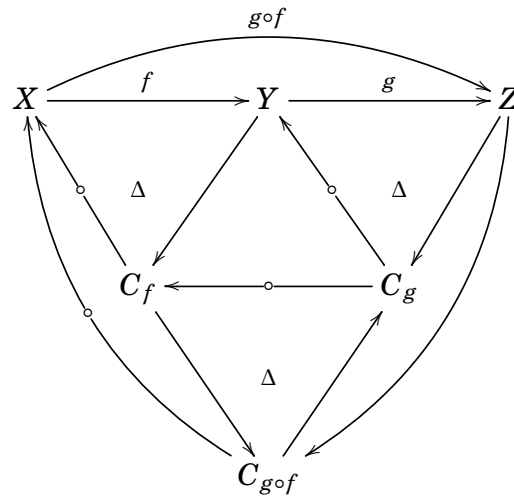
and

$$X \xrightarrow{g \circ f} Z \longrightarrow C_{g \circ f} \longrightarrow \Sigma X.$$

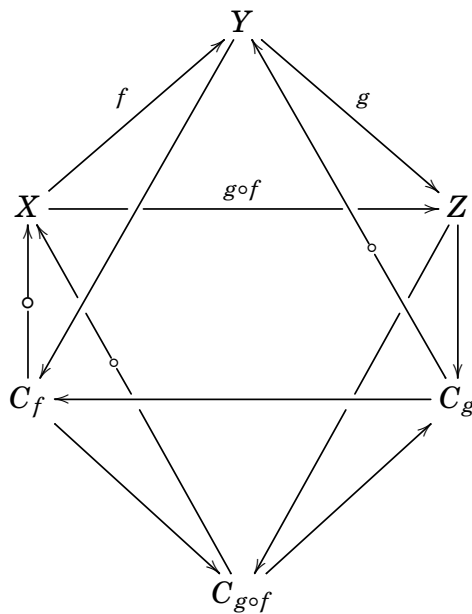
Then there exists an exact triangle

$$C_f \longrightarrow C_{g \circ f} \longrightarrow C_g \longrightarrow \Sigma C_f$$

such that the unlabelled regions of the following diagram commute



and such that the two paths from  $Y$  to  $C_{g \circ f}$  coincide; that is to say, the composite  $Y \rightarrow C_f \rightarrow C_{g \circ f}$  coincides with the composite  $Y \rightarrow Z \rightarrow C_{g \circ f}$ . This is known as the “octahedral axiom” because if one grasps  $Y$  and pulls the diagram out of the page one obtains an octahedron, four of whose faces commute and four of whose faces are exact triangles. (One of the faces is produced by the region of the page outside the diagram, which is the exact triangle for  $g \circ f$ .)



*Remark 3.1.2.* The self-equivalence  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  is called “suspension.” The notation and terminology are due to examples arising in topology where  $\Sigma$  is related to the (reduced) suspension of a (based) topological space; see for example the discussion on “stable homotopy theory” in Section 3.8. In some sources,  $\Sigma$  is assumed to be an isomorphism rather than merely an equivalence. This simplifies exposition but there are important examples where  $\Sigma$  is only an equivalence and in this case we use  $\Sigma^{-1}$  to denote a specific choice of quasi-inverse. In any case, we’ll sometimes abuse notation and make statements that only strictly make sense if  $\Sigma$  is an isomorphism, but such abuses can be easily rectified by utilizing the isomorphisms  $\eta : \text{id}_{\mathcal{T}} \xrightarrow{\sim} \Sigma \Sigma^{-1}$  and  $\epsilon : \Sigma^{-1} \Sigma \xrightarrow{\sim} \text{id}_{\mathcal{T}}$ . For example, it follows from the rotation axiom [TR2] that if the triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  is exact then so is

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z \quad (3.1.3)$$

except that (3.1.3) is not actually a triangle because  $\Sigma \Sigma^{-1} Z \neq Z$ . The precise statement is that the triangle

$$\Sigma^{-1} Z \xrightarrow{-\epsilon_X \circ \Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{\eta_Z \circ g} \Sigma \Sigma^{-1} Z \quad (3.1.4)$$

is exact. This follows from the following isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \parallel & & \downarrow \sim \eta_Z & & \downarrow \sim \text{id} \\ X & \xrightarrow{f} & Y & \xrightarrow{\eta_Z \circ g} & \Sigma \Sigma^{-1} Z & \xrightarrow{\Sigma \epsilon_X \circ \Sigma \Sigma^{-1} h} & \Sigma X \end{array}$$

where the last square commutes using the naturality of  $\eta$ :

$$\Sigma \epsilon_X \circ \Sigma \Sigma^{-1} h \circ \eta_Z = \Sigma \epsilon_X \circ \eta_{\Sigma X} \circ h = \text{id}_{\Sigma X} \circ h = h.$$

The bottom triangle is exact since every triangle isomorphic to an exact triangle is exact [TR0] and hence the rotation axiom [TR2] implies that the triangle (3.1.4) is exact. In any case, these are unimportant details that we will not dwell upon any longer.

*Remark 3.1.5.* Besides the choice of whether to require  $\Sigma$  to be an equivalence or an isomorphism, different sources (e.g. [Nee01, Ver96, Wei94]) give slightly different statements of the axioms. Nevertheless, they all produce the same notion of a triangulated category. For example, [Hub] discusses seven statements which are all equivalent to the octahedral axiom. For additional discussion on the choice of axioms see [May], [May01] and [Nee01, Remarks 1.1.3 and 1.4.7].

*Remark 3.1.6.* The axioms of a triangulated category are self-dual in the sense that if  $\mathcal{T}$  is a triangulated category then the opposite category  $\mathcal{T}^{\text{op}}$  inherits the structure of a triangulated category. The suspension on  $\mathcal{T}^{\text{op}}$  is taken to be  $\Sigma^{\circ} := \Sigma^{-1}$  regarded as a functor  $\mathcal{T}^{\text{op}} \rightarrow \mathcal{T}^{\text{op}}$  and a triangle  $X^{\circ} \xrightarrow{f^{\circ}} Y^{\circ} \xrightarrow{g^{\circ}} Z^{\circ} \xrightarrow{h^{\circ}} \Sigma^{\circ}(X^{\circ})$  is an exact triangle in  $\mathcal{T}^{\text{op}}$  if  $\Sigma^{-1}X \xrightarrow{-h} Z \xrightarrow{g} Y \xrightarrow{f} X$  is an exact triangle in  $\mathcal{T}$ .

**Lemma 3.1.7.** *Let  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  be an exact triangle. Then  $g \circ f = 0$  and  $g$  is a weak cokernel of  $f$ . That is to say, if  $u : Y \rightarrow W$  is a map such that  $u \circ f = 0$  then there exists a map  $v : Z \rightarrow W$  such that  $u = v \circ g$ . The map  $v$  is not required to be unique. Similarly,  $f$  is a weak kernel of  $g$ .*

*Proof.* Applying [TR0] and [TR3], we obtain a factorization

$$\begin{array}{ccccccc} X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f & & \vdots & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma X \end{array}$$

of  $g \circ f$  through zero. To show that  $g$  is a weak cokernel of  $f$ , we can apply [TR3] to

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow & & \downarrow u & & \downarrow v & & \downarrow \\ 0 & \longrightarrow & W & \xrightarrow{1} & W & \longrightarrow & 0 \end{array}$$

in order to obtain the required map  $v$ . A similar argument can be used to show that  $f$  is a weak kernel of  $g$ . □

**Lemma 3.1.8.** *An endomorphism of an exact triangle*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X. \end{array}$$

*is nilpotent if  $u$  and  $v$  are nilpotent.*

*Proof.* Without loss of generality we may assume that  $u = v = 0$ . Then by Lemma 3.1.7 there exist factorizations

$$\begin{array}{ccc} & Z & \xrightarrow{h} \Sigma X \\ & \downarrow w \\ Y & \xrightarrow{g} & Z \end{array}$$

and hence  $w^2 = 0$  since

$$\begin{array}{ccccc} & Z & & & \\ & \downarrow w & & & \\ Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ & & \downarrow w & & \\ & & Z & & \end{array}$$

exhibits a factorization of  $w^2$  through  $h \circ g = 0$ . □

**Lemma 3.1.9.** *A morphism of exact triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

*is an isomorphism if  $u$  and  $v$  are isomorphisms.*

*Proof.* Applying [TR3] to the diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \xrightarrow{h'} & \Sigma X' \\ \downarrow u^{-1} & & \downarrow v^{-1} & & \downarrow \tilde{w} & & \downarrow \Sigma u^{-1} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

we obtain a map  $\tilde{w} : Z \rightarrow Z$  and one readily checks that

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow 0 & & \downarrow 0 & & \downarrow \tilde{w} \circ w - \text{id}_Z & & \downarrow 0 \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

is a morphism of exact triangles. Applying Lemma 3.1.8, we conclude that  $\tilde{w} \circ w - \text{id}_Z$  is a nilpotent element of the ring  $\text{End}_{\mathcal{T}}(Z)$  and hence  $\tilde{w} \circ w$  is an isomorphism. (If  $x$  is a nilpotent element of a ring then  $1 + x$  is a unit with inverse given by  $1 - x + x^2 - \cdots + (-1)^n x^n$  for some  $n \geq 1$ .) A similar argument establishes that  $w \circ \tilde{w}$  is an isomorphism and it follows that  $w$  itself is an isomorphism. (If  $a$  and  $b$  are two elements of a ring such that  $ab$  and  $ba$  are units then  $a$  is a unit.)  $\square$

*Remark 3.1.10.* The above result is an analogue of the “5-lemma” for abelian categories.

*Remark 3.1.11.* Axiom [TR1] asserts that every morphism  $f : X \rightarrow Y$  appears as the first morphism of an exact triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

Moreover, it follows from [TR3] and Lemma 3.1.9 that the object  $Z$  appearing in such an exact triangle is unique up to isomorphism. It is called the “cone” or “cofiber” of  $f$  and will be denoted by  $\text{cone}(f)$  or  $C_f$ . Again the terminology comes from examples arising in topology (see Section 3.8). However, it is important to bear in mind that although  $\text{cone}(f)$  is unique up to isomorphism there is no canonical choice of isomorphism. This is a fundamental feature of the theory of triangulated categories which ultimately arises from the fact that the morphism asserted to exist in [TR3] need not be unique.

**Lemma 3.1.12.** *A morphism  $f : X \rightarrow Y$  is an isomorphism iff  $\text{cone}(f) = 0$ .*

*Proof.* By [TR3] there is a map  $\alpha : 0 \rightarrow \text{cone}(f)$  filling in the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \parallel & & \downarrow f & & \downarrow \alpha & & \parallel \\ X & \xrightarrow{f} & X & \longrightarrow & \text{cone}(f) & \longrightarrow & \Sigma X \end{array}$$

and Lemma 3.1.9 implies that  $\alpha$  is an isomorphism if  $f$  is an isomorphism. On the other hand, if  $X \xrightarrow{f} X \rightarrow 0 \rightarrow \Sigma X$  is an exact triangle then

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\text{id}} & X & \longrightarrow & \Sigma X \\ \parallel & & \parallel & & \downarrow f & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{f} & X & \longrightarrow & \Sigma X \end{array}$$

is a morphism of exact triangles and hence  $f$  is an isomorphism by Lemma 3.1.9.  $\square$

**Lemma 3.1.13.** *The direct sum of two exact triangles is an exact triangle. More precisely, if*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{and} \quad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$$

*are two exact triangles then the triangle*

$$X \oplus X' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} Z \oplus Z' \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}} \Sigma X \oplus \Sigma X' \cong \Sigma(X \oplus X')$$

*is exact.*

*Proof.* See [Nee01, Proposition 1.2.3].  $\square$

**Corollary 3.1.14.** *For any two objects  $X$  and  $Y$ , the triangle*

$$X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} \Sigma X$$

*is exact, where  $i$  and  $p$  denote the inclusion and projection, respectively.*

**Corollary 3.1.15.** *For any two objects  $X$  and  $Y$ ,  $\text{cone}(X \xrightarrow{0} Y) \simeq \Sigma X \oplus Y$ . More precisely, there is an isomorphism of triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{0} & Y & \longrightarrow & \text{cone}(X \xrightarrow{0} Y) & \longrightarrow & \Sigma X \\ \parallel & & \parallel & & \downarrow \sim & & \parallel \\ X & \xrightarrow{0} & Y & \xrightarrow{i} & \Sigma X \oplus Y & \xrightarrow{p} & \Sigma X. \end{array}$$

**Lemma 3.1.16.** *Consider an exact triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ . Then  $f$  is monic iff  $f$  is split monic iff  $f$  is the inclusion of a direct summand iff  $h = 0$  iff there is an isomorphism of exact triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \parallel & & \downarrow \sim & & \parallel & & \parallel \\ X & \xrightarrow{i} & Z \oplus X & \xrightarrow{p} & Z & \xrightarrow{0} & \Sigma X. \end{array} \quad (3.1.17)$$

*Similarly,  $f$  is epi iff  $f$  is split epi iff  $f$  is the projection onto a direct summand iff  $g = 0$  iff there is an isomorphism of exact triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow \sim & & \parallel & & \parallel & & \downarrow \sim \\ Y \oplus \Sigma^{-1}Z & \xrightarrow{p} & Y & \xrightarrow{0} & Z & \xrightarrow{i} & \Sigma Y \oplus Z. \end{array} \quad (3.1.18)$$

*Proof.* Lemma 3.1.7 implies that  $f \circ \Sigma^{-1}h = 0$ . It follows that  $\Sigma^{-1}h = 0$  if  $f$  is monic and hence that  $h = 0$ . On the other hand, if  $h = 0$  then Corollary 3.1.15 implies that there exists an isomorphism of exact triangles (3.1.17). It follows that  $f$  is an inclusion of a direct summand, hence is split monic. The proof of the dual claim is similar.  $\square$

**Remark 3.1.19.** The above lemma makes it clear that triangulated categories are very rarely abelian. Nevertheless, the exact triangles can be thought of as analogues or replacements for the short exact sequences of abelian categories. Indeed, we have already seen that they have some of the properties of short exact sequences. For further connections see the notion of a homological functor (Definition 3.1.30 below) and the discussion on derived categories in Section 3.8.

**Remark 3.1.20.** Another consequence of Lemma 3.1.16 is that—with the notable exception of products and coproducts—very few limits and colimits exist in a triangulated category. For example, if a morphism  $f : X \rightarrow Y$  admits a cokernel then  $\text{coker}(f)$  is a direct summand of  $Y$  and  $f$  factors through the inclusion of the complementary direct summand. More generally, if  $F : I \rightarrow \mathcal{T}$  is a diagram in  $\mathcal{T}$  such that  $\text{colim}_{i \in I} F(i)$  and  $\coprod_{i \in I} F(i)$  exist then the

canonical epimorphism  $\coprod_{i \in I} F(i) \rightarrow \operatorname{colim}_{i \in I} F(i)$  splits and  $\coprod_{i \in I} F(i)$  is a direct summand of  $\operatorname{colim}_{i \in I} F(i)$ . Nevertheless, most triangulated categories which arise in practice have small coproducts—or at least sit inside a larger triangulated category which has small coproducts.

**Lemma 3.1.21.** *Any triangulated category which admits countable coproducts is idempotent complete.*

*Proof.* Recall from Proposition 2.1.7 that an additive category with countable coproducts is idempotent complete if it has the property that an idempotent  $e$  splits iff  $1 - e$  splits. If the category is triangulated and  $e : A \rightarrow A$  splits as  $A \xrightarrow{f} B \xrightarrow{g} A$  then the map  $f$  is split epi and hence Lemma 3.1.16 implies that we have an isomorphism of exact triangles

$$\begin{array}{ccccccc} C & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \Sigma C \\ \parallel & & \downarrow \sim & & \parallel & & \parallel \\ C & \xrightarrow{i} & B \oplus C & \xrightarrow{p} & B & \xrightarrow{0} & \Sigma C. \end{array}$$

The split idempotent  $e : A \rightarrow A$  becomes  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  under the isomorphism  $A \simeq B \oplus C$  and hence  $1 - e$  becomes the split idempotent  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ .  $\square$

*Remark 3.1.22.* Lemma 3.1.21 was proved in [BN93, Proposition 3.2] using the triangulated category version of a homotopy colimit, but the above proof emphasizes that the result has very little to do with triangulated categories. In any case, the basic idea in both [BN93] and [Fre66] is the “Eilenberg swindle” (a.k.a. telescoping sum construction).

**Definition 3.1.23.** Let  $\mathcal{T}$  be a triangulated category. A *triangulated subcategory* of  $\mathcal{T}$  is a full additive subcategory  $\mathcal{S} \subset \mathcal{T}$  which is closed under isomorphism of objects, (de)suspension, and cofibers. A *thick subcategory* is a triangulated subcategory that is closed under direct summands. A *localizing subcategory* is a thick subcategory that is closed under arbitrary coproducts.

**Remark 3.1.24.** Localizing subcategories are usually only considered in triangulated categories which admit small coproducts. In smaller triangulated categories it is the thick subcategories which are most important. Note that if  $\mathcal{T}$  is a triangulated category which admits small coproducts then any triangulated subcategory  $\mathcal{S} \subset \mathcal{T}$  which is closed under coproducts is automatically thick and hence a localizing subcategory of  $\mathcal{T}$ . This follows from Lemma 3.1.21.

**Notation 3.1.25.** The smallest thick subcategory containing a collection of objects  $\mathcal{G} \subset \mathcal{T}$  is denoted  $\text{thick}\langle \mathcal{G} \rangle$ , while the smallest localizing subcategory containing the collection is denoted  $\text{loc}\langle \mathcal{G} \rangle$ .

**Definition 3.1.26.** An *exact* functor  $F : \mathcal{T} \rightarrow \mathcal{S}$  between triangulated categories is an additive functor  $F : \mathcal{T} \rightarrow \mathcal{S}$  equipped with a natural isomorphism  $\theta : F \circ \Sigma \xrightarrow{\sim} \Sigma \circ F$  with the property that if  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  is an exact triangle in  $\mathcal{T}$  then

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\theta_X \circ Fh} \Sigma FX$$

is an exact triangle in  $\mathcal{S}$ .

**Definition 3.1.27.** Let  $(F, \theta) : \mathcal{T} \rightarrow \mathcal{S}$  and  $(F', \theta') : \mathcal{T} \rightarrow \mathcal{S}$  be two exact functors. A natural transformation  $\alpha : F \rightarrow G$  is said to be a *trinatural transformation* if

$$\begin{array}{ccc} F\Sigma X & \xrightarrow{\alpha_{\Sigma X}} & F'\Sigma X \\ \downarrow \theta_X & & \downarrow \theta'_X \\ \Sigma FX & \xrightarrow{\Sigma \alpha_X} & \Sigma F'X \end{array}$$

commutes for each  $X$  in  $\mathcal{T}$ . These are the appropriate morphisms of exact functors.

**Lemma 3.1.28.** *Let  $F : \mathcal{T} \rightarrow \mathcal{S}$  be an exact functor of triangulated categories. If  $F$  has an adjoint  $G : \mathcal{S} \rightarrow \mathcal{T}$  (either left or right) then  $G$  inherits the structure of an exact functor in such a way that the unit and counit of the adjunction are trinatural.*

*Proof.* See [Nee01, Lemma 5.3.6]. □

**Definition 3.1.29.** An *equivalence of triangulated categories* is an exact functor which is an equivalence of the underlying categories. Lemma 3.1.28 implies that this is the same thing as an equivalence in the 2-category of triangulated categories, exact functors, and trinatural transformations.

**Definition 3.1.30.** Let  $\mathcal{T}$  be a triangulated category. A *homological functor*  $H : \mathcal{T} \rightarrow \mathcal{A}$  is a covariant additive functor from  $\mathcal{T}$  to an abelian category  $\mathcal{A}$  which sends exact triangles to exact sequences. More precisely, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad (3.1.31)$$

is an exact triangle in  $\mathcal{T}$  then the sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

is exact in  $\mathcal{A}$ . A *cohomological functor*  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$  is a contravariant additive functor which sends an exact triangle (3.1.31) to an exact sequence

$$H(Z) \xrightarrow{H(g)} H(Y) \xrightarrow{H(f)} H(X).$$

*Example 3.1.32.* For any object  $X$ , the covariant representable functor  $\text{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \rightarrow \text{Ab}$  is a homological functor, while  $\text{Hom}_{\mathcal{T}}(-, X) : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$  is a cohomological functor (see [Nee01, Lemma 1.1.10]). Further examples will be seen in Section 3.5.

*Remark 3.1.33.* Let  $H : \mathcal{T} \rightarrow \mathcal{A}$  be a homological functor. If we define  $H_i := H \circ \Sigma^{-i}$  for each integer  $i \in \mathbb{Z}$  then any exact triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  gives rise to a long exact sequence

$$\cdots \longrightarrow H_1(Z) \xrightarrow{H_1(h)} H_0(X) \xrightarrow{H_0(f)} H_0(Y) \xrightarrow{H_0(g)} H_0(Z) \xrightarrow{H_0(h)} H_{-1}(X) \longrightarrow \cdots$$

Similarly, if we define  $H^i := H \circ \Sigma^{-i}$  for a cohomological functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$  then the exact triangle gives rise to a long exact sequence

$$\cdots \longrightarrow H^{-1}(X) \xrightarrow{H^0(h)} H^0(Z) \xrightarrow{H^0(g)} H^0(Y) \xrightarrow{H^0(f)} H^0(X) \xrightarrow{H^1(h)} H^1(Z) \longrightarrow \cdots$$

The following terminology is not completely standard but it will be convenient:

**Definition 3.1.34.** A *suspended additive category*  $(\mathcal{A}, \Sigma)$  is an additive category  $\mathcal{A}$  equipped with an auto-equivalence  $\Sigma : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ . A *stable additive functor*  $(F, \theta) : (\mathcal{A}, \Sigma_{\mathcal{A}}) \rightarrow (\mathcal{B}, \Sigma_{\mathcal{B}})$  between suspended additive categories is an additive functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  equipped with a natural isomorphism  $\theta : F \circ \Sigma_{\mathcal{A}} \xrightarrow{\sim} \Sigma_{\mathcal{B}} \circ F$ . A *stable natural transformation*  $\alpha : (F, \theta) \rightarrow (F', \theta')$  between two stable additive functors is a natural transformation such that

$$\begin{array}{ccc} F\Sigma X & \xrightarrow{\alpha_{\Sigma X}} & F'\Sigma X \\ \downarrow \theta_X & & \downarrow \theta'_X \\ \Sigma F X & \xrightarrow{\Sigma \alpha_X} & \Sigma F' X \end{array}$$

commutes for each  $X$  in  $\mathcal{A}$ .

*Example 3.1.35.* Every triangulated category can be regarded as a suspended additive category by forgetting the exact triangles and every exact functor is a stable additive functor between the underlying suspended additive categories.

*Example 3.1.36.* If  $\mathcal{A}$  is an additive category then the category  $\mathcal{A}^{\mathbb{Z}}$  of  $\mathbb{Z}$ -graded objects in  $\mathcal{A}$  is a suspended additive category with suspension defined by  $(\Sigma X)_n := X_{n-1}$ . Similarly, the category of chain complexes  $\text{Ch}(\mathcal{A})$  is a suspended additive category where the differential on  $\Sigma X$  is given by  $d_n^{\Sigma X} := -d_{n-1}^X$ . In the same spirit, if  $R^{\bullet}$  is a graded ring then the category of graded  $R^{\bullet}$ -modules  $R^{\bullet}\text{-grMod}$  is a suspended abelian category with suspension defined by  $(\Sigma M)_n = M_{n-1}$ .

**Definition 3.1.37.** A *stable homological functor*  $H : \mathcal{T} \rightarrow \mathcal{A}$  is a homological functor from a triangulated category to a suspended abelian category which is equipped with a natural isomorphism  $H \circ \Sigma \simeq \Sigma \circ H$ . Similarly, a *stable cohomological functor*  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$  is a cohomological functor equipped with a natural isomorphism  $H \circ \Sigma \simeq \Sigma \circ H$ .

*Example 3.1.38.* Every homological functor  $H : \mathcal{T} \rightarrow \mathcal{A}$  gives rise to a stable homological functor  $H_{\bullet} : \mathcal{T} \rightarrow \mathcal{A}^{\mathbb{Z}}$  by defining  $H_i := H \circ \Sigma^{-i}$  for each  $i \in \mathbb{Z}$ . Similarly, every cohomological

functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$  gives rise to a stable cohomological functor  $H^\bullet : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}^\mathbb{Z}$  by defining  $H^i := H \circ \Sigma^{-i}$  for each  $i \in \mathbb{Z}$ . In Section 3.5, we'll see examples of stable (co)homological functors which do not arise in this way from ordinary (co)homological functors.

**Lemma 3.1.39.** *Let  $\alpha : H \rightarrow H'$  be a stable natural transformation between two stable homological functors  $H : \mathcal{T} \rightarrow \mathcal{A}$  and  $H' : \mathcal{T} \rightarrow \mathcal{A}$ . Then  $\mathcal{J}_\alpha := \{X \mid \alpha_X \text{ is an isomorphism}\}$  is a thick subcategory of  $\mathcal{T}$ . If  $H$  and  $H'$  preserve coproducts then  $\mathcal{J}_\alpha$  is a localizing subcategory of  $\mathcal{T}$ .*

*Proof.* The naturality of  $\alpha$  implies that  $\mathcal{J}_\alpha$  is a replete subcategory of  $\mathcal{T}$ . Next consider a direct sum  $A \oplus B$  in  $\mathcal{T}$ . The fact that  $H$  is additive implies that  $H(A \oplus B)$  is the biproduct  $HA \oplus HB$  with structure maps  $H(i_A)$ ,  $H(i_B)$ ,  $H(p_A)$  and  $H(p_B)$ . Moreover, one can easily check using the naturality of  $\alpha$  that  $\alpha_{A \oplus B} = \alpha_A \oplus \alpha_B$ . Hence  $\alpha_{A \oplus B}$  is an isomorphism iff  $\alpha_A$  and  $\alpha_B$  are isomorphisms. It follows that  $\mathcal{J}_\alpha$  is closed under direct sums and direct summands. The stability of  $\alpha$  shows that  $\mathcal{J}_\alpha$  is closed under suspension and desuspension. Furthermore, if  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  is an exact triangle then we have a commutative diagram

$$\begin{array}{ccccccccc} HX & \xrightarrow{Hf} & HY & \xrightarrow{Hg} & HZ & \xrightarrow{Hh} & H\Sigma X & \xrightarrow{H\Sigma f} & H\Sigma Y \\ \downarrow \alpha_X & & \downarrow \alpha_Y & & \downarrow \alpha_Z & & \downarrow \alpha_{\Sigma X} & & \downarrow \alpha_{\Sigma Y} \\ H'X & \xrightarrow{H'f} & H'Y & \xrightarrow{H'g} & H'Z & \xrightarrow{H'h} & H'\Sigma X & \xrightarrow{H'\Sigma f} & H'\Sigma Y \end{array}$$

and the fact that  $\mathcal{J}_\alpha$  is closed under cofibers follows from the 5-lemma for abelian categories.

Finally, if  $H$  and  $H'$  preserve coproducts then the commutativity of

$$\begin{array}{ccc} \coprod_i H(A_i) & \xrightarrow{\sim} & H(\coprod_i A_i) \\ \downarrow \coprod_i \alpha_{A_i} & & \downarrow \alpha_{\coprod_i A_i} \\ \coprod_i H'(A_i) & \xrightarrow{\sim} & H'(\coprod_i A_i) \end{array} \quad (3.1.40)$$

will show that  $\mathcal{J}_\alpha$  is closed under coproducts. Precomposing the diagram with the canonical map  $H(A_n) \rightarrow \coprod_i H(A_i)$  we get

$$\begin{array}{ccc} H(A_n) & \xrightarrow{H(i_{A_n})} & H(\coprod_i A_i) \\ \downarrow \alpha_{A_n} & & \downarrow \alpha_{\coprod_i A_i} \\ H'(A_n) & \xrightarrow{H'(i_{A_n})} & H'(\coprod_i A_i) \end{array}$$

which commutes by naturality. It follows that (3.1.40) commutes by the universal property of the coproduct  $\coprod_i H(A_i)$ .  $\square$

**Lemma 3.1.41.** *Let  $H : \mathcal{T} \rightarrow \mathcal{A}$  be a stable homological functor. The following are equivalent:*

- (1)  *$H$  is faithful on objects: if  $HX = 0$  then  $X = 0$ .*
- (2)  *$H$  reflects isomorphisms: if  $Hf$  is an isomorphism then  $f$  is an isomorphism.*

*Such a functor is said to be conservative.*

*Proof.* For (1)  $\Rightarrow$  (2), suppose  $f : X \rightarrow Y$  is a morphism such that  $Hf$  is an isomorphism.

Applying  $H$  to an exact triangle for  $f$  we obtain an exact sequence

$$HX \xrightarrow[\sim]{Hf} HY \longrightarrow H\text{cone}(f) \longrightarrow \Sigma HX \xrightarrow[\sim]{\Sigma Hf} \Sigma HY$$

and it follows that  $H\text{cone}(f) = 0$ . Hence  $\text{cone}(f) = 0$  so that  $f$  is an isomorphism. The converse is immediate: if  $HX = 0$  then  $H(0 : X \rightarrow X)$  is an isomorphism. Hence  $0 : X \rightarrow X$  is an isomorphism and therefore  $X = 0$ .  $\square$

**Notation 3.1.42.** The abelian group of morphisms  $A \rightarrow B$  in an additive category will be denoted  $[A, B]$  and the  $\mathbb{Z}$ -graded abelian group of graded morphisms in a suspended additive category will be denoted  $[A, B]_{\bullet}$ . That is,  $[A, B]_i := [\Sigma^i A, B]$ . Note that  $[A, A]$  is a ring and  $[A, A]_{\bullet}$  is a  $\mathbb{Z}$ -graded ring. Also observe that just as an additive functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  between additive categories induces a ring homomorphism  $[X, X] \rightarrow [FX, FX]$  between endomorphism rings, so too a stable additive functor between suspended additive categories induces a graded ring homomorphism between graded endomorphism rings  $[X, X]_{\bullet} \rightarrow [FX, FX]_{\bullet}$ .

**Notation 3.1.43.** If  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  are two endomorphisms, the notation  $f \simeq g$  will signify that there exists an isomorphism  $\alpha : X \xrightarrow{\sim} Y$  such that  $g \circ \alpha = \alpha \circ f$ . Note that if  $f \simeq g$  then  $f = 0$  iff  $g = 0$  and  $f$  is an isomorphism iff  $g$  is an isomorphism.

## 3.2 Compact objects

Many important examples of triangulated categories are “compactly generated.” These are “large” triangulated categories (admitting arbitrary coproducts) which are generated in a suitable sense from a small subcategory of “compact” objects. They satisfy a fundamental result from stable homotopy theory called the Brown Representability Theorem which asserts that product-preserving cohomological functors are representable. This vital result (having origins in [Bro62, Bro63]) is included as one of the axioms in [HPS97]’s notion of an “axiomatic stable homotopy category.” It was [Nee96] who observed that it holds for any compactly generated triangulated category.

**Definition 3.2.1.** Let  $\mathcal{T}$  be a triangulated category. An object  $X$  in  $\mathcal{T}$  is said to be *compact* if the representable functor  $\mathrm{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \rightarrow \mathbf{Ab}$  preserves coproducts. Equivalently, an object  $X$  is compact if any map from  $X$  to an infinite coproduct factors through a finite coproduct.

**Notation 3.2.2.** The collection of all compact objects in  $\mathcal{T}$  forms a thick subcategory  $\mathcal{T}^c \subset \mathcal{T}$ .

**Definition 3.2.3.** A triangulated category  $\mathcal{T}$  is said to be *compactly generated* if it has small coproducts and there exists a set of compact objects  $\mathcal{G}$  with the property that  $\mathcal{T} = \mathrm{loc}\langle G \in \mathcal{G} \rangle$ .

*Remark 3.2.4.* If  $\mathcal{T}$  is a triangulated category with small coproducts and  $\mathcal{G}$  is a set of compact objects in  $\mathcal{T}$  then the condition  $\mathcal{T} = \mathrm{loc}\langle G \in \mathcal{G} \rangle$  is equivalent to the condition that an object  $X \in \mathcal{T}$  is zero iff  $[G, X]_{\bullet} = 0$  for all  $G \in \mathcal{G}$ . A proof is given in [SS03, Lemma 2.2.1]. One direction is easy: for any fixed  $X$ , the collection of all objects  $Y$  such that  $[Y, X]_{\bullet} = 0$  is a localizing subcategory, so if it contains  $\mathcal{G}$  then it contains the whole of  $\mathcal{T}$ . In particular, it would contain  $X$  so that  $[X, X]_{\bullet} = 0$  and hence  $X = 0$ . The other direction is less elementary and uses finite localization with respect to  $\mathcal{G}$ ; cf. Section 3.7 below.

*Remark 3.2.5.* If  $\mathcal{T}$  is a compactly generated triangulated category with set of compact generators  $\mathcal{G}$  then  $\mathcal{T}^c = \text{thick}\langle G \in \mathcal{G} \rangle$ . This follows from [Nee92b, Lemma 2.2].

**Theorem 3.2.6** (Brown Representability). *Let  $\mathcal{T}$  be a compactly generated triangulated category. Every cohomological functor  $H : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$  which takes coproducts in  $\mathcal{T}$  to products in  $\text{Ab}$  is representable.*

*Proof.* See [Nee96, Theorem 3.1]. □

**Corollary 3.2.7.** *Let  $\mathcal{T}$  be a compactly generated triangulated category and let  $\mathcal{S}$  be an arbitrary triangulated category. Every exact functor  $F : \mathcal{T} \rightarrow \mathcal{S}$  which preserves coproducts has a right adjoint.*

*Proof.* See [Nee96, Theorem 4.1]. For any object  $s \in \mathcal{S}$ , the functor  $\text{Hom}_{\mathcal{S}}(F(-), s) : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$  sends coproducts in  $\mathcal{T}$  to products in  $\text{Ab}$ . Hence, by Theorem 3.2.6, there exists an object  $Gs \in \mathcal{T}$  and an isomorphism  $\text{Hom}_{\mathcal{S}}(Ft, s) \simeq \text{Hom}_{\mathcal{T}}(t, Gs)$  natural in  $t \in \mathcal{T}$ . General categorical nonsense implies that  $G$  extends uniquely to a functor  $G : \mathcal{S} \rightarrow \mathcal{T}$  such that these isomorphisms are natural in  $s$  too. □

### 3.3 Tensor triangulated categories

**Definition 3.3.1.** A *tensor triangulated category* is a triangulated category  $\mathcal{T}$  that is also a symmetric monoidal category such that for each object  $a$  in  $\mathcal{T}$  the functors  $a \otimes - : \mathcal{T} \rightarrow \mathcal{T}$  and  $- \otimes a : \mathcal{T} \rightarrow \mathcal{T}$  are exact functors of triangulated categories. This includes the data of natural isomorphisms

$$\lambda_{a,b} : \Sigma a \otimes b \simeq \Sigma(a \otimes b) \quad \text{and} \quad \rho_{a,b} : a \otimes \Sigma b \simeq \Sigma(a \otimes b) \quad (3.3.2)$$

which are required to be “compatible” with the symmetry, associator and unitor isomorphisms of the symmetric monoidal structure. In detail, the three diagrams

$$\begin{array}{ccc}
 \mathbb{1} \otimes \Sigma a & \xrightarrow{\rho_{\mathbb{1}, \Sigma a}} & \Sigma(\mathbb{1} \otimes a) \\
 & \searrow l_{\Sigma a} \quad \swarrow \Sigma(l_a) & \\
 & \Sigma a &
 \end{array} \tag{3.3.3}$$

$$\begin{array}{ccc}
 \Sigma a \otimes (b \otimes c) & \xrightarrow{\lambda_{a, b \otimes c}} & \Sigma(a \otimes (b \otimes c)) \\
 \downarrow \alpha & & \downarrow \Sigma(\alpha) \\
 (\Sigma a \otimes b) \otimes c & \xrightarrow{\lambda_{a, b} \otimes 1} \Sigma(a \otimes b) \otimes c \xrightarrow{\lambda_{a \otimes b, c}} & \Sigma((a \otimes b) \otimes c)
 \end{array} \tag{3.3.4}$$

$$\begin{array}{ccc}
 \Sigma a \otimes b & \xrightarrow{\lambda_{a, b}} & \Sigma(a \otimes b) \\
 \downarrow \tau_{\Sigma a, b} & & \downarrow \Sigma(\tau_{a, b}) \\
 b \otimes \Sigma a & \xrightarrow{\rho_{b, a}} & \Sigma(b \otimes a)
 \end{array} \tag{3.3.5}$$

are required to commute, while the diagram

$$\begin{array}{ccc}
 \Sigma a \otimes \Sigma b & \xrightarrow{\lambda_{a, \Sigma b}} & \Sigma(a \otimes \Sigma b) \\
 \rho_{\Sigma a, b} \downarrow & (-1) & \downarrow \Sigma \rho_{a, b} \\
 \Sigma(\Sigma a \otimes b) & \xrightarrow{\Sigma \lambda_{a, b}} & \Sigma^2(a \otimes b)
 \end{array} \tag{3.3.6}$$

is required to anticommute.

*Remark 3.3.7.* Due to (3.3.5), the isomorphisms  $\lambda$  and  $\rho$  determine each other and the following diagram analogous to (3.3.3) also commutes:

**Lemma 3.3.8.** *Let  $\mathcal{T}$  be a tensor triangulated category. The diagram*

$$\begin{array}{ccc}
 \Sigma a \otimes \mathbb{1} & \xrightarrow{\lambda_{a, \mathbb{1}}} & \Sigma(a \otimes \mathbb{1}) \\
 & \searrow r_{\Sigma a} \quad \swarrow \Sigma(r_a) & \\
 & \Sigma a &
 \end{array} \tag{3.3.9}$$

*commutes for any  $a \in \mathcal{T}$ .*

*Proof.* In the following diagram

$$\begin{array}{ccc}
 \Sigma a \otimes 1 & \xrightarrow{\lambda_{a,1}} & \Sigma(a \otimes 1) \\
 \downarrow \tau & & \downarrow \Sigma(\tau) \\
 1 \otimes \Sigma a & \xrightarrow{\rho_{1,\Sigma a}} & \Sigma(1 \otimes a) \\
 \downarrow l_{\Sigma a} & & \downarrow \Sigma(l_a) \\
 \Sigma a & \xlongequal{\quad} & \Sigma a
 \end{array}
 \begin{array}{c}
 \curvearrowleft r_{\Sigma a} \\
 \curvearrowright \Sigma(r_a)
 \end{array}$$

the left and right regions commute by (2.2.13), the top square commutes by (3.3.5), and the bottom square commutes by (3.3.3).  $\square$

*Remark 3.3.10.* There are two further “associator” coherence diagrams analogous to (3.3.4), which take the suspension out of the second and third coordinate, respectively. The next two lemmas show that these diagrams follow from the axioms.

**Lemma 3.3.11.** *Let  $\mathcal{T}$  be a tensor triangulated category. The diagram*

$$\begin{array}{ccccc}
 a \otimes (\Sigma b \otimes c) & \xrightarrow{1 \otimes \lambda_{b,c}} & a \otimes \Sigma(b \otimes c) & \xrightarrow{\rho_{a,b \otimes c}} & \Sigma(a \otimes (b \otimes c)) \\
 \downarrow \alpha & & & & \downarrow \Sigma(a) \\
 (a \otimes \Sigma b) \otimes c & \xrightarrow{\rho_{a,b} \otimes 1} & \Sigma(a \otimes b) \otimes c & \xrightarrow{\lambda_{a \otimes b, c}} & \Sigma((a \otimes b) \otimes c)
 \end{array}$$

*commutes for any  $a, b, c \in \mathcal{T}$ .*

*Proof.* This is demonstrated by the commutativity of the large diagram on the next page. The left and right arms commute by (2.2.15). The top and bottom large rectangles commute by (3.3.4). The top-right and bottom-left squares commute by (3.3.5). The top-left and bottom-right squares commute by naturality; so does the middle rectangle.  $\square$

$$\begin{array}{ccccc}
a \otimes (\Sigma b \otimes c) & \xrightarrow{1 \otimes \lambda} & a \otimes \Sigma(b \otimes c) & \xrightarrow{\rho} & \Sigma(a \otimes (b \otimes c)) \\
\downarrow \tau & & \downarrow \tau & & \downarrow \Sigma(\tau) \\
(\Sigma b \otimes c) \otimes a & \xrightarrow{\lambda \otimes 1} & \Sigma(b \otimes c) \otimes a & \xrightarrow{\lambda} & \Sigma((b \otimes c) \otimes a) \\
\downarrow \alpha^{-1} & & & & \downarrow \Sigma(\alpha^{-1}) \\
\Sigma b \otimes (c \otimes a) & \xrightarrow{\lambda} & & & \Sigma(b \otimes (c \otimes a)) \\
\downarrow 1 \otimes \tau & & & & \downarrow \Sigma(1 \otimes \tau) \\
\Sigma b \otimes (a \otimes c) & \xrightarrow{\lambda} & & & \Sigma(b \otimes (a \otimes c)) \\
\downarrow \alpha & & & & \downarrow \Sigma(\alpha) \\
(\Sigma b \otimes a) \otimes c & \xrightarrow{\lambda \otimes 1} & \Sigma(b \otimes a) \otimes c & \xrightarrow{\lambda} & \Sigma((b \otimes a) \otimes c) \\
\downarrow \tau \otimes 1 & & \downarrow \Sigma(\tau) \otimes 1 & & \downarrow \Sigma(\tau \otimes 1) \\
(a \otimes \Sigma b) \otimes c & \xrightarrow{\rho \otimes 1} & \Sigma(a \otimes b) \otimes c & \xrightarrow{\lambda} & \Sigma((a \otimes b) \otimes c)
\end{array}$$

$\alpha$  (left curved arrow),  $\Sigma(\alpha)$  (right curved arrow)

$$\begin{array}{ccccc}
a \otimes (b \otimes \Sigma c) & \xrightarrow{1 \otimes \rho} & a \otimes \Sigma(b \otimes c) & \xrightarrow{\rho} & \Sigma(a \otimes (b \otimes c)) \\
\downarrow 1 \otimes \tau & & \downarrow 1 \otimes \Sigma \tau & & \downarrow \Sigma(1 \otimes \tau) \\
a \otimes (\Sigma c \otimes b) & \xrightarrow{1 \otimes \lambda} & a \otimes \Sigma(c \otimes b) & \xrightarrow{\rho} & \Sigma(a \otimes (c \otimes b)) \\
\downarrow \alpha & & & & \downarrow \Sigma(\alpha) \\
(a \otimes \Sigma c) \otimes b & \xrightarrow{\rho \otimes 1} & \Sigma(a \otimes c) \otimes b & \xrightarrow{\lambda} & \Sigma((a \otimes c) \otimes b) \\
\downarrow \tau \otimes 1 & & \downarrow \Sigma \tau \otimes 1 & & \downarrow \Sigma(\tau \otimes 1) \\
(\Sigma c \otimes a) \otimes b & \xrightarrow{\lambda \otimes 1} & \Sigma(c \otimes a) \otimes b & \xrightarrow{\lambda} & \Sigma((c \otimes a) \otimes b) \\
\downarrow \alpha^{-1} & & & & \downarrow \Sigma(\alpha^{-1}) \\
\Sigma c \otimes (a \otimes b) & \xrightarrow{\lambda} & & & \Sigma(c \otimes (a \otimes b)) \\
\downarrow \tau & & & & \downarrow \Sigma \tau \\
(a \otimes b) \otimes \Sigma c & \xrightarrow{\rho} & & & \Sigma((a \otimes b) \otimes c)
\end{array}$$

$\alpha$  (left curved arrow),  $\Sigma(\alpha)$  (right curved arrow)

**Lemma 3.3.12.** *Let  $\mathcal{T}$  be a tensor triangulated category. The diagram*

$$\begin{array}{ccccc}
a \otimes (b \otimes \Sigma c) & \xrightarrow{1 \otimes \rho_{b,c}} & a \otimes \Sigma(b \otimes c) & \xrightarrow{\rho_{a,b \otimes c}} & \Sigma(a \otimes (b \otimes c)) \\
\downarrow \alpha & & & & \downarrow \Sigma(a) \\
(a \otimes b) \otimes \Sigma c & \xrightarrow{\rho_{a \otimes b, c}} & & & \Sigma((a \otimes b) \otimes c)
\end{array}$$

*commutes for any  $a, b, c \in \mathcal{T}$ .*

*Proof.* This is demonstrated by the commutativity of the diagram on the preceding page. The left and right arms commute by (2.2.15). The top-left and middle-left squares commute by (3.3.5); so does the bottom large rectangle. The top-right and middle-right squares commute by naturality. The top large rectangle commutes by Lemma 3.3.11, while the remaining large rectangle commutes by (3.3.4).  $\square$

*Remark 3.3.13.* The compatibility axioms we require between the symmetric monoidal structure and the triangulated structure are quite weak. Most of the axioms merely describe compatibility between the suspension and the monoidal structure and have nothing to do with exact triangles. The only connection with the exact triangles is the requirement that  $a \otimes -$  preserves them. Further axioms connecting the exact triangles with the monoidal structure have been proposed by [May01] and by [KN02] but these are not required for the present development of tensor triangular geometry (Chapter 4) nor for our theory of higher comparison maps (Chapter 5).

*Remark 3.3.14.* Since  $a \otimes -$  is an exact functor, any exact triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$  gives rise to an exact triangle

$$a \otimes X \xrightarrow{1 \otimes f} a \otimes Y \xrightarrow{1 \otimes g} a \otimes Z \xrightarrow{1 \otimes h} a \otimes \Sigma X \simeq \Sigma(a \otimes X).$$

We'll sometimes abuse notation slightly and write

$$a \otimes X \xrightarrow{1 \otimes f} a \otimes Y \xrightarrow{1 \otimes g} a \otimes Z \xrightarrow{1 \otimes h} \Sigma(a \otimes X)$$

for this exact triangle, thereby omitting the relevant suspension isomorphism.

*Remark 3.3.15.* The only mysterious axiom in the definition of a tensor triangulated category is the requirement that diagram (3.3.6) anticommutes. In fact, one could get a perfectly satisfactory theory if we required it to commute, but in all the examples of interest it anticommutes instead. One consequence of the fact that (3.3.6) only commutes up to a sign is that there is ambiguity when we write an isomorphism like  $\Sigma^{i+j}(a \otimes b) \simeq \Sigma^i a \otimes \Sigma^j b$ .

**Convention 3.3.16.** Whenever we write  $\Sigma^{i+j}(a \otimes b) \simeq \Sigma^i a \otimes \Sigma^j b$  we mean the composite  $\Sigma^{i+j}(a \otimes b) \xrightarrow{\sim} \Sigma^j(\Sigma^i a \otimes b) \xrightarrow{\sim} \Sigma^i a \otimes \Sigma^j b$  obtained by first moving all the relevant suspensions onto the first factor and then moving the remaining suspensions onto the second factor.

*Remark 3.3.17.* We can define natural isomorphisms

$$\tilde{\lambda}_{a,b} : \Sigma^{-1} a \otimes b \simeq \Sigma^{-1}(a \otimes b) \quad \text{and} \quad \tilde{\rho}_{a,b} : a \otimes \Sigma^{-1} b \simeq \Sigma^{-1}(a \otimes b)$$

by specifying  $\Sigma \tilde{\lambda}_{a,b}$  and  $\Sigma \tilde{\rho}_{a,b}$  to be

$$\Sigma(\Sigma^{-1} a \otimes b) \xrightarrow{\lambda_{\Sigma^{-1} a, b}^{-1}} \Sigma \Sigma^{-1} a \otimes b \simeq a \otimes b \simeq \Sigma \Sigma^{-1}(a \otimes b)$$

and

$$\Sigma(a \otimes \Sigma^{-1} b) \xrightarrow{\rho_{a, \Sigma^{-1} b}^{-1}} a \otimes \Sigma \Sigma^{-1} b \simeq a \otimes b \simeq \Sigma \Sigma^{-1}(a \otimes b)$$

respectively. This allows us to speak of isomorphisms  $a \otimes \Sigma^i b \simeq \Sigma^i(a \otimes b) \simeq \Sigma^i a \otimes b$  for any integer  $i \in \mathbb{Z}$ .

**Lemma 3.3.18.** *Let  $\mathcal{T}$  be a tensor triangulated category. The diagram*

$$\begin{array}{ccc} \Sigma^i a \otimes \Sigma^j b & \longrightarrow & \Sigma^i(a \otimes \Sigma^j b) \\ \downarrow & (-1)^{ij} & \downarrow \\ \Sigma^j(\Sigma^i a \otimes b) & \longrightarrow & \Sigma^{i+j}(a \otimes b) \end{array}$$

*commutes up to the sign  $(-1)^{ij}$  for any  $a, b \in \mathcal{T}$  and  $i, j \in \mathbb{Z}$ .*

*Proof.* The following diagram establishes the claim when  $i, j \geq 0$ :

$$\begin{array}{ccccccc}
\Sigma^i a \otimes \Sigma^j b & \xrightarrow{\lambda} & \Sigma(\Sigma^{i-1} a \otimes \Sigma^j b) & \xrightarrow{\Sigma\lambda} & \dots & \xrightarrow{\Sigma^{i-2}\lambda} & \Sigma^{i-1}(\Sigma a \otimes \Sigma^j b) & \xrightarrow{\Sigma^{i-1}\lambda} & \Sigma^i(a \otimes \Sigma^j b) \\
\downarrow \rho & & \downarrow \Sigma\rho & & & & \downarrow \Sigma^{i-1}\rho & & \downarrow \Sigma^i\rho \\
\Sigma(\Sigma^i a \otimes \Sigma^{j-1} b) & \xrightarrow{\Sigma\lambda} & \Sigma^2(\Sigma^{i-1} a \otimes \Sigma^{j-1} b) & \xrightarrow{\Sigma^2\lambda} & \dots & \xrightarrow{\Sigma^{i-1}\lambda} & \Sigma^i(\Sigma a \otimes \Sigma^{j-1} b) & \xrightarrow{\Sigma^i\lambda} & \Sigma^{i+1}(a \otimes \Sigma^{j-1} b) \\
\downarrow \Sigma\rho & & \downarrow \Sigma^2\rho & & & & \downarrow \Sigma^i\rho & & \downarrow \Sigma^{i+1}\rho \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\Sigma^{j-1}(\Sigma^i a \otimes \Sigma b) & \xrightarrow{\Sigma^{j-1}\lambda} & \Sigma^j(\Sigma^{i-1} a \otimes \Sigma b) & \xrightarrow{\Sigma^j\lambda} & \dots & \xrightarrow{\Sigma^{i+j-3}\lambda} & \Sigma^{i+j-2}(\Sigma a \otimes \Sigma b) & \xrightarrow{\Sigma^{i+j-2}\lambda} & \Sigma^{i+j-1}(a \otimes \Sigma b) \\
\downarrow \Sigma^{j-1}\rho & & \downarrow \Sigma^j\rho & & & & \downarrow \Sigma^{i+j-2}\rho & & \downarrow \Sigma^{i+j-1}\rho \\
\Sigma^j(\Sigma^i a \otimes b) & \xrightarrow{\Sigma^j\lambda} & \Sigma^{j+1}(\Sigma^{i-1} a \otimes b) & \xrightarrow{\Sigma^{j+1}\lambda} & \dots & \xrightarrow{\Sigma^{i+j-2}\lambda} & \Sigma^{i+j-1}(\Sigma a \otimes b) & \xrightarrow{\Sigma^{i+j-1}\lambda} & \Sigma^{i+j}(a \otimes b).
\end{array}$$

A similar kind of diagram can be used to establish the  $i \geq 0, j < 0$  case using the diagram

$$\begin{array}{ccc}
\Sigma a \otimes \Sigma^{-1} b & \xrightarrow{\lambda} & \Sigma(a \otimes \Sigma^{-1} b) \\
\downarrow \tilde{\rho} & & \downarrow \Sigma\tilde{\rho} \\
& & \Sigma\Sigma^{-1}(a \otimes b) \\
& & \downarrow \sim \\
\Sigma^{-1}(\Sigma a \otimes b) & \xrightarrow{\Sigma^{-1}\lambda} & \Sigma^{-1}\Sigma(a \otimes b) \xrightarrow{\sim} a \otimes b
\end{array}$$

analogous to (3.3.6). This last diagram can be shown to commute after applying  $\Sigma$  by expanding out the definition of  $\Sigma\tilde{\rho}$  from Remark 3.3.17:

$$\begin{array}{ccc}
\Sigma(\Sigma a \otimes \Sigma^{-1} b) & \xrightarrow{\Sigma\lambda} & \Sigma^2(a \otimes \Sigma^{-1} b) \\
\downarrow \rho^{-1} & & \downarrow \Sigma\rho^{-1} \\
\Sigma a \otimes \Sigma\Sigma^{-1} b & \xrightarrow{\lambda} & \Sigma(a \otimes \Sigma\Sigma^{-1} b) \\
\downarrow \sim & & \downarrow \sim \\
\Sigma a \otimes b & \xrightarrow{\lambda} & \Sigma(a \otimes b) \\
\downarrow \sim & \searrow \lambda & \downarrow \sim \\
\Sigma\Sigma^{-1}(\Sigma a \otimes b) & \xrightarrow{\Sigma\Sigma^{-1}\lambda} & \Sigma\Sigma^{-1}\Sigma(a \otimes b) \xrightarrow{\sim} \Sigma(a \otimes b).
\end{array}$$

A similar approach can be used to prove the remaining cases:  $i < 0, j \geq 0$  and  $i < 0, j < 0$ .  $\square$

*Remark 3.3.19.* It follows from Lemma 2.2.7 that the endomorphism ring of the unit object in a tensor triangulated category is commutative. The anticommutativity of (3.3.6) ensures that the *graded* endomorphism ring of the unit object is graded-commutative:

**Lemma 3.3.20.** *Let  $\mathcal{T}$  be a tensor triangulated category. The graded endomorphism ring of the unit object is graded-commutative.*

*Proof.* Let  $f : \Sigma^i \mathbb{1} \rightarrow \mathbb{1}$  and  $g : \Sigma^j \mathbb{1} \rightarrow \mathbb{1}$  be two graded endomorphisms of the unit. In the following diagram

$$\begin{array}{ccccc}
 \Sigma^{i+j} \mathbb{1} & \xrightarrow{\Sigma^i g} & & \Sigma^i \mathbb{1} & \\
 \downarrow \Sigma^j f & \swarrow \Sigma^{i+j}(l_{\mathbb{1}}) & \Sigma^{i+j}(1 \otimes \mathbb{1}) \xrightarrow{(-1)^{ij}} \Sigma^i(1 \otimes \Sigma^j \mathbb{1}) & \searrow r_{\Sigma^i \mathbb{1}} & \\
 & \downarrow \Sigma^{i+j}(r_{\mathbb{1}}) & \downarrow & & \\
 & \Sigma^j(\Sigma^i \mathbb{1} \otimes \mathbb{1}) \xrightarrow{1 \otimes g} \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} & \xrightarrow{f \otimes 1} \Sigma^i \mathbb{1} \otimes \mathbb{1} & \xrightarrow{f} \Sigma^i \mathbb{1} & \\
 & \downarrow f \otimes 1 & \downarrow f \otimes 1 & & \\
 & \mathbb{1} \otimes \Sigma^j \mathbb{1} \xrightarrow{1 \otimes g} \mathbb{1} \otimes \mathbb{1} & \xrightarrow{l_{\mathbb{1}}} \mathbb{1} & & \\
 & \swarrow l_{\Sigma^j \mathbb{1}} & \searrow r_{\mathbb{1}} & & \\
 \Sigma^j \mathbb{1} & \xrightarrow{g} & & \mathbb{1} & 
 \end{array} \tag{3.3.21}$$

the right and bottom regions commute by naturality, the top-middle square commutes by Lemma 3.3.18, and the following diagram demonstrates the commutativity of the top region:

$$\begin{array}{ccccc}
 \Sigma^{i+j} \mathbb{1} & \xrightarrow{\Sigma^i g} & & \Sigma^i \mathbb{1} & \\
 \downarrow \Sigma^{i+j}(l_{\mathbb{1}}) & \swarrow \Sigma^i(l_{\Sigma^j \mathbb{1}}) & \Sigma^{i+j}(1 \otimes \mathbb{1}) \xrightarrow{\Sigma^i(1 \otimes g)} \Sigma^i(1 \otimes \mathbb{1}) & \searrow \Sigma^i(l_{\mathbb{1}}) & \\
 & \downarrow \Sigma^{i+j}(r_{\mathbb{1}}) & \downarrow & & \\
 & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \xrightarrow{1 \otimes g} \Sigma^i \mathbb{1} \otimes \mathbb{1} & \xrightarrow{r_{\Sigma^i \mathbb{1}}} \Sigma^i \mathbb{1} & & 
 \end{array} \tag{3.3.22}$$

Here the top region and middle square commute by naturality, while the right triangle

reduces to the commutativity of the following two diagrams:

$$\begin{array}{ccc}
\Sigma(1 \otimes 1) & \xrightarrow{\Sigma(l_1)} & \Sigma 1 \\
\downarrow \lambda^{-1} & \nearrow r_{\Sigma 1} & \\
\Sigma 1 \otimes 1 & & 
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Sigma^{-1}(1 \otimes 1) & \xrightarrow{\Sigma^{-1}(l_1)} & \Sigma^{-1} 1 \\
\downarrow \tilde{\lambda}^{-1} & \nearrow r_{\Sigma^{-1} 1} & \\
\Sigma^{-1} 1 \otimes 1 & & 
\end{array}$$

The commutativity of the first diagram follows immediately from Lemma 3.3.8. To see the commutativity of the second diagram, apply  $\Sigma$  and use the definition of  $\Sigma \tilde{\lambda}$  from Remark 3.3.17:

$$\begin{array}{ccc}
\Sigma \Sigma^{-1}(1 \otimes 1) & \xrightarrow{\Sigma \Sigma^{-1}(l_1) = \Sigma \Sigma^{-1}(r_1)} & \Sigma \Sigma^{-1} 1 \\
\downarrow \sim & & \uparrow \sim \\
1 \otimes 1 & \xrightarrow{r_1} & 1 \\
\downarrow \sim & & \uparrow \sim \\
\Sigma \Sigma^{-1} 1 \otimes 1 & \xrightarrow{r_{\Sigma \Sigma^{-1} 1}} & \Sigma \Sigma^{-1} 1 \\
\uparrow \lambda & \nearrow \Sigma r_{\Sigma^{-1} 1} & \\
\Sigma(\Sigma^{-1} 1 \otimes 1) & & 
\end{array}
\quad \text{id}$$

Here the bottom triangle commutes by Lemma 3.3.8 and the rest commutes by naturality. The left triangle of (3.3.22) can be proved in a similar manner using (3.3.3). Finally, a diagram similar to (3.3.22) demonstrates the commutativity of the bottom-left region in (3.3.21).  $\square$

**Notation 3.3.23.** Let  $\mathcal{T}$  be a tensor triangulated category. For any integer  $i \in \mathbb{Z}$ , we use the notation  $\pi_i : \mathcal{T} \rightarrow \text{Ab}$  to denote the functor  $[\Sigma^i 1, -] : \mathcal{T} \rightarrow \text{Ab}$ . The notation is inspired by the example  $\mathcal{T} = \text{SH}$  (see Section 3.8) in which case  $\pi_i$  gives the  $i$ th stable homotopy groups.

