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Leibniz's Philosophy of Infinity: Comparisons within and across Taxonomies

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Philosophy

by

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DEDICATIONS

To Mom and Dad, for always believing in me.

To Jeremy, for helping me find structure in my unstructured thoughts.

To Ethan and Alysha, for companionship through trying times.

To Phoebe, for stopping my procrastination habits from becoming too unchecked.

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List of Abbreviations

Throughout this dissertation, I use the following abbreviations for primary sources of Leibniz's work.

A G.W. Leibniz, *Sämliche Schriften und Briefe*, Darmstadt/Leipzig/Berlin, Akademie 1923-. Cited as "A [Series.Volume]."

AG *Philosophical Essays*. Translated and edited by Roger Ariew and Dan Garber. Indianapolis: Hackett, 1989.

DLC *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672–1686.* Translated and edited by Richard T. W. Arthur. New Haven, CT: Yale University Press, 2002.

DSR *De Summa Rerurm: Metaphysical Papers 1675-1676*. Translated and Edited by G. H. R. Parkinson. New Haven, CT: Yale University Press, 1992.

GM *Mathematische Schriften*. Edited by C.J. Gerhardt, Berlin/Halle 1849-63. Cited as "GM [Volume]."

GP *Die Philosophischen Schriften*. Edited by C.J. Gerhardt Berlin 1875-90. Cited as "GP [Volume]."

L *Philosophical Papers and Letters*. Translated and edited by Leroy E. Loemker. 2nd ed., Dordrecht: D. Riedell, 1969.

LC Leibniz, G.W. and Clarke, Samuel *Correspondence*. Edited by Roger Ariew, Indianapolis: Hackett, 2000.

LDB *The Leibniz-Des Bosses Correspondence*. Edited and translated by Brandon C. Look and Donald Rutherford. New Haven, CT: Yale University Press, 2007.

LDV *The Leibniz-De Volder Correspondence: With Selections from the Correspondence between Leibniz and Johann Bernoulli*. Edited and translated by Paul Lodge. New Haven: Yale University Press, 2013.

LPP *Leibniz on the Parallel Postulate and the Foundations of Geometry (The Unpublished Manuscripts)*. Translated and edited by Vincenzo De Risi. Berlin: Springer, 2016.

NE *New Essays on Human Understanding*. Translated by Peter Remnant and Jonathan Bennett. Cambridge: Cambridge University Press, 1981

Acknowledgements

A doctoral dissertation is the fruit of years of rigorous intellectual development. Undertaking this scholastic rite of passage is not possible without the guidance, encouragement, and constructive criticism of seasoned academics. For that reason, I would like to extend my sincerest gratitude towards all those who have helped me reach this milestone. First and foremost, this dissertation would not have been possible without the support of my advisor, Jeremy Heis. Six years ago, I attended my first graduate seminar, a course he was teaching on this history of mathematics in the Early Modern Era. The term paper I wrote for this class involved Leibniz’s reaction to a result proven by the mathematician Evangelista Torricelli: one can prove that a certain solid that was infinite in length nevertheless had a finite volume. The research I did for that paper continued to lead me to new questions about Leibniz’s treatment of the infinite over the course of my graduate career but given the massive amount of Leibniz’s writing on the subject, answering these questions and placing those answers into a coherent whole felt like an overwhelming task. It was thanks to the support of Jeremy Heis in both his seminars and one-on-one meetings to discuss my work that I developed the confidence in my philosophical capabilities to achieve this topic.

I would also like to acknowledge the other members of my dissertation committee, Penelope Maddy and Sean Greenberg. During my time at UCI, the seminars and reading groups that I took with Penelope Maddy were among my most memorable. Although none of them involved Leibniz in particular, her seminars fostered my passion for the history of philosophy and the philosophy of mathematics, as well as honing the analytic tools necessary to conduct research in these areas. Sean Greenberg was also essential for the development of this dissertation. Leibniz wrote across many topics, and it is very easy to focus on one area of his work and be blinded by tunnel vision to the rest. I am grateful for his expertise in areas outside of Leibniz’s mathematics. Without his input, this dissertation could have been narrowly inscribed within the confines of Leibniz’s mathematics alone, with little philosophical relevance.

While this dissertation is the culmination of the scholarship conducted during my graduate career, it was built upon the interests and passions I developed during my time as an undergraduate at Macalester. I am extremely grateful for the professors there who encouraged me to pursue philosophy at a graduate level. One of the professors who was influential in my intellectual development at Macalester was Zornitsa Keremidchieva, my first academic advisor. Although I did not remain a Political Science major, she was an important mentor though my time as an undergraduate. I would also like to thank Janet Folina, who first introduced me to logic and the philosophy of mathematics, provided me with the confidence I needed as a budding scholar, and encouraged me to apply to the LPS program at UCI. And last but certainly not least, I extend my gratitude towards Geoffrey Gorham, the advisor who took me in once I declared Philosophy as my major. He encouraged me to apply for summer funding for a student-faculty research grant over the summer before my senior year. We worked together during that summer to investigate various theories about space and time in the Early Modern Era, and this research helped form the basis for my undergraduate honors thesis on Spinoza’s conception of the infinite. My work on this project piqued my interest in historical conceptions of infinity, an interest that obviously stuck with me, given that this present dissertation is about Leibniz’s conception of the infinite.

I would also like to acknowledge Vincenzo De Risi. Going into graduate school, I had no intention whatsoever of studying anything to do with Leibniz. However, Vincenzo presented on Leibniz’s geometric work at LPS’s weekly colloquium series early in my graduate career, and I felt myself soon drawn to the subject. Since then, our paths have crossed numerous times, including a 2016 Leibniz summer school in Leipzig where he was a lecturer and his time as a visiting professor in our department during the spring of 2018. His company has always been

enjoyable, and our conversations about Leibniz's mathematical work have been both inspiring and encouraging.

Finally, I would acknowledge my family and friends for their roles in supporting my graduate career. First and foremost, the emotional and financial support of my parents, Brenda and Wayne Eklund, was vital for my academic achievements. From a young age, they encouraged me to find answers for the questions that made me curious, an attitude necessary to complete a Ph. D. My friends Alysha and Ethan were a life support when I need to vent about the stresses of academia, and my friends Jon, Lauren, and Amy also provided emotional support. Finally, I must thank my wife Phoebe. While many of the themes of this dissertation had been batting around in my head well before we met, I had not put most of them to paper. Her influence over the past two years gave me the necessary encouragement and accountability I needed to see this dissertation through to its completed end.

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Abstract of the Dissertation

Leibniz's Philosophy of Infinity: Comparisons within and across Taxonomies

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Doctor of Philosophy in Philosophy

University of California, Irvine, 2020

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In this dissertation, I analyze the distinction between different types of infinity that Leibniz identifies throughout his philosophical and mathematical works. By attending to these differences, I show how Leibniz's rejection of infinite number as a contradictory notion does not entail the impossibility of infinitely small lines. Chapter 1 explains the differences between three grades of infinity that Leibniz identifies in a 1676 taxonomy on the infinite and how this taxonomy was an attempt to avoid paradoxes of the infinite. It also contains a description of a separate taxonomy of the infinite that Leibniz gave in 1706, in which Leibniz explicitly bans any type of infinity that is a whole composed of infinity many distinct parts. Chapter 2 treats five places where infinity lines arise in Leibniz's mathematics: infinite number, the composition of the continuum, infinite series, infinitesimals and their bounded infinite counterparts, and unbounded infinite lines. Looking at the different ways Leibniz evaluates each of these concepts, we see that infinitesimals and bounded infinite lines stand on firmer conceptual footing then the others from Leibniz's point of view. Chapter 3 argues that Leibniz's claims that infinitesimals are "fictions" or "impossible," this is in reference to a specific type of impossibility that he calls the impossible *per accidens*. Unlike the absolutely impossible, this type of impossibility is not rooted in contradiction, but conflict with metaphysical principles that bar their existence in the order of created things. Hence, Leibniz's stance towards infinitesimals allows them to be perfectly coherent entities for the purposes of geometric reasoning, despite his ban on their existence within the physical world.

Introduction

Examining the role of the infinite within any philosopher's system is a formidable task, and the sheer quantity of ink Leibniz spilled on the topic of infinity makes analyzing his thoughts in this area particularly difficult. One difficulty is that the immense volume of Leibniz's writings on the infinite is not just a repetition of the exact same tropes and arguments *ad nauseam*. To slightly literalize the metaphor, we can speak of three axes that make Leibniz's work on the subject voluminous. Set on one axis are the pages upon pages Leibniz devoted to the subject. But the sheer quantity of Leibniz's writing alone is just one aspect of this immense volume. For another axis in Leibniz's work on the infinite is the number of diverse subjects that inform and are informed by Leibniz's attitude towards this concept. Examples of subjects alongside a brief reference to just one of ways in which the infinite arises are: logic (in which contingent truths require an infinite analysis); mathematics (where various finite concepts are extended to the domain of the infinite); physics (where the relationship between "dead" and "living" forces is often compared to the difference between an infinitely small motion and a finite one, respectively), biology (where organic bodies are characterized as machines whose parts are smaller organic machines, which are in turn composed of smaller machines, *ad infinitum*), theology (the "omni-" adjectives traditionally ascribed to God all clearly involve an element of transcending the finite), metaphysics (the most famous example being the harmony between the infinitely many monads and their perceptions of one another), and even ethics (where finite humans must reckon with the difficulties in mimicking the infinite benevolence of God), and it is possible to give more examples for each of these domains of inquiry. It would be one thing if the infinite only made an appearance as an ornamentation or a mere afterthought coming from musing on limiting cases or trivial hypotheses in these subjects. But Leibniz does not make such shallow gestures towards the infinite for the sake of simple superficial embellishment; he instead presents an intricate set of thoughts detailing how the infinite plays a foundational role in each of these diverse domains of inquiry. The last axis is temporal. As his career unfolds, Leibniz's categorizations of different types of infinity change, as we will see in the first chapter of this dissertation. Additionally, the infinite is employed in new ways over time, as developments in Leibniz's physics and metaphysics give rise to novel problems and tensions that the infinite is used to help resolve.

Despite these three dimensions of complexity in Leibniz's writing on the infinite, there remains a consistent topic with which he grappled both publicly and privately: defending his newly developed calculus and its apparent reliance on non-finite quantities, also known as infinitesimals. The vast majority of this dissertation will focus on Leibniz's use of the infinite within mathematics, paying specific attention to Leibniz's use of infinitesimals in the foundations of his calculus. The reason for this focus is twofold. First, there is the value in understanding a historical philosophical issue on its own terms. The development of the calculus was a crucial innovation in the history of intellectual achievement and understanding how Leibniz thought of its foundations gives us a greater appreciation of this milestone in human thought. Second, a narrow focus on one area of Leibniz's thought helps us better appreciate his nuances as a thinker and the difficulty in making generalized statements about his philosophical views. Seeing the fine-grained distinctions that Leibniz draws between infinitesimals and various other non-finite entities that he postulated in his mathematical works brings light to the risks of making generalized statements about the role the infinite plays in Leibniz's mathematics. And once we recognize the difficulties with pinning down how the infinite functions in Leibniz's pure mathematics, we see how formidable it is to say something about the use of the infinite across all the intellectual projects with which Leibniz was engaged.

To place the importance of the calculus in context, it is useful to remember that one of the distinctive features of European approaches to natural philosophy during the Early Modern

Era was a growing reliance on increasingly complex mathematical theories to describe and predict phenomena. Galileo, an early proponent of this approach, succinctly captured the spirit of this approach in *The Assayer*, where he states:

> "Philosophy is written in this all-encompassing book that is constantly open before our eyes, that is the universe; but it cannot be understood unless one first learns to understand the language and knows the characters in which it is written. It is written in the mathematical language, and its characters are triangles, circles, and other geometric figures; without these it is humanly impossible to understand a word of it, and one wanders around pointlessly in a dark labyrinth."[1]

In the decades and centuries that followed, natural philosophers took up Galileo's call and studied increasingly diverse chapters of the "book of nature." Luminaries such as Descartes advanced a biology in which the functions of organic body were conceived of as nothing more than the output of a highly complex machine whose parts could in theory be described in purely mathematical terms. Meanwhile, Boyle developed a chemistry that emphasized the importance of quantitative measurements and laws.

Using mathematics to translate new types of phenomena from the book of nature into the realm of human understanding was not the only innovation in natural philosophy following Galileo's description of this approach. Perhaps more significant were a series of mathematical developments that inserted new characters into the language of mathematics. The infinitesimal calculus served as one of these innovations, and enriched mathematics with characters that allowed philosophers to translate the book of nature's passages on motion and rest with greater detail than Galileo was able to achieve with the mathematical language of his era. Despite the utility of the calculus, its introduction to the European academic community was fraught with controversy. Even setting aside the acrimonious dispute between Newtonian and Leibnizian partisans over who had priority in developing the tools of the calculus, one finds a large amount

[1] Quoted in translation from Finocchiaro, *The Essential Galileo,* p. 183.

of ink spilled in the late Seventeenth and early Eighteenth Century in service of defending or criticizing the foundations of the newly-established calculus, and its apparent reliance on infinitely small quantities, commonly known as infinitesimals.

As one of the inventors of the calculus, Leibniz grappled heavily with the foundations of his new method; his mathematical and philosophical manuscripts contain a trove of shifting positions on the status of these infinitely small objects and their relationship to the calculus. Famously derided by George Berkeley in his 1734 book *The Analyst*, these infinitely small quantities possessed many properties deemed to be paradoxical, threatening the reliability of any result reached through a method of reasoning that rested its foundations upon them. However, recent scholarship following the publication of manuscripts unknown to Leibniz's contemporaries and generations of his successors has shown that Leibniz's foundations for the calculus were more sophisticated than the views criticized in Berkeley's polemic. Some authors have argued that Leibniz's appeal to infinitesimals is merely apparent and should instead be seen as a paraphrase for something akin to a contemporary limit concept. Others have argued that Leibniz did appeal to actual infinitely small quantities, but he had careful principles that justified the introduction of these elements, as well as a series of rules that prevented him from falling into contradiction when using such methods.

However, the calculus was not the only area in which Leibniz grappled with the infinite in his mathematics. Space seems to be composed of an infinitude of points, there exist infinite series that converge towards a single finite value, lines can be extended without limit, and one can never specify a number larger than all others. Each of these varied mathematical applications of the infinite raises new paradoxes or puzzles that required Leibniz to concoct different solutions. Some of these cases, such as infinite number, are brushed off as contradictory and hence banished from mathematical considerations. On the other hand, Leibniz frequently refers to infinitesimals as "fictions," signaling a difference between them and ordinary quantities. In this dissertation, I argue that when Leibniz calls infinitesimals "fictions," he does not mean to group them with the types of mathematical reasoning that lead to contradiction. Instead, infinitesimal quantities are "fictional" in the sense that there are no bodies or motions in nature that are infinitely small, even though such infinitesimals may be useful in mathematical models of phenomena. Fully articulating this position requires much groundwork in order to separate what Leibniz says about different types of infinite objects in order to speak comprehensively about infinitesimals without inappropriately importing the properties of other types of infinite entities that Leibniz discusses.

There are three stages in explaining and defending this interpretation of Leibnizian infinitesimals, each of which receives its own chapter. The first is to compare two descriptions Leibniz gives about different grades of infinity in general, one in a 1676 series of notes Leibniz made while reading Spinoza and developing the fundamentals of his calculus and one in 1706 in his correspondence to Des Bosses, representing the fruit of three decades of intellectual maturation. This is the focus of Chapter 1. This analysis is significant because it allows us to situate Leibniz's remarks about infinitesimals within his overall philosophy. In Chapter 1, we see Leibniz's emphasis in classifying the infinite shifts in the 30 years between each taxonomy. Leibniz's characterizations in the earlier taxonomy group the different kinds of infinity by how comprehensive each tier of infinity is. The later taxonomy stresses the relationships that parts have to their whole, a relationship that Leibniz becomes increasingly concerned with when reasoning about the infinite. One major purpose of this chapter is to get a handle on the phrase "syncategorematic," a term from the later taxonomy that frequently arises within the literature of Leibniz's stance towards infinitesimals.

The second stage is to situate infinitesimals within the rest of Leibniz's mathematics. As mentioned above, there are various places in which the infinite arises within mathematics. In Chapter 2, we examine five distinct mathematical applications of the infinite. Through examining the different problems Leibniz encounters with each of these five cases, we again see a concern for how wholes relate to their constituent parts. We also see a discrepancy between how Leibniz treats infinitesimals and other non-finite objects within his mathematics. Some mathematical concepts, such as numbers, become contradictory when extended to the infinite. However, the remarks Leibniz makes about infinitesimals seem to be less condemnatory and put them in a more stable position than other infinite mathematics entities Leibniz considers. In addition to the utility of distinguishing between different aspects of Leibniz's mathematics that are not always treated separately, the work done in this chapter allows us to advance our interpretation of infinitesimals without clouding our judgment from remarks Leibniz makes about other kinds of infinite entities that do not apply to infinitesimals.

The final stage in this argument is to show that when Leibniz calls infinitesimals "fictions," he is not impugning their validity in pure mathematics, and as such should be seen as logically consistent. This is the focus of Chapter 3. Having laid the groundwork of Leibniz's account of the infinite in general within the two taxonomies and specifically within mathematics, this chapter focuses on different interpretations one can have of Leibniz's stance towards infinitesimals. I argue that the concept of an infinitesimal is consistent in and of itself. I then appeal to remarks that Leibniz makes about certain algebraic entities (*e.g.,* imaginary numbers) to distinguish between two different types of impossibility that Leibniz identifies, only one of which involves a contradiction. After showing why any claims about the "impossibility" of infinitesimals are best read as the less severe account of infinitesimals, I consider possible textual evidence against the feasibility of this interpretation. By using language developed in Chapter 1, I show that these damaging quotes only impugn the possibility of infinitesimals existing in nature, leaving them safely as mathematical objects in their own right.

As a result of these considerations, we see an important feature of Leibniz's mathematical practice. In the domain of pure mathematics, Leibniz allows himself to reason with entities that have no correlates in the created world. Leibniz's main concern in mathematics is freedom from contradiction. Seeing philosophy and mathematics as separate enterprises, Leibniz has no interest in letting metaphysical considerations block the free use of consistent mathematical entities. In fact, employing entities that cannot be located within the population of worldly beings and their properties has beneficial uses, for the use of such objects within mathematical reasoning can reveal underlying connections between concepts that were previously seen as disunited. As a result, we see that despite Leibniz's philosophical concerns about the infinite, the differences between the subject matter of philosophy and mathematics allows him to freely employ "impossible" entities, such as infinitesimals, to the advancement of mathematical knowledge.

After this lengthy treatment of Leibniz's use of infinity within pure mathematics, the afterward briefly introduces some of the ways that the infinite arises in other areas of Leibniz's work. There we see ways in which infinitesimals are used to represent what Leibniz calls "dead forces" in physics, and infinite analysis is used in Leibniz's account of contingent truth. I explain the technical ways that these infinite quantities arise, but I do not make any definitive claims about how to interpret Leibniz's use of the infinite in these domains. As we shall see over the course of this dissertation, Leibniz's use of the infinite is highly nuanced even when confined to the realm of pure mathematics, and any explanation of Leibniz's use of the infinite in other areas should be given the same level of scrutiny before coming to any conclusions.

Chapter 1 : Taxonomies of the Infinite

We begin with an analysis of two taxonomies of the infinite that Leibniz constructed in order to understand some of the problems that he saw arising in the context of mathematical uses of the infinite. The first taxonomy comes from a series of notes Leibniz made on Spinoza's philosophy in 1676, during the same time period in which he was developing the basics of the calculus. Here the infinite is divided into three distinct species, each possessing different properties: the Omnia, the Maximum, and the mere infinite. Thirty years later, Leibniz gave a differing categorization in a letter to Des Bosses, a German Jesuit with whom Leibniz had a lengthy correspondence. In this 1706 characterization, Leibniz says that the infinite can be thought of as categorematic, syncategorematic, or hypercategorematic, terms that will be clarified within this taxonomy. Of these three kinds of infinity, the categorematic is rejected completely, leaving only the syncategorematic and the hypercategorematic as viable categories. The reason for selecting these two taxonomies in particular is due to the fact that the earlier taxonomy was developed during Leibniz's initial work on the foundations of the calculus and thus conveys concerns that he will grapple with for the rest of his mathematical career, and the second taxonomy contains terminology that one commonly finds in discussions about Leibniz's stance towards the calculus and infinitesimals.

This chapter focuses on the two different taxonomies in order to track the changes that occurred in Leibniz's thoughts over the 30 years between the two ways that Leibniz presents the distinction between different types of infinity. Ultimately, we see while the 1676 taxonomy is concerned with the issue of one infinite object being greater than another, the 1706 taxonomy reflects a concern with the relationship an infinite whole has to its diverse parts. This concern will resurface in the next chapter, where we will see how Leibniz's concern for possible violations of the axiom that a whole cannot be equal to one of its parts underpin many of his rejections of certain kinds of infinity within pure mathematics.

1.1: The 1676 Taxonomy

In 1672, Leibniz encountered a paradox while reading Galileo's *Two New Sciences*. Galileo noticed that by associating every positive integer with its square, one can say that there are an equal number of squares and positive integers.[2] But given that the perfect squares form a proper subset of the positive integers (*i.e.* every square number is a positive integer, yet not every positive integer is a square), this association between the two groups of numbers violates the ancient axiom that a part is less than its whole, an axiom I refer to as the "part-whole axiom" in the sequel. Galileo's response to this correspondence between the two classes of numbers is to deny that terms like "equal," "greater," or "less" are applicable to the infinite. In his commentary on this section of *Two New Sciences*, Leibniz says that he cannot accept this conclusion. Instead, Leibniz tentatively proposes three alternatives to get around this paradox. First, one could say that the infinite "is not one and not a whole."[3] That is, one can deny that an infinite collection can be spoken of or mathematically operated upon as a completed entity. Another option Leibniz puts forth is to say, "distinguishing among infinities, that the most infinite, *i.e.,* all the numbers, is something that implies a contradiction."[4] Despite his condemnation of infinite numbers, the fact that we are to "distinguish" between different kinds of infinity to find the ones that imply contradiction opens the possibility that "an infinite *X*" will be possible for the right kinds of concepts *X*. The final option is to refrain from any talk of the infinite, "except where

[2] The following description is a cursory treatment of Galileo's paradox and Leibniz's reaction to it in order to show its role in motivating Leibniz's first taxonomy of the infinite. The paradox itself and Leibniz's reaction are treated in greater detail in Section 2.1.
[3] *DLC* p. 9
[4] *ibid*.

there is a demonstration of it."[5] However, it is unclear exactly how this last option would avert Galileo's paradox since there is no explanation of why the numerical correspondence Galileo noted does not count as a demonstration about the infinite.

