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ABSTRACT

If we define a dense plasma to be one in which the effective mean free path of one particle component is small in comparison with the probe dimensions, then Langmuir's theory is not applicable in such a plasma. The presence of the probe causes marked changes of density and potential distributions in the probe environment. We have calculated these effects for insulated probes of various geometries. An exact solution is given for a concentric cylindrical probe. For more general geometries, an appropriate approximation procedure, the "composition method," was developed from variational principles. The effect of probe disturbances on the measurements can be accounted for in terms of an "effective probe position" and a "potential correction." Introduction of the probe also causes changes in the eigenvalue and in the electron temperature. The results allow one to unfold experimental data to find the true plasma qualities. Consideration of the inertia-limited region shows that "Bohm's criterion" is not suitable to judge either the stability or stationarity of the sheath. We find that a stationary inertia-limited region can exist only under certain restricted circumstances.

PROBE THEORY IN A DENSE PLASMA*

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I. INTRODUCTION

It is one of the tasks of plasma physics to determine particle densities, temperatures, and other data, experimentally. There are very few reliable methods for investigating these qualities. One of them is the probe technique.

The theory of probes has been developed by Langmuir and collaborators.[1,2,3] Refinements of this theory have been made by Bohm, Burhop, and Massey,[4] Boyd,[5] and most recently, Bernstein and Rabinowitz,[6] and Hall.[7]

These theories restrict themselves explicitly or implicitly to low-density plasmas. Here "low-density plasma" means that the effective mean free path—in a magnetic field the gyro radius—of all particle components is large compared with the probe dimensions. Nevertheless, these theories have been applied to dense plasmas and to plasmas in strong magnetic fields.

In such plasmas, Langmuir's theory is subject to severe changes. In the following we try to demonstrate, and to account for, these changes which have already been touched upon in some earlier considerations by Davydov and Zmanovskaja,[8] and Boyd.[9]

II. BASIC CONCEPTS AND ASSUMPTIONS

In accordance with the foregoing, we use the term "dense plasma" in this paper for a system in which the effective mean free path λ (in a magnetic field, the gyro radius r_g) of at least one charged-particle component is of the same magnitude as, or smaller than, the characteristic probe dimension ℓ_p.

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The subject of this investigation is an insulated probe with a small sensing element in an appropriate position. We particularly stress that the term probe here means the whole probe body, including the probe support.

The presence of the probe in the plasma causes disturbances of the particle density, temperature, and potential distributions in the probe environment, due to the incidence of charged and neutral particles on the probe surface. To account for these disturbances the following terms are appropriate:

(a) The region of influence is that part of the plasma volume in which a notable change of data due to the presence of the probe can be observed.

(b) The diffusion-disturbed region is that part of the region of influence in which the transport equations with the scalar pressure tensor approximation hold and the condition of proportionality is met.

(c) The inertia-limited region is that part of the region of influence in the immediate neighborhood of the probe where the concept of free fall is applicable.

(d) The transition region is the zone between the diffusion-disturbed and the inertia-limited regions where neither diffusion nor free fall is a good approximation.

(e) The space-charge region is that region where the concept of quasineutrality and the concept of proportionality fail.

(f) The sheath is a collective term which we conveniently use to indicate the whole part of the region of influence not belonging to the diffusion-disturbed region.

It should be noted that these regions may overlap. Moreover, these general definitions may not always agree with the conventionally used terms. For example space-charge region and sheath are not necessarily identical.

The following discussion concentrates on the investigation of the "diffusion-disturbed region" and the "inertia-limited region."

The disturbances produced by the probe depend on the qualities of the plasma and the geometry of the probe.

We consider a steady-state three-component system of neutrals, electrons and singly charged ions. Volume recombination is negligible, recombination taking place only at the walls and at the probe surface. The particle production is proportional to the electron density only. The electron and ion temperatures (T_-, T_+) are assumed to be constant within the diffusion-disturbed region and the plasma volume. We do not consider an external magnetic field. An extension of the methods presented here, including external magnetic fields, is given elsewhere.[10]

III. DIFFUSION-DISTURBED REGION

A. Basic Equations

According to the definition of this region, we describe density n and average velocity v by the transport equations for mass and momentum with the scalar approximation of the pressure tensor. With the assumptions of constant temperature already made, we can omit the energy balance.

The stationarity condition for mass and momentum conservation of the electrons and ions then reads

$$\nabla \cdot \vec{\Gamma}_{\pm} = \nu n_- \tag{1}$$

and

$$\vec{\Gamma}_{\pm} = \pm \mu_{\pm} n_{\pm} \vec{E} - D_{\pm} \nabla n_{\pm} , \tag{2}$$

where $\vec{\Gamma}$ is the particle current density, n the particle density, ν the net rate of ionization, D and μ the diffusion and mobility coefficients, and $\vec{E}$ the electric field. The subscript ± refers to positive ions and electrons respectively. Elimination of $\vec{E}$ from Eqs (1) and (2) by making use of the assumption of proportionality

$$\frac{\nabla n_+}{n_+} = \frac{\nabla n_-}{n_-} \tag{3}$$

results in

$$\nabla \cdot \left[D_s \nabla n_- \right] + \nu n_- = 0 \tag{4}$$

with

$$D_s = \frac{D_+ \mu_- + D_- \mu_+}{\mu_- + \mu_+} \left\{ 1 + \mu_- \frac{(n_+ - n_-)}{\mu_+ n_+ + \mu_- n_-} \right\} . \tag{5}$$

If the condition of quasineutrality is met, then D_s is identical with the ambipolar diffusion coefficient D_{am}.

Occasionally we will find it useful to remember that Eq. (4) may be written in the form

$$\nabla \cdot \vec{u} + (\vec{u})^2 + \nu / D_s = 0 , \tag{6}$$

with

$$\vec{u} = \frac{\nabla n}{n} . \tag{7}$$

The electric potential distribution within the diffusion-disturbed region for our probe with an insulated surface follows readily from Eq. (2) and the requirement of congruence,

$$\vec{\Gamma}_+ = \vec{\Gamma}_- \quad . \tag{8}$$

We have

$$V_2 - V_1 = \frac{D_- - D_s}{\mu_-} \; \ell n \left(\frac{n_{-2}}{n_{-1}} \right) \; ; \tag{9}$$

or, in the quasineutral case,

$$V_2 - V_1 = \frac{D_- - D_+}{\mu_- + \mu_+} \cdot \ell n \left(\frac{n_{-2}}{n_{-1}} \right) \; . \tag{10}$$

B. Boundary Conditions

To define the density distribution from Eq. (4) or (6), one frequently postulates the boundary condition

$$n_0 \approx 0 \tag{11}$$

at the wall of the container (index 0). This is too simple an approximation for our application.

The physical concept governing the boundary condition is the current continuity at the edge of the diffusion-disturbed region.[11] To have stationary conditions, the diffusion current to this edge must be equal to the current entering the sheath. Provided conditions in the sheath are such that no particles entering the sheath return to the plasma, we have to equate the ion-saturation current of the plasma to the diffusion current from the diffusion-disturbed region. If particles do return to the plasma, then a reduced effective-saturation current has to be used. In general we have, therefore,

$$\alpha' (n_+ \bar{v}_+)_s /4 = -(D_s \nabla_\perp n_+)_s \tag{12}$$

where the subscript s means evaluation at the sheath surface and $\nabla_\perp$ is the component of the gradient normal to that surface. The α' is an uncertainty coefficient, discussed below.