**Proposition 3.3.24.** *Let  $\mathcal{T}$  be a tensor triangulated category. The functor  $\pi_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  is a lax symmetric monoidal functor when  $\text{Ab}^{\mathbb{Z}}$  is given the “graded-commutative” symmetric monoidal structure.*

*Proof.* For any  $a, b \in \mathcal{T}$  and  $i, j \in \mathbb{Z}$ , define  $\pi_i(a) \times \pi_j(b) \rightarrow \pi_{i+j}(a \otimes b)$  to be the map which

sends  $(f, g)$  to the composite

$$\Sigma^{i+j}\mathbb{1} \simeq \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \simeq \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \xrightarrow{f \otimes g} a \otimes b.$$

Note that we are implicitly using Convention 3.3.16 in this definition. That this construction is bi-additive follows immediately from the fact that  $-\otimes-$  is additive in each variable. These maps thus provide a homomorphism of graded abelian groups  $\pi_*(a) \otimes \pi_*(b) \rightarrow \pi_*(a \otimes b)$  which is easily checked to be natural in  $a, b \in \mathcal{T}$ . We also have a unit map  $\mathbb{Z} \rightarrow \pi_0(\mathbb{1}) \subset \pi_*(\mathbb{1})$  which sends  $n \in \mathbb{Z}$  to  $n \cdot \text{id}_{\mathbb{1}}$ . Showing that these maps give  $\pi_*$  the structure of a lax symmetric monoidal functor amounts to showing that our “external product”  $\pi_i(a) \otimes \pi_j(b) \rightarrow \pi_{i+j}(a \otimes b)$  is associative, unital and graded-commutative. For three maps  $f \in \pi_i(a)$ ,  $g \in \pi_j(b)$  and  $h \in \pi_k(c)$ , one readily checks that the product  $(f \cdot g) \cdot h$  is a composite

$$\Sigma^{i+j+k}\mathbb{1} \simeq \Sigma^{i+j+k}(\mathbb{1} \otimes \mathbb{1}) \simeq (\Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1}) \otimes \Sigma^k \mathbb{1} \xrightarrow{(f \otimes g) \otimes h} (a \otimes b) \otimes c$$

while  $f \cdot (g \cdot h)$  is a composite

$$\Sigma^{i+j+k}\mathbb{1} \simeq \Sigma^{i+j+k}(\mathbb{1} \otimes \mathbb{1}) \simeq \Sigma^i \mathbb{1} \otimes (\Sigma^j \mathbb{1} \otimes \Sigma^k \mathbb{1}) \xrightarrow{f \otimes (g \otimes h)} a \otimes (b \otimes c).$$

Expanding the definitions, we see that associativity of our external product follows from the commutativity of

$$\begin{array}{ccccc}
\Sigma^{i+j+k}(\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\quad} & \Sigma^{i+j}(\mathbb{1} \otimes \Sigma^k \mathbb{1}) & \xrightarrow{\quad} & \Sigma^{i+j} \mathbb{1} \otimes \Sigma^k \mathbb{1} \\
\downarrow & \searrow & \uparrow & & \downarrow \\
& & \Sigma^{i+j+k}(\mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1})) & \xrightarrow{(\dagger)} & \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \otimes \Sigma^k \mathbb{1} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{i+j}(\mathbb{1} \otimes \Sigma^k \mathbb{1}) & \xrightarrow{\quad} & \Sigma^{i+j}(\mathbb{1} \otimes \Sigma^k(\mathbb{1} \otimes \mathbb{1})) & \xrightarrow{\quad} & \Sigma^{i+j}(\mathbb{1} \otimes (\mathbb{1} \otimes \Sigma^k \mathbb{1})) \xrightarrow{\Sigma^{i+j}\alpha} \Sigma^{i+j}((\mathbb{1} \otimes \mathbb{1}) \otimes \Sigma^k \mathbb{1}) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^i(\mathbb{1} \otimes \Sigma^{j+k} \mathbb{1}) & \xrightarrow{\quad} & \Sigma^i(\mathbb{1} \otimes \Sigma^{j+k}(\mathbb{1} \otimes \mathbb{1})) & \xrightarrow{(\ddagger)} & \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \otimes \Sigma^k \mathbb{1} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^i \mathbb{1} \otimes \Sigma^{j+k} \mathbb{1} & \xrightarrow{\quad} & \Sigma^i \mathbb{1} \otimes \Sigma^{j+k}(\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\quad} & \Sigma^i \mathbb{1} \otimes (\Sigma^j \mathbb{1} \otimes \Sigma^k \mathbb{1}) \xrightarrow{\alpha} (\Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1}) \otimes \Sigma^k \mathbb{1}
\end{array}$$

where the unlabelled morphisms are self-explanatory applications of unitors and suspension isomorphisms. The commutativity of (§) is demonstrated by the following diagram:

$$\begin{array}{ccccccc}
\Sigma^{i+j}(1 \otimes \Sigma^k(1 \otimes 1)) & \longrightarrow & \Sigma^{i+j}(1 \otimes (1 \otimes \Sigma^k 1)) & \longrightarrow & \Sigma^{i+j}((1 \otimes 1) \otimes \Sigma^k 1) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Sigma^i(1 \otimes \Sigma^{j+k}(1 \otimes 1)) & \longrightarrow & \Sigma^i(1 \otimes \Sigma^j(1 \otimes \Sigma^k 1)) & \xrightarrow{(1)} & \Sigma^i(\Sigma^j(1 \otimes 1) \otimes \Sigma^k 1) & \longrightarrow & \Sigma^{i+j}(1 \otimes 1) \otimes \Sigma^k 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \Sigma^i(1 \otimes (\Sigma^j 1 \otimes \Sigma^k 1)) & \longrightarrow & \Sigma^i((1 \otimes \Sigma^j 1) \otimes \Sigma^k 1) & \longrightarrow & \Sigma^i(1 \otimes \Sigma^j 1) \otimes \Sigma^k 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
\Sigma^i 1 \otimes \Sigma^{j+k}(1 \otimes 1) & \longrightarrow & \Sigma^i 1 \otimes (\Sigma^j 1 \otimes \Sigma^k 1) & \longrightarrow & & \longrightarrow & (\Sigma^i 1 \otimes \Sigma^j 1) \otimes \Sigma^k 1.
\end{array}$$

(2)

Here (1) commutes by Lemma 3.3.11 and (2) commutes from (3.3.4); the rest follows from naturality. For the commutativity of (§) we use the following diagram:

$$\begin{array}{ccccccc}
\Sigma^{i+j}(1 \otimes \Sigma^k(1 \otimes 1)) & \xrightarrow{\quad} & \Sigma^{i+j}(1 \otimes \Sigma^k 1) & \xrightarrow{\quad} & \Sigma^{i+j} 1 \otimes \Sigma^k 1 \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
& \Sigma^{i+j+k}(1 \otimes 1) & & & \\
& \swarrow \scriptstyle 1 \otimes l_1^{-1} & \downarrow \scriptstyle r_1^{-1} \otimes 1 & & \\
\Sigma^{i+j+k}(1 \otimes (1 \otimes 1)) & \xrightarrow{\alpha} & \Sigma^{i+j+k}((1 \otimes 1) \otimes 1) & \longrightarrow & \Sigma^{i+j}((1 \otimes 1) \otimes \Sigma^k 1) & \longleftarrow & \Sigma^{i+j}(1 \otimes 1) \otimes \Sigma^k 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{i+j}(1 \otimes \Sigma^k(1 \otimes 1)) & \xrightarrow{\quad} & \Sigma^{i+j}(1 \otimes (1 \otimes \Sigma^k 1)) & & & & 
\end{array}$$

id

Here the top-right square commutes by naturality. The bottom-left rectangle commutes using Lemma 3.3.12. The rest commutes by naturality and (2.2.3). On the other hand, the fact that our product is unital follows from

$$\begin{array}{ccccc}
\Sigma^i(1 \otimes 1) & \longrightarrow & \Sigma^i 1 \otimes 1 & \xrightarrow{f \otimes n.\text{id}_1 = n.(f \otimes \text{id}_1) = (n.f) \otimes \text{id}_1} & a \otimes 1 \\
\downarrow \scriptstyle \Sigma^i(r_1) & \swarrow \scriptstyle r_{\Sigma^i 1} & & & \downarrow \scriptstyle r_1 \\
\Sigma^i 1 & \xrightarrow{n.f} & & & a \\
\uparrow \scriptstyle \Sigma^i(l_1) & \swarrow \scriptstyle l_{\Sigma^i 1} & & & \uparrow \scriptstyle l_1 \\
\Sigma^i(1 \otimes 1) & \longrightarrow & 1 \otimes \Sigma^i 1 & \xrightarrow{(n.\text{id}_1) \otimes f = n.(\text{id}_1 \otimes f) = \text{id}_1 \otimes (n.f)} & 1 \otimes a
\end{array}$$

while graded-commutativity follows from

$$\begin{array}{ccccc}
\Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) & \xlongequal{\quad} & \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\sim} & \Sigma^i(\mathbb{1} \otimes \Sigma^j \mathbb{1}) \\
\downarrow \sim & & \downarrow \sim & \scriptstyle (-1)^{ij} & \downarrow \sim \\
\Sigma^j(\mathbb{1} \otimes \Sigma^i \mathbb{1}) & \xrightarrow{\Sigma^j(\tau)} & \Sigma^j(\Sigma^i \mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\sim} & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \\
\downarrow \sim & & \nearrow \tau & & \downarrow f \otimes g \\
\Sigma^j \mathbb{1} \otimes \Sigma^i \mathbb{1} & \xrightarrow{g \otimes f} & b \otimes a & \xrightarrow{\tau} & a \otimes b.
\end{array}$$

For this last diagram, note that  $\tau_{\mathbb{1}, \mathbb{1}} = \text{id}_{\mathbb{1} \otimes \mathbb{1}}$  by (2.2.13) and Lemma 2.2.6. The graded-commutativity of the top-right square comes from Lemma 3.3.18.  $\square$

*Remark 3.3.25.* Since  $\pi_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  is a lax symmetric monoidal functor, it preserves commutative monoids by Remark 2.2.31. The unit object  $\mathbb{1} \in \mathcal{T}$  has an obvious commutative monoid structure and every object  $X \in \mathcal{T}$  is canonically a left  $\mathbb{1}$ -module. Thus  $\pi_\bullet(\mathbb{1})$  has the structure of a graded-commutative ring and  $\pi_\bullet(X)$  has the structure of graded  $\pi_\bullet(\mathbb{1})$ -module. These observations will be considered more systematically in Section 3.5. For now we just want to note that the graded ring structure on  $\pi_\bullet(\mathbb{1})$  inherited via  $\pi_\bullet(-)$  from the commutative monoid structure on  $\mathbb{1}$  agrees with the usual ring structure of  $\pi_\bullet(\mathbb{1}) = [\mathbb{1}, \mathbb{1}]_\bullet$  regarded as a graded ring of endomorphisms. For  $f : \Sigma^i \mathbb{1} \rightarrow \mathbb{1}$  and  $g : \Sigma^j \mathbb{1} \rightarrow \mathbb{1}$  the former product  $f \cdot g$  is

$$\Sigma^{i+j} \mathbb{1} \simeq \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \simeq \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \xrightarrow{f \otimes g} \mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1}$$

while the latter product is

$$\Sigma^{i+j} \mathbb{1} \xrightarrow{\Sigma^i g} \Sigma^i \mathbb{1} \xrightarrow{f} \mathbb{1}.$$

That these coincide is demonstrated by

$$\begin{array}{ccccccc}
\Sigma^{i+j} \mathbb{1} & \xrightarrow{\Sigma^i l^{-1}} & \Sigma^i(\mathbb{1} \otimes \Sigma^j \mathbb{1}) & \xrightarrow{\sim} & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} & \xrightarrow{f \otimes g} & \mathbb{1} \otimes \mathbb{1} \\
\downarrow \Sigma^i g & & \downarrow \Sigma^i(1 \otimes g) & & \downarrow 1 \otimes g & \searrow f \otimes 1 & \downarrow \sim \\
& & \Sigma^i(\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\sim} & \Sigma^i \mathbb{1} \otimes \mathbb{1} & \xrightarrow{f \otimes 1} & \mathbb{1} \otimes \mathbb{1} \\
& & \downarrow \Sigma^i l_1 & & \downarrow l_1 & & \downarrow \sim \\
\Sigma^i \mathbb{1} & \xlongequal{\quad} & \Sigma^i \mathbb{1} & \xlongequal{\quad} & \Sigma^i \mathbb{1} & \xrightarrow{f} & \mathbb{1}.
\end{array}$$

*Example 3.3.26.* Let's consider some basic (non-triangulated) examples to illustrate the anti-commutativity of diagram (3.3.6). First consider the category of graded abelian groups  $\text{Ab}^{\mathbb{Z}}$ . Recall that this is a suspended additive category with  $(\Sigma X)_n := X_{n-1}$  and that the tensor product of abelian groups provides a tensor product:  $(X \otimes Y)_n := \bigoplus_{i+j=n} X_i \otimes Y_j$ . Note that

$$\begin{aligned} (\Sigma(X \otimes Y))_n &= \bigoplus_{i+j=n-1} X_i \otimes Y_j = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \\ (\Sigma X \otimes Y)_n &= \bigoplus_{i+j=n} X_{i-1} \otimes Y_j = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \\ (X \otimes \Sigma Y)_n &= \bigoplus_{i+j=n} X_i \otimes Y_{j-1} = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \end{aligned}$$

and we can take the suspension isomorphisms

$$\Sigma X \otimes Y = \Sigma(X \otimes Y) = X \otimes \Sigma Y$$

to be the identities. In this case diagram (3.3.6) commutes. However, now consider the category of chain complexes  $\text{Ch}(\text{Ab})$ . This is also a suspended additive category with differential defined by  $d_n^{\Sigma X} := -d_{n-1}^X$  and with monoidal structure provided by the tensor product of chain complexes:  $(X \otimes Y)_n := \bigoplus_{i+j=n} X_i \otimes Y_j$  with differential given by  $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ . One can check that the differential

$$\Sigma(X \otimes Y)_n = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \rightarrow \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-2-k} = \Sigma(X \otimes Y)_{n-1}$$

on  $\Sigma(X \otimes Y)$  is given by  $x \otimes y \mapsto -dx \otimes y + (-1)^{k+1} x \otimes dy$  for  $x \in X_k$  and  $y \in Y_{n-1-k}$ . Similarly, the differential

$$(\Sigma X \otimes Y)_n = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \rightarrow \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-2-k} = (\Sigma X \otimes Y)_{n-1}$$

on  $(\Sigma X \otimes Y)$  is also given by  $x \otimes y \mapsto -dx \otimes y + (-1)^{k+1} x \otimes dy$  for  $x \in X_k$  and  $y \in Y_{n-1-k}$ . Thus  $\Sigma X \otimes Y = \Sigma(X \otimes Y)$  as complexes and, as before, the identity gives us the suspension isomorphism. However, the differential

$$(X \otimes \Sigma Y)_n = \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-1-k} \rightarrow \bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-2-k} = (X \otimes \Sigma Y)_{n-1}$$

on  $X \otimes \Sigma Y$  is given by  $x \otimes y \mapsto dx \otimes y + (-1)^{k+1} x \otimes dy$  for  $x \in X_k$  and  $y \in Y_{n-1-k}$ . Thus  $X \otimes \Sigma Y \neq \Sigma(X \otimes Y)$  as complexes and the identity map is not a map of complexes. Instead, we take the suspension isomorphism  $X \otimes \Sigma Y \rightarrow \Sigma(X \otimes Y)$  to be the map which sends  $x \otimes y$  to  $(-1)^k x \otimes y$  for  $x \in X_k$ . One can then check that with these definitions diagram (3.3.6) commutes up to a sign. The situation is as follows

$$\begin{array}{ccc} \Sigma X \otimes \Sigma Y & \xrightarrow{\text{"id"}} & \Sigma(X \otimes \Sigma Y) \\ \downarrow \rho_{\Sigma X, Y} & & \downarrow \Sigma(\rho_{X, Y}) \\ \Sigma(\Sigma X \otimes Y) & \xrightarrow{\text{"id"}} & \Sigma^2(X \otimes Y) \end{array}$$

and the point is that the definition of  $\Sigma \rho_{X, Y} : \Sigma(X \otimes \Sigma Y) \rightarrow \Sigma^2(X \otimes Y)$  on  $x \otimes y$  depends on the degree of  $x$  in  $X$  while the definition of  $\rho_{\Sigma X, Y} : \Sigma X \otimes \Sigma Y \rightarrow \Sigma(\Sigma X \otimes Y)$  on  $x \otimes y$  depends on the degree of  $x$  in  $\Sigma X$ . More precisely,

$$(\Sigma \rho_{X, Y})_n = (\rho_{X, Y})_{n-1} : (X \otimes \Sigma Y)_{n-1} = \bigoplus_{k \in \mathbb{Z}} X_k \otimes (\Sigma Y)_{n-1-k} \rightarrow (X \otimes Y)_{n-2}$$

sends  $x \otimes y \in X_k \otimes (\Sigma Y)_{n-1-k} = X_k \otimes Y_{n-2-k}$  to  $(-1)^k x \otimes y$  while

$$(\rho_{\Sigma X, Y})_n : (\Sigma X \otimes \Sigma Y)_n = \bigoplus_{k \in \mathbb{Z}} (\Sigma X)_k \otimes (\Sigma Y)_{n-k} \rightarrow (\Sigma X \otimes Y)_{n-1}$$

sends  $x \otimes y \in (\Sigma X)_k \otimes (\Sigma Y)_{n-k}$  to  $(-1)^k x \otimes y$  and hence as a map defined on  $\bigoplus_{k \in \mathbb{Z}} X_k \otimes Y_{n-2-k}$  sends  $x \otimes y \in X_k \otimes Y_{n-2-k}$  to  $(-1)^{k+1} x \otimes y$ . A very explicit example can be obtained by taking  $X = Y = \mathbb{1}$ . Recall that the unit  $\mathbb{1}$  is the abelian group  $\mathbb{Z}$  regarded as a complex concentrated in degree zero. Diagram (3.3.6) is then

$$\begin{array}{ccc} \mathbb{Z}_1 \otimes \mathbb{Z}_0 & \xrightarrow{\text{id}} & \mathbb{Z}_0 \otimes \mathbb{Z}_1 \\ \downarrow -\text{id} & & \downarrow \text{id} \\ \mathbb{Z}_1 \otimes \mathbb{Z}_0 & \xrightarrow{\text{id}} & \mathbb{Z}_0 \otimes \mathbb{Z}_0 \end{array}$$

where the subscripts indicate the degree of that factor before taking the tensor product. The horizontal suspension isomorphisms are the identities, but the vertical suspension isomorphisms depend on the degree (according to this indexing scheme) of the first factor. Another

way of appreciating the difference between the suspension isomorphisms  $\Sigma(X \otimes Y) \simeq \Sigma X \otimes Y$  and  $\Sigma(X \otimes Y) \simeq X \otimes \Sigma Y$  in this example is that the second one introduces a sign because it passes the suspension past the factor  $X$ . This corresponds to the usual sign convention we have for the differential of a tensor product of complexes,  $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ , where a sign is introduced when we pass the differential past the first variable. Although  $\text{Ch}(\text{Ab})$  is not a triangulated category, these comments explain why the derived categories of Section 3.8 satisfy the anticommutativity axiom (3.3.6).

*Example 3.3.27.* Topology provides another example of this anticommutative phenomenon. The category of (compactly generated weakly Hausdorff [McC69, §2]) based topological spaces is a symmetric monoidal category under the smash product  $\wedge$  with the 0-sphere  $S^0$  serving as the unit object. There is a natural based homeomorphism  $\Sigma X \cong S^1 \wedge X$  where  $\Sigma X$  denotes reduced suspension, and in this setting

$$\Sigma X \wedge Y \cong S^1 \wedge X \wedge Y \xrightarrow{\text{id}} S^1 \wedge X \wedge Y \cong \Sigma(X \wedge Y)$$

and

$$X \wedge \Sigma Y \cong X \wedge S^1 \wedge Y \xrightarrow{\tau \wedge 1} S^1 \wedge X \wedge Y \cong \Sigma(X \wedge Y)$$

provide suspension homeomorphisms. The category of based spaces is not an additive category, so we can't speak of diagram (3.3.6) anticommuting; nevertheless, we see that

$$\begin{array}{ccc} S^1 \wedge X \wedge S^1 \wedge Y & \xlongequal{\quad} & S^1 \wedge X \wedge S^1 \wedge Y \\ \downarrow \tau_{S^1 \wedge X, S^1 \wedge Y} & & \downarrow S^1 \wedge \tau_{X, S^1 \wedge Y} \\ S^1 \wedge S^1 \wedge X \wedge Y & \xlongequal{\quad} & S^1 \wedge S^1 \wedge X \wedge Y \end{array}$$

can't possibly commute because the symmetry map  $S^1 \wedge S^1 \xrightarrow{\tau} S^1 \wedge S^1$  is not the identity. In fact, it is not hard to see that under the usual identification  $S^1 \wedge S^1 \cong S^2$  the twist map becomes a map  $S^2 \rightarrow S^2$  of degree  $-1$ . It follows that if we pass to the stable homotopy category (see Section 3.8) then  $\tau : S^1 \wedge S^1 \rightarrow S^1 \wedge S^1$  is  $-\text{id}$  and the diagram does in fact anticommute.

*Remark 3.3.28.* In our definition of a tensor triangulated category we required the monoidal structure to be symmetric. This omits some examples from our theory, such as derived categories of bimodules. On the other hand, we don't require the monoidal structure to be closed. If one assumes (as some authors do) that the monoidal structure is a closed symmetric monoidal structure then additional compatibility axioms should be required which relate the internal hom with the triangulated structure; namely, the internal hom  $F(-, -) : \mathcal{T}^{\text{op}} \times \mathcal{T} \rightarrow \mathcal{T}$  should be exact in each variable. (See Remark 3.1.6 for the triangulated structure on  $\mathcal{T}^{\text{op}}$ .) This includes the data of two natural isomorphisms

$$F(a, \Sigma b) \simeq \Sigma F(a, b) \quad \text{and} \quad F(\Sigma a, b) \simeq \Sigma^{-1} F(a, b) \quad (3.3.29)$$

which are further required to be adjoint to the maps

$$\Sigma F(a, b) \otimes a \simeq \Sigma(F(a, b) \otimes a) \xrightarrow{\Sigma(\text{ev})} \Sigma b$$

and

$$\Sigma F(\Sigma a, b) \otimes a \simeq \Sigma(F(\Sigma a, b) \otimes a) \simeq F(\Sigma a, b) \otimes \Sigma a \xrightarrow{\text{ev}} b$$

respectively. (The suspension isomorphisms (3.3.2) for the tensor product determine the suspension isomorphisms (3.3.29) for the internal hom—and vice versa.) There is a slightly subtle point here. Indeed, if  $\mathcal{T}$  is any compactly generated tensor triangulated category whose tensor product preserves coproducts (a very mild assumption) then Brown representability (Corollary 3.2.7) implies that every functor  $- \otimes b$  has a right adjoint and hence there exists an internal hom by Remark 2.2.45. However, it does not follow from this abstract argument that the resulting internal hom need preserve exact triangles. This is quite unfortunate because such compatibility between the internal hom and the triangulated structure is actually needed to prove some basic results; for example, see the proof of Proposition 3.4.5 below.

**Definition 3.3.30.** A morphism of tensor triangulated categories (or tensor triangulated functor) is an exact functor of triangulated categories  $(F, \theta) : \mathcal{T} \rightarrow \mathcal{S}$  which is also a strong symmetric monoidal functor  $(F, \varphi, \varphi_0) : \mathcal{T} \rightarrow \mathcal{S}$  such that the following diagram commutes

$$\begin{array}{ccccc}
 F\Sigma a \otimes Fb & \xrightarrow{\theta_a \otimes Fb} & \Sigma Fa \otimes Fb & \xrightarrow{\lambda_{Fa, Fb}} & \Sigma(Fa \otimes Fb) \\
 \downarrow \varphi_{\Sigma a, b} & & & & \downarrow \Sigma \varphi_{a, b} \\
 F(\Sigma a \otimes b) & \xrightarrow{F\lambda_{a, b}} & F\Sigma(a \otimes b) & \xrightarrow{\theta_{a \otimes b}} & \Sigma F(a \otimes b)
 \end{array} \tag{3.3.31}$$

for any  $a, b \in \mathcal{T}$ .

**Lemma 3.3.32.** Let  $F : \mathcal{T} \rightarrow \mathcal{S}$  be a tensor triangulated functor. Then the diagram

$$\begin{array}{ccccc}
 Fa \otimes F\Sigma b & \xrightarrow{Fa \otimes \theta_b} & Fa \otimes \Sigma Fb & \xrightarrow{\rho_{Fa, Fb}} & \Sigma(Fa \otimes Fb) \\
 \downarrow \varphi_{a, \Sigma b} & & & & \downarrow \Sigma \varphi_{a, b} \\
 F(a \otimes \Sigma b) & \xrightarrow{F(\rho_{a, b})} & F\Sigma(a \otimes b) & \xrightarrow{\theta_{a \otimes b}} & \Sigma F(a \otimes b)
 \end{array}$$

commutes for any  $a, b \in \mathcal{T}$ .

*Proof.* This is demonstrated by the following diagram:

$$\begin{array}{ccccccc}
 Fa \otimes F\Sigma b & \xrightarrow{\quad} & Fa \otimes \Sigma Fb & \xrightarrow{\quad} & \Sigma(Fa \otimes Fb) \\
 \downarrow \varphi_{a, \Sigma b} & \searrow \tau & \downarrow \tau & & \swarrow \Sigma \tau \\
 & F\Sigma b \otimes Fa & \longrightarrow & \Sigma Fb \otimes Fa & \longrightarrow & \Sigma(Fb \otimes Fa) \\
 & \downarrow & & \downarrow \Sigma \varphi_{b, a} & & \downarrow \Sigma \varphi_{a, b} \\
 & F(\Sigma b \otimes a) & \longrightarrow & F\Sigma(b \otimes a) & \xrightarrow{\theta_{b \otimes a}} & \Sigma F(b \otimes a) \\
 & \swarrow F\tau & \downarrow F\Sigma \tau & & \searrow \Sigma F\tau & \\
 F(a \otimes \Sigma b) & \xrightarrow{F\rho} & F\Sigma(a \otimes b) & \xrightarrow{\theta_{a \otimes b}} & \Sigma F(a \otimes b)
 \end{array}$$

The left and right squares commute using the fact that  $F$  is *symmetric* monoidal functor.

The middle square is axiom (3.3.31), while the top-right and bottom-left commute by (3.3.5).

The top-left and bottom-right commute by naturality.  $\square$

*Remark 3.3.33.* The naturality of our graded comparison maps introduced in Chapter 5 will depend on the compatibility axiom (3.3.31) for morphisms of tensor triangulated functors.

**Definition 3.3.34.** An *equivalence of tensor triangulated categories* is a tensor triangulated functor which is an equivalence of the underlying categories. According to Definition 3.1.29 and Remark 2.2.41 this is the same thing as an equivalence in the 2-category of tensor triangulated categories, tensor triangulated functors, and monoidal trnatural transformations.

*Remark 3.3.35.* The opposite category  $\mathcal{T}^{\text{op}}$  of a tensor triangulated category  $\mathcal{T}$  inherits a canonical tensor triangulated structure with essentially the same exact triangles; cf. Remark 2.2.17 and Remark 3.1.6. Just bear in mind that passing to the opposite category exchanges suspension with desuspension and modifies some signs.

*Remark 3.3.36.* In the setting of tensor triangulated categories, we'll often use the terms *tensor category* and *tensor functor*—abbreviated  $\otimes$ -category and  $\otimes$ -functor—as synonyms for “symmetric monoidal category” and “strong symmetric monoidal functor.” Nevertheless, we'll often write “strong  $\otimes$ -functor” for emphasis.

**Definition 3.3.37.** A full replete subcategory  $\mathcal{S} \subset \mathcal{T}$  of a tensor triangulated category  $\mathcal{T}$  is said to be  $\otimes$ -*ideal* if  $a \in \mathcal{T}$  and  $b \in \mathcal{S}$  implies that  $a \otimes b \in \mathcal{S}$ . A thick subcategory that is also a  $\otimes$ -ideal is simply called a *thick  $\otimes$ -ideal*. The smallest thick  $\otimes$ -ideal containing a collection of objects  $\mathcal{G} \subset \mathcal{T}$  is denoted  $\text{thick}_{\otimes} \langle \mathcal{G} \rangle$ .

### 3.4 Rigid categories

Recall from Section 2.2 that if every object in a symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbb{1})$  is dualizable then by choosing duals we obtain an equivalence  $D : \mathcal{C}^{\text{op}} \xrightarrow{\sim} \mathcal{C}$  which is unique up to isomorphism and which is a strong  $\otimes$ -functor when  $\mathcal{C}^{\text{op}}$  is given the induced monoidal structure:  $D\mathbb{1} \simeq \mathbb{1}$  and  $D(X \otimes Y) \simeq DX \otimes DY$ . In fact, when  $\mathcal{C}$  has additional structure, the functor  $D$  often preserves it. For example, if  $\mathcal{C}$  is a tensor triangulated category then the compatibility requirements between the suspension and the monoidal structure imply that

the duality functor  $D$  is a stable additive functor provided that  $\mathcal{C}^{\text{op}}$  is given the opposite suspended structure. In particular,  $D(\Sigma X) \simeq \Sigma^{-1}DX$  and  $D(X \oplus Y) \simeq DX \oplus DY$ . In contrast, the compatibility requirements between the triangulated structure (more precisely, the exact triangles) and the monoidal structure are fairly weak (cf. Remark 3.3.13) and there seems to be no reason why  $D$  would automatically preserve exact triangles (cf. Remark 3.3.28). Therefore, we make this condition part of our definition:

**Definition 3.4.1.** A tensor triangulated category  $\mathcal{K}$  is *rigid* if there exists an exact functor  $D : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$  and a natural isomorphism  $[X \otimes Y, Z] \simeq [X, DY \otimes Z]$ .

*Remark 3.4.2.* Rigidity is a *property* of a tensor triangulated category. Any functor between triangulated categories which is naturally isomorphic to an exact functor inherits the structure of an exact functor. Thus, since the duality  $D : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$  is unique up to isomorphism (if it exists), we can ask whether one (hence any) choice of duality functor admits the structure of an exact functor. In other words, a rigid tensor triangulated category is a tensor triangulated category in which every object is dualizable and such that any functorial choice of duals  $D : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$  admits the structure of an exact functor.

The following definition is the tensor-triangulated analogue of the notion of a compactly generated triangulated category:

**Definition 3.4.3.** A tensor triangulated category is said to be *rigidly-compactly generated* if

- (1)  $\mathcal{T}$  is compactly generated by a set of dualizable objects.
- (2) The unit is compact.
- (3) The symmetric monoidal structure admits a closed monoidal structure compatible with the triangulation.

*Remark 3.4.4.* This (slightly awkward) definition can be rephrased in several cosmetically different (but logically equivalent) ways. The main point of the definition is that it should be a compactly generated tensor triangulated category whose compact objects form a rigid tensor triangulated subcategory. For this to be the case, the  $\otimes$ -unit has to be compact and, moreover, under the very mild assumption that  $- \otimes -$  preserves coproducts in each variable, it follows that every dualizable object is compact.

**Proposition 3.4.5.** *If  $\mathcal{T}$  is a rigidly-compactly generated tensor triangulated category then the compact objects coincide with the dualizable objects. They form an essentially small, idempotent complete, rigid tensor triangulated category.*

*Proof.* The fact that  $\mathbb{1}$  is compact and  $- \otimes b$  preserves coproducts (since it has a right adjoint) implies that every dualizable object is compact. Indeed, one just needs to check that the diagram

$$\begin{array}{ccc}
 \oplus_i [X, Z_i] & \longrightarrow & [X, \coprod_i Z_i] \\
 \downarrow \sim & & \downarrow \sim \\
 & & [\mathbb{1}, DX \otimes \coprod_i Z_i] \\
 & & \downarrow \sim \\
 \oplus_i [\mathbb{1}, DX \otimes Z_i] & \longrightarrow & [\mathbb{1}, \coprod_i (DX \otimes Z_i)]
 \end{array}$$

commutes for every set of objects  $(Z_i)_{i \in I}$  in  $\mathcal{T}$ . This is readily accomplished by precomposing with  $[X, Z_n] \rightarrow \oplus_i [X, Z_i]$  and using the universal properties defining the maps in the diagram. On the other hand, since  $\mathcal{T}$  is closed symmetric monoidal, the dualizable objects can be characterized as those objects  $X \in \mathcal{T}$  for which the map  $v : F(X, \mathbb{1}) \otimes Z \rightarrow F(X, Z)$  from Remark 2.2.59 is a natural isomorphism. (This was proved in Lemma 2.2.61.) Using this characterization, we can show that the collection of dualizable objects forms a thick subcategory of  $\mathcal{T}$ . The hard part is showing that the cofiber of a map of dualizable objects is again dualizable. However,  $F(-, Z)$  preserves exact sequences by the assumed compatibility between

the closed structure and the tensor structure (cf. Remark 3.3.28) so if  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$  is an exact triangle then

$$\begin{array}{ccccccc}
F(C, \mathbb{1}) \otimes Z & \xrightarrow{F(g,1) \otimes 1} & F(B, \mathbb{1}) \otimes Z & \xrightarrow{F(f,1) \otimes 1} & F(A, \mathbb{1}) \otimes Z & \xrightarrow{F(\Sigma^{-1}h,1) \otimes 1} & F(\Sigma^{-1}C, \mathbb{1}) \otimes Z \\
\downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu \\
F(C, Z) & \xrightarrow{F(g,1)} & F(B, Z) & \xrightarrow{F(f,1)} & F(A, Z) & \xrightarrow{F(\Sigma^{-1}h,1)} & F(\Sigma^{-1}C, Z)
\end{array}$$

is a morphism of exact triangles. It follows from Lemma 3.1.9 that if two of the objects in an exact triangle are dualizable then so is the third. By assumption,  $\mathcal{T}$  is compactly generated by a set of dualizable objects  $\mathcal{G}$ . Remark 3.2.5 implies that  $\mathcal{T}^c = \text{thick}\langle \mathcal{G} \rangle$  but since the generating objects are dualizable and the dualizable objects form a thick subcategory we conclude that  $\mathcal{T}^c = \text{thick}\langle \mathcal{G} \rangle$  is contained in the collection of dualizable objects. This proves that the compact objects coincide with the dualizable objects. In any triangulated category the compact objects form a thick subcategory, while the dualizable objects in any symmetric monoidal category form a monoidal subcategory. It follows that the compact/dualizable objects form a tensor triangulated subcategory of  $\mathcal{T}$ . It is essentially small because it is generated by a set of objects. It is idempotent complete because it is a *thick* subcategory of  $\mathcal{T}$  and it is rigid because all the objects are dualizable. (The duality functor  $D(-) \simeq F(-, \mathbb{1})$  preserves exact triangles by assumption.)  $\square$

*Remark 3.4.6.* As in Remark 3.4.2 being rigidly-compactly generated is a property of a tensor triangulated structure since any two choices of internal hom are necessarily isomorphic and the compatibility conditions mentioned in Remark 3.3.28 can (1) be expressed purely in terms of the structure of the tensor triangulated category and (2) are invariant under replacing the internal hom with a naturally isomorphic bifunctor.

*Remark 3.4.7.* A rigidly-compactly generated tensor triangulated category is essentially the same thing as a “unital algebraic stable homotopy category” in the terminology of [HPS97].

That reference contains many interesting results about such categories. Dualizable objects in tensor triangulated categories are also discussed in [Bal07, Section 2].

*Remark 3.4.8.* Many of the tensor triangulated categories studied by tensor triangular geometry are rigid categories sitting as the compact/dualizable objects in a larger rigidly-compactly generated category. We'll see some examples in Section 3.8.

### 3.5 Homology theories

**Definition 3.5.1.** Let  $\mathcal{T}$  be a tensor triangulated category. Any object  $\mathbb{E} \in \mathcal{T}$  gives rise to an associated stable homological functor  $\mathbb{E}_\bullet : \mathcal{T} \rightarrow \mathbf{Ab}^{\mathbb{Z}}$  defined by  $\mathbb{E}_i(X) := \pi_i(\mathbb{E} \otimes X) = [\Sigma^i \mathbb{E}, X]$  and an associated stable cohomological functor  $\mathbb{E}^\bullet : \mathcal{T}^{\text{op}} \rightarrow \mathbf{Ab}^{\mathbb{Z}}$  defined by  $\mathbb{E}^i(X) := [X, \Sigma^i \mathbb{E}]$ . We call these the “homology theory” and “cohomology theory” associated to  $\mathbb{E}$ . Note that if  $X$  is dualizable then  $\mathbb{E}_\bullet(X) \cong \mathbb{E}^{-\bullet}(DX)$  and  $\mathbb{E}^\bullet(X) \cong \mathbb{E}_\bullet(DX)$ .

*Remark 3.5.2.* Recall from Proposition 3.3.24 that  $\pi_\bullet : \mathcal{T} \rightarrow \mathbf{Ab}^{\mathbb{Z}}$  is a lax symmetric monoidal functor when  $\mathbf{Ab}^{\mathbb{Z}}$  is equipped with the “graded-commutative” symmetric monoidal structure:

$$[\Sigma^i \mathbb{1}, X] \otimes [\Sigma^j \mathbb{1}, Y] \xrightarrow{\otimes} [\Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1}, X \otimes Y] \xrightarrow{\sim} [\Sigma^{i+j} \mathbb{1}, X \otimes Y].$$

Hence by Remark 2.2.31, if  $\mathbb{E} \in \mathcal{T}$  has the structure of a ring object then  $\mathbb{E}_\bullet := \mathbb{E}_\bullet(\mathbb{1}) \simeq \pi_\bullet(\mathbb{E})$  inherits the structure of a graded ring. Moreover, for any object  $X \in \mathcal{T}$ , the graded abelian group  $\mathbb{E}_\bullet(X)$  inherits the structure of a graded module over  $\mathbb{E}_\bullet$ :

$$\pi_\bullet(E) \otimes \pi_\bullet(E \otimes X) \rightarrow \pi_\bullet(E \otimes E \otimes X) \xrightarrow{\pi_*(\mu \otimes 1)} \pi_\bullet(\mathbb{E} \otimes X).$$

Indeed, if  $\mathbb{E}$  is a ring object in  $\mathcal{T}$  then the stable homological functor  $\mathbb{E}_\bullet : \mathcal{T} \rightarrow \mathbf{Ab}^{\mathbb{Z}}$  lifts to a stable homological functor  $\mathbb{E}_\bullet : \mathcal{T} \rightarrow \mathbb{E}_\bullet\text{-grMod}$ . The graded ring  $\mathbb{E}_\bullet = \pi_\bullet(\mathbb{E})$  is called the “coefficient ring” of the homology theory  $\mathbb{E}_\bullet(-)$ . Similarly, the graded abelian group  $\mathbb{E}^\bullet(X)$

inherits the structure of a graded module over the graded ring  $\mathbb{E}^\bullet := \mathbb{E}^\bullet(\mathbb{1}) = \mathbb{E}_\bullet(\mathbb{1}) = \mathbb{E}_\bullet$  and the cohomological functor  $\mathbb{E}^\bullet : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}^{\mathbb{Z}}$  lifts to a stable cohomological functor  $\mathbb{E}^\bullet : \mathcal{T}^{\text{op}} \rightarrow \mathbb{E}^\bullet\text{-grMod}$ . Note that the coefficient ring  $\mathbb{E}^\bullet$  of the cohomology theory is just  $\mathbb{E}_\bullet$  with the opposite grading.

*Remark 3.5.3.* If  $\mathcal{T} = \text{SH}$  is the stable homotopy category then a stable homological functor  $H_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  is essentially the same thing as a (reduced) generalized homology theory in the sense of the Eilenberg-Steenrod axioms. More precisely, it is the analogue of a (reduced) generalized homology theory defined on the stable homotopy category rather than on (say) the category of based CW-complexes. Similarly, stable cohomological functors  $H^\bullet : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}^{\mathbb{Z}}$  are essentially the same thing as (reduced) generalized cohomology theories. The idea that an arbitrary object  $\mathbb{E} \in \mathcal{T}$  should represent a cohomology theory  $\mathbb{E}^\bullet : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}^{\mathbb{Z}}$  was a fundamental idea lying at the heart of stable homotopy theory, but it was [Whi62] who first emphasized that  $\mathbb{E}$  also provides a homology theory  $\mathbb{E}_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$ . These homology theories are extremely important in stable homotopy theory and will play an important role in Chapter 6.

*Remark 3.5.4.* If  $\mathcal{T} = \text{SH}$  then the coefficient ring  $\mathbb{E}_\bullet = \mathbb{E}_\bullet(\mathbb{1})$  is the value of the reduced generalized homology theory on the 0-sphere  $S^0$  which is the same thing as the value of the corresponding unreduced generalized homology theory on the point  $*$ . The original axioms for unreduced (co)homology put forward by Eilenberg and Steenrod included a “dimension axiom” which asserted that the (co)homology of a point was concentrated in a single degree. Generalized (co)homology theories are theories satisfying all of the Eilenberg-Steenrod axioms except for the dimension axiom.

*Remark 3.5.5.* In this context it is an interesting question to ask whether a given stable (co)homological functor (a.k.a. generalized (co)homology theory) is the (co)homology theory associated to an object  $\mathbb{E} \in \mathcal{T}$ . A necessary condition for a stable cohomological functor  $H^\bullet$  to

be represented by an object  $\mathbb{E}$  is that  $H^\bullet$  must send coproducts to products: an analogue of Milnor's Wedge Axiom. Moreover, it follows from Brown Representability (Theorem 3.2.6) that this necessary condition is also sufficient. In fact this holds in any compactly generated category since these categories satisfy the Brown Representability theorem. To be clear: if  $H^\bullet : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}^{\mathbb{Z}}$  is a stable cohomological functor defined on a compactly generated triangulated category  $\mathcal{T}$  which sends coproducts in  $\mathcal{T}$  to products in  $\text{Ab}^{\mathbb{Z}}$  then there exists an object  $\mathbb{E} \in \mathcal{T}$  and a stable natural isomorphism  $H^\bullet(X) \simeq \mathbb{E}^\bullet(X)$ . However, representability for homology theories is more subtle. Note that we are not asking for a homology theory  $H_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  to be a representable functor in the categorical sense; rather, we are asking for the existence of an object  $\mathbb{E}$  such that  $H_\bullet(X) \simeq [\Sigma^\bullet \mathbb{1}, \mathbb{E} \otimes X]$ . This question is related to a different kind of representability called “Brown-Adams representability.” Adams [Ada71] showed (when  $\mathcal{T} = \text{SH}$ ) that every cohomological functor

$$H : (\mathcal{T}^c)^{\text{op}} \rightarrow \text{Ab}$$

defined on the subcategory of compact objects  $\mathcal{T}^c \subset \mathcal{T}$  is the restriction of a representable functor  $[-, \mathbb{E}] : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$  and, moreover, every natural transformation  $[-, \mathbb{E}]|_{\mathcal{T}^c} \rightarrow [-, \mathbb{F}]|_{\mathcal{T}^c}$  between two such functors is induced by a map  $\mathbb{E} \rightarrow \mathbb{F}$  between the representing objects. This can be used to answer our question in the following way: given a stable homological functor  $H_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  one can use duality to define a stable cohomological functor  $H^\bullet : (\mathcal{T}^c)^{\text{op}} \rightarrow \text{Ab}^{\mathbb{Z}}$  on the subcategory of compact objects by  $H^i(X) := H_{-i}(DX)$ . It follows from Adams theorem that there exists an object  $\mathbb{E} \in \mathcal{T}$  such that  $H^\bullet(X) \simeq \mathbb{E}^\bullet(X)$  for each *compact* object  $X \in \mathcal{T}^c$  and hence that  $H_\bullet(X) \simeq \mathbb{E}_\bullet(X)$  for each compact  $X$ . One then needs a homological analogue of Milnor's axiom to ensure that this equivalence extends from compact objects to the whole category. The correct analogue is the requirement that our original homological functor  $H_\bullet$  satisfy

$$\text{colim}_\alpha H_\bullet(X_\alpha) \xrightarrow{\sim} H_\bullet(X) \tag{3.5.6}$$

for every object  $X$  where the colimit is over all compact objects  $X_\alpha \rightarrow X$  mapping into  $X$ . The statement then becomes: any stable homological functor  $H_\bullet : \text{SH} \rightarrow \text{Ab}^\mathbb{Z}$  which satisfies (3.5.6) for every object  $X \in \text{SH}$  is representable. Condition (3.5.6) can be generalized satisfactorily to any tensor triangulated category generated in a nice enough way from a good collection of objects—e.g., for a rigidly-compactly generated tensor triangulated category (see e.g. [HPS97, Definition 2.3.7 and Section 4.1]). However, Adams theorem rarely holds for general triangulated categories (see for example [Nee97]). Thus tensor triangulated categories satisfying homological representability are much rarer than those satisfying cohomological representability. Those rare categories which do satisfy homological representability (such as  $\text{SH}$ ) are called “Brown categories” in the axiomatic framework of [HPS97].

We finish this section with a minor technical remark which is needed to understand some comments made in Chapter 6.

**Lemma 3.5.7.** *Let  $\mathbb{E}$  be a commutative ring object in a tensor triangulated category  $\mathcal{T}$ . There is a natural transformation*

$$\mathbb{E}_\bullet(X) \otimes_{\mathbb{E}_\bullet} \mathbb{E}_\bullet(Y) \rightarrow \mathbb{E}_\bullet(X \otimes Y).$$

*defined for any two objects  $X, Y \in \mathcal{T}$ , which is induced by the map*

$$\pi_\bullet(\mathbb{E} \otimes X) \otimes \pi_\bullet(\mathbb{E} \otimes Y) \xrightarrow{\varphi} \pi_\bullet(\mathbb{E} \otimes X \otimes \mathbb{E} \otimes Y) \simeq \pi_\bullet(\mathbb{E} \otimes \mathbb{E} \otimes X \otimes Y) \xrightarrow{\pi_\bullet(\mu \otimes 1 \otimes 1)} \pi_\bullet(\mathbb{E} \otimes X \otimes Y). \quad (3.5.8)$$

*Note that  $\mathbb{E}_\bullet(X)$  can be regarded as a right  $\mathbb{E}_\bullet$ -module since  $\mathbb{E}_\bullet$  is a commutative monoid in a symmetric monoidal category.*

*Proof.* We need to check that there is an induced map on the following coequalizer

$$\begin{array}{ccc} \pi_\bullet(\mathbb{E} \otimes X) \otimes \pi_\bullet(\mathbb{E}) \otimes \pi_\bullet(\mathbb{E} \otimes Y) & \xrightarrow[1 \otimes \lambda]{\rho \otimes 1} & \pi_\bullet(\mathbb{E} \otimes X) \otimes \pi_\bullet(\mathbb{E} \otimes Y) \longrightarrow \pi_\bullet(\mathbb{E} \otimes X) \otimes_{\pi_\bullet(\mathbb{E})} \pi_\bullet(\mathbb{E} \otimes Y) \\ & & \downarrow \hat{\varphi} \\ & & \pi_\bullet(\mathbb{E} \otimes X \otimes Y) \end{array}$$

(A dotted arrow points from the top right object to the bottom object.)

where  $\rho$  and  $\lambda$  denote the right and left actions of  $\mathbb{E}$ . and  $\hat{\varphi}$  denotes the composite (3.5.8). In other words, we need to check that  $\hat{\varphi} \circ (\rho \otimes 1) = \hat{\varphi} \circ (1 \otimes \lambda)$ . To this end, consider the following diagram

$$\begin{array}{ccccc}
\pi(E \otimes X) \otimes \pi(E) \otimes \pi(E \otimes Y) & \xrightarrow{1 \otimes \varphi} & \pi(E \otimes X) \otimes \pi(E \otimes E \otimes Y) & \xrightarrow{1 \otimes \pi(\mu \otimes 1)} & \pi(E \otimes X) \otimes \pi(E \otimes Y) \\
\downarrow \varphi \otimes 1 & & \downarrow \varphi & & \downarrow \varphi \\
\pi(E \otimes X \otimes E) \otimes \pi(E \otimes Y) & \xrightarrow{\varphi} & \pi(E \otimes X \otimes E \otimes E \otimes Y) & \xrightarrow{\pi(1 \otimes 1 \otimes \mu \otimes 1)} & \pi(E \otimes X \otimes E \otimes Y) \\
\downarrow \pi(1 \otimes \tau) \otimes 1 & & \downarrow \pi(1 \otimes \tau \otimes 1) & & \downarrow \pi(1 \otimes \tau \otimes 1) \\
\pi(E \otimes E \otimes X) \otimes \pi(E \otimes Y) & \xrightarrow{\pi(1 \otimes \tau \otimes 1 \otimes 1)} & \pi(E \otimes E \otimes E \otimes X \otimes Y) & \xrightarrow{\pi(1 \otimes \mu \otimes 1 \otimes 1)} & \pi(E \otimes E \otimes X \otimes Y) \\
\downarrow \varphi & & \downarrow \pi(\mu \otimes 1 \otimes 1 \otimes 1) & & \downarrow \pi(\mu \otimes 1 \otimes 1) \\
\pi(E \otimes E \otimes X \otimes E \otimes Y) & \xrightarrow{\pi(1 \otimes 1 \otimes \tau \otimes 1)} & \pi(E \otimes E \otimes E \otimes X \otimes Y) & \xrightarrow{\pi(1 \otimes \mu \otimes 1 \otimes 1)} & \pi(E \otimes E \otimes X \otimes Y) \\
\downarrow \pi(\mu \otimes 1 \otimes 1 \otimes 1) & & \downarrow \pi(\mu \otimes 1 \otimes 1 \otimes 1) & & \downarrow \pi(\mu \otimes 1 \otimes 1) \\
\pi(E \otimes X \otimes E \otimes Y) & \xrightarrow{\pi(1 \otimes \tau \otimes 1)} & \pi(E \otimes E \otimes X \otimes Y) & \xrightarrow{\pi(\mu \otimes 1 \otimes 1)} & \pi(E \otimes X \otimes Y)
\end{array}$$

where we have written  $\pi$  for  $\pi_*$  in order to simplify notation. Going along the top and right-hand side we get  $\hat{\varphi} \circ (1 \otimes \lambda)$ , while

$$\begin{array}{ccccc}
\pi(E \otimes X) \otimes \pi(E) \otimes \pi(E \otimes X) & \xrightarrow{\varphi \otimes 1} & \pi(E \otimes X \otimes E) \otimes \pi(E \otimes X) & & \\
\downarrow \tau \otimes 1 & & \downarrow \pi(1 \otimes \tau) \otimes 1 & & \\
\pi(E) \otimes \pi(E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\pi(\tau) \otimes 1} & \pi(E \otimes E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\varphi} & \pi(E \otimes E \otimes X \otimes E \otimes Y) \\
\downarrow \varphi \otimes 1 & & \downarrow \pi(\mu \otimes 1) \otimes 1 & & \downarrow \pi(\mu \otimes 1 \otimes 1 \otimes 1) \\
\pi(E \otimes E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\pi(\tau \otimes 1) \otimes 1} & \pi(E \otimes E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\varphi} & \pi(E \otimes E \otimes X \otimes E \otimes Y) \\
\downarrow \pi(\mu \otimes 1) \otimes 1 & & \downarrow \pi(\mu \otimes 1) \otimes 1 & & \downarrow \pi(\mu \otimes 1 \otimes 1 \otimes 1) \\
\pi(E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\varphi} & \pi(E \otimes X) \otimes \pi(E \otimes X) & \xrightarrow{\varphi} & \pi(E \otimes X \otimes E \otimes Y).
\end{array}$$

shows that  $\hat{\varphi} \circ (1 \otimes \rho)$  is obtained by going down the left-hand side and along the bottom. Note that in the bottom-left square of the last diagram we have used the fact that  $\mathbb{E}$  is commutative.  $\square$

### 3.6 Verdier localization

The most fundamental construction in the theory of triangulated categories is the *Verdier quotient*. Before introducing this construction, we recall the general notion of localization in category theory.

**Definition 3.6.1.** Let  $S$  be a collection of morphisms in a category  $\mathcal{C}$ . The *localization*  $\mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  is defined to be the universal functor out of  $\mathcal{C}$  which inverts the morphisms in  $S$ . More precisely,  $\mathcal{C}[S^{-1}]$  is a category equipped with a functor  $q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  such that

- (1)  $q(s)$  is an isomorphism for every  $s \in S$ ; and
- (2) for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that  $F(s)$  is an isomorphism for every  $s \in S$ , there exists a unique functor  $\bar{F} : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  such that  $\bar{F} \circ q = F$ .

*Remark 3.6.2.* The above definition can be rephrased as follows: condition (1) is equivalent to the statement that for any category  $\mathcal{D}$ , the functor  $- \circ q : \mathcal{D}^{\mathcal{C}[S^{-1}]} \rightarrow \mathcal{D}^{\mathcal{C}}$  factors through the full subcategory  $\mathcal{D}^{(\mathcal{C}, S)} \subset \mathcal{D}^{\mathcal{C}}$  consisting of those functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  which invert the morphisms in  $S$ , and condition (2) is the statement that the functor  $- \circ q : \mathcal{D}^{\mathcal{C}[S^{-1}]} \rightarrow \mathcal{D}^{(\mathcal{C}, S)}$  is a bijection on objects. The point we would like to make is that it actually follows from the definition that  $- \circ q : \mathcal{D}^{\mathcal{C}[S^{-1}]} \rightarrow \mathcal{D}^{(\mathcal{C}, S)}$  is fully faithful. In other words, natural transformations  $\alpha : F_1 \rightarrow F_2$  between functors  $F_i : \mathcal{C} \rightarrow \mathcal{D}$  which invert  $S$  correspond bijectively with natural transformations between the induced functors  $\bar{F}_i : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$ . In order to explain this fact, let  $\text{Arr}(\mathcal{D})$  denote the “arrow category” whose objects are the morphisms in  $\mathcal{D}$  and whose morphisms are the commutative squares of morphisms in  $\mathcal{D}$ . There are two “projection” functors  $p_1, p_2 : \text{Arr}(\mathcal{D}) \rightarrow \mathcal{D}$  and one sees that natural transformations  $\alpha : F_1 \rightarrow F_2$  correspond uniquely to functors  $\hat{\alpha} : \mathcal{C} \rightarrow \text{Arr}(\mathcal{D})$  such that  $p_i \circ \hat{\alpha} = F_i$  for  $i = 1, 2$ . If the  $F_i$  invert  $S$  then one easily observes that  $\hat{\alpha} : \mathcal{C} \rightarrow \text{Arr}(\mathcal{D})$  also inverts  $S$  and hence there exists a unique

functor  $\beta: \mathcal{C}[S^{-1}] \rightarrow \mathbf{Arr}(\mathcal{D})$  such that  $\beta \circ q = \hat{\alpha}$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\hat{\alpha}} & \mathbf{Arr}(\mathcal{D}) \xrightleftharpoons[p_2]{p_1} \mathcal{D} \\ q \downarrow & \nearrow \exists! \beta & \\ \mathcal{C}[S^{-1}] & & \end{array}$$

This functor  $\beta$  describes a natural transformation  $\bar{F}_1 \rightarrow \bar{F}_2$  and one readily checks that this construction produces a map

$$\mathcal{D}^{(\mathcal{C}, S)}(F_1, F_2) \rightarrow \mathcal{D}^{\mathcal{C}[S^{-1}]}(\bar{F}_1, \bar{F}_2)$$

which is inverse to the map

$$\mathcal{D}^{\mathcal{C}[S^{-1}]}(\bar{F}_1, \bar{F}_2) \xrightarrow{- \circ q} \mathcal{D}^{(\mathcal{C}, S)}(F_1, F_2).$$

The author thanks John Bourke for explaining this point to him.

*Remark 3.6.3.* If the localization  $\mathcal{C}[S^{-1}]$  exists then it is unique up to isomorphism; however, the existence of  $\mathcal{C}[S^{-1}]$  involves some set-theoretic issues. There is a simple formal construction of  $\mathcal{C}[S^{-1}]$  obtained by taking the same objects as  $\mathcal{C}$  and by taking morphisms to be equivalence classes of zig-zags of morphisms in  $\mathcal{C}$  (see [GZ67, Section I.1] for a precise statement) but this category need not be locally small. More seriously, there is no feasible way to work with this formal construction—it is useless for all practical purposes. Fortunately, Gabriel and Zisman showed that the localization  $\mathcal{C}[S^{-1}]$  has a much more accessible construction when the collection of morphisms  $S$  satisfies the conditions of the following definition. We follow the terminology of [GZ67, Section I.2].

**Definition 3.6.4.** A collection of morphisms  $S$  in a category  $\mathcal{C}$  is said to *admit a calculus of left fractions* if

- (1) The collection  $S$  is closed under composition and contains all the identity morphisms.

(2) Any diagram  $X' \xleftarrow{s} X \xrightarrow{f} Y$  in  $\mathcal{C}$  with  $s \in S$  can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow s & & \downarrow s' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

such that  $s' \in S$ .

(3) Consider two parallel morphisms  $f, g : X \rightarrow Y$  in  $\mathcal{C}$ . If there is a morphism  $s : X' \rightarrow X$  in  $S$  with  $f \circ s = g \circ s$  then there exists a morphism  $t : Y \rightarrow Y'$  in  $S$  with  $t \circ f = t \circ g$ .

*Remark 3.6.5.* A “left fraction” is a diagram  $X \xrightarrow{f} Y' \xleftarrow{s} Y$  with  $s \in S$ . We abbreviate it  $(f, s)$  and think of it as a “fraction”  $s^{-1}f$ . The significance of the second axiom is that  $f s^{-1}$  can be replaced by  $(s')^{-1}f'$ ; hence, in a zig-zag  $f_1 s_1^{-1} f_2 s_2^{-1} \cdots f_n s_n^{-1}$  all the  $s_i^{-1}$  can be “moved to the left” so that zig-zags reduce to left fractions. Indeed, when  $S$  admits a calculus of left fractions the formal construction of  $\mathcal{C}[S^{-1}]$  in terms of “zig-zags” has the following description in terms of “fractions”:

**Theorem 3.6.6** (Gabriel-Zisman). *Let  $S$  be a collection of morphisms in a category  $\mathcal{C}$  admitting a calculus of left fractions. Define  $S^{-1}\mathcal{C}$  to be the category which has the same objects as  $\mathcal{C}$  and whose morphisms  $X \rightarrow Y$  are the equivalence classes of “left fractions”  $X \xrightarrow{f} Y' \xleftarrow{s} Y$  with  $f \in \text{Mor}(\mathcal{C})$  and  $s \in S$ , where two left fractions  $X \xrightarrow{f_1} Y_1 \xleftarrow{s_1} Y$  and  $X \xrightarrow{f_2} Y_2 \xleftarrow{s_2} Y$  are equivalent if there exists a commutative diagram*

$$\begin{array}{ccccc} & & Y_1 & & \\ & \nearrow f_1 & \downarrow & \nwarrow s_1 & \\ X & \xrightarrow{f_3} & Y_3 & \xleftarrow{s_3} & Y \\ & \searrow f_2 & \uparrow & \swarrow s_2 & \\ & & Y_2 & & \end{array}$$

with  $s_3 \in S$ . Two equivalence classes  $[f, s]$  and  $[g, t]$  are composed by using axiom (2) to

construct a diagram

$$\begin{array}{ccccc}
 & & Z'' & & \\
 & g' \nearrow & & \nwarrow s' & \\
 & Y' & & Z' & \\
 f \nearrow & & & & \nwarrow t \\
 X & & Y & & Z
 \end{array}$$

where  $s' \in S$ , and defining the composite to be  $[g' \circ f, s' \circ t]$ . This is well-defined. There is a canonical functor  $\mathcal{C} \rightarrow S^{-1}\mathcal{C}$  which is the identity on objects and sends  $X \xrightarrow{f} Y$  to  $[f, \text{id}_Y]$ . It satisfies the universal property for the localization of  $\mathcal{C}$  with respect to  $S$  and thus can be regarded as the localization  $\mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ .

*Proof.* For full details (e.g., that composition is well-defined and that  $\mathcal{C} \rightarrow S^{-1}\mathcal{C}$  satisfies the universal property) see [GZ67, §I.2].  $\square$

*Remark 3.6.7.* Even if  $S$  is a set of morphisms there is no reason to expect that the equivalence classes of fractions  $\text{Hom}_{S^{-1}\mathcal{C}}(X, Y)$  between any two objects need form a set. However, if  $\mathcal{C}$  is *essentially small* then the localization  $\mathcal{C}[S^{-1}] \cong S^{-1}\mathcal{C}$  exists for any collection of morphisms  $S$  which is closed under isomorphisms and admits a left calculus of fractions. Indeed, choose a small skeleton  $\mathcal{C}_0 \subset \mathcal{C}$ . Given any left fraction  $(X \xrightarrow{f} Y_1 \xleftarrow{s} Y)$  we can choose an isomorphism  $\alpha : Y_1 \xrightarrow{\sim} Y_0$  for some  $Y_0 \in \mathcal{C}_0$ . Then the diagram

$$\begin{array}{ccccc}
 & & Y_1 & & \\
 & f \nearrow & \parallel & \nwarrow s & \\
 X & \xrightarrow{f} & Y_1 & \xleftarrow{s} & Y \\
 \alpha \circ f \searrow & & \uparrow \alpha^{-1} & & \swarrow \alpha \circ s \\
 & & Y_0 & & 
 \end{array}$$

shows that  $[(X \xrightarrow{f} Y_1 \xleftarrow{s} Y)] = [(X \xrightarrow{\alpha \circ f} Y_0 \xleftarrow{\alpha \circ s} Y)]$ . Thus  $\text{Hom}_{S^{-1}\mathcal{C}}(X, Y)$  is a quotient of a subset of the set  $\bigcup_{Y_0 \in \mathcal{C}_0} \text{Hom}_{\mathcal{C}}(X, Y_0) \times \text{Hom}_{\mathcal{C}}(Y, Y_0)$  and hence is a set.

**Definition 3.6.8.** Let  $\mathcal{T}$  be a triangulated category and let  $\mathcal{S} \subset \mathcal{T}$  be a triangulated subcategory. The *Verdier quotient*  $\mathcal{T}/\mathcal{S}$  is defined to be the localization

$$\mathcal{T}/\mathcal{S} := \mathcal{T}[S^{-1}]$$

where  $S := \{f \in \text{Mor}(\mathcal{T}) \mid \text{cone}(f) \in \mathcal{S}\}$ .