This first option, denying the oneness and wholeness of a completed infinite, is the path Leibniz eventually takes, and we will see it reflected in his rejection of the "categorematic" infinite in the taxonomy of 1706. However, four years after his initial encounter with Galileo's paradox, it seems that Leibniz was still experimenting with the second option that he proposed: distinguishing among infinities and avoiding those that imply a contradiction. In February of 1676, Leibniz gives three grades of infinity in notes he made in Spinoza's *Ethics* and "Letter on the Infinite."[6] I will refer to this exposition of the 1676 taxonomy as passage **1**:

> I usually say that there are three degrees of infinity. The lowest is, for the sake of example, like that of the asymptote of a hyperbola; and this I usually call the mere infinite. It is greater than any assignable, as can also be said of all the other degrees. The second is that which is greatest in its own kind, as for example the greatest of all extended things is the whole of space, the greatest of all successives is eternity. The third degree of infinity, and this is the highest degree, is *everything*, and this kind of infinite is in God, since he is all one; for in him are contained the requisites for the existing of all the others. I make these comments in passing.[7]

In the second half of April 1676, Leibniz wrote a series of notes on a copy of Spinoza's "Letter on the Infinite," where he gives the same taxonomy, but elaborates further on certain concepts. I will refer to the following as passage **2** in the sequel:

> I have always distinguished the Immensum from the Unbounded, i.e., that which has no bound. And that to which nothing can be added from that which exceeds an assignable number. Briefly, I set in order of degree: Everything [*Omnia*], Maximum [*Maximum*], Infinity [*Infinitum*]. Whatever contains *everything* is maximum in entity; just as a space unbounded in every dimension is maximum in extension. Likewise, that which contains everything is the most infinite, as I am accustomed to call it, or the absolute infinite. The

[5] *ibid.*
[6] The "Letter on the Infinite" is Spinoza's April 20 1663 letter to Lodewijk Meyer. Printed as Letter 12 in Spinoza's *Opera Posthuma*.
[7] *DLC* pp. 42-43.

> *maximum* is *everything* of its kind, i.e., that to which nothing can be added, for instance a line unbounded on both sides, which is obviously also infinite, for it contains every length. Finally, those things are *infinite the lowest degree* whose magnitude is greater than we can expound by an assignable ratio to sensible things, even though there exists something greater than these things. In just this way, there is the infinite space comprised between Apollonius's Hyperbola and its asymptote, which is one of the most moderate of infinities, to which there somehow corresponds in numbers the sum of this space: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, which is $\frac{1}{0}$.[8] Only let us understand this 0, or nought, or rather instead a quantity infinitely or inassignably small, to be greater or smaller according as we have assumed the last denominator of this infinite series of fractions, which is itself also infinite, smaller or greater. For a maximum does not apply in the case of numbers.[9]

We briefly examine the three types of infinity present in the taxonomy of 1676 before explaining how the category of the Maximum is a continuation of the second option Leibniz proposed in response to his encounter with Galileo's paradox: distinguishing among infinities to accept only those that do not lead to contradiction. We now move to analysis of each of these categories, ascending from the lowest grade of infinity to the highest.

The first category is the mere infinite [*tantum infinitum*]. This category contains quantities that are greater than any finite quantity. Contrary to Galileo, Leibniz believes that "greater" and "lesser" can be spoken of with connection to this type of infinity. The examples Leibniz gives of instances of this type of infinity are geometric in nature. If we consider a hyperbola [Figure 1.1], the area in the section marked by the solid lines [**A**] will be infinite and the area under the dashed lines [**B**] will be finite. But this infinite section **A** can be seen as a part of another infinite area, such as the area consisting of both the solid and dashed lines [**A+B**]. Since the former is a proper part of the latter, Leibniz's adherence to the axiom that the whole is greater than any of its parts would lead him to conclude that **A** is less than **A+B**, even though both **A** and **A+B** both have an infinite area. This falls into the lowest grade of infinity, for although the areas **A** and

[8] This is known as the "harmonic series," a series that does not converge to a finite value.

[9] *DLC* pp. 114-115. Leibniz gives a condensed taxonomy of these three kinds of infinity in a different set of notes on Spinoza. Those remarks can be found in A vi.3 pp. 384-385 and in translation in *DLC* pp. 40-43.

A+B are greater than any finite quantity, we can continue to add finite areas such that we have figures that encompasses more space than the previous figures and are thus larger by the part-whole axiom.

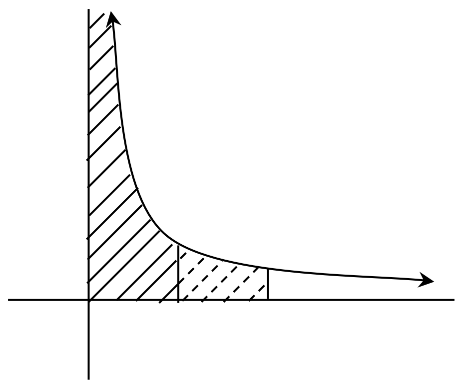

Figure 1.1: The area under a hyperbola as merely infinite

The next grade of infinity in this taxonomy is the Maximum.[10] A straight line unbounded on both sides is the paradigm case Leibniz gives of this kind of infinity in the taxonomy quoted above. Leibniz notes that such a line contains every length. In this statement, it is important that "any length" refers to both finite and infinite lengths. Suppose we had a line that starts at point **B** and proceeds unbounded in only one direction, call this infinite line **BC…** [Figure 1.2]. Then for any finite length, we could find a point **C** on the line such that the finite line **BC** is equal to that length. But given that the line is unbounded in only one direction, we could find another line unbounded on one side that starts at point **A**, such that our first line unbounded on one side is a proper part of this second line. Call this new line **AC…**. Because **AC…** contains **BC…** as a proper part, Leibniz would claim that **AC…** is longer than **BC…**. Thus, **BC…** will not contain all infinite lengths, for it does not contain the length corresponding to **AC…**. So a line unbounded on one side is a mere infinite, for one can always obtain a longer line by adding a line segment

[10] I capitalize "Maximum" here and in the sequel to make it clear that I'm referring to this specific category of infinity.

on the bounded side of the line.[11] But this is not the case for lines unbounded on both sides. In passage **1**, Leibniz also lists the whole of space as a Maximum in extension and eternity as the maximum of extension. While every object in this second category is the greatest of its kind, not all kinds are capable of possessing Maxima, as Leibniz makes clear when he denies the existence of a greatest number at the end of passage **2**.

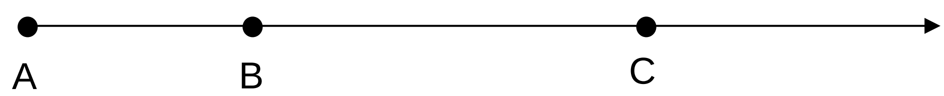

Figure 1.2: Lines unbounded on one side

The final category in Leibniz's 1676 classification is Omnia (Latin for "all"), also known as the absolute infinite. This is the category of infinity that applies to God. Although the description of Omnia that Leibniz makes in passage **2** is sparse, Leibniz elaborates his reasoning for using "Omnia" to describe this highest grade of infinity in **1**. There he says, "for in [God] are contained the requisites for existing of all the others."[12] There are strong reasons to hold that the Omnia applies only to God, as God is the only perfect substance. This means created substances lack something that God possesses, barring them from having all (*omnia*) positive properties.[13]

A Maximum is the greatest infinite with respect to a given kind, and Leibniz's initial phrasing appears to suggest that the Omnia is like a Maximum, only the kind in question is "entity." This may make it seem like the Omnia is just a species of the Maximum, but there are compelling reasons to think otherwise. At the beginning of the block passage quoted above, Leibniz draws a distinction between the Immensum and the Unbounded [*Interminato*]. Ohad

[11] Much of this reasoning can be found in the April 1676 note "Infinite Lines." The mathematical importance of this passage is treated below in Section 2.5.
[12] *DLC* pp. 42-43
[13] Nachtomy 2011. pp 957-958.

Nachtomy argues that this is the distinction between extension as it pertains to Omnia and Maxima, respectively. While "unbounded" means a line or space metrically greater than all others, "Immensum" surpasses any metric categories.[14] Thus, Omnia is a way for Leibniz to ascribe infinity to God but in a way that keeps God from being divisible like an infinite line. If a line unbounded on both sides is divisible and a Maximum, but the nature of Omnia rules divisibility out entirely, then Leibniz's talk of Omnia as being the Maximum of the category "entity" appears to be metaphorical, rather than a claim that the Omnia is a just a species of the Maximum.

Leibniz ends this 1676 passage by noting that "a maximum does not apply in the case of numbers." This is similar to his remarks on *Two New Sciences* from 1672, where Leibniz notes that there is no greatest number. If the completed set of all positive integers were a possible entity, it would have to be a Maximum, for it would contain every entity of a certain kind (namely the kind "positive integer"). Presumably we would have to treat the set of all square numbers as an impossible Maximum as well. Otherwise, we could easily run another version of Galileo's paradox; if we associate every perfect square with every number that is a fourth power (i.e., n^2 with n^4 for all positive integers n), we obtain another bijection between a set of numbers and a proper part of that set (since every n^4 will be a n^2, but not *vice versa*). So, when Leibniz says that a Maximum does not apply in the case of numbers, any consistent interpretation of this position would have to apply to any infinite subset of numbers as well. As mentioned above, one of the solutions that Leibniz offered in his analysis of Galileo's paradox was to allow some kinds of infinity but ban others. Thus, we can see the 1676 taxonomy as a continuation of this line of thought, for it explicitly notes that only some kinds are capable of possessing Maxima. Unfortunately for any practicing mathematician who would want to heed this advice, there appear to be no criteria beyond discovering a contradiction during the course of one's reasoning

[14] Ibid. p. 946

that could decide in advance whether or not a given concept is capable of maximization. It is now time to move on to the 1706 taxonomy, where the concern for which concepts are capable of having a greatest of their kind has dropped from the taxonomic structure.

1.2: The 1706 Taxonomy

Thirty years later, Leibniz gives a different taxonomy of the infinite in an unsent supplement to a September 1, 1706 letter to the German Jesuit Des Bosses. If sent, this taxonomy would have been a response to Des Bosses's August 20, 1706 letter that asked for Leibniz's opinion of various Cartesian positions that had been recently condemned by the Jesuits. One of these was the proposition: "our mind, insofar as it is finite, can know nothing certain about the infinite; consequently, we should never engage in disputes about it."[15] In the letter as sent, Leibniz addresses this proposition by saying that mathematics has already revealed many truths about the infinite, but he affirms that "an infinite composed of parts is neither one nor a whole."[16] We receive more detail about this claim in the unsent supplement. In this taxonomy, the category of the Maximum seems to vanish. Furthermore, the 1676 taxonomy is organized according to whether or not a type of infinity is capable of being lesser or greater than another; a mere infinite can always be surpassed by another object that contains more than itself, a Maximum cannot be exceeded by anything else of the same kind, and as the name suggests, the Omnia contains everything and thus surpasses any Maxima. The focus of Leibniz's later taxonomy is structured according to how parts relate to their wholes:

> *There is a syncategorematic infinite* or passive power having parts, namely, the possibility of further progress by dividing, multiplying, subtracting, or adding. In addition, *there is a hypercategorematic infinite*, or potestative infinite, an active power having, as it

[15] *LDB*, p. 49.
[16] Ibid. p. 53.

> were, parts eminently but not formally or actually. This infinite is God himself. But *there is not a categorematic infinite* or one actually having infinite parts formally.
>
> There is also an actual infinite in the sense of a distributive whole but not a collective one. Thus, something can be stated about all numbers, but not collectively. In this way it can be said that for every even number there is a corresponding odd number, and vice versa; but it is not therefore accurately said that there is an equal multitude of even and odd numbers.[17]

"Categorematic" and "syncategorematic" are terms of art in scholastic usage that Leibniz is adopting here. "Categorematic" refers to a term that fits into one of the categories given by Aristotle. These terms have significance on their own, like "red" and "rose." Meanwhile, a term is used in a syncategorematic way when it only has significance in the context of a complete proposition. So, in the sentence "not all roses are red," the quantifier "not all" and the copula "are" are syncategorematic terms, with "roses" and "red" remaining categorematic.[18] "Hypercategorematic" appears to be original to Leibniz.[19] Leibniz adopts two other technical terms from Scholastic philosophy in describing the part-whole relations that occur within each type of infinity: "formal" and "eminent." Something is formally contained in something else when it exists in its usual and proper way of being as flame is contained in heat. One the other hand, something is eminently contained in another eminently if (a) it does not exist in its actual form but in a form that contains additional perfections, (b) this higher form can produce all the effects of the first alongside additional perfections.[20] With these terms defined, we can discuss each kind of infinity in turn.

Like the Omnia of 1676, the hypercategorematic infinite of the 1706 taxonomy applies to God alone. The description of God as containing parts "eminently, but not formally or actually," implies that the perfections of God are found in God, but in a way that bars them from existing in

[17] *LDB* pp. 52-53. Emphasis in original.
[18] Antognazza 2015, p. 6.
[19] Antognazza 2015, p. 25, footnote 27.
[20] Antognazza 2015, p. 12.

divisible states, even conceptually. As mentioned above, God's simultaneous infinity and indivisibility are present in the description of the Omnia in the 1676 taxonomy. Since a special grade of indivisible infinity is reserved for God in both taxonomies, and this category is not applicable to mathematics, we do not further analyze the category of the hypercategorematic here.

The next item from this taxonomy to be analyzed is the syncategorematic infinite. This is an infinite that always possesses (or can possess) more parts than any finite number, but there is no infinite number that represents a gathering of all these parts into one whole. For instance, to say that a line is infinitely divisible in a syncategorematic sense is to say that for any finite number of parts the line is divided into, we can find another division of the line with a greater number of parts, but there is no stage at which we contemplate a complete infinity of distinct parts at once. The same is true of numbers: for every number there exists infinitely many numbers greater than it, but Leibniz does not believe that there is a number infinitely greater than all others. Formally, this is the difference between ($\forall x \exists y\ y>x$) and ($\exists y \forall x\ y>x$).[21] There is some controversy about whether the second paragraph of this taxonomy describes a distinct kind of infinity from the first. The argument for their difference is that the first describes ideal objects, such as lines, whose parts exist only when specified in some manner and can be increased indefinitely, much like the Aristotelian infinite.[22] Meanwhile, the second paragraph describes collections whose parts are actual and cannot be gathered into one whole.[23] The argument that these do not describe two different types of infinity is that the actuality or ideality of the parts is immaterial to its classification among the infinite. What is relevant is that in neither

[21] Arthur 2001, p. 107.
[22] Section 2.2 below contains a more detailed analysis of Leibniz's claim that the parts of a line do not exist until specified.
[23] Antognazza 2015, p. 9.

case there exist no infinite parts that are gathered into one completed whole.[24] In this dispute, I am inclined to see the second paragraph as referring to another use for the syncategorematic infinite, rather than being a wholly different species of infinity.

The final category of the 1676 categorization is the categorematic infinite. This is described as an infinity "actually having infinite parts formally," in contrast to the eminent containment of parts of the hypercategorematic infinite. As mentioned earlier, formal containment means something exists according to its usual way of being. For instance, a (finite) flock of sheep contains each individual sheep as a part formally, since each sheep is present in its natural form as a member of the flock. Furthermore, these divisions are actual rather than potential, for the reality of the flock is formed by aggregating sheep, rather than the sheep gaining their individual existences from dividing a pre-existing flock into varying parts. When he denies the categorematic infinite, Leibniz denies infinite collections that contain each member in its usual form (possessing formal parts and pre-divided (possessing actual parts), e.g., an infinite flock of sheep. There are no such unities. On the other hand, God is an infinite unity because the varying parts are contained eminently rather than formally; they exist in God as a higher way of being that precludes their divisibility.[25]

We have seen two taxonomies of the infinite that Leibniz professed in his life: mere infinite, Maximum, and Omnia; categorematic, syncategorematic, and hypercategorematic. We can now put each taxonomy against each other in order to see what differences exist. I pass over comparisons between the hypercategorematic and the Omnia, for as mentioned above, there are compelling reasons to see the former as a continuation of the latter. On the other hand, there is not a continuation between the remaining categories of each taxonomy. This is because while the 1676 taxonomy was concerned with which categories are capable of having

[24] Arthur 2018.
[25] Antognazza 2015 p. 15.

a greatest of their kind, the 1706 taxonomy is explicitly centered on the relation a whole has to its parts.

Maria Rosa Angotnazza argues that the syncategorematic infinite of 1706 can be identified with the mere infinite of 1676.[26] However, there are good reasons to think that this is not the case. To see why, we can look to the following example. As mentioned in the discussion of the 1676 taxonomy, a line bounded on one side would be a mere infinite because one could add more to the line by moving the bounding endpoint, meaning this line is not the greatest of its kind. On the other hand, a line unbounded on both sides is a Maximum, for there can exist no straight line that is longer. However, both of these would be classified as syncategorematic in the 1706 taxonomy. To use the language of this latter taxonomy, the parts of each of these lines are various line segments. These are contained formally inside each line because there is no change in the segment itself regardless of whether other lines adjoin one or both of its endpoints. However, these parts are either potential because the divisions of the line that define them have not yet been specified, or they are actual, but finite in number. Since a line unbounded on one side and a line unbounded on both sides would be categorized as a mere infinite and a Maximum in 1676, but both are syncategorematic, it is clear that there is no mapping between these non-God types of infinity across the two taxonomies. The reason these two types of lines are no longer distinguished in the 1706 taxonomy is that the nature of the relationship between parts and their various wholes is emphasized, rather than under what conditions one infinity can be called greater than another, and this part-whole relationship is the same for lines unbounded on either one or both sides.

A similarity between the two taxonomies does occur within the categorematic infinite: the denial of a type of infinity that applies to numbers. In 1676, Leibniz was concerned with blocking

[26] Antognazza 2015, p. 18.

Galileo's paradox, and the way he eventually blocked this paradox was by denying that any infinite whole can be composed of distinct parts. Because numbers are understood as unities (or wholes) that are composed of distinct parts, this means there are no infinite numbers, for such an entity would have to be both a whole/unity and composed of infinitely many distinct parts. And Leibniz explicitly denies the possibility of an infinite number when elaborating on the 1676 taxonomy. And as we shall see in Chapter 2.1, Leibniz ties the impossibility of infinite number to the impossibility of any infinite collection composed of distinct parts. For this reason, the denial of infinite number in 1676 resurfaces as the denial of the categorematic infinite in 1706, for both are concerned with the denial of infinite wholes. This is another case where the 1706 taxonomy focuses more specifically on the nature of the relation between parts that compose wholes.

Despite Leibniz's stern denial of infinite numbers, he has a more permissive attitude towards other non-finite objects within his mathematics. Examining his taxonomies of the infinite is instructive in understanding his mathematical work, for it tells us to avoid dealing with wholes composed of infinitely many actual and formal parts. But as we saw in the case of lines bounded on one or both sides, the mathematical features of certain infinite objects are not always distinguished in Leibniz's philosophical discussion of the infinite. For this reason, the next chapter examines case studies to show exactly how Leibniz applies the infinite to his mathematics, and the nuanced ways various results and concerns emerge from extending different kinds of mathematical objects to the realm of the infinite. As we will see, Leibniz sees logical impossibilities in extending some types of mathematical objects to the domain of the infinite, but infinitesimals do not give rise to these same problems and are spoken of in more laudable terms than infinite numbers and other mathematically impossible entities.

Chapter 2 : Applications of the Infinite in Leibniz's Mathematics

The phrase "the infinite in Leibniz's mathematics" has a tidy ring to it, but one purpose of this chapter is to show that there is not a unified application of infinity that neatly captures all of Leibniz's mathematical work. Instead of a single kind of infinite object or singular use for such objects, one finds a broad variety of non-finite entities within Leibniz's mathematics, each with different properties, applications, and philosophical entanglements. There are of course some common characteristics among these different uses of the infinite, such as the requirement that any posited entity still respects the maxim that a whole must be greater than its parts. For some proposed infinite entities, this axiom is used to deny the coherence of the mathematical object in question. However, this same axiom is also used for other proposed infinite entities to derive the properties they would have, even if Leibniz is agnostic or denialist about the existence of such objects in the created world. Thus, while there are commonalities among infinite entities in Leibniz's mathematics, accounts of this area of Leibniz's work must be careful to not to impose a faulty sense of unity onto Leibniz's multifaceted approach to the infinite within his mathematics.

To take one concrete example of how Leibniz's mathematical treatment of the infinite is relatively disunited, this chapter shows how he takes a much firmer stance against the very coherence of infinite numbers than he does against infinitely long lines. Furthermore, Leibniz sometimes distinguishes between "bounded" and "unbounded" infinite lines, and each of these receive distinct treatments, with the former type of infinitely long line giving rise to a proposed arithmetic of the infinite to which the latter cannot be subjected. In this section, I go through the properties of five distinct applications of the infinite within Leibniz's pure mathematics, noting the similarities and differences among each area. These five applications are: (1) infinite totalities/numbers; (2) the composition of the continuum; (3) infinite series; (4) unbounded infinite lines; and (5) bounded infinite lines as well as infinitesimals. This taxonomy is not meant to be exhaustive (*e.g*., it does not include Leibniz's discussion of points "at infinity"), and much of it centers around Leibniz's early work in mathematics in the 1670s, when he was developing the rudiments of the infinitesimal calculus. When constructing this list, I chose to focus primarily on this early period of Leibniz's career because this is where one sees Leibniz freely postulating the existence of new types of infinitary mathematical entities and then speculatively deriving facts about these proposed objects. In some cases, such as unbounded infinite lines, the objects in question have properties that are in stark contrast to their finite counterparts, but these differences are initially treated as oddities, rather than absolute contradictions that demonstrate their utter impossibility. In other cases, logical contradictions are immediately derived from an infinitary object's supposed existence, with infinite number serving as the paradigm case of an absolutely impossible infinite entity. After proceeding through these five cases, I present difficulties in interpreting Leibniz's treatment of bounded infinite lines and infinitesimals.

