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A QUANTAL RESPONSE EQUILIBRIUM MODEL
OF ORDER STATISTIC GAMES

BY

KANG-OH YI

A Quantal Response Equilibrium Model
of Order Statistic Games[*]

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Abstract

This paper applies quantal response equilibrium (QRE) models (McKelvey and Palfrey, *Games and Economic Behavior* 10 (1995), 6-38) to a wide class of symmetric coordination games in which each player's best response is determined by an order statistic of all players' decisions, as in the classic experiments of Van Huyck, Battalio, and Beil (*American Economic Review* 80 (1990), 234-248; *Quarterly Journal of Economics* 106 (1991), 885-910), but players have a bounded continuum of decisions, which approximates to Van Huyck, Battalio, and Rankin's (1996) environment. Generalizing the results of Anderson, Goeree, and Holt (1998) with a quadratic payoff function, I show that as the noise vanishes the QRE approaches the most efficient equilibrium as a unique limit for all order statistics, including the minimum.

JEL Classification Number: C79, C92.

Key Words: Equilibrium selection, coordination, strategic uncertainty, logit equilibrium.

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1 Introduction

This paper applies "quantal response equilibrium" (henceforth, QRE) models, proposed by McKelvey and Palfrey (1995), to a wide class of symmetric coordination games in which each player's best response is determined by an order statistic of all players' decisions, as in the classic experiments of Van Huyck, Battalio, and Beil (1990, 1991), but players have a bounded continuum of decisions, which approximates to Van Huyck, Battalio, and Rankin's (1996) environment.

The notion of QRE is motivated to model human subjects' imperfectly optimizing behavior, which often observed in experiments. A QRE is defined as an equilibrium in which players choose their strategies stochastically, with strategies that have higher expected payoffs chosen with higher probabilities. In the equilibrium the players take the noise in each other's strategy choices rationally into account, choosing strategies with probabilities that are increasing in their expected payoffs based on the distributions of others' strategies. Thus, the QRE describes imperfectly optimizing behavior by assuming a stochastic decision rule, but maintains most of the parsimony of an equilibrium analysis.

QRE allows a wide class of probabilistic choice rules to be substituted for perfect maximizing behavior in an equilibrium context. In this paper, the QRE is specialized to the case of the logistic response to determine a "logit equilibrium", in which the amount of strategic uncertainty is measured by a single parameter. I call a limit point of a sequence of logit equilibria as the noise vanishes a "limiting logit equilibrium." Since a limiting logit equilibrium gives the clearest results, the present analysis focuses on a limiting logit equilibrium.[1]

Stochastic choice models like multinomial logit and multinomial probit that underlie QRE are well known in econometrics, but in games players' choice probabilities interact in ways that make the analysis considerably more difficult. As a result, most applications have relied on numerical solutions. Using such methods, McKelvey and Palfrey (1995, 1998) have shown that, for values of the error parameter estimated from experimental data, the logit equilibrium often gives an accurate description of experimental outcomes. In those

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[1]McKelvey and Palfrey (1995) used term "limiting logit equilibrium" for the limit point of a specific sequence of logit equilibria, which McKelvey and Palfrey (1996) called "logit solution," and they did not make a distinction between "limiting logit equilibrium" and "logit solution." In this paper, limiting logit equilibrium is used to denote a limit point of a sequence of logit equilibria. See footnote 9 for a informal definition of logit solution.

papers, a limiting logit equilibrium (strictly speaking, logit solution) is used as a benchmark to compare it with other refinements. McKelvey and Palfrey (1995, 1998) suggested the possibility of using the limiting logit equilibrium as an equilibrium selection criterion (see footnote 9), but very little is known about the implications of this notion in games with multiple equilibria.

Anderson, Goeree and Holt (1998b) have recently applied the notion of limiting logit equilibrium to analyze some coordination games that have been studied experimentally by Van Huyck, Battalio, and Beil (henceforth "VHBB") (1990). They showed that the limiting logit equilibrium not only selects a unique equilibrium in those games but also describes the limiting outcomes in VHBB's (1990) experiments with surprising accuracy. Their result suggests that logit equilibrium may also determine a unique equilibrium that describes the experimental results on the closely related games studied experimentally by VHBB (1991) and Van Huyck, Battalio, and Rankin (henceforth "VHBR") (1996).

In VHBB's (1990, 1991) and VHBR's (1996) games, each of a group of symmetric players simultaneously chose among pure strategies called "efforts," and players' payoffs and best responses were determined by their own efforts and simple summary statistics of their own and the other players' efforts. In each case, any configuration in which all players choose the same effort is a strict, symmetric, pure-strategy equilibrium, and these equilibria are Pareto-ranked. Other things equal, the closer subjects' efforts were to the summary statistic, the higher their payoffs, with all players preferring equilibria with higher efforts to those with lower efforts. These games are of particular economic interest because they capture some essential features of the equilibrium selection problem in important applications. Bryant (1983) and Cooper and John (1988), for instance, have used this kind of game as a model of Keynesian effective demand failures.[2]

In VHBB (1990), each player chose among seven efforts and a given set of subjects participated in a sequence of treatments. In the large group minimum treatments, A and B, "large" groups consisted of 14-16 subjects played minimum games. Small group treatment, C, used "small" groups of 2 subjects, randomly selected from that set. There were two versions of small group minimum treatment. In one, here called "C_d", subjects pairing were

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[2]VHBB (1990, 1991), VHBR (1996), and Crawford (1991) describe the design and the results in more detail. My discussion of VHBB's designs and results follows Crawford (1991).

different in each stage; in the other, here called "C_f", they were fixed. Treatments A and C combined a preference for a higher minimum effort, other things equal, with increasingly severe penalties for being further and further away from the minimum. In treatment B, the effort cost was set equal to 0, making highest effort (and the only one consistent with efficiency) a weakly dominant strategy.

In VHBB's (1990) experiments, the effects of strategic uncertainty showed up especially clearly in the dynamics and limiting outcomes. Although the efficient high-effort equilibrium is best for all players, its payoffs are more sensitive to coordination failures than other equilibria. As a result, there were interesting, systematic tendencies in the dynamics, and subjects often converged to inefficient equilibria, even though they would all have preferred to coordinate on higher common effort level. In treatment A, the play gravitated toward an inefficient Nash outcome without an exception. By contrast, in treatment B, almost all subjects (96%) reached the highest effort by the last round. The outcomes were very different in the small groups of minimum treatments C_d and C_f, which had the same payoff function as treatment A. In treatment C_d, subjects' choices drifted over time with no clearly discernible trend with a median effort level of 5 in the 1-7 scale, while in treatment C_f most subjects (90%) reached the highest effort by the last round.

Anderson, Goeree, and Holt (1998b) characterized logit equilibrium in games that maintain the general features of the payoff structures of VHBB's (1990) minimum games, but approximate the strategy space with a bounded continuum of efforts. They showed that logit equilibria are unique for any given value of the error parameter (despite the games' continua of strict Nash equilibria without errors), and for reasonable values of the error parameter, surprisingly close to the last period outcomes of VHBB's (1990) treatments.[3] Their model also yields comparative statics results that the efforts decrease stochastically with increases in effort costs and the number of players. This resembles the effects of those changes across VHBB's (1990) treatments; between treatments A and B, and between treatments A and

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[3]Logit equilibrium of a stage game does not discriminate the fixed- pair 2-person minimum treatment and the random-pairing minimum treatment, and in both treatments the limiting logit equilibrium strategy is translated to an effort level of 4 in the 1-7 scale used by VHBB (1990). In random-pairing minimum treatment the median effort level was 5 which is very close to the prediction, but in the fixed-pair 2-person minimum treatment most subjects played the highest possible effort level, which is far from the logit equilibrium. This result suggests that a kind of dynamic game aspects of the fixed-pair treatment facilitated coordination, which is not modeled in stage game logit equilibrium.

C_d.

In VHBB's (1991) experiment, with a fixed group playing in each stage, groups of 9 subjects played median games that had similar payoff structures as in VHBB (1990), but each player's payoff decreased quadratically in the distance between the median and his own effort.[4] The play converged to the initially determined median, which is usually inefficient. Such history dependence was not observed in VHBB's (1990) experiments.

In VHBR's (1996) experiment, the subjects played 5- and 7-person second-order statistic games and fourth-order statistic games with 101 effort levels and with the identical payoff functional form that used in VHBB (1991), but with different payoff parameters. Interestingly, changing the grid size of effort levels made the result strikingly different from those in VHBB (1990, 1991). A most striking contrast to VHBB (1990, 1991) is that each treatment elicited several different patterns of convergence. Since in every treatment in VHBB (1990, 1991) plays exhibited similar dynamics, the changing grid size seem to have an important influence on behavior. Moreover, in the 7-person second order statistic game, 2 out of 10 groups were converging to the most efficient outcome which requires that at least 6 players should coordinate on the higher effort while 2 players are enough to lower the order statistic.[5]

Although VHBB's (1991) median games are very similar in structure to VHBB's (1990) minimum games, and Anderson, Goeree, and Holt's (1998) continuous strategy spaces bring their analysis much closer to VHBR's (1996) environment, Anderson, Goeree, and Holt's (1998) analysis is confined to minimum games, and they do not attempt to use their results to suggest an interpretation of the results in VHBB's (1991) and VHBR's (1996) experiments.

In this paper, I extend Anderson, Goeree, and Holt's analysis to fill that gap, characterizing the limiting logit equilibrium in a class of order statistic games with continuous strategy spaces that includes both minimum and median games. The results can be used to evalu-

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[4]In the experiment, they used various payoff functions which satisfy the basic properties described before. In treatment Γ, a player's payoff is linearly increasing in the group summary statistic and quadratically decreasing in the difference between the summary statistic and his own effort choice. In treatment Φ, a higher median does not raise payoff, but one who mismatches the median is penalized in the same way in treatment Γ. In treatment Ω, a player's payoff increases linearly in the median effort, but the payoff is 0 when his effort is different from the order statistic. In the text, I focus on the baseline payoff function which used in treatment Γ.

[5]VHBR (1996) suggested "Reducing the opportunity cost of local exploration increases subjects' propensity to experiment with actions slightly higher than last period's order statistic." The analysis in this paper shows that the logit equilibrium effort density induces similar behavior.

ate the ability of the logit equilibrium and related notions to describe limiting outcomes in VHBB's (1991) and VHBR's (1996) experiments, as well as in VHBB's (1990) experiments. The main results are that, in every order statistic game in the class under consideration, the most efficient Nash equilibrium is the unique limiting logit equilibrium.[6] This result suggests that the logit equilibrium fails to capture some important aspects of subjects' responses to VHBB's (1991) median games, but that it has the potential to explain some of the otherwise extremely puzzling outcomes of VHBR's (1996) experiments.

In their analysis, Anderson, Goeree, and Holt (1998b) focused on the effect of costs and number of players, and paid little attention to the difference between their continuous strategy spaces and the discrete strategy spaces used in the experiment. With the payoff function used in the minimum treatments, the continuous approximation to the coarse strategy space does not distort the game structure very much. However, the present analysis shows that with the payoff function used in VHBB (1991) and VHBR (1996), the continuous approximation changes the game structure to the extent that even some refinement criteria like risk dominance also make different predictions with finer grid size of the effort level, even though the set of Nash equilibria is not affected by such a change. This suggests that, by taking the coarseness of the strategy space into account, the logit equilibrium could explain most of the experimental results reported in VHBB (1990, 1991) and VHBR (1996), but I do not pursue that line of research in this paper. Instead I use simple calculations to demonstrate the effect of the grid size on the individual subjects' incentive problems in VHBB's (1990, 1991) and VHBR's (1996) experiments.

The rest of the paper is organized as follows. Section 2 introduces a broad class of order statistic games with bounded, continuous strategy spaces, and defines the notions of logit equilibrium and limiting logit equilibrium for those games. In Section 3, I adapt Anderson, Goeree, and Holt's (1998b) methods to characterize the limiting logit equilibrium in those games. I illustrate by means of examples the effect of the continuity and functional form of the payoff function on behavior. Section 3 concludes by discussing the usefulness of the logit equilibrium in describing subjects' behavior in VHBB's and VHBR's experiments. Section 4 studies a variant of logit equilibrium that I call the "competitive logit equilibrium,"

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[6]When a higher order statistic does not raise payoffs ($a = 0$ in Eq.(1)) as in treatment Φ in VHBB (1991), all equilibria are equally efficient.

which is plausible in games with larger numbers of players, in which people often seem to use a "competitive" approximation to the effects of others' strategies on their own payoffs. The competitive logit equilibrium in order-statistic games is defined like the original logit equilibrium, but with players ignoring their own influences on the order statistic. One might expect that a Nash equilibrium of a game with a large finite number of players is close to the analogous notion in which players ignore their own influences, the limiting competitive logit equilibrium can be very different from the limiting logit equilibrium, even in the nonpathological class of order-statistic games studied here. Section 5 is the conclusion.

2 Order Statistic Games and Logit Equilibrium

In an n-person order statistic game, a player's payoff is determined by his own effort and an order statistic of his own and the other players' efforts. Each player chooses an effort level $x_i \in [0, \bar{x}], i = 1, \cdots, n$, and $\bar{x}$ is a finite maximum effort level. Each player is assumed to have a risk-neutral preference. Let $u_i(x_i, m)$ be the player i's payoff when he plays x_i and the prespecified order statistic is m. The basic structure of payoff function in the economic literature is that $u_i(m, m) \geq u_i(m', m')$ for all $m > m'$, and $u_i(x_i, m) > u_i(x_i', m)$ for all $|x_i - m| < |x_i' - m|$.[7] In this paper, I use a specific functional form which has been used in VHBB (1991) and VHBR (1996).

$$(1) \quad u_i(x_i, m_{j:n}) = am_{j:n} - b(m_{j:n} - x_i)^2 + c, \quad a, b, c \geq 0$$

where $m_{j:n}$ is the jth inclusive order statistic which is defined by $m_{1:n} \leq m_{2:n} \leq \cdots \leq m_{n:n}$, where the $m_{j:n}$ is the jth element of choice combinations $\{x_1, \cdots, x_n\}$ arranged in increasing order. When no confusion will arise, $u_i(x_i, m)$ is denoted by $u_i(x_i)$.

With a strictly positive a, it is always best to coordinate on the highest effort, $\bar{x}$. To realize the efficient outcome, however, players must overcome a subtle incentive problem. For instance, in a minimum game, the higher effort's higher payoff when all choose it must be traded off against its greater risk of lower payoffs if others do not. That risk is entirely due to strategic uncertainty. For a player to find it rational to choose $\bar{x}$, he must believe that the correctness of this choice is sufficiently obvious that it is likely that "all" of the other players will believe that its correctness is sufficiently obvious to all. The strategic

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[7]In some treatments in VHBB (1990, 1991), $u_i(x_i, m) = u_i(x_i', m)$ if $x_i \neq m$ and $x_i' \neq m$.

uncertainty that underlies this incentive problem can profoundly affect behavior. In the order statistic games, however, the present analysis shows that the restriction of QRE on belief and behavior removes the indeterminacy of conditional probabilities of outcomes and eliminates strategy uncertainty.