**Theorem 3.6.9** (Verdier). *Let  $\mathcal{T}$  be a triangulated category and let  $\mathcal{S} \subset \mathcal{T}$  be a triangulated subcategory.*

- (1) *The collection  $S := \{f \in \text{Mor}(\mathcal{T}) \mid \text{cone}(f) \in \mathcal{S}\}$  admits a left calculus of fractions.*
- (2) *The Verdier quotient  $\mathcal{T}/\mathcal{S}$  admits the structure of a triangulated category such that  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  is an exact functor. The exact triangles in  $\mathcal{T}/\mathcal{S}$  are those triangles isomorphic to the image of an exact triangle in  $\mathcal{T}$ .*
- (3) *The Verdier quotient  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  is the universal exact functor out of  $\mathcal{T}$  which kills the objects of  $\mathcal{S}$ .*
- (4) *The kernel of  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  is the thick subcategory generated by  $\mathcal{S}$ .*

*Proof.* Since  $\mathcal{S}$  is triangulated, the octahedral axiom implies that  $S$  is closed under composition. Next, given  $X' \xleftarrow{s} X \xrightarrow{f} Y$  we can construct a morphism of exact triangles

$$\begin{array}{ccccccc} \Sigma^{-1}Z & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & Z \\ \parallel & & \downarrow s & & \downarrow s' & & \parallel \\ \Sigma^{-1}Z & \cdots\cdots\cdots & X' & \xrightarrow{f'} & Y' & \cdots\cdots\cdots & Z \end{array}$$

and it follows from Verdier's lemma (a.k.a. the 9-lemma for triangulated categories) that  $\text{cone}(s') \simeq \text{cone}(s)$ ; hence  $s' \in S$ . Finally, given  $X' \xrightarrow{s} X \xrightarrow{f} Y$  we can choose an exact triangle

$$\begin{array}{ccccc} X' & \xrightarrow{s} & X & \longrightarrow & Z \longrightarrow \Sigma X' \\ & & \downarrow f-g & \swarrow u & \\ & & Y & & \end{array}$$

and obtain a map  $u$  by Lemma 3.1.7. Then choosing an exact triangle for  $u$  we have

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow f-g & & \\ Z & \xrightarrow{u} & Y & \xrightarrow{t} & Y' \longrightarrow \Sigma Z \end{array}$$

and it follows that  $t \circ (f - g) = 0$ . Furthermore, note that  $\text{cone}(t) \simeq \Sigma \text{cone}(s) \in \mathcal{S}$  so  $t \in S$ . This establishes that  $S$  admits a calculus of left fractions. Next we wish to establish that  $\mathcal{T}/S = \mathcal{T}[S^{-1}]$  inherits the structure of a triangulated category from  $\mathcal{T}$ . It follows from [GZ67, Chapter I, Corollary 3.2] that  $\mathcal{T}/S$  is an additive category and that  $q : \mathcal{T} \rightarrow \mathcal{T}/S$  is an additive functor. Moreover, since  $S$  is stable under suspension,  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  induces a unique functor  $\bar{\Sigma} : \mathcal{T}/S \rightarrow \mathcal{T}/S$  such that  $\bar{\Sigma} \circ q = q \circ \Sigma$ . That this is an equivalence of categories follows from Remark 3.6.2. The next step is to show that

- (1) the collection of morphisms  $S$  also admits a “calculus of right fractions” (the notion dual to Definition 3.6.4); and that
- (2) if  $(u, v, w)$  is a morphism of exact triangles with  $u, v \in S$  then there exists a morphism  $(u, v, w')$  with  $w' \in S$ .

These additional facts can be used to show that  $\mathcal{T}[S^{-1}]$  has the structure of a triangulated category when equipped with those triangles that are isomorphic to the image of an exact triangle in  $\mathcal{T}$ . For a proof see [Ver96, Chapter II, Theorem 2.2.6]. In fact, this triangulated structure on  $\mathcal{T}/S$  is the *unique* such structure such that  $q : \mathcal{T} \rightarrow \mathcal{T}/S$  is an exact functor when equipped with the canonical suspension  $\bar{\Sigma}$  and strict suspension isomorphism  $q \circ \Sigma = \bar{\Sigma} \circ q$ . The universal property stated in (3) is easy to establish using the universal property for  $q$  together with Remark 3.6.2. Finally, we will prove that  $\text{Ker } q = \text{thick}\langle S \rangle$ . It is clear that the kernel of an exact functor is a thick subcategory; hence  $\text{thick}\langle S \rangle \subset \text{Ker } q$ . On the other hand, if  $q(X) = 0$  then  $[(X \xrightarrow{0} X \xleftarrow{\text{id}} X)] = [(X \xrightarrow{\text{id}} X \xleftarrow{\text{id}} X)]$  implies that there is a morphism  $s : X \rightarrow Z$

in  $\mathcal{S}$  such that

$$\begin{array}{ccccc}
 & & X & & \\
 & \nearrow \text{id} & \downarrow s & \nwarrow \text{id} & \\
 X & \longrightarrow & Z & \longleftarrow s & X \\
 & \searrow 0 & \uparrow s & \swarrow \text{id} & \\
 & & X & & 
 \end{array}$$

commutes. It follows that  $s$  is the zero morphism from  $X$  to  $Z$  and hence  $\text{cone}(s) \simeq \Sigma X \oplus Z \in \mathcal{S}$  by Corollary 3.1.15. Hence  $X \in \text{thick}\langle \mathcal{S} \rangle$ .  $\square$

*Remark 3.6.10.* Unless  $\mathcal{T}$  is essentially small there is still the issue that  $\mathcal{T}/\mathcal{S}$  might not exist in our universe. Nevertheless, in some situations the quotient functor  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  admits a fully faithful right adjoint in which case  $\mathcal{T}/\mathcal{S}$  is equivalent to a full subcategory of  $\mathcal{T}$  and in particular exists in our universe. This is the topic of Bousfield localization (see Section 3.7).

*Remark 3.6.11.* According to the theorem, the kernel killed by Verdier localization  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  is the thick subcategory generated by  $\mathcal{S}$ . Thus we typically take  $\mathcal{S}$  to be a thick subcategory of  $\mathcal{T}$ . This is one of the reasons why thick subcategories are important in the theory of triangulated categories.

*Remark 3.6.12.* Because the collection of morphisms defining the Verdier quotient admits a calculus of left fractions, the category  $\mathcal{T}/\mathcal{S}$  has a fairly simple description: it has the same objects as  $\mathcal{T}$  and the morphisms  $X \rightarrow Y$  are given by equivalence classes of fractions

$$[(X \xrightarrow{f} Y' \xleftarrow{s} Y)]$$

where  $s : Y \rightarrow Y'$  is a morphism such that  $\text{cone}(s) \in \mathcal{S}$ . It is straightforward to examine how the usual constructions in  $\mathcal{T}$  work in  $\mathcal{T}/\mathcal{S}$ . For example, the suspension of a morphism is given by

$$\Sigma[(X \xrightarrow{f} Y' \xleftarrow{s} Y)] = [(\Sigma X \xrightarrow{\Sigma f} \Sigma Y' \xleftarrow{\Sigma s} \Sigma Y)]$$

and the direct-sum of two morphisms is

$$[(X_1 \xrightarrow{f_1} Y'_1 \xleftarrow{s_1} Y_1)] \oplus [(X_2 \xrightarrow{f_2} Y'_2 \xleftarrow{s_2} Y_2)] = [(X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} Y'_1 \oplus Y'_2 \xleftarrow{\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}} Y_1 \oplus Y_2)].$$

Note that  $\text{cone}\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \simeq \text{cone}(s_1) \oplus \text{cone}(s_2)$  so  $s_1 \oplus s_2 \in S$ . In any case, the only place where we will actually need this concrete description of the Verdier quotient is in Section 5.7.

The following proposition explains how Verdier localization interacts with the tensor structure of a tensor triangulated category.

**Proposition 3.6.13.** *Let  $(\mathcal{T}, \otimes, \mathbb{1})$  be a tensor triangulated category and let  $S \subset \mathcal{T}$  be a thick  $\otimes$ -ideal. The Verdier quotient  $\mathcal{T}/S$  inherits a canonical tensor triangulated structure  $(\mathcal{T}/S, \hat{\otimes}, \mathbb{1})$  such that  $q : \mathcal{T} \rightarrow \mathcal{T}/S$  is a tensor triangulated functor. The tensor product  $-\hat{\otimes}-$  is defined on objects by  $A \hat{\otimes} B = A \otimes B$  and on morphisms by*

$$[X_1 \xrightarrow{f_1} Y'_1 \xleftarrow{s_1} Y_1] \hat{\otimes} [X_2 \xrightarrow{f_2} Y'_2 \xleftarrow{s_2} Y_2] = [X_1 \otimes X_2 \xrightarrow{f_1 \otimes f_2} Y'_1 \otimes Y'_2 \xleftarrow{s_1 \otimes s_2} Y_1 \otimes Y_2].$$

*Proof.* Let  $S := \{f \in \text{Mor}(\mathcal{T}) \mid \text{cone}(f) \in S\}$ . Note that  $S \times S$  is a collection of morphisms in  $\mathcal{T} \times \mathcal{T}$ . It is easy to check that  $(\mathcal{T} \times \mathcal{T})[(S \times S)^{-1}] = \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}]$  with the canonical map given by  $q \times q : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}]$ . Now consider two morphisms  $s : X \rightarrow Y$  and  $t : A \rightarrow B$  in  $S$ . Since  $s \otimes t = (s \otimes B) \circ (X \otimes t)$ , the octahedral axiom implies that there is an exact triangle

$$X \otimes \text{cone}(t) \longrightarrow \text{cone}(s \otimes t) \longrightarrow \text{cone}(s) \otimes B \longrightarrow \Sigma X \otimes \text{cone}(t).$$

It follows that  $s \otimes t \in S$  since  $S$  is assumed to be a  $\otimes$ -ideal. Therefore, by the universal property of localization there exists a unique functor  $-\hat{\otimes}- : \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \rightarrow \mathcal{T}[S^{-1}]$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{T} & \xrightarrow{-\otimes-} & \mathcal{T} \\ q \times q \downarrow & & \downarrow q \\ \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & \xrightarrow{-\hat{\otimes}-} & \mathcal{T}[S^{-1}]. \end{array} \tag{3.6.14}$$

It follows immediately from this diagram that  $A \hat{\otimes} B = qA \hat{\otimes} qB = q(A \otimes B) = A \otimes B$  for any two objects  $A, B \in \mathcal{T}[S^{-1}]$ . Moreover, note that every morphism in  $\mathcal{T}[S^{-1}]$  is of the form  $q(s)^{-1} \circ q(f)$  for some  $f \in \text{Mor}(\mathcal{T})$  and  $s \in S$ . If  $f : X \rightarrow Y'$ ,  $s : Y \rightarrow Y'$ ,  $g : A \rightarrow B'$ ,  $t : B \rightarrow B'$  are morphisms in  $\mathcal{T}$  with  $s, t \in S$  then the commutativity of (3.6.14) implies that

$$\begin{aligned}
(q(s)^{-1} \circ q(f)) \hat{\otimes} (q(t)^{-1} \circ q(g)) &= ((q(s)^{-1} \circ q(f)) \hat{\otimes} B) \circ (X \hat{\otimes} (q(t)^{-1} \circ q(g))) \\
&= (q(s)^{-1} \hat{\otimes} B) \circ (q(f) \hat{\otimes} B) \circ (X \hat{\otimes} q(t)^{-1}) \circ (X \hat{\otimes} q(g)) \\
&= (q(s)^{-1} \hat{\otimes} B) \circ (X \hat{\otimes} q(t)^{-1}) \circ (q(f) \hat{\otimes} B) \circ (X \hat{\otimes} q(g)) \\
&= (q(s)^{-1} \hat{\otimes} q(t)^{-1}) \circ (q(f) \hat{\otimes} q(g)) \\
&= (q(s) \hat{\otimes} q(t))^{-1} \circ (q(f) \hat{\otimes} q(g)) \\
&= q(s \otimes t)^{-1} \circ q(f \otimes g).
\end{aligned}$$

In other words, the tensor product  $-\hat{\otimes}-$  is given on morphisms by

$$\begin{aligned}
[(X \xrightarrow{f} Y' \xleftarrow{s} Y)] \otimes [(A \xrightarrow{g} B' \xleftarrow{t} B)] \\
&= [(Y \otimes B \xrightarrow{s \otimes t} Y' \otimes B' \xleftarrow{\text{id}} Y' \otimes B')]^{-1} \circ [(X \otimes A \xrightarrow{f \otimes g} Y' \otimes B' \xleftarrow{\text{id}} Y' \otimes B')] \\
&= [(Y' \otimes B' \xrightarrow{\text{id}} Y' \otimes B' \xleftarrow{s \otimes t} Y \otimes B)] \circ [(X \otimes A \xrightarrow{f \otimes g} Y' \otimes B' \xleftarrow{\text{id}} Y' \otimes B')] \\
&= [(X \otimes A \xrightarrow{f \otimes g} Y' \otimes B' \xleftarrow{s \otimes t} Y \otimes B)].
\end{aligned}$$

Next consider the associator, unitor and symmetry isomorphisms of the symmetric monoidal structure on  $\mathcal{T}$ . Note that the associator  $\alpha$  is a natural transformation from the functor

$$\mathcal{T} \times \mathcal{T} \times \mathcal{T} \xrightarrow{\text{id} \times \otimes} \mathcal{T} \times \mathcal{T} \xrightarrow{\otimes} \mathcal{T} \quad (3.6.15)$$

to the functor

$$\mathcal{T} \times \mathcal{T} \times \mathcal{T} \xrightarrow{\otimes \times \text{id}} \mathcal{T} \times \mathcal{T} \xrightarrow{\otimes} \mathcal{T} \quad (3.6.16)$$

and that these two functors induce the analogous functors on  $\mathcal{T}[S^{-1}]$ . For example, the

following diagram

$$\begin{array}{ccccc}
\mathcal{T} \times \mathcal{T} \times \mathcal{T} & \xrightarrow{\text{id} \times \otimes} & \mathcal{T} \times \mathcal{T} & \xrightarrow{\otimes} & \mathcal{T} \\
\downarrow q \times q \times q & & \downarrow q \times q & & \downarrow q \\
\mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & \xrightarrow{\text{id} \times \hat{\otimes}} & \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & \xrightarrow{\hat{\otimes}} & \mathcal{T}
\end{array}$$

indicates that (3.6.15) descends to the analogous functor on  $\mathcal{T}[S^{-1}]$ ; similarly for (3.6.16). By

Remark 3.6.2,  $\alpha$  corresponds to a natural isomorphism  $\bar{\alpha}$  between these induced functors.

The relationship between  $\alpha$  and  $\bar{\alpha}$  is that the natural transformation

$$\begin{array}{ccc}
\mathcal{T}^{\times 3} & \xrightarrow{q^{\times 3}} & \mathcal{T}[S^{-1}]^{\times 3} \\
& & \begin{array}{c} \xrightarrow{\hat{\otimes} \circ (\text{id} \times \hat{\otimes})} \\ \Downarrow \bar{\alpha} \\ \xrightarrow{\hat{\otimes} \circ (\hat{\otimes} \times \text{id})} \end{array} \\
& & \mathcal{T}[S^{-1}]
\end{array}$$

coincides with the natural transformation

$$\begin{array}{ccc}
\mathcal{T}^{\times 3} & \begin{array}{c} \xrightarrow{\otimes \circ (\text{id} \times \otimes)} \\ \Downarrow \alpha \\ \xrightarrow{\otimes \circ (\otimes \times \text{id})} \end{array} & \mathcal{T} \\
& & \xrightarrow{q} \mathcal{T}[S^{-1}].
\end{array}$$

Similar statements hold for the unitor and symmetry isomorphisms. Now note that each of the axioms for a symmetric monoidal category is an assertion that two particular natural transformations coincide. By using the isomorphism of categories

$$\mathcal{T}[S^{-1}]\mathcal{T}[S^{-1}] \xrightarrow[\cong]{q \circ -} \mathcal{T}[S^{-1}]^{(\mathcal{T}, S)}$$

from Remark 3.6.2, we can check that two such natural transformations are equal by checking that they coincide after horizontally precomposing by  $q$  (or  $q^{\times n}$  as the case may be) and this reduces the problem to the corresponding axiom in  $\mathcal{T}$ . For example, the symmetry  $\tau : a \otimes b \xrightarrow{\sim} b \otimes a$  is a natural transformation

$$\begin{array}{ccc}
& \text{id} & \\
& \curvearrowright & \\
\mathcal{T} \times \mathcal{T} & \Downarrow \tau & \mathcal{T} \times \mathcal{T} \\
& \curvearrowleft & \\
& \text{switch} &
\end{array}$$

and axiom (2.2.14) asserts the equality of the following two natural transformations:

$$\begin{array}{ccc}
\mathcal{T} \times \mathcal{T} & \begin{array}{c} \xrightarrow{\text{id}} \\ \Downarrow \text{id} \\ \xrightarrow{\text{id}} \end{array} & \mathcal{T} \times \mathcal{T} \\
& & = \\
\mathcal{T} \times \mathcal{T} & \begin{array}{c} \xrightarrow{\text{id}} \\ \Downarrow \tau \\ \text{switch} \\ \Downarrow \tau \\ \xrightarrow{\text{id}} \end{array} & \mathcal{T} \times \mathcal{T}
\end{array}$$

In order to check that this axiom holds in  $\mathcal{T}[S^{-1}]$ —i.e., in order to check that

$$\begin{array}{ccc} & \xrightarrow{\text{id}} & \\ \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & \xrightarrow{\text{switch}} & \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \\ & \xleftarrow{\text{id}} & \end{array} \quad \begin{array}{c} \Downarrow \bar{\tau} \\ \Downarrow \bar{\tau} \end{array}$$

is the identity natural transformation—it suffices to check that

$$\mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \xrightarrow{q \times q} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \xrightarrow{\text{switch}} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \quad (3.6.17)$$

is the identity natural transformation. But according to the definition of  $\bar{\tau}$ , the natural transformation

$$\mathcal{T} \times \mathcal{T} \xrightarrow{q \times q} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] \xrightarrow{\text{switch}} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}]$$

is equal to

$$\mathcal{T} \times \mathcal{T} \xrightarrow{\text{switch}} \mathcal{T} \times \mathcal{T} \xrightarrow{q \times q} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}]$$

and consequently (3.6.17) is equal to

$$\mathcal{T} \times \mathcal{T} \xrightarrow{\text{switch}} \mathcal{T} \times \mathcal{T} \xrightarrow{q \times q} \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}]$$

which is the identity by axiom (2.2.14) for  $\mathcal{T}$ . In conclusion, this establishes axiom (2.2.14) for  $\mathcal{T}[S^{-1}]$ . The other axioms required of a symmetric monoidal category can be disposed of in a similar fashion by pulling back via  $q^{\times n}$  to reduce them to the corresponding axioms in  $\mathcal{T}$ . The same method can be used to produce suspension isomorphisms for  $\mathcal{T}[S^{-1}]$  and to check axioms (3.3.3)–(3.3.6). Finally, to check that  $a \hat{\otimes} - : \mathcal{T}[S^{-1}] \rightarrow \mathcal{T}[S^{-1}]$  preserves exact triangles it suffices to check that it preserves exact triangles in the image of  $q$ :

$$qX \xrightarrow{qf} qY \xrightarrow{qg} qZ \xrightarrow{qh} q\Sigma X = \bar{\Sigma} qX.$$

Applying  $a\hat{\otimes}-$  and using (3.6.14) we obtain an isomorphism of triangles

$$\begin{array}{ccccccc}
qa\hat{\otimes}X & \xrightarrow{qa\hat{\otimes}qf} & qa\hat{\otimes}qY & \xrightarrow{qa\hat{\otimes}qg} & qa\hat{\otimes}qZ & \xrightarrow{qa\hat{\otimes}qh} & qa\hat{\otimes}q\Sigma X = qa\hat{\otimes}\bar{\Sigma}qX \xrightarrow{\bar{\rho}} \bar{\Sigma}(qa\hat{\otimes}qX) \\
\parallel & & \parallel & & \parallel & & \parallel \\
q(a\otimes X) & \xrightarrow{q(a\otimes f)} & q(a\otimes Y) & \xrightarrow{q(a\otimes g)} & q(a\otimes Z) & \xrightarrow{q(a\otimes h)} & q(a\otimes \Sigma X) \xrightarrow{q(\rho)} q\Sigma(a\otimes X) = \bar{\Sigma}q(a\otimes X).
\end{array}$$

The right square commutes since—according to the definition of  $\bar{\rho} : X\hat{\otimes}\bar{\Sigma}Y \xrightarrow{\sim} \bar{\Sigma}X\hat{\otimes}Y$ —the following natural transformation

$$\begin{array}{ccccc}
& & \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & & \\
& \text{id} \times \bar{\Sigma} \nearrow & & \searrow \hat{\otimes} & \\
\mathcal{T} \times \mathcal{T} & \xrightarrow{q \times q} & \mathcal{T}[S^{-1}] \times \mathcal{T}[S^{-1}] & \Downarrow \bar{\rho} & \mathcal{T}[S^{-1}] \\
& \hat{\otimes} \searrow & & \nearrow \bar{\Sigma} & \\
& & \mathcal{T}[S^{-1}] & & 
\end{array}$$

coincides with

$$\begin{array}{ccccc}
& & \mathcal{T} \times \mathcal{T} & & \\
& \text{id} \times \Sigma \nearrow & & \searrow \otimes & \\
\mathcal{T} \times \mathcal{T} & & & \Downarrow \rho & \mathcal{T} \xrightarrow{q} \mathcal{T}[S^{-1}]. \\
& \otimes \searrow & & \nearrow \Sigma & \\
& & \mathcal{T} & & 
\end{array}$$

In other words,  $q(\rho_{a,b}) = \bar{\rho}_{qa,qb}$ . We conclude that  $\mathcal{T}[S^{-1}]$  is a tensor triangulated category.

Finally, let us observe that  $q : \mathcal{T} \rightarrow \mathcal{T}/S$  is a tensor triangulated functor. We can simply take  $\phi_0 : \mathbb{1} \xrightarrow{\text{id}} q\mathbb{1}$  and  $\phi_{X,Y} : qX\hat{\otimes}qY = q(X\otimes Y) \xrightarrow{\text{id}} q(X\otimes Y)$  to be the identity morphisms.

The axioms for being a tensor triangulated functor are readily checked by writing out the diagrams and using the facts we have used repeatedly in the above discussion: for example, that  $q * \alpha = \bar{\alpha} * q^{\times 3}$ . In conclusion,  $q : \mathcal{T} \rightarrow \mathcal{T}/S$  is a *strict* tensor triangulated functor.  $\square$

### 3.7 Bousfield localization

Before beginning our discussion of Bousfield localization, let us recall a basic fact from category theory:

**Lemma 3.7.1.** *Let  $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$  be an adjunction and let  $S$  denote the collection of morphisms  $f \in \text{Mor}(\mathcal{C})$  such that  $F(f)$  is an isomorphism. The following are equivalent:*

- (1) *The right adjoint  $G$  is fully faithful.*
- (2) *The counit  $\epsilon : FG \rightarrow \text{id}_{\mathcal{D}}$  is an isomorphism.*
- (3) *The induced functor  $\tilde{F} : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  is an equivalence.*

*Proof.* See [GZ67, Chapter I, Proposition 1.3]. □

*Example 3.7.2.* Let  $\mathcal{S}$  be a thick subcategory of a triangulated category  $\mathcal{T}$ . If the Verdier quotient functor  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  has a right adjoint then the right adjoint is automatically fully faithful and hence provides an equivalence between  $\mathcal{T}/\mathcal{S}$  and a full subcategory of  $\mathcal{T}$ . In particular,  $\mathcal{T}/\mathcal{S}$  is locally small. The following notion of “Bousfield localization” is closely related to such Verdier quotients—although at first glance the two notions look quite different.

**Definition 3.7.3.** A *Bousfield localization functor* on a triangulated category  $\mathcal{T}$  is an exact functor  $L : \mathcal{T} \rightarrow \mathcal{T}$  equipped with a natural transformation  $\eta : \text{id}_{\mathcal{T}} \rightarrow L$  such that  $L\eta$  is an isomorphism and  $L\eta = \eta L$ . A morphism  $f : X \rightarrow Y$  is said to be an  $L$ -equivalence if  $L(f)$  is an isomorphism. An object  $X$  is said to be  $L$ -local if  $\eta_X$  is an isomorphism; equivalently,  $X$  is  $L$ -local if it is contained in the essential image  $\text{Im } L$ . Finally, an object  $X$  is said to be  $L$ -acyclic if  $L(X) = 0$ ; that is, if it is contained in the kernel  $\text{Ker } L$ . Note that the collection of  $L$ -acyclic objects forms a thick subcategory of  $\mathcal{T}$  since  $L$  is assumed to be exact.

*Remark 3.7.4.* It is easy to check from the definitions that the map  $\eta_X : X \rightarrow LX$  is the initial map from  $X$  to an  $L$ -local object, and that it is also the terminal  $L$ -equivalence out of  $X$ . In particular,  $\eta_X : X \rightarrow LX$  provides an  $L$ -equivalence from  $X$  to an  $L$ -local object  $LX$ .

*Remark 3.7.5.* If  $(L, \eta)$  is a Bousfield localization functor on a triangulated category  $\mathcal{T}$  then the functor  $L : \mathcal{T} \rightarrow \text{Im} L$  is left adjoint to the inclusion  $\text{Im} L \hookrightarrow \mathcal{T}$ . The adjunction isomorphism  $[LX, Y] \xrightarrow{\sim} [X, Y]$  is defined (for  $X \in \mathcal{T}$  and  $Y \in \text{Im} L$ ) by  $f \mapsto f \circ \eta_X$ . The fact that this is an isomorphism amounts to the statement that  $\eta_X$  is the initial map from  $X$  to an  $L$ -local object. In any case, by Lemma 3.7.1,  $L : \mathcal{T} \rightarrow \text{Im} L$  induces an equivalence  $\mathcal{T}/\text{Ker} L \xrightarrow{\sim} \text{Im} L$ . In particular, the quotient functor  $q : \mathcal{T} \rightarrow \mathcal{T}/\text{Ker} L$  inherits a (fully faithful) right adjoint  $\mathcal{T}/\text{Ker} L \rightarrow \mathcal{T}$  whose essential image is precisely  $\text{Im} L$ . The functor  $L : \mathcal{T} \rightarrow \mathcal{T}$  may be regarded as either of the composites in the following diagram

$$\begin{array}{ccccc}
 & & \text{Im} L & & \\
 & \nearrow & \uparrow & \nwarrow & \\
 \mathcal{T} & & \sim & & \mathcal{T} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathcal{T}/\text{Ker} L & & 
 \end{array}$$

and we have two complementary ways of thinking about Bousfield localization. The first point of view thinks of “Bousfield localization” in terms of the functor  $\mathcal{T} \rightarrow \text{Im} L$ : it’s a process by which we move from  $\mathcal{T}$  to the category of  $L$ -local objects by replacing each object  $X$  by an  $L$ -local object  $LX$  which is  $L$ -equivalent to  $X$ . The second point of view thinks of “Bousfield localization” in terms of the functor  $\mathcal{T} \rightarrow \mathcal{T}/\text{Ker} L$  whereby we replace  $\mathcal{T}$  by the category  $\mathcal{T}/\text{Ker} L$  obtained by formally inverting the  $L$ -equivalences. Both points of view have their advantages and are related, of course, by the equivalence  $\mathcal{T}/\text{Ker} L \xrightarrow{\sim} \text{Im} L$ .

*Remark 3.7.6.* If  $\mathcal{T}$  is a *tensor* triangulated category and  $\mathcal{S}$  is a thick  $\otimes$ -ideal then  $\mathcal{T}/\mathcal{S}$  is tensor triangulated and  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  is a tensor triangulated functor (cf. Proposition 3.6.13). Consequently, if the kernel  $\text{Ker} L$  of a Bousfield localization functor  $L : \mathcal{T} \rightarrow \mathcal{T}$  is a  $\otimes$ -ideal then the category of  $L$ -local objects inherits a *tensor* triangulated structure via the equivalence  $\mathcal{T}/\text{Ker} L \xrightarrow{\sim} \text{Im} L$ . This is one example where there is some advantage in regarding the Bousfield localization of  $\mathcal{T}$  as being  $\mathcal{T}/\text{Ker} L$  rather than  $\text{Im} L$ . Although the category  $\text{Im} L$  is a tensor triangulated category it is *not* a tensor triangulated subcategory of  $\mathcal{T}$ ; rather, it

inherits the  $\otimes$ -structure of  $\mathcal{T}$  indirectly via the equivalence  $\mathcal{T}/\text{Ker}L \xrightarrow{\sim} \text{Im}L$ . Note that this equivalence sends an object  $X \in \mathcal{T}/\text{Ker}L$  to  $LX \in \text{Im}L$ . If we unwind how the  $\otimes$ -structure on  $\mathcal{T}/\text{Ker}L$  gets transferred to  $\text{Im}L$  via Proposition 2.2.42 we see that the unit is  $L1$  and that  $X \otimes_L Y := L(X \otimes Y)$  for two  $L$ -local objects  $X$  and  $Y$ . All algebraic constructions in  $\mathcal{T}$  pass to the  $L$ -local category via the strong  $\otimes$ -functor  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S} \cong \text{Im}L$ . For example, the  $L$ -localization of a ring object in  $\mathcal{T}$  is a ring object in the  $L$ -local category. If you would like to work exclusively with  $\text{Im}L$  without using the Verdier quotient then the yoga to keep in mind is that you can perform all the constructions you want in  $\mathcal{T}$  but at the end of the day you must apply  $L$  to bring yourself back to the  $L$ -local category.

**Definition 3.7.7.** For any collection of objects  $\mathcal{E}$  in a triangulated category  $\mathcal{T}$ , we define the “left orthogonal” of  $\mathcal{E}$  to be the subcategory

$${}^\perp\mathcal{E} = \{X \in \mathcal{T} \mid [X, E] = 0 \text{ for all } E \in \mathcal{E}\}$$

and we define the “right orthogonal” of  $\mathcal{E}$  to be the subcategory

$$\mathcal{E}^\perp = \{X \in \mathcal{T} \mid [E, X] = 0 \text{ for all } E \in \mathcal{E}\}.$$

These are both thick subcategories of  $\mathcal{T}$ . Moreover,  ${}^\perp\mathcal{E}$  is localizing if  $\mathcal{T}$  has coproducts; dually,  $\mathcal{E}^\perp$  is colocalizing if  $\mathcal{T}$  has products.

**Warning.** Some authors use the opposite convention and write  ${}^\perp\mathcal{E}$  for our  $\mathcal{E}^\perp$ .

**Lemma 3.7.8.** *Let  $(L, \eta)$  be a Bousfield localization functor on a triangulated category  $\mathcal{T}$ . Then  $\text{Im}L = (\text{Ker}L)^\perp$  and  $\text{Ker}L = {}^\perp(\text{Im}L)$ . That is to say, an object  $X$  is  $L$ -local iff  $[A, X] = 0$  for every  $L$ -acyclic object  $A$  and an object  $A$  is  $L$ -acyclic iff  $[A, X] = 0$  for every  $L$ -local object  $X$ .*

*Proof.* It is easy to check that any map  $A \rightarrow X$  from an  $L$ -acyclic object  $A$  to an  $L$ -local object  $X$  is zero. The two inclusions  $\text{Im}L \subset (\text{Ker}L)^\perp$  and  $\text{Ker}L \subset {}^\perp(\text{Im}L)$  immediately follow.

On the other hand, suppose  $A$  is an object such that  $[A, X] = 0$  for all  $L$ -local objects  $X$ . Taking  $X = LA$  we conclude that  $\eta_A = 0$ . It follows that  $LA = 0$  since the isomorphism  $L(\eta_A) : LA \rightarrow L^2A$  is then the zero map. This establishes that  $\text{Ker } L = {}^\perp(\text{Im } L)$  and it remains to show that  $(\text{Ker } L)^\perp \subset \text{Im } L$ . To this end, note that the statement that  $[A, X] = 0$  for all  $L$ -acyclic objects  $A$  is equivalent to the statement that for any  $L$ -equivalence  $f : A \rightarrow B$ , the induced map  $[B, X] \xrightarrow{f^*} [A, X]$  is an isomorphism. (Just apply  $[-, X]$  to an exact triangle for  $f$ .) Taking  $f$  to be  $\eta_X : X \rightarrow LX$  we conclude that there exists a map  $\theta : LX \rightarrow X$  such that  $\theta \circ \eta_X = \text{id}_X$ . On the other hand, using the naturality of  $\eta$  we have  $\eta_X \circ \theta = L\theta \circ \eta_{LX} = L\theta \circ L\eta_X = L(\theta \circ \eta_X) = \text{id}_{LX}$  and we conclude that  $\eta_X$  is an isomorphism and hence  $X$  is  $L$ -local. (The converse also holds: if  $X$  is  $L$ -local then  $[B, X] \xrightarrow{f^*} [A, X]$  is an isomorphism for any  $L$ -equivalence  $f : A \rightarrow B$ . This can be proved using the universal properties of  $\eta_X : X \rightarrow LX$  mentioned in Remark 3.7.4.)  $\square$

*Remark 3.7.9.* A morphism of Bousfield localizations  $(L, \eta) \rightarrow (L', \eta')$  on the same category  $\mathcal{T}$  is a natural transformation  $\alpha : L \rightarrow L'$  such that  $\eta = \eta' \circ \alpha$ . One easily checks using Remark 3.7.4 that if  $(L, \eta)$  and  $(L', \eta')$  are two Bousfield localization functors on  $\mathcal{T}$  having the same local objects then there is a *unique* isomorphism  $(L, \eta) \xrightarrow{\sim} (L', \eta')$ . Thus, a Bousfield localization functor is completely determined by its collection of  $L$ -local objects. Moreover, by Lemma 3.7.8, the  $L$ -local objects of a Bousfield localization functor determine the  $L$ -acyclic objects; vice versa, the  $L$ -acyclic objects determine the  $L$ -local objects and hence completely determine the localization functor. We'll say that a thick subcategory  $\mathcal{S} \subset \mathcal{T}$  “admits” Bousfield localization if there exists an (essentially unique) Bousfield localization functor  $(L, \eta)$  with  $\text{Ker } L = \mathcal{S}$ . The crucial question in the theory of Bousfield localization is to determine when a thick subcategory admits Bousfield localization.

**Proposition 3.7.10.** *Let  $\mathcal{S}$  be a thick subcategory of a triangulated category  $\mathcal{T}$ . The following are equivalent:*

(1) *There exists a Bousfield localization functor  $(L, \eta)$  with  $\text{Ker } L = \mathcal{S}$ .*

(2) *The inclusion  $\mathcal{S}^\perp \hookrightarrow \mathcal{T}$  has a left adjoint and  ${}^\perp(\mathcal{S}^\perp) = \mathcal{S}$ .*

(3) *The quotient  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  has a right adjoint.*

*Proof.* By Lemma 3.7.8,  $(\text{Ker } L)^\perp = \text{Im } L$  and we have already discussed in Remark 3.7.5 and Lemma 3.7.8 how (1) implies (2) and (3). On the other hand, if  $\mathcal{S}^\perp \hookrightarrow \mathcal{T}$  has a left adjoint then the adjunction provides a monad  $(L, \mu, \eta)$  on  $\mathcal{T}$ . Since the right adjoint is fully faithful, the counit is an isomorphism by Lemma 3.7.1 so that  $\mu$  is an isomorphism. In any monad,  $\mu \circ T\eta = \text{id}_T = \mu \circ \eta T$  so since  $\mu$  is invertible we conclude that  $T\eta = \eta T$  and that this is an inverse to  $\mu$ . Thus  $(L, \eta)$  is a localization functor on  $\mathcal{T}$ . The left adjoint of an exact functor is exact (cf. Lemma 3.1.28) so  $L$  is an exact functor (being the composite of such). Moreover, the kernel of  $L$  is just the kernel of the left adjoint  $F : \mathcal{T} \rightarrow \mathcal{S}^\perp$  since the right adjoint  $\mathcal{S}^\perp \hookrightarrow \mathcal{T}$  is faithful. We claim that it is equal to  $\mathcal{S}$ . The adjunction says that  $[FX, Y] = [X, Y]$  for any  $X \in \mathcal{T}$  and  $Y \in \mathcal{S}^\perp$ . It follows that if  $FX = 0$  then  $[X, Y] = 0$  for any  $Y \in \mathcal{S}^\perp$  and hence  $X \in {}^\perp(\mathcal{S}^\perp) = \mathcal{S}$ . On the other hand, if  $X \in \mathcal{S}$  then  $0 = [X, Y] = [FX, Y]$  for any  $Y \in \mathcal{S}^\perp$ ; but  $FX \in \mathcal{S}^\perp$  so taking  $Y = FX$  we conclude that  $FX = 0$ . Similarly, if  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  has a right adjoint then as remarked in Example 3.7.2 the right adjoint is fully faithful, so by the same argument we have just given we obtain an exact localization functor  $(L, \eta)$  on  $\mathcal{T}$ . As before the kernel of  $L$  is the same as the kernel of the left adjoint  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  which is precisely  $\mathcal{S}$ .  $\square$

*Remark 3.7.11.* In order to have much chance of constructing Bousfield localization functors, we typically assume that  $\mathcal{T}$  is “large” in the sense that it has all small coproducts. In this case, by Lemma 3.7.8 and Definition 3.7.7, a necessary condition for  $\mathcal{S} \subset \mathcal{T}$  to admit Bousfield localization is that  $\mathcal{S}$  must be a localizing subcategory of  $\mathcal{T}$ . It is easy to see that the Verdier quotient  $\mathcal{T}/\mathcal{S}$  also has coproducts and that  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$  preserves them. Thus, if  $\mathcal{T}$  is compactly generated and  $\mathcal{T}/\mathcal{S}$  is locally small then Brown Representability (specifically

Corollary 3.2.7) produces the required right adjoint. In other words, if  $\mathcal{T}$  is a compactly generated triangulated category then Bousfield localization exists for a localizing subcategory  $\mathcal{S} \subset \mathcal{T}$  if and only if the Verdier quotient  $\mathcal{T}/\mathcal{S}$  is locally small. It is perhaps not surprising then that the existence of Bousfield localizations is related to foundational questions in set theory. For example, [CGR14] have proved that if a certain large cardinal axiom called “Vopěnka’s principle” is included in the axioms of set theory then every localizing subcategory of  $\mathcal{T}$  admits Bousfield localization provided that  $\mathcal{T}$  is the homotopy category of a stable combinatorial model category. As far as the author is aware, it is not known at present whether Vopěnka’s principle is consistent with ZFC. Nevertheless, in full generality, Bousfield localizations do not always exist. Indeed, as we have already observed, if  $\mathcal{S}$  admits Bousfield localization then in particular  $\mathcal{T}/\mathcal{S}$  must be locally small. However, in Freyd’s book on abelian categories, he gave an example [Fre64, Chapter 6, Exercise A, pp. 131–132] of a certain (locally small) abelian category  $\mathcal{A}$  for which the derived category  $D(\mathcal{A})$  is not locally small. Nevertheless, as observed by [CN09], the homotopy category of acyclic complexes  $K_{ac}(\mathcal{A}) \subset K(\mathcal{A})$  is a localizing subcategory of the homotopy category of complexes  $K(\mathcal{A})$ . Since  $D(\mathcal{A}) = K(\mathcal{A})/K_{ac}(\mathcal{A})$ , this provides an example of a localizing subcategory in a (locally small) triangulated category which does not admit Bousfield localization.

## Finite localization

**Definition 3.7.12.** Let  $L : \mathcal{T} \rightarrow \mathcal{T}$  be a Bousfield localization functor on a triangulated category  $\mathcal{T}$ . We say that  $L$  is a *finite localization* if the subcategory of  $L$ -acyclics is generated by a set of compact objects. Finite localizations are an important class of Bousfield localization functors which have been very useful in applications and which *do* always exist:

**Proposition 3.7.13.** *Let  $\mathcal{T}$  be a compactly generated triangulated category and let  $\mathcal{S} \subset \mathcal{T}$  be a localizing subcategory of  $\mathcal{T}$ . Assume that there exists a set  $\mathcal{G}$  of compact objects in  $\mathcal{T}$  such*

that  $\mathcal{S} = \text{loc}\langle \mathcal{G} \rangle$ . Then  $\mathcal{S}$  admits a Bousfield localization functor.

*Proof.* The proof is not difficult but we won't include it. See [HPS97, Definition 3.3.4], [Kra10, Section 5.6] or [Nee92b].  $\square$

*Remark 3.7.14.* The term “finite localization” was coined by Haynes Miller [Mil92] who studied them in the setting of stable homotopy theory. In Section 3.9, we'll see an example of finite localization in modular representation theory.

*Remark 3.7.15.* Suppose we are in the setting of Proposition 3.7.13. Then  $\mathcal{S}$  is itself compactly generated and it follows from Remark 3.2.5 that  $\mathcal{S}^c = \text{thick}\langle \mathcal{G} \rangle$ . We have a commutative diagram of exact functors

$$\begin{array}{ccccc}
 \mathcal{S}^c & \xrightarrow{\quad} & \mathcal{T}^c & \xrightarrow{\quad} & \mathcal{T}^c/\mathcal{S}^c \\
 \downarrow & & \downarrow & & \downarrow J \\
 \mathcal{S} & \xrightarrow{\quad} & \mathcal{T} & \xrightarrow{\quad} & \mathcal{T}/\mathcal{S}
 \end{array}$$

$(\mathcal{T}/\mathcal{S})^c$

and the following theorem says that the functor  $J$  is “almost” an equivalence.

**Theorem 3.7.16** (Neeman). *Let  $\mathcal{T}$  be a compactly generated triangulated category and let  $\mathcal{S} \subset \mathcal{T}$  be a localizing subcategory which is generated by a set of compact objects. Then the quotient  $\mathcal{T}/\mathcal{S}$  is compactly generated and the functor  $J : \mathcal{T}^c/\mathcal{S}^c \rightarrow (\mathcal{T}/\mathcal{S})^c$  is fully faithful; moreover, every object in  $(\mathcal{T}/\mathcal{S})^c$  is a direct-summand of an object in the essential image of  $J$ . Hence  $J$  induces an equivalence after idempotent completion:  $(\mathcal{T}^c/\mathcal{S}^c)^\# \cong (\mathcal{T}/\mathcal{S})^c$ .*

*Proof.* See [Nee92b].  $\square$

*Remark 3.7.17.* At this point the reader might be refreshed by a retelling of the story of Thomason and Trobaugh. In his work on localization in algebraic  $K$ -theory, Thomason was led to a question concerning the possibility of extending a complex of vector bundles defined

on an open subset to a complex of vector bundles defined on the whole space. To be more precise, fix a quasi-projective noetherian scheme  $X$  and an open subset  $U \subset X$  and let  $Z = X \setminus U$  denote the closed complement. Furthermore, let  $D^b(\text{Vect}/X)$  denote the derived category of bounded complexes of vector bundles on  $X$  and let  $D_Z^b(\text{Vect}/X) \subset D^b(\text{Vect}/X)$  denote the full subcategory consisting of those complexes whose cohomology is supported in  $Z$ ; i.e., those complexes which become acyclic when restricted to  $U$ . This is precisely the kernel of the natural restriction functor  $D^b(\text{Vect}/X) \rightarrow D^b(\text{Vect}/U)$  and hence there is an induced functor

$$\frac{D^b(\text{Vect}/X)}{D_Z^b(\text{Vect}/X)} \rightarrow D^b(\text{Vect}/U). \quad (3.7.18)$$

Thomason was concerned with whether this functor was an equivalence. Unfortunately, it was well-known that this functor is not essentially surjective—for singular varieties it is not possible in general to extend a vector bundle defined on an open subset to a bounded complex of vector bundles on the whole space (although it follows from [Ser55] that this is possible for smooth varieties). In any case, because it was known that (3.7.18) was not essentially surjective most people walked away from the problem confident that the functor couldn't possibly be an equivalence (and they were right of course). Nevertheless, as related in the introduction of [TT90], the ghost of a dead friend came to Thomason in a dream telling him not to give up and this led him to the realization that although (3.7.18) is not essentially surjective it is surjective up to direct summands: every object in the target is a direct summand of an object in the essential image. In fact, it turned out to be fully faithful too and so was an equivalence up to direct-summands—or better, it was an equivalence after idempotent completion:

$$\left( \frac{D^b(\text{Vect}/X)}{D_Z^b(\text{Vect}/X)} \right)^\sharp \cong D^b(\text{Vect}/U).$$

Thomason was so impressed by the ghost's insight and interest in mortal affairs that he added him as a coauthor on his paper. Although Thomason is unfortunately no longer grac-

ing mathematics with his insight, the author can only hope that some day Thomason himself will visit in a dream and provide a profound ethereal epiphany. In any case, Neeman [Nee92b] realized that Thomason's result was just a particular incarnation of the much more general Theorem 3.7.16. Indeed, if  $\mathcal{T} = D(\mathrm{QCoh}/X)$  denotes the derived category of quasi-coherent sheaves on  $X$  then  $\mathcal{S} := D_Z(\mathrm{QCoh}/X) \subset D(\mathrm{QCoh}/X)$  is a localizing subcategory and it is certainly true that  $D(\mathrm{QCoh}/X)/D_Z(\mathrm{QCoh}/X) \cong D(\mathrm{QCoh}/U)$ . The main point is that a quasi-coherent sheaf is a far more flexible notion than a vector bundle and there is no difficulty extending to the whole space: for example, we can take the pushforward of the inclusion map. This is an example of a finite localization and we have a diagram

$$\begin{array}{ccccc}
D_Z^b(\mathrm{Vect}/X) & \hookrightarrow & D^b(\mathrm{Vect}/X) & \longrightarrow & D^b(\mathrm{Vect}/X)/D_Z^b(\mathrm{Vect}/X) \\
\downarrow & & \downarrow & & \downarrow J \\
D_Z(\mathrm{QCoh}/X) & \hookrightarrow & D(\mathrm{QCoh}/X) & \longrightarrow & D(\mathrm{QCoh}/X)/D_Z(\mathrm{QCoh}/X)
\end{array}$$

precisely matching the situation of Theorem 3.7.16. Further discussion of this story can be found in [Nee06, Nee92b, TT90].

## Smashing localizations

**Definition 3.7.19.** A Bousfield localization functor  $L : \mathcal{T} \rightarrow \mathcal{T}$  on a triangulated category  $\mathcal{T}$  is said to be a *smashing localization* if  $L$  preserves coproducts. Let's explain the slightly bizarre terminology. To this end suppose that  $\mathcal{T}$  is a *tensor* triangulated category and that the  $L$ -acyclics form a  $\otimes$ -ideal. Then for any object  $X \in \mathcal{T}$  we can apply the exact functor  $L(- \otimes X)$  to the exact triangle  $W \rightarrow \mathbb{1} \xrightarrow{\eta_1} L\mathbb{1} \rightarrow \Sigma W$  to obtain an exact triangle

$$L(W \otimes X) \rightarrow L(\mathbb{1} \otimes X) \xrightarrow{L(\eta_1 \otimes X)} L(L\mathbb{1} \otimes X) \rightarrow \Sigma L(W \otimes X).$$

Since the  $L$ -acyclics are assumed to be a  $\otimes$ -ideal,  $L(W \otimes X) = 0$  and hence  $L(\eta_1 \otimes X) : L(1 \otimes X) \rightarrow L(L1 \otimes X)$  is an isomorphism. We can then use this isomorphism to construct a natural map  $\alpha_X : L1 \otimes X \rightarrow LX$  as the composite

$$L1 \otimes X \xrightarrow{\eta_{L1 \otimes X}} L(L1 \otimes X) \xrightarrow{L(\eta_1 \otimes X)^{-1}} L(1 \otimes X) \simeq LX.$$

Now we can prove the following:

**Proposition 3.7.20.** *Let  $L : \mathcal{T} \rightarrow \mathcal{T}$  be a Bousfield localization functor on a rigidly-compactly generated tensor triangulated category with the property that the  $L$ -acyclics form a  $\otimes$ -ideal. Then  $L$  is a smashing localization if and only if the natural map  $\alpha_X : L1 \otimes X \rightarrow LX$  is an isomorphism for all  $X \in \mathcal{T}$ .*

*Proof.* One direction is immediate: if  $L1 \otimes X \simeq LX$  then  $L$  preserves coproducts since in a rigidly-compactly generated category tensoring by an object preserves coproducts (since  $a \otimes -$  has a right adjoint). On the other hand, it is easy to check that the set of  $X$  such that  $\alpha_X$  is an isomorphism forms a thick subcategory (compare Lemma 3.1.39). If  $L$  preserves coproducts then it is a localizing subcategory. Since  $\mathcal{T}$  is generated by a set of dualizable objects it suffices to check that  $\alpha_X$  is an isomorphism whenever  $X$  is dualizable. Looking at the definition of  $\alpha_X$ , we see that we need to show that  $L1 \otimes X$  is  $L$ -local when  $X$  is dualizable. Recall from Lemma 3.7.8 that an object  $Y$  is  $L$ -local iff  $[A, Y] = 0$  for every  $L$ -acyclic object  $A$ . Then observe (by Remark 2.2.54 and Lemma 2.2.61) that if  $X$  is dualizable then  $L1 \otimes X \simeq L1 \otimes D^2X \simeq L1 \otimes F(DX, 1) \simeq F(DX, L1)$  and hence  $[A, L1 \otimes X] \simeq [A, F(DX, L1)] = [A \otimes DX, L1] = 0$  since  $A \otimes DX$  is  $L$ -acyclic. This completes the proof.  $\square$

*Remark 3.7.21.* A smashing localization is thus given by tensoring by an object. The tensor product in the stable homotopy category is called the “smash product”: hence the term “smashing localization.”

**Lemma 3.7.22.** *Let  $\mathcal{T}$  be a compactly generated triangulated category. Every finite localization  $L : \mathcal{T} \rightarrow \mathcal{T}$  is a smashing localization.*

*Proof.* We will show that the  $L$ -locals are closed under coproducts; it is a straightforward exercise to show that this implies that  $L$  preserves coproducts. To this end, consider a set of  $L$ -local objects  $X_i$  and let  $\mathcal{G}$  denote a set of compact generators for  $\text{Ker} L = \text{loc}\langle \mathcal{G} \rangle$ . For each  $G \in \mathcal{G}$  we have  $[G, \coprod X_i] = \bigoplus [G, X_i] = 0$  using the fact that  $G$  is compact and  $X_i$  is  $L$ -local. The collection of objects  $A$  such that  $[A, \coprod X_i] = 0$  is a localizing subcategory of  $\mathcal{T}$  and hence  $[A, \coprod X_i] = 0$  for every  $A \in \text{Ker} L$ ; hence  $\coprod X_i$  is  $L$ -local.  $\square$

*Remark 3.7.23.* The “telescope conjecture” is the converse statement: that every smashing localization is a finite localization. It appeared in [Rav84] as one of a collection of seven influential conjectures concerning the structure of the stable homotopy category. The telescope conjecture is the only one of these conjecture which has not been proved; in fact, it is believed by some experts to be false. Nevertheless, the “telescope conjecture” can be asked for any triangulated category. See for example [Kra00] and the references therein.

## Homological localizations

If  $\mathcal{T}$  is a tensor triangulated category then for any object  $\mathbb{E} \in \mathcal{T}$  we can consider the associated homology theory  $\mathbb{E}_\bullet : \mathcal{T} \rightarrow \text{Ab}^{\mathbb{Z}}$  defined in Section 3.5. An object  $X$  is said to be  $\mathbb{E}_\bullet$ -acyclic if  $\mathbb{E}_\bullet(X) = 0$ . The  $\mathbb{E}_\bullet$ -acyclic objects form a thick subcategory of  $\mathcal{T}$  and we can ask whether they admit Bousfield localization. In the specific example  $\mathcal{T} = \text{SH}$ , this was the original question considered by Bousfield from which the term “Bousfield localization” derives. Indeed, he proved in [Bou79] that the  $\mathbb{E}_\bullet$ -acyclics admit Bousfield localization for any spectrum  $\mathbb{E} \in \text{SH}$ . His proof uses non-triangulated model-theoretic methods, but Margolis [Mar83] gave a proof which works in any rigidly-compactly generated tensor triangulated category.

**Theorem 3.7.24** (Bousfield; Margolis). *Let  $\mathcal{T}$  be a rigidly-compactly generated tensor triangulated category. For any object  $\mathbb{E}$  in  $\mathcal{T}$  there exists an exact localization functor  $L_{\mathbb{E}} : \mathcal{T} \rightarrow \mathcal{T}$  whose acyclic objects are precisely those objects  $X \in \mathcal{T}$  such that  $\mathbb{E}_{\bullet}(X) = 0$ .*

*Proof.* See [HPS97, Theorem 3.2.2] and [Mar83, Chapter 7]. □

*Remark 3.7.25.* The  $L_{\mathbb{E}}$ -equivalences are precisely those maps  $f : X \rightarrow Y$  such that  $\mathbb{E}_{\bullet}(f)$  is an isomorphism. Intuitively, passing from  $\mathcal{T}$  to the  $\mathbb{E}_{\bullet}$ -local category focuses on that “part” of  $\mathcal{T}$  that is “seen” by the homology theory  $\mathbb{E}_{\bullet}(-)$ . For example, we wish to make  $L_{\mathbb{E}}$ -equivalences isomorphisms because as far as the homology theory  $\mathbb{E}_{\bullet}(-)$  is concerned  $L_{\mathbb{E}}$ -equivalent objects are the same. The idea of studying the structure of the stable homotopy category  $\mathrm{SH}$  by localizing with respect to various homology theories was emphasized by [Rav84] and will turn up again in Chapter 6.

*Remark 3.7.26.* The term “Bousfield localization” has two meanings in the literature: the theory of Bousfield localization for triangulated categories and the theory of Bousfield localization for model categories. The two usages are related but different. Bousfield localization for model categories is the process by which one can attempt to add additional weak equivalences to an existing model structure; it is described in detail in [Hir03]. This subject is actually quite closely related to the theory of Bousfield localization in triangulated categories. Indeed, Bousfield proved the existence of homological localizations by taking the standard model structure on spectra and “localizing” this model structure to produce a new model structure whose weak equivalences were precisely the  $\mathbb{E}_{\bullet}$ -equivalences. This model-theoretic approach establishes the crucial point of the whole story—that the category obtained by inverting the  $\mathbb{E}_{\bullet}$ -equivalences actually exists—i.e., is locally small (recall Remark 3.7.11). Nevertheless, as implied by the statement of Theorem 3.7.24, the existence of homological localizations can now be established using purely triangulated techniques.

## 3.8 Examples

After one hundred pages of abstract theory, the reader deserves to be presented with some examples of tensor triangulated categories. We will focus on three of the primary examples: derived categories of schemes, stable module categories, and the stable homotopy category. Additional examples are mentioned in [Bal10b] and [HPS97].

### Derived categories

Derived categories were introduced by Jean-Louis Verdier in his 1967 thesis in order to provide the homological algebra necessary for new developments in algebraic geometry—notably the Grothendieck duality theorem [Har66]. These categories were very different from the familiar notion of an abelian category [Gro57] and it was Verdier’s attempt to formalize their basic structure that led him to define the axiomatic notion of a triangulated category. We will first discuss the general notion of the derived category of an abelian category before discussing particular examples arising in algebraic geometry. Standard references include Verdier’s thesis [Ver96], the textbooks [GM03, Wei94] and the survey [Kel96].

**Definition 3.8.1.** Let  $\mathcal{A}$  be an abelian category. The *derived category*  $D(\mathcal{A})$  is the category obtained from the category of chain complexes  $\mathrm{Ch}(\mathcal{A})$  by formally inverting the quasi-isomorphisms:

$$D(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]. \quad (3.8.2)$$

It has the structure of a triangulated category, as we shall see in Remark 3.8.8 below.

*Remark 3.8.3.* Unfortunately, this derived category might not exist in our universe (cf. Remark 3.6.3) and in order to construct derived functors one must use a theory of resolutions of unbounded complexes. For this reason, early authors typically put some boundedness conditions on their complexes. Nevertheless, it is proved in several sources (e.g. [Wei94,

Remark 10.4.5] or [AJS00, Corollary 5.6]) that the unbounded derived category  $D(\mathcal{A})$  exists when  $\mathcal{A}$  is a Grothendieck abelian category—which covers many examples of interest—and resolutions of unbounded complexes are by now well-understood (see e.g. [Spa88, Ser03, AJS00]). A particularly detailed account is [KS06, Chapter 14]. An alternative approach is to use the theory of Quillen model categories. It’s a folklore theorem (proved in [Bek00] and sometimes attributed to Joyal) that if  $\mathcal{A}$  is a Grothendieck abelian category then the category of chain complexes  $\text{Ch}(\mathcal{A})$  admits the structure of a Quillen model category whose weak equivalences are the quasi-isomorphisms and whose cofibrations are the monomorphisms. However, this “injective” model structure is not well-suited for constructing *left* derived functors and consequently doesn’t interact well with  $\otimes$ -product structures that might exist on  $\mathcal{A}$ . For this reason, [Hov01, Gil06, Gil07, CD09] have investigated alternative model structures on  $\text{Ch}(\mathcal{A})$  which have better compatibility with  $\otimes$ -structures on  $\mathcal{A}$ .

*Remark 3.8.4.* The derived category  $D(\mathcal{A})$  may equivalently be defined as the category obtained from the *homotopy category* of chain complexes  $K(\mathcal{A})$  by formally inverting the quasi-isomorphisms:

$$D(\mathcal{A}) = K(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]. \quad (3.8.5)$$

Indeed, any functor on  $\text{Ch}(\mathcal{A})$  which inverts the quasi-isomorphisms must invert homotopy equivalences and hence factors through the quotient functor  $\text{Ch}(\mathcal{A}) \rightarrow K(\mathcal{A})$ . It follows that the composite

$$\text{Ch}(\mathcal{A}) \rightarrow K(\mathcal{A}) \rightarrow K(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]$$

satisfies the universal property used in the definition of

$$\text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]$$

and hence there is a canonical equivalence of categories between definition (3.8.2) and definition (3.8.5). The second definition is the “better” definition for several reasons. For

starters, the collection of quasi-isomorphisms do not generally form a calculus of left fractions in  $\text{Ch}(\mathcal{A})$ . Indeed [GZ67, p. 18] shows that if that were the case then  $\text{Ch}(\mathcal{A})[\text{q.i.}^{-1}]$  would be abelian—but that is extremely rare (cf. Remark 3.1.19). On the other hand, the quasi-isomorphisms do form a calculus of left fractions in  $\text{K}(\mathcal{A})$  (see Remark 3.8.8 below). This discussion is closely related to the second reason why (3.8.5) is preferable to (3.8.2): the triangulated structure of  $\text{D}(\mathcal{A})$  is inherited from a triangulated structure on  $\text{K}(\mathcal{A})$ . In order to define this triangulated structure on  $\text{K}(\mathcal{A})$  we need the following definition:

**Definition 3.8.6** (Mapping cone). Let  $f : X \rightarrow Y$  be a morphism of complexes. The *mapping cone* of  $f$  is the complex  $\text{cone}(f)$  defined by  $\text{cone}(f)_n := X_{n-1} \oplus Y_n$  with differential given by

$$d_n^{\text{cone}(f)} : X_{n-1} \oplus Y_n \xrightarrow{\begin{pmatrix} -d_{n-1}^X & 0 \\ -f_{n-1} & d_n^Y \end{pmatrix}} X_{n-2} \oplus Y_{n-1}.$$

There is an evident short exact sequence of complexes

$$0 \rightarrow Y \rightarrow \text{cone}(f) \rightarrow \Sigma X \rightarrow 0$$

obtained by injecting  $Y$  and then projecting onto  $\Sigma X$ . Using these maps we can associate a triangle

$$X \xrightarrow{f} Y \longrightarrow \text{cone}(f) \longrightarrow \Sigma X$$

in the category  $\text{Ch}(\mathcal{A})$  to every morphism of complexes  $f : X \rightarrow Y$ .

**Proposition 3.8.7** (Verdier; Puppe). *The homotopy category  $\text{K}(\mathcal{A})$  is a triangulated category with suspension given by the usual translation of chain complexes and with exact triangles those triangles  $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$  which are isomorphic (as a triangle in  $\text{K}(\mathcal{A})$ ) to a triangle  $X \xrightarrow{f} Y \rightarrow \text{cone}(f) \rightarrow \Sigma X$  arising from a morphism of complexes  $f : X \rightarrow Y$ .*

*Proof.* See [Ver96, §I.3.3] for Verdier's original proof. Puppe [Pup67, §9] independently proved the result minus the octahedral axiom. □

*Remark 3.8.8.* The derived category  $D(\mathcal{A}) = K(\mathcal{A})[(\text{quasi-isomorphisms})^{-1}]$  is the Verdier quotient  $K(\mathcal{A})/K_{\text{ac}}(\mathcal{A})$  where  $K_{\text{ac}} \subset K(\mathcal{A})$  denotes the thick subcategory of acyclic complexes. This is because a map of complexes is a quasi-isomorphism iff its mapping cone is acyclic. Thus by Theorem 3.6.9,  $D(\mathcal{A})$  inherits a triangulated structure from the triangulated structure on  $K(\mathcal{A})$ . To be precise, the exact triangles are those which are isomorphic (as triangles in  $D(\mathcal{A})$ ) to the images of exact triangles in  $K(\mathcal{A})$ . Furthermore, morphisms in  $D(\mathcal{A})$  can be described as equivalence classes of fractions of morphisms in  $K(\mathcal{A})$  (cf. Remark 3.6.12).