Equation (12) may be written in the form

$$(\mu_\perp)_s = (\nabla_\perp n/n)_s = -\alpha/\lambda_{+s} \tag{13}$$

where for a quasineutral plasma with $T_- \approx T_+$ and $D_s \approx 2D_+$ the factor α is given by

$$\alpha = \alpha' \cdot 3/8 \quad . \tag{14}$$

Of course, all the difficulties of the boundary problem are now included in α. This coefficient is influenced by a large number of parameters--the probe geometry, sheath conditions, particle return, impressed magnetic field, secondary emission, and others. The overall problem of α is much too complicated to be treated in general terms. Its value must be determined for each specific case. Particularly important is the influence of α when the probe dimensions decrease below the effective mean free path. Then many particles enter the sheath, orbit around the probe, and return to the plasma.

Under these circumstances, α goes to zero and we approach the conditions of Langmuir's theory.

If λ_{+s} is much smaller than the characteristic length of the plasma volume, L, then with $\nabla_{\perp} n$ of the order of $\overline{n}/L$ we have the simplified boundary condition

$$n_0 \approx n_s \approx \frac{\overline{n}}{L} \frac{\lambda_{+s}}{\alpha} \approx 0 ; \qquad (15)$$

provided that α does not go to zero. Therefore, at the probe surface the simplified condition (11) can be used only for $\lambda_{+s} << L$ and $\lambda_{+s} << \ell_p$.

C. Solutions of the Continuity Equation

The problem formulated in Eqs. (4) and (6) and (13) and (15) represents the well-known mathematical eigenvalue problem of an elliptical differential equation. The boundary condition (15) is of the "second kind" (Dirichlet type), and the boundary condition (13) is of the socalled "third kind" (mixed type).

As has been shown by variational methods, a complete sytem of eigensolutions of this problem exists for all cases of practical interest.[12] Moreover there is always one solution that is positive definite throughout the whole plasma volume.[13] This condition is essential, because the density n, cannot assume negative values.

However, finding the exact eigensolutions for most geometries is very difficult. There is the possibility of a machine solution by a difference method. Examples for this procedure are known for diffusion problems.[14] But this method has the disadvantages of being quite elaborate and of giving a result only for a specific geometry. Simple solutions do exist for those simple geometries of high symmetry in which the problem has separable solutions. The general procedure for these is well known.

We shall now first examine the exact solution for a cylindrical geometry in which a probe is supported parallel to the plasma in the axis of the column. The sensing elements are located at the surface of the probe on a common diameter of the probe and the plasma. For more general geometries we shall develop an approximation procedure called the "composition method." This procedure satisfies the condition of a simple solution, but still gives results of sufficient accuracy to be of practical interest.

1. Exact Solution for a Cylindrical Probe in the Center

The exact solution for two concentric cylinders of radii r_p and R respectively may be represented in the form

$$n = n_c \left\{ J_0(kr) + c N_0(kr) \right\} , \qquad (16)$$

where $k = (\nu/D_s)^{1/2}$, r_p is the probe radius, R the cylinder radius, n_c and c are arbitrary constants, and J_0 and N_0 are the Bessel and Neumann functions of zero order. Applying the boundary condition (13) at the sheath edge of the probe, we find

$$\frac{a}{\lambda} = \frac{-\left(\frac{\nu}{D_s}\right)^{1/2}\left\{J_1\left[\left(\frac{\nu}{D_s}\right)^{1/2}(r_p+\lambda)\right] + c\,N_1\left[\left(\frac{\nu}{D_s}\right)^{1/2}(r_p+\lambda)\right]\right\}}{J_0\left[\left(\frac{\nu}{D_s}\right)^{1/2}(r_p+\lambda)\right] + c\,N_0\left[\left(\frac{\nu}{D_s}\right)^{1/2}(r_p+\lambda)\right]} \quad (17)$$

assuming that the extension of the sheath, ℓ_s, is defined by the inertia-limited region.

At the outer wall, condition (15) is a sufficient approximation, and we have

$$J_0\left[\left(\frac{\nu}{D_s}\right)^{1/2} R\right] = -c\,N_0\left[\left(\frac{\nu}{D_s}\right)^{1/2} R\right] \quad (18)$$

Equations (17) and (18) are sufficient to determine both c and the eigenvalue $(\nu/D_s)^{1/2}$.

Table I lists the results for these parameters for various a between zero and unity, using the values $r_p = \ell_s = \lambda = 0.02\,R$. Note that even for the Langmuir probe without surface recombination ($a = 0$), the ionization rate ν still must be greater than when the probe is not present, because the probe still occupies a finite volume in the cylinder.

As an illustration of the effect of the probe on the plasma, the density distributions for various values of a are given in Fig. 1. Several important features are immediately apparent from this figure.

(a) The density distribution in the vicinity of the probe is significantly modified for $a \neq 0$, and even for $a = 0$, if the probe is of appreciable dimensions (this latter effect is, however, too small to be shown in the figure for the probe dimension chosen).

(b) The density distribution in the vicinity of the outer walls is changed very little.

(c) The eigenvalues $(\nu/D_s)^{1/2}$ may increase by as much as 20% over that of the unperturbed plasma.

The potential drop in front of the probe within the diffusion-disturbed region can be calculated from formula (10). According to this calculation, in a dense plasma ($a \neq 0$), such a probe should yield particle density and plasma potential measurements which differ appreciably from those predicted by the Langmuir theory. The deviations of the density measurements may be taken from Fig. 1.

Table I. Results $(\nu/D_s)^{1/2}$ and c, for values of α between zero and unity, determined from Eq. (17) and (18).

α	$(\nu/D_s)^{1/2}$	c
1	$\frac{2.91}{R}$	0.569
0.5	$\frac{2.84}{R}$	0.478
0.25	$\frac{2.75}{R}$	0.365
0.1	$\frac{2.61}{R}$	0.216
0.05	$\frac{2.53}{R}$	0.131
0.02	$\frac{2.47}{R}$	0.064
0.01	$\frac{2.44}{R}$	0.037
0	$\frac{2.41}{R}$	0.007

The potential disturbance is best described by the introduction of two appropriate quantities. One is the effective probe position, or effective probe length. This is defined by the point in the plasma at which the density distribution essentially reverts to that of the unperturbed distribution. The other is the potential correction, defined as the potential drop between the effective probe position and the edge of the sheath—as determined from Eq. (10).

2. Approximate Procedure for More General Geometries (Composition Method)

To solve Eq. (4) subject to the boundary conditions of (13) or (15) for more general geometries is a very difficult problem.

Perturbation methods are available if one considers the introduction of the probe as a perturbation of the boundary shape. For homogeneous Neumann or Dirichlet boundary conditions one can obtain expressions for the eigensolutions and eigenvalues in terms of a series involving the unperturbed eigenfunctions and eigenvalues.[15] This procedure has the advantage of being, in principle, very general. However, for a general boundary perturbation, convergence difficulties prevent an explicit expansion. This is particularly true in the neighborhood of the probe surface, which is precisely our region of interest. Consequently, this procedure is not suitable.