In this paper, I focus on a specialized version of QRE where the choice probabilities are analogue of the standard multinomial logit distribution (McFadden, 1974). The probability density of player i's choosing x_i is a function of the expected payoff $\pi_i(x_i)$ and the density of each choice is an increasing function of the expected payoff for that choice:

$$(2) \quad f_i(x_i) = \frac{\exp\left(\lambda \pi_i(x_i)\right)}{\int_0^{\bar{x}} \exp\left(\lambda \pi_i(y)\right) dy}$$

where $0 \leq \lambda < \infty$ measures the amount of noise, or equivalently, the degree of rationality. This functional form is called a logit function where the odds are determined by the exponential transformation of the utility times a given non-negative constant λ. The ratio of probabilities of two different effort choices is $f_i(x_i)/f_i(x_i') = \exp[\lambda(\pi_i(x_i) - \pi_i(x_i'))]$ and the logit function is invariant to the transformation of expected payoffs by changing the origin. As $\lambda \to \infty$, the probability of the choice having the highest expected payoff becomes one, if it is unique, so that the choice behavior becomes best response; when $\lambda = 0$ all choices have equal probability. Throughout this paper I assume that λ is the same for all players and it is common knowledge.[8] Logit equilibrium for λ is defined by a fixed point in these probability distributions with a given λ.

In general, logit equilibrium and limiting logit equilibrium are not unique.[9] In particular, in finite games with multiple strict equilibria, logit equilibrium is not unique for a sufficiently large λ and any strict Nash equilibrium can be found as a limit of logit equilibrium.[10] Therefore, given a sufficiently large λ, the "initial belief" (or "initial play") is crucial in

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[8] In principle, QRE permits different λ's across players (see McKelvey and Palfrey, 1996), but the common knowledge assumption is indispensable for the analysis in this paper.

[9] One can define a unique selection from the set of Nash equilibria by "tracing" the graph of the logit equilibrium correspondence beginning at the centroid of the strategy simplex (the unique solution when $\lambda = 0$) and continuing for larger and larger values of λ. McKelvey and Palfrey (1996) called the limit point of such sequence of logit equilibrium "logit solution" of a game. McKelvey and Palfrey (1995, 1998) showed that logit solution is unique in almost all finite games, but little is known about the properties of logit solution.

[10] By definition, a strict equilibrium is necessarily a pure-strategy equilibrium and strict equilibria remain strict when the payoff are slightly perturbed since the strict inequality remains satisfied.

determining a limiting logit equilibrium. In games with a continuum of Nash equilibria, however, since not every equilibrium can survive small perturbations, the notion of QRE refines the set of equilibria.

Next step is to apply the probabilistic choice rule, Eq.(2), to the payoff structure in Eq.(1). Let $F_i(x)$ denote the cumulative distribution function associated with $f_i(x)$. Let $G^i_{j:n-1}(x)$ be the cumulative distribution function of jth order statistic regarding $\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$ where the x's are drawn from distributions, $\{F_1, \cdots, F_{i-1}, F_{i+1}, \cdots, F_n\}$, respectively. Let $g^i_{j:n-1}(x)$ be the associated probability density function. Given $\{F_1, \cdots, F_{i-1}, F_{i+1}, \cdots, F_n\}$, in the minimum game, $G^i_{1:n-1}(x) = 1 - \prod_{i \neq k}(1 - F_i(x))$ where $\prod_{i \neq k} F_i(x) = F_1(x) \cdots F_{k-1}(x) \cdot F_{k+1}(x) \cdots F_n(x)$, and

$$
\begin{aligned}
\pi_i(x_i) &= \int_0^{x_i} (ay - b(y - x_i)^2) g^i_{1:n-1}(y) dy + ax_i(1 - G^i_{1:n-1}(x_i)) + c \\
&= ax_i G^i_{1:n-1}(x_i) - a \int_0^{x_i} G^i_{1:n-1}(y) dy + 2b \int_0^{x_i} y G^i_{1:n-1}(y) dy \\
&\quad -2bx_i \int_0^{x_i} G^i_{1:n-1}(y) dy + ax_i - ax_i G^i_{1:n-1}(x_i) + c
\end{aligned}
$$

or

$$
(3) \quad \pi_i(x_i) = a \left[x_i - \int_0^{x_i} G^i_{1:n-1}(y) dy \right] + 2b \left[\int_0^{x_i} (y - x_i) G^i_{1:n-1}(y) dy \right] + c
$$

When $2 \leq j \leq n - 1$, given x_i,

$$
\begin{array}{llll}
P(m_{j:n} < y) &= P(m_{j:n-1} < y) &= G^i_{j:n-1}(y), & \text{for } y < x_i \\
P(m_{j:n} < y) &= P(m_{j-1:n-1} < y) &= G^i_{j-1:n-1}(y), & \text{for } y > x_i \\
P(m_{j:n} = y) &= P(m_{j-1:n-1} \leq y \leq m_{j:n-1}) &= G^i_{j-1:n-1}(y) - G^i_{j:n-1}(y), & \text{for } y = x_i
\end{array}
$$

and

$$
G^i_{j-1:n-1}(x) = \sum_{k=j-1}^{n-1} \sum_{S_k} \prod_{l=1}^{k} F_{i_l}(x) \prod_{l=k+1}^{n-1} (1 - F_{i_l}(x))
$$

where the summation S_k extends over all permutations $i_1, \cdots, i_{n-1}$ of $1, \cdots, n-1$ for which $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_{n-1}$. Then the expected payoff becomes

$$
\begin{aligned}
\pi_i(x_i) &= a \int_0^{x_i} y g^i_{j:n-1}(y) dy - b \int_0^{x_i} (y - x_i)^2 g^i_{j:n-1}(y) dy \\
&\quad + a \int_{x_i}^{\bar{x}} y g^i_{j-1:n-1}(y) dy - b \int_{x_i}^{\bar{x}} (y - x_i)^2 g^i_{j-1:n-1}(y) dy \\
&\quad + ax(G^i_{j-1:n-1}(x_i) - G^i_{j:n-1}(x_i)) + c
\end{aligned}
$$

9

$$
\begin{aligned}
= \ & -a\int_0^{x_i} y(g_{j-1:n-1}^i(y) - g_{j:n-1}^i(y))dy \\
& + b\int_0^{x_i}(y - x_i)^2(g_{j-1:n-1}^i(y) - g_{j:n-1}^i(y))dy \\
& + a\int_0^{\bar{x}} y g_{j-1:n-1}^i(y)dy - b\int_0^{\bar{x}}(y - x_i)^2 g_{j-1:n-1}^i(y)dy \\
& + ax(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) + c \\
= \ & -ax_i(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) + a\int_0^{x_i}(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy \\
& + bx_i^2(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) - 2b\int_0^{x_i} y(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy \\
& - 2bx_i^2(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) + 2bx_i\int_0^{x_i}(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy \\
& + bx_i^2(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) + aE_i(m_{j-1:n-1}) - bE_i(m_{j-1:n-1}^2) \\
& + 2bx_i E_i(m_{j-1:n-1}) - bx_i^2 + ax_i(G_{j-1:n-1}^i(x_i) - G_{j:n-1}^i(x_i)) + c \\
= \ & aE_i(m_{j-1:n-1}) - bE_i(m_{j-1:n-1}^2) + 2bE_i(m_{j-1:n-1})x_i - bx_i^2 \\
& + \int_0^{x_i}(a - 2by + 2bx_i)(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy + c
\end{aligned}
$$

Since the logit effort density is invariant to the changing of the origin, the first two terms in the last equation, which are independent of x_i, are irrelevant in determining the equilibrium density function as well as c. $\pi_i(x_i)$ can therefore be rescaled as follows:

$$
(4) \quad \pi_i(x_i) = 2bE_i(m_{j-1:n-1})x_i - bx_i^2 + \int_0^{x_i}(a - 2by + 2bx_i)(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy
$$

The effort density function is constructed by substituting Eq.(3) and Eq.(4) in Eq.(2).

3 Limiting Logit Equilibria of Order Statistic Games

Anderson, Goeree, and Holt (1998b) characterized the logit equilibrium in the minimum game with a continuous strategy space and with a linear payoff function used in VHBB (1990). Their main result is that the logit equilibrium is unique for any given value of λ and the logit equilibrium strategy is stochastically decrease with increases in effort costs and the number of players. They also showed that the efficiency of a limiting logit equilibrium depends on the payoff parameters. In this section, with a quadratic payoff function as in Eq.(1), it is shown that the limiting logit equilibrium is always efficient regardless of the order statistic, the number of players, and the values of payoff parameters as long as $a > 0$ and $b \geq 0$.

For games with a finite discrete strategy space and a finite number of players, McKelvey and Palfrey (1995) proved the existence and the convergence of logit equilibrium to a Nash equilibrium as the λ goes to infinity. In order statistic games, the same properties hold with continuous strategy space.

Proposition 1. There exists a logit equilibrium for every $\lambda \geq 0$.

The proof of Proposition 1 is a simple modification of the existence proof in Anderson, Goeree, and Holt (1998a, Appendix A). For completeness, I provide the proof in Appendix. (Every omitted proof in the text can be found in Appendix.)

Lemma 1. In every order statistic game with Eq.(1), the limiting logit equilibrium is a symmetric pure-strategy Nash equilibrium.

In fact, when the payoff function is given by Eq.(1), there is no mixed-strategy Nash equilibrium while with a linear payoff function there is a continuum of mixed-strategy Nash equilibria (see Anderson, Goeree, and Holt, 1998b).

In order statistic games, not only is the limiting logit equilibrium symmetric, but so is the logit equilibrium for every finite λ. This symmetry is crucial in calculating the limiting logit equilibrium. Since infinite λ is the "rationality" limit, a limiting logit equilibrium gives a selection from Nash equilibria. The way to analyze logit equilibrium is adapted from Anderson, Goeree, and Holt (1998b).

Lemma 2. The logit equilibrium effort density $f_i(x_i)$ is jointly continuous in x_i and λ, and differentiable with respect to x_i.

Proof. In Eq.(3) and Eq.(4), $\pi_i(x_i)$ is jointly continuous in x_i and λ, and differentiable with respect to x_i. In Eq.(2), given strictly positive denominator, for any x_i and λ, $f_i(x_i)$ is the ratio of continuous transformation of $\pi_i(x_i)$ and λ. Q.E.D.

Lemma 3. Any logit equilibrium is symmetric across players.

By Lemma 2 and 3, the first order differential equation of Eq.(2) can be written as:

(5) $f'(x) = \lambda f(x) D_x \pi(x)$

and the first derivatives of expected payoffs in the minimum game and the games with $2 \leq j \leq n-1$ are

$$(6) \quad D_x \pi(x) = a(1 - G_{1:n-1}(x)) - 2b \int_0^x G_{1:n-1}(y)dy$$

$$(7) \quad D_x \pi(x) = 2b \left[E(m_{j-1:n-1}) - x + \int_0^x (G_{j-1:n-1}(y) - G_{j:n-1}(y))dy \right]$$
$$+ a(G_{j-1:n-1}(x) - G_{j:n-1}(x))$$

After substituting Eq.(6) and Eq.(7), integrating Eq.(5) from 0 to x yields,

$$(8) \quad f(x) = f(0) + \lambda a \int_0^x (1 - G_{1:n-1}(y))f(y)dy - 2\lambda b \int_0^x \int_0^y G_{1:n-1}(z)dz\, f(y)dy$$

$$(9) \quad f(x) = f(0) + 2\lambda b \left[E(m_{j-1:n-1})F(x) - \int_0^x y f(y)dy \right.$$
$$+ \int_0^x \int_0^y (G_{j-1:n-1}(z) - G_{j:n-1}(z))dz\, f(y)dy \right]$$
$$+ \lambda a \int_0^x (G_{j-1:n-1}(y) - G_{j:n-1}(y))f(y)dy$$

Proposition 2. In every order statistic game with a quadratic payoff function and $a > 0$, as λ goes to infinity, the logit equilibrium converges to the most efficient Nash equilibrium, $x_i = \bar{x}$ for all i.

Proof. Lemma 1 implies that only one effort level can be played with positive probability in the limit of λ. In order to show that the limiting logit equilibrium is the most efficient Nash equilibrium, it is sufficient to show that $f(\bar{x})$ diverges in the limit.

Consider a minimum game first. From Eq.(8),

$$(10) \quad f(x) = f(0) + \lambda a \int_0^x (1 - G_{1:n-1}(y))f(y)dy - 2\lambda b \int_0^x \int_0^y G_{1:n-1}(z)dz\, f(y)dy$$
$$= f(0) + \frac{\lambda a}{n} \int_0^x \binom{n}{1}(1 - F(y))^{n-2}f(y)dy$$
$$- 2\lambda b \left[\int_0^x G_{1:n-1}(y)dy\, F(x) - \int_0^x G_{1:n-1}(y)F(y)dy \right]$$
$$= f(0) + \frac{\lambda a}{n} G_{1:n}(x) - 2\lambda b \int_0^x G_{1:n-1}(y)[F(x) - F(y)]dy$$

Since $F \in [0,1]$, $(1 - F(x))^n \geq (1 - nF(x))$. Therefore, $G_{1:n-1}(x) \leq nF(x)$.

$$f(\bar{x}) \geq f(0) + \lambda \left[\frac{a}{n} - 2bn \int_0^{\bar{x}} F(y)(1 - F(y))dy \right]$$

12

Suppose $f(x)$ converges to a point-mass at x^*. Since $\pi''(x) < 0$, by Lemma 1, for every $\varepsilon > 0$ and x, there exists a $\lambda_\varepsilon(x)$ such that $f(x) < f(x^* - \varepsilon) < \varepsilon$ for $x \in [0, \max(0, x^* - \varepsilon)]$ and $f(x) < f(x^* + \varepsilon) < \varepsilon$ for $x \in [\min(x^* + \varepsilon, \bar{x}), \bar{x}]$ for every $\lambda > \lambda_\varepsilon$. Then

$$
\begin{aligned}
f(\bar{x}) \ &> \ \lambda_\varepsilon \left[\frac{a}{n} - 2bn \left\{ \int_0^{x^*-\varepsilon} F(y)(1 - F(y))dy \right. \right. \\
&\qquad \left. \left. + \int_{x^*-\varepsilon}^{x^*+\varepsilon} F(y)(1 - F(y))dy + \int_{x^*+\varepsilon}^{\bar{x}} F(y)(1 - F(y))dy \right\} \right] \\
&> \ \lambda_\varepsilon \left[\frac{a}{n} - 2bn \left[(x^* - \varepsilon)\varepsilon + 2\varepsilon + (\bar{x} - x^* - \varepsilon)\varepsilon \right] \right] \\
&> \ \lambda_\varepsilon \left[\frac{a}{n} - 2bn(\bar{x} + 2)\varepsilon \right]
\end{aligned}
$$

Since logit equilibrium converges to a pure-strategy Nash equilibrium, there exists a λ_ε with $\varepsilon < \frac{a}{2bn^2(\bar{x}+2)}$, and $f(\bar{x})$ diverges as λ goes to infinity.