*Remark 3.8.9.* In the derived category  $D(\mathcal{A})$  an object  $A \in \mathcal{A}$  is “identified” with all of its projective resolutions. Moreover, using the description of  $D(\mathcal{A})$  in terms of fractions it is easy to check (cf. [Wei94, Lem. 10.4.6 and Cor. 10.4.7]) that if  $P$  is a bounded below complex of projectives then  $\text{Hom}_{D(\mathcal{A})}(P, X) = \text{Hom}_{K(\mathcal{A})}(P, X)$  for any complex  $X$ . Thus if  $\mathcal{A}$  has enough projectives then for any two objects  $A, B \in \mathcal{A}$  we have

$$\begin{aligned} \text{Hom}_{D(\mathcal{A})}(A, \Sigma^i B) &\simeq \text{Hom}_{D(\mathcal{A})}(P, \Sigma^i B) \simeq \text{Hom}_{K(\mathcal{A})}(P, \Sigma^i B) \\ &= H^i(\text{Hom}_{\text{Ch}(\mathcal{A})}(P, B)) = \text{Ext}_{\mathcal{A}}^i(A, B) \end{aligned}$$

where  $P \rightarrow A$  is any choice of projective resolution.

*Remark 3.8.10.* The homology functors  $H_i : \text{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$  induce homological functors

$$H_i : D(\mathcal{A}) \rightarrow \mathcal{A}.$$

By applying these functors to an exact triangle in  $D(\mathcal{A})$  we obtain a long exact sequence in  $\mathcal{A}$ . Every long exact sequence in homological algebra arises in this manner from an exact triangle in a derived category. The next couple of remarks will clarify this claim.

**Definition 3.8.11** (Mapping cylinder). We saw in Definition 3.8.6 that a morphism of complexes  $f : X \rightarrow Y$  gives rise to a short exact sequence  $0 \rightarrow Y \rightarrow \text{cone}(f) \rightarrow \Sigma X \rightarrow 0$ . Shifting

the second map we obtain a morphism  $\Sigma^{-1}\text{cone}(f) \rightarrow X$ . The *mapping cylinder* of  $f$  is defined to be the mapping cone of this morphism  $\Sigma^{-1}\text{cone}(f) \rightarrow X$ . More concretely, it is a complex  $\text{cyl}(f)$  defined to be  $X \oplus \Sigma X \oplus Y$  with differential given by

$$d_n^{\text{cyl}(f)} : X_n \oplus X_{n-1} \oplus Y_n \xrightarrow{\begin{pmatrix} d_n^X & \text{id}_{n-1}^X & 0 \\ 0 & -d_{n-1}^X & 0 \\ 0 & -f_{n-1} & d_n^Y \end{pmatrix}} X_{n-1} \oplus X_{n-2} \oplus Y_{n-1}.$$

As before it gives rise to a short exact sequence

$$0 \rightarrow X \rightarrow \text{cyl}(f) \rightarrow \text{cone}(f) \rightarrow 0$$

and hence to a triangle

$$X \rightarrow \text{cyl}(f) \rightarrow \text{cone}(f) \rightarrow \Sigma X.$$

**Proposition 3.8.12.** *Let  $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$  be a short exact sequence of chain complexes in an abelian category  $\mathcal{A}$ . There exists a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & \text{cyl}(f) & \longrightarrow & \text{cone}(f) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \longrightarrow 0 \end{array}$$

in  $\text{Ch}(\mathcal{A})$  which has exact rows, such that  $\beta$  is a homotopy equivalence, while  $\gamma$  is a quasi-isomorphism. Furthermore, there is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & \text{cyl}(f) & \longrightarrow & \text{cone}(f) & \longrightarrow & \Sigma X \\ \parallel & & \downarrow \beta & & \parallel & & \parallel \\ X & \xrightarrow{f} & Y & \longrightarrow & \text{cone}(f) & \longrightarrow & \Sigma X \end{array} \quad (3.8.13)$$

in  $\text{K}(\mathcal{A})$  and hence the top row is an exact triangle in  $\text{K}(\mathcal{A})$ . Finally, using the fact that  $\gamma$  is a quasi-isomorphism we can construct a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & \text{cyl}(f) & \longrightarrow & \text{cone}(f) & \longrightarrow & \Sigma X \\ \parallel & & \downarrow \beta & & \downarrow \gamma & & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \cdots \cdots \cdots & \Sigma X \end{array} \quad (3.8.14)$$

in  $\text{D}(\mathcal{A})$  and conclude that there exists an exact triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow \Sigma X$  in  $\text{D}(\mathcal{A})$ .

*Proof.* See [Wei94, §1.5 and §10.4]. □

*Remark 3.8.15.* Thus every short exact sequence of complexes gives rise to an exact triangle in  $D(\mathcal{A})$  which produces the same long exact sequence in homology as the original short exact sequence. Note that the bottom triangle in (3.8.14) is not necessarily an exact triangle in  $K(\mathcal{A})$  because the map  $\gamma$  is only a quasi-isomorphism in general. In fact, according to [Wei94, Exercise 1.5.5], if  $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$  is a short exact sequence of objects in  $\mathcal{A}$  (viewed as complexes concentrated in degree zero) then the map  $\gamma$  is a homotopy equivalence iff the short exact sequence splits. This is the key difference between the triangulated categories  $K(\mathcal{A})$  and  $D(\mathcal{A})$ :  $K(\mathcal{A})$  only has the exact triangles arising from split exact sequences.

## Derived categories of schemes

For any scheme  $X$ , the category of  $\mathcal{O}_X$ -modules  $\text{Mod}(X)$  is a Grothendieck abelian category so we can consider the derived category  $D(X) := D(\text{Mod}(X))$ . It inherits the structure of a tensor triangulated category by taking the derived tensor product  $\otimes_{\mathcal{O}_X}^{\mathbb{L}}$ . (For detailed expositions on how to derive the tensor product in an unbounded situation see [Lip09, §2.5] or [KS06, §18.6].) On the other hand, if we would like to work with quasi-coherent sheaves then we typically assume that  $X$  is quasi-compact and quasi-separated. For example, these are the conditions which ensure that the pushforward of a quasi-coherent sheaf remains quasi-coherent [GD71, Chap. I, Prop. 6.7.1]. For the convenience of the reader, let us recall that a scheme is *quasi-separated* if the intersection of two quasi-compact open subsets is quasi-compact. Their basic properties are studied in [Gro64, §IV.1.2] and [GD71, §I.6.1]. In any case, the assumption that  $X$  be quasi-compact and quasi-separated is a very mild assumption. For example, any noetherian scheme satisfies these conditions, as does any

affine scheme. For  $X$  quasi-compact and quasi-separated we consider the full subcategory

$$D_{\text{qc}}(X) \subset D(X)$$

consisting of those complexes of  $\mathcal{O}_X$ -modules whose cohomology sheaves are quasi-coherent. It is a triangulated subcategory of  $D(X)$  and has arbitrary coproducts. Moreover, the tensor product of  $\mathcal{O}_X$ -modules preserves quasi-coherent sheaves [GD71, Chap. I, Cor. 2.2.2] and hence  $D_{\text{qc}}(X)$  is a tensor triangulated subcategory of  $D(X)$ . However,  $D_{\text{qc}}(X)$  is still a large tensor triangulated category. For the purposes of tensor triangular geometry we want to look at the essentially small “compact part” of  $D_{\text{qc}}(X)$ . To this end, let us recall the notion of a “perfect complex” introduced by Grothendieck and Illusie in [SGA71, Exposés I–III] and revisited by Thomason in [TT90, Section 2].

**Definition 3.8.16.** A *perfect complex* is a complex of  $\mathcal{O}_X$ -modules which is locally quasi-isomorphic to a bounded complex of vector bundles.

*Remark 3.8.17.* A perfect complex is not assumed to be quasi-coherent. However, it follows from [TT90, Prop. 2.2.12 and Rem. 2.2.7] that a perfect complex has quasi-coherent cohomology. Thus the collection of perfect complexes forms a full subcategory  $D_{\text{perf}}(X) \subset D_{\text{qc}}(X)$ .

**Theorem 3.8.18** (Bondal and van den Bergh). *Let  $X$  be a quasi-compact, quasi-separated scheme. Then  $D_{\text{qc}}(X)$  is a rigidly-compactly generated tensor triangulated category whose subcategory of compact-rigid objects is exactly  $D_{\text{perf}}(X)$ . In particular,  $D_{\text{perf}}(X)$  is an essentially small, idempotent complete, rigid tensor triangulated category.*

*Proof.* Bondal and van den Bergh [BB03, Theorem 3.1.1] prove that if  $X$  is quasi-compact and quasi-separated then the compact objects in  $D_{\text{qc}}(X)$  are precisely the perfect complexes and that, moreover,  $D_{\text{qc}}(X)$  is generated by a single perfect complex. On the other hand, it follows from [SGA71, Exposé I, Section 7] that perfect complexes are dualizable so our claims follow from Proposition 3.4.5. □

*Remark 3.8.19.* On the other hand, if  $X$  is quasi-compact and quasi-separated then the abelian category of quasi-coherent sheaves  $\mathrm{QCoh}(X)$  is a Grothendieck abelian category (cf. [SGA71, Exposé II, Lemme 3.1] or [TT90, Appendix B.3]). It is therefore very natural to consider  $D(\mathrm{qc}/X) := D(\mathrm{QCoh}(X))$ .

**Theorem 3.8.20** (Bökstedt and Neeman). *Let  $X$  be a quasi-compact and separated scheme. The functor  $D(\mathrm{qc}/X) \rightarrow D(X)$  induced by the inclusion  $\mathrm{QCoh}(X) \hookrightarrow \mathrm{Mod}(X)$  is fully faithful and produces an equivalence of triangulated categories  $D(\mathrm{qc}/X) \xrightarrow{\sim} D_{\mathrm{qc}}(X)$ .*

*Proof.* This is proved in [BN93]; compare [SGA71, Exposé II, Proposition 3.5]. □

*Remark 3.8.21.* The statement of the theorem is *not* true if  $X$  is only assumed to be quasi-compact and quasi-separated. Indeed, Verdier produces examples in [SGA71, Exposé II, Appendice I] for which the functor  $D(\mathrm{qc}/X) \rightarrow D_{\mathrm{qc}}(X)$  is not fully faithful; see also [SGA71, Exposé II, Remarque 3.6].

*Remark 3.8.22.* By the theorem, if  $X$  is quasi-compact and separated then we can choose to work with  $D(\mathrm{qc}/X)$  rather than  $D_{\mathrm{qc}}(X)$ . For example, in this case  $D_{\mathrm{perf}}(X)$  is equivalent to the full subcategory of  $D(\mathrm{qc}/X)$  consisting of the quasi-coherent perfect complexes.

*Remark 3.8.23.* If  $X$  is a quasi-compact and quasi-separated scheme which admits an ample family of line bundles (cf. [TT90, Def. 2.1.1]) then every perfect complex on  $X$  is *globally* quasi-isomorphic to a bounded complex of vector bundles (cf. [TT90, Prop. 2.3.1]). In this case we have an equivalence  $D_{\mathrm{perf}}(X) \cong D^b(\mathrm{Vect}/X)$ .

*Example 3.8.24.* If  $R$  is a commutative ring then the derived category of perfect complexes  $D_{\mathrm{perf}}(R)$  is equivalent to the bounded derived category of finitely generated projective modules  $D^b(R\text{-proj})$ . In fact, according to Remark 3.8.9, the functor  $K^b(R\text{-proj}) \rightarrow D^b(R\text{-proj})$  is fully faithful. In summary, we have an equivalence of tensor triangulated categories

$$D_{\mathrm{perf}}(R) \cong K^b(R\text{-proj})$$

where the tensor product on  $K^b(R\text{-proj})$  is just the usual tensor product of complexes.

*Example 3.8.25.* If  $X$  is a quasi-projective noetherian scheme then  $D_{\text{perf}}(X) \cong D^b(\text{Vect}/X)$ .

Together with Remark 3.8.22 this explains the discussion in Remark 3.7.17.

## Modular representation theory

Fix a finite group  $G$ , a field  $k$ , and consider the category of  $kG$ -modules  $\text{Mod}(kG)$ . For various purposes in modular representation theory, one wants to “discard” or “ignore” projective  $kG$ -modules. This leads to the following construction:

**Definition 3.8.26.** The *stable module category*  $\text{StMod}(kG)$  is a  $k$ -linear category whose objects are the  $kG$ -modules and whose morphisms, denoted  $\underline{\text{Hom}}_{kG}(M, N)$ , are given by

$$\underline{\text{Hom}}_{kG}(M, N) := \frac{\text{Hom}_{kG}(M, N)}{\text{PHom}_{kG}(M, N)}$$

where  $\text{PHom}_{kG}(M, N)$  is the  $k$ -linear subspace of  $\text{Hom}_{kG}(M, N)$  consisting of those  $kG$ -linear maps which factor through a projective  $kG$ -module. Composition is easily seen to be well-defined.

*Remark 3.8.27.* The stable module category  $\text{StMod}(kG)$  is the quotient of  $\text{Mod}(kG)$  by the “ideal”  $\text{PHom}_k(-, -)$  in the sense of [Mit72]. The quotient functor  $\text{Mod}(kG) \rightarrow \text{StMod}(kG)$  is the universal additive functor which kills the projective modules.

*Remark 3.8.28.* If the characteristic of  $k$  does not divide the order of  $G$  then Maschke’s theorem tells us that every  $kG$ -module is projective; consequently,  $\text{StMod}(kG) = 0$ . We are therefore only interested in the “modular situation” in which  $k$  has positive characteristic dividing the order of  $G$ .

*Remark 3.8.29.* Although  $\text{Mod}(kG)$  is an abelian category, the stable version  $\text{StMod}(kG)$  is never abelian—provided it is non-zero. Nevertheless, it was observed by Dieter Happel [Hap88] that it has the structure of a triangulated category:

**Theorem 3.8.30 (Happel).** *The stable module category  $\text{StMod}(kG)$  is a triangulated category with structure given as follows:*

- (1) *The suspension is usually denoted  $\Omega^{-1}$ . For a  $kG$ -module  $X$ ,  $\Omega^{-1}(X)$  is the cokernel of an embedding of  $X$  into its injective hull. Although this construction is not functorial on  $\text{Mod}(kG)$ , it does provide a functor on the stable category*

$$\Omega^{-1} : \text{StMod}(kG) \rightarrow \text{StMod}(kG)$$

*which is, in fact, an equivalence.*

- (2) *Given a short exact sequence  $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$  in  $\text{Mod}(kG)$ , we can choose an injective hull  $X \hookrightarrow I$  and form a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & I & \longrightarrow & \Omega^{-1}(X) \longrightarrow 0. \end{array}$$

*The map  $Z \rightarrow \Omega^{-1}X$  is uniquely determined in  $\text{StMod}(kG)$  and we obtain a triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Omega^{-1}X$$

*in  $\text{StMod}(kG)$ . The exact triangles in  $\text{StMod}(kG)$  are those triangles which are isomorphic to a triangle arising in this way from a short exact sequence in  $\text{Mod}(kG)$ .*

*Proof.* This is proved in [Hap88]. For further discussion of the stable module category see [Car96] and [BIK12]. □

*Remark 3.8.31.* The inverse of  $\Omega^{-1}$  is a functor  $\Omega : \text{StMod}(kG) \rightarrow \text{StMod}(kG)$  which sends a  $kG$ -module  $X$  to the kernel of a surjective map  $P \rightarrow X$  from a projective module to  $X$ . Schanuel's lemma implies that this definition of  $\Omega(X)$  is well-defined up to projective direct summands—i.e., up to isomorphism in  $\text{StMod}(kG)$ . Showing that  $\Omega$  and  $\Omega^{-1}$  are inverse

equivalences uses the fact that  $kG$  is a Frobenius algebra—i.e., that projective and injective  $kG$ -modules coincide. In fact, the construction of the stable module category works for any Frobenius algebra. See [Hap88] for more details.

*Remark 3.8.32.* The category of  $kG$ -modules  $\text{Mod}(kG)$  is a symmetric monoidal category: the tensor product is given by  $X \otimes_k Y$  with diagonal  $G$ -action and the unit is  $k$  with the trivial  $G$ -action. Frobenius reciprocity with respect to the trivial subgroup can be used to show that the functor  $X \otimes_k - : \text{Mod}(kG) \rightarrow \text{Mod}(kG)$  preserves projective  $kG$ -modules. It follows that the symmetric monoidal structure  $- \otimes_k -$  on  $\text{Mod}(kG)$  descends to a symmetric monoidal structure on  $\text{StMod}(kG)$  and we obtain a tensor triangulated category.

*Remark 3.8.33.* One can just as easily apply the construction to the category  $\text{mod}(kG)$  of all finitely generated  $kG$ -modules—rather than the category  $\text{Mod}(kG)$  of all  $kG$ -modules—and thereby produce a stable module category which we denote by  $\text{stmod}(kG)$ . There is an obvious functor  $\text{stmod}(kG) \rightarrow \text{StMod}(kG)$  which identifies  $\text{stmod}(kG)$  with the full subcategory of  $\text{StMod}(kG)$  consisting of the finitely generated modules. That the functor is faithful follows from the fact that if a map between finitely generated  $kG$ -modules factors through a projective module then it factors through a finitely generated projective module. Thus we have a tensor triangulated subcategory  $\text{stmod}(kG) \subset \text{StMod}(kG)$ . In fact  $\text{StMod}(kG)$  is compactly generated and  $\text{stmod}(kG) = (\text{StMod}(kG))^c$ . The simple  $kG$ -modules provide a generating set of compact objects. If  $G$  is a  $p$ -group then  $k$  is a compact generator since it is the only simple  $kG$ -module.

*Remark 3.8.34.* Another tensor triangulated category related to the representation theory of  $G$  is  $D^b(\text{mod}(kG))$ —the bounded derived category of finitely generated  $kG$ -modules. Note that the tensor product on  $\text{mod}(kG)$  is exact and hence descends very easily to a tensor product on  $D^b(\text{mod}(kG))$ . Rickard’s theorem nicely relates the derived and stable categories:

**Theorem 3.8.35** (Rickard’s theorem). *Let  $G$  be a finite group and let  $k$  be a field. There is*

an equivalence of tensor triangulated categories

$$\text{stmod}(kG) \cong \frac{D^b(\text{mod}(kG))}{D^b(\text{proj}(kG))}$$

where  $D^b(\text{proj}(kG))$  denotes the bounded derived category of finitely generated projective  $kG$ -modules.

*Proof.* The original proof is [Ric89, Theorem 2.1]. The equivalence is induced (recall Remark 3.8.27) by the additive functor  $\text{mod}(kG) \rightarrow D^b(\text{mod}(kG))/D^b(\text{proj}(kG))$  obtained by composing the natural embedding of  $\text{mod}(kG)$  in  $D^b(\text{mod}(kG))$  with the Verdier quotient functor.  $\square$

*Remark 3.8.36.* We have already noted that  $\text{stmod}(kG)$  forms the compact objects of a larger compactly generated category  $\text{StMod}(kG)$ . The situation isn't quite so nice for  $D^b(\text{mod}(kG))$ . Although  $D(\text{Mod}(kG))$  is a compactly generated triangulated category, the subcategory of compact objects is the bounded derived category of finitely generated *projective*  $kG$ -modules:  $D(\text{Mod}(kG))^c \cong D^b(\text{proj}(kG))$ . The correct “large” category for  $D^b(\text{mod}(kG))$  is the homotopy category of injective  $kG$ -modules:  $K(\text{Inj}(kG))$ . Indeed it is proved in [BK08] that  $K(\text{Inj}(kG))$  is compactly generated and that  $K(\text{Inj}(kG))^c \cong D^b(\text{mod}(kG))$ .

*Remark 3.8.37.* The graded endomorphism ring of the unit object in  $D^b(\text{mod}(kG))$  is the group cohomology ring:

$$\text{End}_{D^b(\text{mod}(kG))}^\bullet(\mathbb{1}) = \text{Hom}_{D^b(\text{mod}(kG))}(\Sigma^\bullet k, k) = \text{Ext}_{kG}^{-\bullet}(k, k) = H^{-\bullet}(G, k).$$

On the other hand, as explained in [Car96, Chapter 6], the graded endomorphism ring of the unit object in  $\text{stmod}(kG)$  is the Tate cohomology ring:

$$\text{End}_{\text{stmod}(kG)}^\bullet(\mathbb{1}) = \underline{\text{Hom}}_{kG}(k, \Omega^\bullet k) = \widehat{\text{Ext}}_{kG}^{-\bullet}(M, N) = \widehat{H}^{-\bullet}(G, k).$$

The reason for the opposite grading is due to our choice of “homological” grading for endomorphism rings:  $[-, -]_* = [\Sigma^*(-), -]$ . Our convention matches the usual convention in stable homotopy theory. In any case, keep Remark 2.3.11 in mind.

## Stable homotopy theory

A more topological example of a tensor triangulated category is provided by the stable homotopy category  $\mathrm{SH}$ . In order to motivate this example, we need to recall some classical homotopy theory. To this end, let  $\Sigma X$  denote (for this section only) the reduced suspension of a based space  $X$ , and let  $[X, Y]$  denote the set of based homotopy classes of based maps  $X \rightarrow Y$ . The origins of “stable” homotopy theory lie in the Freudenthal suspension theorem which states that if  $X$  and  $Y$  are based finite CW-complexes then the sequence obtained by iterating the suspension functor

$$[X, Y] \rightarrow [\Sigma X, \Sigma Y] \rightarrow [\Sigma^2 X, \Sigma^2 Y] \rightarrow \cdots$$

eventually becomes an isomorphism. For example, taking  $X$  to be the  $i$ -dimensional sphere  $S^i \cong \Sigma^i S^0$ , the sequence of homotopy groups

$$\pi_i(Y) \rightarrow \pi_{i+1}(\Sigma Y) \rightarrow \pi_{i+2}(\Sigma^2 Y) \rightarrow \cdots$$

eventually stabilizes. The stable value, denoted  $\pi_i^s(Y)$ , is called the  $i$ th *stable* homotopy group of  $Y$ . These “stable” homotopy groups  $\pi_*^s(-)$  have certain advantages over the ordinary homotopy groups  $\pi_*(-)$ ; for example, they satisfy the Eilenberg-Steenrod axioms for a (reduced) generalized homology theory. Nevertheless, they are incredibly difficult to compute: there isn’t a single example of a simply-connected, non-contractible finite CW-complex whose stable homotopy groups are entirely known. In particular, the stable homotopy groups of spheres  $\pi_*^s := \pi_*^s(S^0)$  remain one of the most complicated and mysterious objects in the whole of mathematics.

Besides homotopy groups, there are many other constructions and results in homotopy theory which “stabilize” after suspending enough times, and the study of such “stable” phenomena became known as “stable homotopy theory.” As the subject developed, experts recognized that it would be convenient to have a *category* in which to do stable homotopy theory, but it was not entirely clear what this category should be. The most basic intuition about the “stable homotopy category” is that it should be a place where the suspension functor is invertible. The idea is that objects have been “stabilized” by being suspended infinitely many times, and hence can be infinitely desuspended too. Nevertheless, although various authors proposed possible definitions, all the proposed frameworks for stable homotopy theory were deficient in one way or another. In this search for a good definition of SH, Puppe was led (independently from Verdier) to introduce the axioms of a triangulated category [Pup67]. He emphasized that not only should  $\Sigma$  be invertible, but the stable homotopy category should be additive, and it should have exact triangles arising from cofiber sequences.

In any case, the first satisfactory construction of SH was provided by Boardman (see [Vog70]). Later, several other constructions were proposed (e.g. [Ada74], [Pup73], [BF78]) but they all provided categories equivalent to Boardman’s and it became accepted that any tensor triangulated category equivalent to Boardman’s was “the” stable homotopy category. Despite the variety of constructions, there was always a clear idea of what the subcategory of “finite” or “compact” objects  $\mathrm{SH}^{\mathrm{fin}} \subset \mathrm{SH}$  should be. This category  $\mathrm{SH}^{\mathrm{fin}}$  had a well-established “canonical” construction going back to the early work of Spanier and J.H.C. Whitehead. We’ll describe this construction in Definition 3.8.45 below. Nevertheless, the construction of the larger category SH is a highly non-trivial result. One of the motivations behind the construction of the larger category is that the objects, called “spectra,” should represent different cohomology theories on the category of spaces. For example, the Eilenberg-MacLane spaces  $K(G, n)$ ,  $n \geq 0$  should together produce an object in SH which would represent ordi-

nary cohomology theory  $H^\bullet(-; G)$ . Classical Brown representability should amount to saying that cohomology theories become representable as functors on the stable homotopy category. None of this could be achieved with small finite-dimensional objects or a category like  $\mathrm{SH}^{\mathrm{fin}}$  which does not possess infinite coproducts.

Due to the different possible constructions of  $\mathrm{SH}$ , Margolis wrote his book [Mar83] in a construction-agnostic way by laying out an axiomatic definition for  $\mathrm{SH}$  and showing that one of the constructions satisfies his axioms (namely, Adams' category of CW-spectra [Ada74]). Basically, Margolis' axioms (cf. [Mar83, §II.1]) amount to a tensor triangulated category  $\mathcal{T}$  admitting small coproducts which is compactly generated by the unit  $\mathbb{1}$  and whose subcategory of compact objects  $\mathcal{T}^c$  is equivalent as a tensor triangulated category to the well-established finite stable homotopy category  $\mathrm{SH}^{\mathrm{fin}}$ . This axiomatic point of view was taken very seriously in [HPS97] and has been very influential in bringing techniques from stable homotopy theory into other areas of mathematics.

More recently, the foundations of stable homotopy theory have undergone another shift by the discovery of Quillen model categories of spectra whose homotopy categories are equivalent to  $\mathrm{SH}$  but which, moreover, have compatible monoidal structures at the level of the underlying model categories. This is an extremely important development in algebraic topology and has led to even more constructions for the stable homotopy category—e.g., symmetric spectra [HSS00], orthogonal spectra [MMS01], or the  $S$ -modules of [EKM97]. Nevertheless, for the purposes of this dissertation, we will largely follow Margolis' axiomatic approach. The more recent developments—although important—are not crucial for the results of Chapter 6.

## The Spanier-Whitehead category

There is a well-known formal method for inverting an endofunctor on an arbitrary category. By applying this method to the homotopy category of based CW-complexes  $hCW_*$  one obtains the so-called Spanier-Whitehead category  $SW$ . Although this construction is too naive to be “the stable homotopy category” it serves as a useful example to consider. The following discussion is derived from [Mar83, §I.2] which includes proofs for the following claims.

**Definition 3.8.38.** Let  $SW$  denote the category whose objects are pairs  $(X, n)$  where  $X$  is a based CW-complex and  $n \in \mathbb{Z}$  is any integer with maps defined by

$$\text{Hom}_{SW}((X, n), (Y, m)) := \text{colim}_k [\Sigma^{k+n} X, \Sigma^{k+m} Y]$$

where the colimit is over the suspension maps. There is a canonical functor  $hCW_* \rightarrow SW$  which sends a based CW-complex  $X$  to  $(X, 0)$  and  $SW$  inherits a “geometric suspension”  $\Sigma : SW \rightarrow SW$  from the reduced suspension of based spaces:  $\Sigma(X, n) := (\Sigma X, n)$ . In addition, there is a “formal suspension”  $\Sigma' : SW \rightarrow SW$  defined by  $\Sigma'(X, n) := (X, n + 1)$ . The formal suspension is evidently an automorphism having inverse  $(X, n) \mapsto (X, n - 1)$ .

**Proposition 3.8.39.** *The following hold:*

- (1) *The two suspensions are naturally isomorphic:  $(\Sigma X, n) \simeq (X, n + 1)$ . Consequently, the geometric suspension  $\Sigma : SW \rightarrow SW$  is an equivalence of categories.*
- (2)  *$SW$  is the universal category out of  $hCW_*$  which inverts  $\Sigma : hCW_* \rightarrow hCW_*$ . More precisely, if  $\mathcal{C}$  is any category equipped with an automorphism  $T : \mathcal{C} \rightarrow \mathcal{C}$  and  $F : hCW_* \rightarrow \mathcal{C}$  is a functor such that  $F \circ \Sigma \simeq T \circ F$  then there exists a factorization*

$$\begin{array}{ccc} hCW_* & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow \bar{F} & \\ SW & & \end{array}$$

up to isomorphism such that  $\bar{F} \circ \Sigma' \simeq T \circ \bar{F}$ ; moreover,  $\bar{F}$  is unique up to isomorphism.

(3)  $\mathbf{SW}$  is an additive category. The direct sum is given by

$$(X, m) \oplus (Y, n) \simeq (\Sigma^{k-m} X \vee \Sigma^{k-n} Y, m + n - k)$$

where  $k$  is any integer such that  $k - m, k - n \geq 0$ .

(4)  $\mathbf{SW}$  is a symmetric monoidal category. The tensor product is denoted  $\wedge$  and is derived from the smash product of based spaces:

$$(X, m) \wedge (Y, n) \simeq (X \wedge Y, m + n).$$

The unit is  $(S^0, 0)$ .

(5) If  $Y$  is a finite based CW-complex then  $\mathrm{Hom}_{\mathbf{SW}}((S^0, n), (Y, 0)) \simeq \pi_n^s(Y)$  is the  $n$ th stable homotopy group of  $Y$ .

**Definition 3.8.40.** The (reduced) cone of a based space  $X$  is defined by  $C(X) := I \times X / \sim$  where  $\sim$  collapses  $\{0\} \times X$  and  $I \times \{x_0\}$  to a point. The mapping cone of a map of based spaces  $f : X \rightarrow Y$  is defined by  $C(f) := C(X) \times Y / \sim$  where  $(1, x) \sim f(x)$ . There are canonical maps  $Y \rightarrow C(f)$  and  $C(f) \rightarrow \Sigma X$  and hence every map of based CW-complexes gives rise to a triangle  $X \xrightarrow{f} Y \rightarrow C(f) \rightarrow \Sigma X$  in  $h\mathbf{CW}_*$ . Any triangle in  $h\mathbf{CW}_*$  which is isomorphic to such a triangle is called an “unstable exact triangle.”

**Proposition 3.8.41.** The Spanier-Whitehead category  $(\mathbf{SW}, \Sigma)$  is a triangulated category in which a triangle  $(X, l) \rightarrow (Y, m) \rightarrow (Z, n) \rightarrow \Sigma(X, l)$  is exact if there exists a  $k \geq 0$  such that the sequence of morphisms can be represented by maps

$$\Sigma^{k+l} X \rightarrow \Sigma^{k+m} Y \rightarrow \Sigma^{k+n} Z \rightarrow \Sigma^{k+l+1} X$$

which form an unstable exact triangle in  $h\mathbf{CW}_*$ .

*Remark 3.8.42.* If  $h_\bullet : hCW_* \rightarrow \text{Ab}^{\mathbb{Z}}$  is a reduced homology theory in the classical sense (cf. [Swi02, pp. 109–117]) then it factors through SW by the universal property of Proposition 3.8.41 part (2). In fact, the induced functor  $H_\bullet : \text{SW} \rightarrow \text{Ab}^{\mathbb{Z}}$  is a stable homological functor (in the sense of Definition 3.1.37). Conversely, every stable homological functor  $H_\bullet : \text{SW} \rightarrow \text{Ab}^{\mathbb{Z}}$  gives a reduced homology theory on  $hCW_*$  by precomposing with the canonical map  $hCW_* \rightarrow \text{SW}$ . In fact, we have the following very precise result:

**Proposition 3.8.43.** *Let  $\mathcal{H}_{hCW_*}$  denote the category of reduced homology theories on  $hCW_*$  and let  $\mathcal{H}_{\text{SW}}$  denote the category of stable homological functors  $\text{SW} \rightarrow \text{Ab}^{\mathbb{Z}}$ . Then there is an equivalence of categories  $\mathcal{H}_{hCW_*} \simeq \mathcal{H}_{\text{SW}}$ . Similar statements hold for cohomology theories.*

*Remark 3.8.44.* The construction of the Spanier-Whitehead category is very simple but it has several problems. One of the key problems is that it doesn't have coproducts. For example,  $\coprod_{n \geq 1} (X, -n)$  doesn't exist in SW. Even worse, the canonical functor  $hCW_* \rightarrow \text{SW}$  doesn't preserve the coproducts which exist in  $hCW_*$ . Indeed, given a set  $(X_i)$  of based CW-complexes, their coproduct in  $hCW_*$  is the infinite wedge sum  $\bigvee_i X_i$  but  $\coprod_i (X_i, 0) \neq (\bigvee_i X_i, 0)$  in SW. As a consequence of the lack of coproducts, Brown representability doesn't hold for SW. In other words (cf. Remark 3.8.42), reduced cohomology theories on  $hCW_*$  do not become representable as functors on SW. This is one of the key features that algebraic topologists desired for “the” stable homotopy category and so SW does not give what was desired. Nevertheless, SW does give the appropriate stabilization for *finite* CW-complexes:

**Definition 3.8.45.** The Spanier-Whitehead category of finite CW-complexes  $\text{SW}^{\text{fin}}$  is defined to be the full subcategory of SW consisting of objects  $(X, n)$  where  $X$  is a *finite* CW-complex. This is the canonical description of the tensor triangulated category  $\text{SH}^{\text{fin}} \cong \text{SW}^{\text{fin}}$ .

### 3.9 Classification of thick subcategories

Given a category (such as the category of smooth manifolds) it is highly unlikely that we will be able to classify the objects of that category up to isomorphism (e.g., classify smooth manifolds up to diffeomorphism). Faced with such wild problems, we can instead attempt to provide classification up to a weaker notion of equivalence and there are prominent examples where this approach has been very successful—for example, the classification of manifolds up to (various kinds of) bordism (cf. [Sto68]). In particular, the last 20 years have shown that if the category is triangulated then it is sometimes possible to completely classify the objects of the category up to the naturally available triangulated structure—in other words, classify the objects up to suspensions, cofibers, direct sums, and direct summands. This essentially amounts to a classification of the *thick subcategories* of the triangulated category.

The first theorem of this kind arose in stable homotopy theory. Following on from their work on the Nilpotence Theorem, Hopkins and Smith [HS98] proved a classification theorem for the thick subcategories of the stable homotopy category of finite spectra  $\mathrm{SH}^{\mathrm{fin}}$ . This is quite a remarkable theorem. The category  $\mathrm{SH}^{\mathrm{fin}}$  is extremely complicated—for example, we don't have a complete understanding of the graded morphisms between *any* two non-zero objects. Nevertheless, Hopkins and Smith showed that if one takes a more global approach by stepping back and looking at the whole category then one can obtain a very satisfying classification result.

After this work in stable homotopy theory, Hopkins noticed that an analogous result could be obtained for the derived category of perfect complexes of a commutative ring. Although the proof published in [Hop87] was incorrect, Neeman [Nee92a] showed that the classification result did hold if the ring was noetherian. Later, Thomason [Tho97] extended

this result not just to all commutative rings but more generally to  $D_{\text{perf}}(X)$  for any quasi-compact and quasi-separated scheme  $X$ . Furthermore, in modular representation theory, Benson, Carlson and Rickard [BCR97] obtained an analogous classification theorem for the stable module category of a finite group.

As was made clear by Thomason’s work, the Benson-Carlson-Rickard theorem, and later by Balmer, these results are really not classifications of thick subcategories in triangulated categories—they are really classifications of thick  $\otimes$ -ideals in *tensor* triangulated categories. This fact was hidden in the first two examples ( $\text{SH}^{\text{fin}}$  and  $D_{\text{perf}}(R)$ ) because those two categories are generated by the unit object and this implies that every thick subcategory is automatically a  $\otimes$ -ideal:

**Lemma 3.9.1.** *Let  $\mathcal{K}$  be a tensor triangulated category that is generated by the unit object:  $\mathcal{K} = \text{thick}\langle \mathbb{1} \rangle$ . Then every thick subcategory of  $\mathcal{K}$  is in fact a thick  $\otimes$ -ideal.*

*Proof.* Given a thick subcategory  $\mathcal{S} \subset \mathcal{K}$  one shows using standard techniques that

$$\{a \in \mathcal{K} \mid a \otimes x \in \mathcal{S} \text{ for every } x \in \mathcal{S}\}$$

is again a thick subcategory. It contains  $\mathbb{1}$  and hence is the whole of  $\mathcal{K}$  and we conclude that  $\mathcal{S}$  is a  $\otimes$ -ideal. □

*Remark 3.9.2.* Examples of tensor triangulated categories generated in this way by the unit include: the stable homotopy category of finite spectra  $\text{SH}^{\text{fin}}$ , derived categories of perfect complexes of affine schemes, and stable module categories  $\text{stmod}(kG)$  for  $G$  a  $p$ -group. For such categories a classification of thick subcategories is the same thing as a classification of thick  $\otimes$ -ideals.

*Remark 3.9.3.* If  $\mathcal{T}$  is a tensor triangulated category with the property that  $a \otimes -$  preserves coproducts then a similar statement holds for localizing subcategories: if  $\mathcal{T} = \text{loc}\langle \mathbb{1} \rangle$  then every localizing subcategory is automatically a localizing  $\otimes$ -ideal. The proof is the same.

In the remainder of this section we will briefly recall the classification theorems for  $D_{\text{perf}}(X)$  and  $\text{stmod}(kG)$  as these results set the stage (and provide motivation) for the topic of the next chapter: the theory of tensor triangular geometry. The similarity between the statements of the two theorems should be appreciated. A detailed description of the classification theorem for  $\text{SH}^{\text{fin}}$  will be deferred to Chapter 6.

### The Hopkins-Neeman-Thomason theorem

**Definition 3.9.4.** Let  $X$  be a scheme. For any complex of sheaves of  $\mathcal{O}_X$ -modules  $\mathcal{F}^\bullet$ , define its “cohomological support” to be

$$\text{supp}_X(\mathcal{F}^\bullet) := \{x \in X \mid \text{the stalk complex of } \mathcal{O}_{X,x}\text{-modules } \mathcal{F}_x^\bullet \text{ is not acyclic}\}.$$

**Lemma 3.9.5.** *Let  $\mathcal{F}^\bullet$  be a perfect complex on a quasi-compact, quasi-separated scheme  $X$ . Then for any  $x \in X$ ,  $\mathcal{F}_x^\bullet$  is an acyclic complex of  $\mathcal{O}_{X,x}$ -modules if and only if  $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^\mathbb{L} \kappa(x)$  is an acyclic complex of  $\kappa(x)$ -modules. Thus,  $\text{supp}_X(\mathcal{F}^\bullet) = \{x \in X \mid \mathcal{F}^\bullet \otimes_{\mathcal{O}_X}^\mathbb{L} \kappa(x) \neq 0 \text{ in } D(\kappa(x))\}$ .*

*Proof.* See [Tho97, Lemma 3.3]. □

**Theorem 3.9.6** (Tensor-nilpotence). *Let  $X$  be a quasi-compact and quasi-separated scheme and let  $f : \mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet$  be a morphism in  $D_{\text{qc}}(X)$  with  $\mathcal{E}^\bullet \in D_{\text{perf}}(X)$ . Suppose that for every  $x \in X$ ,  $f \otimes_{\mathcal{O}_X}^\mathbb{L} \kappa(x) = 0$  in  $D(\kappa(x))$ . Then there exists  $n \geq 1$  such that  $f^{\otimes n} = 0$  in  $D_{\text{qc}}(X)$ .*

*Proof.* See [Tho97, Theorem 3.6]. □

**Theorem 3.9.7** (Hopkins-Neeman-Thomason). *Let  $D_{\text{perf}}(X)$  denote the derived category of perfect complexes on a quasi-compact and quasi-separated scheme  $X$ . There is an inclusion-preserving bijection*

$$\{\text{thick } \otimes\text{-ideals of } D_{\text{perf}}(X)\} \longleftrightarrow \{\text{Thomason subsets of } X\}$$

which sends a thick  $\otimes$ -ideal  $\mathcal{J}$  to  $\text{supp}(\mathcal{J}) := \bigcup_{\mathcal{F}^\bullet \in \mathcal{J}} \text{supp}_X(\mathcal{F}^\bullet)$  and which sends a Thomason subset  $Y \subset X$  to the thick  $\otimes$ -ideal  $\{\mathcal{F}^\bullet \in \mathbf{D}_{\text{perf}}(X) \mid \text{supp}_X(\mathcal{F}^\bullet) \subset Y\}$ .

*Proof.* This is [Tho97, Theorem 3.5]. It is easy to check that the two maps are well-defined. The non-formal part of the proof boils down to the following two claims:

1. If  $Z \subset X$  is a closed subset which has quasi-compact complement then there exists a perfect complex  $\mathcal{E}^\bullet$  such that  $\text{supp}_X(\mathcal{E}^\bullet) = Z$ . This is proved in [Tho97, Lemma 3.4] by using a bit of algebraic geometry to reduce to the affine noetherian case and then using the Koszul complex on a set of generators  $Z = V(f_1, \dots, f_n)$  to produce the required perfect complex.
2. If  $\mathcal{E}^\bullet$  and  $\mathcal{F}^\bullet$  are perfect complexes on  $X$  such that  $\text{supp}_X(\mathcal{E}^\bullet) \subset \text{supp}_X(\mathcal{F}^\bullet)$  then  $\mathcal{E}^\bullet$  is contained in  $\text{thick}_\otimes(\mathcal{F}^\bullet)$ . This is the most difficult part and uses Theorem 3.9.6. It is proved in [Tho97, Lemma 3.14].

For a direct proof of the affine noetherian case see [Nee92a]. The relationship between Thomason's proof and the earlier work of Neeman and Hopkins is discussed in [Tho97, Remark 3.17]. □

*Remark 3.9.8.* The key result on which the classification theorem depends is Theorem 3.9.6 which is a direct analogue of the Nilpotence Theorem from stable homotopy theory (cf. Section 6.4). As we shall see, in that theory the analogues of the residue fields  $\kappa(x)$  are the Morava  $K$ -theories  $K(n)$ .

### **The Benson-Carlson-Rickard theorem**

Let  $G$  be a finite group and let  $k$  be a field of positive characteristic dividing the order of  $G$ . Recall that  $\text{Proj}(H^\bullet(G, k))$  is the space of homogeneous prime ideals of  $H^\bullet(G, k)$  equipped

with the Zariski topology and excluding the unique maximal ideal. A classical result of Evens and Venkov states that the group cohomology ring  $H^\bullet(G, k)$  is a finitely generated  $k$ -algebra and hence  $\text{Proj}(H^\bullet(G, k))$  is a noetherian topological space.

*Remark 3.9.9.* In any tensor triangulated category  $\mathcal{T}$ , the graded ring  $[\mathbb{1}, \mathbb{1}]_\bullet$  acts in a natural way on the graded abelian group  $[X, Y]_\bullet$  for any two objects  $X, Y \in \mathcal{T}$ . In particular, recalling Remark 3.8.37, we see that  $H^\bullet(G, k)$  acts on  $\text{Ext}_{kG}^\bullet(M, M)$  for any  $kG$ -module  $M$ .

**Definition 3.9.10.** For any finitely generated  $kG$ -module  $M$ , define its “support” to be

$$\text{supp}_G(M) := \text{Proj}(H^\bullet(G, k)/I_G(M)) \subset \text{Proj}(H^\bullet(G, k))$$

where  $I_G(M) \subset H^\bullet(G, k)$  denotes the annihilator of  $\text{Ext}_{kG}^\bullet(M, M)$  in  $H^\bullet(G, k)$ .

*Remark 3.9.11.* This notion of support satisfies a number of natural properties:

- (1)  $\text{supp}_G(M \oplus N) = \text{supp}_G(M) \cup \text{supp}_G(N)$ ;
- (2)  $\text{supp}_G(M \otimes_k N) = \text{supp}_G(M) \cap \text{supp}_G(N)$ ;
- (3) if  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is a short exact sequence of finitely generated  $kG$ -modules then  $\text{supp}_G(M_i) \subset \text{supp}_G(M_j) \cup \text{supp}_G(M_k)$  for any  $i, j, k \in \{1, 2, 3\}$ ;
- (4)  $\text{supp}_G(k) = \text{Proj}(H^\bullet(G, k))$ ;
- (5)  $\text{supp}_G(M) = \emptyset$  iff  $M$  is projective.

The last condition shows that  $\text{supp}_G(M)$  only depends on the stable isomorphism class of  $M$  and therefore  $\text{supp}_G$  can be regarded as a notion of support defined on  $\text{stmod}(kG)$ . Property (3) shows that if  $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \Omega^{-1}M_1$  is an exact triangle in  $\text{stmod}(kG)$  then  $\text{supp}_G(M_i) \subset \text{supp}_G(M_j) \cup \text{supp}_G(M_k)$  for any  $i, j, k \in \{1, 2, 3\}$ . It also follows that  $\text{supp}_G(M) = \text{supp}_G(\Omega^{-1}M)$ .

*Remark 3.9.12.* The above notion of support for finitely generated  $kG$ -modules is very classical but the proof of the Benson-Carlson-Rickard theorem rests on a well-behaved theory of support for *arbitrary*  $kG$ -modules. Another ingredient in the proof is an elegant application of Bousfield localization in terms of so-called “Rickard idempotents” (introduced in [Ric97]). Conveniently, the notion of support for arbitrary modules can be defined in terms of these Rickard idempotents.

**Definition 3.9.13** (Rickard). Let  $\mathcal{C} \subset \text{stmod}(kG)$  be a thick subcategory. Then we can consider finite localization in  $\text{StMod}(kG)$  with respect to the localizing subcategory  $\text{loc}\langle\mathcal{C}\rangle \subset \text{StMod}(kG)$ . For any  $X$  in  $\text{StMod}(kG)$  we have an exact triangle

$$\Gamma_{\mathcal{C}}X \rightarrow X \rightarrow L_{\mathcal{C}}X \rightarrow \Omega^{-1}\Gamma_{\mathcal{C}}X$$

where  $\Gamma_{\mathcal{C}}X \in \text{loc}\langle\mathcal{C}\rangle$  and  $L_{\mathcal{C}}X \in (\text{loc}\langle\mathcal{C}\rangle)^{\perp}$ . Moreover, if  $\mathcal{C}$  is a thick  $\otimes$ -ideal then  $\text{loc}\langle\mathcal{C}\rangle$  is a localizing  $\otimes$ -ideal (use the proof of Remark 3.9.3) and hence (cf. Lemma 3.7.22 and Proposition 3.7.20) this finite localization is smashing:  $\Gamma_{\mathcal{C}}X \simeq \Gamma_{\mathcal{C}}k \otimes_k X$  and  $L_{\mathcal{C}}X \simeq L_{\mathcal{C}}k \otimes_k X$ . The two  $kG$ -modules  $\Gamma_{\mathcal{C}}k$  and  $L_{\mathcal{C}}k$  are called the “Rickard idempotents” associated to  $\mathcal{C}$ . They are idempotent monoids in  $\text{StMod}(kG)$ . In particular, for a fixed  $\mathfrak{p} \in \text{Proj}(H^*(G, k))$ , let  $\Gamma_{\mathfrak{p}}$  and  $L_{\mathfrak{p}}$  denote the Rickard idempotents for the thick  $\otimes$ -ideal of all  $X \in \text{stmod}(kG)$  such that  $\text{supp}_G(X) \subset \overline{\{\mathfrak{p}\}}$ . Similarly, let  $\tilde{\Gamma}_{\mathfrak{p}}$  and  $\tilde{L}_{\mathfrak{p}}$  denote the Rickard idempotents for the thick  $\otimes$ -ideal consisting of those  $X \in \text{stmod}(kG)$  such that  $\text{supp}_G(X) \subsetneq \overline{\{\mathfrak{p}\}}$ . Finally, define  $\kappa(\mathfrak{p}) := \Gamma_{\mathfrak{p}} \otimes_k \tilde{L}_{\mathfrak{p}}$ .

**Definition 3.9.14.** For any  $kG$ -module  $M$ , define its “support” to be

$$\text{supp}_G(M) := \{\mathfrak{p} \in \text{Proj}(H^*(G, k)) \mid M \otimes_k \kappa_{\mathfrak{p}} \neq 0 \text{ is not projective}\}.$$

If  $M$  is finitely generated then this coincides with the notion of support defined in Definition 3.9.10. However, for arbitrary modules the new definition has better properties. For example,  $\text{supp}_G(M \otimes_k N) = \text{supp}_G(M) \cap \text{supp}_G(N)$  and  $\text{supp}_G(M) = \emptyset$  iff  $M$  is projective.

**Theorem 3.9.15** (Benson-Carlson-Rickard). *Let  $G$  be a finite group and let  $k$  be a field of characteristic  $p$  dividing the order of  $G$ . There is an inclusion-preserving bijection*

$$\{\text{thick } \otimes\text{-ideals of } \text{stmod}(kG)\} \longleftrightarrow \{\text{specialization closed subsets of } \text{Proj}(H^\bullet(G, k))\}$$

*which sends a thick  $\otimes$ -ideal  $\mathcal{I}$  to  $\text{supp}(\mathcal{I}) := \bigcup_{M \in \mathcal{I}} \text{supp}_G(M)$  and which sends a specialization closed subset  $Y \subset \text{Proj}(H^\bullet(G, k))$  to the thick  $\otimes$ -ideal  $\{M \in \text{stmod}(kG) \mid \text{supp}_G(M) \subset Y\}$ .*

*Proof.* As with Theorem 3.9.7, the non-formal part of the proof boils down to the following two claims:

- (1) If  $Z$  is a closed subset of  $\text{Proj}(H^\bullet(G, k))$  then there exists a finitely generated module  $M$  with  $\text{supp}_G(M) = Z$ .
- (2) If  $\mathcal{C}$  is a thick  $\otimes$ -ideal and  $X$  is a finitely generated  $kG$ -module such that  $\text{supp}_G(X) \subset \text{supp}_G(Y)$  for some  $Y \in \mathcal{C}$  then  $X \in \mathcal{C}$ .

The first claim is relatively straightforward. If  $Z = V(\xi_1, \dots, \xi_n)$  then

$$Z = \text{supp}_G(L_{\xi_1} \otimes_k \cdots \otimes_k L_{\xi_n})$$

where the  $L_{\xi_i}$  are the associated “Carlson modules.” They are analogues of the Koszul complexes in the derived category. For the second claim we use finite localization together with our notion of support for infinitely generated modules. Indeed, let  $\Gamma_{\mathcal{C}}k$  and  $L_{\mathcal{C}}k$  denote the Rickard idempotents associated to the thick  $\otimes$ -ideal  $\mathcal{C}$ . Then  $\text{supp}_G(X \otimes_k L_{\mathcal{C}}k) = \text{supp}_G(X) \cap \text{supp}_G(L_{\mathcal{C}}k) \subset \text{supp}_G(Y) \cap \text{supp}_G(L_{\mathcal{C}}k) = \text{supp}_G(Y \otimes_k L_{\mathcal{C}}k) = \emptyset$  since  $Y \otimes_k L_{\mathcal{C}}k = 0$  in  $\text{StMod}(kG)$ . Thus  $X \otimes_k L_{\mathcal{C}}k = 0$  in  $\text{StMod}(kG)$  and hence  $X$  is contained in  $\mathcal{C}$ .  $\square$

*Remark 3.9.16.* The original papers on support for infinitely generated modules [BCR95, BCR96] and the proof of the classification theorem [BCR97] assume that  $k$  is algebraically closed and use the language of maximal ideal spectra. In later work, Friedlander and

Pevtsova [FP07] generalized the classification theorem from finite groups to finite groups schemes and their paper gives a clear version of the proof using more modern scheme-theoretic language and without unnecessary assumptions on the field  $k$ .

## CHAPTER 4

### Tensor triangular geometry

Throughout this chapter let  $\mathcal{K}$  denote an essentially small tensor triangulated category. As we saw in Section 3.4, such categories often arise as the compact-dualizable objects in a larger rigidly-compactly generated tensor triangulated category. In this chapter, we will give a bare-bones account of the spectrum of  $\mathcal{K}$ , as introduced by Paul Balmer. This construction allows for an algebro-geometric approach to the study of tensor triangulated categories which is sometimes referred to as “tensor triangular geometry.” The basic reference is [Bal05] but the theory has been developed in a series of papers. The survey [Bal10b] provides a good overview and for our purposes the papers [Bal07, Bal10a] are particularly relevant.

#### 4.1 Basic definitions

**Definition 4.1.1.** A thick  $\otimes$ -ideal  $\mathcal{J}$  of  $\mathcal{K}$  is a thick triangulated subcategory  $\mathcal{J} \subset \mathcal{K}$  with the property that  $a \in \mathcal{K}$  and  $x \in \mathcal{J}$  implies that  $a \otimes x \in \mathcal{J}$ . A prime ideal of  $\mathcal{K}$  is a thick  $\otimes$ -ideal  $\mathcal{P}$  with the property that  $a \otimes b \in \mathcal{P}$  implies that either  $a \in \mathcal{P}$  or  $b \in \mathcal{P}$ . The *spectrum*  $\mathrm{Spc}(\mathcal{K})$  of  $\mathcal{K}$  is defined to be the set of prime ideals of  $\mathcal{K}$ . It is a set since  $\mathcal{K}$  is essentially small and all subcategories are replete. For each family of objects  $\mathcal{E} \subset \mathcal{K}$ , define

$$Z(\mathcal{E}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{E} \cap \mathcal{P} = \emptyset\}.$$

The collection  $\{Z(\mathcal{E}) \subset \mathrm{Spc}(\mathcal{K}) \mid \mathcal{E} \subset \mathcal{K}\}$  defines the closed sets for a topology on  $\mathrm{Spc}(\mathcal{K})$  called the *Balmer topology*. With this topology  $\mathrm{Spc}(\mathcal{K})$  becomes a spectral space in the sense of Hochster (cf. Section 2.4).

*Remark 4.1.2.* The Balmer topology on  $\mathrm{Spc}(\mathcal{K})$  is *not* the “Zariski” topology one would obtain by mimicing the definition of the Zariski topology on the prime spectrum of a commutative ring. The closed sets in  $\mathrm{Spc}(\mathcal{K})$  are of the form  $Z(\mathcal{E}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{E} \cap \mathcal{P} = \emptyset\}$  whereas the closed sets of the Zariski topology would be of the form  $V(\mathcal{E}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \supset \mathcal{E}\}$ . There is a precise sense in which the Balmer and Zariski topologies on  $\mathrm{Spc}(\mathcal{K})$  are related: both give spectral topologies in the sense of Hochster and the two topologies are Hochster-dual (cf. Remark 2.4.4). The fact that  $\mathrm{Spc}(\mathcal{K})$  has the topology “dual” to the usual Zariski topology familiar to algebraic geometers means that some things in tensor triangular geometry behave a bit differently than one might expect. For example, closure in the Balmer topology goes *down* rather than up:  $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{Q} \subset \mathcal{P}\}$ . In particular, the closed points in  $\mathrm{Spc}(\mathcal{K})$  are the *minimal* primes. Another consequence of the differences between the Balmer and Zariski topologies is that our comparison maps will be inclusion-reversing.

*Remark 4.1.3.* Although the spectrum may be equipped with the structure of a locally ringed space, for our purposes only its topological structure is relevant. This is the reason we use the notation  $\mathrm{Spc}(\mathcal{K})$  rather than  $\mathrm{Spec}(\mathcal{K})$ : the former denotes the spectrum regarded only as a topological space while the latter denotes the spectrum regarded as a locally ringed space.

**Definition 4.1.4.** Let  $F : \mathcal{K} \rightarrow \mathcal{L}$  be a tensor triangulated functor. Define a map

$$\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \rightarrow \mathrm{Spc}(\mathcal{K})$$

by  $\mathrm{Spc}(F)(\mathcal{Q}) := F^{-1}(\mathcal{Q})$ . This is a well-defined spectral map. In this way,  $\mathrm{Spc}(-)$  can be regarded as a contravariant functor from the category of essentially small tensor triangulated categories to the category of spectral spaces.

**Proposition 4.1.5.** *Let  $\mathcal{J} \subset \mathcal{K}$  be a thick  $\otimes$ -ideal and let  $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{J}$  denote the Verdier quotient functor. The map  $\mathrm{Spc}(q) : \mathrm{Spc}(\mathcal{K}/\mathcal{J}) \rightarrow \mathrm{Spc}(\mathcal{K})$  induces a homeomorphism*

$$\mathrm{Spc}(\mathcal{K}/\mathcal{J}) \xrightarrow{\sim} V(\mathcal{J}) \subset \mathrm{Spc}(\mathcal{K})$$

where  $V(\mathcal{J}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \supset \mathcal{J}\}$ .

*Proof.* See [Bal05, Proposition 3.11]. Note that there are no issues of existence with the Verdier quotient  $\mathcal{K}/\mathcal{J}$  since  $\mathcal{K}$  is essentially small (cf. Remark 3.6.7).  $\square$

**Definition 4.1.6.** The *support* of an object  $a \in \mathcal{K}$  is defined to be

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\}.$$

This notion of support satisfies the following properties:

- (1)  $\mathrm{supp}(a)$  is a closed subset of  $\mathrm{Spc}(\mathcal{K})$ .
- (2)  $\mathrm{supp}(0) = \emptyset$  and  $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$ .
- (3)  $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$ .
- (4)  $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$ .
- (5)  $\mathrm{supp}(c) \subset \mathrm{supp}(a) \cup \mathrm{supp}(b)$  whenever there is an exact triangle  $a \rightarrow b \rightarrow c \rightarrow \Sigma a$ .
- (6)  $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$ .

*Remark 4.1.7.* Although the definition of  $\mathrm{Spc}(\mathcal{K})$  as a set of “prime ideals” is aesthetically pleasing to students of algebraic geometry, there is no particular reason why such a definition is at all interesting. A more conceptual definition of  $\mathrm{Spc}(\mathcal{K})$ —which has nothing to do with prime ideals or commutative algebra—is that it is the target of the universal notion of “support” for objects in  $\mathcal{K}$ . The pair  $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$  is the universal notion of support

satisfying properties 1–6 listed above. For a precise statement see [Bal05, Theorem 3.2]. One could thus define  $\mathrm{Spc}(\mathcal{K})$  in terms of a universal property and then use Definition 4.1.1 as a particular construction of this universal object in terms of prime ideals. This point of view is historically accurate as the definition of  $\mathrm{Spc}(\mathcal{K})$  was motivated by various notions of “support” arising in subjects like algebraic geometry and modular representation theory (cf. Definition 3.9.4 and Definition 3.9.10).

*Remark 4.1.8.* An object  $a \in \mathcal{K}$  is  $\otimes$ -nilpotent (i.e.  $a^{\otimes n} = 0$  for some  $n \geq 1$ ) if and only if  $\mathrm{supp}(a) = \emptyset$ . This is proved in [Bal05, Corollary 2.4].

*Remark 4.1.9.* The closed sets  $\mathrm{supp}(a)$  form a basis of closed sets. Moreover, they are precisely the closed sets which have quasi-compact complement:

**Lemma 4.1.10.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $\mathcal{Z}$  be a closed subset of  $\mathrm{Spc}(\mathcal{K})$ . The following are equivalent:*

- (1)  $\mathcal{Z}$  is Thomason (cf. Definition 2.4.3);
- (2)  $\mathcal{Z}$  has quasi-compact complement;
- (3)  $\mathcal{Z} = \mathrm{supp}(a)$  for some  $a \in \mathcal{K}$ .

*Proof.* We will sketch the proof of (1) implies (2) since (2) clearly implies (1) and [Bal05, Proposition 2.14] gives the equivalence of (2) and (3). For any closed subset  $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$ ,  $\mathcal{K}_{\mathcal{Z}} := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset \mathcal{Z}\}$  is a thick  $\otimes$ -ideal and it is easily checked from the definitions that  $\mathrm{Spc}(\mathcal{K}) \setminus \mathcal{Z} \subset \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \supset \mathcal{K}_{\mathcal{Z}}\} =: V(\mathcal{K}_{\mathcal{Z}})$ . On the other hand, if  $\mathcal{Z}$  is Thomason then one readily checks that the reverse inclusion holds using the equivalence of (2) and (3). Thus, if  $\mathcal{Z}$  is Thomason and closed then  $\mathrm{Spc}(\mathcal{K}) \setminus \mathcal{Z} = V(\mathcal{K}_{\mathcal{Z}}) \simeq \mathrm{Spc}(\mathcal{K}/\mathcal{K}_{\mathcal{Z}})$  and the spectrum of any tensor triangulated category is quasi-compact (by [Bal05, Corollary 2.15]).  $\square$

## 4.2 The classification theorem

As we remarked in the introduction, determining  $\mathrm{Spc}(\mathcal{K})$  is a highly non-trivial problem which essentially amounts to classifying the objects of the category up to the naturally available tensor triangulated structure. The precise statement comes from Theorem 4.2.3 below.