On its own, the analysis of each of these different types serves as a study in mathematical practice. We see how Leibniz posits new kinds of mathematical objects and then derives various propositions in order to find which kinds of objects can be projected into the realm of the infinite without contradicting logical truths. We also see how he uses heuristic principles like The Law of Continuity to experiment with transferring as many properties as possible from finite objects to their infinitely large or small counterparts (Section 2.4.4). Although they share some commonalities, the manuscripts where Leibniz treats infinite number, the composition of the continuum, infinite series, unbounded lines, and infinite bounded lines are for the most part separate from one another. And in texts where different types of infinite mathematical objects are posited and reasoned about, Leibniz does not automatically transfer properties from one kind of infinite entity to another. Because they are all treated separately, we can see ways in which Leibniz gives a more favorable appraisal of unbounded infinite lines, bounded infinite lines, and infinitesimals than he does infinite number and infinite series conceived of as completed totalities. A large reason for this discrepancy comes from his solution to the Labyrinth of the Continuum (Section 2.2), where Leibniz develops a theory of the way points relate to their line that does not require the line to be conceived of as an infinite aggregate of actually existing points, but rather as an ideal whole that only has the potentially to have infinitely many points. Understanding the distinction between an infinite which is posterior to its actual parts (infinite number and series) and wholes that are prior to their infinitely many potential parts (the two kinds of infinite lines and the relationship between finite lines and points) allows us to track how Leibniz's attitude towards different kinds of infinity arose from his experience of postulating different properties of infinite objects and seeing if any useful and non-contradictory conclusions can be drawn.

In addition to the value of understanding mathematical practice on its own terms, this chapter sets the stage for the progression of this dissertation's overall argument: Leibniz's remarks about the fictionality of infinitesimals does not impugn their status within pure mathematics. The differences in Leibniz's mathematical work on infinite number and infinite and infinitesimal lines show that Leibniz is much more critical of the former than the latter. Their distinction in mathematical practice help us understand the rhetoric Leibniz gives when describing the status of infinitesimals and allows us to contrast them with the kinds of infinity that Leibniz condemns as logically contradictory, the subject of Chapter 3.

There is one issue worth mentioning before moving on to the five case studies. In the previous chapter, I discussed the two different taxonomies of the infinite that Leibniz presented near the beginning and end of his philosophical career. The various applications of the infinite to Leibniz's mathematics are adjacent to, but distinct from, these taxonomies. For instance, the differences between bounded and unbounded lines matches the distinction between the mere infinite and the *Maxima* of Leibniz's earlier taxonomy. Another connection with these taxonomies comes from what is lacking; there appears to be no infinite mathematical entities that Leibniz would categorize as an instance of *Omnia* or the hypercategorematic, the highest levels of the infinite in the earlier and later taxonomies, respectively. But the taxonomies presented in the previous chapter differ from this collection of infinite mathematical entities in the following regards. First, the list in this chapter is one I compiled from the different ways Leibniz uses the infinite within his mathematics, rather than an analysis of taxonomies Leibniz himself presented. Second, the taxonomies of the last chapter were intended by Leibniz to represent an ascent from lower grades of the infinite to the higher grades of infinite that describe the nature of God. On the other hand, the different uses of the infinite presented in the sequel are not intended to be ranked hierarchically, even though Leibniz has a more permissive attitude towards some mathematical uses of infinity than others. Lastly, the taxonomies from the previous chapter were rooted in the ways that one kind of infinity surpassed others (for the earlier taxonomy) and the way parts factor into some infinite whole (in the case of the later taxonomy). However, the categorization in this chapter is based on which type of entity is being extended into the domain of the infinite and focuses on the mathematical consequences of such extensions. Thus, while some overlap exists between the categorizations of the infinite in these two chapters, enough difference exists in aim and content to justify treating them as separate projects. It is now time to turn to these distinct kinds of infinite mathematical objects in Leibniz's thought.

2.1: Infinite Number

The first instance of infinity in Leibniz's mathematics that is worth noting is the type that applies to infinite totalities and the infinite numbers that would be associated with such infinite collections. As mentioned in the previous chapter, Leibniz's early thoughts on the infinite were strongly influenced by Galileo's *Two New Sciences*, a book which Leibniz read and commented upon in 1672. The previous chapter's discussion of Leibniz's reaction to this text focused on Leibniz's comment that we distinguish among infinities in order to see which concepts were capable of maximization, a proposal that motivates the taxonomy Leibniz made in his 1676 comments on Spinoza's "Letter on the Infinite." Leibniz adopts the position that "infinite number" and "infinite totality" are inconsistent concepts, due to their violation of the axiom that the whole is greater than its parts. In this section, I trace the influence that Galileo's writings on infinite numbers had on Leibniz, Leibniz's response to the paradoxes Galileo describes, and the ways that Leibniz bars infinite numbers from his mathematics.

Let us begin by covering Galileo's paradox regarding the square numbers and the integers in greater detail than the description given in the previous chapter. Galileo shows that there are compelling reasons to believe that there are more positive integers than square numbers, but that we are also pulled to accept that there are just as many positive integers as there are square numbers (square numbers those of the form n^2, where n is an integer).[27] Galileo, through his dialogue spokesman Salviati, first argues that there are more positive integers than squares. The reasoning behind this claim is best spelled out through traditional conceptions of wholes and their parts. When it comes to collections of objects, the following is a natural way to define the terms "part" and "whole": if some collection A contains everything that is in the collection B, and A has some additional element(s) that B lacks, then B is a *part* of the *whole* A. Under this definition, one can see that the square numbers form a part of the positive integers, for there are positive integers that are not squares (*e.g.,* 2, 3, 5, *etc*.). And since all squares are positive integers, but not all positive integers are squares, the squares form a part

[27] *Two New Sciences*, pp. 77-80. I will occasionally follow Galileo and refer to square numbers simply as "squares" throughout this section.

of the positive integers. And a fundamental supposition about the relationships between parts and wholes, dating back to Euclid, is that "the whole is greater than the part," a principle I shall refer to as the part-whole axiom/principle in the sequel.[28] Given this definition of parthood, the part-whole relationship between the squares and positive integers, and the axiom that wholes are greater than their parts, the collection of all positive integers can reasonably be said to be greater than the collection of all squares.[29]

After arguing that the two collections are not equal due to the above-mentioned reasoning, Salviati goes on to give a reason for saying the two collections are in fact equal:

> If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.[30]

Or, to use contemporary terminology, there exists a bijection between the numbers and the squares that assigns each positive integer n to n^2 and each n^2 to its positive root n. A tacit principle underlying this conclusion is that there are "just as many" elements of collection A and collection B just in case one can assign every element from A to exactly one element of B and vice versa. Although this principle (now often given as a definition of *equinumerosity*) is not explicitly asserted or argued for by Salviati, its intuitive appeal means its exclusion should not be treated as a mark against Salviati's argument.

[28] De Risi, 2016, notes that there is a controversy among scholars about whether or not this principle was explicitly given by Euclid himself, or if it was an interpolation added in later editions (p. 596). In that article, De Risi lists the principles employed by many historical editions of the *Elements*, and this part/whole principle is present in all but a few editions. Crucially, it is present in Christopher Clavius's 1589 edition of the *Elements,* and this work was arguably the most influential edition of Euclid in the Early Modern Age.
[29] Or to hedge for those who have a Cantorian bent, "reasonable" from the standpoint of pre-set-theoretic intuitions.
[30] Galileo, *Two New Sciences* p. 79. Quoted from page 32 of Crew and de Salvio's translation. Additionally, by "root," Galileo is referring only to the positive root of a perfect square.

We are thus faced with the competing and incompatible claims: (1) that there are more positive integers than squares; and (2) that there are just as many positive integers as squares. Salviati infers from this that:

> So far as I see we can only infer that the totality of all numbers is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes “equal,” “greater,” and “less,” are not applicable to infinite, but only to finite, quantities.[31]

By withholding the comparative attributes of “greater,” “less,” and “equal” from the domain of the infinite, the dilemma is resolved by restricting the scope of both the part-whole axiom and the definition of equinumerosity to finite quantities. Even if the squares are part of the positive integers, one cannot use the part-whole axiom to derive a comparison of magnitude from the initial mereological fact that there are some integers that are not squares. Similarly, even if what we now call a bijection exists between the squares and positive integers, it would not be licit to then infer equality given Galileo’s proposed restriction of this equinumerosity principle to finite collections.

Galileo’s restriction of the part-whole axiom may superficially suggest a similarity to contemporary set theory, where some infinite sets have proper subsets which are not smaller than the initial set.[32] However, this usage is distinct from the proposal that Galileo is offering, for contemporary set theory still allows some infinite sets to be greater than, equal to, or less than others.[33] That is, Galileo is putting forth the claim that the part-whole axiom fails to apply to infinite sets not because there exist some infinite sets that are in fact equal to some of their

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[31] Galileo, ibid. p. 79. Quoted from pp. 32-33 of Crew and de Salvio’s translation. Prior to this remark on the paradox, Galileo does not make reference to “the number of” either the infinite collections in question.
[32] I say “some” infinite sets have this property rather than “all” because this is technically a definition of Dedekind-infinite sets, rather than infinite sets in general. In systems of set theory with the Axiom of Choice (as well as weaker axioms), this definition is provably equivalent to other standard definitions of infinite sets, but not every axiomatization will have this equivalence.
[33] The caveat “some” refers to the fact that in axiomatization of set theory without the Axiom of Choice, there can be two sets which are neither greater, lesser, nor equal to one another.

proper parts; the failure of traditional part-whole relationships comes from terms like "equal" failing to have meaning in the case of the infinite.

After Salviati gives the remark from the block quote above, Sagredo notes the scope of this denial is larger than it initially seems. Not only is any infinite quantity unable to be called greater than, less than, or equal to another infinite quantity, but one cannot even claim that an infinite quantity is greater than a finite one. Sagredo's reasoning is that in the infinite, the ratio of positive integers to squares seems to be 1, by the mapping of each integer n to n^2. But as one considers finite collections of the first n integers, there is a progressively lower ratio of squares to integers as n increases. For example, three of the first ten positive integers are squares (*i.e.,* 1, 4 and 9) making the ratio of squares to non-squares of 3:10. But only ten of the first hundred positive integers are squares (*i.e.,* 1, 4, …, 81, 100), making the ratio 1:10, a ratio smaller than when one considers only the first 10 positive integers. This leads to the conclusion that "the approach to greater and greater numbers means a departure from infinity."[34] When one considers the first positive integer alone, the ratio of squares to non-squares is 1:1, with this ratio decreasing significantly as one considers increasingly larger sets of the first n positive integers. Since the ratio is equal in the case of the number one and infinitely many numbers but becomes increasingly small as one reaches progressively greater numbers, these larger numbers are then said to be "farther" away from infinity than the number one is. Salviati then approves of this argument from Sagredo, and this nod from Galileo's personal representative in the dialogue shows that it is an argument of which he himself approves.

Leibniz encountered this paradox in 1672, and gave a condensed summary of Galileo's result:

> He [Galileo] thinks that that one infinity is not only not greater than another infinity, but not greater than a finite quantity. And the demonstration is worth noting: Among

[34] Galileo, ibid. p. 79. Quoted from p. 33 of Crew and de Salvio's translation.

> numbers there are infinite roots, infinite squares, and infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore, there are as many squares as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinite itself is nothing, i.e., that it is not one and not a whole [*non esse Unum nec totum*]. Or perhaps we should say, distinguishing among infinites, that the most infinite, i.e., all the numbers, is something that implies a contradiction, for if it were a whole it could be understood as made up of all the numbers continuing to infinity, and would be much greater than all the numbers, that is, greater than the greatest number. Or perhaps we should say that one ought not to say anything about the infinite, as a whole, except where there is a demonstration of it.[35]

Leibniz here identifies Galileo's position as claiming that "in the infinite the whole is not greater than the part," and Leibniz rejects this restriction on the scope of this axiom. Leibniz lists a few options in response to this claim. As noted in the previous chapter, the option that we "distinguish among infinities" motivates his 1676 taxonomy and the category of "Maximum," but most noteworthy for our purposes here is the proposal that the infinite "is not one and not a whole." Rather than restricting the universality of the part-whole axiom, Leibniz simply denies that there are infinite aggregates that form a genuine whole. Thus, the part-whole axiom does not fail to apply to infinite wholes as Galileo thought; there simply exist no infinite wholes consisting of discrete parts to which the axiom can either be applied or denied.

Throughout his career, Leibniz continues to give similar arguments that argue for this conclusion at greater length, such as *Pacidius Philalethi*, an unpublished manuscript on motion written in the Fall of 1676.[36] Before discussing the ways that Leibniz's later presentation and solution of the paradox continues the path set forth in 1672, it is worth noting a shift in Leibniz's presentation of Galileo's paradox that one sees within the later presentation in *Pacidius Philalethi*. Leibniz's initial presentation of the paradox in 1672 is framed through the phrase "as many roots as numbers,"[37] without explicitly referring to the "number of" such collections.

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[35] "Notes on Galileo's Two New Sciences" Quoted from *DLC*, p. 9
[36] Original and translation are in *DLC*, pp. 128-221.
[37] "*Et tot sunt quadrata quot radices*" original and translation in *DLC* pp. 8-9.

Leibniz's earlier presentation follows Galileo's own account. Galileo only speaks of comparisons among the collections in question, without then assuming that there would exist some number corresponding to each collection. But in a few years after his first presentation of Galileo's argument, Leibniz now presents the paradox as a comparison between "the number of all squares" and "the number of all numbers," the latter of which is also referred to as "the number of all unities." Although it is a subtle difference, framing this as a comparison between numbers corresponding to each collection represents a shift from directly comparing the collections themselves, for the new framing involves an abstraction to an infinite number that would correspond to each collection.

One reason for Leibniz's shift towards framing Galileo's paradox as a comparison between numbers corresponding to infinite collections of positive integers and squares can be inferred from a 1677 note. There Leibniz defines number as a "whole composed from unities."[38] Since each number is a discrete entity, the collection of n numbers will be a collection of n distinct units, corresponding to the number n. That is, in the finite case, "a whole composed of n unities," "the collection of the first n numbers," and "the number n" can be substituted for each other. In the infinite case, Leibniz states that the number of all numbers is the same as the number of all unities, for a unit added to any number will lead to a new number. The number of all unities is also claimed to be identical to the greatest number. Leibniz is not explicit on why these three concepts are equivalent, but my best reconstruction of his reasoning is that if a collection has n elements (each conceived of as a unity), then the number corresponding to that collection as a whole is n. Extending this feature to the infinite case explains the equivalence between the number of all unities and the number of all numbers. And the reason both the number of all unities and the number of all numbers are equivalent to the concept "greatest

[38] "Numerus *est totum ex unitatibus compositum*." A VI, 4, p. 31. Emphasis in original. Here and in the sequel, if the Latin original of a quote is in the footnote, the translation is my own unless otherwise noted.

number" most likely comes from the fact that any unit added to a collection of *n* elements yields a new number that is greater than the original *n*, namely *n*+1. But since we are postulating a collection that contains *every* number, any *n*+1 must already be in our initial collection. Hence the number corresponding to the collection of all numbers must contain as many unities as possible, which also makes it the greatest possible number (or would, if such an entity were possible). Despite the shift in framing to include concepts such "number of all numbers," rather than bare associations between two types of numbers, the basic paradox remains: there appear to be as many squares as positive integers, even though the former are a proper part of the latter.

Leibniz's solution to this paradox in *Pacidius Philalethi* picks up on two of the options he presented in his 1672 notes on Galileo's *Two New Sciences*: the infinite "is not one and not a whole" and "distinguishing among infinites … all the numbers is something that implies a contradiction." The claim that the collection of all numbers "is not one and not a whole" is reflected in Leibniz's claim that there can be no number corresponding to the aggregate consisting of all the numbers. For if there were a number corresponding to such a collection, the collection would be a whole, given the definition of number as "a whole composed of unities" and the fact that each number is a distinct unit. Conversely, treating the aggregate of all numbers as a united whole would imply the existence of an infinite number, for the same reasons. This also means that the part-whole axiom does not apply to any other infinite collections of discrete elements, such as a soul belonging to the entirety of the physical world, as there is no whole in question to which some proper part could or could not be equal.[39] That

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[39] The connection between calling an infinite aggregate a "whole"/"totality" and the notion of God as the world soul was discussed by Laurence Carlin, Gregory Brown, and Richard Arthur. This exchange centers on the ways that the rejection of infinite collections plays a role in Leibniz's denial of God as the soul of the world. Carlin and Arthur take the position that Leibniz's denial of infinite collections partially explains why Leibniz denied God as the soul of the world, for God would then be a unity containing an infinite multitude of infinite perceptions. Brown argues that Leibniz's denial of the unity of infinite collections was erroneous in light of contemporary set theory, while Carlin and Arthur emphasize that given Leibniz's other commitments, his argument against God as the soul of the world is consistent. (Carlin 1997, Brown

is, rather than taking Galileo's approach that the axiom does not apply because the comparative terms "greater than," "less than," and "equal to" have no meaning when applied to infinite wholes, Leibniz holds that the axiom does not apply because there are no infinite wholes composed of discrete parts whatsoever.

The second prong of the approach laid out in Leibniz's comments on Galileo's paradox, that we ought to distinguish among infinities in order to weed out uses of the infinite that give rise to contradiction, is also present in the response in *Pacidius Philalethi*. For in this dialogue, Leibniz (via his spokesman Pacidius) asserts "I believe it to be the nature of certain notions that they are incapable of perfection and completion, and also of having a greatest of their kind."[40] He goes on to list the fastest motion as another example of an impossible notion involving the infinite. For if we suppose that the rim of a wheel is moving with the fastest motion, we can imagine one of the spokes being extended. In that case, any point on the extended length of this spoke will be moving faster than any portion of our original wheel, a property that holds for all cases of finite rotation that Leibniz drags into the infinite. And this contradicts the hypothesis that the rim of the wheel was already moving with the fastest possible motion. This leads Leibniz to reject the claim that there can be a fastest possible motion, even conceptually. Similarly, Pacidius notes that there is no number of all curves. Since there are as many degrees of analytic curves as natural numbers (Leibniz here ignores analytic equations whose powers are not natural numbers), then the number of all possible degrees would have to be the same as the number of all possible numbers. Since there are multiple curves of any given degree, the number of analytic curves would be larger than the number of all numbers, and this doesn't even count transcendental curves.[41] This proof relies on two unspoken principles. First, if one

1998, Arthur 1999, Brown 2000, Arthur 2001). See Harmer 2014 for a recent discussion of the connection between infinite numbers, infinite wholes, and the possibility of such collections forming genuine unities.

[40] In *DLC* p. 179.

[41] In *DLC* pp. 179-181.

can establish a bijection between some or all parts of a collection and each positive integer, then the former collection cannot be a whole and cannot be assigned a number. This principle is used in the claim that the number of different possible degrees of analytic curves is the same as the number of all numbers, *i.e.*, for each positive integer *n*, there will be a class of curves of degree *n*. Second, if there is a part of a collection that is equivalent to the number of all numbers, then the original collection itself cannot be taken as a whole that is assigned a specific number. This assumption appears in the inference from the impossibility of numbering all the possible degrees that an analytic curve could possess to the impossibility of numbering every analytic curve. Hence there can be no collection consisting of one analytic curve of each possible degree, and because there are infinitely many analytic curves of curves of each degree, and there can therefore be no collection of every analytic curve, nor can there be any collection of these curves plus the transcendental ones. Notably, these kinds of impossibility arguments are specific to the type of maximal entity being refuted; the claim against the possibility of a fastest motion does not appeal to bijections between the number of all numbers, yet the argument against a number of all curves does. This underscores the need to distinguish between the taxonomies Leibniz gives and their mathematical applications, for the nature of the entity in question determines whether or not it is capable of infinitude or maximization. And Leibniz never develops a procedure to determine whether or not a concept will be contradictory when stretched beyond the finite; he simply postulates the existence of such objects and works by trial and error to see if any contradictory conclusions arise as a result of their supposition.

Before moving to the next type of infinite entity in Leibniz's mathematics, it is worth noting that Leibniz's arguments against infinite wholes and the numbers associated with them remains a stable fixture of his thought as he matures. For instance, in a March 11, 1706 letter to Des Bosses, Leibniz says:

> It must be recognized that an infinite aggregate is in fact not one whole, or endowed with magnitude, and that it cannot be enumerated. And, accurately speaking, in place of "infinite number," we should say that more things are present than can be expressed by any number. … Consequently, even if the world were infinite in magnitude, it would not be one whole, nor could God be imagined to be the soul of the world, as certain ancient authors hold, not only because he is the cause of the world, but also because such a world would not be one body, nor could it be regarded as an animal, and so it would have only a verbal unity. It is therefore a form of shorthand when we say "one" where there are more things than can be comprehended in one specifiable whole, and when we describe as a magnitude something that does not have its properties.[42]

Here Leibniz is reaffirming his view that there is no such thing as an infinite number. He still allows that there can be infinitely many objects in some domain. In such cases, there would be more objects than any number can specify, but there is no number corresponding to such collections because they are unable to be conceived of as a completed whole. That is, there is only an infinity of such objects in a syncategorematic sense. Similarly, such collections cannot form more than a mere verbal unity. Hence, we see that in his later years not only does Leibniz remain opposed to the possibility of infinite numbers, but he continues to tie such opposition to the denial that infinite collections form a whole that could then be numbered.

There is one last aspect of infinite number that ought to be addressed. One may think that problems of the infinite, such as Galilean concerns about square numbers and their roots, are only problems because we are finite beings. Perhaps an infinite intellect could comprehend an infinite whole without running into any contradiction. But Leibniz is clear that not even God can assign a number to the collection of all numbers. When a character in *Pacidius Philalethi* asks "So doesn't even God understand the number of all unities?" Leibniz's spokesman Pacidius responds "How do you suppose he understands what is impossible? Does he comprehend a whole which is equal to its part?"[43] For this reason, God is in no better position than we are to reason about the "number of all numbers" and equivalent notions. The part-whole

[42] In *LDB* pp. 31-33.
[43] In *DLC* p. 181.

axiom applies not only to the reasoning that we finite beings perform, but to God's understanding as well. And because infinite totalities and their corresponding numbers would violate this axiom, they are impossible notions. As we shall see, especially in Chapter 3, infinitesimals appear to lack this strong conceptual impossibility, putting them in a relatively better position than infinite numbers.