Consider $2 \le j \le n - 1$. In Eq.(9), $\int_0^x \int_0^y (G_{j-1:n-1}(z) - G_{j:n-1}(z))dz f(y)dy$ is non-negative for all λ and x. Hence,

$$
\begin{aligned}
(11) \quad f(\bar{x}) \ &\ge \ f(0) + 2\lambda b(E(m_{j-1:n-1}) - E(x)) + \lambda a \int_0^{\bar{x}} (G_{j-1:n-1}(y) - G_{j:n-1}(y))f(y)dy \\
&= \ f(0) + 2\lambda b(E(m_{j-1:n-1}) - E(x)) + \lambda a \int_0^{\bar{x}} \binom{n-1}{j-1}[F(y)]^{j-1}[(1 - F(y))]^{n-j} f(y)dy \\
&= \ f(0) + 2\lambda b(E(m_{j-1:n-1}) - E(x)) + \frac{\lambda a}{n} G_{j:n}(\bar{x}) \\
&= \ f(0) + \lambda \left[2b(E(m_{j-1:n-1}) - E(x)) + \frac{a}{n} \right]
\end{aligned}
$$

By Lemma 1, $|E(m_{j-1:n-1}) - E(x)| \to 0$ as $\lambda \to \infty$. Since $\frac{a}{n}$ is independent of λ, the limiting logit equilibrium is the most efficient Nash equilibrium. Q.E.D.

When the payoff function is quadratic as Eq.(1), the fineness of the strategy space affects the payoff structure with given a and b. Consider the following 2-person Stag-hunt games.

G1	1	$1 - \Delta$
1	a, a	$a(1 - \Delta) - b\Delta^2, a(1 - \Delta)$
$1 - \Delta$	$a(1 - \Delta), a(1 - \Delta) - b\Delta^2$	$a(1 - \Delta), a(1 - \Delta)$

G2	1	$1 - \Delta$
1	a, a	$a(1 - \Delta) - b\Delta, a(1 - \Delta)$
$1 - \Delta$	$a(1 - \Delta), a(1 - \Delta) - b\Delta$	$a(1 - \Delta), a(1 - \Delta)$

where $a, b > 0$ and $\Delta \in (0, 1)$. The payoffs in game G1 correspond to the quadratic payoff function, Eq.(1), when $m_{j:n}$ is the minimum. Let p denote the probability of playing 1, then mixed-strategy Nash equilibrium is $p^* = \frac{b\Delta}{a+b\Delta}$ and, thus, the strategy profile $(1, 1)$ is p-dominant action pair for all $p > p^*$ (Morris, Rob and Shin, 1995). As Δ vanishes, p^* converges to 0 and $(1, 1)$ action pair is becoming 0-dominant, and the game is "almost" dominance solvable. When each player believes that his opponent plays 1 with a probability more than p^*, $1 - \Delta$ becomes a conditionally dominated strategy. A continuous strategy space with a quadratic payoff function makes the effective ratio of a and b infinite. It is not surprising that a unique limiting logit equilibrium is the most efficient Nash equilibrium in every order statistic game.

In contrast, with a linear payoff function as used in VHBB (1990), both the penalty and the benefit increase linearly with an increase in effort. The following game corresponds to the (piecewise) linear payoff function, $u_i(x_i) = am_{j:n} - b|m_{j:n} - x_i| + c$.

In game G2, the mixed-strategy Nash equilibrium strategy is $\frac{b}{a+b}$, which is independent of Δ. The critical value of $\frac{a}{b}$ is not affected by the fineness of the strategy space. The empirical success of logit equilibrium in the minimum game is partly because of the success of continuous approximation to the effort levels which does not change the payoff structure when the payoff function is linear. Anderson, Goeree and Holt (1998) show that, in a minimum game with a linear payoff function, $u_i(x_i) = am_{1:n} - bx_i + c$, the limiting logit equilibrium depends on the value of $\frac{a}{bn}$. The limiting logit equilibria are $x_i = 0$ and $x_i = \bar{x}$ for all i when $\frac{a}{bn} < 1$ and $\frac{a}{bn} > 1$, respectively. If $\frac{a}{bn} = 1$, it is $x_i = \frac{\bar{x}}{n}$. By rescaling the effort according to VHBB's (1990) 1-7 scale, the limiting logit equilibria are 1, 7, and 4 in treatments A, B, and C_d, respectively. Those predictions are surprisingly close to the last period outcomes of VHBB's (1990) treatments.

The prediction of continuous version of logit equilibrium is very poor for VHBB's (1991) median treatments where subjects invariably converged to the equilibrium determined by the initial median which were usually inefficient. Treatment Γ used $a = .1$ and $b = .05$ and treatment Φ used $a = 0$ and $b = .05$. In treatment Ω, payoffs are positive only when a player chooses the median effort level and the equilibrium is Pareto-ranked as in treatment Γ. Using the scale 1-7 as in VHBB (1991), in treatments Γ and Φ, the subjects played effort 4 or 5; in treatment Ω, the final medians were 5 in one run and 7 in two runs. The limiting

logit equilibrium is 4 in all treatments (the limiting logit equilibria (precisely, logit solution) of treatments Φ and Ω are calculated by simulation.)[11] Such a poor prediction is mainly due to the improper continuous approximation to the discrete strategy space. When the strategy space is discrete, QRE has as little predictive power as most traditional equilibrium selection model except notions of risk-dominance and payoff-dominance as I briefly discuss in Section 2. However, by taking the discreteness into account, following exercise shows that the limiting logit equilibrium is quite sensitive to the initial belief.

The strong history dependence showed up in VHBB (1991) can be easily explained in QRE framework by calculating the expected payoff given the amount of noise. Let m^* be the median in the previous period. Suppose, given m^*, players have only two choices in each period; they should choose m^* with probability $1 - p$ and choose $m = m^* + 1$ with probability p.[12] Using the same payoff function used in VHBB (1991), in a 9-person median game (treatment Γ),

$$\pi_i(m^*) = (1 - P_1) \times (.1\ m^*) + P_1 \times [.1\ (m^* + 1) - .05]$$
$$\pi_i(m^* + 1) = (1 - P_2) \times (.1\ m^* - .05) + P_2 \times .1\ (m^* + 1)$$

where $P_1 = \sum_{k=5}^{8} \binom{8}{k} p^k (1 - p)^{8-k}$ and $P_2 = \sum_{k=4}^{8} \binom{8}{k} p^k (1 - p)^{8-k}$ when players choose efforts independently. In order for $\pi_i(m^* + 1) > \pi_i(m^*)$, for all possible values of m^* (the values are almost independent to the level of m^*), p should be greater than .392. When the initial belief is concentrated enough (or λ is sufficiently large), it is hardly expected for players to raise their effort. The strong history dependence observed in VHBB (1991) is a product of j, n, a, b, and the grid size. Other things equal, the value of p for the 9-person median game with 101 effort choices drops to .172.

VHBR (1996) used the same value of $a = .1$ and $b = .05$ as in VHBB (1991), but they transform the effort level using $e = 1 + .06x$, $x \in \{0, 1, \cdots, 100\}$, so that $1 \leq e \leq 7$. Let $m_{j:n}(x)$ be the jth order statistic of $\{x_1, \cdots, x_n\}$. Since $m_{j:n}(e) = \alpha + \beta m_{j:n}(x)$ for $\alpha, \beta > 0$,

$$\pi_i(x_i) = .1\ m_{j:n}(e) - .05\ (m_{j:n}(e) - e_i)^2 + c$$

---

[11] However, the limiting logit equilibrium with the initial belief calculated from the first period play are same as every final outcome in every treatment.

[12] Since the penalty is so severe when $|m_{j:n} - x_i| \geq 2$, it seems very unlikely for a player to choose $m^* + 2$. To justify lowering effort, one needs a large probability that the median will be lower. This simplified setting seems enough to describe the incentive problem that each individual player faced.

$$= [.06 \times .1] \, m_{j:n}(x) - [(.06)^2 \times .05] \, (m_{j:n}(x) - x_i)^2 + c'$$

As a result, the ratio $\frac{a}{b}$ changes from 2 to $\frac{100}{3}$. Literally, with a continuous strategy space, the effective $\frac{a}{b}$ is infinity. In VHBR's (1996) experiment, the play converged to 0 effort only in two runs out of 36. Otherwise, the play either converged to the most efficient outcome (18/36) or exhibited fairly strong history dependence (16/36). Table 1 summarizes the number of runs that converged to the most efficient outcome and the corresponding value of p for each treatment.[13]

| | number of runs that converged to | |
Treatment	most efficient outcome	p
5-person fourth order statistic game	6/8 (75%)	.0074
7-person fourth order statistic game	6/10 (60%)	.103
5-person second order statistic game	4/8 (50%)	.209
7-person second order statistic game	2/10 (20%)	.240

Table 1: Number of Efficient Outcomes

4 Limiting Competitive Logit Equilibria of Order Statistic Games

The analysis of the previous section shows that if the payoff function is quadratic, for any finite n, the limiting logit equilibrium of every order statistic game is efficient. However, when a game involves a large number of players, people often seem to use "competitive" approximation in which players ignore their own influence on "market signal" (here it is the order statistic) as if the market signal is given. Although it usually gives a good approximation to Nash equilibrium in a game with a large number of players, I show in this section that such behavior could lead the play far from the efficient equilibrium in an order statistic game. More importantly, the analysis provides a way to identify the normal limiting logit equilibrium when $a = 0$.[14]

---

[13]In 7-person second order statistic game treatment, the play was "moving toward" the highest effort in 2 runs, which I count as "converged" in Table 1.

[14]In this section, I do not consider a minimum game. First, if players believe that they cannot influence

"Competitive logit equilibrium" is defined similar to the original logit equilibrium except that players ignores their own influence on the order statistic. Let $G_{c,j:n}^i(x)$ denote the player i's expectation of distribution function of $m_{j:n}$ excluding his own choice in an n-person j^{th} order statistic game so that $G_{c,j:n}^i(x)$ maps $F_{-i}(x)$ into $[0,1]$. In an n-person game, however, there are $n-1$ opponents each player should consider and the functional form of $G_{c,j:n}^i(x)$ is not well defined. In the analysis, instead of defining exact functional form of $G_{c,j:n}^i(x)$, I assume that $G_{c,j:n}^i(x)$'s satisfy following properties.

Assumption 1. $G_{c,j:n}^i(x)$'s satisfy basic properties of a probability distribution function and $G_{c,j:n}^i(x)$'s are continuous in F_{-i}.

Assumption 2. Given F_{-i} and $F_{-i'}$, if at least one of the components in F_{-i} first-order stochastically dominates one of the component in $F_{-i'}$ and others are the same, then $G_{c,j:n}^i(x)$ first-order stochastically dominates $G_{c,j:n}^{i'}(x)$.

Assumption 3. Given F_{-i}, $G_{j-1:n-1}^i(x) \geq G_{c,j:n}^i(x) \geq G_{j:n-1}^i(x)$ for all $x \in [0, \bar{x}]$, where $G_{j:n}^i(x)$'s are defined as in Section 2.

Assumption 1 and 2 make $G_{c,j:n}^i(x)$'s satisfy basic properties of order statistics, and competitiveness and Assumption 2 imply identical $G_{c,j:n}^i(x)$'s across players (Lemma 2A) in equilibrium. Assumption 2 is crucial in every proofs in this section on its own. Assumption 3 links players' perceptions of the order statistic to the objective order statistic. Assumption 3 is very restrictive but it is necessary to prove the existence using the same technique in the proof of Proposition 1. For the other results, I use only the relationship,

the minimum, the analysis in this section remains valid. However, this is implausible in a minimum game: if a player believes the expected minimum is $0 < m < \bar{x}$, the assumption requires that he should believe it is m even when he chooses 0. If player i takes the effect of his own choice into account holding $E_i(m_{1:n-1})$ constant, his expected payoff is:

$$\pi_i(x_i) = a\min[x_i, E_i(m_{1:n-1})] - b(\min[x_i, E_i(m_{1:n-1})] - x_i)^2$$

which is not differentiable and that makes the analysis difficult. However, an informal analysis is still possible. In a minimum game, $\pi_i(E_i(m_{1:n-1}) - \varepsilon) = aE_i(m_{1:n-1}) - a\varepsilon$ and $\pi_i(E_i(m_{1:n-1}) + \varepsilon) = aE_i(m_{1:n-1}) - b\varepsilon^2$. Since $E_i(m_{1:n-1})$ depends on the expected payoffs of ε-neighbor of $E_i(m_{1:n-1})$ with sufficiently large λ (because $E_i(m_{1:n-1})$ is a unique best response), a conjecture can be made that a strictly positive a is enough to lead the play to the most efficient Nash equilibrium. As in a logit equilibrium, once players recognize their influence on the order statistic, there is a tension between the benefit and the penalty. In this case, together with the continuous strategy space, the quadratic payoff function would work in favor of more efficient outcomes and that results in the most efficient outcome.

$E(m_{j-1:n-1}) \le E_c^i(m_{j:n}) \le E(m_{j:n-1})$, implied by Assumption 3. Those three assumptions insure the existence of competitive logit equilibrium.

Under the competitiveness assumption, the expected payoff is

$$(12) \quad E_c^i(\pi_{c,i}(x_i)) = aE_c^i(m_{j:n}) - b(E_c^i(m_{j:n}) - x_i)^2 + c$$

Notice that a matters in determining expected payoffs but it does not change the relative expected payoffs. Since the effort density is invariant to the change of the origin, the relevant part of expected payoff is

$$(13) \quad E_c^i(\pi_{c,i}(x_i)) = b[2E_c^i(m_{j:n})x_i - x_i^2]$$

and the effort density is

$$(14) \quad f_c^i(x_i) = \frac{\exp\left[\lambda b\left(2E_c^i(m_{j:n})x_i - x_i^2\right)\right]}{\int_0^{\bar{x}} \exp\left[\lambda b\left(2E_c^i(m_{j:n})y - y^2\right)\right] dy}$$

Because each player thinks that the order statistic is independent of his own choice, his only concern is how close his choice is to $E_c^i(m_{j:n})$. Therefore, the problem each individual player faces is similar to a game with $a = 0$, but the competitive logit equilibrium outcome is still Pareto-ranked as long as $a > 0$.

Consider a finite-person order statistic game with $j < \frac{n+1}{2}$. For the sake of intuition, assume that $E_c^i(m_{j:n}) = E(m_{j:n-1})$ for all i. Suppose $E(m_{j:n-1}) > 0$. The best response in the order statistic game is $E(m_{j:n-1})$ at which the effort density should attain its maximum. Under the competitiveness assumption, since the expected payoff depends only on the distance to $E(m_{j:n-1})$, the effort density is symmetric around $E(m_{j:n-1})$ (Lemma 4A). Moreover, as λ goes to infinity, only the expected payoffs of ε-neighbor of $E(m_{j:n-1})$ are relevant to determine $E(m_{j:n-1})$. By the nature of the order statistic, when j is less than $\frac{n}{2}$, there is a force which pushes $E(m_{j:n-1})$ toward 0. Even though there is no incentive for individual players to change their effort levels, the equilibrium effort density depends on j and n via $E(m_{j:n-1})$.