**Definition 4.2.1.** The *radical*  $\sqrt[n]{\mathcal{I}}$  of a thick  $\otimes$ -ideal  $\mathcal{I} \subset \mathcal{K}$  is defined in the usual way

$$\sqrt[n]{\mathcal{I}} := \{a \in \mathcal{K} \mid a^{\otimes n} \in \mathcal{I} \text{ for some } n \geq 1\}$$

and  $\mathcal{I}$  is said to be *radical*  $\otimes$ -ideal if  $\mathcal{I} = \sqrt[n]{\mathcal{I}}$ .

*Remark 4.2.2.* If  $\mathcal{K}$  is rigid—that is, if every object is dualizable—then *every* thick  $\otimes$ -ideal is automatically radical. See [Bal05, Remark 4.3 and Proposition 4.4]. This should be kept in mind when one reads the following theorem:

**Theorem 4.2.3.** *For any collection of objects  $\mathcal{E} \subset \mathcal{K}$ , define  $\mathrm{supp}(\mathcal{E}) := \bigcup_{a \in \mathcal{E}} \mathrm{supp}(a)$  and for each subset  $Y \subset \mathrm{Spc}(\mathcal{K})$ , define  $\mathcal{K}_Y := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset Y\}$ . These definitions induce order-preserving bijections*

$$\{\text{radical thick } \otimes\text{-ideals of } \mathcal{K}\} \longleftrightarrow \{\text{Thomason subsets of } \mathrm{Spc}(\mathcal{K})\}.$$

*Proof.* See [Bal05, Theorem 4.10]. □

*Remark 4.2.4.* For example, if  $\mathcal{K}$  is a topologically noetherian rigid tensor triangulated category then we have a bijection between the thick  $\otimes$ -ideals of  $\mathcal{K}$  and the specialization-closed subsets of  $\mathrm{Spc}(\mathcal{K})$ . Moreover, if  $\mathcal{K} = \mathrm{thick}\langle \mathbb{1} \rangle$  is generated by the unit then every thick subcategory is automatically a  $\otimes$ -ideal (cf. Lemma 3.9.1) so this actually provides a classification of the thick subcategories of  $\mathcal{K}$ . The reader is urged to compare Theorem 4.2.3 with Theorem 3.9.7 and Theorem 3.9.15 from Section 3.9.

One of the pieces of the proof of Theorem 4.2.3 is the following lemma:

**Lemma 4.2.5.** *If  $\mathcal{J} \subset \mathcal{K}$  is a thick  $\otimes$ -ideal then  $\mathcal{K}_{\text{supp}(\mathcal{J})} = \sqrt[\otimes]{\mathcal{J}}$ .*

*Proof.* See [Bal05, Proposition 4.9]. □

We mention it because we need the following result later:

**Lemma 4.2.6.** *If  $\text{supp}(a) \subset \text{supp}(b)$  then  $a^{\otimes n} \in \text{thick}_{\otimes}\langle b \rangle$  for some  $n \geq 1$ .*

*Proof.* Just observe that if  $\text{supp}(a) \subset \text{supp}(b) \subset \text{supp}(\text{thick}_{\otimes}\langle b \rangle)$  then Lemma 4.2.5 implies that  $a$  is contained in  $\mathcal{K}_{\text{supp}(\text{thick}_{\otimes}\langle b \rangle)} = \sqrt[\otimes]{\text{thick}_{\otimes}\langle b \rangle}$ . □

### 4.3 Local categories

The notion of a *local* tensor triangulated category was introduced in [Bal10a, Section 4]:

**Definition 4.3.1.** For a tensor triangulated category  $\mathcal{K}$  the following conditions are equivalent:

1. The space  $\text{Spc}(\mathcal{K})$  is a local topological space; that is, every open cover  $\text{Spc}(\mathcal{K}) = \bigcup_{i \in I} U_i$  is trivial, in that there exists  $i \in I$  such that  $U_i = \text{Spc}(\mathcal{K})$ .
2. The space  $\text{Spc}(\mathcal{K})$  has a unique closed point.
3. The category  $\mathcal{K}$  has a unique minimal prime.
4. The ideal  $\sqrt[\otimes]{0} \subset \mathcal{K}$  of  $\otimes$ -nilpotent objects is prime (and is the minimal one).
5. For any objects  $a, b \in \mathcal{K}$ , if  $a \otimes b = 0$  then  $a$  or  $b$  is  $\otimes$ -nilpotent.

If these conditions hold we say that  $\mathcal{K}$  is a *local* tensor triangulated category.

*Example 4.3.2.* If  $\mathcal{P} \in \mathrm{Spc}(K)$  is a prime ideal of  $K$  then the category  $K/\mathcal{P}$  is local. Indeed, the ideal  $(0)$  is prime in  $K/\mathcal{P}$ .

*Remark 4.3.3.* Recall from Remark 4.2.2 that in a *rigid* tensor triangulated category every thick  $\otimes$ -ideal is automatically radical:  $a^{\otimes n} \in \mathcal{I} \Rightarrow a \in \mathcal{I}$ . It follows that a rigid tensor triangulated category is local iff  $(0)$  is a prime.

## 4.4 Idempotent completion

Recall the notion of idempotent completion from Section 2.1.

**Theorem 4.4.1** (Balmer-Schlichting). *Let  $\mathcal{T}$  be a triangulated category. The idempotent completion  $\mathcal{T}^\sharp$  admits a unique triangulated category structure such the canonical functor  $i : \mathcal{T} \rightarrow \mathcal{T}^\sharp$  is exact.*

*Proof.* See [BS01]. □

*Remark 4.4.2.* If  $\mathcal{T}$  is a *tensor* triangulated category then  $\mathcal{T}^\sharp$  is also a tensor triangulated category with  $(A, e) \otimes (B, f) := (A \otimes B, e \otimes f)$  and the canonical functor  $\mathcal{A} \rightarrow \mathcal{A}^\sharp$  is a tensor triangulated functor.

**Proposition 4.4.3.** *Let  $K$  be an essentially small tensor triangulated category. The canonical functor  $i : K \rightarrow K^\sharp$  induces a homeomorphism  $\mathrm{Spc}(i) : \mathrm{Spc}(K^\sharp) \xrightarrow{\sim} \mathrm{Spc}(K)$ .*

*Proof.* See [Bal05, Corollary 3.14]. □

## 4.5 Examples

The classification theorems from Section 3.9 can be used in alliance with Theorem 4.2.3 to compute the spectrum in several interesting examples.

**Theorem 4.5.1** (Thomason; Balmer). *Let  $X$  be a quasi-compact, quasi-separated scheme and let  $D_{\text{perf}}(X)$  denote the tensor triangulated category of perfect complexes on  $X$ . There is a homeomorphism*

$$X \xrightarrow{\sim} \text{Spc}(D_{\text{perf}}(X))$$

*which sends  $x \in X$  to  $\{\mathcal{F}^\bullet \in D_{\text{perf}}(X) \mid x \notin \text{supp}_X(\mathcal{F}^\bullet)\}$  where  $\text{supp}_X(-)$  denotes the cohomological support of Definition 3.9.4. Under this homeomorphism the cohomological support in  $X$  of a perfect complex  $\mathcal{F}^\bullet$  coincides with its abstract support in  $\text{Spc}(D_{\text{perf}}(X))$ .*

*Proof.* This was proved for a noetherian scheme  $X$  in [Bal05, Corollary 5.6 and Theorem 6.3] but it was noted in [BKS07] that the proof really works for quasi-compact, quasi-separated schemes (as mentioned, for example, in [Bal10b, Theorem 54]).  $\square$

*Remark 4.5.2.* The homeomorphism is actually an isomorphism of locally ringed spaces when the spectrum is equipped with its locally ringed structure (cf. Remark 4.1.3). Thus, a quasi-compact, quasi-separated scheme can be recovered from its derived category of perfect complexes. This should be contrasted with the fact that such a scheme *cannot* be recovered from its derived category of perfect complexes if we only regard  $D_{\text{perf}}(X)$  as a triangulated category (without the  $\otimes$ -structure); see [Bal10a, Remark 64], for example.

*Remark 4.5.3.* The assumption that the scheme  $X$  is quasi-compact and quasi-separated is necessary for this reconstruction theorem to hold because the spectrum of *any* tensor triangulated category is quasi-compact and quasi-separated.

**Theorem 4.5.4** (Benson-Carlson-Rickard; Balmer). *Let  $G$  be a finite group and let  $k$  be a field. There is a homeomorphism*

$$\text{Proj}(H^\bullet(G, k)) \xrightarrow{\sim} \text{Spc}(\text{stmod}(kG))$$

which sends a point  $\mathfrak{p} \in \text{Proj}(H^\bullet(G, k))$  to  $\{M \in \text{stmod}(kG) \mid \mathfrak{p} \notin \text{supp}_G(M)\}$  where  $\text{supp}_G(-)$  denotes the notion of support from Definition 3.9.10. Under this homeomorphism the support in  $\text{Proj}(H^\bullet(G, k))$  of a  $kG$ -module  $M$  coincides with its abstract support in  $\text{Spc}(\text{stmod}(kG))$ .

*Proof.* See [Bal05, Corollary 5.10 and Theorem 6.3]. □

*Remark 4.5.5.* As with Remark 4.5.2, this homeomorphism is an isomorphism of locally ringed spaces.

## 4.6 The comparison map

The examples  $\text{Spc}(\text{D}_{\text{perf}}(X)) \cong X$  and  $\text{Spc}(\text{stmod}(kG)) \cong \text{Proj}(H^\bullet(G, k))$  from the last section indicate that the spectrum is quite an interesting construction. However, determining the spectrum in those two examples depends on the deep classification theorems discussed in Section 3.9. We would like to go the other way around: develop general techniques for computing the spectrum  $\text{Spc}(\mathcal{K})$  and then obtain a classification of the thick  $\otimes$ -ideals in  $\mathcal{K}$  by invoking Theorem 4.2.3. Balmer took the first step in this direction in [Bal10a] by defining two maps

$$\rho : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}([\mathbb{1}, \mathbb{1}]) \quad \text{and} \quad \rho^\bullet : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}^h([\mathbb{1}, \mathbb{1}]_\bullet)$$

from the tensor triangular spectrum to the Zariski spectrum of the (graded) ring of (graded) endomorphisms of the unit object. Recall that these rings are (graded) commutative by Lemma 2.2.7 and Lemma 3.3.20. The first map is defined by  $\mathcal{P} \mapsto \{f \in [\mathbb{1}, \mathbb{1}] \mid \text{cone}(f) \notin \mathcal{P}\}$  while the graded map sends  $\mathcal{P} \mapsto \{f \in [\mathbb{1}, \mathbb{1}]_i \mid \text{cone}(f) \notin \mathcal{P}\}_{i \in \mathbb{Z}}$ . Balmer establishes two useful surjectivity criteria for these maps:

**Theorem 4.6.1** (Balmer). *If the category  $\mathcal{K}$  is “connective” in the sense that  $\pi_i(\mathbb{1}) = 0$  for  $i < 0$  then the map  $\rho : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}([\mathbb{1}, \mathbb{1}])$  is surjective. If the graded ring  $[\mathbb{1}, \mathbb{1}]_\bullet$  is graded-*

coherent (e.g. graded-noetherian) then both comparison maps  $\rho : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}([\mathbb{1}, \mathbb{1}])$  and  $\rho^\bullet : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}([\mathbb{1}, \mathbb{1}]_\bullet)$  are surjective.

*Proof.* See [Bal10a, Section 7]. □

*Example 4.6.2.* Consider  $\mathcal{K} = D^b(\mathrm{mod}(kG))$  for  $G$  a finite group and  $k$  a field. Recall from Remark 3.8.37 that in this example  $[\mathbb{1}, \mathbb{1}]_\bullet = H^{-\bullet}(G, k)$  is the group cohomology ring. This is known to be graded-noetherian by the classical result of Evens and Venkov. Thus the graded comparison map  $\rho^\bullet : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}^h(H^{-\bullet}(G, k)) = \mathrm{Spec}^h(H^\bullet(G, k))$  is surjective. In fact, Balmer [Bal10a, Proposition 8.5] showed that the map  $\rho^\bullet$  is actually a homeomorphism by using Theorem 4.5.4 and Rickard’s theorem (cf. Theorem 3.8.35) relating  $\mathrm{stmod}(kG)$  and  $D^b(\mathrm{mod}(kG))$ . Unfortunately, this result depends on the Benson-Carlson-Rickard theorem via Theorem 4.5.4. An alternative proof of the injectivity of  $\rho^\bullet$  in this example would provide a new proof of the Benson-Carlson-Rickard theorem.

*Example 4.6.3.* Let  $R$  be a commutative ring and consider  $\mathcal{K} = D_{\mathrm{perf}}(R) = K^b(\mathrm{proj}(R))$ . In this example,  $[\mathbb{1}, \mathbb{1}]_\bullet = [\mathbb{1}, \mathbb{1}] = R$  and so the map  $\rho : \mathrm{Spc}(K^b(\mathrm{proj}(R))) \rightarrow \mathrm{Spec}(R)$  is surjective by the “connective” surjectivity criterion. Using Theorem 4.5.1, one can show that it is a homeomorphism, as indicated, for example, in [Bal10a, Proposition 8.1]. In fact, Balmer (unpublished) has proved the injectivity of  $\rho$  in this example without using Theorem 4.5.1. This provides a new proof of the affine case of the Hopkins-Neeman-Thomason theorem.

*Remark 4.6.4.* These are the maps which provide the starting point for the theory of “higher” comparison maps introduced in the next chapter.

## CHAPTER 5

### Higher comparison maps

#### 5.1 Basic constructions

It is now time to introduce the new comparison maps. As mentioned in the introduction, there are actually several different constructions, but they are closely related and the fundamental ideas are exposed in the simplest example. In all cases, there are graded and ungraded versions. The proofs for the graded constructions are essentially the same as for the ungraded ones, but the ideas are more transparent in the ungraded setting. The notion of a “tensor-balanced” endomorphism will play a central role in these constructions.

**Definition 5.1.1.** An endomorphism  $f : X \rightarrow X$  in a tensor triangulated category is said to be  $\otimes$ -balanced if  $f \otimes X = X \otimes f$  as an endomorphism of  $X \otimes X$ .

*Remark 5.1.2.* The following lemma was established in [Bal10a, Proposition 2.13] in the case when  $f : \mathbb{1} \rightarrow \mathbb{1}$  is an arbitrary endomorphism of the unit and was a crucial technical result used in the construction of the original unit comparison maps. The key to generalizing the result to endomorphisms of an arbitrary object  $X$  is to restrict ourselves to  $\otimes$ -balanced endomorphisms.

**Lemma 5.1.3.** *If  $f : X \rightarrow X$  is a  $\otimes$ -balanced endomorphism then  $f^{\otimes 2} \otimes \text{cone}(f) = 0$ .*

*Proof.* Recall from Lemma 3.1.7 that a map in an exact triangle is a weak kernel of the map following it in the exact triangle and is also a weak cokernel of the map preceding it in the

exact triangle. In particular, the composite of two consecutive maps in an exact triangle is zero. With these facts in mind, start with an exact triangle  $X \xrightarrow{f} X \xrightarrow{g} \text{cone}(f) \xrightarrow{h} \Sigma X$  and observe that in the following morphism of exact triangles

$$\begin{array}{ccccccc}
 X \otimes X & \xrightarrow{X \otimes f} & X \otimes X & \xrightarrow{X \otimes g} & X \otimes \text{cone}(f) & \xrightarrow{X \otimes h} & \Sigma(X \otimes X) \\
 f \otimes X \downarrow & & f \otimes X \downarrow & \searrow 0 & \downarrow f \otimes \text{cone}(f) & \searrow 0 & \downarrow \Sigma(f \otimes X) \\
 X \otimes X & \xrightarrow{X \otimes f} & X \otimes X & \xrightarrow{X \otimes g} & X \otimes \text{cone}(f) & \xrightarrow{X \otimes h} & \Sigma(X \otimes X)
 \end{array}$$

the middle diagonal is zero because  $(X \otimes g) \circ (f \otimes X) = (X \otimes g) \circ (X \otimes f) = X \otimes (g \circ f) = 0$ . It follows that  $f \otimes \text{cone}(f)$  factors through the  $X \otimes h$  of the top row since  $X \otimes h$  is a weak cokernel of  $X \otimes g$ . One may similarly observe that the right diagonal is zero using the fact that  $\Sigma f \circ h = 0$ . It follows that  $f \otimes \text{cone}(f)$  factors through the  $X \otimes g$  of the second row since  $X \otimes g$  is a weak kernel of  $X \otimes h$ . This implies that  $(f \otimes \text{cone}(f))^2 = 0$  since we have a factorization

$$\begin{array}{ccccc}
 & & X \otimes \text{cone}(f) & & \\
 & \swarrow & \downarrow f \otimes \text{cone}(f) & & \\
 X \otimes X & \xrightarrow{X \otimes g} & X \otimes \text{cone}(f) & \xrightarrow{X \otimes h} & \Sigma(X \otimes X) \\
 & & \downarrow f \otimes \text{cone}(f) & & \swarrow \\
 & & X \otimes \text{cone}(f) & & 
 \end{array}$$

through  $(X \otimes h) \circ (X \otimes g) = 0$ . Finally, by observing that  $f^{\otimes 2} = (X \otimes f) \circ (f \otimes X) = X \otimes f^2$  one concludes that  $f^{\otimes 2} \otimes \text{cone}(f) = X \otimes f^2 \otimes \text{cone}(f) = X \otimes (f \otimes \text{cone}(f))^2 = 0$ .  $\square$

**Notation 5.1.4.** Let  $E_X := \{f \in [X, X] \mid f \otimes X = X \otimes f\}$  denote the collection of  $\otimes$ -balanced endomorphisms of  $X$ .

**Proposition 5.1.5.** *For each object  $X$  in a tensor triangulated category  $\mathcal{K}$ ,  $E_X$  is an inverse-closed subring of the endomorphism ring  $[X, X]$ . If  $(0)$  is a prime in  $\mathcal{K}$ , for example if  $\mathcal{K}$  is rigid and local, then  $E_X$  is a local ring provided that  $X \neq 0$ .*

*Proof.* That  $E_X$  is an additive subgroup of  $[X, X]$  follows from the fact that  $X \otimes -$  and  $- \otimes X$  are additive functors:  $(f + g) \otimes X = (f \otimes X) + (g \otimes X) = (X \otimes f) + (X \otimes g) = X \otimes (f + g)$  and  $0_X \otimes X =$

$0_{X \otimes X} = X \otimes 0_X$ . It clearly contains the multiplicative identity ( $\text{id}_X \otimes X = \text{id}_{X \otimes X} = X \otimes \text{id}_X$ ) and

$$(f \circ g) \otimes X = (f \otimes X) \circ (g \otimes X) = (X \otimes f) \circ (X \otimes g) = X \otimes (f \circ g)$$

shows that it is closed under multiplication. Moreover, if  $g \in [X, X]$  is an inverse for  $f \in E_X$  in the full endomorphism ring  $[X, X]$  then

$$\begin{aligned} g \otimes X &= (g \otimes X) \circ \text{id}_{X \otimes X} = (g \otimes X) \circ (X \otimes f) \circ (X \otimes g) \\ &= (g \otimes X) \circ (f \otimes X) \circ (X \otimes g) = \text{id}_{X \otimes X} \circ (X \otimes g) = X \otimes g \end{aligned}$$

shows that  $g$  is also contained in  $E_X$  (so that  $f$  is a unit in  $E_X$ ). On the other hand, suppose that the zero ideal  $(0)$  is a prime in  $\mathcal{K}$  and that  $X \neq 0$ . To prove that the non-zero ring  $E_X$  is local it suffices (by Proposition 2.3.14) to show that the sum of two non-units is again a non-unit. To this end, let  $f_1, f_2 \in E_X$  and suppose that  $f_1 + f_2$  is a unit. By Lemma 5.1.3,  $f_1^{\otimes 2} \otimes \text{cone}(f_1) = 0$  and  $f_2^{\otimes 2} \otimes \text{cone}(f_2) = 0$ . It follows that  $(f_1 + f_2)^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$  for  $n \geq 3$  by expanding  $(f_1 + f_2)^{\otimes n}$  using bilinearity of the  $\otimes$ -product and applying the symmetry. In more detail,  $(f_1 + f_2)^{\otimes n}$  expands to a sum of  $2^n$  endomorphisms of  $X^{\otimes n}$  each of which is of the form  $g_1 \otimes g_2 \otimes \cdots \otimes g_n$  with each  $g_i$  either  $f_1$  or  $f_2$ . By applying a permutation of the factors of  $X^{\otimes n}$  (using the symmetry) we can ensure that all of the  $f_1$ 's are on the left. That is, there exists an isomorphism  $\sigma : X^{\otimes n} \xrightarrow{\sim} X^{\otimes n}$  such that

$$\begin{array}{ccc} X^{\otimes n} & \xrightarrow{g_1 \otimes \cdots \otimes g_n} & X^{\otimes n} \\ \sigma \downarrow \sim & & \sim \downarrow \sigma \\ X^{\otimes n} & \xrightarrow{f_1^{\otimes i} \otimes f_2^{\otimes n-i}} & X^{\otimes n} \end{array}$$

commutes for some  $0 \leq i \leq n$ . Using Notation 3.1.43, we have

$$\begin{aligned} g_1 \otimes \cdots \otimes g_n \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) &\simeq f_1^{\otimes i} \otimes f_2^{\otimes n-i} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) \\ &\simeq f_1^{\otimes i} \otimes \text{cone}(f_1) \otimes f_2^{\otimes n-i} \otimes \text{cone}(f_2) \end{aligned}$$

for some  $0 \leq i \leq n$ . It follows that  $g_1 \otimes \cdots \otimes g_n \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$  if  $n \geq 3$  since in this case either  $i$  or  $n - i$  is greater than or equal to 2. We conclude that

$$(f_1 + f_2)^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$$

for  $n \geq 3$ . However, the unit  $f_1 + f_2$  is a categorical isomorphism. Hence any  $\otimes$ -power  $(f_1 + f_2)^{\otimes n}$  is also an isomorphism and so is  $(f_1 + f_2)^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2)$ . It follows that  $X^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$  for  $n \geq 3$  and hence that  $\text{cone}(f_1) = 0$  or  $\text{cone}(f_2) = 0$  since  $(0)$  is prime and  $X \neq 0$  by assumption. In other words,  $f_1$  or  $f_2$  is an isomorphism (and hence a unit in  $E_X$  since  $E_X$  is an inverse-closed subring of  $[X, X]$ ).  $\square$

*Remark 5.1.6.* Recall from Remark 4.3.3 that a rigid category is local if and only if the  $\otimes$ -ideal  $(0) = \sqrt[0]{(0)}$  is prime. Also recall from Definition 4.3.1 that one of the equivalent conditions for a tensor triangulated category to be local is that  $\text{Spc}(\mathcal{K})$  is a local topological space in the sense that it has no non-trivial open covers. Then for a rigid category we have a slightly stronger statement than the one given above: if  $X$  is a non-zero object in a rigid tensor triangulated category then  $E_X$  is local provided that  $\text{supp}(X)$  is local as a topological space. On the other hand, it is known that  $E_X$  local does not imply that  $\text{supp}(X)$  is local, even when  $X = \mathbb{1}$  (see [Bal10a, Example 4.6]).

**Lemma 5.1.7.** *If  $F : \mathcal{K} \rightarrow \mathcal{L}$  is a morphism of tensor triangulated categories then the induced ring homomorphism  $[X, X]_{\mathcal{K}} \rightarrow [FX, FX]_{\mathcal{L}}$  restricts to a ring homomorphism  $E_{\mathcal{K}, X} \rightarrow E_{\mathcal{L}, FX}$ .*

*Proof.* This follows from the fact that  $F : \mathcal{K} \rightarrow \mathcal{L}$  is (by definition) a strong  $\otimes$ -functor. Indeed, if  $f : X \rightarrow X$  is a  $\otimes$ -balanced endomorphism in  $\mathcal{K}$  then the diagram

$$\begin{array}{ccc} FX \otimes FX & \xrightarrow{Ff \otimes FX} & FX \otimes FX \\ \wr & \xrightarrow{F(f \otimes X)} & \wr \\ \text{id}_{FX \otimes FX} \swarrow & \xrightarrow{F(X \otimes f)} & \searrow \text{id}_{FX \otimes FX} \\ FX \otimes FX & \xrightarrow{FX \otimes Ff} & FX \otimes FX \end{array}$$

demonstrates that  $Ff : FX \rightarrow FX$  is again  $\otimes$ -balanced.  $\square$

*Remark 5.1.8.* These results reveal the crucial properties that are secured by restricting ourselves to  $\otimes$ -balanced endomorphisms: they provide us with rings of endomorphisms that are local when the category is local, behave well with respect to tensor triangular functors, and have the property that the units are the elements that are categorical isomorphisms. However, these rings are not necessarily commutative.

**Theorem 5.1.9.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $X$  be an object in  $\mathcal{K}$ . For any commutative ring  $A$  and ring homomorphism  $\alpha : A \rightarrow E_X$  there is an inclusion-reversing, spectral map*

$$\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$$

*defined by  $\rho_{X,A}(\mathcal{P}) := \{a \in A \mid \text{cone}(\alpha(a)) \notin \mathcal{P}\}$ .*

*Proof.* Recall that  $\text{supp}(X) = \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid X \notin \mathcal{P}\}$ . By Lemma 5.1.7, the Verdier quotient  $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$  induces a ring homomorphism  $E_{\mathcal{K},X} \rightarrow E_{\mathcal{K}/\mathcal{P},q(X)}$  and since  $X \notin \mathcal{P}$  the target ring  $E_{\mathcal{K}/\mathcal{P},q(X)}$  is a local ring by Proposition 5.1.5. For any element  $f \in E_{\mathcal{K},X}$  observe that  $\text{cone}(f) \notin \mathcal{P}$  iff  $q(f)$  is not an isomorphism in  $\mathcal{K}/\mathcal{P}$  iff  $q(f)$  is a non-unit in the local ring  $E_{\mathcal{K}/\mathcal{P},q(X)}$ . Since the non-units in a local ring form a two-sided ideal, it follows that the preimage  $\{f \in E_{\mathcal{K},X} \mid \text{cone}(f) \notin \mathcal{P}\}$  is a two-sided ideal of  $E_{\mathcal{K},X}$ . Moreover, this ideal is “prime” in the sense that  $\text{cone}(f \cdot g) \notin \mathcal{P}$  implies that  $\text{cone}(f) \notin \mathcal{P}$  or  $\text{cone}(g) \notin \mathcal{P}$ . Indeed, the octahedral axiom applied to the composite  $f \circ g = f \cdot g$  implies that there is an exact triangle of the form  $\text{cone}(g) \rightarrow \text{cone}(g \cdot f) \rightarrow \text{cone}(f) \rightarrow \Sigma \text{cone}(g)$  and our claim follows from the fact that  $\mathcal{P}$  is a triangulated subcategory. In any case, this “prime” ideal of the non-commutative ring  $E_{\mathcal{K},X}$  pulls back via  $\alpha$  to a genuine prime ideal  $\rho_{X,A}(\mathcal{P})$  of the commutative ring  $A$ . This establishes that the map  $\rho_{X,A}$  is well-defined and it is clear from the definition that it is inclusion-reversing.

An arbitrary closed set for the Zariski topology on  $\text{Spec}(A)$  is of the form

$$V(\mathcal{E}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supset \mathcal{E}\}$$

for some subset  $\mathcal{E} \subset A$ . One readily checks that  $\rho_{X,A}^{-1}(V(\mathcal{E})) = \bigcap_{a \in \mathcal{E}} \text{supp}(\text{cone}(\alpha(a)))$  and we conclude that  $\rho_{X,A}$  is continuous. Moreover, if  $V(\mathcal{E})$  has quasi-compact complement then  $V(\mathcal{E}) = V(a_1, \dots, a_n)$  for some finite collection  $a_1, \dots, a_n \in A$  and the preimage

$$\rho_{X,A}^{-1}(V(a_1, \dots, a_n)) = \bigcap_{i=1}^n \text{supp}(\text{cone}(\alpha(a_i))) = \text{supp}(\text{cone}(\alpha(a_1)) \otimes \dots \otimes \text{cone}(\alpha(a_n)))$$

also has quasi-compact complement by Lemma 4.1.10. □

*Remark 5.1.10.* Keep in mind that  $\text{supp}(X) = \emptyset$  if and only if  $X$  is  $\otimes$ -nilpotent (cf. Remark 4.1.8) which if  $\mathcal{K}$  is rigid is the same thing as saying that  $X = 0$  (cf. Remark 4.2.2). On the other hand,  $E_X$  is non-zero precisely when  $X \neq 0$ .

*Remark 5.1.11.* If  $X = \mathbb{1}$  then the condition  $f \otimes X = X \otimes f$  is always satisfied. Indeed, if  $l_a : \mathbb{1} \otimes a \simeq a$  and  $r_a : a \otimes \mathbb{1} \simeq a$  denote the left and right unitors of the monoidal structure then for any endomorphism  $f : \mathbb{1} \rightarrow \mathbb{1}$  the diagram

$$\begin{array}{ccc} \mathbb{1} \otimes \mathbb{1} & \xrightarrow{f \otimes \mathbb{1}} & \mathbb{1} \otimes \mathbb{1} \\ \downarrow r_{\mathbb{1}} & & \downarrow r_{\mathbb{1}} \\ \mathbb{1} & \xrightarrow{f} & \mathbb{1} \\ \uparrow l_{\mathbb{1}} & & \uparrow l_{\mathbb{1}} \\ \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\mathbb{1} \otimes f} & \mathbb{1} \otimes \mathbb{1} \end{array} \quad \begin{array}{c} \text{id} \swarrow \\ \searrow \text{id} \end{array}$$

commutes because  $l_{\mathbb{1}} = r_{\mathbb{1}}$  (cf. Lemma 2.2.6). Thus  $E_{\mathbb{1}} = [\mathbb{1}, \mathbb{1}]$  and this ring is commutative by Lemma 2.2.7. Taking  $X = \mathbb{1}$  and  $\alpha = \text{id}_{[\mathbb{1}, \mathbb{1}]}$  one sees that the construction of Theorem 5.1.9 recovers the original unit comparison map from [Bal10a] that we described in Section 4.6.

All the results above have corresponding graded analogues. For a graded endomorphism  $f : \Sigma^k X \rightarrow X$  we abuse notation and write  $f \otimes X = X \otimes f$  when we really mean that the

following diagram commutes:

$$\begin{array}{ccc}
\Sigma^k(X \otimes X) & \simeq & \Sigma^k X \otimes X \\
\wr & & \downarrow f \otimes X \\
X \otimes \Sigma^k X & \xrightarrow{X \otimes f} & X \otimes X
\end{array} \tag{5.1.12}$$

**Proposition 5.1.13.** *A graded subring  $E_X^\bullet$  of the graded endomorphism ring  $[X, X]_\bullet$  is defined by setting  $E_X^i := \{f \in [X, X]_i \mid f \otimes X = X \otimes f\}$ . It has the property that a homogeneous element is a unit in  $E_X^\bullet$  iff it is a unit in  $[X, X]_\bullet$  iff it is a categorical isomorphism. Moreover, if  $(0)$  is a prime in  $\mathcal{K}$ , for example if  $\mathcal{K}$  is rigid and local, then  $E_X^\bullet$  is graded-local provided that  $X \neq 0$ .*

*Proof.* The proof is similar to the ungraded version, one just needs to take the relevant suspension isomorphisms into account. In particular, the result of Lemma 5.1.3 holds for a graded endomorphism  $f : \Sigma^k X \rightarrow X$  satisfying  $f \otimes X = X \otimes f$ . On the other hand, one could save time and conclude that  $E_X^\bullet$  is graded-local simply by invoking the fact that the  $\mathbb{Z}$ -graded ring  $E_X^\bullet$  is graded-local iff  $E_X^0 = E_X$  is local (see Proposition 2.3.15 and [Li12, Theorem 2.5]). For the sake of illustration, let us demonstrate the proof that  $E_X^\bullet$  defines an inverse-closed graded subring of  $[X, X]_\bullet$ . To see that  $E_X^i$  is an additive subgroup of  $[X, X]_i$  baptize the canonical suspension isomorphisms

$$\alpha : \Sigma^i(X \otimes X) \simeq \Sigma^i X \otimes X \quad \text{and} \quad \beta : \Sigma^i(X \otimes X) \simeq X \otimes \Sigma^i X$$

and observe that

$$\begin{aligned}
((f + g) \otimes X) \circ \alpha &= (f \otimes X + g \otimes X) \circ \alpha \\
&= (f \otimes X) \circ \alpha + (g \otimes X) \circ \alpha \\
&= (X \otimes f) \circ \beta + (X \otimes g) \circ \beta = (X \otimes (f + g)) \circ \beta
\end{aligned}$$

for any  $f, g \in E_X^i$ .

On the other hand, recall that if  $f : \Sigma^i X \rightarrow X$  and  $g : \Sigma^j X \rightarrow X$  are homogeneous elements of the graded endomorphism ring  $[X, X]$ , then their product  $f \cdot g$  is by definition  $f \circ \Sigma^i g$ . The diagram

$$\begin{array}{ccccc}
\Sigma^{i+j}(X \otimes X) & \simeq & \Sigma^i(\Sigma^j X \otimes X) & \simeq & \Sigma^{i+j} X \otimes X \\
\wr & & \downarrow \Sigma^i(g \otimes X) & & \downarrow \Sigma^i g \otimes X \\
\Sigma^i(X \otimes \Sigma^j X) & \xrightarrow{\Sigma^i(X \otimes g)} & \Sigma^i(X \otimes X) & \simeq & \Sigma^i X \otimes X \quad (f \cdot g) \otimes X \\
\wr & & \wr & & \downarrow f \otimes X \\
X \otimes \Sigma^{i+j} X & \xrightarrow{X \otimes \Sigma^i g} & X \otimes \Sigma^i X & \xrightarrow{X \otimes f} & X \otimes X \\
& \searrow X \otimes (f \cdot g) & & & \uparrow \\
& & & & X \otimes X
\end{array}$$

shows that if  $f \in E_X^i$  and  $g \in E_X^j$  then the product  $f \cdot g \in [X, X]_{i+j}$  is contained in  $E_X^{i+j}$ . Here the top-right and bottom-left squares commute by naturality, the bottom-right square is an incarnation of (5.1.12), and the top-left square is  $\Sigma^i(-)$  applied to an incarnation of (5.1.12).

Now let  $f \in E_X^i$  be a unit in  $[X, X]$ , say with inverse  $g : \Sigma^{-i} X \rightarrow X$ . We claim that  $g \in E_X^{-i}$ . The idea for the proof is simple enough (cf. the degree zero case in the proof of Proposition 5.1.5), but things look a bit more complicated when the suspensions are taken into account. In any case, since  $\Sigma^i g : X \rightarrow \Sigma^i X$  is a categorical inverse for  $f$ , the following diagram commutes:

$$\begin{array}{ccc}
& X \otimes X & \\
& \downarrow X \otimes \Sigma^i g & \searrow \text{id} \\
\Sigma^i(X \otimes X) & \xrightarrow{\sim} & X \otimes \Sigma^i X \\
\downarrow \sim & & \downarrow X \otimes f \\
\Sigma^i X \otimes X & \xrightarrow{f \otimes X} & X \otimes X \\
& \searrow \text{id} & \downarrow \Sigma^i g \otimes X \\
& & \Sigma^i X \otimes X
\end{array}$$

In other words, we have a commutative diagram

$$\begin{array}{ccc}
 X \otimes X & \xrightarrow{\Sigma^i g \otimes X} & \Sigma^i X \otimes X \xrightarrow{\sim} \Sigma^i(X \otimes X) \\
 & \searrow X \otimes \Sigma^i g & \nearrow X \otimes \Sigma^i X \xrightarrow{\sim} \\
 & & 
 \end{array} \tag{5.1.14}$$

and the fact that  $g \in E_X^{-i}$  is established by the diagram

$$\begin{array}{ccc}
 & \xrightarrow{\sim} \Sigma^{-i} X \otimes X \xrightarrow{g \otimes X} & \\
 \nearrow \Sigma^{-i}(\Sigma^i g \otimes X) & \Sigma^{-i}(\Sigma^i X \otimes X) \xrightarrow{\sim} & \\
 \Sigma^{-i}(X \otimes X) & & X \otimes X \\
 \searrow \Sigma^{-i}(X \otimes \Sigma^i g) & \Sigma^{-i}(X \otimes \Sigma^i X) \xrightarrow{\sim} & \\
 & X \otimes \Sigma^{-i} X \xrightarrow{X \otimes g} & 
 \end{array}$$

where the center diamond is  $\Sigma^{-i}(-)$  applied to (5.1.14), and the outer edges commute by naturality. □

**Lemma 5.1.15.** *If  $F : \mathcal{K} \rightarrow \mathcal{L}$  is a morphism of tensor triangulated categories then the induced graded ring homomorphism  $[X, X]_{\mathcal{K}, \bullet} \rightarrow [FX, FX]_{\mathcal{L}, \bullet}$  restricts to a graded ring homomorphism  $E_{\mathcal{K}, X}^\bullet \rightarrow E_{\mathcal{L}, FX}^\bullet$ .*

*Proof.* This involves verifying that a diagram commutes using the monoidal nature of the functor. The subtle point is that because of the suspension isomorphisms involved in (5.1.12) one must utilize the compatibility axioms for morphisms of tensor triangulated categories (cf. Definition 3.3.30 and Lemma 3.3.32). Getting down to business, let  $f : \Sigma^i X \rightarrow X$  be a

$\otimes$ -balanced endomorphism. The diagram that we must show commutes is

$$\begin{array}{ccc}
\Sigma^i(FX \otimes FX) & \xrightarrow{\sim} & \Sigma^i FX \otimes FX \\
\downarrow \sim & & \downarrow \sim \\
FX \otimes \Sigma^i FX & & F\Sigma^i X \otimes FX \\
\downarrow \sim & & \downarrow Ff \otimes 1 \\
FX \otimes F\Sigma^i X & \xrightarrow{1 \otimes Ff} & FX \otimes FX.
\end{array}$$

To see this consider the following diagram:

$$\begin{array}{ccccc}
\Sigma^i(FX \otimes FX) & \xrightarrow{\sim} & \Sigma^i FX \otimes X & & \\
\downarrow \sim & \searrow \sim & & \downarrow \sim & \\
& \Sigma^i F(X \otimes X) & & & \\
& \searrow \sim & & & \\
& F(\Sigma^i(X \otimes X)) & \xrightarrow{\sim} & F(\Sigma^i X \otimes X) & \xrightarrow{\sim} & F\Sigma^i X \otimes FX \\
& \downarrow \sim & & \downarrow F(f \otimes 1) & & \downarrow Ff \otimes 1 \\
& F(X \otimes \Sigma^i X) & \xrightarrow{F(1 \otimes f)} & F(X \otimes X) & \searrow \sim & \\
& \downarrow \sim & & & & \\
FX \otimes \Sigma^i FX & \xrightarrow{\sim} & FX \otimes F\Sigma^i X & \xrightarrow{1 \otimes Ff} & FX \otimes FX.
\end{array}$$

$(\dagger)$  is marked above the top-right square, and  $(\dagger)$  is marked to the left of the bottom-left square.

The regions marked  $(\dagger)$  commute by the compatibility axiom. The central region is  $F(-)$  applied to (5.1.12) and the remaining two regions commute by naturality. Note that in the creation of the above diagram we have used the fact that  $F$  is a strong  $\otimes$ -functor.  $\square$

**Theorem 5.1.16.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $X$  be an object in  $\mathcal{K}$ . For any (graded-)commutative graded ring  $A^\bullet$  and graded ring homomorphism  $\alpha : A^\bullet \rightarrow E_X^\bullet$  there is an inclusion-reversing, spectral map*

$$\rho_{X,A^\bullet}^\bullet : \text{supp}(X) \rightarrow \text{Spec}^h(A^\bullet)$$

defined by  $\rho_{X,A^\bullet}^\bullet(\mathcal{P}) := \{a \in A^i \mid \text{cone}(\alpha(a)) \notin \mathcal{P}\}_{i \in \mathbb{Z}}$ .

*Proof.* Let  $\mathcal{P} \in \text{supp}(X)$ . By Lemma 5.1.15, the localization functor  $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$  induces a graded ring homomorphism  $E_{\mathcal{K},X}^\bullet \rightarrow E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$ . Since  $X \notin \mathcal{P}$ , Proposition 5.1.13 implies that the target ring  $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$  is graded-local. The homogeneous non-units in  $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$  therefore form a two-sided ideal (recall Proposition 2.3.15). Moreover this ideal is “prime” in the sense that the product of two homogeneous units is again a unit. Indeed, since  $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$  is an inverse-closed subring of the graded endomorphism ring  $[q(X), q(X)]_\bullet$ , a homogeneous element of  $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$  is a unit iff it is a categorical isomorphism; thus if  $f \in E_{\mathcal{K}/\mathcal{P},q(X)}^i$  and  $g \in E_{\mathcal{K}/\mathcal{P},q(X)}^j$  are homogeneous units then so too is  $f \cdot g = \Sigma^j f \circ g$ . In any case, this homogeneous “prime” ideal pulls back via the graded ring homomorphism  $E_{\mathcal{K},X}^\bullet \rightarrow E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$  to a homogeneous “prime” ideal of the non-commutative ring  $E_{\mathcal{K},X}^\bullet$  which in turn pulls back via  $\alpha : A^\bullet \rightarrow E_X^\bullet$  to a genuine homogeneous prime ideal of  $A^\bullet$  and this is exactly what  $\rho_{X,A^\bullet}^\bullet(\mathcal{P})$  is defined to be. This shows that the map  $\rho_{X,A^\bullet}^\bullet$  is well-defined and it is clear from the definition that it is inclusion-reversing. Showing that it is spectral involves an argument similar to the one given in the proof of Theorem 5.1.9. For example, if  $D(a) = \{\mathfrak{p} \in \text{Spec}^h(A^\bullet) \mid a \notin \mathfrak{p}\}$  denotes the principal open subset of  $\text{Spec}^h(A^\bullet)$  defined by a homogeneous element  $a \in A^i$  then  $(\rho_{X,A^\bullet}^\bullet)^{-1}(D(a)) = \{\mathcal{P} \in \text{supp}(X) \mid \text{cone}(\alpha(a)) \in \mathcal{P}\} = \text{supp}(X) \setminus \text{supp}(\text{cone}(\alpha(a)))$ . (Note that  $\text{supp}(\text{cone}(\alpha(a))) \subset \text{supp}(\Sigma^i X) \cup \text{supp}(X) = \text{supp}(X)$  from the exact triangle  $\Sigma^i X \xrightarrow{\alpha(a)} X \rightarrow \text{cone}(\alpha(a)) \rightarrow \Sigma^{i+1} X$ .)  $\square$

*Remark 5.1.17.* It is clear from the definitions that there is a commutative diagram

$$\begin{array}{ccc} \text{supp}(X) & \xrightarrow{\rho_{X,A^\bullet}^\bullet} & \text{Spec}^h(A^\bullet) \\ & \searrow \rho_{X,A^0} & \downarrow (-)^0 \\ & & \text{Spec}(A^0) \end{array}$$

where  $(-)^0$  is the surjective spectral map  $\mathfrak{p}^\bullet \mapsto \mathfrak{p}^\bullet \cap A^0$  considered in Lemma 2.3.7.

*Example 5.1.18.* Any (graded-)commutative graded subring of  $E_X^\bullet$  yields an associated com-

parison map. Obvious examples include the graded-center of  $E_X^\bullet$  and

$$\{f \in \text{Center}[X, X]_\bullet \mid f \otimes X = X \otimes f\}.$$

A more exotic example is given by

$$\{f \in \text{Center}[X, X]_\bullet \mid f \otimes X \in \text{Center}[X^{\otimes 2}, X^{\otimes 2}]_\bullet\}. \quad (5.1.19)$$

For this third example, note that if  $f \otimes X \in \text{Center}[X^{\otimes 2}, X^{\otimes 2}]_\bullet$  then it follows from the fact that the symmetry  $\tau : X \otimes X \xrightarrow{\sim} X \otimes X$  is in  $[X^{\otimes 2}, X^{\otimes 2}]$  that  $f \in E_X^\bullet$ ; so (5.1.19) does indeed give a graded subring of  $E_X^\bullet$ .

*Example 5.1.20.* If  $X = \mathbb{1}$  then the condition  $f \otimes X = X \otimes f$  holds for any graded endomorphism and the ring  $E_{\mathbb{1}}^\bullet = [\mathbb{1}, \mathbb{1}]_\bullet$  is graded-commutative by Lemma 3.3.20. The map  $\rho_{\mathbb{1}, [\mathbb{1}, \mathbb{1}]_\bullet}^\bullet$  is the original graded comparison map from [Bal10a] which we discussed in Section 4.6.

*Example 5.1.21.* Recall the notion of the graded-center  $Z^\bullet(\mathcal{T})$  of a triangulated category  $\mathcal{T}$  (see [KY11], for example). For a *tensor* triangulated category  $\mathcal{T}$  one can define a graded-commutative graded subring of  $Z^\bullet(\mathcal{T})$  by setting

$$Z_\otimes^i(\mathcal{T}) := \{\alpha \in Z^i(\mathcal{T}) \mid X \otimes \alpha_Y = \alpha_{X \otimes Y} \text{ for every } X, Y \in \mathcal{T}\}$$

for each  $i \in \mathbb{Z}$ . Observe that any  $\alpha \in Z_\otimes^i(\mathcal{T})$  is completely determined by  $\alpha_{\mathbb{1}}$  and there is an obvious isomorphism  $Z_\otimes^\bullet(\mathcal{T}) \xrightarrow{\sim} [\mathbb{1}, \mathbb{1}]_\bullet$ . However, the definition makes sense for any thick  $\otimes$ -ideal  $\mathcal{J} \subseteq \mathcal{T}$  and  $Z_\otimes^\bullet(\mathcal{J})$  is not obviously so trivial for  $\mathcal{J} \subsetneq \mathcal{T}$ . For any object  $X \in \mathcal{J}$  there is a graded ring homomorphism  $Z_\otimes^\bullet(\mathcal{J}) \rightarrow E_X^\bullet$  given by  $\alpha \mapsto \alpha_X$  and so we obtain a map  $\text{supp}(X) \rightarrow \text{Spec}^h(Z_\otimes^\bullet(\mathcal{J}))$ . Explicitly, it maps a prime  $\mathcal{P}$  to  $\{\alpha \in Z_\otimes^\bullet(\mathcal{J}) \mid \text{cone}(\alpha_X) \notin \mathcal{P}\}$ . However, for a fixed prime  $\mathcal{P}$  and a fixed  $\alpha \in Z_\otimes^\bullet(\mathcal{J})$  the set  $\{Y \in \mathcal{J} \mid \text{cone}(\alpha_Y) \in \mathcal{P}\}$  is readily checked to be a thick  $\otimes$ -ideal. It follows that if  $X$  generates  $\mathcal{J}$  as a thick  $\otimes$ -ideal then  $\{\alpha \in Z_\otimes^\bullet(\mathcal{J}) \mid \text{cone}(\alpha_X) \notin \mathcal{P}\}$  is the same as  $\{\alpha \in Z_\otimes^\bullet(\mathcal{J}) \mid \text{cone}(\alpha_Y) \notin \mathcal{P} \text{ for some } Y \in \mathcal{J}\}$ . In other words, if  $\mathcal{J}$  is generated as a

thick  $\otimes$ -ideal by a single object (equivalently, by a finite number of objects) then every generator gives the exact same comparison map. In conclusion, every finitely generated thick  $\otimes$ -ideal  $\mathcal{J}$  has an associated (generator-independent) comparison map  $\text{supp}(\mathcal{J}) \rightarrow \text{Spec}^h(Z_\otimes^\bullet(\mathcal{J}))$  which sends a prime  $\mathcal{P}$  to  $\{\alpha \in Z_\otimes^\bullet(\mathcal{J}) \mid \text{cone}(\alpha_Y) \notin \mathcal{P} \text{ for some } Y \in \mathcal{J}\}$ .

*Remark 5.1.22.* In the proof of Theorem 5.1.9, we saw how to associate a “prime” ideal of the non-commutative ring  $E_X$  to any prime  $\mathcal{P} \in \text{supp}(X)$  which was then pulled back to a genuine prime ideal of a commutative ring  $A$  via a map  $A \rightarrow E_X$ . A suitable theory of spectra for non-commutative rings might allow us to work directly with the ring  $E_X$  but this avenue has not been pursued (and such a theory may not exist—cf. [Rey12]). In any case, taking commutative rings mapping into  $E_X$  is a flexible approach which provides for some interesting examples not obviously tied to the ring  $E_X$  (e.g., Example 5.1.21 above). On the other hand, although the maps  $\rho_{X,A}$  are useful for some purposes, they will not typically be natural with respect to tensor triangular functors. The problem is that although the construction of the ring  $E_X$  is functorial (recall Lemma 5.1.7), the construction of various commutative rings  $A$  mapping into  $E_X$  will typically not be. For example, the center of  $E_X$  is not a functorial construction, nor is the graded-center of a triangulated category. In the next section, we will replace  $E_X$  by a functorial commutative ring  $R_X$  and obtain a comparison map  $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$  which is natural with respect to tensor triangular functors. In fact, the construction of  $\rho_X$  will be a special case of a much more general construction, which will provide us with additional examples of natural comparison maps.

## 5.2 Natural constructions

Let  $\Phi$  be a non-empty set of objects in a tensor triangulated category  $\mathcal{K}$  that is closed under the  $\otimes$ -product ( $a, b \in \Phi \Rightarrow a \otimes b \in \Phi$ ). For any object  $X \in \mathcal{K}$  recall that  $E_X$  denotes the ring of

$\otimes$ -balanced endomorphisms of  $X$ .

**Lemma 5.2.1.** *Suppose  $f : X \rightarrow X$  is an endomorphism of  $X$  and  $g : Y \rightarrow Y$  is a  $\otimes$ -balanced endomorphism of  $Y$ . If  $X \otimes f \otimes g = f \otimes X \otimes g$  then  $f \otimes g$  is a  $\otimes$ -balanced endomorphism of  $X \otimes Y$ .*

*Proof.* The commutativity of the diagram

$$\begin{array}{ccc}
 X \otimes Y \otimes X \otimes Y & \xrightarrow{X \otimes Y \otimes f \otimes g} & X \otimes Y \otimes X \otimes Y \\
 \downarrow \tau \otimes X \otimes Y & & \downarrow \tau \otimes X \otimes Y \\
 \text{id} \swarrow & Y \otimes X \otimes X \otimes Y \xrightarrow{Y \otimes X \otimes f \otimes g} Y \otimes X \otimes X \otimes Y & \searrow \text{id} \\
 & \downarrow Y \otimes f \otimes X \otimes g & \\
 & X \otimes Y \otimes X \otimes Y \xrightarrow{f \otimes Y \otimes X \otimes g} X \otimes Y \otimes X \otimes Y & \\
 \downarrow X \otimes \tau \otimes Y & & \downarrow X \otimes \tau \otimes Y \\
 \text{id} \swarrow & X \otimes X \otimes Y \otimes Y \xrightarrow{f \otimes X \otimes Y \otimes g} X \otimes X \otimes Y \otimes Y & \searrow \text{id} \\
 & \downarrow f \otimes X \otimes g \otimes Y & \\
 & X \otimes Y \otimes X \otimes Y \xrightarrow{f \otimes g \otimes X \otimes Y} X \otimes Y \otimes X \otimes Y & \\
 \downarrow X \otimes \tau \otimes Y & & \downarrow X \otimes \tau \otimes Y
 \end{array}$$

verifies that  $f \otimes g \in E_{X \otimes Y}$ . □

**Corollary 5.2.2.** *For any pair of objects  $X, Y \in \mathcal{K}$  the functors  $- \otimes Y$  and  $X \otimes -$  induce ring homomorphisms  $E_X \rightarrow E_{X \otimes Y}$  and  $E_Y \rightarrow E_{X \otimes Y}$ .*

*Proof.* If  $f \in E_X$  then the  $g = \text{id}_Y$  case of Lemma 5.2.1 establishes that  $f \otimes Y \in E_{X \otimes Y}$ . Similarly, if  $g \in E_Y$  then  $X \otimes g \in E_{X \otimes Y}$  by taking  $f = \text{id}_X$  in Lemma 5.2.1. □

**Definition 5.2.3.** Define  $R_\Phi$  to be the set  $\{(X, f) : X \in \Phi, f \in E_X\} / \sim$  where  $\sim$  is the smallest equivalence relation such that  $(X, f) \sim (a \otimes X, a \otimes f)$  and  $(X, f) \sim (X \otimes a, f \otimes a)$  for every  $a \in \Phi$ .

**Notation 5.2.4.** A subscript  $f_X$  will indicate that  $f$  is an endomorphism of  $X$  and  $[f_X]$  will denote the image of  $(X, f)$  in  $R_\Phi$ .

**Lemma 5.2.5.** *For any isomorphism  $\alpha : X \xrightarrow{\sim} Y$  in  $\mathcal{K}$ , the isomorphism of rings  $[X, X] \xrightarrow{\sim} [Y, Y]$  given by  $f \mapsto \alpha \circ f \circ \alpha^{-1}$  restricts to give an isomorphism  $\alpha_* : E_X \xrightarrow{\sim} E_Y$ . If  $f \in E_X$  then  $f \otimes Y = X \otimes \alpha_*(f)$  as an endomorphism of  $X \otimes Y$ .*

*Proof.* This is routine from the definitions. If  $f \in E_X$  then the diagram

$$\begin{array}{ccc}
 Y \otimes Y & \xrightarrow{\alpha f \alpha^{-1} \otimes Y} & Y \otimes Y \\
 \downarrow \alpha^{-1} \otimes \alpha^{-1} & & \downarrow \alpha^{-1} \otimes \alpha^{-1} \\
 X \otimes X & \xrightleftharpoons[f \otimes X]{X \otimes f} & X \otimes X \\
 \downarrow \alpha \otimes \alpha & & \downarrow \alpha \otimes \alpha \\
 Y \otimes Y & \xrightarrow{Y \otimes \alpha f \alpha^{-1}} & Y \otimes Y
 \end{array}
 \begin{array}{c}
 \text{id} \swarrow \quad \searrow \text{id} \\
 \text{ }
 \end{array}$$

shows that  $\alpha \circ f \circ \alpha^{-1} \in E_Y$ . A similar diagram shows the converse. Also, if  $f \in E_X$  then  $X \otimes \alpha_*(f) = (X \otimes \alpha) \circ (X \otimes f) \circ (X \otimes \alpha^{-1}) = (X \otimes \alpha) \circ (f \otimes X) \circ (X \otimes \alpha^{-1}) = f \otimes (\alpha \circ \alpha^{-1}) = f \otimes Y$ .  $\square$

**Notation 5.2.6.** For two endomorphisms  $f_X$  and  $g_Y$  the notation  $f_X \simeq g_Y$  will indicate that there exists an isomorphism  $\alpha : X \xrightarrow{\sim} Y$  such that  $\alpha_*(f_X) = g_Y$ . This coincides with our earlier Notation 3.1.43.

*Remark 5.2.7.* The above lemma implies that in  $R_\Phi$  endomorphisms of  $X$  are identified with endomorphisms of  $Y$  via all isomorphisms  $X \xrightarrow{\sim} Y$ . In particular, an endomorphism  $f_X$  is identified with the “twisted” version  $\sigma \circ f \circ \sigma^{-1}$  for every automorphism  $\sigma : X \xrightarrow{\sim} X$ . Because of these identifications, an essentially equivalent approach to the construction of  $R_\Phi$  could be obtained by taking  $\Phi$  to be a set of isomorphism classes of objects closed under the  $\otimes$ -product. However, such an approach would obscure the fact that these identifications up to isomorphism are forced by the innocuous identifications  $f \sim a \otimes f$  and  $f \sim f \otimes a$ .

**Lemma 5.2.8.** *Two endomorphisms  $f_X$  and  $g_Y$  are equivalent in  $R_\Phi$  if and only if there exist objects  $a, b \in \Phi$  such that  $a \otimes f_X \simeq b \otimes g_Y$  if and only if there exists an object  $c \in \Phi$  such that  $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$ .*

*Proof.* The proof of this lemma is straightforward from the definitions once one appreciates Remark 5.2.7. Since the lemma is important we will give more details than are really necessary. First we check that the third condition (existence of  $c \in \Phi$  such that

$f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$ ) defines an equivalence relation on  $\{(X, f) : X \in \Phi, f \in E_X\}$ . For reflexivity, we take  $c = X$  and use the fact that  $f$  is  $\otimes$ -balanced:  $f_X \otimes X \otimes X = X \otimes f \otimes X = X \otimes X \otimes f$ . For symmetry, it suffices to show that if  $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$  then  $g_Y \otimes c \otimes X = Y \otimes c \otimes f_X$ . This is evident from the commutativity of

$$\begin{array}{ccc}
 Y \otimes c \otimes X & \xrightarrow{g \otimes c \otimes X} & Y \otimes c \otimes X \\
 \text{id} \swarrow & \text{ } & \searrow \text{id} \\
 & \text{ } & \\
 X \otimes c \otimes Y & \xrightleftharpoons[f \otimes c \otimes Y]{X \otimes c \otimes g_Y} & X \otimes c \otimes Y \\
 & \text{ } & \\
 Y \otimes c \otimes X & \xrightarrow{Y \otimes c \otimes f} & Y \otimes c \otimes X
 \end{array}$$

where the vertical isomorphisms are those isomorphisms induced by the monoidal symmetry which swap the appropriate  $\otimes$ -factors.

Transitivity is fairly immediate: if  $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$  and  $g_Y \otimes d \otimes Z = Y \otimes d \otimes h_Z$  for some  $c, d \in \Phi$  then  $f_X \otimes c \otimes Y \otimes d \otimes Z = X \otimes c \otimes g_Y \otimes d \otimes Z = X \otimes c \otimes Y \otimes d \otimes h_Z$ . To show that this equivalence relation is stronger than the equivalence relation defining  $R_\Phi$  we need to show that  $f \in E_X$  is identified with  $a \otimes f$  and  $f \otimes a$  for any  $a \in \Phi$ . Indeed, the fact that  $f_X$  is  $\otimes$ -balanced implies that  $f \otimes c \otimes a \otimes X = X \otimes c \otimes a \otimes f$  and  $f \otimes c \otimes X \otimes a = X \otimes c \otimes f \otimes a$  for any object  $c \in \mathcal{K}$ . For example:

$$\begin{array}{ccc}
 X \otimes c \otimes X \otimes a & \xrightarrow{f \otimes c \otimes X \otimes a} & X \otimes c \otimes X \otimes a \\
 \text{id} \swarrow & \text{ } & \searrow \text{id} \\
 & \text{ } & \\
 c \otimes X \otimes X \otimes a & \xrightleftharpoons[c \otimes X \otimes f \otimes a]{c \otimes f \otimes X \otimes a} & c \otimes X \otimes X \otimes a \\
 & \text{ } & \\
 X \otimes c \otimes X \otimes a & \xrightarrow{X \otimes c \otimes f \otimes a} & X \otimes c \otimes X \otimes a.
 \end{array}$$

These kinds of tricks moving around  $\otimes$ -balanced endomorphisms using the symmetry have already been seen in the proof of Lemma 5.2.1 and will be used in the sequel without further comment. On the other hand,  $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$  clearly implies that  $[f_X] = [g_Y]$  in  $R_\Phi$  and we conclude that the equivalence relation given by the third condition is precisely the equivalence relation defining  $R_\Phi$ .