2.2: The Composition of the Continuum

As seen in the above discussion regarding infinite number, Leibniz believed that infinite collections could neither form a genuine whole nor be assigned a number, on pain of violating the part-whole axiom. But geometry appears to present us with objects that are best understood as wholes that are composed of an infinitude of elements. The most obvious example is that for any line, no matter how small, one cannot claim that there are only a finite number of points on the line.[44] The same is true of the relation between surfaces and the lines that lie on them, as well as solids and their planar cross-sections. When considering these objects, it feels natural to consider every *n* dimensional object as a whole consisting of an infinitude of *n-1* dimensional parts, which may in turn be composed of an infinitude of parts of a lower dimension, until one arrives at indivisible points. But if this were the case, it would violate Leibniz's strong proscriptions against wholes consisting of an infinitude of elements using principles identified in his argument against such collections. One of those principles came about in Leibniz's rejection of the number of all curves: if there are at least as many objects as there are positive integers, then these objects cannot form a genuine whole. It is rather simple to show that the number of points on any finite line would be at least as many as the number of all positive integers. Let the first point identified be the line's midpoint, the second point be the midpoint of one of the divided

[44] Or to be more precise: one cannot claim that a line is composed of a mere finite number of points without massively upending traditional conceptions of points and lines. Famously, George Berkeley argued for this revisionary approach.

segments, the third point be the midpoint of one of the segments created from the previous division, and so on. Even though this process will not exhaust each possible point of the line, it will never end, and hence there are at least as many points on the line as there are positive integers.[45] This quick argument for the impossibility of lines being composed of an infinite aggregate of points relies on reducing this case to the argument against infinite number. But this is not the only argument that poses problems for views of lines as infinite aggregates of points; Leibniz presents additional paradoxes that arise from principles germane to geometry itself. Leibniz refers to these difficulties in clarifying the relationship between a continuous geometric object and its coincident parts of a lower dimension as creating a "Labyrinth of the continuum."[46]

In this section, I briefly summarize some of the paradoxes Leibniz encountered when wandering this labyrinth, as well as the path that eventually led him out of its paradoxical twists and turns. The exit strategy from this labyrinth is to deny that lines are *composed* of points, planar figures of lines, and solids of planar cross-sections. Instead, lines are posited as wholes and the points on them only exist once specified, and similar reasoning applies to higher dimension loci and the lower dimensional objects contained within them. This position is subtle, and I explain it in greater detail at the end of this section. First, I address the paradoxes Leibniz encountered in his initial attempts to characterize the nature of the continuum.

As with the number of all numbers, the paradoxes that emerge from the composition of the continuum are rooted in the part-whole axiom, and Leibniz's firm adherence to that axiom

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[45] This argument shows that there are *at least as many* natural numbers as there are specifiable points on a line. As Georg Cantor would go on to show, the principles of set theory show that there are *more* points on a line than positive integers. However, the above presentation follows principles we have already seen Leibniz affirm, and hence avoids ahistoricity.

[46] Leibniz uses a metaphor of a labyrinth to describe the continuum in a number of places. It occurs in "*De Usu Geometriae*" in 1676 (A VI.3 p. 449). He later repeats the phrase in both the January 21, 1704 letter to De Volder (GP II p. 262) and the July 31, 1709 letter to Des Bosses (in LDB, p. 141). Leibniz also mentions the Labyrinth of the Continuum in his *Theodicy*, where he also refers to the problem of human freedom as another great labyrinth of philosophy (GP VI p. 29). Leibniz believes that the latter labyrinth is a more pressing issue, for all of humanity cares for freedom, whereas only a narrow segment of the population worries themselves about the intricacies of mathematical foundations.

causes him to search for alternate premises to reject when resolving paradoxes. One clear example of the part-whole axiom leading to a paradox in the composition of the continuum comes from the previously-mentioned dialogue *Pacidius Philalethi*, where he presents what is sometimes called the Diagonal Paradox.[47] In this text, Leibniz ultimately blames the paradox on the assumption that lines are infinite aggregates of uniform points, a hypothesis that I dub the *aggregate-composition* principle. In this proof, we will see one additional premise that I shall call the *aggregate-magnitude hypothesis*. The aggregate-magnitude hypothesis holds that two lines are composed of the same (infinite) amount of uniform points if and only if the lines in question are equal in length. Although Leibniz does not explicitly acknowledge and name this second principle, it plays a crucial role in his proof. I follow the spirit of Leibniz's presentation of the paradox in *Pacidius Philalethi*, but I have numbered each step and explicitly drawn out the implicit assumptions that ground this *reductio* argument.[48]

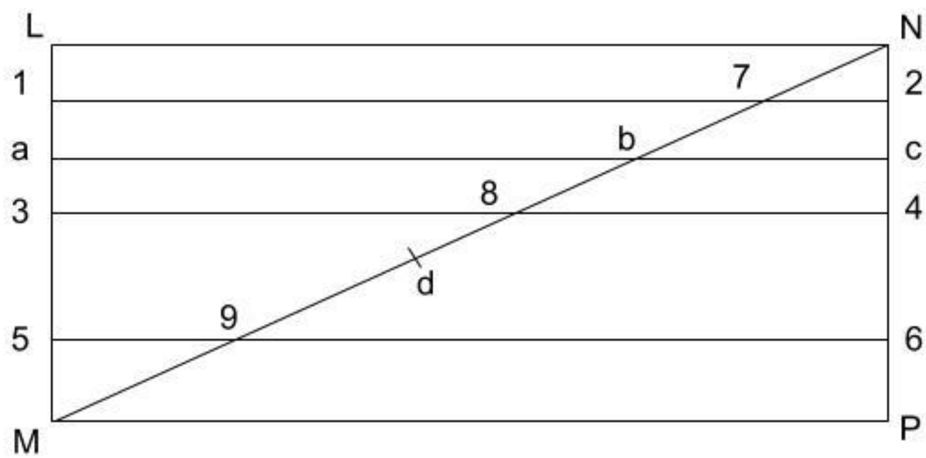

Figure 2.1: The Diagonal Paradox

[47] Not to be confused with Georg Cantor's argument that there are sets that cannot be put in a bijection with the natural numbers, a proof which is commonly known as his Diagonal Argument.
[48] For the original proof, see *DLC* pp. 174-177 (Latin and English translation) or A VI.3 pp. 549-550 (Latin only).

The Diagonal Paradox: Proof

1. Consider the rectangle *LNPM* [Figure 2.1]. Sides *LM* and *NP* are equal in length, so by the aggregate-magnitude principle, each line will be composed from an equal aggregate of points.

2. For each point on *LM*, such as 1, 3, 5, *etc*. draw a straight line parallel to line *LN*. Each line will intersect line *NP* at a point corresponding to the points on *LM*, such as 2, 4, 6, *etc*., respectively.

3. Each of these lines (labeled 1–2, 3–4, 5–6, *etc*.) will then intersect the diagonal *MN* at some other point. For the lines 1–2, 3–4, 5–6, these are points 7, 8, and 9, respectively.

4. Suppose that some point *b* can be taken on *MN* that isn't among the points 7, 8, 9, *etc*. that are intersected by the lines 1–2, 3–4, 5–6 *etc*.. Draw a straight line passing through this point *b* parallel to *LN*. It will intersect line *LM* at point *a*.

5. Point *a* cannot be one of the points 1, 3, 5 *etc*. on *LM*. The reason for this is that because *LM* and *NP* are equal in length, the aggregate-magnitude principle says that they must have the same amount of points. Thus, there will be some point *c* on line *NP* that corresponds to point *a* on *LM*. And line *a*–*c* will intersect the diagonal *MN* at point *b*, making *b* one of the points 7, 8, 9, *etc.*, contrary to our assumption.

6. Hence, we must reject the assumption made in step 4. That is, if lines are drawn from every point of *LM* to their corresponding points in *NP*, then there will be no point on the diagonal *MN* that is also not on the lines *LM* and *NP*. Leibniz (via the character Charinus in the dialogue) infers from this rejection that "It is therefore clear that we must understand there to be just as many points in *LM* and *NP* as in [*MN*], so that if these

lines are mere aggregates of points, the smaller line will be equal to the greater."[49] This is another tacit appeal to the aggregate-magnitude principle, as the equality of these lines is said to follow from each being composed of an equal aggregate of points.

7. Find the point *d* on the diagonal *MN* such that the line *Md* is equal in length to the lines *LM* and *NP*.

8. Since *Md* was specified as being equal to these sides, there will be as many points in *LM* as there are in *Md* by the aggregate-magnitude principle.

9. And since step 6 showed that there are as many points in *LM* (and hence *Md*) as there are in *MN*, "*MN and Md* will also have the same number of points, part and whole alike, which is absurd. Whence it is established that lines are not composed of points."[50] That is, we reject the aggregate-composition principle.

One interesting feature of this proof is that Leibniz could have stopped at line 6. It is an elementary fact that the diagonal of a rectangle is always larger than any of its sides, so the conclusion that these two unequal lines must be equal is sufficiently contradictory to conclude the *reductio* proof. Had Leibniz ended at this stage, he could have held onto the aggregate-composition principle while rejecting the aggregate-magnitude principle. That is, he could claim that while lines are composed of an infinite aggregate of points, the amount of points present within a line does not determine a line's length.[51] But the way in which Leibniz continues his reasoning shows why this position is untenable. At step 6, Leibniz has established that *LM* and

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[49] Translation from *DLC*, p. 177. The original Latin is on p. 176: "*Patet ergo tot necessario intelligi puncta in* LM*, et* NP *quot in* NM *adeoque si hae lineae mera sunt aggregata punctorum, esse lineam minorem aequalem majori."* It may be possible that the claim of equality in the last clause is only a claim about equality between the amount of constituent points, rather than the further claim that they therefore are equal in length. But the wording more naturally suggests my reading that this further claim is being asserted. Additionally, I changed the "*NM*" to "*MN*" for consistency in naming.

[50] *DLC* p. 177.

[51] This would be the approach taken by contemporary mathematics, which separates the cardinality of a set of points from a metric function defined on the set.

MN must have the same amount of points, despite the differences in length of these two lines. But via the consideration of line *Md*, Leibniz has transferred this discrepancy-in-length-but-equality-in-points from the non-overlapping *MN* and *LM* to the whole of *MN* and its proper part *Md*. Since there are points that *MN* has that *Md* lacks, the aggregate of points on *MN* should be greater than *Md*, according to the part-whole axiom. But we have already established that there must be the same amount of points on both lines. Hence even if Leibniz were to give up the aggregate-magnitude principle as a response to step 6, the conclusion to his proof shows that there would still be a conflict with the part-whole axiom. And as we have already seen, rejecting this axiom is a line Leibniz refuses to cross. Therefore, Leibniz identifies the culprit of this paradox as the aggregate-composition principle. And rejecting this principle causes the aggregate-magnitude principle to vanish in a puff of smoke. If lines are not composed through an aggregation of points, then any statements about the relationship between such aggregates and a line's length become moot.

Holding firm to the part-whole axiom, Leibniz rejects the claim that a line is composed of an aggregate of uniform and indivisible points, but there are various ways one can reject this claim. Leibniz's position in *Pacidius Philalethi* is that lines are not *composed* of uniform and indivisible points, a position that remains stable for the rest of his career. Notably, four years prior, in a 1672 manuscript titled "On Minimum and Maximum," Leibniz presents the Diagonal Paradox in a substantially similar way as *Pacidius Philalethi*.[52] But here he denies that lines are composed of *uniform and indivisible* points. That is, he holds that lines are composed of points, but such points can differ in size and can be divided. After repeating the argument that the number of all numbers is an impossible notion because it would have to be equal to one of its parts, Leibniz asserts: "We therefore hold that two things are excluded from the realm of intelligibles: minimum, and maximum; the indivisible, or what is entirely *one*, and *everything*;

[52] A VI.3 p. 97-101. Original and translation is also in *DLC* pp. 8-19.

what lacks parts, and what cannot be part of another."[53] Instead of indivisible entities, Leibniz holds that the continuum contains infinitely small objects that are each divisible and capable of size comparison. For instance, he says that one point can be smaller than another, for the vertex of one angle can be smaller than the vertex of another angle. In fact, one point can be infinitely smaller than another, which is the case when comparing the angle of contact between a line and circle with an angle composed of two straight lines.[54] Leibniz does not spell out how the denial of minima is meant to block the paradox. One likely option is that if we are dealing with infinitely small intervals instead of indivisible points, then the intervals on lines *LM* and *NP* would not be equal to the corresponding intervals on line *MN*. That is, referring again to Figure 2.1, suppose that line 3–5 represents an infinitely small interval on line *LM*, the kind of infinitely small quantity meant to replace points. Drawing parallel lines 3–4 and 5–6 gives the interval 4–6 on line *NP*. Then, we no longer consider a single point of intersection on line *MN*, but the interval 8–9. One can clearly see that when the intervals are finite, line 8–9 will be larger than the 3–4 and 5–6, and this will also be true when the intervals become infinitely small.[55] One could still conclude that under this division, there are as many "points" on line *LM* as *NM*. But it will not follow that these two lines should be equal in length, for the "points" into which each line is composed are unequal and thus the lines they compose can be unequal as well.

Blocking paradoxes regarding the composition of the continuum by positing unequal and divisible points that are capable of composing lines is not the path Leibniz took in *Pacidius Philalethi* and later texts. In *Pacidius Philalethi*, uniform and indivisible points have been

[53] *DLC* pp. 12-13. Emphasis in original.
[54] It ought to be noted that equating points and angles in this way was also a short-lived supposition. For instance, in the April 10 1676 text "Infinite Numbers" (A VI.3 pp. 496-504/*DLC* pp.82-101), Leibniz says "an angle is not the quantity of a point" (translation quoted from *DLC* p.89). One reason for this reversal is that in "Infinite Numbers," Leibniz no longer thinks the notion of an absolute minima (*i.e.* points as traditionally conceived) is unintelligible. In this later text, points are defined as those whose parts are nothing, yet one angle can be a part of another.
[55] The justification for this claim would rely on the "Law of Continuity," which I address in detail in Section 2.5 and in Chapter 3.

rehabilitated as coherent geometric notions. Even though points in their traditional conception are now considered to be respectable mathematical entities, Leibniz still denies that such points *compose* the continuum. Similarly, surfaces and solids are not composed of lines and surfaces, respectively. This position is presented by the character Charinus, and met with approval by Leibniz's spokesman Pacidius:

> [W]e will say that there are no points before they are designated. If a sphere touches a plane, the locus of contact is a point; if a body is intersected by another body, or a surface by another surface, then the locus of intersection is a surface or a line, respectively. But there are no points, lines, or surfaces anywhere else, and in general the only extrema are those made by an act of dividing: nor are there any parts in the continuum before they are produced by a division. But all the divisions that can be made are never in fact made. Rather, the number of possible divisions is no more than the number of possible entities, which coincides with the number of all numbers.[56]

This view is subtle, and it is one that remains relatively stable throughout his career. When Leibniz speaks of "an act of dividing," he is not talking about the divisions that occur when one chops wood. The claim that there are no extrema prior to an "act of dividing" means that there are no physical entities that lack either breadth, depth, length, or a combination of those dimensions. Instead, such entities only come into being during a mental act of division and are thus purely ideal. In later writings, Leibniz will go on to explicitly deny that such extrema form genuine parts of geometric objects, for his later definition of "part" requires that it be homogenous to any claimed whole.[57] In the example from Figure 2.1, this priority of the whole over any of its possible parts blocks the inference that there are "as many" points on lines *LM*

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[56] In *DLC* p. 181.

[57] This view shows up in the text "*Specimen Calculi Coincidentium et Inexistentium*," which the Akademie editors date between Spring 1686 and Early 1687 (A VI.6 pp. 830-845. It appears in translation under the title "A Study in the Logical Calculus" in *L* pp. 371-381). There, Leibniz says the relation between "container" and "contained" is broader than that of "whole" and "part," for the latter has the homogeneity requirement. He explicitly uses this to say that a point is not a part of a line (*L* p. 379). The idea that points are not strictly "parts" of lines due to their inhomogeneity surfaces again in an important note on mathematics, dated sometime after 1714: the "Initia Rerum Mathematicarum Metaphysica" (GM VII pp. 17-29, translated as "The Metaphysical Foundations of Mathematics" in *L* pp. 666-673). There the *inesse* is posited as a generalization of the part-whole relation that does not have the homogeneity requirement that true parts and wholes have.

and *MN*. If points only result from an "act of dividing," one could associate points 7, 8, and 9 on the diagonal *MN* with points 1, 3, and 5 on the side *LM*, respectively. And one could add more parallel lines to create new points of intersection. But we will be unable to fully equate *all* the points of the two lines because as finite beings, we never exhaust all possible divisions of the lines.

This passage in the previous block quote contains another important element of Leibniz's beliefs concerning the continuum that appears to originate during the spring of 1676 and remains with Leibniz throughout his career: "nor are there any parts in the continuum before they are produced by a division." When Leibniz describes the "Labyrinth of the Continuum," he typically emphasizes the relationship between a geometric object and minimum parts of a lower dimension that can be found within that object. However, there is still an issue that arises even when one considers homogenous parts of a greater continuous whole: a line can still be divided into different lines in infinitely many different ways. For instance, a line could be evenly divided into halves, thirds, quarters, *etc*.; it could be divided into an arbitrary amount of even and uneven segments that are commensurable with each other; or it could be divided into parts that are incommensurable with each other (*i.e.,* those whose proportion cannot be expressed by a rational number). And clearly such smaller lines would be parts of the larger line. By claiming that such parts do not exist prior to any specifying acts of division, Leibniz avoids the conclusion that an infinite number of parts actually exist within a given line.[58] Instead, for continuous quantity, "the whole is prior to its parts," a claim that appears in April 1676 and is repeatedly

[58] This story becomes more complicated once one takes the physical world into account. In the actual world, Leibniz thinks that matter is divided into an infinite number of smaller and smaller parts, based on the motion of the parts within parts. But even though actual bodies are divided into infinitely many parts, the claim that "all the divisions that can be made are never in fact made" means that there is some specific pattern of motion into which matter is divided, and a different division would result if some other pattern of motion were to be instituted.

referenced throughout the remainder of his career.[59] In a note from April 15, 1767, Leibniz explicitly contrasts this feature of continuous quantity with numbers, where the parts that are numbered exist prior to the whole.[60] Although infinitely divisible, the fact that lines are not in fact infinitely divided allows one to posit a line without also positing a whole composed of an infinite number of parts. For this reason, the nature of the continuum does not consist in a completed infinity that has been gathered into a whole. It is instead a whole that precedes any parts but contains the potential for indefinitely many parts to be specified. This priority of the whole over its parts is not a possible way to salvage the concept of an infinite number; his definition of number requires wholes formed from a determinate amount of discrete units, unlike the continuous objects of geometry. However, the infinitude of the natural numbers as well as the parts of the continuum are both syncategorematic in that there is never a whole consisting of infinitely many parts. What marks continuous objects as unique is that the whole is prior to indefinitely many parts, rather than being infinitely many existing parts that are unable to be considered as a completed totality.

2.3: Infinite series

Another part of mathematics in which Leibniz wrestled with difficulties in applying infinitary concepts comes from his earliest original work in mathematics: developing techniques to sum convergent infinite series.[61] In 1672, Huygens gave the 26-year-old Leibniz a problem to test his mathematical acuity: to find the sum of an infinite series consisting of the reciprocals of

[59] This claim occurs as quoted in "Infinite Numbers" (quoted from *DLC* p. 97). For an example of Leibniz repeating this claim later, see Leibniz's letter to Des Bosses from July 31, 1709: "In actual things, simples are prior to aggregates; in ideal things, the whole is prior to the part. Neglect of this consideration has produced the labyrinth of the continuum." (Quoted from *LDB* p. 141). In "Infinite Numbers" and other texts from 1676, Leibniz uses the priority of the whole over its parts to characterize entities that are maxima. Later, this is used to classify the distinction between actual and ideal things, and it becomes an important part of his philosophy of mathematics.
[60] From "On Body, Space, and the Continuum" *DLC* pp. 120-121.
[61] See Hoffman 1974 for an excellent analysis of Leibniz's work with infinite series from 1672-1676.

the so-called “triangular numbers.” That is, the series $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots+\frac{2}{n^2+n}+\cdots$, whose sum is expressed using contemporary formal notation by: $\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$. While summation techniques for infinite series are not often ranked among Leibniz’s memorable mathematical achievements, the procedures Leibniz developed to sum the inverse triangular numbers and other infinite series led to insights that lead to the development of his infinitesimal calculus.[62]

Leibniz’s early experiments in solving the problem posed to him by Huygens are primarily located in Series 7, Volume 3 of the Akademie Edition. One of the first strategies Leibniz adopts is to search for ways to express a given series in relation to another series. This can mean discovering that one series is generated through adding, subtracting, multiplying, or dividing the successive terms of another series. Or it could mean one series is the result of multiplying or dividing each term of another series by some constant. Leibniz’s intent in performing these manipulations was to find some pattern among these numbers that would relate the series to some other series whose sum is either already known or can be algebraically manipulated to help find the sum of the original series in question. The notes of Leibniz’s earliest attempts at summing the triangular numbers show Leibniz carrying out all sorts of transformations in order to reach a solution, writing the terms of each series out horizontally or vertically on the page, and then writing the transformed series next to the original. A short note from the fall of 1672 shows this experimental method yielding a discovered connection between the series of the reciprocal triangular numbers, and what is called the harmonic series: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n}+\ldots$ for every natural number *n*.[63] To explain the relationship Leibniz

[62] Bos 1974 presents the technical similarities between the “method of differences” that Leibniz developed to sum infinite series and the techniques used in his early presentations of the differential calculus.
[63] “Differentiae Numerorum Harmonicorum Et Reciprocorum Triangularium,” A VII.3, n. 2, pp. 10-16.

found between these two series, and how this connection led him to find the sum requested by Huygens, I use the following abbreviations and contemporary notation:

For the terms of the various sequences:

H is the Harmonic Sequence:

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ *i.e.*, $a_n = \frac{1}{n}$

ΔH is the difference between successive terms in H:

$\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots, \frac{1}{n} - \frac{1}{n+1}, \ldots$ *i.e.*, $a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

T is the sequence of reciprocal triangular numbers:

$\frac{1}{1}, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \ldots, \frac{2}{n(n+1)}, \ldots$ *i.e.*, $a_n = \frac{2}{n(n+1)}$

For the infinite sums of each series:

S(H) is the sum of the terms of H: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots = \sum_{i=1}^{\infty} \frac{1}{n}$

S(ΔH) is the sum of the terms of ΔH: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{n(n+1)} + \ldots = \sum_{i=1}^{\infty} \frac{1}{n(n+1)}$

S(T) is the sum of the terms of T: $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{2}{n(n+1)} + \ldots = \sum_{i=1}^{\infty} \frac{2}{n(n+1)}$

For partial sums of the first *n* terms of a series:

P_n(H) = The sum of the harmonic series:[64] $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots = \sum_{i=1}^{n} \frac{1}{n}$

P_n(ΔH) = The sum of the series: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{n(n+1)} + \ldots = \sum_{i=1}^{n} \frac{1}{n(n+1)}$

P_n(T) = The sum of the triangular series: $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{2}{n(n+1)} + \ldots = \sum_{i=1}^{n} \frac{2}{n(n+1)}$

[64] As in the case of the infinite series, I listed the first few terms of each series to remind the reader of what the first few numbers in each series will look like. Obviously, if *n* is one of the first few numbers, P_n(H) may not have all of the terms listed here. The same is true for the other partial sums.