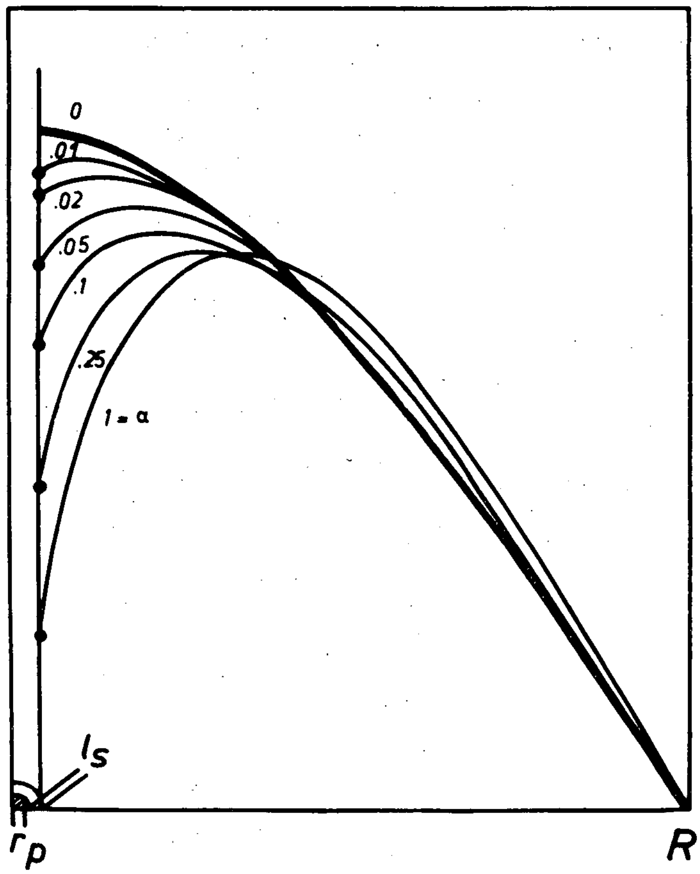

Fig. 1. Density distributions for a coaxial cylindrical probe of radius r_p and sheath thickness ℓ_s for various values of the boundary parameter α.

In the attempt to develop an appropriate procedure it seems advisable to recall our intentions. First, it is meaningless to seek a solution of Eqs. (4) and (13) with an accuracy higher than that already limited by the other assumptions of our model. Secondly, we need a mathematical approach that is simple enough so that it can be handled easily by the experimental investigator, and at the same time accurate enough so that the corrections are of value. A very difficult mathematical procedure, which indeed would be of great interest in principle, would not serve this latter purpose.

We therefore aim—as in the case of the exact solution for the concentric cylindrical probe—to describe our corrections in terms of an "effective probe position" and a "potential correction." This is possible if we know the extent of the "region of influence," and the density distribution within the diffusion-disturbed region.

The effective probe position and the potential correction can be approximately determined by the following "Composition Method."

For the "Composition Method" we first define a trial solution. We subdivide the whole plasma volume into two regions (I, II) by an appropriate interface (see Fig. 2), each region containing one part of the boundary (B_I, B_{II}). We choose the subdivision so that in each region we can find an exact solution of the Helmholtz equation (4) satisfying condition (13) or (15) for the part of the boundary belonging to this region. The same eigenvalue underlies both regions. As can be seen from Eq. (7), the two solutions each contain an arbitrary factor. For physical reasons we require

$$\int_\sigma n_{I\sigma} d\sigma = \int_\sigma n_{II\sigma} d\sigma \qquad (19)$$

where the index (σ) indicates the dividing interface.

So far, the composition of the trial solution is completely arbitrary. Equations (6) and (13) show that we can expect a good fit of this trial solution in regions where one or the other boundary part (B_I, B_{II}) is dominating. But serious deviations occur where B_I and B_{II} are of equal importance; the value of the trial solution, therefore, depends on the position and shape of the interface, and we consequently need a criterion for choosing these quantities.

Such a criterion may be derived from the variational principle for eigenvalue problems. For a mixed boundary condition,

$$\nabla_\perp n + \epsilon n = 0 , \qquad (20)$$

this principle requires, in the case of the Helmholtz equation,

$$\delta \left\{ \frac{\int (\nabla n)^2 dV + \int \epsilon n^2 d\sigma}{\int n^2 dV} \right\} = \delta \left[k^2\right] = 0 \qquad (21)$$

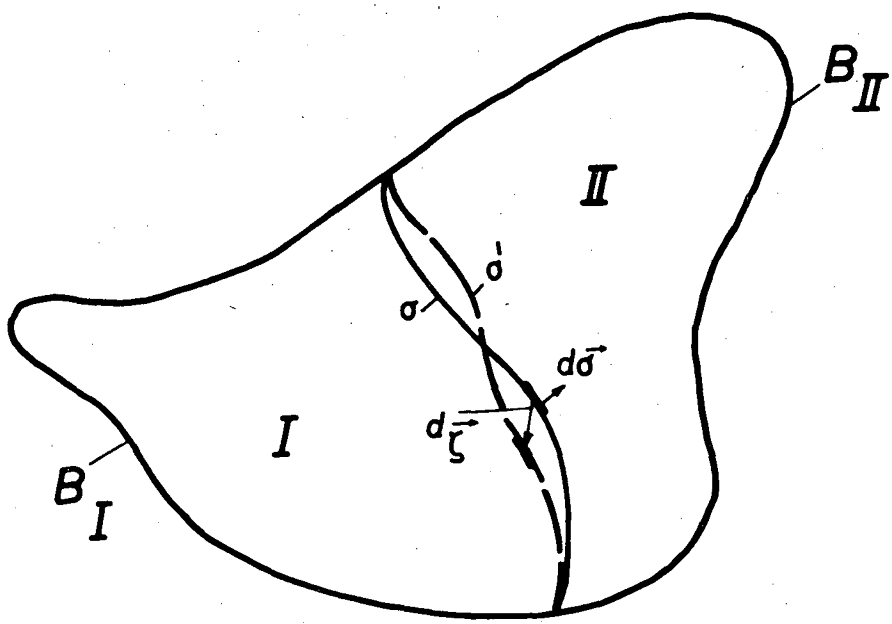

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Fig. 2. Illustration of the subdivision used in the "composition method."

where all trial solutions should satisfy the boundary condition (20). For these trial solutions there exists an absolute minimum of $[k^2] = [(\nu/D_s)]$. The trial solution producing this minimum value is the exact solution.

We will now introduce our composition-trial solution into Eq. (21), applying

$$(\nabla n)^2 = \nabla \cdot (n \nabla n) - n \nabla^2 n , \qquad (22)$$

and the Gaussian theorem. The partial solutions satisfy the Helmholtz equation with the same eigenvalue k_0^2 and condition (20). Thus we arrive at

$$\delta \left\{ k_0^2 + \frac{\sigma \int [n_I \nabla n_I - n_{II} \nabla n_{II}]\, d\vec{\sigma}}{\int_I n_I^2\, dV + \int_{II} n_{II}^2\, dV} \right\} = \delta \left[k_0^2 + \frac{Z}{N} \right] \qquad (23)$$

or

$$\frac{\delta Z}{Z} - \frac{\delta N}{N} = 0 . \qquad (24)$$

Our variation consists of a deformation of the shape of the interface from σ to σ' described by $\vec{d\zeta}$. Consequently we have, with Eq. (19),

$$\delta Z = (\vec{d\zeta})^2 \cdot \int_\sigma (n_I \nabla n_I - n_{II} \nabla n_{II})\, \vec{d\sigma} \qquad (25)$$

The second term in Eq. (24) is small of higher order, and in application to our composition method the variation principle therefore requires

$$\int_\sigma (n_I \nabla n_I - n_{II} \nabla n_{II})\, d\vec{\sigma} = 0; \qquad (26)$$

or, with the definition of the mean value,

$$\overline{\nabla n} = \frac{\int_\sigma n \nabla n\, d\vec{\sigma}}{\int_\sigma n\, d\sigma} , \qquad (27)$$

the principle may be written

$$\overline{n_I}\ \overline{\nabla n_I} - \overline{n_{II}}\ \overline{\nabla n_{II}} = 0 \qquad (28)$$

That means the best approximation is achieved if

$$\frac{\overline{\nabla n_I}}{\overline{n_{II}}} = \frac{\overline{\nabla n_{II}}}{\overline{n_I}} \qquad (29)$$

is fulfilled. Or more conveniently, remembering Eq. (19),

$$\frac{\overline{\nabla n_I}}{\overline{n_I}} = \frac{\overline{\nabla n_{II}}}{\overline{n_{II}}} . \qquad (30)$$

With this criterion we are able to select compositions which give a good fit to the physical situation.