Next we consider the second criterion in the statement of the lemma: the existence of

objects  $a, b \in \Phi$  such that  $a \otimes f_X \simeq b \otimes g_Y$ . According to Notation 5.2.4 this means that there exists an isomorphism  $\alpha : a \otimes X \xrightarrow{\sim} b \otimes Y$  such that  $\alpha \circ (a \otimes f_X) = (b \otimes g_Y) \circ \alpha$ . First we will check that this second criterion defines an equivalence relation on  $\{(X, f) : X \in \Phi, f \in E_X\}$ . Reflexivity is immediate. For symmetry: if  $\alpha \circ (a \otimes f_X) = (b \otimes g_Y) \circ \alpha$  then  $\alpha^{-1} \circ (b \otimes g_Y) = (a \otimes f_X) \circ \alpha^{-1}$  so taking the inverse  $\alpha^{-1} : b \otimes Y \xrightarrow{\sim} a \otimes X$  we have that  $b \otimes g_Y \simeq a \otimes f_X$ . Finally, for transitivity, if  $\beta : b \otimes Y \xrightarrow{\sim} c \otimes Z$  is an isomorphism such that  $\beta \circ (b \otimes g_Y) = (c \otimes h_Z) \circ \beta$  then  $\beta \circ \alpha : a \otimes X \xrightarrow{\sim} c \otimes Z$  is an isomorphism with  $\beta \circ \alpha \circ (a \otimes f_X) = \beta \circ (b \otimes g_Y) \circ \alpha = (c \otimes h_Z) \circ \beta \circ \alpha$  so that  $a \otimes f_X \simeq c \otimes h_Z$ . Next we wish to show that this equivalence relation is stronger than the one defining  $R_\Phi$ . Given  $f_X$  and  $a \in \Phi$  want to show that  $f_X \sim a \otimes f_X$ . Well,  $a \otimes f_X \simeq 1 \otimes a \otimes f_X$  so taking  $b = 1 \otimes a$  gives what we want. For  $f_X \sim f_X \otimes b$  we can similarly take  $a = 1 \otimes b$ . On the other hand, if  $a \otimes f_X \simeq b \otimes g_Y$  then by definition there exists an isomorphism  $\alpha : a \otimes X \xrightarrow{\sim} b \otimes Y$  such that  $b \otimes g_Y = \alpha_*(a \otimes f_X)$ . Lemma 5.2.5 then implies that  $a \otimes f_X \otimes b \otimes Y = a \otimes X \otimes b \otimes Y$  so that  $[f_X] = [g_Y]$  in  $R_\Phi$ . We conclude that the equivalence relation given by the second condition is precisely the equivalence relation defining  $R_\Phi$ .  $\square$

**Proposition 5.2.9.** *The set  $R_\Phi$  is a commutative ring with addition and multiplication defined by  $[f_X] + [g_Y] := [f_X \otimes Y + X \otimes g_Y]$  and  $[f_X] \cdot [g_Y] := [(f_X \otimes Y) \circ (X \otimes g_Y)] = [f_X \otimes g_Y]$ . The zero element is  $[0_X]$  for any  $X \in \Phi$  and the identity element is  $[\text{id}_X]$  for any  $X \in \Phi$ . More generally,  $[f_X] = 0$  iff there exists  $a \in \Phi$  such that  $a \otimes f_X = 0$ . Similarly,  $[f_X] = 1$  iff there exists  $a \in \Phi$  such that  $a \otimes f_X = \text{id}_{a \otimes X}$ . In addition,  $[f_X]$  is a unit iff there exists  $a \in \Phi$  such that  $a \otimes f_X$  is an isomorphism.*

*Proof.* Armed with Lemma 5.2.8, it is a long but straightforward exercise to establish that addition and multiplication are well-defined and endow  $R_\Phi$  with a ring structure. Let's begin by demonstrating that multiplication is well-defined. If  $[f_X] = [f'_X]$  and  $[g_Y] = [g'_Y]$

then by the second criterion of Lemma 5.2.8 there exist isomorphisms

$$\alpha : a \otimes X \xrightarrow{\sim} a' \otimes X' \quad \text{and} \quad \beta : b \otimes Y \xrightarrow{\sim} b' \otimes Y'$$

for some objects  $a, a', b, b' \in \Phi$  such that  $\alpha_*(a \otimes f) = a' \otimes f'$  and  $\beta_*(b \otimes g) = b' \otimes g'$ . Defining  $\gamma : a \otimes b \otimes X \otimes Y \xrightarrow{\sim} a' \otimes b' \otimes X' \otimes Y'$  to be the composite isomorphism

$$a \otimes b \otimes X \otimes Y \xrightarrow{1 \otimes \tau \otimes 1} a \otimes X \otimes b \otimes Y \xrightarrow{\alpha \otimes \beta} a' \otimes X' \otimes b' \otimes Y' \xrightarrow{1 \otimes \tau \otimes 1} a' \otimes b' \otimes X' \otimes Y'$$

we claim that  $\gamma_*(a \otimes b \otimes f \otimes g) = a' \otimes b' \otimes f' \otimes g'$  so that  $[f \otimes g] = [f' \otimes g']$  in  $R_\Phi$ . Indeed,

$$\begin{aligned} \gamma \circ (a \otimes b \otimes f \otimes g) &= (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \circ (a \otimes b \otimes f \otimes g) \\ &= (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (a \otimes f \otimes b \otimes g) \circ (1 \otimes \tau \otimes 1) \\ &= (1 \otimes \tau \otimes 1) \circ (a' \otimes f' \otimes b' \otimes g') \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\ &= (a' \otimes b' \otimes f' \otimes g') \circ (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\ &= (a' \otimes b' \otimes f' \otimes g') \circ \gamma \end{aligned}$$

or, diagrammatically,

$$\begin{array}{ccc} a \otimes b \otimes X \otimes Y & \xrightarrow{1 \otimes 1 \otimes f \otimes g} & a \otimes b \otimes X \otimes Y \\ \downarrow 1 \otimes \tau \otimes 1 & & \downarrow 1 \otimes \tau \otimes 1 \\ a \otimes X \otimes b \otimes Y & \xrightarrow{1 \otimes f \otimes 1 \otimes g} & a \otimes X \otimes b \otimes Y \\ \downarrow \alpha \otimes \beta & & \downarrow \alpha \otimes \beta \\ a' \otimes X' \otimes b' \otimes Y' & \xrightarrow{1 \otimes f' \otimes 1 \otimes g'} & a' \otimes X' \otimes b' \otimes Y' \\ \downarrow 1 \otimes \tau \otimes 1 & & \downarrow 1 \otimes \tau \otimes 1 \\ a' \otimes b' \otimes X' \otimes Y' & \xrightarrow{1 \otimes 1 \otimes f' \otimes g'} & a' \otimes b' \otimes X' \otimes Y'. \end{array}$$

$\gamma$  (curved arrow from top-left to bottom-left)       $\gamma$  (curved arrow from top-right to bottom-right)

Similarly, to show that addition is well-defined it suffices to show that

$$\begin{aligned}
\gamma \circ (a \otimes b \otimes (f_X \otimes Y + X \otimes g_Y)) &= \gamma \circ (a \otimes b \otimes f_X \otimes Y) + \gamma \circ (a \otimes b \otimes X \otimes g_Y) \\
&= (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \circ (a \otimes b \otimes f_X \otimes Y) \\
&\quad + (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \circ (a \otimes b \otimes X \otimes g_Y) \\
&= (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (a \otimes f_X \otimes b \otimes Y) \circ (1 \otimes \tau \otimes 1) \\
&\quad + (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (a \otimes X \otimes b \otimes g_Y) \circ (1 \otimes \tau \otimes 1) \\
&= (1 \otimes \tau \otimes 1) \circ (a' \otimes f'_{X'} \otimes b' \otimes Y') \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\
&\quad + (1 \otimes \tau \otimes 1) \circ (a' \otimes X' \otimes b' \otimes g'_{Y'}) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\
&= (a' \otimes b' \otimes f'_{X'} \otimes Y') \circ (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\
&\quad + (a' \otimes b' \otimes X' \otimes g'_{Y'}) \circ (1 \otimes \tau \otimes 1) \circ (\alpha \otimes \beta) \circ (1 \otimes \tau \otimes 1) \\
&= (a' \otimes b' \otimes f'_{X'} \otimes Y') \circ \gamma + (a' \otimes b' \otimes X' \otimes g'_{Y'}) \circ \gamma \\
&= (a' \otimes b' \otimes (f'_{X'} \otimes Y' + X' \otimes g'_{Y'})) \circ \gamma.
\end{aligned}$$

Having shown that addition and multiplication are well-defined it is straightforward to check that  $R_\Phi$  satisfies the axioms of a ring. There are several ways to see that this ring structure is commutative. For example, the fact that  $f_X$  is  $\otimes$ -balanced implies that  $f \otimes g \otimes X = X \otimes g \otimes f$  and so  $[f] \cdot [g] = [f \otimes g] = [f \otimes g \otimes X] = [X \otimes g \otimes f] = [g \otimes f] = [g] \cdot [f]$ . The remaining statements (e.g., that  $[f] = 0$  iff  $a \otimes f = 0$  for some  $a \in \Phi$ ) are easily verified.  $\square$

*Remark 5.2.10.* The ring  $R_\Phi$  is the colimit of a diagram of rings consisting of  $E_X$  for each  $X \in \Phi$  with maps generated by  $a \otimes - : E_X \rightarrow E_{a \otimes X}$  and  $- \otimes a : E_X \rightarrow E_{X \otimes a}$ . Although the index category on which this diagram is defined is not technically a filtered category (because there are parallel arrows that are not coequalized in the category),  $\text{colim}_{X \in \Phi} E_X$  is still a filtered colimit in the more general sense of [Sta, Chapter 4, Section 17] which is sufficient for the colimit to be created in the category of sets. Rather than define  $R_\Phi$  to be this colimit (which

would necessitate a longer discussion of these technicalities) we have opted for the concrete description given above.

The following definition will be important in Theorem 5.2.15 below.

**Definition 5.2.11.** Define a closed subset  $\mathcal{Z}_\Phi$  of the tensor triangular spectrum  $\mathrm{Spc}(\mathcal{K})$  by

$$\mathcal{Z}_\Phi := \bigcap_{a \in \Phi} \mathrm{supp}(a).$$

**Lemma 5.2.12.** *The commutative ring  $R_\Phi$  is zero iff  $\Phi$  contains zero iff  $\Phi$  contains an object which is  $\otimes$ -nilpotent iff  $\mathcal{Z}_\Phi = \emptyset$ .*

*Proof.* Fix an  $X \in \Phi$  and note that  $R_\Phi = 0$  iff  $[\mathrm{id}_X] = [0_X]$  iff  $a \otimes X = 0$  for some  $a \in \Phi$  iff  $\Phi$  contains zero. Since  $\Phi$  is closed under  $\otimes$ -product, it contains zero iff it contains an object which is  $\otimes$ -nilpotent. To complete the proof we need to recall from Remark 4.1.8 that  $\mathrm{supp}(X) = \emptyset$  iff  $X$  is  $\otimes$ -nilpotent. Thus,  $\mathcal{Z}_\Phi = \emptyset$  if  $\Phi$  contains a  $\otimes$ -nilpotent object. Conversely, if  $\mathcal{Z}_\Phi = \bigcap_{a \in \Phi} \mathrm{supp}(a) = \emptyset$  then the quasi-compactness of  $\mathrm{Spc}(\mathcal{K})$  implies that

$$\mathrm{supp}(a_1) \cap \cdots \cap \mathrm{supp}(a_n) = \emptyset$$

for some finite set of objects  $a_1, \dots, a_n \in \Phi$  and the object  $a_1 \otimes \cdots \otimes a_n \in \Phi$  is  $\otimes$ -nilpotent since  $\mathrm{supp}(a_1 \otimes \cdots \otimes a_n) = \mathrm{supp}(a_1) \cap \cdots \cap \mathrm{supp}(a_n) = \emptyset$ .  $\square$

**Proposition 5.2.13.** *Let  $\Phi$  be a non-empty  $\otimes$ -multiplicative set of objects in a tensor triangulated category  $\mathcal{K}$ . If  $(0)$  is a prime in  $\mathcal{K}$ , for example if  $\mathcal{K}$  is rigid and local, then  $R_\Phi$  is a local ring provided that it is non-zero.*

*Proof.* The proof closely mirrors the proof that  $E_X$  is local (Proposition 5.1.5). Indeed, if  $[f_X] + [g_Y]$  is a unit in  $R_\Phi$  then  $a \otimes (f \otimes Y + X \otimes g)$  is an isomorphism for some  $a \in \Phi$ . It follows that  $a^{\otimes n} \otimes (f \otimes Y + X \otimes g)^{\otimes n} \otimes \mathrm{cone}(f) \otimes \mathrm{cone}(g)$  is both zero and an isomorphism for  $n \geq 3$ . This implies that  $a^{\otimes n} \otimes (X \otimes Y)^{\otimes n} \otimes \mathrm{cone}(f) \otimes \mathrm{cone}(g) = 0$  for  $n \geq 3$  and since  $(0)$  is prime and

$X, Y, a \in \Phi$  are non-zero we conclude that  $f$  or  $g$  is an isomorphism (and hence  $[f]$  or  $[g]$  is a unit in  $R_\Phi$ ).  $\square$

**Proposition 5.2.14.** *Let  $F : \mathcal{K} \rightarrow \mathcal{L}$  be a morphism of tensor triangulated categories. Suppose  $\Phi \subset \mathcal{K}$  and  $\Psi \subset \mathcal{L}$  are non-empty  $\otimes$ -multiplicative subsets such that  $F(\Phi) \subset \Psi$ . Then  $[f] \mapsto [F(f)]$  defines a ring homomorphism  $R_{\mathcal{K}, \Phi} \rightarrow R_{\mathcal{L}, \Psi}$ .*

*Proof.* This is a routine verification using Lemma 5.2.8 and the fact that  $F : \mathcal{K} \rightarrow \mathcal{L}$  is a strong  $\otimes$ -functor. Indeed, if  $[f_X] = [f'_{X'}]$  then there exists an object  $c \in \Phi$  such that  $f \otimes c \otimes X' = X \otimes c \otimes f'$  and the diagram

$$\begin{array}{ccc}
 FX \otimes Fc \otimes FX' & \xrightarrow{Ff \otimes Fc \otimes FX'} & FX \otimes Fc \otimes FX' \\
 \wr \downarrow & & \wr \downarrow \\
 F(X \otimes c \otimes X') & \xrightarrow[F(X \otimes c \otimes f')]{F(f \otimes c \otimes f')} & F(X \otimes c \otimes X') \\
 \wr \downarrow & & \wr \downarrow \\
 FX \otimes Fc \otimes FX' & \xrightarrow{FX \otimes Fc \otimes Ff'} & FX \otimes Fc \otimes FX'
 \end{array}
 \begin{array}{c}
 \text{id} \swarrow \\
 \searrow \text{id}
 \end{array}$$

demonstrates that  $[Ff] = [Ff']$  in  $R_{\mathcal{L}, \Psi}$ .  $\square$

**Theorem 5.2.15.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $\Phi \subset \mathcal{K}$  be a non-empty set of objects closed under the  $\otimes$ -product. Let  $\mathcal{Z}_\Phi := \bigcap_{X \in \Phi} \text{supp}(X)$ . There is an inclusion-reversing, spectral map*

$$\rho_\Phi : \mathcal{Z}_\Phi \rightarrow \text{Spec}(R_\Phi)$$

*defined by  $\mathcal{P} \mapsto \{[f] \in R_\Phi \mid \text{cone}(f) \notin \mathcal{P}\}$ .*

*Proof.* The first point to make is that for  $\mathcal{P} \in \mathcal{Z}_\Phi$  the condition  $\text{cone}(f) \notin \mathcal{P}$  does not depend on the choice of representative of  $[f] \in R_\Phi$ . Indeed, if  $[f_X] = [g_Y]$  then  $a \otimes f \simeq b \otimes g$  for some  $a, b \in \Phi$ . It follows that  $\text{cone}(a \otimes f) \simeq \text{cone}(b \otimes g)$ . Indeed, if  $\alpha : a \otimes X \xrightarrow{\sim} b \otimes Y$  is an isomorphism

such that  $\alpha_*(a \otimes f) = b \otimes g$  then we can construct a morphism of exact triangles

$$\begin{array}{ccccccc}
a \otimes X & \xrightarrow{a \otimes f} & a \otimes X & \longrightarrow & \text{cone}(a \otimes f) & \longrightarrow & \Sigma a \otimes X \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \exists \beta & & \downarrow \Sigma \alpha \\
b \otimes Y & \xrightarrow{b \otimes g} & b \otimes Y & \longrightarrow & \text{cone}(b \otimes g) & \longrightarrow & \Sigma b \otimes Y
\end{array}$$

and Lemma 3.1.9 implies that  $\beta : \text{cone}(a \otimes f) \rightarrow \text{cone}(b \otimes g)$  is an isomorphism. Furthermore,  $\text{cone}(a \otimes f) \simeq a \otimes \text{cone}(f)$  and  $\text{cone}(b \otimes g) \simeq b \otimes \text{cone}(g)$  since  $a \otimes -$  and  $b \otimes -$  are exact functors. If  $\mathcal{P} \in \mathcal{Z}_\Phi$  then  $\mathcal{P} \in \text{supp}(\text{cone}(f))$  iff  $\mathcal{P} \in \text{supp}(a) \cap \text{supp}(\text{cone}(f)) = \text{supp}(b) \cap \text{supp}(\text{cone}(g))$  iff  $\mathcal{P} \in \text{supp}(\text{cone}(g))$ . The second point to make is that for any prime  $\mathcal{P} \in \text{Spc}(\mathcal{K})$  the quotient functor  $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$  induces a ring homomorphism  $R_{\mathcal{K},\Phi} \rightarrow R_{\mathcal{K}/\mathcal{P},q(\Phi)}$  whose target ring is local provided that  $0 \notin q(\Phi)$ ; in other words, provided that  $\mathcal{P} \cap \Phi = \emptyset$  which is equivalent to saying that  $\mathcal{P} \in \mathcal{Z}_\Phi$ . With these facts in mind the proof is similar to the proof of Theorem 5.1.9. For an element  $[f] \in R_{\mathcal{K},\Phi}$ ,  $[q(f)]$  is a unit in  $R_{\mathcal{K}/\mathcal{P},q(\Phi)}$  iff there exists  $a \in \Phi$  such that  $q(a \otimes f)$  is an isomorphism in  $\mathcal{K}/\mathcal{P}$  iff there exists  $a \in \Phi$  such that  $a \otimes \text{cone}(f) \in \mathcal{P}$  iff  $\text{cone}(f) \in \mathcal{P}$ . Thus  $\rho_\Phi(\mathcal{P})$  is the pullback of the collection of non-units in  $R_{\mathcal{K}/\mathcal{P},q(\Phi)}$  and since the non-units in a local ring form a (two-sided) ideal, this establishes that  $\rho_\Phi(\mathcal{P})$  is an ideal. It is prime since  $[f_X] \cdot [g_Y] \in \rho_\Phi(\mathcal{P})$  implies that  $\mathcal{P} \in \text{supp}(\text{cone}(f \otimes g)) \subset \text{supp}(\text{cone}(f \otimes Y)) \cup \text{supp}(\text{cone}(X \otimes g)) \subset \text{supp}(\text{cone}(f)) \cup \text{supp}(\text{cone}(g))$ . That  $\rho_\Phi$  is a spectral map follows from similar modifications to the argument given in the proof of Theorem 5.1.9.  $\square$

*Example 5.2.16.* For any object  $X \in \mathcal{K}$ , taking  $\Phi := \{X^{\otimes n} \mid n \geq 1\}$  provides a comparison map  $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$ . These are the “object” comparison maps mentioned in the introduction.

*Example 5.2.17.* For any closed set  $\mathcal{Z} \subset \text{Spc}(\mathcal{K})$ , taking  $\Phi := \{a \in \mathcal{K} \mid \text{supp}(a) \supset \mathcal{Z}\}$  gives a comparison map  $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Spec}(R_{\mathcal{Z}})$ . These are the “closed set” comparison maps mentioned in the introduction. Note that  $\mathcal{Z}_\Phi = \mathcal{Z}$  because  $\{\text{supp}(a) : a \in \mathcal{K}\}$  forms a basis of closed sets for the topology on  $\text{Spc}(\mathcal{K})$ . Also note that if the category  $\mathcal{K}$  is not small then there is the

unfortunate detail that  $\Phi$  might not be a set; however, we do not need to worry about this technicality because of Remark 5.2.7.

*Example 5.2.18.* For any Thomason closed set  $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$ , taking  $\Phi := \{a \in \mathcal{K} \mid \mathrm{supp}(a) = \mathcal{Z}\}$  provides another comparison map defined on  $\mathcal{Z}$ . However, the ring  $R_\Phi$  is canonically isomorphic to the one in Example 5.2.17 and under this identification the two comparison maps coincide. In other words, when  $\mathcal{Z}$  is Thomason we can take the target ring of the “closed set” comparison map  $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathrm{Spec}(R_{\mathcal{Z}})$  to be defined using only those objects  $X$  for which  $\mathrm{supp}(X) = \mathcal{Z}$ .

*Example 5.2.19.* Another candidate to consider would be  $\Phi := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset \mathcal{Z}\}$  but in this case  $\mathcal{Z}_\Phi = \emptyset$  and  $R_\Phi = 0$  as indicated by Lemma 5.2.12.

*Remark 5.2.20.* There are other examples that could be considered, such as the collection of  $\otimes$ -invertible objects, or the collection of objects that are isomorphic to a direct sum of suspensions of  $\mathbb{1}$ .

**Proposition 5.2.21.** *Let  $F : (\mathcal{K}, \Phi) \rightarrow (\mathcal{L}, \Psi)$  be a morphism of tensor triangulated categories  $\mathcal{K} \rightarrow \mathcal{L}$  such that  $F(\Phi) \subset \Psi$ . Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{L}) \supset \mathcal{Z}_{\mathcal{L}, \Psi} & \xrightarrow{\rho_{\mathcal{L}, \Psi}} & \mathrm{Spec}(R_{\mathcal{L}, \Psi}) \\ \downarrow & & \downarrow \\ \mathrm{Spc}(\mathcal{K}) \supset \mathcal{Z}_{\mathcal{K}, \Phi} & \xrightarrow{\rho_{\mathcal{K}, \Phi}} & \mathrm{Spec}(R_{\mathcal{K}, \Phi}) \end{array} \quad (5.2.22)$$

*in the category of spectral spaces.*

*Proof.* A ring homomorphism  $R_{\mathcal{K}, \Phi} \rightarrow R_{\mathcal{L}, \Psi}$  is provided by Proposition 5.2.14 and the rest is a routine recollection of the relevant definitions. Let  $\phi : \mathrm{Spc}(\mathcal{L}) \rightarrow \mathrm{Spc}(\mathcal{K})$  be the map on spectra induced by the tensor triangulated functor  $F$ . Recall that for any  $X \in \mathcal{K}$ ,  $\mathrm{supp}_{\mathcal{L}}(FX) = \phi^{-1}(\mathrm{supp}_{\mathcal{K}}(X))$ . Since  $F(\Phi) \subset \Psi$  we then have

$$\mathcal{Z}_{\mathcal{L}, \Psi} = \bigcap_{Y \in \Psi} \mathrm{supp}_{\mathcal{L}}(Y) \subset \bigcap_{X \in \Phi} \mathrm{supp}_{\mathcal{L}}(FX) = \phi^{-1}\left(\bigcap_{X \in \Phi} \mathrm{supp}_{\mathcal{K}}(X)\right) = \phi^{-1}(\mathcal{Z}_{\mathcal{K}, \Phi})$$

and so  $\phi$  restricted to  $\mathcal{Z}_{\mathcal{L},\Psi}$  lands in  $\mathcal{Z}_{\mathcal{K},\Phi}$ . Note that this restriction is a spectral map by Proposition 2.4.9. Consider  $\mathcal{Q} \in \mathcal{Z}_{\mathcal{L},\Psi}$ . Then the top-right edge sends it to

$$\{[f] \in R_{\mathcal{K},\Phi} \mid \text{cone}(Ff) \notin \mathcal{Q}\} = \{[f] \in R_{\mathcal{K},\Phi} \mid F(\text{cone}(f)) \notin \mathcal{Q}\}.$$

On the other hand,  $\phi(\mathcal{Q}) = F^{-1}(\mathcal{Q})$  so  $\mathcal{Q}$  is sent along the bottom-right to

$$\{[f] \in R_{\mathcal{K},\Phi} \mid \text{cone}(f) \notin F^{-1}(\mathcal{Q})\} = \{[f] \in R_{\mathcal{K},\Phi} \mid F(\text{cone}(f)) \notin \mathcal{Q}\}$$

and we conclude that the diagram commutes.  $\square$

*Example 5.2.23.* If  $\mathcal{Z}_1 \subset \mathcal{Z}_2$  is an inclusion of closed subsets then there is a ring homomorphism  $R_{\mathcal{Z}_2} \rightarrow R_{\mathcal{Z}_1}$  and a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_2 & \xrightarrow{\rho_{\mathcal{Z}_2}} & \text{Spec}(R_{\mathcal{Z}_2}) \\ \cup & & \uparrow \\ \mathcal{Z}_1 & \xrightarrow{\rho_{\mathcal{Z}_1}} & \text{Spec}(R_{\mathcal{Z}_1}). \end{array}$$

*Example 5.2.24.* If  $\mathcal{Z}$  is a Thomason closed subset then for any object  $X$  with  $\text{supp}(X) = \mathcal{Z}$  there is a ring homomorphism  $R_X \rightarrow R_{\mathcal{Z}}$  and a commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\rho_{\mathcal{Z}}} & \text{Spec}(R_{\mathcal{Z}}) \\ & \searrow \rho_X & \downarrow \\ & & \text{Spec}(R_X). \end{array}$$

It is worth explicitly stating the naturality in the case of the object and closed set comparison maps:

**Proposition 5.2.25.** *If  $F : \mathcal{K} \rightarrow \mathcal{L}$  is a morphism of tensor triangulated categories and  $X \in \mathcal{K}$  then there is a commutative diagram*

$$\begin{array}{ccccc} \text{Spc}(\mathcal{L}) \supset \text{supp}_{\mathcal{L}}(FX) & \xrightarrow{\rho_{\mathcal{L},FX}} & \text{Spec}(R_{\mathcal{L},FX}) \\ \downarrow & & \downarrow \\ \text{Spc}(\mathcal{K}) \supset \text{supp}_{\mathcal{K}}(X) & \xrightarrow{\rho_{\mathcal{K},X}} & \text{Spec}(R_{\mathcal{K},X}) \end{array}$$

*in the category of spectral spaces, where the left square is cartesian.*

**Proposition 5.2.26.** *If  $F : \mathcal{K} \rightarrow \mathcal{L}$  is a morphism of small tensor triangulated categories and  $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$  is a closed subset then there is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spc}(\mathcal{L}) \supset \phi^{-1}(\mathcal{Z}) & \xrightarrow{\rho_{\mathcal{L}, \phi^{-1}(\mathcal{Z})}} & \mathrm{Spec}(R_{\mathcal{L}, \phi^{-1}(\mathcal{Z})}) \\ \downarrow \phi & & \downarrow \\ \mathrm{Spc}(\mathcal{K}) \supset \mathcal{Z} & \xrightarrow{\rho_{\mathcal{K}, \mathcal{Z}}} & \mathrm{Spec}(R_{\mathcal{K}, \mathcal{Z}}) \end{array}$$

*in the category of spectral spaces, where the left square is cartesian.*

*Remark 5.2.27.* Considering  $\mathrm{supp}_{\mathcal{K}}(X)$  and  $\mathrm{Spec}(R_{\mathcal{K}, X})$  as contravariant functors from the category of essentially small tensor triangulated categories with chosen object to the category of spectral spaces, the object comparison maps  $\rho_{\mathcal{K}, X}$  can be regarded as a natural transformation  $\mathrm{supp}_{\mathcal{K}}(X) \rightarrow \mathrm{Spec}(R_{\mathcal{K}, X})$ . Similarly, there is a contravariant “forgetful” functor  $(\mathcal{K}, \mathcal{Z}) \mapsto \mathcal{Z}$  from the category of small tensor triangulated categories with chosen closed subset of their spectrum to the category of spectral spaces, and the closed set comparison maps  $\rho_{\mathcal{K}, \mathcal{Z}}$  form a natural transformation from this functor to the functor  $(\mathcal{K}, \mathcal{Z}) \mapsto \mathrm{Spec}(R_{\mathcal{K}, \mathcal{Z}})$ . Finally, the general comparison map  $\rho_{\mathcal{K}, \Phi}$  can be regarded as a natural transformation from  $(\mathcal{K}, \Phi) \mapsto \mathcal{Z}_{\Phi}$  to  $(\mathcal{K}, \Phi) \mapsto \mathrm{Spec}(R_{\mathcal{K}, \Phi})$ .

*Remark 5.2.28.* It is straightforward to develop the graded version of these constructions. One checks that the graded analogue of Corollary 5.2.2 holds and then defines  $R_{\Phi}^{\bullet}$  to be the colimit of the diagram of graded rings generated by the maps  $E_X^{\bullet} \rightarrow E_{X \otimes Y}^{\bullet}$  and  $E_Y^{\bullet} \rightarrow E_{X \otimes Y}^{\bullet}$ . One checks that this is a filtered colimit (in the weak sense—see Remark 5.2.10) and it is easily determined how filtered colimits of graded rings are constructed. To be clear, the abelian group  $R_{\Phi}^i$  is the filtered colimit of abelian groups  $\mathrm{colim}_{X \in \Phi} E_X^i$  and thus consists of equivalence classes  $[f]$  where  $f \in E_X^i$  for some  $X \in \Phi$ . The product on  $R_{\Phi}^{\bullet}$  is given by

$$\begin{aligned} \mathrm{colim}_{X \in \Phi} E_X^i \times \mathrm{colim}_{Y \in \Phi} E_Y^j &\longrightarrow \mathrm{colim}_{Z \in \Phi} E_Z^{i+j} \\ ([f_X], [g_Y]) &\longmapsto [(f_X \otimes Y) \cdot (X \otimes g_Y)] \end{aligned}$$

where  $(f_X \otimes Y) \cdot (X \otimes g_Y)$  is the graded product in  $E_{X \otimes Y}^\bullet$ . Note that  $R_\Phi^0$  is exactly the ungraded ring  $R_\Phi$  from Definition 5.2.3. It is straightforward to show that  $R_\Phi^\bullet$  is graded-commutative although one needs to be clear about our abuses of notation concerning the suspension isomorphisms. Ultimately the graded-commutativity comes from the anti-commutativity of diagram (3.3.6) in the axioms of a tensor triangulated category.

The proof of the following theorem is very similar to the proof of Theorem 5.1.16 just with the kind of modifications we saw in the proof of Theorem 5.2.15.

**Theorem 5.2.29.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $\Phi \subset \mathcal{K}$  be a non-empty set of objects closed under the  $\otimes$ -product. There is a graded-commutative graded ring  $R_\Phi^\bullet$  and an inclusion-reversing, spectral map*

$$\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \mathrm{Spec}^h(R_\Phi^\bullet)$$

defined by  $\rho_\Phi^\bullet(\mathcal{P}) := \{[f] \in R_\Phi^i \mid \mathrm{cone}(f) \notin \mathcal{P}\}_{i \in \mathbb{Z}}$ . The ring  $R_\Phi$  is precisely  $R_\Phi^0$  and  $\mathfrak{p}^\bullet \mapsto \mathfrak{p}^\bullet \cap R_\Phi^0$  defines a surjective spectral map  $\mathrm{Spec}^h(R_\Phi^\bullet) \rightarrow \mathrm{Spec}(R_\Phi)$  such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{Z}_\Phi & \xrightarrow{\rho_\Phi^\bullet} & \mathrm{Spec}^h(R_\Phi^\bullet) \\ & \searrow \rho_\Phi & \downarrow (-)^0 \\ & & \mathrm{Spec}(R_\Phi). \end{array} \tag{5.2.30}$$

*Remark 5.2.31.* The graded comparison maps have the same kind of naturality properties as the ungraded comparison maps (cf. Remark 5.2.27).

### 5.3 Object comparison maps

In this section we will establish some of the basic features of the natural “object” comparison maps  $\rho_X : \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_X)$  defined in Example 5.2.16. More specifically, our primary goal is to establish that  $\rho_X$  is invariant under some natural operations that can be

performed on the object  $X$  such as taking duals, or suspensions, or  $\otimes$ -powers, etc. Before we begin proving such results, let us remark that for  $X = \mathbb{1}$  the canonical map  $[\mathbb{1}, \mathbb{1}] = E_{\mathbb{1}} \rightarrow R_{\mathbb{1}}$  is an isomorphism and under this identification  $\rho_{\mathbb{1}} : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(R_{\mathbb{1}})$  is the original unit comparison map from [Bal10a]; similarly for the graded version.

**Proposition 5.3.1.** *There is a canonical isomorphism of rings*

$$R_X \cong \mathrm{colim}(E_X \xrightarrow{X \otimes -} E_{X^{\otimes 2}} \xrightarrow{X \otimes -} E_{X^{\otimes 3}} \xrightarrow{X \otimes -} E_{X^{\otimes 4}} \xrightarrow{X \otimes -} \dots) \quad (5.3.2)$$

induced by the canonical maps  $E_{X^{\otimes n}} \rightarrow R_X$ .

*Proof.* Let  $n \geq 1$ . It is straightforward to check that the canonical map  $E_{X^{\otimes n}} \rightarrow R_X$  is a ring homomorphism. Indeed,  $[f] + [g] = [f \otimes X^{\otimes n} + X^{\otimes n} \otimes g] = [X^{\otimes n} \otimes f + X^{\otimes n} \otimes g] = [X^{\otimes n} \otimes (f + g)] = [f + g]$  and  $[f] \cdot [g] = [(f \otimes X^{\otimes n}) \circ (X^{\otimes n} \otimes g)] = [(X^{\otimes n} \otimes f) \circ (X^{\otimes n} \otimes g)] = [X^{\otimes n} \otimes (f \circ g)] = [f \circ g]$  for  $f, g \in E_{X^{\otimes n}}$ . Now, it follows from Corollary 5.2.2 that we have a filtered diagram

$$E_X \xrightarrow{X \otimes -} E_{X^{\otimes 2}} \xrightarrow{X \otimes -} E_{X^{\otimes 3}} \xrightarrow{X \otimes -} E_{X^{\otimes 4}} \xrightarrow{X \otimes -} \dots$$

and since  $[f_{X^{\otimes n}}] = [X \otimes f_{X^{\otimes n}}]$  in  $R_X$ , the maps  $E_{X^{\otimes n}} \rightarrow R_X$  induce a ring homomorphism  $\mathrm{colim}_{n \geq 1} E_{X^{\otimes n}} \rightarrow R_X$ . This homomorphism is evidently surjective. On the other hand, if  $[f_{X^{\otimes n}}] = 0$  in  $R_X$  then by Proposition 5.2.9 there exists  $m \geq 1$  such that  $X^{\otimes m} \otimes f = 0$  so that  $f = 0$  in  $\mathrm{colim}_{n \geq 1} E_{X^{\otimes n}}$ .  $\square$

In the rest of this section we will often tacitly make the identification  $R_X = \mathrm{colim}_{n \geq 1} E_{X^{\otimes n}}$ .

**Lemma 5.3.3.** *If  $f \in E_X$  then  $\mathrm{thick}_{\otimes} \langle \mathrm{cone}(f) \rangle \subset \{a \in \mathcal{K} \mid a \otimes f^{\otimes n} = 0 \text{ for some } n \geq 1\}$ .*

*Proof.* We check that the right-hand side is a thick  $\otimes$ -ideal. That it is closed under  $\Sigma$ ,  $\Sigma^{-1}$ ,  $\oplus$ -sums, and  $\oplus$ -summands is clear from  $\Sigma a \otimes f^{\otimes n} \simeq \Sigma(a \otimes f^{\otimes n})$  and  $(a \oplus b) \otimes f^{\otimes n} \simeq (a \otimes f^{\otimes n}) \oplus (b \otimes f^{\otimes n})$ . On the other hand, suppose  $a \rightarrow b \rightarrow c \rightarrow \Sigma a$  is an exact triangle and that  $a$  and

$b$  are contained in the right-hand side. Then there exists  $n \geq 1$  such that  $a \otimes f^{\otimes n} = 0$  and  $b \otimes f^{\otimes n} = 0$ . We then have an endomorphism of exact triangles

$$\begin{array}{ccccccc} a \otimes X^{\otimes n} & \longrightarrow & b \otimes X^{\otimes n} & \longrightarrow & c \otimes X^{\otimes n} & \longrightarrow & \Sigma(a \otimes X^{\otimes n}) \\ \downarrow a \otimes f^{\otimes n} & & \downarrow b \otimes f^{\otimes n} & & \downarrow c \otimes f^{\otimes n} & & \downarrow \Sigma(a \otimes f^{\otimes n}) \\ a \otimes X^{\otimes n} & \longrightarrow & b \otimes X^{\otimes n} & \longrightarrow & c \otimes X^{\otimes n} & \longrightarrow & \Sigma(a \otimes X^{\otimes n}) \end{array}$$

where three of the vertical morphisms are zero. By the proof of Lemma 3.1.8, it follows that the remaining vertical arrow squares to zero:  $(c \otimes f^{\otimes n})^2 = 0$ . Since

$$\begin{aligned} c \otimes f^{\otimes 2n} &= (c \otimes f^{\otimes n} \otimes X^{\otimes n}) \circ (c \otimes X^{\otimes n} \otimes f^{\otimes n}) \\ &= (c \otimes f^{\otimes n} \otimes X^{\otimes n}) \circ (c \otimes f^{\otimes n} \otimes X^{\otimes n}) \\ &= (c \otimes f^{\otimes n})^2 \otimes X^{\otimes n} \\ &= 0 \end{aligned}$$

we conclude that  $c$  is also in the right-hand side. This establishes that it is closed under cofibers and it is evidently also a  $\otimes$ -ideal. In summary, the right-hand side is a thick  $\otimes$ -ideal. It contains  $\text{cone}(f)$  by Lemma 5.1.3 and the inclusion in the statement follows.  $\square$

**Proposition 5.3.4.** *An isomorphism  $\alpha : X \xrightarrow{\sim} Y$  in  $\mathcal{K}$  induces an isomorphism of rings*

$$\alpha_* : R_X \xrightarrow{\sim} R_Y$$

*and under this identification  $\rho_X$  coincides with  $\rho_Y$ .*

*Proof.* This is routine from the definitions. One readily verifies that the induced isomorphisms  $(\alpha^{\otimes n})_* : E_{X^{\otimes n}} \rightarrow E_{Y^{\otimes n}}$  given by Lemma 5.2.5 commute with the  $X \otimes -$  and  $Y \otimes -$  maps, and hence induce an isomorphism  $\text{colim}_{n \geq 1} E_{X^{\otimes n}} \rightarrow \text{colim}_{n \geq 1} E_{Y^{\otimes n}}$ . Verifying that  $\rho_X = \rho_Y$  under this identification amounts to checking the definitions and noting that if  $f$  is an endomorphism of an object  $X$  and  $\alpha : X \xrightarrow{\sim} Y$  is an isomorphism then  $\text{cone}(f) \simeq \text{cone}(\alpha \circ f \circ \alpha^{-1})$ .  $\square$

**Proposition 5.3.5.** *Tensoring on the right by an object  $Y$  induces a ring homomorphism  $R_X \rightarrow R_{X \otimes Y}$  and a commutative diagram*

$$\begin{array}{ccc} \text{supp}(X) & \xrightarrow{\rho_X} & \text{Spec}(R_X) \\ \uparrow & & \uparrow \\ \text{supp}(X) \cup \text{supp}(Y) & \xrightarrow{\rho_{X \otimes Y}} & \text{Spec}(R_{X \otimes Y}). \end{array} \quad (5.3.6)$$

*If  $\text{supp}(X) \subset \text{supp}(Y)$  then the kernel of the map  $R_X \rightarrow R_{X \otimes Y}$  consists entirely of nilpotents. There is a similar result for tensoring on the left.*

*Proof.* Note that there is a canonical isomorphism  $X^{\otimes n} \otimes Y^{\otimes n} \xrightarrow{\sim} (X \otimes Y)^{\otimes n}$  obtained from the symmetry that preserves the order of the  $X$ 's and the order of the  $Y$ 's. One can then define a ring homomorphism  $E_{X^{\otimes n}} \rightarrow E_{(X \otimes Y)^{\otimes n}}$  as the composition of  $- \otimes Y^{\otimes n} : E_{X^{\otimes n}} \rightarrow E_{X^{\otimes n} \otimes Y^{\otimes n}}$  and the isomorphism  $E_{X^{\otimes n} \otimes Y^{\otimes n}} \xrightarrow{\sim} E_{(X \otimes Y)^{\otimes n}}$  and one readily verifies that these maps induce a homomorphism  $R_X \rightarrow R_{X \otimes Y}$ . That (5.3.6) commutes follows from the definitions, observing that if  $\mathcal{P} \in \text{supp}(Y)$  then  $a \in \mathcal{P}$  iff  $a \otimes Y \in \mathcal{P}$ . Finally, if  $[f] \in R_X$  is mapped to zero in  $R_{X \otimes Y}$  then  $X^{\otimes i} \otimes Y^{\otimes j} \otimes f = 0$  for some  $i, j \geq 1$ . It follows from Corollary 3.1.15 and the condition  $\text{supp}(X) \subset \text{supp}(Y)$  that  $\text{supp}(X) \cap \text{supp}(\text{cone}(f)) = \text{supp}(X)$  and hence that  $\text{supp}(X) \subset \text{supp}(\text{cone}(f))$ . It then follows from Lemma 4.2.6 that  $X^{\otimes k} \in \text{thick}_{\otimes} \langle \text{cone}(f) \rangle$  for some  $k \geq 1$ . Lemma 5.3.3 then implies that  $X^{\otimes k} \otimes f^{\otimes n} = 0$  for some  $n \geq 1$  and we conclude that  $[f]$  is a nilpotent element of  $R_X$ .  $\square$

*Example 5.3.7.* For every object  $X$  we have a map  $[\mathbb{1}, \mathbb{1}] \rightarrow R_X$  whose kernel consists of nilpotent elements and a commutative diagram:

$$\begin{array}{ccc} \text{Spc}(\mathcal{K}) & \longrightarrow & \text{Spec}([\mathbb{1}, \mathbb{1}]) \\ \uparrow & & \uparrow \\ \text{supp}(X) & \longrightarrow & \text{Spec}(R_X). \end{array}$$

**Proposition 5.3.8.** *For every pair of objects  $X$  and  $Y$  and every integer  $k \geq 1$ , tensoring on the left by  $X^{\otimes(k-1)}$  induces an isomorphism  $R_{X \otimes Y} \xrightarrow{\sim} R_{X^{\otimes k} \otimes Y}$ . Under this identification the maps  $\rho_{X \otimes Y}$  and  $\rho_{X^{\otimes k} \otimes Y}$  coincide.*

*Proof.* Since the homomorphisms  $R_{X \otimes Y} \hookrightarrow R_{X^{\otimes 2} \otimes Y} \hookrightarrow R_{X^{\otimes 3} \otimes Y} \hookrightarrow \dots$  induced by  $X \otimes -$  are evidently injective, the problem reduces to showing that  $R_{X \otimes Y} \rightarrow R_{X^{\otimes 2} \otimes Y}$  is surjective. In other words, we need to show that every  $f \in E_{(X^{\otimes 2} \otimes Y)^{\otimes n}}$  is equivalent in  $R_{X^{\otimes 2} \otimes Y}$  to an element coming from  $R_{X \otimes Y}$ . We'll give the proof under the assumption that  $n = 1$ . The proof for arbitrary  $n \geq 1$  is similar.

Consider the element  $g \in E_{(X \otimes Y)^{\otimes 3}}$  corresponding to  $X \otimes f \otimes Y^{\otimes 2} \in E_{X^{\otimes 3} \otimes Y^{\otimes 3}}$ . Under the map  $R_{X \otimes Y} \rightarrow R_{X^{\otimes 2} \otimes Y}$ ,  $[g]$  is sent to the image in  $R_{X^{\otimes 2} \otimes Y}$  of the element in  $E_{(X^{\otimes 2} \otimes Y)^{\otimes 3}}$  corresponding to  $X^{\otimes 4} \otimes f \otimes Y^{\otimes 2} \in E_{X^{\otimes 6} \otimes Y^{\otimes 3}}$ . We claim that this element in  $E_{(X^{\otimes 3} \otimes Y)^{\otimes 3}}$  equals  $(X^{\otimes 2} \otimes Y)^{\otimes 2} \otimes f$  so that the image of  $[g]$  in  $R_{X^{\otimes 2} \otimes Y}$  equals  $[f]$ . This is not completely obvious and involves an unilluminating trick. In order to describe this trick, let's write  $a := X^{\otimes 2}$  and  $b := Y$  for simplicity of notation; so  $f \in E_{a \otimes b}$ . We will use subscripts to indicate position and we'll drop the tensors from the notation. Consider the diagram

$$\begin{array}{ccc}
 a_1 b_1 a_2 b_2 a_3 b_3 & \xrightleftharpoons[\text{abfab}]{\text{ababf}} & a_1 b_1 a_2 b_2 a_3 b_3 \\
 \downarrow \sim & & \downarrow \sim \\
 a_1 a_2 b_2 a_3 b_1 b_3 & \xrightleftharpoons[\text{a}^2 b f b]{\text{a f a b}^2} & a_1 a_2 b_2 a_3 b_1 b_3 \\
 \downarrow \sim & & \downarrow \sim \\
 a_1 a_2 a_3 b_1 b_2 b_3 & \xrightarrow{\text{a}^2 f b^2} & a_1 a_2 a_3 b_1 b_2 b_3
 \end{array}$$

where the vertical maps are the indicated permutations of the factors induced by the symmetry. Note that the composition of the two vertical permutations is the unique permutation from the source to the target that preserves the order of the  $a$ 's and the order of the  $b$ 's. The top arrow is  $(X^2 \otimes Y)^{\otimes 2} \otimes f$  and the bottom arrow is the morphism that it corresponds to in  $E_{X^{\otimes 6} \otimes Y^{\otimes 3}}$ . The commutativity of the diagram verifies that this equals  $X^{\otimes 4} \otimes f \otimes Y^{\otimes 2}$  as

claimed. □

**Proposition 5.3.9.** *If  $\mathcal{K}$  is a rigid tensor triangulated category then for every object  $X$  in  $\mathcal{K}$  there is a canonical isomorphism  $R_X \simeq R_{DX}$  under which the map  $\rho_X$  coincides with  $\rho_{DX}$ .*

*Proof.* The duality functor  $D : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$  gives a ring isomorphism  $[X, X] \xrightarrow{\sim} [DX, DX]^{\text{op}}$  and an easy application of the fact that  $D$  is a strong  $\otimes$ -functor shows that the isomorphism restricts to an isomorphism  $E_X \xrightarrow{\sim} E_{DX}^{\text{op}}$ . It is straightforward but tedious to verify that these isomorphisms induce an isomorphism  $R_X \xrightarrow{\sim} R_{DX}^{\text{op}} = R_{DX}$ . Showing that  $\rho_X$  and  $\rho_{DX}$  correspond amounts to showing that  $\text{supp}(\text{cone}(Df)) = \text{supp}(D(\text{cone}(f)))$ . Here we use the fact that  $D$  is an exact functor of triangulated categories and the fact that  $\text{supp}(DX) = \text{supp}(X)$  in any rigid category (cf. [Bal07, Proposition 2.7]). □

Our next goal is to establish that  $\rho_X = \rho_{X \oplus X}$ . Note that under the usual identification of  $(X \oplus X)^{\otimes n}$  with a  $\oplus$ -sum of  $2^n$  copies of  $X^{\otimes n}$  an endomorphism  $(X \oplus X)^{\otimes n} \rightarrow (X \oplus X)^{\otimes n}$  can be regarded as a  $2^n \times 2^n$  matrix  $(f_{ij})$  with entries  $f_{ij} : X^{\otimes n} \rightarrow X^{\otimes n}$ . We will make such identifications without further comment.

**Lemma 5.3.10.** *An endomorphism  $f = (f_{ij}) : a_1 \oplus \cdots \oplus a_n \rightarrow a_1 \oplus \cdots \oplus a_n$  is contained in  $E_{a_1 \oplus \cdots \oplus a_n}$  if and only if*

1.  $a_i \otimes f_{jj} = f_{ii} \otimes a_j$  for all  $i, j$ , and
2.  $f_{ij} \otimes (a_1 \oplus \cdots \oplus a_n) = 0$  for  $i \neq j$ .

*Proof.* Observe that  $f \otimes (a_1 \oplus \cdots \oplus a_n)$ , viewed as an  $n^2 \times n^2$  matrix, consists of  $n \times n$  blocks, each of which is diagonal, while  $(a_1 \oplus \cdots \oplus a_n) \otimes f$  consists of  $n \times n$  blocks, arranged along the diagonal. Equating the off-diagonal blocks gives the condition that  $f_{ij} \otimes a_k = 0$  for all  $k$  if  $i \neq j$ , which is equivalent to condition (2). Similarly, equating the off-diagonals of the

diagonal blocks gives the equivalent condition that  $a_k \otimes f_{ij} = 0$  for all  $k$  if  $i \neq j$ . On the other hand, the diagonal of the  $i$ th diagonal block gives the condition that  $f_{ii} \otimes a_j = a_i \otimes f_{jj}$  for all  $j$ .  $\square$

**Corollary 5.3.11.** *If  $f \in E_{(X \oplus X)^{\otimes n}}$  then there exists some  $\alpha \in E_{X^{\otimes n}}$  such that  $(X \oplus X)^{\otimes n} \otimes f$  (regarded as a matrix of endomorphisms of  $X^{\otimes 2n}$ ) is diagonal with copies of  $X^{\otimes n} \otimes \alpha$  along the diagonal.*

*Proof.* This follows from Lemma 5.3.10. We can take  $\alpha := f_{11}$  for example.  $\square$

**Proposition 5.3.12.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $X$  be an object in  $\mathcal{K}$ . There is a canonical isomorphism  $R_X \simeq R_{X \oplus X}$  under which  $\rho_X$  coincides with  $\rho_{X \oplus X}$ .*

*Proof.* Invoking Lemma 5.3.10, we see that there is a ring homomorphism  $\Delta : E_{X^{\otimes n}} \hookrightarrow E_{(X \oplus X)^{\otimes n}}$  for each  $n \geq 1$  which sends  $f \in E_{X^{\otimes n}}$  to the diagonal matrix consisting of copies of  $f$  on the diagonal. One checks that these maps commute with  $X \otimes -$  and  $(X \oplus X) \otimes -$  and therefore induce an injection  $R_X \hookrightarrow R_{X \oplus X}$ . Surjectivity of this map follows from Corollary 5.3.11. That  $\rho_X$  and  $\rho_{X \oplus X}$  correspond boils down to the definitions and the fact that  $\text{cone}(f \oplus \cdots \oplus f) \simeq \text{cone}(f) \oplus \cdots \oplus \text{cone}(f)$ .  $\square$

A similar argument shows that  $\rho_X = \rho_{X \oplus \Sigma X}$  after a canonical identification  $R_X \simeq R_{X \oplus \Sigma X}$ . More generally:

**Proposition 5.3.13.** *Let  $\mathcal{K}$  be a tensor triangulated category and let  $X$  be an object of  $\mathcal{K}$ . If  $Y$  is a  $\oplus$ -sum of suspensions of  $X$  then  $\rho_X = \rho_Y$  after a canonical identification  $R_X \simeq R_Y$ .*

*Proof.* The proof is a more advanced version of the proof of Proposition 5.3.12. Observe that  $(\Sigma^{i_1} X \oplus \cdots \oplus \Sigma^{i_k} X)^{\otimes n}$  may be identified with a  $\oplus$ -sum of suspensions of  $X^{\otimes n}$ . One may define a “diagonal” map  $E_{X^{\otimes n}} \hookrightarrow E_{(\Sigma^{i_1} X \oplus \cdots \oplus \Sigma^{i_k} X)^{\otimes n}}$  which sends  $f$  to a diagonal matrix whose diagonal

entries are copies of  $f$  suspended the appropriate numbers of times. One checks that these maps induce a map  $R_X \hookrightarrow R_{(\Sigma^{i_1} X \oplus \dots \oplus \Sigma^{i_k} X)}$  and a similar argument shows that this map is in fact surjective.  $\square$

*Remark 5.3.14.* There are graded versions of all of the above results, establishing that  $\rho_X^\bullet$  is invariant under suspension, tensor powers, and so on. The only result for which we should be careful is taking duals. The duality induces an isomorphism  $R_X^\bullet \simeq R_{DX}^{\bullet, \text{op}}$  but we can't remove the op because  $R_{DX}^\bullet$  is only graded-commutative. In any case, there is a canonical homeomorphism  $\text{Spec}^h(R_{DX}^{\bullet, \text{op}}) \simeq \text{Spec}^h(R_{DX}^\bullet)$  and under these identifications  $\rho_X^\bullet$  coincides with  $\rho_{DX}^\bullet$ .

*Example 5.3.15.* For any object  $X \in \mathcal{K}$  there is a homomorphism  $[\mathbb{1}, \mathbb{1}]_\bullet \rightarrow E_X^\bullet$  which sends  $\alpha$  to  $\alpha \otimes X = X \otimes \alpha$ . These induce a homomorphism  $R_\mathbb{1}^\bullet \rightarrow R_\Phi^\bullet$  for any  $\otimes$ -multiplicative set  $\Phi \subset \mathcal{K}$  whose kernel consists of nilpotents (cf. Proposition 5.3.5). If  $\Phi$  is taken to be the collection of objects that are isomorphic to direct sums of suspensions of  $\mathbb{1}$  then this map is an isomorphism (cf. Proposition 5.3.13). For example, if  $\mathcal{K} = \text{D}_{\text{perf}}(k)$  for a field  $k$  then every object is a direct sum of suspensions of  $\mathbb{1}$  and the “only” comparison map is the original unit comparison map  $\rho_\mathbb{1}^\bullet : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}^h([\mathbb{1}, \mathbb{1}]_\bullet)$  which, moreover, coincides with the ungraded unit comparison map  $\rho_\mathbb{1} : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}([\mathbb{1}, \mathbb{1}])$ . More generally, it would be interesting to know whether  $R_\mathbb{1}^\bullet \rightarrow R_\Phi^\bullet$  is an isomorphism (under suitable generation hypotheses) when  $\Phi$  is the collection of solid objects—those objects  $X$  with  $\text{supp}(X) = \text{Spc}(\mathcal{K})$ . In other words, it would be interesting to know whether the closed set comparison map  $\rho_{\text{Spc}(\mathcal{K})}^\bullet$  associated with the whole spectrum reduces to  $\rho_\mathbb{1}^\bullet$  under suitable hypotheses.

*Remark 5.3.16.* Recall from the proof of Theorem 5.1.9 that under the unnatural comparison map  $\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$  the preimage of a Thomason closed subset  $V(a_1, \dots, a_n) \subset \text{Spec}(A)$  is exactly the support of the “Koszul” object  $\text{cone}(\alpha(a_1)) \otimes \dots \otimes \text{cone}(\alpha(a_n))$ . On the other hand, for our natural comparison map  $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$  the elements of  $R_X$

are equivalence classes of endomorphisms, but one still finds that

$$\rho_X^{-1}(V([f_1], \dots, [f_n])) = \text{supp}(\text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n))$$

independent of the choice of representatives  $f_i$ . Nevertheless, a different set of representatives  $[f'_1], \dots, [f'_n]$  gives a different Koszul object  $\text{cone}(f'_1) \otimes \dots \otimes \text{cone}(f'_n)$  and there is no reason *a priori* for the comparison maps of these two Koszul objects to coincide. However,  $X^{\otimes i} \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n) \simeq X^{\otimes j} \otimes \text{cone}(f'_1) \otimes \dots \otimes \text{cone}(f'_n)$  for some  $i, j \geq 1$  and it follows from Proposition 5.3.8 that the comparison map for  $X \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n)$  does not depend on the choice of representatives  $f_i$ . Thus when one decides to examine a closed set  $\text{supp}(X_0)$  more closely by choosing generators of a Thomason closed subset  $V([f_1], \dots, [f_n]) \subset \text{Spec}(R_{X_0})$ , it is advisable to take the generator of the preimage  $\rho_{X_0}^{-1}(V([f_1], \dots, [f_n]))$  to be  $X_1 := X_0 \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n)$ . A serendipitous consequence of including  $X_0$  as a  $\otimes$ -factor is that we then have a ring homomorphism  $R_{X_0} \rightarrow R_{X_1}$  and a commutative diagram

$$\begin{array}{ccc} \text{supp}(X_0) & \xrightarrow{\rho_{X_0}} & \text{Spec}(R_{X_0}) \\ \uparrow & & \uparrow \\ & V([f_1], \dots, [f_n]) & \\ \text{supp}(X_1) & \xrightarrow{\rho_{X_1}} & \text{Spec}(R_{X_1}). \end{array}$$

On the other hand, this procedure still apparently depends on the choice of generators for the Thomason closed subset  $V([f_1], \dots, [f_n])$ .

## 5.4 Universal property

For any commutative ring  $A$  and ring homomorphism  $A \rightarrow R_\Phi$  one obtains an inclusion-reversing, spectral map  $\mathcal{Z}_\Phi \rightarrow \text{Spec}(A)$  by composing  $\rho_\Phi$  with the induced map  $\text{Spec}(R_\Phi) \rightarrow \text{Spec}(A)$ . For example, the comparison maps  $\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$  defined in Section 5.1 are recovered from  $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$  by composing  $A \rightarrow E_X$  with the canonical map

$E_X \rightarrow R_X$ . In fact, the ring  $R_\Phi$  can be regarded as a colimit of all the commutative rings  $A$  mapping into the rings  $E_X$  for  $X \in \Phi$ . The purpose of this section is to make this precise. To this end, consider triples  $(A, \alpha, X)$  where  $A$  is a commutative ring,  $\alpha : A \rightarrow E_X$  is a ring homomorphism, and  $X$  is an object of  $\Phi$ . Define a morphism  $(A, \alpha, X) \rightarrow (A', \alpha', X')$  to be a morphism  $u : A \rightarrow A'$  such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & E_X \\ u \downarrow & & \searrow \\ A' & \xrightarrow{\alpha'} & E_{X'} \end{array} \quad \begin{array}{c} \\ \\ \nearrow \\ \end{array} \quad \begin{array}{c} \\ \\ E_{X \otimes a \otimes X'} \end{array}$$

commutes for some object  $a \in \Phi$ . (If the object  $a$  were not included then one would run into difficulties composing such morphisms.) Note that there is a functor  $(A, \alpha, X) \mapsto A$  from the category of triples to the category of rings and we can consider the ring  $\text{colim}_{(A, \alpha, X)} A$ . The following proposition establishes that this is a filtered colimit.

**Proposition 5.4.1.** *The category of triples  $(A, \alpha, X)$  is a filtered category.*

*Proof.* Given two triples  $(A, \alpha, X)$  and  $(A', \alpha', X')$ , consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & E_X \\ & & \searrow - \otimes X' \\ & & E_{X \otimes X'} \\ A' & \xrightarrow{\alpha'} & E_{X'} \\ & & \nearrow X \otimes - \end{array}$$

The image of  $A$  in  $E_{X \otimes X'}$  is a commutative subring consisting of endomorphisms of the form  $f \otimes X'$  while the image of  $A'$  in  $E_{X \otimes X'}$  is a commutative subring consisting of endomorphisms of the form  $X \otimes g$ . Since such endomorphisms commute, the subring  $B$  generated by the images of  $A$  and  $A'$  in  $E_{X \otimes X'}$  is commutative. It provides a triple  $(B, i, X \otimes X')$  which admits morphisms  $(A, \alpha, X) \rightarrow (B, i, X \otimes X')$  and  $(A', \alpha', X') \rightarrow (B, i, X \otimes X')$ . Next consider two parallel morphisms  $u, v : (A, \alpha, X) \rightarrow (B, \beta, Y)$  in the category of triples. Then we can construct a

commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\alpha} & E_X & \searrow & \\
 u \downarrow & & & & E_{X \otimes a \otimes Y} \\
 B & \xrightarrow{\beta} & E_Y & \searrow & \\
 v \uparrow & & & & E_{Y \otimes b \otimes X} \\
 A & \xrightarrow{\alpha} & E_X & \searrow & \\
 & & & & E_{X \otimes a \otimes Y \otimes b \otimes X}
 \end{array}$$

Let  $\omega : B \rightarrow E_{X \otimes a \otimes Y \otimes b \otimes X}$  denote the composite from  $E_Y$  to  $E_{X \otimes a \otimes Y \otimes b \otimes X}$  displayed in the diagram. Then  $(\text{im } \omega, i, X \otimes a \otimes Y \otimes b \otimes X)$  is a triple with an obvious morphism  $(B, \beta, Y) \rightarrow (\text{im } \omega, i, X \otimes a \otimes Y \otimes b \otimes X)$  and one readily checks that  $\omega \circ u = \omega \circ v$ .  $\square$

**Proposition 5.4.2.** *The ring homomorphisms  $A \xrightarrow{\alpha} E_X \rightarrow R_\Phi$  induce an isomorphism of rings  $\text{colim}_{(A, \alpha, X)} A \rightarrow R_\Phi$ .*

*Proof.* That the maps  $A \xrightarrow{\alpha} E_X \rightarrow R_\Phi$  induce a map  $\text{colim}_{(A, \alpha, X)} A \rightarrow R_\Phi$  is made clear by the commutativity of

$$\begin{array}{ccccc}
 A & \longrightarrow & E_X & \searrow & \\
 u \downarrow & & & & E_{X \otimes a \otimes X'} \\
 A' & \longrightarrow & E_{X'} & \searrow & \\
 & & & & R_\Phi
 \end{array}$$

We claim that the induced ring homomorphism  $\text{colim}_{(A, \alpha, X)} A \rightarrow R_\Phi$  is an isomorphism. For surjectivity, one observes that if  $[f_X] \in R_\Phi$  then the subring  $\mathbb{Z}[f_X] \subset E_X$  generated by  $f_X$  is commutative and provides a triple  $(\mathbb{Z}[f_X], i, X)$ . By definition, the diagram

$$\begin{array}{ccc}
 \text{colim}_{(A, \alpha, X)} A & \longrightarrow & R_\Phi \\
 \uparrow & & \uparrow \\
 \mathbb{Z}[f_X] & \xrightarrow{i} & E_X
 \end{array}$$

commutes, which shows that  $[f_X]$  is in the image of the map  $\text{colim}_{(A, \alpha, X)} A \rightarrow R_\Phi$ . To show that the map is injective we will use the fact that the category of triples is filtered. It follows that every element  $[x] \in \text{colim}_{(A, \alpha, X)} A$  is the image of an element  $x \in A$  under the canonical

map  $A \rightarrow \operatorname{colim}_{(A,\alpha,X)} A$  associated with a triple  $(A, \alpha, X)$ . If the element  $[x] \in \operatorname{colim}_{(A,\alpha,X)} A$  goes to zero in  $R_\Phi$  then  $[\alpha(x)] = 0$  in  $R_\Phi$  and so there is some  $a \in \Phi$  such that  $a \otimes \alpha(x) = 0$ . Then let  $\beta$  denote the composite  $A \xrightarrow{\alpha} E_X \xrightarrow{a \otimes -} E_{a \otimes X}$  and consider the following diagram

$$\begin{array}{ccccc}
 \operatorname{colim}_{(A,\alpha,X)} A & \xrightarrow{\quad} & R_\Phi & & \\
 & \swarrow & \nearrow & & \\
 & A & \xrightarrow{\alpha} & E_X & \\
 & \downarrow \text{id} & & \downarrow a \otimes - & \\
 & A & \xrightarrow{\beta} & E_{a \otimes X} & \\
 & \downarrow \pi & & \downarrow \text{id} & \\
 & A/\ker(\beta) & \xrightarrow{\bar{\beta}} & E_{a \otimes X} & 
 \end{array}$$

The induced map  $\bar{\beta} : A/\ker \beta \rightarrow E_{a \otimes X}$  provides a triple  $(A/\ker \beta, \bar{\beta}, a \otimes X)$  while the quotient map  $\pi : A \rightarrow A/\ker \beta$  defines a map of triples  $(A, \alpha, X) \rightarrow (A/\ker \beta, \bar{\beta}, a \otimes X)$ . Since the canonical map  $A \rightarrow \operatorname{colim}_{(A,\alpha,X)} A$  for the triple  $(A, \alpha, X)$  factors through the above map of triples, we conclude that the image of  $x$  in  $\operatorname{colim}_{(A,\alpha,X)} A$  is zero. That is  $[x] = [\pi(x)] = 0$ .  $\square$

*Remark 5.4.3.* From this perspective, the maps defined in Section 5.1 are obtained from  $\rho_\Phi$  by pulling back via the canonical map  $A \rightarrow \operatorname{colim}_{(A,\alpha,X)} A \simeq R_\Phi$  associated to a triple  $(A, \alpha, X)$ .