After many false starts, Leibniz's method of playing around with the series in order to find usable patterns lead him to consider ΔH, the sequence generated by subtracting the successive terms of the harmonic sequence H from each other. That is, the first term of term of ΔH is $\frac{1}{2} = \frac{1}{1} - \frac{1}{2}$ (the difference between the first two terms of H), and the second term of ΔH is $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$ (the difference between the second and third terms of H). And it turns out that the sequence ΔH can also be obtained by dividing each term of the triangular series T by 2, and T is the series that Leibniz originally wanted to sum. Leibniz then notes that subtracting each term in H by the corresponding term in ΔH results in a sequence identical to H but starting with the second term of H: $\frac{1}{2}$. Written out, this is:

H - ΔH = $(\frac{1}{1} - \frac{1}{2}), (\frac{1}{2} - \frac{1}{6}), (\frac{1}{3} - \frac{1}{12}), \ldots (\frac{1}{n} - \frac{1}{n(n+1)}), \ldots$.= $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}, \ldots$.

Or to use sequence notation:

H - ΔH = the sequence $a_n = \frac{1}{n+1}$

Turning to sum of each series, Leibniz shows that:

S(H) - S(ΔH) = $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{6}) + (\frac{1}{3} - \frac{1}{12}) + \ldots + (\frac{1}{n} - \frac{1}{n(n+1)}) + \ldots$.= S(H) - 1

Subtracting S(H) from both sides of the equation and dividing both sides of the equation by -1 yields S(ΔH) = 1. And because ΔH is the result of dividing each term in T by 2:

S(T) = 2(ΔH) = 2.

And this is the solution to the problem set forth by Huygens.[65]

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[65] A VII.3 contains Leibniz's work on this subject. Items 1, 2, 11, and 35 in this volume inform my presentation here. The first two manuscripts are dated Fall 1672, number 11 comes from early Spring 1673, and number 35 is thought to be from either August or September of 1674.

However, a contemporary reader may immediately notice a crucial flaw in this reasoning. The harmonic series that Leibniz appeals to in this demonstration is divergent; S(H) has no finite sum and thus cannot be treated as a well-defined value that can be subtracted from both sides of an equation. Furthermore, Leibniz himself cannot posit the existence of some infinite number that represents the summation S(H) because as we have already seen, Leibniz's exposure to Galileo's *Two New Sciences* in the Fall of 1672 solidified his belief that the notion of an infinite number is inconsistent. However, an alternate interpretation of these results leads us out of this problem and shows us how Leibniz handled the problem of how to interpret convergent infinite summations.

This interpretation is alluded to in the 1676 note "Infinite Numbers," where Leibniz says "Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like."[66] That is, we focus on the partial sums of these series that are represented by P_n (H), P_n(ΔH), and P_n(T), rather than the infinite sums of S(H), S(ΔH), and S(T). In this case, the "same rule" is referring to the formulas that generate the terms of S(H), S(ΔH), and S(T), such as $a_n = \frac{1}{n}$. And the "error" is the difference between a partial sum of the first *n* terms and the proposed sum that represents the summation of the whole infinite series. In the case of the reciprocal triangular numbers, this means that the claim that its sum is 2 reduces to the claim that P_n(T) will be equal to 2 - *x*, and that *x* becomes as small as one wishes as one picks successively large values for *n*, *i.e.,* adds together more and more of the reciprocal triangular numbers. But before we can conclude this, we need some method to establish that the partial sum of the first *n* triangular numbers will be equal to 2 - *x,* as well as specify how to calculate this *x* for different values of *n*.

[66] *DLC*, pp. 98-99.

Leibniz does not explicitly show how to do this in the case of the reciprocal triangular numbers, but it is not difficult to reconstruct how the argument could go from statements he makes elsewhere. First, we make use of what Richard Arthur calls the Difference Principle in his review of Volume 3, Series 7 of the Akademie Edition: "the sum of the differences is the difference between the first term and the last."[67] In this case, the "differences" refer to $P_n(\Delta H)$, the sum of the first *n* terms of ΔH. The "first term and the last" refer to the terms of P_{n+1} (H), or 1 and $\frac{1}{n+1}$, respectively.[68] By the difference principle, the sum of $P_n(\Delta H)$ will be $1 - \frac{1}{n+1}$. Now, Leibniz would be able to apply the interpretation of infinite summations set out in "Infinite Numbers" to show that any proposed difference between S(ΔH) and 1 is an overestimation. Suppose one were to say that S(ΔH) was actually $\frac{999}{1000}$, meaning there would be an error of $\frac{1}{1000}$ between the originally proposed value of 1 and this newly-claimed value. In that case, one would note that the sum of the 1000 terms of $P_{1000}(\Delta H)$ is equal to $1 - \frac{1}{1001} = \frac{1000}{1001}$, by the Difference Principle. This shows that $\frac{1}{1000}$ was an overestimate of the difference between S(ΔH) and 1 because $\frac{1}{1001}$ is less than the proposed error of $\frac{1}{1000}$. Taking increasingly large values of n will allow us to find a value smaller than any other other proposed error.[69] Hence we can treat S(ΔH) as having a value of 1, even if strictly speaking, there is no such thing as a summation of an infinitude of terms. And S(T) would then have a value of 2, since each term of S(T) is twice that of the corresponding term in S(ΔH).

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[67] Lebiniz makes this claim in "De Progressionibus et de Arithmetica Infinitorum" A vii. 3, p. 95: "*Hinc summa differentiarum est differentia inter terminum primum et ultimum*." Arthur discusses it in Arthur 2006, p. 221.

[68] The reason "*n* + 1" appears in the index P_{n+1} (H) and in the denominator of the corresponding last term of that series is that each term *n* of ΔH is the difference between the *n*-th and *n*+1-th terms of the original series H.

[69] Strictly speaking, we've shown that 1 cannot be an *overestimate* for the value of S(ΔH), which leaves open the possibility that S(ΔH) is greater than 1. But this possibility can be refuted by showing there cannot exist a natural number *n* such that 1 - [1/(n+1)] is greater than 1.

This view of infinite series is a type of syncategorematic approach. In the 1706 taxonomy, a syncategorematic approach to the infinite meant there is never a completed whole composed of an infinitude of parts, yet one is always able to specify larger and larger collections of finite parts. In the case of infinite series, this means that there is never a stage at which there is a complete summation (the whole) of every possible term in the series (the infinitude of parts). Instead, the kind of infinitude involved in convergent infinite series means that, for any summation of the first n terms of a series, one is able to specify a summation of the first m (where $m>n$) terms that is closer to the convergent sum than the sum of the first n terms was. Leibniz's reason for picking a syncategorematic approach that uses partial sums over a categorematic approach in which the sum of a convergent series would be seen as an actually completed summation of infinitely many parts is tied to his rejection of infinite number. The reasoning is as follows. If one views the sum of an infinite series as the sum of an infinite number of terms, Leibniz says that one would have to ascribe a last term to this series. And this term would be a_n for some number n. It cannot be a finite number because then the series would not be completed contrary to our hypothesis, so n would have to be an infinite number. And as we have previously seen, Leibniz thinks such a notion is contradictory. Therefore, we reject the original supposition that the sum of an infinite convergent series is an actual and complete summation of infinitely many terms.

As with Leibniz's belief that an infinite number is impossible and that the whole is prior to the parts for mathematical objects, this syncategorematic interpretation of infinite series is present in the April 1676 text "Infinite Numbers," and remains a fixture of his thought. As an example of his later adherence to this position, in a February 21, 1699 letter to Johann Bernoulli, Leibniz tells Johann Bernoulli: "I concede the infinite plurality of terms, but this plurality itself does not constitute a number or a single whole. It means nothing, in fact, but that there are more terms than can be designated by a number. Just so there is a plurality or a complex of all numbers, but this plurality is not a number or a single whole."[70] That is, one can continually specify more and more finite terms of an infinite series to one's heart's content, but one is unable to claim that there exists a sum of an actual infinitude of terms.

The syncategorematic interpretation of infinite series that Leibniz adopts is similar to that of infinite number inasmuch as they are both interpreted such they are never treated as actual completed entities consisting of infinitely many parts. However, they differ in that one infinite series can be said to be twice another, as in the case of S(ΔH) and S(T), but Leibniz is not inclined to say that one infinite collection is twice as large as another. Another way to categorize this difference is that even though S(H) is divergent, and S(ΔH) is convergent, they are both equally worrisome when considering infinite number, for interpreting them both as completed sums of an infinitude of terms invites the same worry about the implications that there is some last term. And this worry would persist regardless of whether we consider the sums S(H) and S(ΔH) or the unsummed lists of sequences H and ΔH. But series possess the additional properties of being convergent or divergent, and Leibniz must develop tools to distinguish the one kind of series from another, a task that does not arise when considering these objects without performing the operation of addition on them. For this reason, infinite number and infinite series are in fact non-finite mathematical concepts that deserve separate analyses. Having given overviews of both of these concepts, it is time to move on to yet another type of infinite object one encounters in Leibniz's mathematics, the infinitesimal and the bounded infinite line.

2.4: Infinitesimals and bounded infinite lines

The infinite mathematical entities that generated the greatest amount of controversy during Leibniz's life and the following centuries come from the foundations of the calculus. In

[70] Quoted in *L* p. 514.

Leibniz's presentation, there appears to be a reliance on infinitely small lines that are used in his procedures for constructing tangents and calculating areas; these infinitely small lines also go by the name infinitesimals. In this section, I address formal properties of Leibniz's infinitesimals and their reciprocal counterparts: so-called "bounded infinite lines." In his seminal study of Leibniz's technical use of infinitesimals, Bos groups Leibniz's approaches to the infinitesimal calculus into two methods.[71] Sometimes infinitesimals are presented by Leibniz as mere abbreviations for an ancient technique known as the Method of Exhaustion. Other times, Leibniz treats infinitesimals as the result of applying what he calls the Law of Continuity. I give some of Leibniz's justifications using these methods here. Leibniz frequently refers to these infinitesimals as "fictions," and I postpone my analysis of what this claim entails to the next chapter. In this section, the focus is on the mathematical properties of these non-finite lines. I begin with a description of the Method of Indivisibles, a procedure that was in many ways a precursor to the calculus, and how Leibniz's infinitesimals have mathematical properties that are distinct from indivisibles. I then describe some of the mathematical properties of these infinitely small lines and their reciprocal infinitely large counterparts. I end with a description of the Method of Exhaustion and the Law of Continuity as supposed grounds for the methods of the calculus.

2.4.1: Infinitesimals as replacements for indivisibles

As many commentators have noted, the development of the calculus was heavily influenced by the pioneering work of Bonaventura Cavalieri, who developed a technique known as the Method of Indivisibles. This method is a way to prove that two figures have the same area (or that two solids have the volume). A proof using this method has the following form. One starts with two geometric objects and then defines a rectilinear motion such that each point of that motion corresponds to a line in both two-dimensional figures (or to planar figures in both

[71] Bos, 1974, p. 55.

solids). The cross-sections defined by the motion are called the "indivisibles" of the larger figures. If it can be established that at every point of the motion, the indivisibles of the two larger figures are equal in quantity, then the figures as a whole will be equal.[72]

In *Two New Sciences*, Galileo points to a paradox that results from utilizing this method, in an attempt to highlight the dangers of reasoning about the infinite. He takes a cone and a bowl-shaped figure of the same height and shows that they have the same volume without using the Method of Indivisibles. Galileo then defines a motion in accordance with the Method of Indivisibles, so that at every point of the motion, any cross-section of the cone has the same area as the corresponding cross-section of the bowl. Or to be more accurate, the two corresponding cross-sections will be equal at every instant except for the last point of the defined motion; the indivisible section at the apex of the cone is a point, but the indivisible corresponding to the bowl at this stage is a circular line. Given the independently-proven equality between the two solids and the equality of each indivisible up to the pinnacle, Galileo (via his stand-in Salviati) concludes that there are compelling reasons to accept the equality between a point and a circular line, a paradoxical result that he sees as casting doubt on the Method of Indivisibles.[73] In his notes on *Two New Sciences* from 1672, Leibniz says that "These things demonstrate well enough that points are nothing, and that only bodies smaller than any given must be used."[74] Points possess no parts and have no extension, whereas "bodies smaller than any given," i.e. infinitesimals, would be understood to still have parts and extension, even though they are infinitely smaller than any finite quantity. Leibniz again emphasizes this point against points in a 1676 manuscript on the calculus called *De Quadratura*

[72] Mancosu 1996 provides an in-depth account of Cavalieri's Method of Indivisibles on pp. 39-50. He is one of the commentators who highlights the connections between Cavalieri's method and the development of the infinitesimal calculus.
[73] Ibid. pp. 120-122.
[74] *DLC* p. 7

Arithmetica, but this is far from the only place where he is clear on this topic.[75] That is, rather than using the approach of the Method of Indivisibles (performing a quadrature by summing every possible line within a given figure), Leibniz holds that one ought instead to use a method of integration wherein one sums a series of (infinitely many) rectangles of infinitely small width.[76]

One important use for infinitesimals is in tangent construction, and I now present a quick overview of how they function in the infinitesimal calculus in order to demonstrate their utility. In the 1684 text "*Nova Methodus pro Maximis et Minimis*," Leibniz's first actual published work on the calculus, he says "in general, to find a *tangent* is to produce a straight line, which joins two points of the curve having an infinitely small distance, or the produced side of an infinitely-angled polygon, which is equivalent to the *curve* for us."[77] The meaning of this quote and its relation to the infinitesimal method is best illustrated by a quick informal example, and I discuss the justifications for the various claims made later in this section. Consider the parabola expressed by the equation $y = x^2$ [Figure 2.2]. Using the methods of the infinitesimal calculus, $dy = 2xdx$ will be the derivative of this equation, where dx and dy are the differences that two infinitely close points have along the x- and y- axes, respectively. If one picks the point such that $y = x^2 = x = 1$, then one has $dy = 2dx$. And $\frac{dy}{dx} = 2$ represents the slope of the line that connects this point and a point infinitely close to it. Since the lines dx and dy form a right triangle, the Pythagorean theorem can be used to calculate the length of ds, the straight line that connects our two points: $ds = \sqrt{(dy)^2 + (dx)^2}$. Despite stipulating that dx, dy, and ds are infinitely small lines, it is crucially important that they are able to bear different proportions to each other. If we pick a different point on the parabola, say $y = x^2 = 4$, then $x = 2$, and $\frac{dy}{dx} = 4$.

[75] A vii 6, pp. 548-549.
[76] See Knobloch 2002 for examples of how Leibniz uses such infinitesimal rectangles to calculate the area under a curve.
[77] "*In genere,* tangentem *invenire esse rectam ducere, quae duo curave puncta distantiam infinite parvam habentia jungat, seu latus productam polygoni infinitanguli*, *quod nobis* curvae *aequivalet.*" (GM V p. 223, emphases in original).

If infinitely small lines were the same as points, we cannot make sense of the difference between these two cases because one point cannot be smaller than another point.[78] Of course, the presentation I just gave does not explain how we arrived at the equation $dy = 2xdx$, does not argue for the cogency of operating on infinitely small lines, nor explain how we can treat a tangent as an infinitely small line that connects two infinitely close points on a curve. But setting these problems aside shows us an upshot from Leibniz's transition from the Method of Indivisibles to the method of infinitesimals, as he crucially makes use of the fact that infinitely small lines can stand in different proportions to one another.

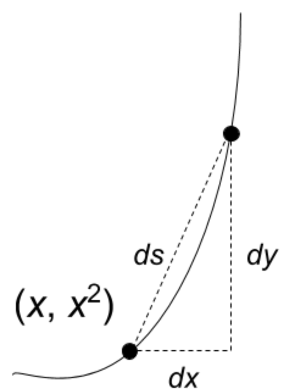

Figure 2.2: An infinitely small triangle

The above considerations show that it is important that we treat an infinitesimal as capable of being twice or four times larger than another, or even stand in a more complicated proportion to other specified infinitesimals, as in the case of ds being equal to the square root of the sum of the squares of two other infinitesimals. However, Leibniz also allows that one infinitesimal can be infinitely larger or smaller than another infinitesimal. This partly stems from

[78] As previously mentioned, Leibniz did experiment with the idea that points can be larger or smaller than other points. See "On Minimum and Maximum: On Bodies and Minds" in *DLC* pp. 8-19, written in 1672 or early 1673. But Leibniz abandoned this approach relatively quickly, and for the rest of his career he favors infinitesimals as the infinitely small entities that can be larger or smaller than one another.

the fact that the operation of differentiation can be applied repeatedly. If $dx$is an infinitesimal quantity, Leibniz says:

> ddx is the element of the element, or the *difference of the differences*, for the quantity dx itself is not always constant, but usually increases or decreases continually. And in the same way one may proceed to $dddx$ or d^3x and so forth.[79]

To return to the case of $y = x^2$, the value of $dy = 2xdx$ is not constant, for it changes with the values of the variable x. If one supposes that the values of dx in this equation are constant, then the rate of change of dy is represented by the equation $ddy = 2(dx)^2$, where ddy is a line infinitely small in comparison to the initial dy.[80] This leads to what Bos refers to as different orders of infinitesimal, with the first differential of a finite quantity being infinitely larger than the second differential, which is infinitely larger than the third, and so on.[81]

2.4.2: Infinitely small lines imply infinitely large lines

Additionally, Leibniz often takes infinitesimal lines to be on the same footing as what he calls “bounded infinite lines,” or *linea infinita interminata*. In an example from a footnote to Leibniz’s 1676 masterwork on the calculus, *De Quadratura Arithmetica*, Leibniz considers a rectangular hyperbola and shows how a bounded infinite line can arise as the reciprocal of an infinitely small line. By the nature of a hyperbola, any chosen ordinate will have a corresponding abscissa such that the rectangle formed from this hyperbola has an area equal to some given constant, no matter what initial ordinate is assumed. In the simplest case, if one considers the hyperbola $y = \frac{1}{x}$, then for any value of x, we have $xy = 1$. Leibniz then says that if one were to take an infinitely small abscissa, *i.e.,* a point on the x-axis infinitesimally far away from the

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[79] From “*Monitum de characteribus algebraicis*.” (GM VII pp. 155-160.) Quoted in translation from Bos 1974 p. 19.

[80] In the Leibnizian calculus, it is important that one specify which variable remains constant when working with higher order derivatives. If one held that *dy* or *ds* remain constant, then we would arrive at a different equation than the one here. See Bos pp. 29-31 for more on this aspect of Leibniz’s calculus.

[81] Bos 1974 pp. 22-23.

origin, the corresponding ordinate would have to be infinitely long in order for the rectangle's area to have the same finite value of all the other rectangles formed from finite abscissae and ordinates.[82] That is, the infinitely small quantity x cannot be multiplied by any finite y to yield an area of 1, but it can for infinite values of y. However, this infinite line y has a starting point on the x-axis and has an endpoint where it touches the hyperbola, and for smaller values of x, the corresponding infinite line will be greater. Hence Leibniz calls this the bounded infinite. This hyperbola example is provided to illustrate a claim that finite quantity is the median between infinitesimal and bounded infinite quantities, for the reciprocal of an infinitesimal quantity will be an infinite quantity, and the product of the two will be a finite quantity.

However, just as there are different orders of infinitesimal, there are differing orders of bounded infinite lines. The following example from a manuscript reproduced by O. Bradley Bassler shows this result.[83] In this text, Leibniz starts with a normal finite line and an infinitesimal line, and then produces another infinitesimal line and two infinite lines from them. He does this by taking AB to be an infinitesimal straight line, and CD to be a fine straight line [Figure 2.3]. Leibniz then gives the mean proportional between these two lines and calls this line EF. That is, he presents the line EF such that $\frac{AB}{EF} = \frac{EF}{CD}$. This new line EF must be infinitely small compared to CD , yet the infinitesimal line AB must be infinitely small with respect to that same EF. For if $\frac{AB}{EF}$ were a finite ratio, then $\frac{EF}{CD}$ would be a non-finite ratio, and the equality $\frac{AB}{EF} = \frac{EF}{CD}$ could not hold. Similarly, if $\frac{EF}{CD}$ were a finite ratio, then EF would be finite, the proportion $\frac{AB}{EF}$ would not be finite, and $\frac{AB}{EF} = \frac{EF}{CD}$ would fail to be true as well. Hence, EF must be a quantity that is both

[82] A VII.6 p. 549.