Proposition 3. If $G_{c,j:n}^i(x)$'s satisfy Assumption $1 - 3$, as λ goes to infinity, the competitive logit equilibrium converges to the point-mass at 0 when $j \le \frac{n+1}{2} - 1$ and it converges to $\bar{x}$ when $j \ge \frac{n+1}{2} + 1$.

When $\frac{n}{2} \leq j \leq \frac{n+2}{2}$, the limiting competitive logit equilibrium depends on the players' perceptions of the order statistic. The result is summarized in following corollary.

Corollary 1. Under Assumption $1-3$, as λ goes to infinity, if $E_c^i(m_{j:n}) < E(m_{\frac{n}{2}:n-1})$, the competitive quantal response equilibrium converges to the least efficient Nash equilibrium; if $E_c^i(m_{j:n}) > E(m_{\frac{n}{2}:n-1})$, it converges to the most efficient one; if $E_c^i(m_{j:n}) = E(m_{\frac{n}{2}:n-1})$ or $E_c^i(m_{j:n}) = E(m_{\frac{n-1}{2}:n-2})$,[15] it converges to a point-mass at $\frac{\bar{x}}{2}$.

The prediction of the competitive logit equilibrium is the similar to that of Kandori, Mailath and Rob's (1993) evolutionary model. Robles (1997) has applied Kandori, Mailath and Rob's model to order statistic games and has shown that when players do not consider the effect of their own choice on the order statistic, that is, $E_c^i(m_{j:n}) = E(m_{j:n})$; if $j < \frac{n+1}{2}$ $(j > \frac{n+1}{2})$, the evolutionary model selects a unique equilibrium where the effort choice is 0 $(\bar{x})$; when $j = \frac{n+1}{2}$, every strict Nash equilibrium has the same size of basin of attraction and it does not shrink the set of Nash equilibria. That is not a coincidence. While the competitive logit equilibrium depend on the statistical property of order statistics, the long-run equilibrium depends on sizes of basins of attraction which are determined by the order statistic.

Proposition 2 states that it is sufficient for efficiency that a finite n and sophisticated enough players to process their own chances of affecting the order statistic, while Proposition 3 states that the competitive behavior could result in inefficiency. A question arises if the competitiveness assumption can be justified in the limit of n. If λ grows faster than n, or if players choices converges to the best response fast enough, the conclusion of Proposition 2 remains valid.

Proposition 4. In order statistic games with $2 \leq j \leq n-1$, if n_t and λ_t go to infinity with $\frac{a\lambda_t}{n_t} \to \infty$ and $\frac{j_t}{n_t+1} \equiv q$ where $q = \frac{j}{n+1}$, the logit equilibrium converges to the most efficient Nash equilibrium.

Proof. Choosing $n_t = t(n+1)$ makes j_t and n_t integers. When $j \geq 2$, from Eq.(11),

$$f_t(\bar{x}) \geq f_t(0) + \lambda_t \left[2b(E_t(m_{j_t-1:n_t-1}) - E_t(x)) + \frac{a}{n_t} \right]$$

---

[15] This is the case where $E_c^i(m_{j:n}) = \frac{1}{2}[E(m_{\frac{n-1}{2}:n-1}) + E(m_{\frac{n+1}{2}:n-1})]$ when n is odd.

Since the variance of x, denoted by s^2, is order of λ^{-2}, and $-s\sqrt{\frac{n-j}{j}} \leq E(m_{j_t-1:n_t-1}) - E_t(x) \leq s\sqrt{\frac{j-1}{n-j+1}}$ (Wolkowicz and Styan, 1979) or $-s\sqrt{\frac{1-q}{q}} \leq E(m_{j_t-1:n_t-1}) - E_t(x) \leq s\sqrt{\frac{q}{1-q}}$, $E(m_{j_t-1:n_t-1}) - E_t(x)$ is, at least, of order $O(\lambda^{-1})$ and by the assumption $\frac{a}{n_t}$ is of order $o(\lambda^{-\varepsilon})$ for some $0 < \varepsilon < 1$. Therefore, $f_t(\bar{x})$ diverges and the result follows. Q.E.D.

Proposition 4 implies that if λ grows fast enough so that $\frac{\partial E_i(m_{j:n})}{\partial x_i}$ remains big enough, the limiting logit equilibrium should be efficient. Therefore, although a large number of players is not sufficient for a justification for the use of the competitive model, if n grows quickly enough relative to λ, the individual influences become too small to affect the resulting order statistic, and the logit equilibrium and the competitive logit equilibrium converge to the same Nash equilibrium in the limit of λ. To prove this claim, after defining a competitive logit equilibrium in the limit of n, I show a logit equilibrium converges to that.

As n goes to infinity, an order statistic can be represented by a quantile. Given q-quantile for $0 < q < 1$, let f_q and F_q be the common equilibrium effort density function and cumulative distribution function, respectively. Then the limit of the sequence of competitive logit equilibria as the number of players goes to infinity is well defined. In this case, the expected payoff is given by

(15) $\pi_q^e(x) = am_q - b(m_q - x)^2 + c$

where $m_q = F_q^{-1}(q)$. Since the competitive logit equilibrium effort density function is invariant to the changing the origin and m_q is independent of x, $\pi_q^e(x)$ can be rescaled as follows:

(16) $\pi_q^e(x) = b[2m_q x - x^2]$

Then the competitive logit equilibrium effort density in q-quantile game is

(17) $f_q(x) = \dfrac{\exp\left[\lambda b\left(2m_q x - x^2\right)\right]}{\int_0^{\bar{x}} \exp\left[\lambda b\left(2m_q y - y^2\right)\right] dy}$

or

(18) $f_q(x) = f_q(0) + 2\lambda b\left[m_q F_q(x) - \int_0^x y f_q(y) dy\right]$

The equilibrium is a solution, $f_q(x)$, to Eq.(18).

In a quantile game, given λ, it is the best to set the effort to the quantile of current distribution of the order statistic because each individual player cannot affect the quantile. However, as λ grows, by construction, $F_q(m_q)$ must be q in the competitive logit equilibrium and $m_q = \text{argmax}_x f_q(x)$. These two conditions make the competitive logit equilibrium converge to the extreme Nash equilibria except in the median game.

Lemma 4. Given q, as λ goes to infinity, the q-quantile competitive logit equilibrium effort density converges to a point-mass at 0, $\frac{\bar{x}}{2}$ and $\bar{x}$ when $q < \frac{1}{2}$, $q = \frac{1}{2}$ and $q > \frac{1}{2}$, respectively.

Proposition 5. In order statistic games with $2 \leq j \leq n - 1$ and $j \neq \frac{n+1}{2}$, if n_t and λ_t go to infinity with $n_t > \lambda_t^{2+\varepsilon}$ for any $\varepsilon > 0$ and $\frac{j_t}{n_t+1} \equiv q$ where $q = \frac{j}{n+1}$, the logit equilibrium effort density of a order statistic game converges to the limiting competitive logit equilibrium of corresponding q-quantile game.

Proposition 5 leaves the limiting logit equilibrium of a median game unidentified. This is because bounds for a logit equilibrium in the proof is (too) wide. Since $\int_0^{\bar{x}} (G_{j-1:n-1}(y) - G_{j:n-1}(y)) dy = E(m_{j:n-1}) - E(m_{j-1:n-1})$, Eq.(7) can be rewritten as:

$$
\begin{aligned}
D_x \pi^e(x) &= 2b \left[E(m_{j:n-1}) - x - \int_x^{\bar{x}} (G_{j-1:n-1}(y) - G_{j:n-1}(y)) dy \right] \\
&\quad + a(G_{j-1:n-1}(x) - G_{j:n-1}(x))
\end{aligned}
$$

Then

$$
\begin{aligned}
(19) \quad f(x) &= f(0) + 2\lambda b \left[E(m_{j:n-1}) F(x) - \int_0^x y f(y) dy \right. \\
&\quad \left. - \int_0^x \int_y^{\bar{x}} (G_{j-1:n-1}(z) - G_{j:n-1}(z)) dz f(y) dy \right] \\
&\quad + \lambda a \int_0^x (G_{j-1:n-1}(y) - G_{j:n-1}(y)) f(y) dy
\end{aligned}
$$

Since both $\int_0^x \int_0^y (G_{j-1:n-1}(z) - G_{j:n-1}(z)) dz f(y) dy$ in Eq.(9) and $\int_0^x \int_y^{\bar{x}} (G_{j-1:n-1}(z) - G_{j:n-1}(z)) dz f(y) dy$ in Eq.(19) vanishes as λ and n increase, we need the exact value of $\int_0^x \int_0^y (G_{j-1:n-1}(z) - G_{j:n-1}(z)) dz f(y) dy$ to calculate limiting logit equilibrium of a median game.

When $a = 0$, the expected payoff is $\pi(x) = -b(E(m_{j:n}) - x)^2$. In this case, the best response is to place the choice between $E(m_{j-1:n-1})$ and $E(m_{j:n-1})$. Following proposition

21

uses the result in Corollary 1 and characterizes the limiting logit equilibrium except for games with $\frac{n}{2} \leq j \leq \frac{n}{2} + 1$. In a minimum game, since 0 is a unique weakly dominant strategy and there is no mixed-strategy Nash equilibrium, the limiting logit equilibrium should be $x_i = 0$ for all i as λ goes to infinity.

Proposition 6. When $a = 0$, the limiting logit equilibrium for $j < \frac{n}{2}$ is $x_i = 0$ and for $j > \frac{n}{2} + 1$, it is $x_i = \bar{x}$ for all i.[16]

5 Conclusion

This paper considers logit equilibrium as a way of modeling players' behavior in a class of order-statistic coordination games studied experimentally by VHBB (1990, 1991) and VHBR (1996), and theoretically by Anderson, Goeree, and Holt (1998). The standard notion of equilibrium requires common knowledge of rationality and the structure of the game. It also requires mutually consistent beliefs, which is a particularly strong assumption when there are multiple equilibria. Those assumptions, however, imply only the iterated elimination of strategies that are never weakly best replies, which in many games yields no useful restrictions on behavior. The QRE also assumes common knowledge of rationality and of the structure, but models the strategic uncertainty that such games often create by assuming that each player forms beliefs about the others' strategies based on a stochastic decision rule, and requiring mutually consistent beliefs about the distribution of the resulting decision errors.[17] The QRE, by assuming noisy but equilibrium based responses to systematic decision errors, allows players to use a common principle to make independent predictions about others' strategies. As a result, indeterminacy of conditional probabilities of outcomes is largely eliminated.

---

[16] When $j = \frac{n}{2}$, $x_i \leq \frac{\bar{x}}{2}$ and when $j = \frac{n}{2} + 1$, $x_i \geq \frac{\bar{x}}{2}$ for all i.

[17] QRE also requires players' abilities to process a good deal of information to calculate "correct" beliefs. This assumption can be relaxed by replacing equilibrium beliefs with adaptive learning models like fictitious play. Chen, Friedman, and Thisse (1997) identified conditions such that for a given λ the path of choices over time converges to a QRE when the players update their beliefs based on fictitious play.

Appendix

Proof of Proposition 1

This is a slight modification of the existence proof in Anderson, Goeree and Holt (1996). When $j = 1$ and $j = n$, the proofs are identical with appropriate expected payoffs.

When $2 \leq j \leq n - 1$, the expected payoff is given by

$$\pi_i^e(x_i) = 2bE_i(m_{j-1:n-1})x_i - bx_i^2 + \int_0^{x_i}(a - 2by + 2bx_i)G_{j-1:n-1}^i(y)dy$$

After substituting π_i^e in Eq.(2) with above expected payoff, integrating both side yields the equilibrium cumulative distribution function $F_i(x_i)$ as a fixed point of the operator $\mathbf{T}$;

$$\mathbf{TF}_i(x_i) = \frac{\int_0^{x_i}\exp\left[\lambda\left(2bE_i(m_{j-1:n-1})x_i - by^2 + \int_0^y(a - 2bz + 2by)G^i(z)dz\right)\right]dy}{\int_0^{\bar{x}}\exp\left[\lambda\left(2bE_i(m_{j-1:n-1})x_i - by^2 + \int_0^y(a - 2bz + 2by)G^i(z)dz\right)\right]dy}$$

where $\mathbf{F}$ is a vector, $(F_1, \cdots, F_n)$, F_i is an ith element of $\mathbf{F}$, $G^i(x) \equiv G_{j-1:n-1}^i(x) - G_{j:n-1}^i(x)$ and the definition of $G_{j:n-1}^i(x)$ is in Section 2. If a fixed point exists, we know that $F_i(x_i)$ is continuous, so is $F(x)$. Therefore we can restrict the solution set to the set of continuous function on $[0, \bar{x}]$, denoted by $\mathbf{C}[0, \bar{x}]$. In particular, consider the set: $\mathbf{S} \equiv \{\mathbf{F} \in \mathbf{C}[0, \bar{x}] \mid \|\mathbf{F}\| \leq 1\}$, where $\|\cdot\|$ denotes the sup norm. The set, which includes all continuous cumulative distributions, is an infinite dimensional unit ball, and is thus a closed and convex subset of $\mathbf{C}[0, \bar{x}]$. Since $\mathbf{S}$ is not compact I will use the following fixed point theorem (Griffel, 1985, p158).

Schauder's Second Theorem If $\mathbf{S}$ is a closed convex set of a normed space and $\mathbf{R}$ is a relatively compact subset of $\mathbf{S}$, then every continuous mapping of $\mathbf{S}$ to $\mathbf{R}$ has a fixed point.

To apply the theorem, we need to prove that

(1) $\mathbf{R} \equiv \{\mathbf{TF}|\mathbf{F} \in \mathbf{S}\}$ is relatively compact, and
(2) $\mathbf{T}$ is a continuous mapping from $\mathbf{S}$ to $\mathbf{R}$.

(1) Proof of the relative compactness of R

A theorem due to Arzela-Ascoli (Griffel, 1985, p156) states that a set of function in $C[0, \bar{x}]$, with sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0, \bar{x}]$. The set $\mathbf{R} \equiv \{\mathbf{TF}|\mathbf{F} \in \mathbf{S}\}$ is uniformly bounded if there exists a number K such that $\|\mathbf{F}(x)\| \leq K$ for all $\mathbf{F} \in \mathbf{S}$ and $\mathbf{x} \in [0, \bar{x}]$. Since mapping $\mathbf{TF}(\mathbf{x})$ is positive and nondecreasing, $\|\mathbf{TF}(\mathbf{x})\| \leq \|\mathbf{TF}(\bar{x})\| = 1$ for all $\mathbf{F} \in \mathbf{S}$ and $\mathbf{x} \in [0, \bar{x}]$. So $\mathbf{TF}$ is uniformly bounded for all $\mathbf{F} \in \mathbf{S}$.