## 5.5 Idempotent completion

Recall from Section 4.4 that every tensor triangulated category  $\mathcal{K}$  may be embedded into an idempotent complete tensor triangulated category  $\mathcal{K}^\natural$  and that the embedding  $i : \mathcal{K} \hookrightarrow \mathcal{K}^\natural$  induces a homeomorphism of spectra (cf. Proposition 4.4.3). There is a precise sense in which  $\mathcal{K}$  and  $\mathcal{K}^\natural$  admit “the same” theory of higher comparison maps. We begin with the following unsurprising result.

**Proposition 5.5.1.** *For any non-empty  $\otimes$ -multiplicative subset  $\Phi \subset \mathcal{K}$ , there is a canonical identification  $R_{\mathcal{K},\Phi} \simeq R_{\mathcal{K}^\natural,i(\Phi)}$  while  $\mathcal{Z}_{\mathcal{K},\Phi} \simeq \mathcal{Z}_{\mathcal{K}^\natural,i(\Phi)}$  under the homeomorphism  $i^* : \operatorname{Spc}(\mathcal{K}^\natural) \xrightarrow{\sim}$*

$\mathrm{Spc}(\mathcal{K})$ . Under these identifications,  $\rho_{\mathcal{K},\Phi} = \rho_{\mathcal{K}^\natural, i(\Phi)}$ .

*Proof.* Recall that objects of  $\mathcal{K}^\natural$  are pairs  $(X, e)$  where  $X$  is an object of  $\mathcal{K}$  and  $e : X \rightarrow X$  is an idempotent, while the morphisms  $(X, e) \rightarrow (Y, f)$  are the morphisms  $\alpha : X \rightarrow Y$  in  $\mathcal{K}$  such that  $f \circ \alpha = \alpha = \alpha \circ e$ . The functor  $i$  sends an object  $X$  to  $(X, \mathrm{id}_X)$ , while the tensor is defined by  $(X, e) \otimes (Y, f) = (X \otimes Y, e \otimes f)$ . It is then clear that  $[X, X] = [iX, iX]$ , that  $E_X = E_{iX}$  and more generally that  $R_\Phi = R_{i(\Phi)}$  (note that  $i$  is a strict  $\otimes$ -functor). From the fact that  $\mathrm{supp}_{\mathcal{K}^\natural}(iX) = (i^*)^{-1} \mathrm{supp}_{\mathcal{K}}(X)$  it is clear that  $(i^*)^{-1}(\mathcal{Z}_{\mathcal{K},\Phi}) = \mathcal{Z}_{\mathcal{K}^\natural, i(\Phi)}$  while the fact that  $\mathrm{supp}(\mathrm{cone}(i(\alpha))) = \mathrm{supp}(i(\mathrm{cone}(\alpha))) = (i^*)^{-1}(\mathrm{supp}(\mathrm{cone}(\alpha)))$  for any  $[\alpha] \in R_\Phi$  makes it clear that  $\rho_\Phi = \rho_{i(\Phi)}$ .  $\square$

In particular, this tells us that the object comparison maps for objects in  $\mathcal{K}$  are unaffected when we pass to  $\mathcal{K}^\natural$ . But it still could be possible that in passing to  $\mathcal{K}^\natural$  we get new object comparison maps coming from the new objects in  $\mathcal{K}^\natural$ . However, this is not the case:

**Proposition 5.5.2.** *For any object  $X \in \mathcal{K}^\natural$ , the object  $X \oplus \Sigma X$  is contained in  $\mathcal{K} \subset \mathcal{K}^\natural$ . There is a canonical isomorphism  $R_{\mathcal{K}^\natural, X} \simeq R_{\mathcal{K}, X \oplus \Sigma X}$  and after this identification  $\rho_{\mathcal{K}^\natural, X} = \rho_{\mathcal{K}, X \oplus \Sigma X}$ .*

*Proof.* That  $X \oplus \Sigma X$  is contained in  $\mathcal{K}$  is a well-known fact; see the proof of [Bal05, Proposition 3.13] for example. Our result then follows from Proposition 5.5.1 together with the result of Proposition 5.3.13 which told us that  $\rho_X = \rho_{X \oplus \Sigma X}$ .  $\square$

Next we can ask about the closed set comparison maps.

**Proposition 5.5.3.** *Let  $\mathcal{Z}$  be a closed subset of  $\mathrm{Spc}(\mathcal{K})$  and let  $\mathcal{Z}' = (i^*)^{-1}(\mathcal{Z})$  be the corresponding closed subset of  $\mathrm{Spc}(\mathcal{K}^\natural)$ . There is a canonical isomorphism  $R_{\mathcal{K}, \mathcal{Z}} \simeq R_{\mathcal{K}^\natural, \mathcal{Z}'}$  such that with the identification  $\mathcal{Z} \simeq \mathcal{Z}'$  the comparison map  $\rho_{\mathcal{K}, \mathcal{Z}}$  coincides with  $\rho_{\mathcal{K}^\natural, \mathcal{Z}'}$ .*

*Proof.* Let  $\Phi = \{x \in \mathcal{K} \mid \text{supp}_{\mathcal{K}}(x) \supset \mathcal{Z}\}$  and let  $\Phi' = \{a \in \mathcal{K}^{\natural} \mid \text{supp}_{\mathcal{K}^{\natural}}(a) \supset \mathcal{Z}'\}$ . Then  $i(\Phi) \subset \Phi'$  and we have an induced ring homomorphism  $R_{i(\Phi)} \rightarrow R_{\Phi'}$ . If  $a \in \mathcal{K}^{\natural}$  then  $a \oplus \Sigma a \in i(\mathcal{K})$  as before and it follows that if  $a \in \Phi'$  then  $a \oplus \Sigma a \in i(\Phi)$ . We claim that for any  $a \in \Phi'$ , the diagram

$$\begin{array}{ccc} R_{i(\Phi)} & \longrightarrow & R_{\Phi'} \\ \uparrow & \nearrow & \uparrow \\ R_{a \oplus \Sigma a} & \xleftarrow{\sim} & R_a \end{array}$$

commutes, where the bottom row is the isomorphism obtained in Proposition 5.3.13; it will follow that the map  $R_{i(\Phi)} \rightarrow R_{\Phi'}$  is surjective. The commutativity of the top triangle is immediate. On the other hand, consider some  $[f] \in R_a$ , say with  $f \in E_{a^{\otimes n}}$ . Suppose for starters that  $n = 1$ . Then  $[f]$  maps to  $[f \oplus \Sigma f]$  in  $R_{a \oplus \Sigma a}$  and so the question (in this case) is whether  $[f \oplus \Sigma f] = [f]$  in  $R_{\Phi'}$ . This is readily verified:

$$[f \oplus \Sigma f] = [(f \oplus \Sigma f) \otimes a] = [(f \otimes a) \oplus (\Sigma f \otimes a)] = [(a \otimes f) \oplus (\Sigma a \otimes f)] = [(a \oplus \Sigma a) \otimes f] = [f].$$

For general  $n \geq 1$ , regarding  $(a \oplus \Sigma a)^{\otimes n}$  as a  $\oplus$ -sum of suspensions of  $a \otimes a$ , an element  $f \in E_{a^{\otimes n}}$  is sent to a  $\oplus$ -sum of suspensions of  $f$  and it comes down to showing that  $[f] = [\Sigma^{i_1} f \oplus \Sigma^{i_2} f \oplus \cdots \oplus \Sigma^{i_k} f]$  in  $R_{\Phi'}$  (which can be verified in a similar manner).

On the other hand, if  $[f] \in R_{i(\Phi)}$  is an element that is sent to zero in  $R_{\Phi'}$  then  $a \otimes f = 0$  for some  $a \in \Phi'$  and  $(a \oplus \Sigma a) \otimes f \simeq (a \otimes f) \oplus \Sigma(a \otimes f) = 0$  shows that  $[f] = 0$  in  $R_{i(\Phi)}$ . Therefore the map  $R_{i(\Phi)} \rightarrow R_{\Phi'}$  is also injective. It is clear that under this isomorphism  $\rho_{i(\Phi)} = \rho_{\Phi'}$  while Proposition 5.5.1 implies that  $\rho_{\Phi} = \rho_{i(\Phi)}$  after identifying  $\mathcal{Z} \simeq \mathcal{Z}'$ .  $\square$

In other words, we have established that  $\mathcal{K}$  and  $\mathcal{K}^{\natural}$  give precisely the same object comparison maps and precisely the same closed set comparison maps.

## 5.6 Topological results

Throughout this section let  $\mathcal{K}$  be a tensor triangulated category and let  $\Phi \subset \mathcal{K}$  be a non-empty set of objects closed under the  $\otimes$ -product.

**Proposition 5.6.1.** *If  $\mathcal{Z}_\Phi$  is connected then  $\text{Spec}(R_\Phi)$  is connected.*

*Proof.* By Proposition 5.5.1, it suffices to prove the result under the additional hypothesis that  $\mathcal{K}$  is idempotent-complete. If  $\text{Spec}(R_\Phi)$  is disconnected then there is a non-trivial idempotent  $[e_X]$  in the ring  $R_\Phi$ . By Lemma 5.2.8,  $[e_X] = [e_X]^2 = [e_X^2]$  implies that there is an  $a \in \Phi$  such that  $e_X \otimes a \otimes X = X \otimes a \otimes e_X^2 = (X \otimes a \otimes e_X)^2$  while  $e_X \otimes a \otimes X = X \otimes a \otimes e_X$  since  $e_X$  is  $\otimes$ -balanced. Moreover,  $[e_X] \neq 0$  implies  $X \otimes a \otimes e_X \neq 0$  and  $[e_X] \neq 1$  implies  $X \otimes a \otimes e_X \neq \text{id}_{X \otimes a \otimes X}$ , so  $f := X \otimes a \otimes e_X$  is a non-trivial idempotent endomorphism of  $X \otimes a \otimes X$ .

Since  $\mathcal{K}$  is idempotent-complete, there is an associated decomposition  $X \otimes a \otimes X \simeq a_1 \oplus a_2$  for two non-zero objects  $a_1$  and  $a_2$  such that  $f$  becomes the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . In particular,  $\text{cone}(f) \simeq \Sigma a_2 \oplus a_2$  and  $\text{cone}(\text{id}_{X \otimes a \otimes X} - f) \simeq \Sigma a_1 \oplus a_1$  (cf. Lemma 3.1.13 and Corollary 3.1.15). It follows that

$$\mathcal{Z}_\Phi \cap \text{supp}(a_1) \cap \text{supp}(a_2) = \emptyset$$

because otherwise there would be a  $\mathcal{P} \in \mathcal{Z}_\Phi$  with  $\text{cone}(f) \notin \mathcal{P}$  and  $\text{cone}(\text{id}_{X \otimes a \otimes X} - f) \notin \mathcal{P}$ , which would imply that both  $[f]$  and  $1 - [f]$  are contained in the prime  $\rho_\Phi(\mathcal{P})$ . From the definition,  $\mathcal{Z}_\Phi \subset \text{supp}(X \otimes a \otimes X)$ , and we conclude that

$$\begin{aligned} \mathcal{Z}_\Phi &= \mathcal{Z}_\Phi \cap \text{supp}(X \otimes a \otimes X) \\ &= \mathcal{Z}_\Phi \cap \text{supp}(a_1 \oplus a_2) \\ &= \mathcal{Z}_\Phi \cap (\text{supp}(a_1) \cup \text{supp}(a_2)) \\ &= (\mathcal{Z}_\Phi \cap \text{supp}(a_1)) \sqcup (\mathcal{Z}_\Phi \cap \text{supp}(a_2)) \end{aligned}$$

is a disjoint union of closed sets. It remains to show that  $\mathcal{Z}_\Phi \cap \text{supp}(a_i) \neq \emptyset$  for  $i = 1, 2$ .

If  $\mathcal{Z}_\Phi \cap \text{supp}(a_1)$  were empty then the quasi-compactness of  $\text{Spc}(\mathcal{K})$  would imply that there are  $c_1, \dots, c_n \in \Phi$  such that  $\text{supp}(c_1) \cap \dots \cap \text{supp}(c_n) \cap \text{supp}(a_1) = \emptyset$ . Taking  $c := c_1 \otimes \dots \otimes c_n$  we obtain an object  $c \in \Phi$  such that  $\text{supp}(c \otimes a_1) = \emptyset$  and hence that  $c \otimes a_1$  is  $\otimes$ -nilpotent. But observe that under the identification  $X \otimes a \otimes X \simeq a_1 \oplus a_2$ , the endomorphism  $f^{\otimes n}$  becomes a matrix with all zero entries except for  $\text{id}_{a_1^{\otimes n}}$  at one position along the diagonal. From this it is clear that  $c^{\otimes n} \otimes f^{\otimes n} = 0$  for some  $n \geq 1$  since  $c \otimes a_1$  is  $\otimes$ -nilpotent. But then  $[e_X] = [f] = [f^n] = [f^{\otimes n}] = [c^{\otimes n} \otimes f^{\otimes n}] = 0$  in the ring  $R_\Phi$ , which contradicts the fact that  $[e_X]$  is nontrivial. A similar argument shows that if  $\mathcal{Z}_\Phi \cap \text{supp}(a_2)$  were empty then  $[e_X] = 1$ .  $\square$

For the converse of the above result, we need to add additional assumptions.

**Proposition 5.6.2.** *Suppose  $\mathcal{K}$  is rigid and  $\mathcal{Z}_\Phi$  is Thomason. If  $\text{Spec}(R_\Phi)$  is connected then  $\mathcal{Z}_\Phi$  is connected.*

*Proof.* By passing to the idempotent completion it suffices to prove the result under the additional hypothesis that  $\mathcal{K}$  is idempotent-complete (cf. Proposition 5.5.1 and note that  $\mathcal{K}$  rigid implies that  $\mathcal{K}^\natural$  is also rigid [Bal07, Proposition 2.15(i)]). Suppose  $\mathcal{Z}_\Phi = Y_1 \sqcup Y_2$  is a disjoint union of non-empty closed sets. Each  $Y_i$  is quasi-compact (being closed) and it follows from the fact that they are disjoint and that  $\mathcal{Z}_\Phi$  is Thomason that each  $Y_i$  is Thomason. For example,  $\text{Spc}(\mathcal{K}) \setminus Y_1 = (Y_1^c \cap \mathcal{Z}_\Phi) \sqcup (Y_1^c \cap (\text{Spc}(\mathcal{K}) \setminus \mathcal{Z}_\Phi)) = Y_2 \sqcup (\text{Spc}(\mathcal{K}) \setminus \mathcal{Z}_\Phi)$  is the union of two quasi-compact sets, and hence is quasi-compact.

It also follows from the definition  $\mathcal{Z}_\Phi := \bigcap_{X \in \Phi} \text{supp}(X)$  and the fact that  $\text{Spc}(\mathcal{K}) \setminus \mathcal{Z}_\Phi$  is quasi-compact that  $\mathcal{Z}_\Phi = \text{supp}(a)$  for some  $a \in \Phi$ . Since  $\mathcal{K}$  is rigid and idempotent-complete, the generalized Carlson theorem [Bal07, Remark 2.12] implies that there exist  $a_1, a_2 \in \mathcal{K}$  such that  $a \simeq a_1 \oplus a_2$  and  $\text{supp}(a_i) = Y_i$  for  $i = 1, 2$ . Since  $\mathcal{K}$  is rigid and  $\text{supp}(a_1) \cap \text{supp}(a_2) = \emptyset$  it follows [Bal07, Corollary 2.8] that  $[a_i, a_j] = 0$  for  $i \neq j$ . This implies that the idempotent  $f := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  is  $\otimes$ -balanced ( $f \otimes a = a \otimes f$ ) and hence provides

an idempotent element  $[f]$  of the ring  $R_\Phi$ . If  $[f]$  was a trivial idempotent it would follow that  $b \otimes a_1 = 0$  or  $b \otimes a_2 = 0$  for some  $b \in \Phi$ . But  $Y_i = \text{supp}(a_i) = \text{supp}(b \otimes a_i)$  since  $\text{supp}(a_i) \subset \mathcal{Z}_\Phi \subset \text{supp}(b)$  and so  $b \otimes a_i = 0$  contradicts the fact that  $Y_i$  is non-empty. We conclude that  $[f]$  is a nontrivial idempotent of the commutative ring  $R_\Phi$  and hence that  $\text{Spec}(R_\Phi)$  is disconnected.  $\square$

*Remark 5.6.3.* Example 5.6.13 below will provide some (non-rigid) examples of tensor triangulated categories for which the conclusion of Proposition 5.6.2 does not hold.

*Remark 5.6.4.* For any (graded-)commutative graded ring  $R^\bullet$ , the space  $\text{Spec}^h(R^\bullet)$  is connected if and only if  $\text{Spec}(R^0)$  is connected (cf. Lemma 2.3.8), so the graded version of the statements of Proposition 5.6.1 and Proposition 5.6.2 also hold.

**Lemma 5.6.5.** *If  $c$  is an object in  $\mathcal{K}$  with  $\mathcal{Z}_\Phi \subset \text{supp}(c)$  then there is some  $a \in \Phi$  such that  $\text{supp}(a) \subset \text{supp}(c)$ .*

*Proof.* Since  $\text{Spc}(\mathcal{K}) \setminus \text{supp}(c)$  is quasi-compact (recall Lemma 4.1.10), it follows from the definition  $\mathcal{Z}_\Phi := \bigcap_{X \in \Phi} \text{supp}(X)$  that there is a finite collection of objects  $X_1, \dots, X_n \in \Phi$  such that  $\text{supp}(X_1) \cap \dots \cap \text{supp}(X_n) \subset \text{supp}(c)$  and we can take  $a := X_1 \otimes \dots \otimes X_n \in \Phi$ .  $\square$

**Proposition 5.6.6.** *Consider the map  $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \text{Spec}^h(R_\Phi^\bullet)$  and let  $\mathcal{W} \subset \text{Spec}^h(R_\Phi^\bullet)$  be any closed set. If  $(\rho_\Phi^\bullet)^{-1}(\mathcal{W}) = \mathcal{Z}_\Phi$  then  $\mathcal{W} = \text{Spec}^h(R_\Phi^\bullet)$ . In other words, the preimage of a proper closed set remains proper.*

*Proof.* Let  $\mathcal{W} = V(I^\bullet)$  for some homogeneous ideal  $I \subset R_\Phi^\bullet$  and let  $[f]$  be an arbitrary homogeneous element of  $I^\bullet$ . It follows from our hypothesis that  $[f] \in \rho_\Phi^\bullet(\mathcal{P})$  for every  $\mathcal{P} \in \mathcal{Z}_\Phi$  and hence that  $\mathcal{Z}_\Phi \subset \text{supp}(\text{cone}(f))$ . Lemma 5.6.5 then implies that there is some  $a \in \Phi$  such that  $\text{supp}(a) \subset \text{supp}(\text{cone}(f))$ . It follows (cf. Lemma 4.2.6) that  $a^{\otimes i} \in \text{thick}_\otimes \langle \text{cone}(f) \rangle$  for some  $i \geq 1$  and hence that  $a^{\otimes i} \otimes f^{\otimes j} = 0$  for some  $i, j \geq 1$  by Lemma 5.3.3. But then

$[f]^j = [f^{\otimes j}] = [a^{\otimes i} \otimes f^{\otimes j}] = 0$  so that  $[f]$  is a nilpotent element of  $R_\Phi$ . Since every homogeneous element of  $I$  is nilpotent,  $\mathcal{W} = V(I^\bullet) = \text{Spec}^h(R_\Phi^\bullet)$ .  $\square$

*Remark 5.6.7.* One easily verifies that the ungraded maps  $\rho_\Phi$  also have the property that preimages of proper closed subsets remain proper, either by carrying out an ungraded version of the above proof, or as a corollary of the above proposition by observing that the surjective map  $(-)^0 : \text{Spec}^h(R_\Phi^\bullet) \rightarrow \text{Spec}(R_\Phi)$  has this property and using diagram (5.2.30).

**Corollary 5.6.8.** *The maps  $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \text{Spec}^h(R_\Phi^\bullet)$  and  $\rho_\Phi : \mathcal{Z} \rightarrow \text{Spec}(R_\Phi)$  have dense images. Consequently, if  $\mathcal{Z}_\Phi$  is irreducible then  $\text{Spec}^h(R_\Phi^\bullet)$  and  $\text{Spec}(R_\Phi)$  are also irreducible.*

**Lemma 5.6.9.** *If  $\mathcal{K}$  is rigid and  $[f]$  is a homogeneous non-unit in the ring  $R_\Phi^\bullet$  then there is some  $\mathcal{P} \in \mathcal{Z}_\Phi$  such that  $[f] \in \rho_\Phi^\bullet(\mathcal{P})$ .*

*Proof.* If there were no such  $\mathcal{P}$  then  $\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f)) = \emptyset$  which implies by the quasi-compactness of  $\text{Spec}(\mathcal{K})$  that  $\text{supp}(a) \cap \text{supp}(\text{cone}(f)) = \emptyset$  for some  $a \in \Phi$ . But

$$\text{supp}(a \otimes \text{cone}(f)) = \emptyset$$

implies by the rigidity of  $\mathcal{K}$  that  $a \otimes \text{cone}(f) = 0$ . Thus  $a \otimes f$  is invertible and hence  $[f]$  is a unit in  $R_\Phi^\bullet$ .  $\square$

**Proposition 5.6.10.** *Suppose  $\mathcal{K}$  is rigid. If the map  $\rho_\Phi : \mathcal{Z}_\Phi \rightarrow \text{Spec}(R_\Phi)$  is a constant map then  $\text{Spec}(R_\Phi)$  is a point. Similarly, if the map  $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \text{Spec}^h(R_\Phi^\bullet)$  is a constant map then  $\text{Spec}^h(R_\Phi^\bullet)$  is a point.*

*Proof.* Suppose  $\mathfrak{p}$  is a prime in  $R_\Phi$  such that  $\rho_\Phi(\mathcal{P}) = \mathfrak{p}$  for all  $\mathcal{P} \in \mathcal{Z}_\Phi$ . Lemma 5.6.9 then implies that  $\mathfrak{p}$  contains every non-unit of  $R_\Phi$ . It follows that  $R_\Phi$  is local with  $\mathfrak{p}$  its unique maximal ideal. On the other hand, Corollary 5.6.8 implies that  $\{\mathfrak{p}\}$  is dense, and so  $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = \text{Spec}(R_\Phi)$ . An identical argument works in the graded case.  $\square$

*Remark 5.6.11.* The theorems proved in this section are new even in the case of Balmer's original map  $\rho_1$ . For example, Corollary 5.6.8 implies that  $\rho_1$  always has dense image without any assumptions on the category  $\mathcal{K}$ . This result is particularly interesting in light of the surjectivity criteria for  $\rho_1$  established in [Bal10a].

*Remark 5.6.12.* One strategy for studying the spectrum is to iteratively build a filtration of closed subsets by pulling back filtrations via our closed set comparison maps. More precisely, we begin with the trivial filtration  $\{\mathrm{Spc}(\mathcal{K})\}$  and at each iterative step we consider every closed set  $\mathcal{Z}$  in the filtration (or only those that were newly added at the last step) and refine the filtration below  $\mathcal{Z}$  by pulling back the filtration of closed subsets of  $\mathrm{Spec}^h(R_{\mathcal{Z}}^\bullet)$  via the map  $\rho_{\mathcal{Z}}^\bullet : \mathcal{Z} \rightarrow \mathrm{Spec}^h(R_{\mathcal{Z}}^\bullet)$ . A key result for this idea is Proposition 5.6.6 which asserts that proper closed subsets of  $\mathrm{Spec}^h(R_{\mathcal{Z}}^\bullet)$  pull back to proper subsets of  $\mathcal{Z}$ . This implies that the process continues to refine the spectrum for as long as the spaces  $\mathrm{Spec}^h(R_{\mathcal{Z}}^\bullet)$  are non-trivial. However, an obstacle is the possibility that  $\mathrm{Spec}^h(R_{\mathcal{Z}}^\bullet)$  might be trivial for some non-trivial closed set  $\mathcal{Z}$ , in which case the internal structure of  $\mathcal{Z}$  would remain hidden. In fact, there are examples of tensor triangulated categories for which it seems the process may hit the wall at the very first step (cf. Example 5.6.13 below). Nevertheless, these examples are non-rigid and there is some hope (cf. Proposition 5.6.10) that under suitable hypotheses this obstacle might disappear.

*Example 5.6.13.* Let  $P$  be a finite poset,  $k$  a field, and let  $\mathcal{K} := D^b(\mathrm{rep}_k(P))$  be the derived category of finite-dimensional  $k$ -linear representations of  $P$ . The abelian category  $\mathrm{rep}_k(P)$  has an exact vertex-wise  $\otimes$ -structure and  $\mathcal{K}$  inherits the structure of a tensor triangulated category. Recognizing that representations of  $P$  are the same thing as quiver representations of the associated Hasse diagram with full commutativity relations, one sees that the work of [LS13] completely describes the spectrum  $\mathrm{Spc}(\mathcal{K})$ . It turns out to be rather trivial: a discrete space with points corresponding to the elements of  $P$ . For a representation

$V$  regarded as a complex concentrated in degree 0,  $\text{supp}(V) = \{x \in P \mid V_x \neq 0\}$ . If  $P$  has a least element then  $\mathbb{1}$  is projective (so  $[\mathbb{1}, \mathbb{1}]_i = 0$  for  $i \neq 0$ ) and  $[\mathbb{1}, \mathbb{1}] \simeq k$  by inspection. There are thus many examples where  $\text{Spec}^h(R_1^*)$  is trivial (in particular, connected) but  $\text{Spc}(\mathcal{K})$  is disconnected. This doesn't contradict Proposition 5.6.2 because these examples of derived quiver representations are not rigid. For example, consider the simplest non-trivial example:  $P = (1 \rightarrow 2)$ . Let  $S_1 = (k \rightarrow 0)$  and  $S_2 = (0 \rightarrow k)$  be the two simple representations. There is an obvious exact triangle  $S_2 \rightarrow \mathbb{1} \rightarrow S_1 \rightarrow \Sigma S_2$ . If  $\mathcal{K}$  were rigid then the fact that  $\text{supp}(S_1) \cap \text{supp}(\Sigma S_2) = \text{supp}(S_1) \cap \text{supp}(S_2) = \emptyset$  implies that  $[S_1, \Sigma S_2] = 0$ , and hence that the exact triangle splits:  $\mathbb{1} \simeq S_1 \oplus S_2$  which is evidently not the case.

*Remark 5.6.14.* A topological criterion was given in [DS14, Proposition 3.11] for the injectivity of the comparison maps introduced in that paper. Their proof gives a similar criterion for our comparison maps. Recall that  $\{\text{supp}(a) \mid a \in \mathcal{K}\}$  is a basis of closed sets for  $\text{Spc}(\mathcal{K})$ .

**Proposition 5.6.15.** *Let  $\Phi$  be a non-empty  $\otimes$ -multiplicative set of objects in a tensor triangulated category  $\mathcal{K}$ . If the collection of subsets*

$$\mathcal{B} = \{\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f)) \mid [f] \in R_\Phi\}$$

*is a basis of closed sets for  $\mathcal{Z}_\Phi$  then the comparison map  $\rho_\Phi$  is injective, and furthermore a homeomorphism onto its image.*

*Proof.* Note that  $\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f))$  doesn't depend on the choice of representative for  $[f]$ . Let  $\mathcal{P}, \mathcal{Q} \in \mathcal{Z}_\Phi$  and suppose  $\rho_\Phi(\mathcal{P}) = \rho_\Phi(\mathcal{Q})$ . To show that  $\mathcal{P} = \mathcal{Q}$  it suffices to show that  $\overline{\{\mathcal{P}\}} = \overline{\{\mathcal{Q}\}}$  since  $\text{Spc}(\mathcal{K})$  is  $T_0$ . Note that  $\rho_\Phi(\mathcal{P}) = \rho_\Phi(\mathcal{Q})$  says that  $\mathcal{P} \in \text{supp}(\text{cone}(f))$  iff  $\mathcal{Q} \in \text{supp}(\text{cone}(f))$  for any  $[f] \in R_\Phi$ . Using the fact that  $\mathcal{B}$  is a basis we then have that

$$\overline{\{\mathcal{P}\}} = \bigcap_{\substack{[f] \in R_\Phi \\ \mathcal{P} \in \text{supp}(\text{cone}(f))}} \text{supp}(\text{cone}(f)) = \bigcap_{\substack{[f] \in R_\Phi \\ \mathcal{Q} \in \text{supp}(\text{cone}(f))}} \text{supp}(\text{cone}(f)) = \overline{\{\mathcal{Q}\}}.$$

Now for  $[f] \in R_\Phi$  one easily checks that

$$\rho_\Phi(\mathcal{Z}_\Phi \setminus (\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f)))) = \rho_\Phi(\mathcal{Z}_\Phi \setminus \rho_\Phi^{-1}(V([f]))) = \rho_\Phi(\mathcal{Z}_\Phi \cap D([f]))$$

where  $D([f]) := \{\mathfrak{p} \in \text{Spec}(R_X) \mid [f] \notin \mathfrak{p}\}$  is the usual distinguished open in  $\text{Spec}(R_\Phi)$ . Since the sets  $\mathcal{Z}_\Phi \setminus (\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f)))$  form an open basis for  $\mathcal{Z}_\Phi$  it follows that  $\rho_\Phi$  is a homeomorphism onto its image.  $\square$

## 5.7 Algebraic localization

An important feature of the original unit comparison maps developed in [Bal10a] is that it is possible to localize the category  $\mathcal{K}$  with respect to primes in the ring  $[\mathbb{1}, \mathbb{1}]$ , which enables one to reduce questions about the unit comparison maps to the case where the target ring is local. Fortunately, one may establish such a localization technique for our more general comparison maps.

**Theorem 5.7.1.** *Let  $\mathcal{K}$  be a tensor triangulated category,  $\Phi \subset \mathcal{K}$  a non-empty set of objects closed under the  $\otimes$ -product,  $S \subset R_{\mathcal{K}, \Phi}^\bullet$  a multiplicative set of even (hence central) homogeneous elements, and  $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{J}$  the Verdier quotient of  $\mathcal{K}$  with respect to the thick  $\otimes$ -ideal  $\mathcal{J} := \text{thick}_\otimes \langle \text{cone}(s) \mid [s] \in S \rangle$ . Then  $R_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet$  is isomorphic to the ring-theoretic localization  $S^{-1}(R_{\mathcal{K}, \Phi}^\bullet)$  and we have a diagram*

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}/\mathcal{J}, \Phi} & \xhookrightarrow{\quad} & \mathcal{Z}_{\mathcal{K}, \Phi} \\ \downarrow \rho_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet & & \downarrow \rho_{\mathcal{K}, \Phi}^\bullet \\ \text{Spec}^h(R_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet) = \text{Spec}^h(S^{-1}R_{\mathcal{K}, \Phi}^\bullet) & \xhookrightarrow{\quad} & \text{Spec}^h(R_{\mathcal{K}, \Phi}^\bullet). \end{array} \quad (5.7.2)$$

that is commutative and cartesian:  $\mathcal{Z}_{\mathcal{K}/\mathcal{J}, \Phi} \cong \{\mathcal{P} \in \mathcal{Z}_{\mathcal{K}, \Phi} \mid \rho_{\mathcal{K}, \Phi}^\bullet(\mathcal{P}) \cap S = \emptyset\}$ .

**Remark 5.7.3.** If  $S \subset R^0$  then  $(S^{-1}R^\bullet)^0 = S^{-1}R^0$  and one readily verifies that applying  $(-)^0$

to the bottom row of (5.7.2) yields a commutative, cartesian diagram

$$\begin{array}{ccc}
\mathcal{Z}_{\mathcal{K}/\mathcal{J},\Phi} & \xrightarrow{\quad} & \mathcal{Z}_{\mathcal{K},\Phi} \\
\downarrow \rho_{\mathcal{K}/\mathcal{J},\Phi} & & \downarrow \rho_{\mathcal{K},\Phi} \\
\mathrm{Spec}(R_{\mathcal{K}/\mathcal{J},\Phi}) = \mathrm{Spec}(S^{-1}R_{\mathcal{K},\Phi}) & \xrightarrow{\quad} & \mathrm{Spec}(R_{\mathcal{K},\Phi}).
\end{array} \tag{5.7.4}$$

This gives the ungraded version of our localization result.

The remainder of this section is devoted to proving the theorem. For purposes of clarity we will prove the ungraded version—the graded result stated in Theorem 5.7.1 is proved in the same way but the notation gets more cumbersome. Thus, for the rest of the section we fix a multiplicative set  $S \subset R_{\mathcal{K},\Phi}$ . For an element  $[s] \in S$ , we'll use the notation  $X_s$  to indicate that the representative  $s$  is an endomorphism of  $X_s$ . For morphisms in  $\mathcal{K}/\mathcal{J}$  we'll use the left fractions discussed in Section 3.6. It is immediate from the definition of  $\mathcal{J}$  that the canonical map  $R_{\mathcal{K},\Phi} \rightarrow R_{\mathcal{K}/\mathcal{J},q(\Phi)}$  factors as

$$R_{\mathcal{K},\Phi} \xrightarrow{\epsilon} S^{-1}(R_{\mathcal{K},\Phi}) \xrightarrow{i} R_{\mathcal{K}/\mathcal{J},q(\Phi)}$$

where  $\epsilon$  is the canonical localization map. In order to show that  $i$  is an isomorphism we will need the following three lemmas:

**Lemma 5.7.5.** *If  $a \in \mathcal{J}$  then there is a representative  $s$  of an element  $[s] \in S$  with the property that  $a \otimes s = 0$ .*

*Proof.* One verifies using standard techniques that the collection of objects  $a \in \mathcal{K}$  for which there is a representative  $s$  of an element  $[s] \in S$  such that  $a \otimes s^{\otimes n} = 0$  for some  $n \geq 1$  forms a thick  $\otimes$ -ideal of  $\mathcal{K}$ . It contains each cone( $s$ ) by Lemma 5.1.3 and hence it contains  $\mathcal{J}$ . Since  $S$  is multiplicative,  $[s^{\otimes n}] = [s]^n \in S$ . □

**Lemma 5.7.6.** *If  $f : a \rightarrow b$  and  $g : c \rightarrow d$  are morphisms such that  $\mathrm{cone}(f) \otimes g = 0$  then there exists a morphism  $u : b \otimes c \rightarrow a \otimes d$  such that  $a \otimes g = u \circ (f \otimes c)$  and a morphism  $v : b \otimes c \rightarrow a \otimes d$  such that  $b \otimes g = (f \otimes d) \circ v$ .*

*Proof.* The morphisms  $u$  and  $v$  are obtained from the morphism of exact triangles

$$\begin{array}{ccccccc}
a \otimes c & \xrightarrow{f \otimes c} & b \otimes c & \longrightarrow & \text{cone}(f) \otimes c & \longrightarrow & \Sigma(a \otimes c) \\
a \otimes g \downarrow & & \downarrow b \otimes g & & \downarrow \text{cone}(f) \otimes g = 0 & & \downarrow \Sigma(a \otimes g) \\
a \otimes d & \xrightarrow{f \otimes d} & b \otimes d & \longrightarrow & \text{cone}(f) \otimes d & \longrightarrow & \Sigma(a \otimes d)
\end{array}$$

$\swarrow \begin{smallmatrix} u \\ v \end{smallmatrix}$

by the weak kernel and cokernel properties of exact triangles.  $\square$

**Lemma 5.7.7.** *Let  $f : a \rightarrow a$  be an endomorphism in a tensor triangulated category. If there exists a  $\otimes$ -balanced automorphism  $\sigma : a \rightarrow a$  such that  $f \circ \sigma$  is  $\otimes$ -balanced then  $f$  is  $\otimes$ -balanced.*

*Proof.* This is easily verified from the definitions recalling that  $E_a$  is an inverse closed subring of  $[a, a]$ .  $\square$

**Proposition 5.7.8.** *The map  $i : S^{-1}(R_{\mathcal{K}, \Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$  is injective.*

*Proof.* Consider an element  $[f]/[s] \in S^{-1}(R_{\mathcal{K}, \Phi})$ . If  $i([f]/[s]) = 0$  then

$$[(X_f \otimes X_s \xrightarrow{f \otimes 1} X_f \otimes X_s \xleftarrow{1 \otimes s} X_f \otimes X_s)] = 0$$

in  $R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$  which implies that there is an  $a \in \Phi$  such that

$$(a \otimes X_f \otimes X_s \xrightarrow{1 \otimes f \otimes 1} a \otimes X_f \otimes X_s \xleftarrow{1 \otimes s \otimes 1} a \otimes X_f \otimes X_s) = 0$$

as a morphism in  $\mathcal{K}/\mathcal{J}$ . It follows that there is a morphism  $k : a \otimes X_f \otimes X_s \rightarrow b$  in  $\mathcal{K}$  with  $\text{cone}(k) \in \mathcal{J}$  such that  $k \circ (a \otimes f \otimes X_s) = 0$ . By Lemma 5.7.5 there is some  $[t] \in S$  such that  $\text{cone}(k) \otimes t = 0$  and hence by Lemma 5.7.6 there is a morphism  $u : b \otimes X_t \rightarrow a \otimes X_f \otimes X_s \otimes X_t$  such that  $u \circ (k \otimes X_t) = a \otimes X_f \otimes X_s \otimes t$ . In the ring  $R_{\mathcal{K}, \Phi}$  we then have

$$[t] \cdot [f] = [(a \otimes X_f \otimes X_s \otimes t) \circ (a \otimes f \otimes X_s \otimes X_t)] = 0$$

and we conclude that  $[f]/[s] = ([t] \cdot [f])/([t] \cdot [s]) = 0$  in  $S^{-1}(R_{\mathcal{K}, \Phi})$ .  $\square$

**Proposition 5.7.9.** *The map  $i : S^{-1}(R_{\mathcal{K},\Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J},\Phi}$  is surjective.*

*Proof.* Consider an element  $[(a \xrightarrow{f} b \xleftarrow{\sigma} a)] \in R_{\mathcal{K}/\mathcal{J},q(\Phi)}$ . By Lemma 5.7.5 there is some  $[s] \in S$  such that  $\text{cone}(\sigma) \otimes s = 0$ . It then follows from Lemma 5.7.6 that there exists a morphism  $u : b \otimes X_s \rightarrow a \otimes X_s$  such that  $u \circ (\sigma \otimes X_s) = a \otimes s$ . We thus have an equality of left fractions

$$(a \otimes X_s \xrightarrow{f \otimes 1} b \otimes X_s \xleftarrow{\sigma \otimes 1} a \otimes X_s) = (a \otimes X_s \xrightarrow{u \circ (f \otimes 1)} a \otimes X_s \xleftarrow{1 \otimes s} a \otimes X_s).$$

The left-hand side is an element of  $E_{\mathcal{K}/\mathcal{J},a \otimes X_s}$  and so it follows from Lemma 5.7.7 that

$$(a \otimes X_s \xrightarrow{u \circ (f \otimes 1)} a \otimes X_s \xleftarrow{\text{id}} a \otimes X_s)$$

is an element of  $E_{\mathcal{K}/\mathcal{J},X \otimes X_s}$ . The claim then follows from Lemma 5.7.10 below.  $\square$

**Lemma 5.7.10.** *If  $f : a \rightarrow a$  is an endomorphism of an object  $a \in \Phi$  such that  $q(f)$  is tensor-balanced as an endomorphism in  $\mathcal{K}/\mathcal{J}$  then  $[q(f)] \in R_{\mathcal{K},q(\Phi)}$  is contained in the image of  $i : S^{-1}(R_{\mathcal{K},\Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J},\Phi}$ .*

*Proof.* It follows from the equality of fractions

$$(a \otimes a \xrightarrow{a \otimes f} a \otimes a \xleftarrow{\text{id}} a \otimes a) = (a \otimes a \xrightarrow{f \otimes a} a \otimes a \xleftarrow{\text{id}} a \otimes a)$$

that there is a morphism  $\tau : a \otimes a \rightarrow b$  in  $\mathcal{K}$  such that  $\text{cone}(\tau) \in \mathcal{J}$  and  $\tau \circ (a \otimes f) = \tau \circ (f \otimes a)$ .

By Lemma 5.7.5 and Lemma 5.7.6 there is a  $[t] \in S$  such that  $\text{cone}(\tau) \otimes t = 0$  and a morphism  $u : b \otimes X_t \rightarrow a \otimes a \otimes X_t$  such that  $u \circ (\tau \otimes X_t) = a \otimes a \otimes t$ . It follows that  $a \otimes f \otimes t = f \otimes a \otimes t$  and we conclude using Lemma 5.2.1 that  $f \otimes t$  is an element of  $E_{a \otimes X_t}$ . Thus

$$[(a \xrightarrow{f} a \xleftarrow{\text{id}} a)] \cdot [(X_t \xrightarrow{t} X_t \xleftarrow{\text{id}} X_t)] = [(a \otimes X_t \xrightarrow{f \otimes t} a \otimes X_t \xleftarrow{\text{id}} a \otimes X_t)]$$

is contained in the image of  $i$ .  $\square$

In summary, we have established that the canonical map  $R_{\mathcal{K},\Phi} \rightarrow R_{\mathcal{K}/J,q(\Phi)}$  factors as

$$R_{\mathcal{K},\Phi} \xrightarrow{\epsilon} S^{-1}(R_{\mathcal{K},\Phi}) \cong R_{\mathcal{K}/J,q(\Phi)}$$

where  $\epsilon$  is the canonical localization map. The commutativity of (5.7.4) then follows from Proposition 5.2.21. Moreover, one readily checks from the definitions that the image of  $\mathcal{Z}_{\mathcal{K}/J,\Phi}$  in  $\mathrm{Spc}(\mathcal{K})$  is precisely  $V(J) \cap \mathcal{Z}_{\mathcal{K},\Phi}$ . If  $\mathcal{P} \in \mathcal{Z}_{\mathcal{K},\Phi}$  is a prime such that  $\rho_{\mathcal{K},\Phi}(\mathcal{P}) \cap S = \emptyset$  then  $\mathrm{cone}(s) \in \mathcal{P}$  for all  $[s] \in S$  so that  $J \subset \mathcal{P}$ . Hence  $\mathcal{P} \in V(J) \cap \mathcal{Z}_{\mathcal{K},\Phi} \cong \mathcal{Z}_{\mathcal{K}/J,\Phi}$ . This establishes that diagram (5.7.4) is cartesian and the proof is complete.

## CHAPTER 6

### Stable homotopy theory

The purpose of this chapter is to apply the theory of higher comparison maps to the stable homotopy category of finite spectra. This will be carried out in Section 6.6. In the preceding sections we will set the stage by recalling some relevant notions from chromatic homotopy theory. Much of this is very well-known to the homotopy theorist but we have included some elementary material for the benefit of those less familiar with the subject.

Ultimately, our ability to apply the theory of higher comparison maps in this example rests on the seminal work of Devinatz, Hopkins and Smith on the Ravenel conjectures and, in particular, on the Nilpotence and Periodicity Theorems. Although we will discuss some aspects of this work, we will only scratch the surface and we refer the reader to [Rav92] and [Rav86] for further information about this beautiful subject. In particular, we say almost nothing about formal group laws and very little about the motivation for Ravenel’s conjectures stemming from periodic phenomena in the stable homotopy groups of spheres and the Adams-Novikov spectral sequence.

Throughout this chapter,  $\mathrm{SH}$  will denote the stable homotopy category of spectra and  $\mathrm{SH}^{\mathrm{fin}}$  will denote its full subcategory of finite spectra. We refer the reader to Margolis’ book [Mar83] for a discussion of these categories that takes a fairly categorical (less topological) point of view—also see [HPS97]. From these references recall that  $\mathrm{SH}$  is a tensor triangulated category with arbitrary coproducts whose unit, the sphere spectrum, is a compact,

graded weak generator and that  $\mathrm{SH}^{\mathrm{fin}}$  is exactly the subcategory of compact objects in  $\mathrm{SH}$ . Also recall that  $\mathrm{SH}^{\mathrm{fin}}$  is the thick subcategory generated by the unit and that  $\mathrm{SH}$  is the localizing subcategory generated by the unit:  $\mathrm{SH}^{\mathrm{fin}} = \mathrm{thick}\langle \mathbb{1} \rangle$ ,  $\mathrm{SH} = \mathrm{loc}\langle \mathbb{1} \rangle$ . It follows (recall Lemma 3.9.1) that every thick subcategory of  $\mathrm{SH}^{\mathrm{fin}}$  and every localizing subcategory of  $\mathrm{SH}$  is automatically a  $\otimes$ -ideal. As usual we denote the suspension spectrum of a finite based CW-complex  $X$  by  $\Sigma^\infty X$  and the smash product of spectra by  $\wedge$ . We'll occasionally use the term “selfmap” to refer to a graded endomorphism  $\Sigma^d X \rightarrow X$  as this terminology is common in the homotopy theory literature. Finally, for any integer  $n \in \mathbb{Z}$  and spectrum  $X$ , we'll use  $n : X \rightarrow X$  to denote the endomorphism  $n \cdot \mathrm{id}_X$ .

## 6.1 Stable homotopy groups

Recall that the stable homological functor  $\pi_\bullet(-) = [\Sigma^\bullet \mathbb{1}, -] : \mathrm{SH} \rightarrow \mathrm{Ab}^\mathbb{Z}$  associated to the sphere spectrum  $\mathbb{1} = \Sigma^\infty(S^0)$  gives the *stable* homotopy groups. In other words,  $\pi_\bullet(\Sigma^\infty X) = \pi_\bullet^s(X)$  for a based finite CW-complex  $X$  (cf. the discussion in Section 3.8). In particular, the stable homotopy groups of spheres  $\pi_\bullet(\mathbb{1}) = \pi_\bullet^s(S^0)$  form a graded-commutative graded ring. In this section, we'll recall some classical facts about these stable homotopy groups. For further discussion and references to the classical literature see [Rav86, §1.1].

**Theorem 6.1.1** (Hurewicz).  $\pi_0(\mathbb{1}) \cong \mathbb{Z}$  and  $\pi_i(\mathbb{1}) = 0$  for  $i < 0$ .

**Theorem 6.1.2** (Nishida). *Every element of positive degree in  $\pi_\bullet(\mathbb{1})$  is nilpotent.*

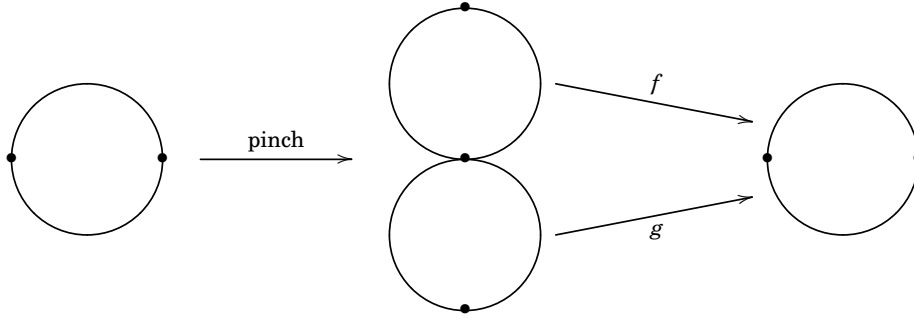
*Remark 6.1.3.* These theorems imply that—despite their complexity—the groups  $\pi_i(\mathbb{1})$  are very simple up to multiplicative nilpotent phenomena. In particular, the map

$$(-)^0 : \mathrm{Spec}^h(\pi_\bullet(\mathbb{1})) \xrightarrow{\sim} \mathrm{Spec}(\pi_0(\mathbb{1}))$$

is a homeomorphism and the graded comparison map  $\rho^\bullet : \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}^h(\pi_\bullet(\mathbb{1}))$  gives us

nothing more than the ungraded comparison map  $\rho : \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}(\pi_0(\mathbb{1}))$ . Moreover, since  $\mathrm{Spec}(\pi_0(\mathbb{1})) = \mathrm{Spec}(\mathbb{Z})$ , the map  $\rho$  is very far from injective and tells us very little about  $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$ . We need techniques for studying the fibers of this map.

*Remark 6.1.4.* Under the isomorphism  $\pi_0(\mathbb{1}) \simeq \mathbb{Z}$ , the integer  $n \in \mathbb{Z}$  corresponds to  $n \cdot \mathrm{id}_{\mathbb{1}}$ . It is not hard to see that the stable map  $n \cdot \mathrm{id}_{\mathbb{1}}$  is represented by the degree  $n$  map from the sphere to the sphere. For example, recall that two endomorphisms of the 1-sphere  $f, g : S^1 \rightarrow S^1$  are added in the group of homotopy classes of maps  $S^1 \rightarrow S^1$  by pinching the sphere and letting  $f$  and  $g$  act on the resulting two spheres separately:



One then readily observes (for example) that the sum  $\mathrm{id}_{\mathbb{1}} + \mathrm{id}_{\mathbb{1}}$  is (homotopic) to the degree 2 map. More generally, the sum of a degree  $n$  map and a degree  $m$  map is readily seen to be a degree  $n + m$  map for any  $n, m \in \mathbb{Z}$ .

**Theorem 6.1.5** (Serre). *For any integer  $i > 0$ ,  $\pi_i(\mathbb{1})$  is a finite abelian group.*

**Corollary 6.1.6.** *For any finite spectrum  $X$ ,  $\pi_i(X)$  is a finitely generated abelian group for all  $i \in \mathbb{Z}$ .*

*Proof.* Note that finitely generated modules over a noetherian ring (like  $\mathbb{Z}$ ) satisfy the two-out-of-three property for short exact sequences. It follows that the collection of spectra  $X$  with the property that  $\pi_i(X)$  is a finitely generated abelian group for all  $i \in \mathbb{Z}$  forms a thick subcategory of  $\mathrm{SH}$ . It contains  $\mathbb{1}$  and hence contains  $\mathrm{SH}^{\mathrm{fin}} = \mathrm{thick}\langle \mathbb{1} \rangle$ .  $\square$

*Remark 6.1.7.* If  $X$  and  $Y$  are any two finite spectra then the abelian groups  $[X, Y]_i$  are also finitely generated because  $[X, Y]_i = \pi_i(DX \wedge Y)$ .

*Remark 6.1.8.* The fact that  $\mathbb{1}$  generates  $\text{SH}$  implies that  $\pi_\bullet(-) : \text{SH} \rightarrow \text{Ab}^\mathbb{Z}$  is a conservative functor. In other words, it is faithful on objects. By Lemma 3.1.41, this is equivalent to saying that  $\pi_\bullet(-)$  reflects isomorphisms.

**Notation 6.1.9.** For any abelian group  $A$ , we write  $HA$  for the Eilenberg-MacLane spectrum representing ordinary cohomology  $H^\bullet(-; A)$  with coefficients in  $A$ . If  $R$  is a (commutative) ring then the Eilenberg-MacLane spectrum  $HR$  has the structure of a (commutative) ring spectrum.

**Definition 6.1.10** (Hurewicz map). For any ring spectrum  $\mathbb{E}$  and spectrum  $X$  we can define a map

$$X \simeq \mathbb{1} \wedge X \xrightarrow{\eta \wedge 1} \mathbb{E} \wedge X.$$

By applying  $\pi_\bullet(-)$  we obtain a map

$$\pi_\bullet(X) \rightarrow \pi_\bullet(\mathbb{E} \wedge X) = \mathbb{E}_\bullet(X)$$

called the *Hurewicz map* associated to the ring spectrum  $\mathbb{E}$ .

**Theorem 6.1.11** (Serre). *The Hurewicz map*

$$\pi_\bullet(\mathbb{1}) \rightarrow H\mathbb{Z}_\bullet(\mathbb{1})$$

*is a rational isomorphism.*

**Corollary 6.1.12.** *For any spectrum  $X$ , there is an isomorphism*

$$\pi_\bullet(X) \otimes \mathbb{Q} \simeq H_\bullet(X; \mathbb{Q})$$

*between the rational stable homotopy and the rational homology.*

*Proof.* The Hurewicz map  $\alpha_X : \pi_*(X) \rightarrow H_*(X; \mathbb{Z})$  gives a stable natural transformation (cf. Definition 3.1.34) between two stable homological functors. Since  $- \otimes \mathbb{Q}$  is an exact functor we thereby obtain a stable natural transformation  $\alpha_X \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \rightarrow H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \simeq H_*(X; \mathbb{Q})$ . Since the two homological functors preserve coproducts, Lemma 3.1.39 implies that the collection of spectra  $X$  such that  $\alpha_X \otimes \mathbb{Q}$  is an isomorphism forms a localizing subcategory of SH. It contains  $\mathbb{1}$  by Serre's result and hence contains the whole of  $\text{SH} = \text{loc}\langle \mathbb{1} \rangle$ .  $\square$

*Remark 6.1.13.* By Corollary 6.1.6, the stable homotopy groups  $\pi_i(X)$  of a finite spectrum  $X$  are finitely generated abelian groups. Moreover, by Corollary 6.1.12, their ranks are given simply by rational homology. The complexity of stable homotopy groups lies in their torsion. In fact, McGibbon and Neisendorfer [MN84] showed using Miller's solution of the Sullivan conjecture [Mil84] that if  $X$  is a simply-connected finite CW-complex with  $\tilde{H}(X; \mathbb{Z}/p) \neq 0$  then  $\pi_i^s(X)$  contains  $p$ -torsion for infinitely many  $i$ . This was conjectured by Serre [Ser53] at least for  $p = 2$ . For example, there is an infinite amount of  $p$ -torsion in the stable homotopy groups of spheres for any prime  $p$ . These results are pretty convincing indicators of the complexity of stable homotopy groups.<sup>1</sup> In any case, it is typical in the subject to fix a prime  $p$  and focus on the  $p$ -torsion, which leads naturally to the topic of  $p$ -localization.

## 6.2 Rationalization and $p$ -localization

First let us recall some basic notions.

**Definition 6.2.1.** An abelian group  $A$  is said to be  *$p$ -local* if the map  $n : A \rightarrow A$  is an isomorphism for every integer  $n$  such that  $p \nmid n$ . Equivalently,  $q : A \rightarrow A$  is an isomorphism for all primes  $q \neq p$ . Equivalently,  $A$  admits the (unique) structure of a  $\mathbb{Z}_{(p)}$ -module. Equivalently,

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<sup>1</sup>The McGibbon-Neisendorfer result is actually a theorem about unstable homotopy groups, but it implies the stable result that we have mentioned. A different way to obtain stable results of this nature is to use Ravenel's results on harmonic and dissonant spectra in [Rav84].

the map  $A \simeq A \otimes \mathbb{Z} \rightarrow A \otimes \mathbb{Z}_{(p)}$  is an isomorphism.

*Example 6.2.2.* The ring  $\mathbb{Z}_{(p)}$  is  $p$ -local; so is  $\mathbb{Z}/p^t$  for  $t \geq 1$ . On the other hand, if  $p \nmid n$  then  $\mathbb{Z}/n \otimes \mathbb{Z}_{(p)} = 0$ .

*Remark 6.2.3.* A finitely generated abelian group is  $p$ -local iff it is torsion and all its torsion is  $p$ -torsion. The functor  $- \otimes \mathbb{Z}_{(p)}$  turns an abelian group into a  $p$ -local abelian group and can be viewed as a process which kills all the  $q$ -torsion for primes  $q \neq p$ .

**Definition 6.2.4.** For a fixed prime number  $p$ , the  $p$ -localization of spectra refers to Bousfield localization (cf. Section 3.7) with respect to the homological functor  $\pi_*(-) \otimes \mathbb{Z}_{(p)}$ . This is a homological functor since  $\mathbb{Z}_{(p)}$  is torsion-free. The  $p$ -local spectra—i.e., the local objects for this Bousfield localization—are those spectra whose homotopy groups are  $p$ -local abelian groups (see Lemma 6.2.5 below). As usual, define the stable homotopy category of  $p$ -local spectra  $\mathrm{SH}_{(p)}$  to be the full subcategory of  $\mathrm{SH}$  consisting of the  $p$ -local spectra, and define  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  to be the full subcategory of all spectra isomorphic to the  $p$ -localization of a finite spectrum. Note that  $\mathrm{SH}_{(p)}^{\mathrm{fin}} \subsetneq \mathrm{SH}^{\mathrm{fin}}$ . For example, the  $p$ -local sphere spectrum  $\mathbb{1}_{(p)}$  has 0th homotopy group equal to  $\mathbb{Z}_{(p)}$  which is not a finitely generated abelian group—so the  $p$ -localization of a finite spectrum is not a finite spectrum in general. Finally, we'll use the notation  $(-)_{(p)} : \mathrm{SH} \rightarrow \mathrm{SH}_{(p)}$  to denote the  $p$ -localization functor. Good references for  $p$ -localization include [Mar83, Chapter 8] and [Bou79, Section 2].

**Lemma 6.2.5.** *For any spectrum  $X$ , the following are equivalent:*

- (1)  $X$  is a  $p$ -local spectrum.
- (2)  $\pi_i(X)$  is a  $p$ -local abelian group for all  $i \in \mathbb{Z}$ .
- (3) The map  $q : X \rightarrow X$  is an isomorphism for all primes  $q \neq p$ .