[83] Bassler, 2008. The manuscript in question is LH XXV, VIII, f. 37, and as of this writing has not yet been published in the Akademie editions. As Bassler explains, there is some controversy over the dating of the text, with André Robinet claiming it was written during around 1702, during the correspondence with Varignon. On the other hand, Enrico Pasini claims it comes from Leibniz's first period of residence in Hanover, placing the text somewhere within the years of 1676 to 1687.

infinitely smaller than the finite CD while also being infinitely larger than the infinitesimal AB. Under this interpretation, neither $\frac{AB}{EF}$ nor $\frac{EF}{CD}$ would be finite ratios, but they would still be equal to each other. Leibniz then tells us to consider the line GH that satisfies the equation $\frac{EF}{CD} = \frac{CD}{GH}$. This line GH must be infinitely larger than the finite line CD, for reasoning analogous to why EF must be infinitely small compared to CD and infinitely large compared to AB. Finally, Leibniz defines line IK through the proportion $\frac{CD}{GH} = \frac{GH}{IK}$. While both GH and IK are infinite lines, GH and IK cannot be in a finite proportion to each other, since $\frac{CD}{GH}$ is not a finite ratio. So IK must be an infinite line that is infinitely longer than the already infinite line GH. In this way, one has differing orders of infinite quantities, each of which can be multiplied by an infinitesimal of a corresponding order to yield a finite quantity.

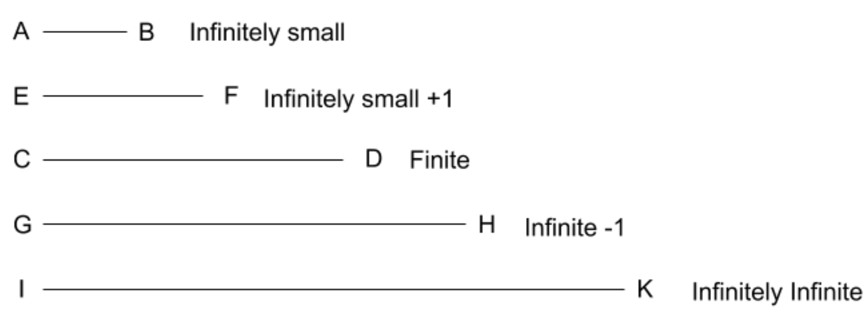

Figure 2.3: Differing orders of infinitesimal and infinite lines[84]

Now that we have seen the ways infinitesimals are (1) placed in a finite proportion to each other (*e.g.,* $\frac{dy}{dx} = 2$); (2) can be infinitely smaller than one another (*e.g.,* dx and ddx); and

[84] This is similar to the diagram presented in Bassler 2008, with one change. In the original diagram AB and EF are both labeled "infinitely small." I chose to label EF as "infinitely small +1" to indicate that it is an infinitesimal line segment that is incomparably large with respect to AB, just as GH was already labeled "infinite -1" to signify its incompatibility with respect to infinite line IK.

(3) can lead to bounded infinite lines, it is time to address Leibniz's justification for their use within the calculus. As mentioned at the start of this section, Leibniz has two general approaches to justify the use of infinitesimals. The first is to claim infinitesimals are just a shorthand for an older technique known as the Method of Exhaustion. The second is to claim that their use is warranted by a principle known as the Law of Continuity. We begin with a description of this first line of justification.

2.4.3: Infinitesimals and the Method of Exhaustion

This section deals with Leibniz's argument concerning the Method of Exhaustion. This method dates back to Euclid and Archimedes; both geometers used this method to evaluate the areas and volumes of certain geometric objects. Proofs using the method of exhaustion tend to follow a particular pattern. First, some method is given for constructing polygons or polyhedrons that increasingly approximate the geometric object(s) in question. Then, derived facts about these approximations are used to show that the quantity in question cannot be greater than a certain quantity. Finally, either the same construction procedure or a new one is used to explain how the approximation cannot be less than the quantity proposed in the first half of the proof. Since it is shown that the object in question cannot be greater or lesser than the proposed quantity, they must be equal to it. Leibniz claims that the dx's and dy's of the calculus can be taken as shorthands for these procedures of taking consistently smaller finite quantities rather than standing for lines that are actually infinitely small. To understand what this means and why one would want a shorthand for this method, an example of the Method of Exhaustion in action is worth examining.

As an example of this method, consider Proposition 10 of Book XII of Euclid's *Elements*. This is the proof that the volume of a cone is equal to one third the volume of a cylinder with the same base and height.[85] The proof begins with the (to be rejected) hypothesis that the volume of the cylinder is greater than thrice the volume of the cone with the same base and height. Labeling O as the volume of the cylinder and V the volume of the cone, this hypothesis is the assertion that $O > 3V$. Assuming this inequality means that there will be some remaining value R such that $O - 3V = R$. The point of this first half of the proof is to inscribe a polygon within the circle and construct a prism that has this polygon as its base and has the same height as our initial cone and cylinder such that the prism's volume is between O and 3V. Calling the volume of this prism P, we want a prism such that $O - P < R$. Since we do not know the value of hypothesized error R, Euclid makes use of Proposition X.1, which says that given some magnitude and a magnitude less than it, if at least half of the original magnitude is subtracted, and at least half of that remainder is subtracted, and so on, eventually there will be a stage where the result of this division yields a magnitude smaller than the original one. In this case, the original magnitude is the difference between the volumes of the cylinder and an inscribed prism ($O - P$), and we want it to be less than the hypothesized difference between the volumes of the cylinder and thrice the cone (R). Euclid does this by considering a series of inscribed polygons that form the bases of prisms of increasingly-many sides, rather than contemplating the properties of a polygon with a fixed number of sides.

The way these polygons are constructed starts with inscribing the square ABCD inside the circle ABCD [Figure 2.4]. Notably, the area of the square is greater than half that of the circle in which it is inscribed, and if a rectangular prism is constructed on square ABCD, its volume will be greater than half the volume of a cylinder with the same height constructed on

[85] This example is also presented in Mancosu 1996 pp. 36-38, although my presentation more closely follows the one given in Heath's edition of the *Elements*. Mancosu's book also contains an excellent discussion of concerns about epistemic status of proofs by contradiction in the Renaissance and the Early Modern Era. Mancosu then traces the ways in which proofs using the infinitary techniques developed in the Seventeenth Century were said to be of greater epistemic virtues than the same results proven by the double *reductio* present in proofs by exhaustion.

the circle ABCD. This fact will be used to ensure the applicability of Proposition X.1. After considering this square, Euclid bisects each of the four arcs of the circle between the points A, B, C, and D, leading to the points E, F, G, and H, that form the regular octagon AEBFCGDH. Then, Euclid shows that the congruent triangles AEB, BFC, CGD, and DHA, are all greater than half the area of the segments of the circle that enclose them (i.e., segments, AEB, BFC, etc.). Euclid also notes that the triangular prisms constructed on these triangles will be greater than half the volume of the portion of the cylinder whose base is the segment of the circle that encloses the triangles. The same process that was used to construct the octagon from the square would then be used to construct a 16-gon and a 32-gon, continuing to any arbitrary 2^n-gons (where *n* is an integer greater than 2). When continuing this process of generating new polygons, the difference between the additional area covered by the polygon and the remaining segments of the circle will always be greater than half the area of the previous uncovered segments of the circle, and this property will also apply to the corresponding prisms that we erect. This finally allows Euclid to apply Proposition X.1, so that regardless of the value of *R*, we are guaranteed that there will be a stage at which the difference between the volume of the cylinder *O* and the prism *P* is less than the value *R*, implying that $O > P > 3V$. Then, Euclid notes that the pyramid with the same base as the prism and the same vertex as the original cone will have a volume of ⅓ *P*. And since $P > 3V$, ⅓ $P > V$, i.e., the pyramid's volume is greater than the cone. But the pyramid would be enclosed within the cone, giving the cone a greater volume than the pyramid. Having arrived at the contradiction, Euclid concludes that the volume of the original cylinder cannot be greater than three times the volume of the cone.

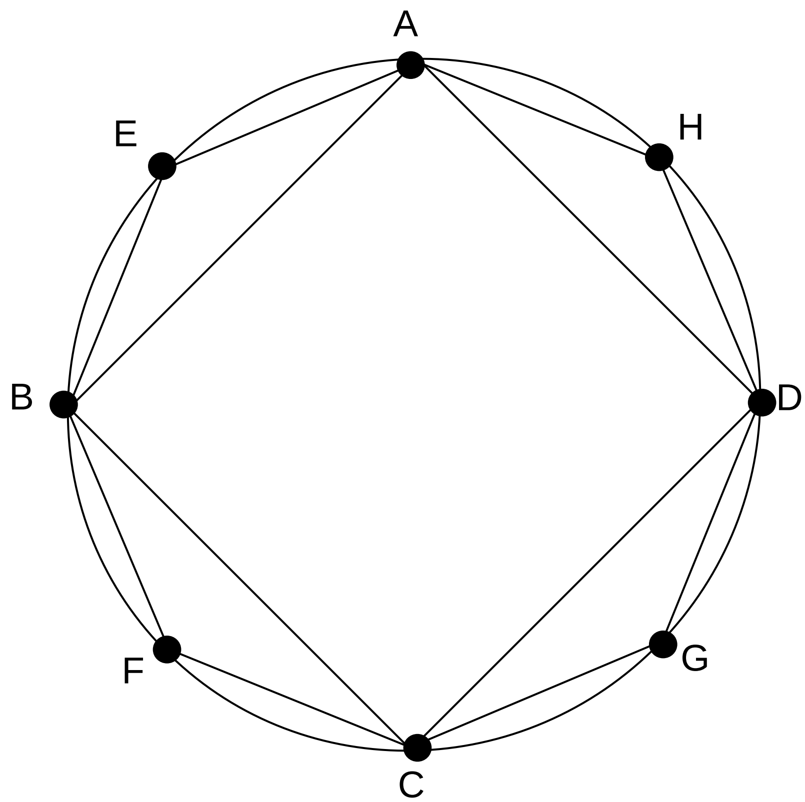

Figure 2.4: The diagram for Euclid XII.10

But the proof does not end here, for it must still be shown that the volume of the cylinder cannot be less than thrice the cone either. This portion of the proof follows a similar strategy. If $O < 3V$, then $V < \frac{1}{3} O$. This implies that there is some remaining volume R such that $V - \frac{1}{3} O = R$. And the proof then proceeds in a manner similar to the previous portion of the proof, only now pyramids are constructed on the successively constructed polygons instead of prisms. Since the difference between each new pyramid and the cone is always greater than half of the difference between the previous pyramid and the cone, we eventually will reach a point where that difference becomes less than R. Calling the volume of this pyramid Π, we have $V > \Pi > \frac{1}{3}O$. And since pyramids have one third the volume of prisms of corresponding height, the volume of a prism on the same base as the pyramid is 3Π. And because $\Pi > \frac{1}{3}O$, it must be the case that $3\Pi > O$. But this inequality says that the volume of the prism is greater than the volume of the cylinder in which it is contained, a contradiction. Hence, we have shown that the volume of the cylinder can be neither greater nor lesser than three times the volume of a cone with the same base and height. Since both these options have been rejected, Euclid concludes that the volumes are in fact equal.[86] What is notable about this method, is that the supposed error *R* between the quantities that are compared is never specified. Instead, it is simply shown that whatever that error may be, the approximation process will reach a stage where the difference between the approximation and the relevant volume becomes less than the supposed error.

With this exhausting detour through the Method of Exhaustion, we see how the method works in practice and why one would be eager to find an alternative proof method. But knowing how this method works illuminates what Leibniz means when he justifies infinitesimals through the method of exhaustion. This line of defense claims that one can replace instances of infinitesimal quantities with a progression of ever-diminishing finite quantities, like the increasingly diminishing area of the circle that was not included in the inscribed polygons in the previous proof. By representing this progression of shrinking finite lines with a fixed symbol such as dx, one has a more efficient presentation of the method of exhaustion than constantly referring to some process that generates this diminishing progression. Leibniz also believes that the infinitesimal notation provides a better way to discover new results than the Method of Exhaustion.[87] The ancient technique may work as a rigorous (although cumbersome) proof method, but it requires one to know ahead of time what the quantities being compared are. For

[86] It should be noted that this example differs from the method as employed by Archimedes. In the hands of Euclid, both hypotheses are rejected by means of increasingly many-sided inscribed polygons. For Archimedes, one hypothesis is rejected via inscribing polygons, and the other by a series of circumscribing polygons. See Heath’s commentary on Proposition XII.2 for more on the difference between both geometers’ methods.

[87] Bos pp. 55-56.

if one were asked to find the volume of a cone, Euclid's proof by Exhaustion would not occur to anyone unless they already knew it was one third the volume of a corresponding cylinder.[88]

2.4.4: Infinitesimals and the Law of Continuity

The second line of reasoning Leibniz uses to support his infinitesimal methods comes from a principle he calls the Law of Continuity. Leibniz has a few different ways in which he phrases this principle. In some cases, it is presented as a claim that if there is some process in which an input diminishes to an arbitrary degree, then the outputs will diminish accordingly:

> When the difference between two instances in a given series or that which is presupposed can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be diminished or become less than any given quantity whatever.[89]

At other times, it is presented through general language that makes no mention as to whether we are dealing with a process of increments or decrements: "In any supposed continuous transition, stopping in any *terminus*, one is allowed to establish a common reasoning, in which the ultimate *terminus* may also be included."[90] This difference in formulation is important because in some cases, Leibniz will invoke the Law of Continuity to justify treating parallel lines as those whose point of intersection is located infinitely far away, and it is difficult to see how one can justify this claim using the formulation of the principle that references inputs and outputs.[91] Another characterization is simple: it is a law "excluding a leap in changing." After formulating the Law in this way in 1695's *Specimen Dynamicum*, Leibniz goes on to list some of the consequences of the law: one can treat rest as a special case of motion, or equality as a

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[88] See Hoffmann 1974 p. 63 *et passim* for remarks about the calculus as an *ars invendi*, *i.e.* a method for discovering new truths rather than simply a new way to rigorously prove what is already known.
[89] From the 1687 text "A Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature Through a Consideration of the Divine Wisdom…" in *L* p. 351.
[90] "*Proposito quocunque transitu continuo in aliquem terminum desinente, liceat ratioci rationem communem instituere, qua ultimus terminus comprehendatur*." Gerhardt 1846, p 40.
[91] *Cum Prodiisset* in Gerhardt 1846, pp. 40-41 is one place where this argument occurs.

case of “vanishing” inequality.[92] Another important way Leibniz characterizes his Law of Continuity is as a bridge between finite and infinite cases: “the rules of the finite are found to succeed in the infinite… And conversely the rules of the infinite apply to the finite.”[93] Regardless of how he phrases the Law of Continuity, Leibniz is clear that it is a principle that justifies the use of infinitesimal reasoning. The next chapter delves further into Leibniz’s use of this principle and its implications for the reality of infinitesimals. But for now, it is worth noting how Leibniz uses it to justify the methods of the infinitesimal calculus. To do so, I return to the case of constructing a line tangent to a parabola, as well as show an example of how Leibniz sees the Law of Continuity as being connected with algebraic reasoning.

One example of using the Law of Continuity to justify a method for constructing a tangent to a parabola comes from the 1710 text *Cum Prodiisset*.[94] In this case AY is the parabola in question, where A is the vertex and AX is the axis tangent to the parabola at the point A, and we will refer to this as the x-axis [Figure 2.5].[95] For any point Y on the parabola, let y be the *ordinate*; *i.e.* the line XY, where X is the point of intersection between the x-axis and the line that passes through Y and is normal to the x-axis. Similarly, let x be the the *abscissa* corresponding to point Y, *i.e.*, the line AX, where X is still the point of intersection described in the previous sentence. If a stands for the *latus rectum*[96] of the parabola, then the equation $x^2 = ay$ describes the relationship between the abscissae and ordinates for all points Y on the parabola.

Leibniz then picks an arbitrary point Y_1 on the parabola, and then lets $x = AX_1$ and $y = X_1 Y_1$. He then picks another point Y_2 on the parabola, and this determines the new point X_2 as

[92] AG p. 133.
[93] Feb 2, 1702 “Letter to Varignon,” in Loemker p. 544.
[94] In Gerhardt 1846, pp 39-50.
[95] I follow Leibniz’s presentation where the *x*-axis is vertical and the *y*-axis is horizontal. Additionally, I adopt his terminology of using “ordinate” to refer to the length along the *x*-axis and “abscissa” to refer to the length along the *y*-axis, even though these terms are usually switched in contemporary use.
[96] The *latus rectum* of a parabola is the line segment that is perpendicular to parabola’s vertex, passes through the parabola’s focus, and whose endpoints are the points this line touches on the parabola itself.

one end of the ordinate and abscissa of Y_2. From point Y_1, Leibniz draws a line perpendicular to the abscissa X_1Y_1, and uses D as the name of the point of intersection between this perpendicular and the abscissa X_2Y_2. He then uses dx to refer to the line segment X_1X_2, the difference between the abcissae of Y_1 and Y_2. Correspondingly, dy refers to the line DY_2, the difference between the ordinates of Y_1 and Y_2 (due to the way the lines are defined, X_1Y_1= X_2D and $X_2Y_2 - X_2D \ = X_2Y_2 - X_1Y_1 = DY_2$). Using these variable assignments, the abscissa and ordinate corresponding to point Y_2 are $x + dx$ and $y + dy$, respectively. Entering these into the equation for the parabola results in the equation: $a(y + dy) = (x + dx)^2 = x^2 + 2dx + dxdx$. Dividing both sides by a gives us $y + dy = \frac{x^2+2dx\ +dxdx}{a}$. Because $ay = x^2$, $y = \frac{x^2}{a}$. Subtracting y from the left side of $y + dy = \frac{x^2+2dx\ +dxdx}{a}$ and$\frac{x^2}{a}$from the right and then dividing the whole equation by dx leads to $\frac{dy}{dx} = \frac{2x+dx}{a}$. This equation expresses what we would now call the slope of the secant Y_1Y_2. Leibniz calls it "a general rule, expressing the ratio of the difference of the ordinates to the difference of the abscissae," and he says this will also be the same as the ratio between the ordinate X_1Y_1 and the sub-secant X_1T, where T is the point of intersection between $Y_1Y_2$and the x-axis.

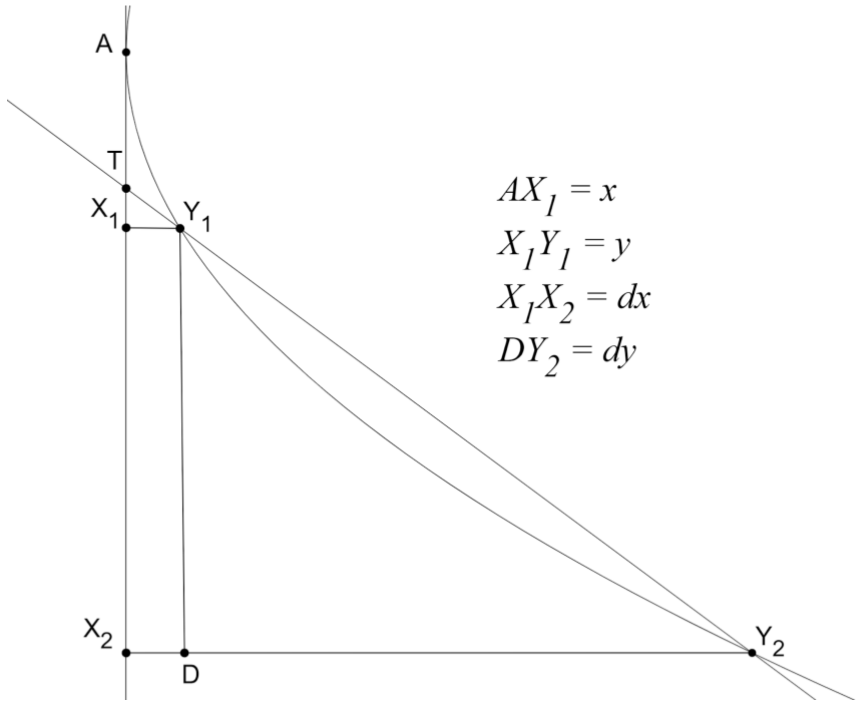

Figure 2.5: Tangent construction in Cum Prodiisset[97]

The equation $\frac{dy}{dx} = \frac{2x+dx}{a}$ describes the slope of any secant, but we were promised a method to construct tangents. To transition from secant to tangent, Leibniz invokes the Law of Continuity, which he postulates at the beginning of *Cum Prodiisset*. In this text, the law is worded: “In any supposed continuous transition, stopping in any *terminus*, one is allowed to establish a common reasoning, in which the ultimate *terminus* may also be included.”[98] In this case, the continuous transition would be points Y_1 and Y_2 moving closer and closer to each other, or as Lebiniz describes it, the ordinate X_2Y_2 will move towards the ordinateX_1Y_1 . The general reasoning is the formula $\frac{dy}{dx} = \frac{2x+dx}{a}$. And the *terminus* is the case in which the two lines coincide, causing dx to shrink to 0. But we are able to imagine a moment *just* before the final

[97] This image is similar to the figure as presented in Gerhardt 1846, p. 44, with three notational differences added for the sake of legibility. First, I replaced Leibniz’s $_1X$, $_2X$, etc. with X_1, X_2, etc.. Second, I added a key to the diagram to remind the reader of the connections between the variable letters and the line segments to which they refer. Third, I added small circles to specify the points of intersection.

[98] “*Proposito quocunque transitu continuo in aliquem terminum desinente, liceat ratioci rationem communem instituere, qua ultimus terminus comprehendatur*.” Gerhardt 1846, p 40.

vanishing or evanescence of dx, and in this case, we can discard the dx in $\frac{2x+dx}{a}$, since $\frac{dx}{a}$ will be a quantity infinitely small in comparison with the rest of the equation (since $\frac{dx}{dy}$ is a proportion between two infinitely small quantities of the same order, it will be finite).[99] In this way, one passes from the finite case, where dx and dy are finite lines, to the non-finite case, where dx and dy are infinitely small. And one uses the same rules for both cases, a move justified by the Law of Continuity.