To prove equicontinuity of $\mathbf{R}$, I have to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathbf{TF}(\mathbf{x_1}) - \mathbf{TF}(\mathbf{x_2})\| < \varepsilon$ whenever $|\mathbf{x_1} - \mathbf{x_2}| < \delta$, for all $\mathbf{TF} \in \mathbf{R}$ and $\mathbf{x_1}, \mathbf{x_2} \in [0, \bar{x}]$.

$$\|\mathbf{TF}(\mathbf{x_1}) - \mathbf{TF}(\mathbf{x_2})\|$$

$$< \frac{\max_i |\int_{x_1}^{x_2} \exp\left[\lambda\left(2bm_{j-1:n-1}^i y - by^2 + \int_0^y (a - 2bz + 2by)G^i(z)dz\right)\right]dy|}{\min_i \int_0^{\bar{x}} \exp\left[\lambda\left(2bm_{j-1:n-1}^i y - by^2 + \int_0^y (a - 2bz + 2by)G^i(z)dz\right)\right]dy}$$

$$< \frac{|x_2 - x_1|\exp[\lambda(4b\bar{x}^2 + a\bar{x})]}{\bar{x}\exp(-b\bar{x}^2)} = \frac{|x_2 - x_1|}{\bar{x}}\exp[\lambda(5b\bar{x}^2 + a\bar{x})]$$

The second inequality holds because $\int_0^x (a - 2by + 2bx)G^i(y)dy > 0$ for all x. Thus the difference in the value of the $\mathbf{TF}$ operator is ensured to be less than ε by setting $|\mathbf{x_1} - \mathbf{x_2}| < \delta$, where $\delta = \bar{x}\exp[-\lambda(5b\bar{x}^2 + a\bar{x})]\varepsilon$. Therefore $\mathbf{TF}$ is equicontinuous for all $\mathbf{F} \in \mathbf{S}$.

(2) **Proof of the continuity of T**

The mapping $\mathbf{T}$ is continuous if for all $\mathbf{F^1}, \mathbf{F^2} \in \mathbf{S}$ and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\mathbf{TF^1} - \mathbf{TF^2}\| < \varepsilon$ whenever $\|\mathbf{F^1} - \mathbf{F^2}\| < \delta$. In order to get a bound on $\|\mathbf{TF^1} - \mathbf{TF^2}\|$, we need to compare $E_i^1(m_{j:n})$ and $E_i^2(m_{j:n})$, and $G_1^i(x)$ and $G_2^i(x)$.

Let's write $\mathbf{F^1}(\mathbf{x}) = \mathbf{F^2}(\mathbf{x}) + \mathbf{h}(\mathbf{x})$ with $\|\mathbf{h}(\mathbf{x})\| < \delta$ for all $\mathbf{x} \in [0, \bar{x}]$. Then $\mathbf{1} - \mathbf{F^1}(\mathbf{x}) = \mathbf{1} - \mathbf{F^2}(\mathbf{x}) - \mathbf{h}(\mathbf{x})$, which is greater than $\mathbf{1} - \mathbf{F^2}(\mathbf{x}) - \delta$ and less than $\mathbf{1} - \mathbf{F^2}(\mathbf{x}) + \delta$ for all $\mathbf{x}$. Using the upper bound,

$$G_{1,j-1:n-1}^i < \sum_{k=j-1}^{n-1} \sum_{S_k} \prod_{l=1}^k (F_{i_l}^1(x) + \delta) \prod_{l=k+1}^{n-1} (1 - F_{i_l}^1(x) + \delta)$$

Using following relation,

$$(F_1^1 + \delta)(F_2^1 + \delta)\cdots(F_k^1 + \delta) \leq \left[F_1^1(F_2^1 + \delta)\cdots(F_k^1 + \delta)\right] + \delta(1 + \delta)^{k-1}$$

$$\leq \prod_{l=1}^k F_l^1 + (1 + \delta)^k - 1$$

we have

$$G_{1,j-1:n-1}^i < G_{2,j-1:n-1}^i + ((1 + \delta)^{n-1} - 1)\sum_{k=j-1}^{n-1} \sum_{S_k} 1$$

and, similarly, using $1 - F_1(x) = 1 - F_2(x) - h(x) > 1 - F_2(x) - \delta$, we derive

$$G_{1,j-1:n-1}^i > G_{2,j-1:n-1}^i + ((1 - \delta)^{n-1} - 1)\sum_{k=j-1}^{n-1} \sum_{S_k} 1$$

This yields

$$E_2^i(m_{j:n}) - \bar{x}((1 + \delta)^{n-1} - 1)\sum_{k=j-1}^{n-1} \sum_{S_k} 1 < E_1^i(m_{j:n})$$

$$< E_2^i(m_{j:n}) - \bar{x}((1 - \delta)^{n-1} - 1)\sum_{k=j-1}^{n-1} \sum_{S_k} 1$$

24

and

$$G_2^i + ((1-\delta)^{n-1} - 1) \sum_{S_{j-1}} 1 < G_1^i < G_2^i + ((1+\delta)^{n-1} - 1) \sum_{S_{j-1}} 1$$

In the definition of the operator $\mathbf{TF^1}$, we can use the upper bound for the integral in the numerator and the lower bound for the internal in the denominator to obtain

$$TF_i^1(x_i) < \frac{\int_0^{x_i} \exp\left[\lambda\left(2bE_2^i(m_{j-1:n-1})y - by^2 + \int_0^y (a - 2bz + 2by)G_2^i(z)dz\right) + K_1\right] dy}{\int_0^{\bar{x}} \exp\left[\lambda\left(2bE_2^i(m_{j-1:n-1})y - by^2 + \int_0^y (a - 2bz + 2by)G_2^i(z)dz\right) + K_2\right] dy}$$

where

$$K_1(\delta) = -2b\bar{x}((1-\delta)^{n-1} - 1) \sum_{k=j-1}^{n-1} \sum_{S_k} 1 + \bar{x}(a + 2b\bar{x})((1+\delta)^{n-1} - 1) \sum_{S_{j-1}} 1$$

$$K_2(\delta) = -2b\bar{x}((1+\delta)^{n-1} - 1) \sum_{k=j-1}^{n-1} \sum_{S_k} 1 + \bar{x}(a + 2b\bar{x})((1-\delta)^{n-1} - 1) \sum_{S_{j-1}} 1$$

Since $K_1 > 0$ and $K_2 < 0$,

$$\mathbf{TF^1} < K(\delta)\mathbf{TF^2}, \quad \text{where} \quad K(\delta) = \frac{\exp\left(\lambda\bar{x}K_1(\delta)\right)}{\exp\left(\lambda\bar{x}K_2(\delta)\right)}$$

The same approach can be used to show that $TF^1 > K(\delta)^{-1}TF^2$. Thus we conclude that

$$K(\delta)^{-1}\mathbf{TF_2} < \mathbf{TF_1} < K(\delta)\mathbf{TF_2}$$

Notice that $K(\delta)$ is strictly increasing for $\delta > 0$, with $K(0) = 1$. The final step is to obtain a bound on $\|\mathbf{TF_2} - \mathbf{TF_1}\|$. Suppose that the supremum is obtained at some x^* at which $\mathbf{TF_1} > \mathbf{TF_2}$, without loss of generality. On the other hand, $\mathbf{TF_1} < K(\delta)\mathbf{TF_2}$. Now we have

$$\|\mathbf{TF_1} - \mathbf{TF_2}\| = \|\mathbf{TF_1}(x^*) - \mathbf{TF_2}(x^*)\| < (K(\delta) - 1)\|\mathbf{TF_2}\| < K(\delta) - 1$$

Since $K(\delta)$ is continuous and increasing in δ with $K(0) = 1$, there exists a $\delta^*(\epsilon)$ such that $K(\delta) - 1$ is less than ϵ for all $0 < \delta < \delta^*(\epsilon)$. Hence $\mathbf{T}$ is a continuous mapping from $\mathbf{S}$ to $\mathbf{R}$. Q.E.D.

Proof of Lemma 1

For the convergence of logit equilibrium to a Nash equilibrium, Theorem 2 and Lemma 2 in McKelvey and Palfrey (1995)) extend to games with continuous strategy spaces provided that a logit equilibrium exists (the proof does not require a discrete strategy space.)

Theorem (McKelvey and Palfrey, 1995; Chen, Friedman, and Thisse, 1997) Let $\{\lambda_1, \lambda_2, \cdots\}$ be a sequence such that $\lambda_t > 0$ and $\lim_{t\to\infty} \lambda_t = \infty$. In a finite-person game, for each λ_t, let f_t be a fixed point (logit equilibrium). Then the sequence of $\{f_1, f_2, \cdots\}$ has at least one point of accumulation, f^*, and any such point of accumulation is a Nash equilibrium of the game.

Given the convergence result, it is sufficient to show that every Nash equilibrium is a symmetric pure-strategy equilibrium. Consider a pure-strategy Nash equilibrium first. In the minimum game, suppose a pure-strategy equilibrium is not symmetric. Let x_m and x_M be the minimum and the maximum effort levels of the Nash equilibrium strategy profile. In this case, the minimum is always x_m, and x_M cannot be the best response as long as $x_m \neq x_M$. For the games other than the minimum game, similar arguments hold.

Next, I need to show that there is no mixed-strategy Nash equilibrium. Suppose that there is a mixed-strategy Nash equilibrium, which can have three forms; either the effort levels played with positive probability are isolated or the support involves contiguous intervals or both. In any case, the expected payoffs are given by Eq.(3) and Eq.(4). That is,

$$
\begin{aligned}
\pi_i^e(x_i) &= a\left[x_i - \int_0^{x_i} G_{1:n-1}^i(y)dy\right] + 2b\left[\int_0^{x_i}(y - x_i)G_{1:n-1}^i(y)dy\right] \\
\pi_i^e(x_i) &= 2bE_i(m_{j-1:n-1})x_i - bx_i^2 \\
&\quad + \int_0^{x_i}(a - 2by + 2bx_i)(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy
\end{aligned}
$$

Let x_m and $x_{m'}$ be two effort levels in the same interval such that $x_m < x_{m'}$ and $f_i(x)$ and $F_i(x)$ be player i's effort density and effort distribution functions, respectively. Then $f_i(x) > 0$ and $0 < F_i(x) < 1$ on $(x_m, x_{m'})$.

First, consider the minimum game. If the support of a equilibrium strategy involves an interval $[x_m, x_{m'}]$, then a player's expected payoff must be constant at all effort levels. Then the first derivative of expected payoff exists and must be zero, and so does the second derivative. For $x \in [x_m, x_{m'}]$, by the construction,

$$
D_x^2 \pi_i^e(x) = -(a + 2b)g_{1:n-1}^i(x) < 0
$$

This is a contradiction. Hence the equilibrium density can only involve atoms.

Suppose there are more than two atoms: $x_a < x_b < x_c < \cdots$.

$$
\begin{aligned}
\pi_i(x_a) &= ax_a \\
\pi_i(x_b) &= p_a[ax_a - b(x_a - x_b)^2] + (1 - p_a)ax_b \\
\pi_i(x_c) &= p_a[ax_a - b(x_a - x_c)^2] + p_b[ax_b - b(x_b - x_c)^2] + (1 - p_a - p_b)ax_c
\end{aligned}
$$

where p_k is the probability that x_k is the minimum. By equating $\pi_i(x_a)$ and $\pi_i(x_b)$, we have $a(1 - p_a) = bp_a(x_b - x_a)$. Equating last two equations gives

$$
bp_a(x_c + x_b - 2x_a) + bp_b(x_c - x_b) = a(1 - p_a)
$$

Substituting $a(1 - p_a) = bp_a(x_b - x_a)$ yields

$$
p_a(x_c - x_a) + p_b(x_c - x_b) = 0
$$

This is a contradiction. Hence a mixed-strategy Nash equilibrium could involve only two atoms. Choose x^* such that $x^* \in (x_a, x_b)$.

$$
P_i(m_{1:n} = x_a | x_i = x_a) = 1,
$$

$$P_i(m_{1:n} = x_a | x_i = x^*) = P_i(m_{1:n-1} = x_a) \equiv p$$
$$P_i(m_{1:n} = x^* | x_i = x^*) = P_i(m_{1:n-1} \geq x^*) \equiv 1 - p$$

and

$$P_i(m_{1:n} = x_a | x_i = x_b) = P_i(m_{1:n-1} = x_a) = p$$
$$P_i(m_{1:n} = x_b | x_i = x_b) = P_i(m_{1:n-1} \geq x_b) = 1 - p$$

By equating $\pi_i(x_a)$ and $\pi_i(x_b)$, $p = \frac{a}{a+b(x_b-x_a)}$. By the equilibrium condition,

$$\pi_i(x_a) \geq \pi_i(x^*) \quad \rightarrow \quad ax_a \geq p(ax_a - b(x_a - x^*)^2) + (1-p)ax^*$$
$$\rightarrow \quad p \geq 1 - \frac{b(x^*-x_a)}{a}$$
$$\rightarrow \quad \frac{a}{a+b(x_b-x_a)} \geq \frac{a-b(x^*-x_a)}{a}$$

By choosing $x^* < x_a + \frac{a(x_b-x_a)}{a+b(x_b-x_a)}$, we have $\frac{a}{a+b(x_b-x_a)} < \frac{a-b(x^*-x_a)}{a}$, a contradiction.

Next, consider games with $2 \leq j \leq n-1$. If the support of strategy space involves an interval, for all $x \in [x_m, x_{m'}]$,

$$\pi_i^e(x_m) = 2bE_i(m_{j-1:n-1})x_m - bx_m^2$$
$$+ \int_0^{x_m} (a - 2by + 2bx_m)(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy$$
$$\pi_i^e(x_{m'}) = 2bE_i(m_{j-1:n-1})x_{m'} - bx_{m'}^2$$
$$+ \int_0^{x_{m'}} (a - 2by + 2bx_{m'})(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy$$

Since $\int_0^{x_m} (a - 2by + 2bx_m)(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy < \int_0^{x_{m'}} (a - 2by + 2bx_{m'})(G_{j-1:n-1}^i(y) - G_{j:n-1}^i(y))dy$, by equating those two expected payoffs, we have $E_i(m_{j-1:n-1}) < \frac{x_m + x_{m'}}{2}$. Then we are left with only two possibilities, $E_i(m_{j-1:n-1}) = x_m$ and $E_i(m_{j-1:n-1}) < x_m$. Since $D_{x_i}\pi_i(E_i(m_{j-1:n-1})) > 0$, x_m cannot be $E_i(m_{j-1:n-1})$. Therefore, $E_i(m_{j-1:n-1}) < x_m$ and this implies that the minimum effort level in a Nash equilibrium strategy profile is isolated. Let player i' be the one who play the minimum effort level, x_a. Since at least player i puts positive probability on x_m which is greater than x_a, $x_a < E_{i'}(m_{j-1:n-1})$ and x_a cannot be a weakly best response for player i'. This is a contradiction. Hence a equilibrium mixed-strategy could involve only atoms.