In our problem we have some simplifying facts. The appropriate composition is prescribed. It consists of a part solution belonging to the probe enviroment and another part solution belonging to the rest of the plasma volume. For the second part we use the unperturbed eigensolutions of the problem without the probe. The first part, in the enviroment of the probe, depends on the special probe geometry. In general, the probe volume will be small in comparison with the plasma volume. It is then sufficient to use the unperturbed eigenvalue (without probe) as an approximation for k_0^2.

The position of the interface governed by Eq. (30) defines the extent of the "region of influence" and thus the "effective probe position." With the density distribution within the region of influence, one can easily evaluate the potential drop within the diffusion-disturbed region with Eq. (10).

A rigorous evaluation of the criterion (30) is still a difficult problem, but it is not required. Within the frame of our accuracy, one can find simple procedures to determine the position of the interface. Either we choose the interface in a position where $\overline{\nabla n_I}/\overline{n_I}$ is large, but approximately of the same direction and magnitude as $\overline{\nabla n_{II}}/\overline{n_{II}}$ (Form A), or we choose—if such a region exists—the interface in the position where the two gradients both become small (Form B).

3. Results of the Composition Method for the Displaced Cylindrical Probe and the Radial Prolate Probe

For the displaced cylindrical probe we have used the familiar unperturbed solution

$$n = n_c J_0(kr) \tag{31}$$

for the region outside the probe zone of influence. In the vicinity of the probe we used combination (16). We have applied criterion (30) in the form B, and in particular we have limited our calculations to the distribution along the common diameter of cylinder and probe.

In Fig. 3 we show the distributions vs the probe position for a constant parameter value α. The exact solution for the coaxial probe is indicated together with the approximate solution, which allows an estimate of the error of our approximation. As expected, the error of the approximation appears to be greatest near the interface of the composition. The magnitude of error decreases as we approach the probe. As the densities enter ΔV only logarithmically, the potential correction is accurate within the general limitations of the model.

In Fig. 4 we show the effective probe position as a function of the true probe position for various values of α. The corresponding potential drop is given in Fig. 5. Note that both effects are asymmetric. Figure 6 demonstrates for an assumed true potential the potentials that would be measured by such a cylindrical probe.

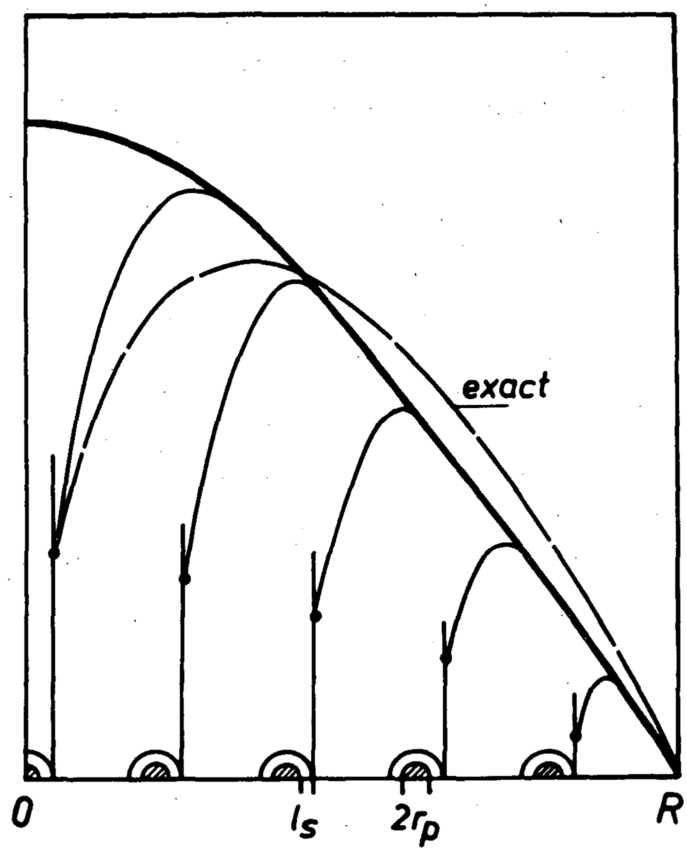

Fig. 3. Density distributions calculated by the "composition method" for a cylindrical probe of radius r_p and sheath thickness ℓ_s at various positions in a cylindrical discharge with a Bessel distribution. The boundary parameter α is chosen to be 0.5. The exact solution is indicated for the coaxial probe.

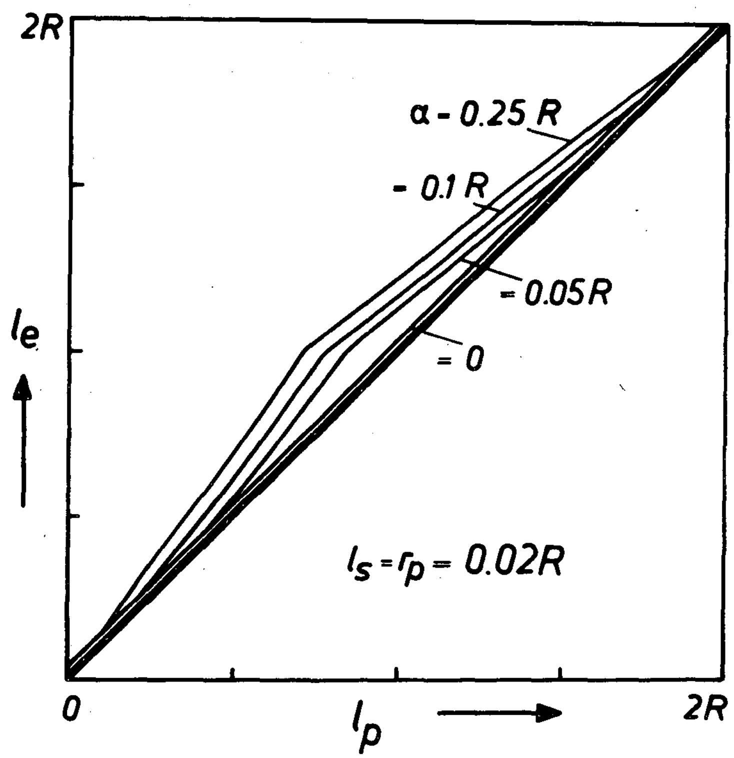

Fig. 4. The effective probe position ℓ_e is plotted vs the true probe position ℓ_p for various values of the boundary parameter α.