*Proof.* Note that (3) immediately implies (2) since applying the additive functor  $\pi_i(-)$  to  $q : X \rightarrow X$  produces  $q : \pi_i(X) \rightarrow \pi_i(X)$ . To see (2)  $\Rightarrow$  (3), consider the exact triangle

$$X \xrightarrow{q} X \rightarrow X/q \rightarrow \Sigma X$$

and the associated long exact sequence

$$\pi_i(X) \xrightarrow{q_\cdot} \pi_i(X) \rightarrow \pi_i(X/q) \rightarrow \pi_{i-1}(X) \xrightarrow{q_\cdot} \pi_{i-1}(X).$$

If each  $\pi_i(X)$  is  $p$ -local then the maps  $q : \pi_i(X) \rightarrow \pi_i(X)$  are isomorphisms; hence  $\pi_i(X/q) = 0$  for all  $i$ . Since  $\pi_\bullet(-)$  is conservative, it follows that  $X/q = 0$  and hence that  $q : X \rightarrow X$  is an isomorphism. On the other hand, note that for any spectrum  $A$ , the map  $q : A \rightarrow A$  is a  $p$ -local equivalence. Consequently, the induced map  $[A, X] \xrightarrow{-\circ q} [A, X]$  is an isomorphism for any  $p$ -local spectrum  $X$ . Taking  $A = \Sigma^i \mathbb{1}$  and noting that  $[\Sigma^i \mathbb{1}, X] \xrightarrow{-\circ q} [\Sigma^i \mathbb{1}, X]$  is just  $q : \pi_i(X) \rightarrow \pi_i(X)$ , we see that  $q : \pi_i(X) \rightarrow \pi_i(X)$  is an isomorphism when  $X$  is  $p$ -local. This proves (1)  $\Rightarrow$  (2). Finally, to prove (2)  $\Rightarrow$  (1) consider the  $p$ -localization exact triangle

$$W \longrightarrow X \xrightarrow{\eta} X_{(p)} \longrightarrow \Sigma W.$$

Applying  $\pi_\bullet(-) \otimes \mathbb{Z}_{(p)}$  and using the fact that  $W$  is  $p$ -acyclic we obtain from the long exact sequence that

$$\pi_i(X) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_i(\eta) \otimes \mathbb{Z}_{(p)}} \pi_i(X_{(p)}) \otimes \mathbb{Z}_{(p)}$$

is an isomorphism for all  $i$ . If  $\pi_i(X)$  is  $p$ -local then we have

$$\begin{array}{ccc} \pi_i(X) \otimes \mathbb{Z}_{(p)} & \xrightarrow{\pi_i(\eta) \otimes \mathbb{Z}_{(p)}} & \pi_i(X_{(p)}) \otimes \mathbb{Z}_{(p)} \\ \downarrow \sim & & \downarrow \sim \\ \pi_i(X) & \xrightarrow{\pi_i(\eta)} & \pi_i(X_{(p)}) \end{array}$$

and hence  $\pi_i(\eta)$  is an isomorphism for all  $i$ . Since  $\pi_\bullet$  reflects isomorphisms, we conclude that  $\eta : X \rightarrow X_{(p)}$  is an isomorphism; hence  $X$  is  $p$ -local. □

**Lemma 6.2.6.** *For any spectrum  $X$ ,  $\pi_i(X_{(p)}) \simeq \pi_i(X) \otimes \mathbb{Z}_{(p)}$ .*

*Proof.* Applying  $\pi_i(-) \otimes \mathbb{Z}_{(p)}$  to the exact triangle

$$W \longrightarrow X \xrightarrow{\eta} X_{(p)} \longrightarrow \Sigma W$$

we obtain an isomorphism  $\pi_i(X) \otimes \mathbb{Z}_{(p)} \xrightarrow{\sim} \pi_i(X_{(p)}) \otimes \mathbb{Z}_{(p)}$  and since  $\pi_i(X_{(p)})$  is  $p$ -local (by Lemma 6.2.5) we have an isomorphism  $\pi_i(X_{(p)}) \otimes \mathbb{Z}_{(p)} \simeq \pi_i(X_{(p)})$ .  $\square$

**Definition 6.2.7.** For any integer  $n \in \mathbb{Z}$ , the Moore spectrum  $M(n)$  is defined by the exact triangle

$$\mathbb{1} \xrightarrow{n} \mathbb{1} \rightarrow M(n) \rightarrow \Sigma \mathbb{1}$$

in the stable homotopy category. Note that  $\pi_0(M(n)) \simeq \mathbb{Z}/n$  and  $\pi_i(M(n)) = 0$  for  $i < 0$ .

**Proposition 6.2.8.** *For any prime  $p$ ,  $p$ -localization is a finite localization in the sense of Definition 3.7.12. In other words, the localizing subcategory of  $p$ -acyclics is generated (as a localizing subcategory) by the finite  $p$ -acyclics.*

*Proof.* Note that if  $p \nmid n$  then the Moore spectrum  $M(n)$  is  $p$ -acyclic. Hence  $\text{loc}\langle M(n) : p \nmid n \rangle$  is contained in the localizing subcategory of  $p$ -acyclic spectra. To see that these subcategories coincide we use the fact that the fiber of the localization map  $\mathbb{1} \rightarrow \mathbb{1}_{(p)}$  is the homotopy colimit  $\text{hocolim} \Sigma^{-1} M(n)$  over the diagram of natural numbers coprime to  $p$  with divisibility defining the maps. In other words, we have an exact triangle

$$\text{hocolim} \Sigma^{-1} M(n) \longrightarrow \mathbb{1} \longrightarrow \mathbb{1}_{(p)} \longrightarrow \Sigma \text{hocolim} \Sigma^{-1} M(n).$$

It is easy to see that  $p$ -localization is a smashing localization (since the subcategory of  $p$ -local spectra is evidently closed under coproducts) and thus if  $X$  is a  $p$ -acyclic spectrum then we have an isomorphism  $X \wedge \text{hocolim} \Sigma^{-1} M(n) \simeq X$ . Hence  $X$  is contained in  $\text{loc}\langle M(n) : p \nmid n \rangle$ .  $\square$

*Remark 6.2.9.* We have defined the  $p$ -local stable homotopy category  $\mathrm{SH}_{(p)}$  in the usual way—as a subcategory of  $\mathrm{SH}$ —but for some purposes it is convenient to regard it as the Verdier quotient  $\mathrm{SH}/(p\text{-acyclic spectra})$  instead (cf. Remark 3.7.5). For example, the  $p$ -local stable homotopy category inherits its structure as a *tensor* triangulated category via the equivalence  $\mathrm{SH}/(p\text{-acyclic spectra}) \cong \mathrm{SH}_{(p)}$  (cf. Remark 3.7.6). From this point of view, we would like to also realize the stable homotopy category of *finite*  $p$ -local spectra  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  as a Verdier quotient of the stable homotopy category of *finite* spectra  $\mathrm{SH}^{\mathrm{fin}}$ :

**Proposition 6.2.10.** *The stable homotopy category of finite  $p$ -local spectra  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  is the subcategory of compact objects in  $\mathrm{SH}_{(p)}$ . Moreover, there is an equivalence of tensor triangulated categories*

$$\mathrm{SH}_{(p)}^{\mathrm{fin}} \cong \mathrm{SH}^{\mathrm{fin}} / (\text{finite } p\text{-acyclic spectra})$$

*under which the  $p$ -localization functor  $\mathrm{SH}^{\mathrm{fin}} \rightarrow \mathrm{SH}_{(p)}^{\mathrm{fin}}$  becomes the Verdier quotient functor  $\mathrm{SH}^{\mathrm{fin}} \rightarrow \mathrm{SH}^{\mathrm{fin}} / (\text{finite } p\text{-acyclic spectra})$ .*

*Proof.* Because  $p$ -localization is a finite localization (cf. Proposition 6.2.8) we can use the techniques of Neeman-Thomason (i.e. Theorem 3.7.16). It follows from the fact that the quotient functor  $q : \mathrm{SH} \rightarrow \mathrm{SH}_{(p)}$  preserves compactness that  $\mathrm{SH}_{(p)}$  is compactly generated by  $\mathbb{1}_{(p)}$  and that we have the following commutative diagram of exact functors:

$$\begin{array}{ccccc} (\text{finite } p\text{-acyclics}) & \hookrightarrow & \mathrm{SH}^{\mathrm{fin}} & \longrightarrow & \mathrm{SH}^{\mathrm{fin}} / (\text{finite } p\text{-acyclics}) \\ \downarrow & & \downarrow & & \downarrow J \\ & & \mathrm{SH} & \longrightarrow & \mathrm{SH}_{(p)}^c \\ & & \downarrow & & \downarrow \\ (p\text{-acyclics}) & \hookrightarrow & \mathrm{SH} & \longrightarrow & \mathrm{SH}_{(p)}. \end{array}$$

Theorem 3.7.16 implies that  $J$  is fully faithful. It induces an equivalence between  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$ , which is the essential image of  $J$  essentially by definition, and  $\mathrm{SH}^{\mathrm{fin}} / (\text{finite } p\text{-acyclics})$ . Moreover, Theorem 3.7.16 states that every object in  $\mathrm{SH}_{(p)}^c$  is a direct summand of an object

in  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$ . In other words,  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  is a *dense* triangulated subcategory of  $\mathrm{SH}_{(p)}^c$ . One way to see that  $\mathrm{SH}_{(p)}^{\mathrm{fin}} = \mathrm{SH}_{(p)}^c$  is to use Thomason's correspondence between the dense triangulated subcategories of an (essentially small) triangulated category  $\mathcal{T}$  and the subgroups of the Grothendieck group  $K_0(\mathcal{T})$ . Indeed, under the correspondence (see [Tho97, Theorem 2.1]) a dense triangulated subcategory  $i : \mathcal{S} \hookrightarrow \mathcal{T}$  corresponds to the subgroup

$$\mathrm{im}(i : K_0(\mathcal{S}) \rightarrow K_0(\mathcal{T})) \leq K_0(\mathcal{T})$$

and so in particular  $\mathcal{S} = \mathcal{T}$  iff the map  $i : K_0(\mathcal{S}) \rightarrow K_0(\mathcal{T})$  is surjective. In our example, to establish the surjectivity of  $K_0(\mathrm{SH}_{(p)}^{\mathrm{fin}}) \rightarrow K_0(\mathrm{SH}_{(p)}^c)$  it suffices to establish the surjectivity of the composite  $K_0(\mathrm{SH}^{\mathrm{fin}}) \rightarrow K_0(\mathrm{SH}_{(p)}^{\mathrm{fin}}) \rightarrow K_0(\mathrm{SH}_{(p)}^c)$ . Using the fact that  $\mathrm{SH}^{\mathrm{fin}} = \mathrm{SH}^c = \mathrm{thick}\langle \mathbb{1} \rangle$  and  $\mathrm{SH}_{(p)}^c = \mathrm{thick}\langle \mathbb{1}_{(p)} \rangle$ , one can show (see for instance [Che06, Section 3]) that the Euler characteristic  $X \mapsto \sum_{i \in \mathbb{Z}} \dim H\mathbb{Q}_i(X)$  provides isomorphisms  $K_0(\mathrm{SH}^{\mathrm{fin}}) \xrightarrow{\sim} \mathbb{Z}$  and  $K_0(\mathrm{SH}_{(p)}^c) \xrightarrow{\sim} \mathbb{Z}$ . The abelian group homomorphism  $\mathbb{Z} \simeq K_0(\mathrm{SH}^{\mathrm{fin}}) \rightarrow K_0(\mathrm{SH}_{(p)}^c) \simeq \mathbb{Z}$  is then surjective as it sends  $1 = [\mathbb{1}]$  to  $1 = [\mathbb{1}_{(p)}]$ .  $\square$

**Definition 6.2.11.** Rationalization of spectra refers to Bousfield localization with respect to the homological functor  $\pi_*(-) \otimes \mathbb{Q}$  which by Corollary 6.1.12 is the same thing as rational homology  $H_*(-; \mathbb{Q})$ . The acyclic objects form a localizing subcategory  $\mathrm{SH}_{\mathrm{tor}} \subset \mathrm{SH}$  of *torsion spectra*, as well as a thick subcategory  $\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}} \subset \mathrm{SH}^{\mathrm{fin}}$  of *finite torsion spectra*. A finite spectrum is a torsion spectrum iff the abelian group  $\pi_i(X)$  is torsion for all  $i \in \mathbb{Z}$ .

The last goal for this section is to establish that our notion of algebraic localization (cf. Section 5.7) in the example  $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$  is just the same thing as  $p$ -localization and rationalization.

**Proposition 6.2.12.** *Algebraic localization of  $\mathrm{SH}^{\mathrm{fin}}$  at the prime  $(p) \subset \mathbb{Z} \cong \mathrm{End}_{\mathrm{SH}^{\mathrm{fin}}}(\mathbb{1})$  is  $p$ -localization. Algebraic localization of  $\mathrm{SH}^{\mathrm{fin}}$  at the prime  $(0)$  is rationalization.*

*Proof.* In light of Proposition 6.2.10, proving the first statement amounts to showing that the thick  $\otimes$ -ideal  $\text{thick}_{\otimes} \langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid p \nmid d \rangle$  is precisely the set of finite  $p$ -acyclic spectra. One inclusion is easily obtained by applying  $\pi_{\bullet}(-) \otimes \mathbb{Z}_{(p)}$  to an exact triangle for  $d : \mathbb{1} \rightarrow \mathbb{1}$ . In fact, we've used a similar argument in the proof that  $p$ -localization is a finite localization (cf. Proposition 6.2.8). Note that  $\text{cone}(d : \mathbb{1} \rightarrow \mathbb{1})$  is the Moore spectrum  $M(d)$  and that every thick subcategory is a thick  $\otimes$ -ideal in  $\text{SH}^{\text{fin}}$ . On the other hand, if  $X$  is a finite  $p$ -acyclic spectrum then  $\pi_i(X)$  is finite with no  $p$ -torsion for all  $i \in \mathbb{Z}$ . For any finite spectrum  $X$  it is straightforward to check that

$$\mathcal{I}_X := \{Y \in \text{SH}^{\text{fin}} \mid [Y, X]_i \text{ is finite with no } p\text{-torsion for all } i \in \mathbb{Z}\}$$

is a thick subcategory of  $\text{SH}^{\text{fin}}$ . If  $X$  is finite  $p$ -acyclic then  $\mathcal{I}_X$  contains  $\mathbb{1}$  and hence contains the whole of  $\text{SH}^{\text{fin}}$ . In particular  $\mathcal{I}_X$  contains  $X$  itself and we conclude that  $\text{id}_X$  has finite order  $d$  prime to  $p$ . Then  $\Sigma X \oplus X \simeq \text{cone}(d \cdot \text{id}_X) \simeq X \otimes \text{cone}(d \cdot \text{id}_{\mathbb{1}})$  establishes that  $X$  is contained in  $\text{thick}_{\otimes} \langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid p \nmid d \rangle$ . A similar approach can be used to prove that  $\text{SH}_{\text{tor}}^{\text{fin}}$  is equal to  $\text{thick}_{\otimes} \langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid d \neq 0 \rangle$  and just as in Proposition 6.2.10, rationalization of finite spectra can be regarded as the Verdier quotient  $\text{SH}^{\text{fin}} \rightarrow \text{SH}^{\text{fin}} / \text{SH}_{\text{tor}}^{\text{fin}}$ .  $\square$

### 6.3 Complex cobordism and Morava $K$ -theory

The three most fundamental cohomology theories in algebraic topology are indisputable: ordinary cohomology, topological  $K$ -theory, and complex cobordism. The importance of complex cobordism  $MU^*(-)$  can be appreciated from several points of view, not least of which is the fact that it is the universal complex-oriented cohomology theory and is thereby the breeding ground for a host of cohomology theories intimately related to the theory of formal group laws. It is represented in the stable homotopy category by the Thom spectrum  $MU$  and an indication of its strength is provided by the Nilpotence Theorem (cf. Theorem 6.4.1

below) which implies that  $MU^*(-)$  detects nilpotent maps between finite spectra. Its coefficient ring  $\pi_*(MU)$  classifies the cobordism classes of almost complex manifolds (see [Sto68, Chapter VII]) and was shown by Milnor and Novikov to be a polynomial ring

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$$

in an infinite number of variables with  $|x_i| = 2i$ . Brown and Peterson [BP66] were the first to observe that the  $p$ -localization of  $MU$  splits as a direct sum of shifted copies of a “smaller”  $p$ -local ring spectrum  $BP$ :

**Theorem 6.3.1** (Brown-Peterson; Quillen). *Fix a prime  $p$ . There exists a unique  $p$ -local ring spectrum  $BP$  which is a direct summand of  $MU_{(p)}$  such that the map  $MU_{(p)} \rightarrow BP$  is a ring homomorphism. The coefficient ring is a polynomial ring*

$$BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

with  $|v_n| = 2(p^n - 1)$ . In fact,  $MU_{(p)}$  splits as a direct sum of shifted copies of  $BP$ .

*Remark 6.3.2.* Two references for understanding this construction are [Ada74, §II.15] and [Wil82]. Although the spectrum  $BP$  was first constructed by Brown and Peterson, it was clarified by Quillen [Qui69]. He showed that  $BP$  can be defined as the image of a multiplicative idempotent  $MU_{(p)} \xrightarrow{e} MU_{(p)}$  now called the “Quillen idempotent.” By general nonsense (cf. Lemma 3.1.21), the idempotent splits

$$\begin{array}{ccc} MU_{(p)} & \xrightarrow{e} & MU_{(p)} \\ & \searrow & \nearrow \\ & BP & \end{array}$$

and  $BP$  is a direct-summand of  $MU_{(p)}$ . Moreover, since  $MU_{(p)}$  splits as a direct sum of shifted copies of  $BP$ , results for  $MU_{(p)}$  can generally be expressed in terms of  $BP$ —and vice versa. Indeed there is some convenience when working  $p$ -locally to use the “smaller” spectrum  $BP$  rather than  $MU_{(p)}$ .

*Remark 6.3.3.* The Brown-Peterson spectrum  $BP$  gives rise to a whole collection of different cohomology theories. The basic idea is to attempt to realize algebraic operations that one can perform on the coefficient ring  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$  in the category of spectra. For example, given a homogeneous ideal  $I \subset BP_*$  we can ask whether there exists a spectrum  $BP/I$  with  $\pi_*(BP/I) = \pi_*(BP)/I$ . Similarly, given a homogeneous element  $v \in BP_*$  we can ask whether there exists a spectrum  $v^{-1}BP$  such that  $\pi_*(v^{-1}BP) = BP_*[v^{-1}]$ . Historically, such spectra were constructed using the Baas-Sullivan theory of manifolds with singularities together with the Landweber exact functor theorem. We'll follow the historical approach (as it is the method used in the fundamental papers on the Ravenel conjectures) but in Remark 6.3.18 we will mention more recent developments. The basic idea of the Baas-Sullivan theory is that by choosing the types of singularities allowed in our manifolds we can construct cobordism theories with prescribed properties—e.g., which realize a certain formal group law, or which have a certain coefficient ring, etc. The interested reader can look at [Baa73] for an original source and [Rud98, Chapter IX] for a more recent exposition. On the other hand, the Landweber exact functor theorem [Lan76] is far more algebraic in nature. It has versions both for  $MU$  and  $BP$  but we'll just describe the  $BP$  version. To this end, define homogeneous ideals  $I_n := (p, v_1, \dots, v_{n-1}) \subset BP_*$  for each  $n \geq 1$ .

**Theorem 6.3.4** (Landweber exact functor theorem). *Let  $M$  be a (graded)  $BP_*$ -module. The functor  $BP_*(-) \otimes_{BP_*} M$  is a stable homological functor iff for all  $n \geq 1$ , multiplication by  $v_n$  on  $M \otimes_{BP_*} BP_*/I_n$  is a monomorphism. In particular, for such an  $M$  there exists a  $BP$ -module spectrum  $E$  with  $\pi_*(E) \simeq M$ .*

*Remark 6.3.5.* Recall from Remark 3.5.5 that every stable homological functor on the stable homotopy category is represented by a spectrum—i.e. SH is a Brown category in the terminology of [HPS97]. This explains why the second statement in the theorem follows from the first.

*Remark 6.3.6.* A  $BP_*$ -module is said to be *Landweber exact* if it satisfies the conditions of the theorem. Note that if  $E$  is any  $BP$ -module spectrum then we can construct a natural map of spectra  $BP \wedge E \wedge X \xrightarrow{\rho \wedge 1} E \wedge X$  where  $\rho : BP \wedge E \rightarrow E$  is the module structure map. Applying  $\pi_*$  and using Lemma 3.5.7 we obtain a stable natural transformation  $BP_*(X) \otimes_{BP_*} E_* \rightarrow E_*(X)$  given by

$$\pi_*(BP \wedge X) \otimes_{BP_*} \pi_*(E) \rightarrow \pi_*(BP \wedge X \wedge E) \simeq \pi_*(BP \wedge E \wedge X) \rightarrow \pi_*(E \wedge X).$$

If the coefficient module  $E_*$  is Landweber exact then this is a stable natural transformation between stable homological functors which preserve coproducts. It is an isomorphism when  $X = \mathbb{1}$  and hence by Lemma 3.1.39 it is a natural isomorphism for all  $X$ . This explains that although Landweber exactness is just a condition on the coefficient module  $E_*$  it implies that the homology theory  $E_*(X) \simeq BP_*(X) \otimes_{BP_*} E_*$  is just Brown-Peterson homology tensored with  $E_*$ .

*Example 6.3.7.* The  $BP_*$ -modules  $BP_*[v_n^{-1}]$  and  $\mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$  are both Landweber exact. Indeed, in both cases  $M/I_{n+1} = 0$  so the condition for  $v_i$  is automatically satisfied for  $i \geq n+1$ . Moreover, it evidently holds for  $i = 1, \dots, n$ . Thus, there exist  $BP$ -module spectra  $v_n^{-1}BP$  and  $E(n)$  such that  $\pi_*(v_n^{-1}BP) = BP_*[v_n^{-1}]$  and  $\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$ . On the other hand, other interesting examples of  $BP_*$ -modules such as

- $BP_*/I_n$
- $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$
- $v_n^{-1}BP_*/I_n$
- $\mathbb{F}_p[v_n, v_n^{-1}]$

are not Landweber exact. Although there do exist  $BP$ -module spectra which realize these

$BP_\bullet$ -modules, they cannot be obtained from the Landweber exact functor theorem and their associated homology theories do not take the form  $BP_\bullet(-) \otimes_{BP_\bullet} E_\bullet$ .

**Theorem 6.3.8** (Johnson-Wilson; Würgler). *Fix a prime number  $p$ . For each  $n \geq 1$  there is a  $BP$ -module spectrum  $P(n)$  such that  $\pi_\bullet(P(n)) = BP_\bullet/I_n$ . If  $p$  is odd then  $P(n)$  admits a unique structure of a commutative ring spectrum compatible with the  $BP$ -module structure. If  $p = 2$  then  $P(n)$  admits two noncommutative multiplications  $\mu_1$  and  $\mu_2$  which are opposite to each other:  $\mu_1 = \mu_2 \circ \tau$ .*

*Proof.* These spectra were first constructed by Johnson and Wilson [JW75] using the Baas-Sullivan theory. Würgler studied their multiplicative structures: for odd  $p$  in [Wur77] and for  $p = 2$  in [Wur86]. □

*Remark 6.3.9.* Yagita [Yag76] observed that an analogue of the Landweber exact functor theorem holds for the spectra  $P(n)$ :

**Theorem 6.3.10** (Yagita). *Let  $M$  be a (graded)  $P(n)_\bullet$ -module. The functor  $P(n)_\bullet(-) \otimes_{P(n)_\bullet} M$  is a stable homological functor iff for each  $m > n$ , multiplication by  $v_m$  on  $M \otimes_{P(n)_\bullet} BP_\bullet/I_m$  is a monomorphism.*

*Example 6.3.11.* This condition is satisfied, for example, by the  $P(n)_\bullet$ -modules  $\mathbb{F}_p[v_n, v_n^{-1}]$  and  $v_n^{-1}BP_\bullet/I_n$ . The first example provides the celebrated Morava  $K$ -theories:

**Theorem 6.3.12.** *Fix a prime  $p$ . For each  $n \geq 1$  there is a  $p$ -local ring spectrum  $K(n)$ , called the  $n$ th Morava  $K$ -theory spectrum (for the prime  $p$ ), whose coefficient ring is the graded-field  $K(n)_\bullet = \mathbb{F}_p[v_n, v_n^{-1}]$  with  $|v_n| = 2(p^n - 1)$ . For  $p > 2$  the ring structure on  $K(n)$  is unique and commutative. For  $p = 2$  there are two noncommutative multiplications  $\mu_1$  and  $\mu_2$  which are opposite to each other:  $m_1 = m_2 \circ \tau$ .*

*Proof.* Jack Morava was the first to construct and exploit these theories, although he never published his results. His work was heavily infused with algebraic geometry and [JW75] provided a construction of the Morava  $K$ -theories using more standard algebraic topology techniques. See [Wur91] for an old survey on the subject.  $\square$

*Remark 6.3.13.* The Morava  $K$ -theories are called  $K$ -theories because they have periodic phenomena analogous to Bott periodicity. Indeed,  $v_n \in K(n)_*$  is an invertible homogeneous element and hence multiplication by  $v_n$  provides an isomorphism of graded modules

$$v_n : K(n)_*(X) \xrightarrow{\sim} K(n)_{*+|v_n|}(X)$$

for any spectrum  $X$ . Thus, the functor  $K(n)_*(-)$  is periodic with period  $2(p^n - 1)$ . Similarly, the coefficient ring for complex  $K$ -theory is the Laurent polynomial ring  $\mathbb{Z}[t, t^{-1}]$  with  $|t| = 2$  and multiplication by  $t$  amounts to Bott periodicity.

**Definition 6.3.14.** The collection of Morava  $K$ -theories  $K(n)$  is completed by defining

$$K(0) := H\mathbb{Q} \quad \text{and} \quad K(\infty) := H\mathbb{F}_p.$$

Note that these are also  $p$ -local ring spectra since  $\mathbb{Q}$  and  $\mathbb{F}_p$  are  $p$ -local abelian groups. The following lemma justifies the definition of  $K(\infty)$ :

**Lemma 6.3.15.** *Let  $X$  and  $Y$  be finite  $p$ -local spectra. For  $m \gg 0$ ,  $K(m)_*X \simeq K(m)_* \otimes H\mathbb{F}_p_*X$ ,  $K(m)_*Y \simeq K(m)_* \otimes H\mathbb{F}_p_*Y$ , and  $K(m)_*f \simeq K(m)_* \otimes H\mathbb{F}_p_*f$  for any map  $f : X \rightarrow Y$ .*

*Remark 6.3.16.* The Morava  $K$ -theories have a number of special properties. For example, they satisfy a Künneth formula:

**Lemma 6.3.17.** *For any two spectra  $X$  and  $Y$ , the natural transformation*

$$K(n)_*X \otimes_{K(n)_*} K(n)_*Y \rightarrow K(n)_*(X \wedge Y)$$

is a natural isomorphism. In other words, the stable homological functor

$$K(n)_\bullet : \mathbf{SH} \rightarrow K(n)_\bullet\text{-grMod}$$

is a strong  $\otimes$ -functor.

*Proof.* The natural transformation is the one defined in Lemma 3.5.7. The key point is that  $K(n)_\bullet X$  is a flat (in fact free)  $K(n)_\bullet$ -module since  $K(n)_\bullet$  is a graded-field. Thus

$$K(n)_\bullet X \otimes_{K(n)_\bullet} K(n)_\bullet (-) \rightarrow K(n)_\bullet (X \wedge -)$$

is a stable natural transformation of stable homological functors. It is an isomorphism on  $\mathbb{1}$  and hence is an isomorphism for all  $Y$ .  $\square$

*Remark 6.3.18.* The method we have outlined for constructing the Morava  $K$ -theories using the Landweber exact functor theorem and the Baas-Sullivan theory of manifolds with singularities is the classical approach used in the literature from the 1970s and 1980s. However, the new foundations for stable homotopy theory invented in the 1990s provide for a newer approach. For example, [HS99, §1.1] discusses how to construct the Morava  $K$ -theory spectrum  $K(n)$  using the  $S$ -modules of [EKM97]. The basic idea is that if  $E$  is a commutative  $S$ -algebra then there is a well-behaved category of  $E$ -module spectra in which one can perform module-theoretic constructions that mirror the constructions we desire on the coefficient ring. However, for these more recent techniques, one must use  $MU_{(p)}$  instead of  $BP$  because the Brown-Peterson spectrum is *not* known to have an  $E_\infty$ -structure and thus cannot be regarded as a commutative  $S$ -algebra. In contrast, the Thom spectrum  $MU$  has a canonical  $E_\infty$ -structure—indeed, it is one of the prototypical examples of an  $E_\infty$ -ring spectrum [May77]—and  $MU_{(p)}$  inherits such a structure. In fact, according to [JN10] there is a fundamental incompatibility between  $BP$  and the  $E_\infty$ -structure on  $MU$ : even if  $BP$  had the

structure of an  $E_\infty$ -ring spectrum, the canonical map  $MU \rightarrow BP$  would not be  $E_\infty$ . In any case, at present  $BP$  is only known to be  $E_4$ ; see [BM13].

## 6.4 The nilpotence theorem

In the influential paper [Rav84], Doug Ravenel studied Bousfield localization with respect to the various  $BP$ -module spectra mentioned in the last section and formulated seven conjectures concerning the structure of the stable homotopy category (cf. the preface of [Rav92]). The conjecture which turned out to be the most important was the so-called “nilpotence conjecture” which stated that complex bordism detects nilpotent maps between finite spectra. The proof of this conjecture by Devinatz, Hopkins and Smith [DHS88] led to the solution of all but one of Ravenel’s conjectures in the follow-up work by Hopkins and Smith [HS98]. The nilpotence theorem which lies at the heart of these results actually has three forms depending on what one means by nilpotent:

**Theorem 6.4.1** (Devinatz-Hopkins-Smith). *The following three “nilpotence” theorems hold:*

(1) *Let  $R$  be a ring spectrum. The kernel of the  $MU$  Hurewicz map*

$$\pi_* R \rightarrow MU_* R \tag{6.4.2}$$

*consists of nilpotent elements.*

(2) *Let  $f : X \rightarrow Y$  be a map from a finite spectrum  $X$  to an arbitrary spectrum  $Y$ . If  $MU \wedge f = 0$  then  $f$  is smash-nilpotent: there exists  $n \geq 1$  such that  $f^{\wedge n} = 0$ .*

(3) *Let  $f : \Sigma^k X \rightarrow X$  be a selfmap of a finite spectrum  $X$ . If  $MU_*(f) = 0$  then  $f$  is nilpotent: there exists  $n \geq 1$  such that  $f^n = 0$ .*

*Proof.* This is a hard topological result and the paper [DHS88] is devoted to proving it. Let

us merely indicate how (2) implies (1) and how (1) implies (3). Suppose  $\alpha \in \pi_i R$  is in the kernel of the  $MU$  Hurewicz map (6.4.2). In other words, the composite

$$\Sigma^i \mathbb{1} \xrightarrow{\alpha} R \simeq \mathbb{1} \wedge R \xrightarrow{\eta \otimes 1} MU \wedge R$$

is equal to zero. One readily checks that this composite is equal to

$$\Sigma^i \mathbb{1} \simeq \mathbb{1} \wedge \Sigma^i \mathbb{1} \xrightarrow{\eta \wedge 1} MU \wedge \Sigma^i \mathbb{1} \xrightarrow{MU \wedge \alpha} MU \wedge R$$

but smashing this map with  $MU$  and using the fact that  $MU$  is a ring spectrum we observe

$$\begin{array}{ccccccc} MU \wedge \Sigma^i \mathbb{1} & \xrightarrow{\sim} & MU \wedge \mathbb{1} \wedge \Sigma^i \mathbb{1} & \xrightarrow{MU \wedge \eta \wedge 1} & MU \wedge MU \wedge \Sigma^i \mathbb{1} & \xrightarrow{MU \wedge MU \wedge \alpha} & MU \wedge MU \wedge R \\ & \searrow \text{id} & & & \downarrow \mu \wedge 1 & \nearrow MU \wedge \alpha & \\ & & & & MU \wedge \Sigma^i \mathbb{1} & & \end{array}$$

and conclude that  $MU \wedge \alpha = 0$ . Hence (2) implies that that  $\alpha^{\wedge n} = 0$  for some  $n \geq 1$ . But according to the definition of multiplication in the ring  $\pi_*(R)$ , the product  $\alpha^n$  is the composite

$$\Sigma^{in} \mathbb{1} \simeq (\Sigma^i \mathbb{1})^{\wedge n} \xrightarrow{\alpha^{\wedge n}} R^{\wedge n} \xrightarrow{\mu} R$$

so indeed  $\alpha^n = 0$ . This proves that (2) implies (1). Now for (1) implies (3). If  $\pi_*(MU \wedge f) = 0$  then

$$\begin{aligned} 0 &= \text{colim} \left( \pi_*(MU \wedge X) \xrightarrow{(MU \wedge f)_*} \pi_*(MU \wedge \Sigma^{-k} X) \xrightarrow{(MU \wedge f)_*} \dots \right) \\ &= \pi_*(\text{hocolim}(MU \wedge X \xrightarrow{MU \wedge f} MU \wedge \Sigma^{-k} X \xrightarrow{MU \wedge f} \dots)) \end{aligned}$$

which implies that  $\text{hocolim}(MU \wedge X \xrightarrow{MU \wedge f} MU \wedge \Sigma^{-k} X \xrightarrow{MU \wedge f} \dots) = 0$ . Since  $X$  is finite (a.k.a compact) it follows that

$$X \xrightarrow{\eta \otimes 1} MU \otimes X \xrightarrow{MU \wedge f} MU \wedge \Sigma^{-k} X \xrightarrow{MU \wedge f} \dots \xrightarrow{MU \wedge f} MU \wedge \Sigma^{-kn} X$$

is zero for  $n$  large enough, which is the same map as

$$\Sigma^{kn} X \xrightarrow{f^n} X \xrightarrow{\eta \wedge 1} MU \wedge X.$$

Replacing  $f$  by  $f^n$  we can assume without loss of generality that  $n = 1$ . Now  $X$  is dualizable, so we can consider the ring spectrum  $X \wedge DX$ . We have an isomorphism of rings  $[X, X]_{\bullet} \simeq \pi_{\bullet}(X \wedge DX)$  and the following diagram commutes

$$\begin{array}{ccc} [X, X]_{\bullet} & \xrightarrow{\sim} & [\mathbb{1}, X \wedge DX]_{\bullet} \\ (\eta \wedge 1) \circ - \downarrow & & \downarrow (\eta \wedge 1 \wedge 1) \circ - \\ [X, MU \wedge X]_{\bullet} & \xrightarrow{\sim} & [\mathbb{1}, MU \wedge X \wedge DX]_{\bullet} \end{array}$$

where the right-hand vertical map is the Hurewicz map for the ring spectrum  $X \wedge DX$ . Thus the map  $\tilde{f} \in \pi_{\bullet}(X \wedge DX)$  corresponding to  $f \in [X, X]_{\bullet}$  is in the kernel of the Hurewicz map—hence is nilpotent by (1)—and we conclude that  $f$  is nilpotent in  $[X, X]_{\bullet}$ .  $\square$

*Remark 6.4.3.* The nilpotence theorem holds  $p$ -locally and in fact  $MU$  can be replaced by  $BP$ . Furthermore, as shown in [HS98], all three forms of the nilpotence theorem can be expressed in terms of the Morava  $K$ -theories:

**Theorem 6.4.4.** *The following three “nilpotence” theorems hold:*

- (1) *Let  $R$  be a  $p$ -local ring spectrum. An element  $\alpha \in \pi_{\bullet}(R)$  is nilpotent iff  $K(n)_{\bullet}(\alpha)$  is nilpotent for all  $0 \leq n \leq \infty$ .*
- (2) *A map  $f : X \rightarrow Y$  from a finite  $p$ -local spectrum to an arbitrary  $p$ -local spectrum is smash-nilpotent iff  $K(n)_{\bullet}f = 0$  for all  $0 \leq n \leq \infty$ .*
- (3) *A graded endomorphism  $f : \Sigma^d X \rightarrow X$  of a finite  $p$ -local spectrum is nilpotent iff  $K(n)_{\bullet}f$  is nilpotent for all  $0 \leq n < \infty$ .*

*Remark 6.4.5.* For a graded endomorphism  $f : \Sigma^d X \rightarrow X$  of a finite  $p$ -local spectrum,  $K(\infty)_{\bullet}f$  is nilpotent iff  $K(n)_{\bullet}f$  is nilpotent for all  $n \gg 0$ . See [HS98, Corollary 2.2]. This explains that in statement (3) the  $n = \infty$  case could be included as well.

*Remark 6.4.6.* The strongest form of the theorem is the “smash-nilpotence” result (2); cf. the proof of Theorem 6.4.1. As we shall discuss in the next section, this form of the nilpotence theorem provided [HS98] with a classification of the thick subcategories in the category of finite  $p$ -local spectra  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$ . Recall from Section 3.9 that Thomason’s classification of the thick  $\otimes$ -ideals in  $\mathrm{D}_{\mathrm{perf}}(X)$  depended on an analogous “tensor-nilpotence” theorem in that setting.

## 6.5 Classification of thick subcategories

Fix a prime  $p$ . For each  $0 \leq n \leq \infty$ , the  $n$ th Morava  $K$ -theory spectrum  $K(n)$  provides a stable homological functor

$$K(n)_{\bullet}(-) : \mathrm{SH}_{(p)}^{\mathrm{fin}} \rightarrow K(n)_{\bullet}\text{-grMod}$$

which is in fact a strong  $\otimes$ -functor. It follows, using the fact that  $K(n)_{\bullet}$  is a graded-field, that the kernel of this functor is more than a thick subcategory—it is a prime  $\otimes$ -ideal; i.e. a point of  $\mathrm{Spc}(\mathrm{SH}_{(p)}^{\mathrm{fin}})$ . Conforming to the notation of [HS98] we define  $\mathcal{C}_{\infty} := \ker(K(\infty)_{\bullet}(-))$ ,  $\mathcal{C}_n := \ker(K(n-1)_{\bullet}(-))$  for  $n \geq 1$ , and  $\mathcal{C}_0 := \mathrm{SH}_{(p)}^{\mathrm{fin}}$ .

**Theorem 6.5.1.** *These categories fit into a filtration*

$$0 = \mathcal{C}_{\infty} \subsetneq \cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0 = \mathrm{SH}_{(p)}^{\mathrm{fin}}.$$

*Proof.* The inclusions  $\mathcal{C}_{n+1} \subset \mathcal{C}_n$  follow from a theorem of Ravenel [Rav84, Theorem 2.11] and Mitchell [Mit85] proved that they are strict. Moreover, it follows from Lemma 6.3.15 that  $\mathcal{C}_{\infty} = \bigcap_{n \geq 1} \mathcal{C}_n$  and it is elementary that  $H\mathbb{F}_p_{\bullet} X = 0$  implies that  $X = 0$  for a finite  $p$ -local spectrum.  $\square$

**Theorem 6.5.2** (Hopkins-Smith). *If  $\mathcal{C} \subset \mathrm{SH}_{(p)}^{\mathrm{fin}}$  is a thick subcategory of the stable homotopy category of finite  $p$ -local spectra then  $\mathcal{C} = \mathcal{C}_n$  for some  $0 \leq n \leq \infty$ .*

*Proof.* This is [HS98, Theorem 7]. It is an consequence of the Nilpotence Theorem.  $\square$

*Remark 6.5.3.* It follows that every thick subcategory in  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  is not only a  $\otimes$ -ideal—it is in fact *prime*. Note also that since  $(0)$  is a prime, the tensor triangulated category  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$  is a *local* tensor triangulated category (cf. Section 4.3).

*Remark 6.5.4.* From a different point of view, these results show that  $\mathrm{Spc}(\mathrm{SH}_{(p)}^{\mathrm{fin}})$  consists of a sequence of points

$$\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \mathcal{C}_3 \rightarrow \cdots \rightarrow \mathcal{C}_n \rightarrow \mathcal{C}_{n+1} \rightarrow \cdots \rightarrow \mathcal{C}_\infty = (0)$$

where  $\rightarrow$  indicates specialization:  $\overline{\{\mathcal{C}_n\}} = \{\mathcal{C}_i \mid i \geq n\}$ .

*Remark 6.5.5.* A classification of the thick subcategories in the finite stable homotopy category  $\mathrm{SH}^{\mathrm{fin}}$  can be deduced from the classification of thick subcategories in the  $p$ -local categories  $\mathrm{SH}_{(p)}^{\mathrm{fin}}$ . A particularly nice exposition is given in [Bal10a, Section 9]. We'll just summarize the result and let the reader refer to [Bal10a] for proofs. Be warned, however, that Balmer writes  $\mathcal{C}_n$  for what we write  $\mathcal{C}_{n+1}$ . (Our notation follows [HS98].)

**Corollary 6.5.6.** *For each prime  $p$ , consider the localization functor*

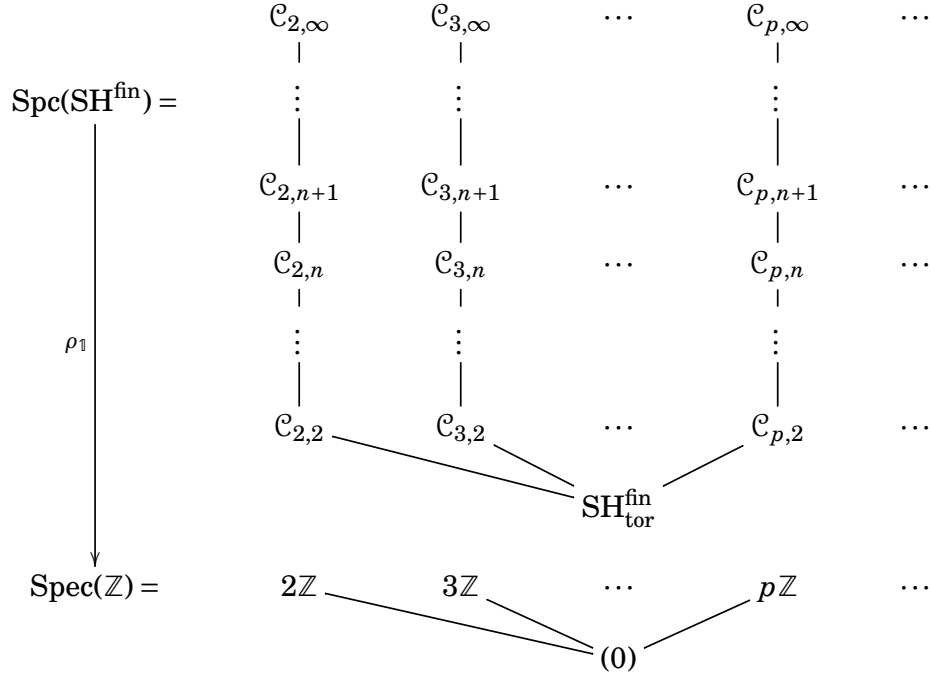
$$q_p : \mathrm{SH}^{\mathrm{fin}} \rightarrow \mathrm{SH}_{(p)}^{\mathrm{fin}}$$

*and define  $\mathcal{C}_{p,n} := q_p^{-1}(\mathcal{C}_n)$  for  $1 \leq n \leq \infty$ . They are points in  $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$ . Note that  $q_p^{-1}(\mathcal{C}_1) = \mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$  is the subcategory of finite torsion spectra—it is independent of  $p$ . The space  $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$  is displayed in the diagram on the following page. The unit comparison map*

$$\rho_{\mathbb{1}} : \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}(\mathbb{Z})$$

*sends  $\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$  to  $(0)$  and sends  $\mathcal{C}_{p,n}$  to  $(p)$  for  $2 \leq n \leq \infty$ . The space  $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$  is irreducible with unique generic point  $\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$  and with one closed point  $\mathcal{C}_{p,\infty}$  for each prime number  $p$ . In*

general, the closure of  $\mathcal{C}_{p,n}$  is  $\overline{\mathcal{C}_{p,n}} = \{\mathcal{C}_{p,m} \mid n \leq m \leq \infty\}$ .



The fiber above  $(0)$  consists of the single point  $\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$  while the fiber above each  $(p)$  is an infinite tower of points corresponding to the Morava  $K$ -theories at  $p$ .

## 6.6 Higher comparison maps

The purpose of this section is to illustrate the iterative method for examining fibers of comparison maps in the example of the stable homotopy category of finite spectra  $\mathrm{SH}^{\mathrm{fin}}$ . This will depend on a description of the graded centers of endomorphism rings of finite  $p$ -local spectra provided by [HS98] which affords a description of the ring

$$A_X^\bullet := \mathrm{Center}([X, X]_\bullet) \cap E_X^\bullet \quad (6.6.1)$$

for every finite  $p$ -local spectrum  $X$ , but not of the non-commutative ring  $E_X^\bullet$ , nor the ring  $R_X^\bullet = \mathrm{colim}_{n \geq 1} E_{X^{\otimes n}}^\bullet$ . For this reason, we'll have to settle for the unnatural comparison maps of Section 5.1.

The main results from [HS98] which allow for a description of our higher comparison maps arise from their study of non-nilpotent (graded) endomorphisms of finite  $p$ -local spectra. Recall that one statement of the Nilpotence Theorem (cf. Theorem 6.4.4) is that an endomorphism  $f : \Sigma^d X \rightarrow X$  of a finite  $p$ -local spectrum is nilpotent iff  $K(n)_*(f)$  is nilpotent for all  $0 \leq n < \infty$ . This motivates the following definition which aims to pin down those non-nilpotent endomorphisms that are as simple as possible.

**Definition 6.6.2.** [HS98, Definition 8] Let  $n \geq 1$  and let  $X$  be a finite  $p$ -local spectrum. An endomorphism  $f : \Sigma^d X \rightarrow X$  is a  $v_n$ -selfmap if  $K(n)_*(f)$  is an isomorphism, and  $K(i)_*(f)$  is nilpotent for  $i \neq n$ .

*Remark 6.6.3.* It follows from the definitions (and the Nilpotence Theorem) that if  $X$  is contained in  $\mathcal{C}_{n+1}$  then  $v_n$ -selfmaps are the same thing as nilpotent selfmaps but that if  $X \notin \mathcal{C}_{n+1}$  then  $v_n$ -selfmaps are never nilpotent. It is also easily shown that  $X \in \mathcal{C}_n$  is a necessary condition for the existence of a  $v_n$ -selfmap. Thus, the notion is mostly of interest for  $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$ . The first result of substance is that  $v_n$ -selfmaps exist:

**Theorem 6.6.4** (Hopkins-Smith). *A finite  $p$ -local spectrum  $X$  admits a  $v_n$ -selfmap if and only if  $X \in \mathcal{C}_n$ .*

*Proof.* This is [HS98, Theorem 9]. □

*Remark 6.6.5.* The most important properties about  $v_n$ -selfmaps, other than the fact that they exist, are that they are asymptotically unique and asymptotically central:

**Proposition 6.6.6** (Hopkins-Smith). *If  $f$  and  $g$  are two  $v_n$ -selfmaps of  $X$  then  $f^i = g^j$  for some  $i, j \geq 1$ .*

*Proof.* This is [HS98, Corollary 3.7]. □

**Proposition 6.6.7** (Hopkins-Smith). *If  $f$  is a  $v_n$ -selfmap of  $X$  then some power of  $f$  is contained in the center of  $[X, X]_{\bullet}$ .*

*Proof.* This can be derived from [HS98, Corollary 3.8] but let us give a direct proof. If  $f : \Sigma^k X \rightarrow X$  is a  $v_n$ -selfmap then it follows from the fact that the Morava  $K$ -theories are strong  $\otimes$ -functors that  $\Sigma^k Df \wedge X$  and  $DX \wedge f$  are two  $v_n$ -selfmaps of  $DX \wedge X$ . Invoking Proposition 6.6.6, there is an  $i \geq 1$  such that  $(\Sigma^k Df \wedge X)^i$  coincides with  $(DX \wedge f)^i$  as an element of  $[DX \wedge X, DX \wedge X]_{ik}$ . Noting that  $(\Sigma^k Df \wedge X)^i = \Sigma^{ki} D(f^i) \wedge X$  and  $(DX \wedge f)^i = DX \wedge f^i$ , we are reduced to the following claim: if a map  $f : \Sigma^k X \rightarrow X$  satisfies  $\Sigma^k Df \wedge X = DX \wedge f$  then  $f$  is in the graded center of  $[X, X]_{\bullet}$ . Given another endomorphism  $g : \Sigma^l X \rightarrow X$ , the equality  $f \cdot g = (-1)^{kl} g \cdot f$  can be checked in the ring  $\pi_{\bullet}(DX \wedge X) \simeq [X, X]_{\bullet}$ . Under this isomorphism, the map  $f \cdot g = f \circ \Sigma^k g$  corresponds to

$$\Sigma^{k+l} \mathbb{1} \xrightarrow{\Sigma \eta} \Sigma^{k+l}(DX \wedge X) \simeq DX \wedge \Sigma^{k+l} X \xrightarrow{DX \wedge \Sigma^k g} DX \wedge \Sigma^k X \xrightarrow{DX \wedge f} DX \wedge X$$

while the map  $g \cdot f = g \circ \Sigma^l f$  corresponds to

$$\Sigma^{k+l} \mathbb{1} \xrightarrow{\Sigma \eta} \Sigma^{k+l}(DX \wedge X) \simeq DX \wedge \Sigma^{k+l} X \xrightarrow{DX \wedge \Sigma^l f} DX \wedge \Sigma^l X \xrightarrow{DX \wedge g} DX \wedge X.$$

It thus follows from

$$\begin{array}{c}
\begin{array}{ccccccc}
& & \text{id} & & & & \\
& \nearrow & & \searrow & & & \\
DX \wedge \Sigma^{k+l} X & \xrightarrow{\sim} & \Sigma^l(DX \wedge \Sigma^k X) & \xrightarrow{\sim} & DX \wedge \Sigma^{k+l} X & \xrightarrow{1 \wedge \Sigma^k g} & DX \wedge \Sigma^k X \\
\downarrow 1 \wedge \Sigma^l f & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
& & \Sigma^{k+l}(DX \wedge X) & \xrightarrow{\sim} & \Sigma^k(DX \wedge \Sigma^l X) & \xrightarrow{\Sigma^k(1 \wedge g)} & \Sigma^k(DX \wedge X) \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
& & \Sigma^l(\Sigma^k DX \wedge X) & \xrightarrow{\sim} & \Sigma^k DX \wedge \Sigma^l X & \xrightarrow{1 \wedge g} & \Sigma^k DX \wedge X \\
& & \downarrow \Sigma^l(\Sigma^k Df \wedge 1) & & \downarrow \Sigma^k Df \wedge 1 & & \downarrow \Sigma^k Df \wedge 1 \\
DX \wedge \Sigma^l X & \xrightarrow{\sim} & \Sigma^l(DX \wedge X) & \xrightarrow{\sim} & DX \wedge \Sigma^l X & \xrightarrow{1 \wedge g} & DX \wedge X
\end{array}
\end{array}$$

id

that indeed  $f \cdot g = (-1)^{kl} g \cdot f$  in  $\pi_{k+l}(DX \wedge X)$ . □

For our purposes, we need to include the following result:

**Lemma 6.6.8.** *If  $f$  is a  $v_n$ -selfmap of  $X$  then some power of  $f$  is  $\otimes$ -balanced.*

*Proof.* This is straightforward in light of Proposition 6.6.6 by recognizing that  $f \otimes X$  and  $X \otimes f$  are two  $v_n$ -selfmaps of  $X \otimes X$ . More precisely, if  $f : \Sigma^k X \rightarrow X$  is a  $v_n$ -selfmap then

$$\Sigma^k(X \wedge X) \simeq \Sigma^k X \wedge X \xrightarrow{f \wedge X} X \wedge X$$

and

$$\Sigma^k(X \wedge X) \simeq X \wedge \Sigma^k X \xrightarrow{X \wedge f} X \wedge X$$

are two  $v_n$ -selfmaps of  $X \otimes X$ . One easily checks that their  $i$ th powers are the maps

$$\Sigma^{ik}(X \wedge X) \simeq \Sigma^{ik} X \wedge X \xrightarrow{f^i \wedge X} X \wedge X$$

and

$$\Sigma^{ik}(X \wedge X) \simeq X \wedge \Sigma^{ik} X \xrightarrow{X \wedge f^i} X \wedge X$$

and the equality of these two maps is what it means for the graded endomorphism  $f^i$  to be  $\otimes$ -balanced (cf. diagram (5.1.12) on p. 155). □

The notion of a  $v_n$ -selfmap leads to a very complete description of the centers of graded endomorphism rings of finite  $p$ -local spectra—up to nilpotents.

**Theorem 6.6.9** (Hopkins-Smith). *Let  $X$  be a finite  $p$ -local spectrum and let  $f$  be a graded endomorphism of  $X$  which is in the center of  $[X, X]_\bullet$ . If  $f$  is degree 0 then some power of  $f$  is a multiple of the identity; otherwise,  $f$  is nilpotent or a  $v_n$ -selfmap.*

*Proof.* This is established in [HS98, Corollary 5.5, Proposition 5.6]. □

Recall that the commutative graded ring  $A_X^\bullet$  was defined by  $A_X^\bullet := \text{Center}([X, X]_\bullet) \cap E_X^\bullet$ . The above theorem enables us to completely describe the space  $\text{Spec}^h(A_X^\bullet)$ :

**Corollary 6.6.10.** *Let  $n \geq 1$  and  $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$ . The space  $\mathrm{Spec}^h(A_X^\bullet)$  consists of two points: a generic point consisting of the homogeneous nilpotents and a closed point consisting of the homogeneous non-units. The closed point is of the form  $\sqrt{(f)}$  for any  $\otimes$ -balanced  $v_n$ -selfmap  $f : \Sigma^d X \rightarrow X$ .*

*Proof.* If  $f \in A_X^0$  then  $f^k = m \cdot \mathrm{id}_X$  for some  $k \geq 1$  and  $m \in \mathbb{Z}$ . If  $p \mid m$  then it follows from the Nilpotence Theorem and the fact that  $X \in \mathcal{C}_1$  that  $f$  is nilpotent, while if  $p \nmid m$  then  $m \cdot \mathrm{id}_X$  is an isomorphism. Thus, every element of degree zero in  $A_X^\bullet$  is either nilpotent or a unit. On the other hand, every element of non-zero degree is either nilpotent or a  $v_n$ -selfmap; moreover,  $v_n$ -selfmaps are not nilpotent since  $X \notin \mathcal{C}_{n+1}$ , nor are they units. It follows that the homogeneous non-units form an ideal  $\mathfrak{m}$  which is necessarily the unique maximal homogeneous ideal of  $A_X^\bullet$ . On the other hand, the ideal of homogeneous nilpotents  $\mathfrak{n}$  is readily seen to be prime just using the fact that the product of two  $v_n$ -selfmaps is again a  $v_n$ -selfmap and the product of a unit and a  $v_n$ -selfmap is again a  $v_n$ -selfmap. Since a  $v_n$ -selfmap exists in  $A_X^\bullet$  (by Lemma 6.6.8 and Theorem 6.6.4),  $\mathfrak{n} \subsetneq \mathfrak{m}$ . Moreover, if  $\mathfrak{p}$  is a homogeneous prime ideal then  $\mathfrak{n} \subsetneq \mathfrak{p}$  implies that there is a  $v_n$ -selfmap  $f$  contained in  $\mathfrak{p}$ . By the asymptotic uniqueness of  $v_n$ -selfmaps, it follows that every  $v_n$ -selfmap is contained in  $\mathfrak{p}$ , so that  $\mathfrak{p} = \mathfrak{m}$ . We conclude that the only homogeneous primes of  $A_X^\bullet$  are  $\mathfrak{n}$  and  $\mathfrak{m}$ . Finally note that the asymptotic uniqueness of  $v_n$ -selfmaps implies that  $\mathfrak{m} = \sqrt{(f)}$  for any  $\otimes$ -balanced  $v_n$ -selfmap  $f : \Sigma^d X \rightarrow X$ .  $\square$

**Lemma 6.6.11.** *Let  $n \geq 1$  and  $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$ . If  $f : \Sigma^d X \rightarrow X$  is a  $v_n$ -selfmap then  $\mathrm{cone}(f)$  is contained in  $\mathcal{C}_{n+1} \setminus \mathcal{C}_{n+2}$ .*

*Proof.* This follows from the long exact sequence obtained by applying Morava  $K$ -theory to an exact triangle for  $f : \Sigma^d X \rightarrow X$ . In more detail, the fact that  $K(n)_*(f)$  is an isomorphism implies that  $K(n)_*(\mathrm{cone}(f)) = 0$  so that  $\mathrm{cone}(f) \in \mathcal{C}_{n+1}$ . On the other hand, if  $\mathrm{cone}(f) \in \mathcal{C}_{n+2}$

then  $K(n+1)_\bullet(\text{cone}(f)) = 0$  which implies that  $K(n+1)_\bullet(f)$  is an isomorphism. But since  $K(n+1)_\bullet(f)$  is also nilpotent, it follows that  $K(n+1)_\bullet(X) = 0$ . Hence  $X \in \mathcal{C}_{n+2} \subset \mathcal{C}_{n+1}$  which would contradict our assumption that  $X \notin \mathcal{C}_{n+1}$ .  $\square$

We are now in a position to examine the structure of  $\text{SH}^{\text{fin}}$  via higher comparison maps. The starting point is the comparison map for the unit object: the sphere spectrum. It is well-known that the endomorphism ring of the sphere spectrum is  $\text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1}) \cong \mathbb{Z}$ . On the other hand,  $\pi_i(\mathbb{1}) = 0$  for  $i < 0$  and all the graded endomorphisms of positive degree are nilpotent by Nishida's theorem. It follows that  $\text{Spec}^h(\text{End}_{\text{SH}^{\text{fin}}}^\bullet(\mathbb{1})) \simeq \text{Spec}(\text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1}))$  and that the graded and ungraded comparison maps coincide. Moreover, algebraic localization with respect to  $\mathfrak{p}^\bullet \subset \text{End}_{\text{SH}^{\text{fin}}}^\bullet(\mathbb{1})$  is the same as algebraic localization with respect to  $\mathfrak{p}^0 \subset \text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1})$ . Then consider the map  $\rho_1 : \text{Spc}(\text{SH}^{\text{fin}}) \rightarrow \text{Spec}(\mathbb{Z})$ . Algebraic localization with respect to the generic point  $(0) \in \text{Spec}(\mathbb{Z})$  gives a map

$$\text{Spc}(\text{SH}^{\text{fin}}/\text{SH}_{\text{tor}}^{\text{fin}}) \rightarrow \text{Spec}(\mathbb{Q})$$

and we conclude that the fiber over  $(0)$  is  $V(\text{SH}_{\text{tor}}^{\text{fin}}) \subset \text{SH}^{\text{fin}}$ . In fact, one can show that  $\text{SH}^{\text{fin}}/\text{SH}_{\text{tor}}^{\text{fin}} \simeq D^b(\mathbb{Q})$  (see [Mar83, page 113]) and hence the spectrum of  $\text{SH}^{\text{fin}}/\text{SH}_{\text{tor}}^{\text{fin}}$  is a single point. Moreover,  $\text{SH}_{\text{tor}}^{\text{fin}}$  itself is prime and so we conclude that the fiber over  $(0)$  is the single point  $\{\text{SH}_{\text{tor}}^{\text{fin}}\}$ .

Next consider the fiber over a closed point  $(p) \in \text{Spec}(\mathbb{Z})$ . Algebraic localization provides a map  $\text{Spc}(\text{SH}_{(p)}^{\text{fin}}) \rightarrow \text{Spec}(\mathbb{Z}_{(p)})$  and the fiber over the unique closed point  $(p) \in \text{Spec}(\mathbb{Z}_{(p)})$  is  $\text{supp}(\text{cone}(p.\text{id}_1)) = \overline{\{\mathcal{C}_2\}}$ . In other words, the fiber includes everything with the exception of a single point:  $\mathcal{C}_1$ . The next step is to define  $X_1 := \text{cone}(p.\text{id}_1)$  and consider

$$\rho_{X_1, A_{X_1}^\bullet}^\bullet : \overline{\{\mathcal{C}_2\}} = \text{supp}(X_1) \rightarrow \text{Spec}^h(A_{X_1}^\bullet).$$

By Corollary 6.6.10 the unique closed point of  $\text{Spec}^h(A_{X_1}^\bullet)$  is of the form  $\sqrt{(f)}$  for any tensor-balanced  $v_2$ -selfmap  $f$  of  $X_1$  and by Lemma 6.6.11 the fiber over this point is  $\text{supp}(\text{cone}(f)) =$

$\overline{\{\mathcal{C}_3\}}$ . Again the fiber consists of everything except for one point and the process continues.

At the  $n$ th step we have an object  $X_n$  and a map

$$\rho_{X_n, A_{X_n}^\bullet}^\bullet : \overline{\{\mathcal{C}_{n+1}\}} = \text{supp}(X_n) \rightarrow \text{Spec}^h(A_{X_n}^\bullet).$$

The unique closed point is generated as a radical ideal by any  $\otimes$ -balanced  $v_{n+1}$ -selfmap  $f_n$  and the fiber over this point is  $\text{supp}(\text{cone}(f_n)) = \overline{\{\mathcal{C}_{n+2}\}}$ . Altogether this gives a filtration of the fiber

$$\rho_{\text{SH}_{(p)}^{\text{fin}}, \mathbb{1}}^{-1}((p)) = \overline{\{\mathcal{C}_2\}} \supset \overline{\{\mathcal{C}_3\}} \supset \overline{\{\mathcal{C}_4\}} \supset \dots \quad (6.6.12)$$

where exactly one point is removed at each step. All of this may be better appreciated by considering the picture of  $\text{Spc}(\text{SH}^{\text{fin}})$  displayed on page 221. Algebraic localization at  $p$  focuses on a single branch and then each successive comparison map chops off the root heading towards  $\mathcal{C}_{p, \infty}$ . Note that we obtain every irreducible closed subset of the fiber (6.6.12) except for the closed point  $\overline{\{\mathcal{C}_\infty\}} = \{\mathcal{C}_\infty\}$ . The fact that this point is missed shouldn't be alarming since it corresponds to an irreducible closed subset which is not Thomason. If all the rings involved are noetherian then we can only expect to obtain Thomason closed subsets since our comparison maps are spectral—when using these strategies we should take arbitrary intersections of all of the closed subsets that we obtain. Keep in mind that the Thomason closed subsets are a basis of closed sets, so if we can obtain all the Thomason closed subsets of the spectrum then we have obtained the entire spectrum.

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