Let x_a denote the minimum effort level and let x_b be the second largest effort level to be played with positive probability (x_b could be x_m). Choose x^* such that $x_a < x^* < E_i(m_{j-1:n-1}) < x_b$. Then,

$$P_i(m_{j:n} = x_a | x_i = x_a) = P_i(m_{j-1:n-1} = x_a)$$
$$P_i(m_{j:n} \leq x_b | x_i = x_a) = P_i(m_{j-1:n-1} \leq x_b)$$

and

$$P_i(m_{j:n} = x_a | x_i = x^*) = P_i(m_{j:n-1} = x_a)$$
$$= P_i(m_{j-1:n-1} = x_a) + P_i(m_{j:n-1} = x_a) - P_i(m_{j-1:n-1} = x_a)$$
$$P_i(m_{j:n} = x^* | x_i = x^*) = P_i(m_{j-1:n-1} \leq x^*)$$
$$= P_i(m_{j-1:n-1} = x_a) - P_i(m_{j:n-1} = x_a)$$
$$P_i(m_{j:n} \leq x_b | x_i = x^*) = P_i(m_{j-1:n-1} \leq x_b)$$

Since $P_i(m_{j:n} \leq x | x_i = x_a) = P_i(m_{j:n} \leq x | x_i = x^*)$ for all $x \geq x_b$,

$$
\begin{aligned}
\pi_i^e(x_a) &= E_i(am_{j:n} - b(m_{j:n} - x_a)^2) \\
\pi_i^e(x^*) &= E_i(am_{j:n} - b(m_{j:n} - x^*)^2) \\
&\quad + (P_i(m_{j:n-1} = x_a) - P_i(m_{j-1:n-1} = x_a))(ax_a - b(x_a - x^*)^2) \\
&\quad + (P_i(m_{j-1:n-1} = x_a) - P_i(m_{j:n-1} = x_a))(ax^*) \\
&= E_i(am_{j:n} - b(m_{j:n} - x^*)^2) \\
&\quad + (P_i(m_{j-1:n-1} = x_a) - P_i(m_{j:n-1} = x_a))(ax^* - ax_a + b(x_a - x^*)^2)
\end{aligned}
$$

Since $P_i(m_{j-1:n-1} = x_a) - P_i(m_{j:n-1} = x_a) > 0$, in order for a strategy profile to be a mixed strategy equilibrium,

$$
\begin{aligned}
\pi_i^e(x_a) \geq \pi_i^e(x^*) &\rightarrow E_i(am_{j-1:n-1} - b(m_{j-1:n-1} - x_a)^2) \geq E_i(am_{j-1:n-1} - b(m_{j-1:n-1} - x^*)^2) \\
&\rightarrow 2b E_i(m_{j-1:n-1})(x_a - x^*) - b(x_a^2 - x_*^2) \geq 0 \\
&\rightarrow x_a + x^* \geq 2E_i(m_{j-1:n-1})
\end{aligned}
$$

This is a contradiction and there is no mixed-strategy Nash equilibrium. Q.E.D.

Proof of Lemma 3

Suppose there be two different logit equilibrium densities, $f_1(x)$ and $f_2(x)$. Suppose $f_1(x) > f_2(x)$ on (x_a, x_b) where x_a is the smallest one from which two densities have different values and x_a could be 0. Since they have to cross before $\bar{x}$, there exists $x_b < \bar{x}$ such that $f_1(x_b) = f_2(x_b)$, $f_1'(x_b) < f_2'(x_b)$, and $F_1(x_b) > F_2(x_b)$.

Consider a minimum game first. From Eq.(6),

$$
D_x \pi_1^e(x) = a \prod_{i \neq 1}(1 - F_i(x)) - 2b \int_0^x \left[1 - \prod_{i \neq 1}(1 - F_i(y))\right] dy
$$

and, in Eq.(5), f_1' is decreasing in F_2. This is a contradiction.

For games with $2 \leq j \leq n - 1$, regardless of what are the equilibrium densities of players i_1 and i_2, they share the beliefs about the other $(n - 2)$ players. Let $Q_{j:n-2}(x)$ be the probability that exactly j of the x_i's are less than or equal to x and $n - j - 2$ are greater than or equal to x in the random sample $n - 2$. Then

$$
G_{j:n}^1(x) = Q_{j-1:n-2}(x)F_2(x) + \sum_{k=j}^{n-2} Q_{k:n-2}(x)
$$

and

$$
G_{j-1:n}^1(x) - G_{j:n}^1(x) = Q_{j-1:n-2}(x) + F_2(x)\left[Q_{j-2:n-2}(x) - Q_{j-1:n-2}(x)\right]
$$

where $Q_{0:n-2}(x) = 1$.

First, consider a case that $F_1(x) \geq F_2(x)$ for all x. Then $F_2(x)$ first-order stochastically dominates $F_1(x)$, and $E_1(m_{j-1:n-1}) < E_2(m_{j-1:n-1})$. However, since $E_1(m_{j-1:n-1})$ depends on F_2 not on F_1, $E_1(m_{j-1:n-1}) > E_2(m_{j-1:n-1})$, which is a contradiction.

$F_1(x)$ and $F_2(x)$ have to cross more than once and $f_1(x)$ and $f_2(x)$ cross at x_b and x_c. Let $H(x) \equiv \frac{1}{\lambda}\left(\frac{f_1'(x)}{f_1(x)} - \frac{f_2'(x)}{f_2(x)}\right)$. Then $H(x_b) < 0 < H(x_c)$. From Eq.(5) and Eq.(7),

$$
\begin{aligned}
H(x) &= 2b\Big[E_1(m_{j-1:n-1}) - E_2(m_{j-1:n-1}) \\
&\quad + \int_0^x (F_2(y) - F_1(y))(Q_{j-2:n-2}(y) - Q_{j-1:n-2}(y))dy\Big] \\
&\quad + a(F_2(x) - F_1(x))(Q_{j-2:n-2}(x) - Q_{j-1:n-2}(x))
\end{aligned}
$$

Since $F_2(x) - F_1(x) \leq 0$ for all $x \in [0, x^*]$ for some $x^* > x_b$, $D_x H(0) < 0$ and $H(x)$ attains its local minimum at $x_b < x^* < x_c$ where $F_2(x^*) - F_1(x^*) = 0$. The first order condition is:

$$
\begin{aligned}
D_x H(x^*) &= 2b(F_2(x^*) - F_1(x^*))(Q_{j-2:n-2}(x^*) - Q_{j-1:n-2}(x^*)) \\
&\quad + a(F_2(x^*) - F_1(x^*))(D_x Q_{j-2:n-2}(x^*) - D_x Q_{j-1:n-2}(x^*)) \\
&\quad + a(f_2(x^*) - f_1(x^*))(Q_{j-2:n-2}(x^*) - Q_{j-1:n-2}(x^*)) \\
&= a(f_2(x^*) - f_1(x^*))(Q_{j-2:n-2}(x^*) - Q_{j-1:n-2}(x^*)) = 0
\end{aligned}
$$

When $a > 0$, since $Q_{j-2:n-2}(x) - Q_{j-1:n-2}(x) > 0$ for all $x \in (0, \bar{x})$, the condition implies $f_2(x^*) - f_1(x^*) = 0$. This is a contradiction.

When $a = 0$, by inspection, $H(x)$ attains a strictly positive local maximum at x_c. This implies that $D_x H(x_c) = 0$ and $F_2(x_c) - F_1(x_c) = 0$. This is a contradiction. Q.E.D.

Proof of Proposition 3

Let's define some notation for the proof. Let $f_{c,j:n}^i(x)$ and $F_{c,j:n}^i(x)$ be player i's competitive equilibrium effort density and distribution function in the game with j and n, respectively. $E_f(m_{j:n})$ and $E_f(x)$ denote unconditional expectations of order statistic $m_{j:n}$ and x with given f. Let $f_{j:n-1}(x)$ and $f_{j-1:n-1}(x)$ denote the equilibrium effort densities associated with $E_c(m_{j:n}) = E(m_{j-1:n-1})$ and $E_c(m_{j:n}) = E(m_{j-1:n-1})$, respectively. $F_{j:n-1}(x)$ and $F_{j-1:n-1}(x)$ denote corresponding cumulative distribution functions.

Lemma 1A. Given λ, for any i_1, i_2, j_1, j_2, and n_1, n_2, if $E_c^{i_1}(m_{j_1:n_1}) > E_c^{i_2}(m_{j_2:n_2})$ then $F_{c,j_1:n_1}^{i_1}(x)$ first-order stochastically dominates $F_{c,j_2:n_2}^{i_2}(x)$, and if $E_c^{i_1}(m_{j1:n1}) = E_c^{i_2}(m_{j2:n1})$, then $F_{c,j_1:n_1}^{i_1}(x) = F_{c,j_2:n_2}^{i_2}(x)$ for all x.

Proof. From Eq.(14), $f_{c,j:n}^{i'}(x_i) = 2\lambda b f_{c,j:n}^i(x_i)(E_c^i(m_{j:n}) - x_i)$, and

$$
\frac{f_{c,j_1:n_1}^{i_1'}(x)}{f_{c,j_1:n_1}^{i_1}(x)} - \frac{f_{c,j_2:n_2}^{i_2'}(x)}{f_{c,j_2:n_2}^{i_2}(x)} = 2\lambda b\left[E_c^{i_1}(m_{j_1:n_1}) - E_c^{i_2}(m_{j_2:n_2})\right] > 0
$$

and $f_{c,j:n}^i(0)$ is decreasing in $E_c^i(m_{j:n})$. Thus they cannot cross more than once and the first result follows. The second result is direct from Eq.(14). The converse is true when $\lambda > 0$. Q.E.D.

Lemma 2A. Every competitive logit equilibrium is symmetric across players.

Proof. By Lemma 1A, if two players have different $E_c^i(m_{j:n})$'s, then the effort densities should be different. Suppose $E_c^1(m_{j:n}) > E_c^2(m_{j:n})$. $f_{c,j:n}^1(x)$ depends on $f_{c,j:n}^2(x)$ as well as the other $f_{c,j:n}^i(x)$'s but not on itself, so does $f_{j:n}^2(x)$. By Lemma 1A, $F_{c,j:n}^1(x)$ first-order stochastically dominates $F_{c,j:n}^2(x)$. Therefore, by Assumption 2, $G_{c,j:n}^2(x)$ first-order stochastically dominates $G_{c,j:n}^1(x)$. This is a contradiction. Q.E.D.

By Lemma 2A, the superscript i can be suppressed.

Lemma 3A. Given λ, for any $j_1 < j_2$, $E_c^1(m_{j_1:n}) < E_c^2(m_{j_2:n})$ in a competitive logit equilibrium.

Proof. For a given λ, in order to reach a contradiction, suppose $E_c^1(m_{j_1:n}) > E_c^2(m_{j_2:n})$ in a equilibrium. When $\lambda = 0$, by Assumption 3, $f_{c,j:n}(x)$ is uniformly distributed and $E_c^1(m_{j_1:n}) < E_c^2(m_{j_2:n})$. Since $E_c(m_{j:n})$ is continuous in λ, there must exist a λ^* such that $E_c^1(m_{j_1:n}) = E_c^2(m_{j_2:n})$. Then $f_{c,j_1:n}^1(x) = f_{c,j_2:n}^2(x)$ by Lemma 1A. However, if $f_{c,j_1:n}^1(x) = f_{c,j_2:n}^2(x)$, $E_c^1(m_{j_1:n}) < E_c^2(m_{j_2:n})$ by Assumption 3. This is a contradiction. Q.E.D.

Lemma 3A enable to compare competitive logit equilibrium effort densities based only on the $E_c(m_{j:n})$ given λ. Since there is a unique "best response", $E_c(m_{j:n})$, from Eq.(12), $f_{c,j:n}(x)$ converges to a point-mass at $E_c(m_{j:n})$. Together with Assumption 3, by comparing unconditional expectations $E(m_{j:n})$'s, one can "order" $E_c(m_{j:n})$'s because equilibrium $E(m_{j:n})$ is well ordered by Lemma 3A. In other words, in order to prove Proposition 3, it is sufficient to show that $E(m_{j-1:n-1})$ goes to $\bar{x}$ and $E(m_{j:n-1})$ goes to 0 as λ goes to infinity.

Lemma 4A. Given λ, for any $2 \leq j \leq n-1$ and n, $f_{c,j:n}(x)$ is symmetric around $E_c(m_{j:n})$. That is, if $(2E_c(m_{j:n}) - x) \in [0, \bar{x}]$, then $f_{c,j:n}(x) = f_{c,j:n}(2E_c(m_{j:n}) - x)$.

Proof. If $(2E_c(m_{j:n}) - x) \in [0, \bar{x}]$,

$$f_{c,j:n}(2E_c(m_{j:n}) - x) = \frac{\exp\left[\lambda b \left(2E_c(m_{j:n})(2E_c(m_{j:n}) - x) - (2E_c(m_{j:n}) - x)^2\right)\right]}{\int_0^{\bar{x}} \exp\left[\lambda b \left(2E_c(m_{j:n})y - y^2\right)\right] dy}$$

Q.E.D.

Lemma 5A (Ali and Chen, 1965). If a distribution function, $F(x)$, is symmetric, continuous, strictly positive and unimodal, for $j > \frac{n+1}{2}$, $E(m_{j:n}) \geq F^{-1}(\frac{j}{n+1})$ and for $j = \frac{n+1}{2}$ $E(m_{j:n}) = F^{-1}(\frac{j}{n+1})$.

where F is unimodal if there exists at least one real c such that F^{-1} is concave for $x < c$ and convex for $x > c$.

30

Lemma 6A. In a competitive logit equilibrium, for all λ, if $j \leq \frac{n+1}{2} - 1$, then $E_c(m_{j:n}) < \frac{\bar{x}}{2}$; if $j \geq \frac{n+1}{2} + 1$, then $E_c(m_{j:n}) > \frac{\bar{x}}{2}$.

Proof. By Lemma 3A and Assumption 2, it is sufficient to show that $E_{f_c}(m_{j-1:n-1}) > \frac{\bar{x}}{2}$ when $j \geq \frac{n+1}{2} + 1$, and $E_{f_c}(m_{j:n-1}) < \frac{\bar{x}}{2}$ when $j \leq \frac{n+1}{2} - 1$.