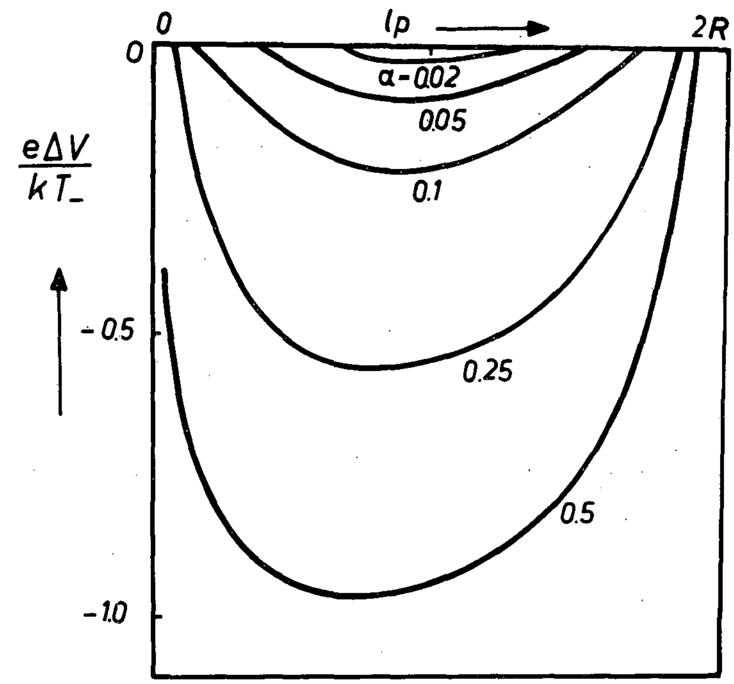

Fig. 5. The potential correction ΔV is plotted vs the true probe position ℓ_p for various values of the parameter α and conditions corresponding to those in Fig. 4.

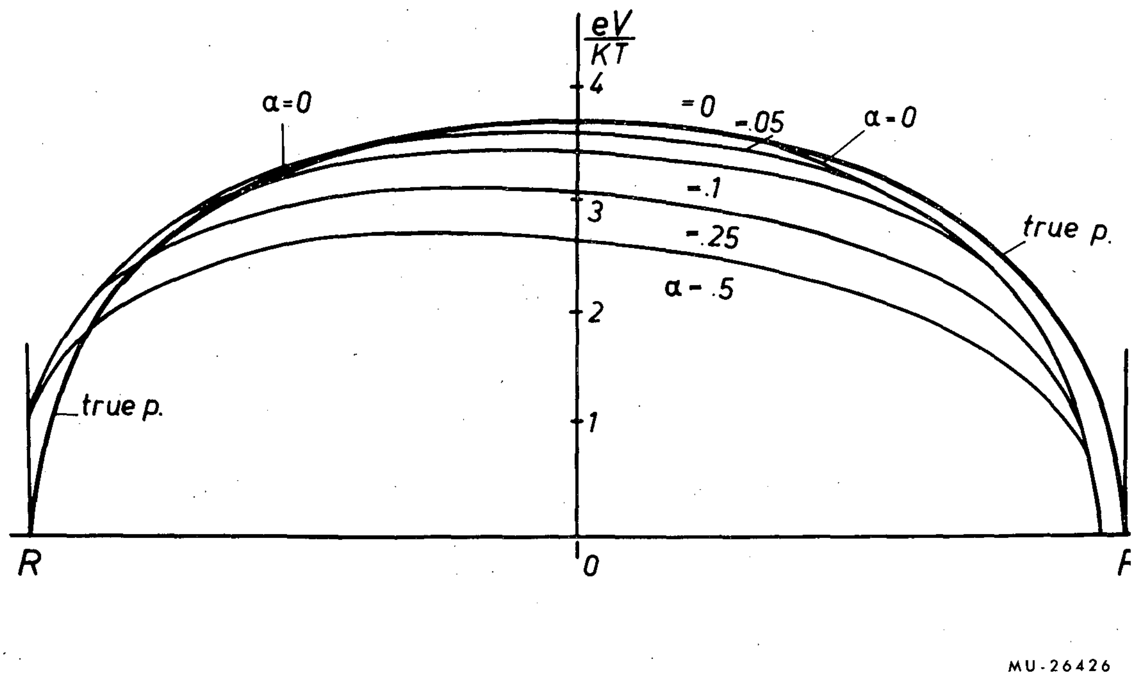

Fig. 6. For a true potential true p), this figure gives the distorted results that should be expected from potential measurements with a probe corresponding to those of Figs. 4 and 5.

For the prolate probe we again used the unperturbed distribution (31). Prolate spheroidal wave functions were applied in obtaining solutions within the region of influence of the probe (trigonometric functions could be used in this region as an order-of-magnitude estimate). The spheroidal solutions are

$$n = n_0 \quad S_{01}(c, \eta) \left[R_{01}^{(1)} (c, \xi) + B\, R_{01}^{(2)} (c, \xi) \right] \tag{32}$$

with

$$c \approx \frac{d}{2} k \approx \frac{2.4}{R} \cdot \ell_p \tag{33}$$

where ξ and η are prolate elliptical coordinates, $d/2$ is the length ℓ_p of the probe. $S_{01}^{(1)}$ is the spheroidal angle function of the first kind, and $R_{01}^{(1),(2)}$ are the spheroidal radial functions of the first and second kind. These functions are discussed and tabulated for a small range of the argument by Flammer.[16] An extension has been given elsewhere.[10]

We performed the evaluation for the prolate probe using procedure (A) for the boundary condition (15) at the probe surface.

Figure 7 shows the effect of probes of various radii on the density distribution along the diameter coinciding with the probe axis. Figure 8 gives the same distribution for various depths of probe penetration, and Figs. 9 and 10 show the effective probe length as a function of the actual probe length, and the potential correction vs the effective probe length. Here the asymmetry is even more striking.

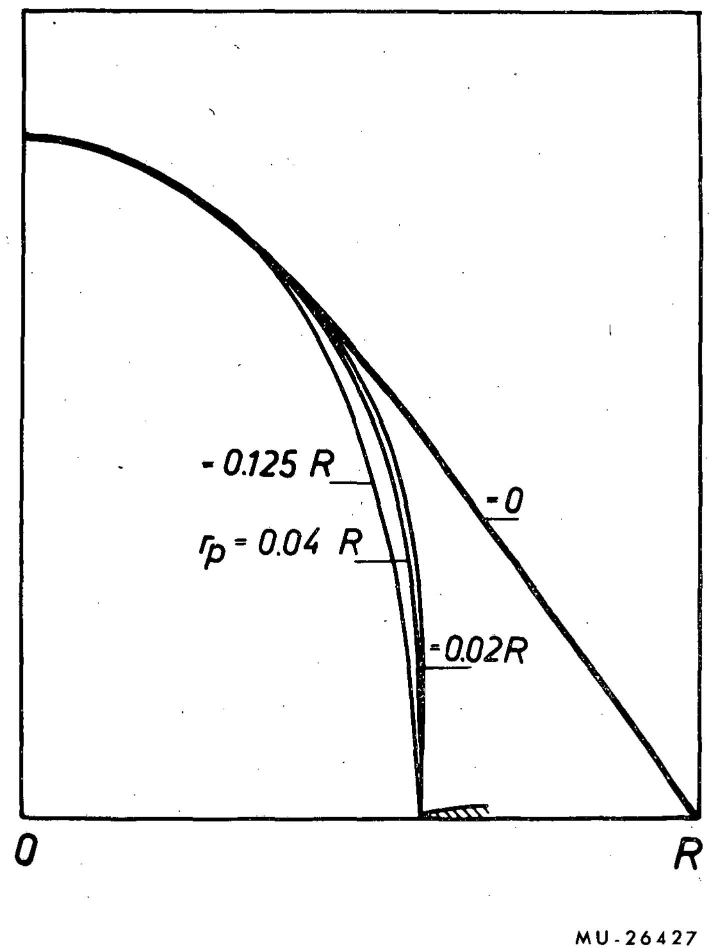

Fig. 7. Density distributions for prolate probes of various radii penetrating radially into a cylindrical discharge with a Bessel distribution.