There is one more justification for Leibniz's use of infinitesimal reasoning that is important to take into consideration. This is Leibniz's brief note "Justification of the Infinitesimal Calculus by that of Ordinary Algebra," published January 1701 in the *Mémoires de Trevoux*, as well as sent to Leibniz's correspondents Pinson and Varignon.[100] The example is meant to show how one can be led to infinitely small quantities by means of a continuous change, and why such quantities retain properties that they had throughout earlier moments in the change. Consider straight lines AX and EY that meet at point C, and let EA and XY be both perpendicular to line AX [Figure 2.6]. Leibniz then lets e be EA, c be AC, x be AX, and y be XY. Due to their constructions, triangles CAE and CXY are similar, and hence $\frac{x-c}{y} = \frac{c}{e}$. For reasons that will be apparent later, Leibniz tells us to assume that angles ECA and XCY are not 45°, i.e., that the ratio $\frac{x-c}{y} = \frac{c}{e}$ is not equal to 1. If one imagines the line EY moving parallel to itself towards point A, the points of intersection between the line EY and the lines EA, AX, and XY change as well, represented on the right-hand figure by points E_2, C_2, and Y_2, respectively. As the line EY moves towards point A in this manner, the values e and c continually decrease, and the values of $x-c$ and y increase. Yet because triangles E_2AC_2 and Y_2XC_2 are similar both to each other and the

[99] Katz and Sherry (2013) say that this is justified by the Law of Transcendental Homogeneity: one can discard terms in an equation that are infinitely smaller than the rest of the terms in that equation.
[100] In Loemker, pp. 545-546.

original triangles EAC and XYC, the equality $\frac{x-c}{y} = \frac{c}{e}$ remains constant. And the ratios of these changing quantities are not only equal to each other at any one stage in the transition but remain the same throughout the transition. Next, Leibniz considers the case where line EY passes through the point A. In this case, e and c vanish, so $\frac{x-c}{y} = \frac{c}{e}$ becomes $\frac{x}{y} = \frac{c}{e}$. Although he does not explicitly cite it, this reasoning is an application of the Transcendental Law of Homogeneity, a law that allows one to discard terms infinitely small in comparison to the other terms of the equation.[101] But at this moment of transition, Leibniz says e and c don't become "absolutely nothing." For then $\frac{c}{e}$ becomes $\frac{0}{0}$, a value which Leibniz claims is equal to 1. And should $\frac{x}{y} = \frac{c}{e} = 1$, that would contradict the earlier hypothesis that angles ECA and XCY are not 45°. Hence in this final case, c and e are distinct from absolute zeros for they bear a proportion to each other, and yet they are treated as zero with respect to the finite quantities x and y. Thus, c and e are treated as infinitely small lines that are capable of being in proportion to one another. This justification may not be satisfactory for many people, and the amphibolous behavior of these infinitely small lines was famously skewered by George Berkeley in *The Analyst*. But it shows how Leibniz uses the Law of Continuity to argue for the validity of infinitesimal reasoning.

[101] See Katz and Sherry 2013 for more on the Transcendental Law of Homogeneity and its use in discarding quantities from equations.

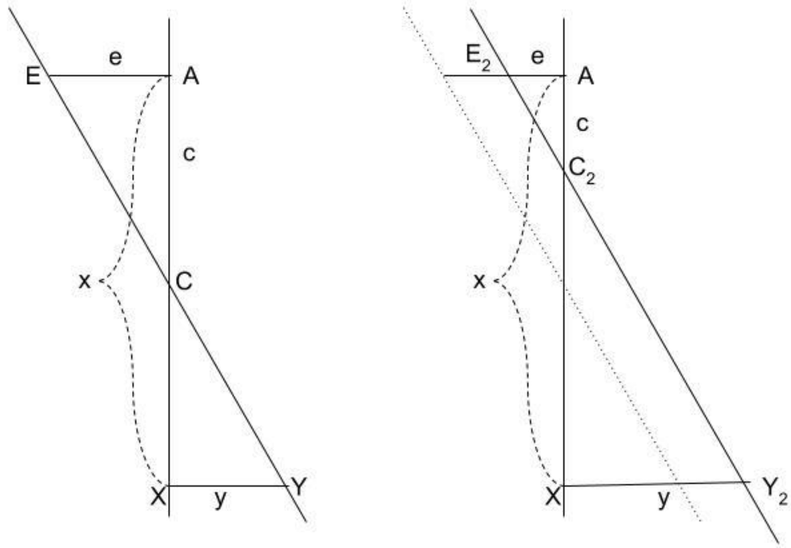

Figure 2.6: Shrinking triangles[102]

The proper way to interpret the difference between Leibniz's justificational approaches stemming from the Method of Exhaustion and the Law of Continuity, and whether or not there are indeed different approaches has been the subject of much discussion in the secondary literature and will be addressed in the next chapter. Furthermore, there is the topic of exactly how one is to interpret infinitesimal lines. It seems that Leibniz rejects their reality, but there is a dispute over when and why he did this, as well as what his rejection actually entails. Before turning to these matters, there is one more infinite object that Leibniz worked with in texts from the mid-1670s: unbounded infinite lines.

[102] Leibniz only has one diagram; the one on the left with a dotted line to represent the movement of line EY. For ease of presentation, I added the second line, as well as labeling the new points of intersection with subscripts.

2.5: Unbounded infinite lines

The bounded infinite lines that are posited as the reciprocals of infinitesimal lines are not what one usually thinks when they hear the phrase "infinite line." Typically, one conceives of infinite lines as those that proceed without any end whatsoever, as opposed to having a supposed endpoint that is an infinite distance away. We have already seen Leibniz's treatment of the bounded infinite lines in conjunction with infinitesimal lines. While Leibniz was reckoning with the geometric properties of such lines in the mid-1670s, he also devoted time to investigating the properties of unbounded infinite lines. In these texts we once again see how the part-whole axiom is at the root of paradoxes of the infinite. These paradoxes are unique to unbounded lines, even though offences against the part-whole axiom occur among other uses of the infinite in Leibniz's mathematics, as documented in the previous sections of this chapter. In 1676, Leibniz works his way through these paradoxes by positing that an infinite line is immovable, but his mature position is to adopt a syncategorematic approach, in which an unbounded infinite line is not treated as constituting a whole, much like the position he adopted towards infinite number in the early 1670s.

An early presentation of problems that arise when considering the properties of unbounded infinite lines comes in a brief note from January 3, 1676: "An Infinite Line is Immovable."[103] Here, Leibniz posits an infinite straight line unbounded on one side: *AB…* which starts at point *A* and proceeds in the direction of point *B* without end, where the ellipses indicate the direction in which it infinitely extends [Figure 2.7].[104] Suppose *AB…* is to be rotated about the fixed endpoint point A, causing it to coincide with the unbounded line *AC… .* Let *DE…* be a

[103] In *DLC* pp. 40-41.

[104] In this text, Leibniz refers to unbounded lines without using ellipses to indicate that it is unbounded on one side (i.e. he writes *AB*). However, in texts from later in 1676, he adds ellipses for this purpose, and I use this later notation here. Additionally, when Leibniz refers to unbounded lines in these notes, his arguments seem to tacitly assume that such lines are straight. In this section, I also use "line" to mean "straight line."

line parallel to *AC*… and let the distance between the two be finite. When *AC…* and *AB*… coincide, the entirety of *AC*… will be below the line *DE*…. But at any intermediate stage of the rotation, such as when the line is in position *AF*…, the line will intersect *DE*… at some point, and there will remain some portion of the line that remains above *DE*…. Since there is no last point of the unbounded line *AB*…, there will be no moment in which *AB*… goes from being partially above *DE*… to fully below it. Leibniz says that even if the angle *AFC* were infinitely small, there would still be an infinite part of the line above the parallel line *DE*…. Leibniz claims that this shows that there must be some point at which the whole of the portion of *AB*… that is above *DE*… would descend below that line at the same time, which he says is an absurdity. Hence, he concludes that an unbounded line is immovable.

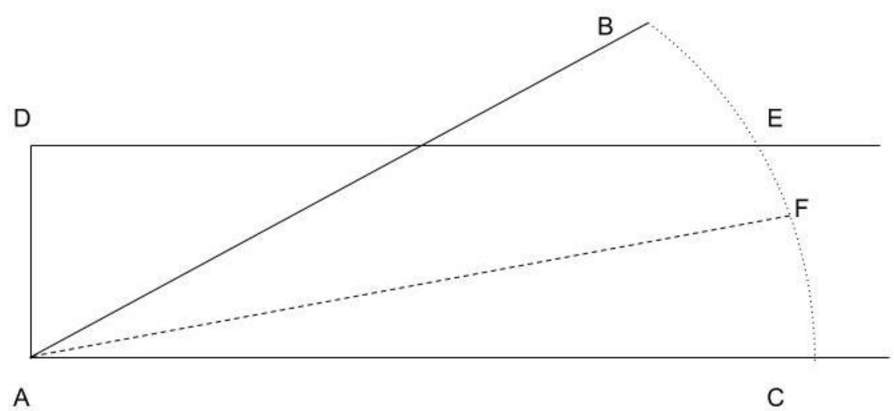

Figure 2.7: First proof against the movability of unbounded lines

There are two aspects of this proof worth noting. First, the contradiction reached in the end appears to tacitly refer to some early instinct that will eventually evolve in Leibniz's Law of Continuity, rather than the part-whole axiom. For we have a continuous transition (the rotation of line *AB*… and the corresponding decrease in the portion of the line that is above *DE*…); a *terminus* (the eventual coincidence of *AB…* and *AC…,* as well as the non-existence of a portion of the line above *DE*…); and the supposition that there is a moment in which a decreasing quantity vanishes (the moment in which the portion of the line above *DE*… goes from having some length to having no length). Second, as we will see with the other paradoxes associated with unbounded infinite lines, such an argument does not hold if one replaces the unbounded infinite line *AB*… with a bounded infinite line. If one supposes that *AB* is a bounded infinite line, then it would have *B* as an endpoint, even though the endpoint would be an infinite distance away from *A*. Since *AB* would have an endpoint, we could suppose some moment at which that point passes from above *DE*… to below it, and since *AB* would end at point *B*, the entirety of the line would be below *DE*… after this moment of transition. This inference was not possible in the case of unbounded line *AB*…, for we explicitly posited that it had no endpoint. I mention this fact to highlight the distinct features that separate unbounded lines from bounded yet infinite lines.

A few months later, Leibniz addresses the issue of unbounded lines again in a note from April 1676: "*Linea Interminata*."[105] Here, part-whole reasoning plays a critical role in leading to a paradox. Consider the unbounded lines *CB*… and *EB*… [Figure 2.8]. There exists a part of the line *CE*… that is not present in *EB*…, namely the bounded line segment *CE*.[106] This means *CB*… is a whole of which *EB*… is a proper part, and by the part-whole axiom, *CB*… must be larger than *EB*…. Now suppose one translated the shorter line *EB*… so that it would start at point *C*. Then one would have two infinite lines starting at point *C* and proceeding without end in the direction of point *B*: the original line *CB*… and some line *CB'...* that is equal to *EB*…. Implicitly invoking a converse of the part-whole axiom, Leibniz concludes that since *CB*… is longer than *CB'...*, there must be some part of the former that is not a part of the latter. But both lines start at the same point *C,* and since they are supposed to be unbounded, there exists no point at which *CB'*… stops and *CB*… continues. So, there can be no portion of *CB…* that is not also a part of *CB'*…, violating the fact that the former is supposed to be longer than the latter.

[105] In *DLC* pp. 64-75.

[106] Leibniz never explicitly says that *CE* or any of the other bounded lines mentioned in this note must be finite, and it seems that the reasoning of this note would hold in cases where the bounded quantities are infinite or infinitesimal.

From this, Leibniz concludes that from one point, only one unbounded line can be drawn in a given direction. It also follows that an unbounded line cannot be translated laterally along itself to a new starting point, for then a line such as *EB…* would have to change its size during its motion to a new point like C, a possibility that Leibniz implicitly rejects.

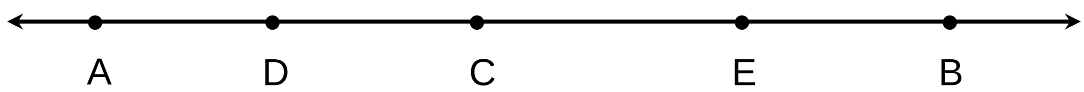

Figure 2.8: Second proof against the movability of unbounded lines

A separate argument shows that a line unbounded on both sides possesses a unique midpoint. Let *…AB…* be a line unbounded on both sides. Let it be divided into two parts, …*AE* and *EB*…, each of which is a line unbounded on one side. As seen from his previous argument against a certain kind of movement of an unbounded line, Leibniz thinks it is possible for two lines unbounded on one side to be unequal in length. Thus, Leibniz is able to suppose that a division is possible such that *…AE* is larger than *EB….* Due to this inequality, Leibniz says there will exist some bounded line segment that can be subtracted from *…AE* to make the resulting line equal to the smaller *EB*…. Leibniz calls this segment *DE*, and the line *…AD* is the part of *…AE* equal to *EB….* Since *DE* is bounded, we can find its midpoint, *C*. Since *DC* = *CE* and *…AD* = *EB…,* one has that *…AD* + *DC* = *CE* + *EB*…, i.e., …*AC* = *CE*…. Hence, *C* divides the doubly-unbounded *… AB…* into two equal lines unbounded on one side. Leibniz then says that the rotation of a line unbounded on both sides is impossible unless the rotation is around its midpoint. For suppose *…AB*… were rotated along the non-midpoint *E*, assuming again that *EB…* < …*AE* . After being rotated 180°, *EB…* would occupy the position of *…AE*. And since there can only be one unbounded line originating in one point and going towards another, *EB…*

and …*AE* would have to be equal when they coincide like this. But it was already supposed that the two lines were unequal. Hence, such rotation around a point other than the midpoint is impossible. Leibniz then considers the bisection of other unbounded quantities, namely unbounded planes and the entire universe, as well as ways in which other unbounded lines could complicate the rotation of line unbounded on both sides along its own midpoint.

These proofs from “Unbounded Lines” rely more explicitly on part-whole relations than the proof against motion found in “An Unbounded Line is Immovable.” Specifically, Leibniz supposes that if two lines unbounded on one side overlap, save for some bounded segment that one has and the other lacks, then the former line must be greater than the larger line. And from this fact, Leibniz allows for comparisons of size among these infinite lines. However, unlike finite lines, there appears to be no way for us to make these comparisons unless the lines already partially coincide. For when Leibniz supposes that when a line unbounded on both sides is divided in a point other than its midpoint, some line segment is assumed to exist which is the difference between the two unbounded lines, and the midpoint of this bounded line allows one to find the midpoint of the original line unbounded on both sides. But Leibniz gives no procedure for determining the length of this bounded segment. When a line unbounded on both sides is divided, relative sizes of the two resulting lines appear to be a brute fact undiscoverable by us. A traditional way of comparing the sizes of lines would be to move them and see whether or not their endpoints coincide. But since Leibniz believes motion is in most cases impossible for unbounded lines, this method will be of no assistance.[107]

Additionally, comparisons of size seem independent of the concept of number in this note. For as we have already seen, Leibniz rejected the notion of infinite number in 1672, and if

[107] Strictly speaking, Leibniz believes that an unbounded line can move parallel to itself under specific conditions. But this will not help us to compare the sizes of unbounded lines that are not parallel to each other.

comparisons among unbounded lines requires infinite numbers, Leibniz would not have begun such investigations into unbounded lines in the first place. Even though Leibniz holds that there is some determinate length that an unbounded line would have to retain during a supposed motion, this length would not be associated with a number, on pain of the whole exercise collapsing into futility. Here again, the priority of the whole of a line over its parts plays a key role, for an unbounded line can be understood as posited without also positing a determinate number of discrete parts. But unlike bounded infinite lines, unbounded lines cannot be subjected to arithmetic operations. In a footnote to his Summer 1676 masterwork on the calculus, *De Quadratura Arithmetica*, Leibniz notes that finite quantity is a middle between infinitesimals and bounded infinite quantities, for an infinitesimal quantity times a bounded infinite quantity will be finite when the two non-finite quantities are of the corresponding order. But bounded quantity is only the middle of *minima* (i.e., points) and *maxima* (i.e., unbounded lines, or "*linea interminata*") through a loose way of speaking:

> For the magnitude of an unbounded line is in no way subjected to the considerations of Geometers, no more than the magnitude of a point. For, just as it is useless for points (let them even be infinite in number[108]) to be added or subtracted from a bounded line, so too a bounded line cannot make or exhaust an unbounded line, no matter how many times it is repeated one after another.[109]

While Leibniz developed an arithmetic of the infinite in *De Quadratura Arithmetica*, this note makes it clear that this arithmetic is meant to only apply to bounded infinite quantities. The fact that one can use algebraic operations that connect infinitesimal, finite, and bounded infinite quantities shows that the difference between such quantities is a difference of degree. But since

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[108] Later remarks in this footnote make it clear that Leibniz is referring to as many points as one wishes, rather than the impossible notion of an actual infinite number.

[109] "Nam *lineae interminatae magnitudo nullo modo Geometricis considerationibus subdita est, non magis quam puncti. Qvemadmodum enim puncta, licet numero infinita, frustra adduntur aut subtrahuntur lineae terminatae, ita linea terminata, qvotcunqve licet vicibus repetita interminatam facere aut exhaurire non potest.*" (A VII.6 p. 549). There are two extant versions of this note, with the differences between the two notes marked by the Akademie. I chose to present the second version.

one cannot use such operations to move from points to finite quantities to unbounded infinite quantities shows that such entities truly differ in kind.

Additionally, it may seem as if Leibniz's claim that "the magnitude of an unbounded line is in no way subjected to the consideration of Geometers" means that he had abandoned the work from "Unbounded Lines." After all, there Leibniz appears to be a geometer inspecting the magnitude of unbounded lines when he judges one to be greater than another. However, I do not think we need to posit that Leibniz changed his mind between April 1676 and the composition of *De Quadratura Arithmetica* in the Summer of 1676. Consider the context of the remark from *De Quadratura Arithmetica*. It is a footnote in which Leibniz remarks upon results of other geometers that showed certain infinite solids/figures have a finite volume/area, such as Evangelista Torricelli's result that a certain hyperbolic solid is equal to a finite cylinder.[110] In this note, Leibniz says that such results are not as miraculous as they seem, for one can replace the infinities involved with bounded infinite quantities, rather than genuinely unbounded figures. He says, "I would be more amazed, if someone were able to reduce the absolutely unbounded space between a curve and a completed asymptote to a finite area."[111] Hence Leibniz's remarks about the magnitude of an unbounded line can be read as a claim that there can be no finite quadrature of an unbounded infinite region. We can still make comparisons of the magnitudes of unbounded lines using congruence, for equal lines unbounded on one end can be superimposed, so they originate at the same point and proceed without end.[112] And such superposition is not possible in the case of unequal unbounded lines. But since one cannot

[110] See Mancosu 1996, pp. 129-139 for a presentation of Torricelli's result and the reaction it received in Early Modern Europe.
[111] "*Magis mirarer, si qvis ipsum spatium absolute interminatum inter curvam atque perfectam asymptoton interjectum; ad finitum spatium re- ducere posset.* "A VII.6 p. 549. Again, I have chosen to reproduce only the second version of the footnote.
[112] Such a superimposition would have to be understood as more of a teleportation than the result of continuous motion, given the arguments against the motion of an unbounded line.

apply the arithmetic of the infinite developed in *De Quadratura Arithmetica* to such lines, their magnitude is inaccessible to geometers.

After 1676, Leibniz appears to retain the conceptual distinction between unbounded and bounded infinite lines. For instance, in his Feb 2, 1702 letter to Varignon, Leibniz says that calculus does not depend on the metaphysical existence of infinitesimal lines or infinite lines. After mentioning infinite lines, Leibniz adds the following remark that was omitted from the letter as sent: "yet with ends; this is important inasmuch as it has seemed to me that the infinite, taken in a rigorous sense, must have its source in the unterminated; otherwise, I see no way of finding an adequate ground for distinguishing it from the finite."[113] Here, Leibniz notes that the infinite lines that are reciprocals of infinitesimals must have ends, i.e., be bounded. The claim that the infinite in a "rigorous sense must originate in the unterminated" reflects the trend towards treating all infinite entities as unbounded in Leibniz's later thought, as Bradley O. Bassler has noted in various articles.[114] This is in part due to difficulties in distinguishing the finite from the infinite if the infinite were treated as bounded, an issue I address at the end of Section 3.4. What this passage seems to indicate is that the notion of an unbounded infinite line has gained priority over the notion of a bounded infinite line as Leibniz's thought develops.

However, even if the notion of an unbounded infinite line occupies a more prominent position than the bounded but infinite in certain areas of Leibniz's thought, he still denies that the latter can be taken as a genuine whole. For he tells Des Bosses on March 11, 1706 that one should say "in place of 'infinite straight line,' that a line is extended beyond any specifiable magnitude, so that there always remains a longer and longer line. It is of the essence of number, of line, and of any whole whatsoever to be bounded."[115] The arguments from 1676

[113] In *L*. p. 543.
[114] Bassler 2008 p. 144 gives this interpretation of the footnote. Bassler 1998 catalogues Leibniz's development towards understanding all infinities as unterminated or indefinite.
[115] *LDB* p. 33.

concerning unbounded lines rely on treating such lines as if they were wholes possessing some fixed length that could be greater or lesser than other unbounded lines. But since Leibniz eventually says that all lines must be bounded, the arguments against the motion of infinite lines fizzle away.

2.6: Conclusion

In this chapter, I have presented five distinct appearances of the non-finite in Leibniz's mathematics: (1) infinite number, (2) the composition of the continuum, (3) Infinite series, (4) infinitely small lines and their bounded infinite reciprocals, and (5) unbounded infinite lines. Leibniz found paradoxes lurking within many of these concepts, leading him to reject a categorematic interpretation of infinite wholes composed of distinct parts. This meant rejecting infinite number because Leibniz explicitly defines number as a whole composed of parts. Similar reasoning also led Leibniz to say that even convergent infinite series cannot be considered as a whole whose parts can be summed, for this would then imply the existence of an infinite number that marks the supposed last term in the infinite series.

Leibniz also rejects the idea that any continuous geometric object, such as a line, is composed of an infinitude of indivisible parts. Interestingly, Leibniz does not transfer over the arguments against infinite number when he makes this rejection. Rather than arguing that the problem with such a view is that it would imply the existence of an infinite *number* of parts, Leibniz gives distinctly geometric arguments for why the continuum cannot be composed from an infinitude of points. Despite the arguments here having a different structure than the ones against infinite number, the axiom that the part must be less than the whole still plays a fundamental role. Additionally, paradoxes involving unbounded infinite lines rely on geometric assumptions, such as the possibility of rotation and the fact that movement should not alter the size of a geometric object. However, the part-whole axiom still played a role in the paradoxes that arise from unbounded infinite lines. In these geometric cases, as well as the arithmetic objects listed above, logical contradictions involving the part-whole axiom led Leibniz to banish these objects from his mathematics.