When n is even, consider the case of $j = \frac{n}{2} + 1$. Suppose $f_{j-1:n-1}(x)$ is the competitive logit equilibrium effort density and $E_f(m_{j-1:n-1}) < \frac{\bar{x}}{2}$. (This is a median game in terms of players' perceptions.) Let's define a density;

$$(20) \quad f^*(x) = f_{j-1:n-1}(x) + \frac{1}{2E_f(m_{j-1:n-1})} \int_{2E_f(m_{j-1:n-1})}^{\bar{x}} f_{j-1:n-1}(y)dy$$
$$\text{if } x \leq 2E_f(m_{j-1:n-1})$$
$$= 0 \quad \text{if } x > 2E_f(m_{j-1:n-1})$$

By Lemma 4A, $f^*(x)$ satisfies the conditions in Lemma 5A and $E_f(m_{j-1:n-1}) = E_{f^*}(m_{j-1:n-1})$. If $E_f(m_{j-1:n-1}) < \frac{\bar{x}}{2}$, $f_c(x)$ first-order stochastically dominates $f^*(x)$ and it contradicts to the equality. Therefore $E_f(m_{j-1:n-1}) \geq \frac{\bar{x}}{2}$. Since $j = \frac{n}{2} + 1 < \frac{n+1}{2} + 1$, $\frac{n}{2} + 2$ is the smallest integer which is greater than $\frac{n+1}{2} + 1$. By Assumption 3,

$$\frac{\bar{x}}{2} \leq E(m_{\frac{n}{2}:n-1}) < E(m_{\frac{n}{2}+1:n-1}) \leq E_c(m_{\frac{n}{2}+2:n})$$

and the result follows. The same arguments hold for $j = \frac{n}{2}$ with

$$(21) \quad f^{**}(x) = f_{j:n-1}(x) + \frac{1}{2(\bar{x} - E_f(m_{j:n-1}))} \int_0^{2E_f(m_{j:n-1})-\bar{x}} f_{j:n-1}(y)dy$$
$$\text{if } x \geq 2E_f(m_{j:n-1}) - \bar{x}$$
$$= 0 \quad \text{if } x < 2E_f(m_{j:n-1}) - \bar{x}$$

If n is odd, consider the case of $j = \frac{n+1}{2} + 1$. Suppose $E_f(m_{j-1:n-1}) \leq \frac{\bar{x}}{2}$, then by Lemma 4A and 5A, $E_{f^*}(m_{j-1:n}) = E_f(m_{j-1:n-1})$. If $E_f(m_{j-1:n-1}) = \frac{\bar{x}}{2}$, $f_{j-1:n-1}(x)$ and $f^*(x)$ are identical, but by the definition of order statistic it is not the case. Therefore $F_{j-1:n-1}(x)$ should first-order stochastically dominates $F^*(x)$. Since $E_{f^*}(m_{j-1:n}) < E_{f^*}(m_{j-1:n-1})$, $E_{f^*}(m_{j-1:n-1}) > E_f(m_{j-1:n-1})$. However, $F_{j-1:n-1}(x)$ first-order stochastically dominates $F^*(x)$, which is a contradiction. The proof for $j = \frac{n+1}{2} - 1$ is identical with $f^{**}(x)$. Q.E.D.

For the final result, we need Lemma 4A, 5A and 6A, but Lemma 5A is valid only for $j \geq \frac{n+1}{2}$. Following lemma takes care of the cases $j \leq \frac{n+1}{2} - 1$.

Lemma 7A. In competitive logit equilibria, given λ, for any j_1 and j_2, if $E_c(m_{j_1:n}) = E(m_{j-1:n-1})$ and $E_c(m_{j_2:n}) = E(m_{n-j+1:n-1})$ for some j, then $E_c(m_{j_1:n}) = \bar{x} - E_c(m_{j_2:n})$.

Proof. Let $F_{j-1:n-1}(x)$ be a competitive logit equilibrium effort distribution function in a game with j and n.

$$F_{j-1:n-1}(x) = \frac{\int_0^x \exp\left[\lambda b\left(2E(m_{j-1:n-1})y - y^2\right)\right]dy}{\int_0^{\bar{x}} \exp\left[\lambda b\left(2E(m_{j-1:n-1})y - y^2\right)\right]dy}$$

31

By change of variable,

$$F_{j-1:n-1}(x) = 1 - \frac{\int_0^{\bar{x}-x} \exp\left[\lambda b \left(2(\bar{x} - E(m_{j-1:n-1}))y - y^2\right)\right] dy}{\int_0^{\bar{x}} \exp\left[\lambda b \left(2(\bar{x} - E(m_{j-1:n-1}))y - y^2\right)\right] dy}$$

Let

$$F^*(x) = \frac{\int_0^x \exp\left[\lambda b \left(2(\bar{x} - E(m_{j-1:n-1}))y - y^2\right)\right] dy}{\int_0^{\bar{x}} \exp\left[\lambda b \left(2(\bar{x} - E(m_{j-1:n-1}))y - y^2\right)\right] dy}$$

Then $F^*(x) = 1 - F_{j-1:n-1}(\bar{x} - x)$. Next, one needs to show that $F^*(x)$ is a fixed point. From the above definition, it suffices to show that $E^*(m_{n-j+1:n-1}) = \bar{x} - E(m_{j-1:n-1})$.

$$
\begin{aligned}
E^*(m_{n-j+1:n-1}) &= \bar{x} - \int_0^{\bar{x}} G^*_{c,n-j+1:n-1}(y) dy \\
&= \bar{x} - \int_0^{\bar{x}} \sum_{k=n-j+1}^{n-1} B(n-1,k)[F^*(y)]^k [1 - F^*(y)]^{n-k-1} dy \\
&= \bar{x} - \int_0^{\bar{x}} \sum_{k=n-j+1}^{n-1} B(n-1,k)[1 - F_{j-1:n-1}(\bar{x}-y)]^k [F_{j-1:n-1}(\bar{x}-y)]^{n-k-1} dy \\
&= \bar{x} - \int_0^{\bar{x}} \sum_{k=n-j+1}^{n-1} B(n-1,k)[1 - F_{j-1:n-1}(y)]^k [F_{j-1:n-1}(y)]^{n-k-1} dy
\end{aligned}
$$

Since $B(n-1,k) = B(n-1, n-k-1)$, by substituting $r = n - k - 1$,

$$
\begin{aligned}
E^*(m_{n-j+1:n-1}) &= \bar{x} - \int_0^{\bar{x}} \sum_{r=0}^{j-2} B(n-1,r)[F_{j-1:n-1}(y)]^r [1 - F_{j-1:n-1}(y)]^{n-r-1} dy \\
&= \int_0^{\bar{x}} \left[1 - \sum_{r=0}^{j-2} B(n-1,r)[F_{j-1:n-1}(y)]^r [1 - F_{j-1:n-1}(y)]^{n-r-1}\right] dy
\end{aligned}
$$

By using $\sum_{r=0}^{n-1} B(n-1,r)[F_{j-1:n-1}(z)]^{n-r-1}[1 - F_{j-1:n-1}(z)]^r = 1$,

$$
\begin{aligned}
E^*(m_{n-j+1:n-1}) &= \int_0^{\bar{x}} \sum_{r=j-1}^{n-1} B(n-1,r)[F_{j-1:n-1}(y)]^{n-r-1}[1 - F_{j-1:n-1}(y)]^r dy \\
&= \bar{x} - E(m_{j-1:n-1})
\end{aligned}
$$

Q.E.D.

Proof of Proposition 3. When $j \geq \frac{n+1}{2} + 1$, by Assumption 2 and Lemma 3A, it is sufficient to show that $E(m_{j-1:n-1}) = E(m_{\frac{n+1}{2}:n-1})$ converges to $\bar{x}$. By Lemma 6A, $E(m_{j-1:n-1})$ should be greater than $\frac{\bar{x}}{2}$. By Lemma 4A,

(22) $(2E_f(m_{j-1:n-1}) - \bar{x})f_{j-1:n-1}(\bar{x}) = (2E_f(m_{j-1:n-1}) - \bar{x})f_{j-1:n-1}(2E_f(m_{j-1:n-1}) - \bar{x})$

and

$$(23) \quad \int_0^{2E_f(m_{j-1:n-1})-\bar{x}} f_{j-1:n-1}(y)dy = 1 - \int_{2E_f(m_{j-1:n-1})-\bar{x}}^{\bar{x}} f_{j-1:n-1}(y)dy$$

$$= 1 - 2\int_{E_f(m_{j-1:n-1})}^{\bar{x}} f_{j-1:n-1}(y)dy$$

$$= 1 - 2(1 - F_{j-1:n-1}(E_f(m_{j-1:n-1}))$$

$$= 2F_{j-1:n-1}(E_f(m_{j-1:n-1}) - 1$$

Suppose $f_{j-1:n-1}(x)$ converges to a mass-point at $m < \bar{x}$, then $f_{j-1:n-1}(\bar{x})$ converges to 0 and $f_{j-1:n-1}(2E_f(m_{j-1:n-1}) - \bar{x})$ converges to 0. That implies that for every $\varepsilon > 0$, there exists a λ^* such that $|f_{j-1:n-1}(x) - f^{**}(x)| < \varepsilon$ for all $\lambda > \lambda^*$ where $f^{**}(x)$ is defined as in Eq.(21). Let $|F_{j-1:n-1}(E_f(m_{j-1:n-1})) - F^{**}(E_{f^{**}}(m_{j-1:n-1}))| = \delta_\lambda$, then $\delta_\lambda < \bar{x}f_c(\bar{x})$. Since $f_{j-1:n-1}(\bar{x}) \geq f_{j-1:n-1}(x)$ for every $x \in [0, 2E_f(m_{j-1:n-1}) - \bar{x}]$, from Eq.(22) and Eq.(23),

$$(2E_f(m_{j-1:n-1}) - \bar{x})f_{j-1:n-1}(\bar{x}) \geq \int_0^{2E_f(m_{j-1:n-1})-\bar{x}} f_{j-1:n-1}(y)dy$$

$$= 2F_{j-1:n-1}(E_f(m_{j-1:n-1})) - 1$$

$$= 2F^{**}_{j-1:n-1}(E_{f^{**}}(m_{j-1:n-1})) - 1 - 2\delta_\lambda$$

By Lemma 5A, $F^{**}_{j-1:n-1}(E_{f^{**}_c}(m_{j-1:n-1})) \geq \frac{j-1}{n}$, and

$$(2E_f(m_{j-1:n-1}) - \bar{x})f_{j-1:n-1}(\bar{x}) + 2\delta_\lambda > \frac{2(j-1)}{n} - 1 = \frac{1}{n}$$

By the hypothesis, $f_{j-1:n-1}(\bar{x})$ and δ_λ vanish as λ goes to infinity. This is a contradiction.

When $j \leq \frac{n+1}{2} - 1$, by Assumption 3 and Lemma 3A, it is sufficient to show that $E_f(m_{j:n-1}) = E_f(m_{\frac{n+1}{2}-1:n-1})$ converges to 0. By Lemma 7A, $E_f(m_{\frac{n+1}{2}:n-1}) = \bar{x} - E_f(m_{\frac{n+1}{2}-1:n-1})$. By taking smallest $E_f(m_{\frac{n+1}{2}:n-1})$, if not unique, $E_f(m_{\frac{n+1}{2}-1:n-1})$ becomes the greatest equilibrium value. The result follows from that $E_f(m_{\frac{n+1}{2}:n-1})$ goes to $\bar{x}$ as λ increases. Q.E.D.

Proof of Lemma 4

Lemma 8A. Given λ, for any q, $f_q(x)$ is symmetric around m_q. That is, if $(2m_q - x) \in [0, \bar{x}]$, then $f_q(x) = f_q(2m_q - x)$.

Proof. Identical to the proof of Lemma 4A.

Lemma 9A. In a competitive equilibrium, for all λ, if $q < \frac{1}{2}$, then $m_q < \frac{\bar{x}}{2}$; if $q > \frac{1}{2}$, then $m_q > \frac{\bar{x}}{2}$; When $q = \frac{1}{2}$, $m_q = \frac{\bar{x}}{2}$.

Proof. By Lemma 8A, $F_q(m_q) < \frac{1}{2}$ if $m_q < \frac{\bar{x}}{2}$; if $F_q(m_q)q > \frac{1}{2}$ if $m_q > \frac{\bar{x}}{2}$; $F_q(m_q) = \frac{1}{2}$ if $m_q = \frac{\bar{x}}{2}$. Q.E.D.

Proof of Lemma 4. When $q < \frac{1}{2}$, by Lemma 8A, $f_q(2m_q) > f_q(x)$ for all $x \in (2m_q, \bar{x}]$ and

$$0 < 1 - 2q = 1 - F_q(2m_q) = \int_{2m_q}^{\bar{x}} f_q(y)dy < (\bar{x} - 2m_q)f_q(0)$$

This implies $f_q(0) > 0$. In Eq.(17), only $f_q(m_q)$ can have positive value in the limit of λ and $f_q(m_q) \to \infty$ as $\lambda \to \infty$. Since $0 < f_q(0) = f_q(2m_q) \le f_q(m_q)$, $2m_q \to 0$ and the result follows. Similarly, when $q > \frac{1}{2}$,

$$0 < 2q - 1 = F_q(2m_q - \bar{x}) = \int_0^{2m_q - \bar{x}} f_q(y)dy < (2m_q - \bar{x})f_q(\bar{x})$$

When $q = \frac{1}{2}$, the result follows from the convergence of $f_q(x)$ to a point mass. Q.E.D.

Proof of Proposition 5

The proof consists of two parts. After showing a logit equilibrium effort density converges uniformly on $[0, \bar{x}]$ to the competitive logit equilibrium effort density as λ and n go to infinity, I show that it converges uniformly to that of the q-quantile game. Then the result follows from Lemma 4.

Let $q = \frac{j}{n+1}$ and (j_t, n_t) be increasing sequences such that $\frac{j_t}{n_t+1} \equiv q$, $t = 1, 2, \cdots$, with $n_t = t(n+1)$. Then j_t and n_t are integers. Let's define a competitive QRE where $E_c(m_{j:n}) = E(m_{j:n})$.

(24) $\pi_{c,t}^e(x) = b[2E_t(m_{j_t:n_t})x - x^2]$

The corresponding equilibrium effort density satisfies

(25) $f_{c,t}(x) = f_{c,t}(0) + 2\lambda_t b \left[E_t(m_{j_t:n_t})F_{c,t}(x) - \int_0^x y f_{c,t}(y)dy \right]$

A logit equilibrium effort density is

(26) $f_t(x) = f_t(0) + 2\lambda_t b \left[E_t(m_{j_t-1:n_t-1})F_t(x) - \int_0^x y f_t(y)dy \right]$

$$+ 2\lambda_t b \int_0^x \int_0^y (G_{j_t-1:n_t-1}^t(z) - G_{j_t:n_t-1}^t(z))dz f_t(y)dy + \frac{a\lambda_t}{n_t}G_{j_t:n_t}^t(x)$$

and the last term vanishes under the assumption. For the convergence, it is sufficient to show that $\lambda_t|E_t(m_{j_t-1:n_t-1})) - E_t(m_{j_t:n_t-1})| \to 0$ because, then, from Eq.(9) or Eq.(19), $f_t(x)$ converges to either

$$f_t(x) \rightarrow f_t(0) + 2\lambda_t b \left[E_t(m_{j_t-1:n_t-1})F_t(x) - \int_0^x y f_t(y)dy \right]$$

$$f_t(x) \rightarrow f_t(0) + 2\lambda_t b \left[E_t(m_{j_t:n_t-1})F_t(x) - \int_0^x y f_t(y)dy \right]$$

Since $E_t(m_{j_t-1:n_t})) < E_t(m_{j_t-1:n_t-1}) < E_t(m_{j_t:n_t})) < E_t(m_{j_t:n_t-1})) < E_t(m_{j_t+1:n_t})$, I show that $\lambda_t|E_t(m_{j_t-1:n_t})) - E_t(m_{j_t+1:n_t})| \to 0$. Then, $f_t(x)$ converges to the corresponding competitive logit equilibrium.