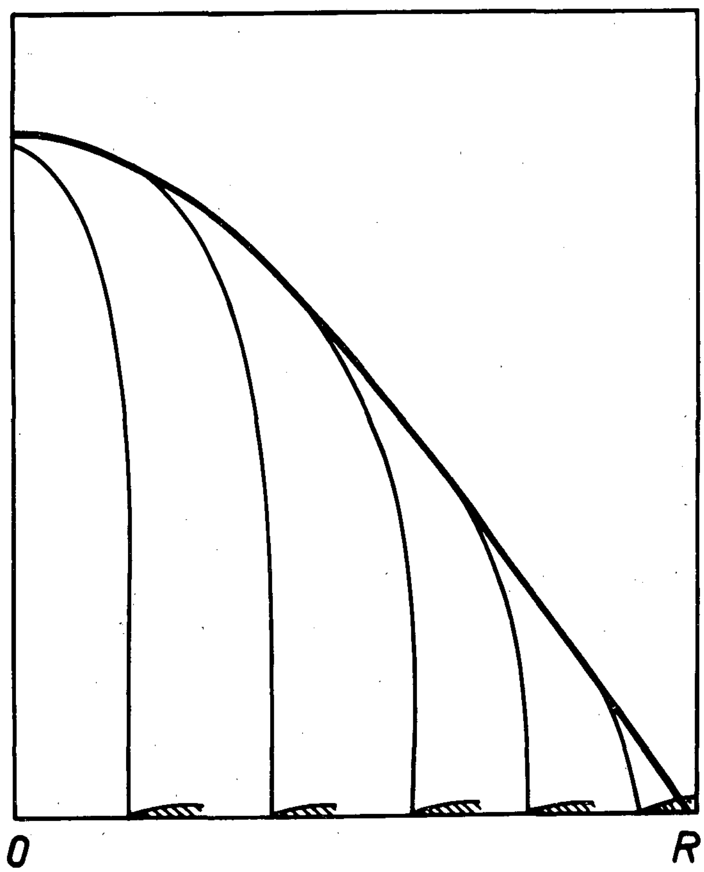

Fig. 8. Density distribution for prolate probes of various depths of probe penetration. Here probe radius $r_p = 0.02$ R. Boundary condition (15) was used.

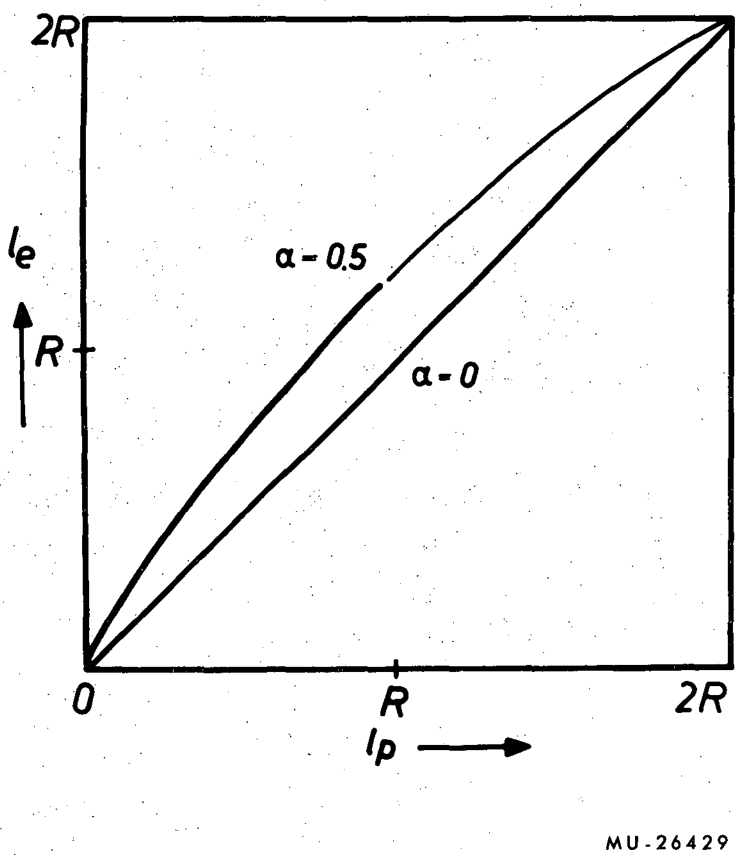

Fig. 9. Effective probe length vs true probe length for a prolate probe in a cylindrical Bessel distribution, where probe radius $r_p = 0.02$ R sheath thickness $\ell_s = 0.01$ R.

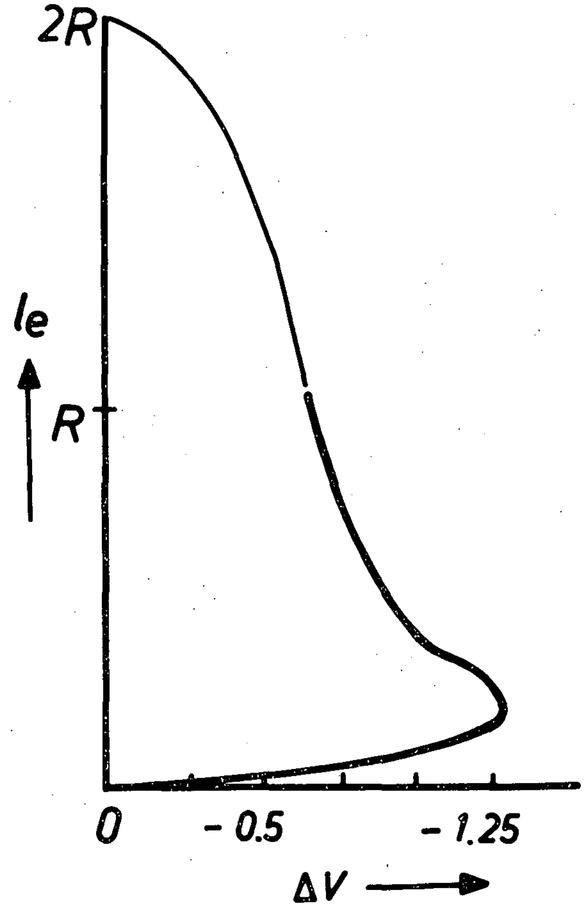

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Fig. 10. Potential drop for a prolate probe under conditions corresponding to those in Fig. 9. The unit is KT_-/e.

IV. THE INERTIA-LIMITED REGION

Since the extension of the inertia-limited region is small compared with the probe dimensions, it is sufficient to consider an infinite insulated plane wall bounding an infinite plasma. This problem has already been the subject of earlier investigations.[17, 18, 19]

The extension of the space-charge region may be either larger or smaller than that of the inertia-limited zone. The ion and electron densities at the edge of the free-fall zone are taken to be n_+^0, n^0 respectively. We describe the electron density at a point inside the free-fall region by the Boltzmann relation. This is correct provided that the isotropic current density of the electrons in the plasma is much larger than the current density going to the wall.

The ions enter the free-fall zone with initial velocity v_+^0. Their motion in the free-fall region is inertia-limited.

Under these assumptions the description of the inertia-limited region is simple and given by the time-dependent equations

$$\partial^2 V/\partial x^2 = 4\pi e (n_+ - n_-) \tag{34}$$

$$n_- = n_-^0 \exp[-eV/kT_-] \tag{35}$$

$$\partial (n_+ v_+)/\partial x = -\partial n_+/\partial t \tag{36}$$

and

$$\partial v_+/\partial t + v_+ \partial v_+/\partial x = (e/M)\ \partial V/\partial x \tag{37}$$

where V is the negative of the electrostatic potential.

In the stationary situation, all time derivatives in (34) through (37) are identically zero and all partial space derivatives may be replaced by total derivatives. This system of equations yields the well-known equation of the "inertia-limited sheath,"

$$d^2\eta/ds^2 = (1 + 2\eta/\gamma)^{-1/2} - \delta \exp[-\eta] \tag{38}$$

where we have used

$$\eta = \frac{eV}{kT_-}\ ;\ \gamma = \frac{M(v_+^0)^2}{kT_-}\ ;\ \delta = \frac{n_-^0}{n_+^0}\ ;\ s = \frac{x}{\ell_D}\ ;\ \ell_D^2 = \frac{kT_-}{4\pi e^2 n_+^0} \tag{39}$$

This equation may be integrated at once and yields

$$(d\eta/ds)^2 = 2\gamma\left[(1 + 2\eta/\gamma)^{1/2} - 1\right] + 2\delta(\exp[-\eta] - 1) + \epsilon_0^2 \tag{40}$$

where we used

$$\eta_0 = 0\ ;\ (d\eta/ds)_0 = \epsilon_0 \tag{41}$$

From this result, Bohm[17] concluded his well-known "sheath stability criterion" assuming $\delta = 1$ and $\epsilon_0 = 0$. Because of the positive definite character of the left-hand side, it follows from an expansion of the right-hand side in powers of η that

$$\gamma \geqslant 1 \ . \tag{42}$$

Hall[19] has pointed out the unreliability of this criterion, since ϵ_0 was not taken into account.

In the following we investigate whether Bohm's criterion is necessary and sufficient. To see whether it is necessary, we omit the assumptions $\delta = 1$ and $\epsilon_0 = 0$. We introduce a critical value δ_c which is chosen so that the right-hand side of Eq. (40) at its minimum value is equal to zero. This condition may be written in the form

$$\delta_c = \frac{e^{\eta^*}}{(1 + 2\eta^*/\gamma)^{1/2}} \tag{43}$$

where η^* is the root of equation

$$\gamma \left[1 + 2\eta^*/\gamma\right] - (\gamma - \epsilon_0{}^2/2)(1 + 2\eta^*/\gamma)^{1/2} + (1 - \epsilon^{\eta^*}) = 0 \tag{44}$$

The result of this calculation is demonstrated in Fig. 11. In addition, Fig. 11 shows a curve designated nsp. This curve separates the region with negative space charge from that with positive space charge only. This relation is easily obtained by finding the value of η for which the right-hand side of the space-charge equation (38) has a minimum, and by adjusting δ so that at this minimum the space charge is equal to zero.

The results presented in Fig. 11 may be summarized as follows. According to "Bohm's stability criterion" there should be no stationary solution within the whole range of this figure. We find, however, by accounting for the variations of δ and ϵ_0, that stationary solutions exist for $\gamma < 1$. For a given value of ϵ_0 all combinations of γ and δ to the right and below the curve, with the index ϵ_0, give stationary solutions. If the chosen combination of γ and δ lies between the curve and the nsp curve, we must expect a partial negative space charge. If a chosen combination of γ and δ lies to the right of and below the nsp curve, then we have positive space charge only.

The distinction of partial negative space charge contributions is essential, since it has been argued that such configurations should be excluded[18, 20].

Bohm's criterion is not necessary because there are stationary solutions without negative space charge in the area to the right and below the curve nsp.

A justifiable question is whether the values of ϵ_0 and δ in our calculations correspond to physical reality. There can be no doubt that at the edge of the inertia-limited region the statements $\delta \neq 1$ and $\epsilon_0 \neq 0$ are correct.

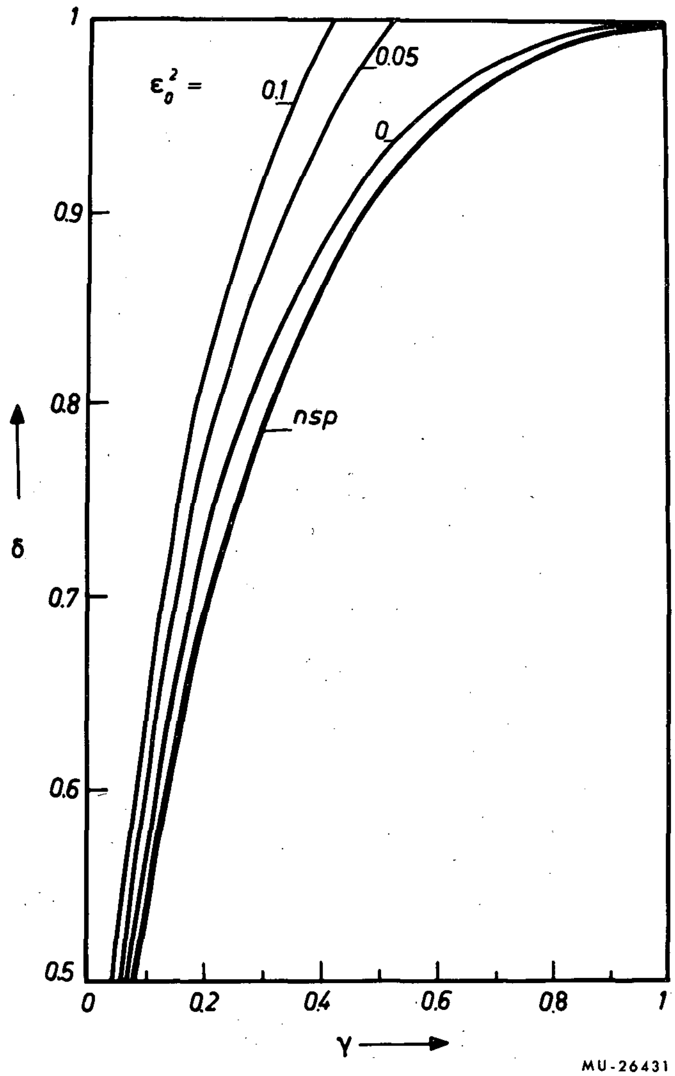

Fig. 11. This figure demonstrates how the validity of Bohm's criterion is affected by an initial field (ϵ_0) and a density ratio δ at the edge of the inertia-limited region. The symbols and meaning of the curves are explained in the text.

The point is only how large these deviations are. This can be estimated by the results from the calculation of the diffusion-disturbed region.