In the proof, the expected value of each order statistic is approximated by Taylor series expansion. The precision in terms of n is shown in David and Johnson (1954), but for the precision in terms of λ, following exercise is necessary.

The probability integral transformation, $u = F(x)$, transforms the order statistic $m_{j:n}$ from a continuous population with distribution function $F(x)$ into the uniform order statistic $U_{j:n}$ on $[0, 1]$. Hence, by inverting the above transformation, we have

$$m_{j:n} = F^{-1}(U_{j:n}) = Q(U_{j:n})$$

By Taylor's theorem, there exists a $\tilde{q} \in [\min[U_{j:n}, q], \max[U_{j:n}, q]]$ such that

$$(27) \quad m_{j:n} = Q(q) + Q'(q)(U_{j:n} - q) + \frac{1}{2}Q''(\tilde{q})(U_{j:n} - q)^2$$

The central moments of uniform order statistics are

$$E(U_{j:n}) = q, \qquad E(U_{j:n} - E(U_{j:n}))^2 = \frac{q(1-q)}{n+2}$$

By taking expectation on both sides of Eq.(27) and using the values of central moments,

$$(28) \quad E(m_{j:n}) = Q(q) + \frac{q(1-q)}{2(n+2)}Q''(\tilde{q})$$

and in a logit equilibrium,

$$Q'(q) = \frac{1}{f(Q(q))}, \qquad Q''(\tilde{q}) = -\frac{f'(Q(\tilde{q}))Q'(\tilde{q})}{f(Q(\tilde{q}))^2} = -\frac{\lambda\pi'(Q(\tilde{q}))}{f(Q(\tilde{q}))^2}$$

Therefore,

$$(29) \quad E_t(m_{j_t-1:n_t}) - E_t(m_{j_t+1:n_t}) = Q_t(q - \frac{1}{n_t+1}) - Q_t(q + \frac{1}{n_t+1}) + \frac{q(1-q)}{2(n_t+2)}[Q_t''(\hat{q}) - Q_t''(\check{q})]$$

where

$$\hat{q} \in \left[\min[F_t(E(m_{j_t-1:n_t})), q], \max[F_t(E(m_{j_t-1:n_t})), q]\right]$$

$$\check{q} \in \left[\min[F_t(E(m_{j_t+1:n_t})), q], \max[F_t(E(m_{j_t+1:n_t})), q]\right]$$

Since $F_t(Q_t(q)) > 0$, $f_t(Q_t(q))$.[18] $f_t(E_t(m_{j_t:n_t}))$ is strictly positive for all j_t and n_t and $\pi'(x)$ is bounded. Thus both $Q_t''(\hat{q})$ and $Q_t''(\check{q})$ are of order $O(\lambda)$ and $O(n^0)$. Therefore, under

---

[18]By letting $x^* = \text{argmax}_x f(x)$, when $Q(q) < x^*$, $F(Q(q)) = \int_0^{Q(q)} f(y)dy < Q(q)f(Q(q))$ and $f(Q(q)) > 0$. Similarly, $f(Q(q)) > 0$ when $Q(q) > x^*$.

the assumption that $n > \lambda^{2+\varepsilon}$, if $\lambda[Q_t(q - \frac{1}{n_t+1}) - Q_t(q + \frac{1}{n_t+1})] \to 0$, $\lambda[E_t(m_{jt-1:n_t})) - E_t(m_{jt+1:n_t})] \to 0$. Since

$$Q_t(q + \frac{1}{n_t + 1}) = Q_t(q) + \frac{1}{n_t + 1}Q_t'(q) + \frac{1}{2(n_t + 1)^2}Q_t''(\bar{q})$$

$$Q_t(q - \frac{1}{n_t + 1}) = Q_t(q) - \frac{1}{n_t + 1}Q_t'(q) + \frac{1}{2(n_t + 1)^2}Q_t''(\breve{q})$$

where $q \leq \bar{q} \leq q + \frac{1}{n+1}$ and $q - \frac{1}{n+1} \leq \breve{q} \leq q$, we have

$$Q_t(q + \frac{1}{n_t + 1}) - Q_t(q - \frac{1}{n_t + 1}) = \frac{2}{n_t + 1}\frac{1}{f_t(Q(q))} + \frac{1}{(n_t + 1)^2}[Q_t''(\bar{q}) - Q_t''(\breve{q})]$$

Using the same argument above, $Q_t''(\bar{q})$ and $Q_t''(\breve{q})$ are of order $O(\lambda)$, and the result follows.

Next, I need to show that the competitive effort density converges uniformly to that of q-quantile game as t goes to infinity, that is, for every $\varepsilon > 0$ there exists a $T > 0$ such that $|E_{t_1}(m_{jt_1:n_{t_1}}) - E_{t_2}(m_{jt_2:n_{t_2}})| < \varepsilon$ for all $T < t_1 < t_2$.

Suppose $E_{t_1}(m_{jt_1+1:n_{t_1}}) < E_{t_2}(m_{jt_2:n_{t_2}})$ for a given λ. Since $f_{c,t}$ is uniformly distributed when $\lambda = 0$, $E_t(m_{jt:n_t}) = \bar{x}\frac{jt}{n_t+1}$ and $E_{t_1}(m_{jt_1+1:nt1}) > E_{t_2}(m_{jt_2:n_{t_2}})$. Since $E_t(m_{jt:n_t})$ is continuous in λ, there exists a λ^* such that $E_{t_1}(m_{jt_1+1:n_{t_1}}) = E_{t_2}(m_{jt_2:n_{t_2}})$, and f_{c,t_1} and f_{c,t_2} are identical.

By Taylor's theorem,

$$Q_{c,t}(q + \frac{1}{n_{t_1} + 1}) = Q_{c,t}(q) + \frac{1}{(n_{t_1} + 1)}\frac{1}{f_{c,t}(Q_{c,t}(q))} + \frac{1}{2(n_t + 1)^2}Q_{c,t}''(\hat{q})$$

By substituting this into Eq.(28),

(30) $E_{t_1}(m_{jt_1+1:n_{t_1}}) = Q_{ct,jt_1+1:n_{t_1}}(q) + \dfrac{1}{(n_{t_1} + 1)f_{c,t}(Q_{ct,jt_1+1:n_{t+1}}(q))}$

$$+ \frac{1}{2(n_{t_1} + 1)^2}Q_{ct,jt_1+1:n_{t_1}}''(\hat{q}) + \frac{1}{2(n_{t_1} + 2)^2}Q_{ct,jt_1+1:n_{t_1}}''(\breve{q})$$

(31) $\quad E_{t_2}(m_{jt_2:n_{t_2}}) = Q_{ct,jt_2:n_{t_2}}(q) + \dfrac{1}{2(n_{t_2} + 2)^2}Q_{ct,jt_1+1:n_{t_2}}''(\breve{q})$

Following identical argument before, one can show that $f_{ct_1}(Q_{ct,jt_1+1:n_{t_1}}(\hat{q}))$, $f_{ct_1}(Q_{ct,jt_1+1:n_{t_1}}(\breve{q}))$, and $f_{ct_2}(Q_{ct,jt_2+1:n_{t_2}}(\breve{q}))$ are strictly positive. Since f_{c,t_1} and f_{c,t_2} are identical, $Q_{ct,jt_1+1:n_{t_1}}(q) = Q_{ct,jt_2:n_{t_2}}(q)$. Comparing the orders of the terms in the right hand sides of Eq.(30) and Eq.(31) shows that there exists a T such that for every $T < t_1 < t_2$, $E_{t_1}(m_{jt_1+1:n_{t_1}}) > E_{t_2}(m_{jt_2:n_{t_2}})$. This is a contradiction. Therefore, for a sufficiently large T, $E_{t_1}(m_{jt_1+1:n_{t_1}}) \geq E_{t_2}(m_{jt_2:n_{t_2}})$ for all $T < t_1 < t_2$.

Similarly, $E_{t_1}(m_{jt_1-1:n_{t_1}}) \leq E_{t_2}(m_{jt_2:n_{t_2}})$ and we have $E_{t_1}(m_{jt_1-1:n_{t_1}}) \leq E_{t_2}(m_{jt_2:n_{t_2}}) \leq E_{t_1}(m_{jt_1+1:n_{t_1}})$. By combining this with $E_{t_1}(m_{jt_1-1:n_{t_1}}) \leq E_{t_1}(m_{jt_1:n_{t_1}}) \leq E_{t_1}(m_{jt_1+1:n_{t_1}})$,
we have $|E_{t_1}(m_{jt_1:n_{t_1}}) - E_{t_2}(m_{jt_2:n_{t_2}})| \leq E_{t_1}(m_{jt_1+1:n_{t_1}}) - E_{t_1}(m_{jt_1-1:n_{t_1}})$.

Using identical steps before, one can show that $\lambda|E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) - E_{t_1}(m_{j_{t_1}-1:n_{t_1}})| \to 0$ as $T \to \infty$. From Eq.(28), $\lambda|F_{c,t}^{-1}(q) - E_t(m_{j_t:n_t})| \to 0$ as $t \to \infty$ and $f_{c,t}(x)$ converges uniformly to the competitive logit equilibrium of corresponding q-quantile game.

Since the competitive logit equilibrium converges under the assumption, the final step is to determine the value of $F_q^{-1}(q)$ in the limit. The convergence results implies that if $f_{c,t}(m_{j_t:n_t-1})$ converges to a point-mass at 0, then so does $f_{j_t:n_t}(x)$. By Lemma 4 and the convergence results, if $j_t < \frac{n_t}{2}$, then $f_{c,t}(m_{j_t:n_t-1})$ converges to a point-mass at 0. Moreover,

$$\frac{j_t}{n_t} < \frac{1}{2} \Rightarrow \frac{q(n_t+1)}{n_t} < \frac{1}{2} \Rightarrow \frac{q}{1-\frac{1}{n_t+1}} < \frac{1}{2}$$

and for every finite j and n with $j < \frac{n+1}{2}$, there exists a sufficiently large t which satisfies above inequality. The same argument applies to $j > \frac{n+1}{2}$. This completes the proof. Q.E.D.

Proof of Proposition 6

Consider a game with following expected payoff.

$$(32) \quad \pi(x;\alpha) = 2bE(m_{j-1:n-1})x - bx^2 + 2\alpha b \int_0^x (x-y)(G_{j-1:n-1}(y) - G_{j:n-1}(y))dy$$

where $\alpha \in [0,1]$ is a constant. When $\alpha = 1$, Eq.(32) is identical to Eq.(4). Let

$$F_{j:n}(x;\alpha) = \frac{\int_0^x \exp(\lambda\pi(y;\alpha))\,dy}{\int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\,dy}$$

Since the proof of Proposition 1 is valid for all α, there exists a logit equilibrium for this game. Let $F_{j:n}^c(x)$ denote the competitive logit equilibrium effort distribution for games with j and n. Then the logit equilibrium effort distribution is $F_{j:n}(x;\alpha=1)$ and $F_{j:n}(x;\alpha=0) = F_{j-1:n-1}^c(x)$.

Differentiating $F_{j:n}(x;\alpha)$ with respect to α yields:

$$
\begin{aligned}
D_\alpha F_{j:n}(x;\alpha) &= \left[\int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\,dy\right]^{-2} \times \\
&\quad \left[\int_0^x \exp(\lambda\pi(y;\alpha))\lambda\frac{\partial\pi(y;\alpha)}{\partial\alpha}dy \times \int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\,dy\right. \\
&\quad \left. - \int_0^x \exp(\lambda\pi(y;\alpha))\,dy \times \int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\lambda\frac{\partial\pi(y;\alpha)}{\partial\alpha}dy\right] \\
&= \left[\int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\,dy \times \int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\lambda\frac{\partial\pi(y;\alpha)}{\partial\alpha}dy\right]^{-1} \times \\
&\quad \left[\frac{\int_0^x \exp(\lambda\pi(y;\alpha))\lambda\frac{\partial\pi(y;\alpha)}{\partial\alpha}dy}{\int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\lambda\frac{\partial\pi(y;\alpha)}{\partial\alpha}dy} - \frac{\int_0^x \exp(\lambda\pi(y;\alpha))\,dy}{\int_0^{\bar{x}} \exp(\lambda\pi(y;\alpha))\,dy}\right]
\end{aligned}
$$

Since $\frac{\partial\pi(y;\alpha)}{\partial\alpha}$ is strictly positive for $x > 0$ and strictly increasing in x, $D_\alpha F_{j:n}(x;\alpha) \leq 0$ for all x and α. Therefore, $F_{j:n}(x;\alpha=1) \leq F_{j:n}(x;\alpha=0) = F_{j-1:n-1}^c(x)$.

From Eq.(4) with $a = 0$,

$$
\begin{aligned}
\pi(x) &= 2bE(m_{j-1:n-1})x - bx^2 + 2b\int_0^x (x-y)(G_{j-1:n-1}(y) - G_{j:n-1}(y))dy \\
&= 2bE(m_{j:n-1})x - bx^2 - 2b\int_0^x y(G_{j-1:n-1}(y) - G_{j:n-1}(y))dy \\
&\quad -2bx\int_x^{\bar{x}} G_{j-1:n-1}(y) - G_{j:n-1}(y)dy
\end{aligned}
$$

Consider following game for the lower bound.

$$
\begin{aligned}
(33)\quad \pi(x;\beta) &= 2bE(m_{j:n-1})x - bx^2 - 2\beta b\int_0^x y(G_{j-1:n-1}(y) - G_{j:n-1}(y))dy \\
&\quad -2\beta bx\int_x^{\bar{x}} G_{j-1:n-1}(y) - G_{j:n-1}(y)dy
\end{aligned}
$$

where $\beta \in [0,1]$ is a constant. When $\beta = 1$, Eq.(33) is equivalent to Eq.(4). Let $F_{j:n}(x;\beta)$ be associated distribution function. Then the logit equilibrium distribution is $F_{j:n}(x;\beta = 1)$ and $F_{j:n}(x;\beta = 0) = F_{j:n-1}^c(x)$. Since $D_\beta\pi(x;\beta) \leq 0$ and it is decreasing ($D_\beta^2\pi(x;\beta) < 0$), following the identical steps above, we have $F_{j:n}(x;\beta = 1) > F_{j:n}(x;\beta = 0) = F_{j:n-1}^c(x)$.

$F_{j-1:n-1}^c(x)$ and $F_{j:n-1}^c(x)$ serve the lower-bound and the upper-bound of a logit equilibrium effort distribution with $a = 0$. Combining those bounds and the result in Corollary 1 yields the result. Q.E.D.

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