For example, if we look at the solution of the cylindrical positive column at a distance λ from the walls, and assume that the equations of ambipolar diffusion apply up to this point, we easily obtain

$$\epsilon_0 \approx \frac{\ell_D}{\lambda} \left(1 + \frac{2\lambda}{R}\right)^{1/2} ; \quad \delta \approx 1 - \epsilon_0^{\,2} \tag{45}$$

where we have taken $\ell_D < \lambda << R$. By use of the relation (45), Eq. (40) was integrated with the aid of an analogue computer. The results for the potential distribution for $\gamma = 0.6$ and various values of ϵ_0 are shown in Fig. 12. This figure shows quite clearly the effects discussed in connection with Fig. 11.

Now that we have seen that the criterion $\gamma \geqslant 1$ is not necessary, we wish to investigate whether it is sufficient to ensure a stationary solution. The argument that it is not sufficient can be given in general terms. Integrating Eq. (40), we get a value V_w for the wall potential. This value depends on the parameters occurring in Eq. (40), which in turn depend on the plasma parameters $n_+^{\,0}$, T_-, E_0 at the edge of the inertia-limited region. We then get the wall potential as a function of these plasma parameters in the general form

$$V_w = V_w(n_+^{\,0}, \; T_-, E_0) \; . \tag{46}$$

Now we have assumed an insulated wall, and consequently the net electric current to the wall in the stationary state must be zero. Expressing the wall current in terms of the plasma data $n_+^{\,0}$, T_-, E_0, and using the equation for the description of the inertia-limited region, we get another function that says

$$J(n_+^{\,0}, \; T_-, \; E_0, \; V_w) = 0 \tag{47}$$

should hold. The elimination of V_w between Eqs. (46) and (47) then yields the condition

$$G(n_+^{\,0}, \; T_-, E_0) = 0 \tag{48}$$

There is no reason, in general, why a plasma should fulfill this relation at the edges of either the diffusion-disturbed region or the inertia-limited region. Consequently, our only conclusion can be that certain bounded plasmas cannot have a stationary sheath at an insulated wall. This statement is true even if Bohm's criterion is fulfilled. Consequently, Bohm's criterion is not sufficient.

One might ask why this consequence does not arise in Bohm's discussion. It is because Bohm considers the space-charge region and not, as in our case, the inertia-limited region. For his space-charge region the extension is a parameter that he can dispose of. In our calculation, the extension of the inertia-limited region is a predetermined quantity.

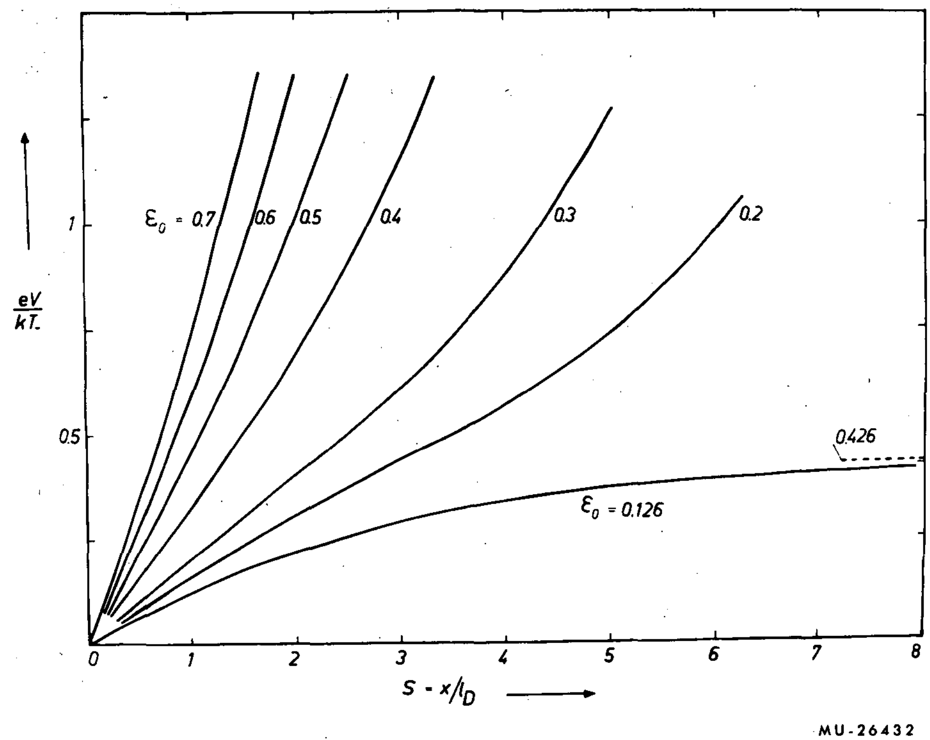

Fig. 12. Potential distribution within the inertia-limited region calculated for various initial-field values ϵ_0 is related to δ by the solution of the diffusion-disturbed region. Solutions with $\epsilon_0 < 0.126$ are periodic, and consequently do not correspond to stationary states.

A remark on the commonly used term "stability" is here appropriate. The foregoing description shows that the criterion that the right-hand side of (40) be positive does not state the sheath to be stable, but only that there is a stationary solution for the sheath. So it is actually not a "stability criterion" but a "stationarity criterion."

Whether the stationary state is stable is still an open question requiring consideration of the time dependence of small perturbations of this stationary state.

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REFERENCES

1. I. Langmuir and H. Mott-Smith, Jr., Gen. Elec. Rev. 27, 449, 538, 616, 762, 810 (1924).
2. H. Mott-Smith, Jr. and I. Langmuir, Phys. Rev. 28, 727 (1926).
3. L. Tonks and I. Langmuir, Phys. Rev. 33, 1070 34, 876 (1929).
4. D. Bohm, E. H. S. Burhop, and H. S. W. Massey, in The Characteristics of Electrical Discharge in Magnetic Field, edited by A. Guthrie and R. K. Wakerling, (Mc Graw-Hill Book Company, Inc., New York, 1949), Chap. II.
5. R. L. F. Boyd, Proc. Roy. Soc. (London) A 201, 329 (1950).
6. I. B. Bernstein and I. N. Rabinowitz, Phys. Fluids, 2, 112 (1959).
7. Lawrence S. Hall, Probes and Magnetic Pumping in Plasma, Lawrence Radiation Laboratory Report UCRL-6535, July 19, 1961.
8. B. Davydov and L. Zmanovskaja, Soviet Phys. -Tech. Phys. 38, 715 (1936).
9. R. L. F. Boyd, Proc. Phys. Soc. (London) B64, 795 (1951).
10. K. S. Masterson, (Thesis), United States Naval Postgraduate School, Monterey, California, 1961.
11. G. Ecker, Proc. Phys. Soc. (London) 67, 485 (1954).
12. R. Courant and D. Hilbert, Methoden der mathematischen Physik, (Springer Verlag, Berlin, 1937), Vol. II, Chap. VII.
13. Ibid., Vol. 1 p. 392.
14. W. C. Sangrem, Digital Computer and Nuclear Reactor Calculations (John Wiley & Sons, New York, 1960).
15. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (Mc Graw-Hill Book Company, Inc., New York, 1953), Vol. 2 p. 1062.
16. C. Flammer, Spheroidal Wave Functions (Stanford University Press, 1957).
17. D. Bohm, in the Characteristics of Electrical Discharges in Magnetic Fields, edited by A. Guthrie and R. K. Wakerling, (Mc Graw-Hill Book Company, Inc., New York, 1949), Chap. 3.
18. G. J. Schulz and S. C. Brown, Phys. Rev. 98, 1642 (1955).
19. L. S. Hall, Phys. Fluids 4, 388 (1961).
20. J. E. Allen and P. C. Thonemann, Proc. Phys. Soc. (London) B67, 769 (1954).