This leaves the infinitesimals and bounded infinite lines. Unlike the other cases, I have not been able to find an argument that such objects violate the part-whole axiom. In fact, I am not aware of *any* arguments that infinitesimals lead to a logical contradiction.[116] Nevertheless, Leibniz often hedges on the status of infinitesimals, leading many to say that in his final years, he adopted a “fictionalist” stance in which infinitesimals are merely useful for calculations but have no independent status beyond this. The exact details of what Leibniz’s fictionalism entails vary from author to author. In the following chapter, I argue that Leibniz held that there are no infinitesimals in the physical world. Nevertheless, I believe Leibniz’s position was that infinitesimals are perfectly consistent from a *logical* standpoint, and yet nevertheless are impossible in the physical world due to violating metaphysical principles. A focal point of this claim is Leibniz’s distinction between two types of impossibility, only one of which involves contradiction. Under my reading, even if infinitesimals are fictions, the fact that they are consistent, while the other four uses of infinity lead to logical paradoxes, gives infinitesimals a privileged position within Leibniz’s mathematical treatment of the infinite.

[116] Levey, 2008 notes the point that no such proof has yet surfaced in Leibniz’s extant writings (p. 116), and I am unaware of any such arguments have surfaced since Levey made that claim.

Chapter 3 : Interpreting Infinitesimals

In the previous chapter, we saw a variety of different ways that Leibniz pushed traditional finite mathematical objects into the realm of the infinite and the special considerations that arose within each of these extensions beyond the finite. For some objects, such as numbers, paradoxes that clashed against the axiom that a whole is always greater than its parts caused Leibniz to reject the coherence of such forays into the infinite. In the preceding chapter, discussion of infinitesimals and their reciprocal infinite lines was limited to their uses and properties within pure mathematics. In this chapter, I wade into the discussion of how to interpret Leibniz's commitment to the reality of infinitesimals. While there is a consensus that Leibniz held a "fictionalist" stance in which talk of infinitesimals can (and in some highly rigorous contexts should) be replaced by series of diminishing finite quantities, there are many interpretations of what Leibniz means when he refers to infinitesimals as "fictions," and his motivation for adopting this position. In this chapter, I present strong evidence that Leibniz's references to infinitesimals as "fictions" does not imply that such entities are contradictory. I argue that contrary to some commentators, the concept of an infinitely small is a mathematically acceptable notion. Leibniz's arguments for the impossibility of infinitesimals should thus be read as concerning metaphysical considerations that bar them from actual existence, rather than their absolute impossibility. Due to the separation between mathematics and metaphysics, infinitesimals in and of themselves can be consistently introduced as purely geometric objects without entering into a full-blown contradiction.

Section 3.1 reviews how others have parsed Leibniz's fictionalism. Here we see many different interpretations divided along various fault lines. The section makes clear how little agreement there is about what the term "fictionalism" means in relation to Leibniz's views of infinitesimals. I argue that when Leibniz calls infinitesimals "fictions," this claim should be interpreted such that infinitesimals are consistent in and of themselves, while remaining agnostic to many of the other questions that divide interpreters. In Section 3.2, I present four texts in chronological order that appear to show Leibniz vacillating on the consistency of infinitesimals between each one: "*Elementa Nova Matheseos Universalis*"*;* his November 18, 1698 letter to Johann Bernoulli; the Feb 2, 1702 letter to Varignon; and passages from the *New Essays on Human Understanding*. The remarks in these four texts highlight how difficult it is to thread the needle when trying to pin down Leibniz's views of infinitesimals. In Section 3.3, I examine the comparison Leibniz makes between imaginary roots and infinitesimals in the *"Elementa Nova*" and associated texts.[117] Much of this distinction rests on the difference between two types of impossibility that Leibniz identifies, only one of which involves a contradiction. Leibniz says that imaginary numbers are "impossible" in his non-contradictory sense of the term, and textual evidence heavily implies that Leibniz thinks infinitesimals ought to be viewed in the same way. Section 3.4 contains an explanation of how the remarks in Leibniz's correspondence with Bernoulli that reject the possibility of infinitesimals can be read as an argument against the possibility of infinitesimals in the physical world, rather than an argument against their logical consistency within pure mathematics. Finally, Section 3.5 explains how remarks against infinite space and infinitesimals that Leibniz makes in the *New Essays* have a limited scope that prevents mathematical uses of such concepts from being contradictory. This reading hinges on the distinction between real objects (whose parts are prior to any unification into a whole) and ideal objects (where the part precedes the whole), a distinction Leibniz used in his discussion of the continuum (Section 2.2); I argue that Leibniz's remarks in the *New Essays* only apply to the notion of a real infinite space united into a whole, rather than an ideal one. In the end, we see that whatever reservations lead Leibniz to discount infinitesimals from the

[117] Throughout this chapter, I follow Leibniz and use "imaginary roots/numbers" to refer to both pure imaginary numbers of the form *bi* (where *b* is a nonzero real number and *i* is the square root of -1) and complex numbers of the form *a* + *bi* (where both *a* and *b* are both nonzero real numbers and *i* is the same as before).

physical world are not to be read as marks against their consistency and use within pure mathematics.

3.1: Flavors of Fictionalism

As mentioned in Section 2.4, Leibniz was frequently forced to defend the foundations of his infinitesimal calculus from critics. In defending his use of infinitesimals, Leibniz repeatedly refers to non-finite quantities as fictions throughout his career. For instance, in late 1676, the Scholium to Proposition 23 of *De Quadratura Arithmetica* states that even if infinitesimals and infinite quantities do not exist in nature, one can still reason about them, for "it suffices that they be introduced by a fiction."[118] And 30 years later in a March 11, 1706 letter to Des Bosses, Leibniz says "I consider both [the infinitely large and small] to be fictions of the mind, due to abbreviated ways of speaking, which are suitable for calculation, in the way that imaginary roots in algebra are."[119] Even though Leibniz asserts that infinitesimals and the corresponding reciprocal infinite lines are fictitious, it is not explicitly clear what motivates this position, what it means to call a quantity fictitious, and how labeling these quantities as fictitious helps secure the foundations of the calculus. Leibniz obviously intends the label of "fiction" to draw some distinction between such quantities and their finite counterparts, and various commentators have given their interpretation of the work this phrase does for Leibniz. In this section, we examine these accounts before I advance my reading of the fictionality of infinitesimals. I will argue that this label means that although they cannot be found in the phenomenal world, they remain consistent entities to postulate within mathematical contexts.

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[118] The full quote is: "*Quae de infinitis atque infinite parvis huc usque diximus, obscura quibusdam videbuntur, ut omnia nova; sed mediocri meditatione ab unoquoque facile percipientur: qui vero perceperit, fructum agnoscet. Nec refert an tales quantitates sint in rerum natura, sufficit enim fictione introduci, cum loquendi cogitandique, ac proinde inveniendi pariter ac demonstrandi compendia praebeant, ne semper inscriptis vel circumscriptis uti, et ad absurdum ducere, et errorem assignabili quovis minorem ostendere necesse sit*." A VII.6 p. 585.

[119] *LDB* p. 33.

As mentioned both above and in the previous chapter, one can group Leibniz's defenses of the calculus into two different categories. The first is to claim that the methods of the infinitesimal calculus are simply an improvement upon the ancient Method of Exhaustion, a method detailed in Section 2.4.3 of the previous chapter. In place of actual non-finite quantities, we use variable quantities that can be taken to be as small (or large) as desired in order to construct tangents and quadratures. In his later years, Leibniz refers to this as the "method of incomparables." One prominent place in which Leibniz gives a defense through the method of incomparables occurs in the February 2, 1702 letter to Varignon. There, Leibniz gives the ratio of a grain of sand to the entirety of our planet and our planet to the entirety of the cosmos as an example of incomparable magnitudes. These two quantities are actually "comparable" under the traditional definition according to the Archimedean Axiom: two quantities are said to be comparable if there exists some finite natural number *n* such that when repeated *n* times, the smaller quantity will eclipse the larger one. A grain of sand can in theory be multiplied to the point where it surpasses Earth's volume. Although this repetition would literally be astronomical in scope, precise mathematical rigor would not allow us to discard such small finite quantities. For this reason, Leibniz says that these "incomparable" quantities are not taken to be static and fixed; we can take progressively smaller and smaller finite quantities in place of infinitely small quantities (or larger and larger quantities in place of the infinitely large). Like the Method of Exhaustion, these ever-diminishing or increasing quantities are used to show that for any proposed error, there will exist some stage where the diminishing finite quantities show that the estimated error is too large. Leibniz says that such incomparable and variable finite quantities "have the effect of the infinitely small in the rigorous sense," and "it follows from this [method of incomparables] that even if someone refuses to admit infinite and infinitesimal lines in a rigorous metaphysical sense and as real things, he can still use them with confidence as ideal concepts which shorten his reasoning."[120]

The other type of justification comes from the Law of Continuity, a law that often is expressed in highly abstract terms. However, what is common to the different presentations of this principle is the belief that one can apply the general law governing some continuous transition to the *terminus* of that transition. Leibniz uses an example from physics, noting that a body's motion can continuously become slower and slower, approaching a state of absolute rest. But if this state of rest is approached in a continuous way (and in fact Leibniz believes that all changes in motion are continuous processes), Leibniz holds that we can treat a body in rest as a special case of our regular laws of motion; only this motion is now treated as infinitely small rather than absolutely zero. And if our laws governing motion fail to adequately handle states of rest conceived of as being states of infinitely small motions, Leibniz thinks we should take that as evidence of the inadequacy of the proposed laws of motion, rather than as evidence against the Law of Continuity.[121] And while it is not "rigorously true" that when a series approaches a *terminus*, the *terminus* itself is included in the series, Leibniz says we are warranted in using our reasoning to it as if it were included.[122]

When it comes to a philosophical analysis of Leibniz's fictionalism, one helpful framework for how to categorize the various attitudes within the secondary literature comes from Mikhail Katz and David Sherry. They divide views on fictionalism by the ways commentators address the relationship between the Law of Continuity and the Method of Exhaustion/Incomparables. One class of views argues that the Law of Continuity provides a

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[120] Feb 2, 1702 Letter to Varignon. Quoted from Loemker p. 543.
[121] In "Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature Through a Consideration of the Divine Wisdom…" Published July 1687 in *Nouvelles de la République des Lettres*. Translated In Loemker pp. 351-354.
[122] In the 1701 text, "Justification of the Infinitesimal Calculus by Means of Ordinary Algebra" in Loemker, p. 546.

distinct justification for the calculus, separate from the method of incomparable finite quantities. The other approach sees the Law of Continuity as reducible to the method of variable finite quantities.[123] Labeling this latter fictionalism as "logical" or "reductive," Katz and Sherry summarize it as a claim that "[p]ropositions that refer apparently to fictions may be reduced to propositions that refer only to standard mathematical entities."[124] The reductive nature of these interpretations of fictionalism resides in the claim that any apparent references to non-finite quantities are illusory. Under this reading, treating infinitesimals and the infinitely large as referring to a special type of quantity may aid our calculations, but strictly speaking, any seeming reference to a non-finite entity would actually be a reference to a process wherein a sequence of diminishing or increasing quantities is used in the manner of Archimedes. Such interpretations of non-finite quantities are also called "syncategorematic," for it implies that non-finite quantities have no meaning apart from designating these limiting processes that only rely on finite quantities.[125]

One proponent of a reading in which Leibniz adopted the purely syncategorematic interpretation of infinitesimals is Richard Arthur. Consider the example of tangent construction in *Cum Prodiisset*, presented in Section 2.4.4. In that example, an equation for the slope of a secant of a parabola was given. One then moves to consider the case in which the distance between the two points becomes infinitely small and applies the same slope equation that governed the cases where the distance between the points was finite. Then one passes from this secant to the tangent, for the difference between this secant and a genuine tangent is infinitely smaller than any of the other relevant quantities. After presenting the justification by the Law of Continuity, Leibniz says that one could represent the ratio $\frac{dy}{dx}$as a ratio of finite lines that

[123] Katz and Sherry (2013) pp. 567-568 & 587.
[124] Katz and Sherry (2013). p. 587.
[125] See Section 1.2 for an explanation of the term "syncategorematic."

exist in the same ratio as the supposed infinitesimal quantities that form the slope of the secant, calling these finite lines (dx) and (dy). In his presentation of this example, Arthur argues that this shows that the variables dx and dy at no point stand for *bona fide* infinitely small lines, but instead always represent variable finite line segments that can be made arbitrarily small. In the cases in which dx and dy seem to vanish (as when a secant becomes a tangent), the auxiliary lines (dx) and (dy) serve as "finite surrogates" that express the ratio that is approached as the line dx becomes arbitrarily small.[126] Leibniz introduces this method of finite surrogates in *Cum Prodiisset* with the motivation: "But if it is desired to retain dy and dx in the calculation, so that they may represent non-evanescent quantities even in the ultimate case," before describing how to employ such finite proxies.[127]

Under a reading that sees Leibniz having two distinct approaches to the foundations of the calculus, Leibniz's remark before introducing the method of finite proxies would be meant to show how one *could* replace appeals to a ratio between two infinitely small quantities with the same ratio between two finite quantities, but such a substitution is unnecessary. However, under Arthur's reading and most other reductive accounts, the method of finite proxies is not just a technique that we *can* invoke should we be struck by the fancy to do so; it represents what is *actually* going on when one appeals to the Law of Continuity. For him and other proponents of this reading, the Law of Continuity serves as a convenient way to abbreviate the cumbersome process of always specifying the shrinking/increasing finite surrogates. The fictional nature of infinitesimal quantities is thus the fiction of taking this shorthand at face-value: "[i]nfinitesimals are fictions in the sense that the terms designating them can be treated as if they refer to entities incomparably smaller than finite quantities, but really stand for variable finite quantities

[126] Arthur 2013b pp. 564-567.
[127] Translation quoted from Child p. 152. Original Latin is: "*Quod si velimus in calculo retinere dx et dy, ita ut significent quantitates non evanescentes etiam in ultimi casu…*" in Gerhardt 1846 p. 45.

that can be taken as small as desired."[128] The position that apparent mentions of non-finite quantities actually refer to a proposition with a more complex logical value originates with Hidé Ishiguro. She compares Leibniz's approach with Bertrand Russell's analysis of "the present king of France is bald."[129] For Russell, this proposition seems to be a simple attribution of a unary predicate to a subject ("baldness" to "present king of France") but is actually a complex expression with existential and universal quantification that asserts the supposed existence and uniqueness of the "present king of France," alongside the prediction of baldness to this individual. In this way, we see a difference between the patterns in reductive views that Katz and Sherry noted and those presented here. Katz and Sherry seem to be primarily concerned with questions of *method reductionism*, *i.e.,* the possibility of reducing a method that involves infinitesimals (the law of continuity) to methods that only involve manipulations of finite quantities (the method of exhaustion/incomparables). On the other hand, the positions mentioned in this paragraph seem to concern themselves more with a *reference reductionism*, *i.e.,* the position that any use of infinitesimals in a given discourse must be taken as referring to an appropriate series of shrinking values rather than a quantity intended to actually be infinitely smaller than another.

But even if a method reductive reading of Leibniz's fictionalism is a correct reflection of Leibniz's own position, and questions about the consistency of infinitesimals is thus irrelevant to mathematical practice, it seems a stretch to say that *every* appeal Leibniz makes to infinitesimals must be syncategorematic (the reference reductive position). It could be the case that in the context of mathematical reasoning, we must treat infinitesimal terms as referring only to the process of taking variable finite surrogates. But there are cases where Leibniz seems to reference non-finite quantities in ways that become meaningless if we are forced to read them

[128] Arthur 2013, p. 554.
[129] Ishiguro 1990, pp. 97-99.

syncategorematically. For instance, in the opening of the Feb 2, 1702 letter to Varignon, Leibniz says that the mathematical use of non-finite quantities is independent from the metaphysical question of the existence of lines in nature infinitely smaller (or larger) than ordinary lines in a rigorous sense.[130] If we read the references to non-finite lines here as merely a paraphrase of the process of taking ever-diminishing or increasing lines, the passage loses its obvious intended meaning. In this passage and others like it, Leibniz is talking about the possibility of genuine non-finite quantities, not the possibility of finding a series of finite lines larger or smaller than some given line. Syncategorematic readings of fictionalism may give us reason to suppose that non-finite quantities are not intended to be referential in mathematical contexts, but in order to retain plausibility, such views should admit that there are contexts where Leibniz treats them as at least *attempting* to refer to genuinely non-finite quantities. Whether or not such phrases succeed at referring to non-finite quantities is a further question settled by whether or not such entities are possible, but it is a question that must at least be possible to pose. Thus, I think a broad and unqualified reference reduction cannot be what Leibniz means when he calls infinitesimals "fictions."

Arthur's reductive reading of Leibniz's fictionalism is similar to that advanced by Samuel Levey. Levey argues that Leibniz's fictional stance towards infinitesimals was firmly established in the middle of 1676, when Leibniz discovered methods for finding the areas of various conic sections. In these quadrature problems, the curves in question are divided into sections that are approximated by rectangles whose sides are allowed to become as narrow as one wishes. During this process, the sums of the rectangles become better approximations as one considers more and more rectangles with increasingly narrow bases. By treating the area under a curve as the sum of these shrinking rectangles, Leibniz derives a technique that often draws

[130] In Loemker, pp. 542-543.

comparisons with Riemannian integration.[131] And just as in the Method of Exhaustion, if one believes that the sum arrived at by this method differs from the true quadrature by some proposed quantity, one can take the sides of these rectangles to be sufficiently small (but still finite) so that the proposed error is shown to be larger than the summation process allows. And one can continue this process if an even smaller error is then proposed. Levey argues that the infinitesimal calculus is the only reason one would contemplate the existence of non-finite quantities.[132] Once techniques to replace them with variable finite quantities in the manner of the Method of Exhaustion are discovered, that motivation vanishes. According to this interpretation, without any reason to suppose their existence, Leibniz defaults to the position that there are no infinitely small quantities, and the references to them that appear in the calculus remain as convenient abbreviations. While Katz and Sherry present Levey's account of fictionalism as collapsing justifications stemming from the Method of Exhaustion with those that originate in the Law of Continuity, Levey actually grants that the Law of Continuity may serve as a separate and non-reductive justification for some types of mathematical fictions, such as fictional points at infinity that are meant to represent the points of intersection between parallel lines.[133] This justification would be distinct from the method employed to eliminate references to infinitesimals in the calculus. But in the case of infinitesimals and the bounded infinite, justification through a reduction to diminishing finite values takes precedence and appeals to the Law of Continuity only justify a separate class of fictions for which reductive techniques have not been established. For this reason, Levey says it may be more appropriate to speak of Leibniz's "fictionalisms," rather than a single overarching theory of fictions in his mathematics.[134] Despite

[131] Levey 2008, pp. 118-119. Knobloch 2002 also draws comparisons between Leibniz's work in this period and Riemannian integration.
[132] It is worth noting that the so-called "horn angle," or the angle of contact between a circle and straight line, could have lead to another source to motivation the introduction of infinitesimal quantities, since one cannot compare them with angles between two straight lines using finite ratios.
[133] This example will resurface later in the chapter, in the discussion of Figure 3.2.
[134] Levey 2008, p. 131. The case of parallel lines intersecting at a point at infinity is treated in our next section.

the separate justification that the Law of Continuity serves in other contexts, Levey's belief that infinitesimals are to always be interpreted syncategorematically places his views in the reductive camp.

Disagreeing with accounts that merge justifications from the Law of Continuity with the Method of Exhaustion, Douglas Jesseph argues that these methods must be kept separate. In his analysis, claims about the fictional nature of infinitesimals refers to the fact that we do not treat them as having "serious ontological import."[135] Jesseph focuses on the fact that Leibniz does not call non-finite quantities "fictions" without qualification; Leibniz refers to them "useful" or "well-founded" fictions. Jesseph's disagreement with the above-mentioned positions of Arthur and Levey are two-fold. First, Jesseph denies that Leibniz's position on the fictional nature of infinitesimals was firmly in place by mid-1676 and remained a stable position until Leibniz's death. Second, he believes that the Law of Continuity constitutes a separate method for justifying infinitesimal techniques. Jesseph notes that the method Leibniz gives in 1676 to eliminate infinitesimal quantities is presented within the context of the quadrature of conic sections. The method of finite proxies that Leibniz uses to eliminate appeals to non-finite quantities relies on our ability to construct tangents to a given curve. In the case of conic sections, such constructions are easy to produce. But given that Leibniz wanted a broader application for his calculus than these relatively simple conic problems, the arguments of Arthur and Levey cannot explain Leibniz's full stance towards the non-finite quantities in his calculus.[136] While holding that the Law of Continuity is a separate method of justification, Jesseph is pessimistic about its ability to serve as a genuine foundation over and above a method of finite proxies. Leibniz uses examples of how reasoning rooted in the Law of Continuity can deliver us results that we could also attain from independent methods. But he never provides a full-

[135] Jesseph 2008, p. 215.
[136] Jesseph 2011, pp. 195-200.

throated justification that using the Law of Continuity will never lead us into contradiction and error.[137] A rough analogy that Jesseph provides between these two methods and their role in securing the foundations of the calculus is between proof and model theoretic approaches. The method of finite proxies is proof theoretic: it tells us that we can replace the steps in a proof that appeal to non-finite quantities with reasoning that only appeals to finite quantities. The Law of Continuity follows a more model theoretic template, where it is shown that even if the proof cannot omit reference to infinitesimal quantities, the results obtained by these methods that contain no infinitesimal terms will remain true in models with only geometric objects that are finite.[138] Fortunately, Jesseph acknowledges that such analogies must remain at a very general level, given that proof and model theory postdate Leibniz by about 200 years. While he does not think either the reductive approach or the Law of Continuity succeeds in rigorously founding the calculus, Jesseph believes that they serve as distinct approaches that cannot be merged into each other.

Mikhail Katz and David Sherry also believe that the Law of Continuity serves a foundational role logically distinct from approaches that take finite surrogates or appeal to the Method of Exhaustion. They argue that if the Law of Continuity were nothing more than a syncategorematic approach in different clothing, we would be left with the unsolved puzzle of why Leibniz would bother to formulate the law to begin with. If the Law of Continuity were not about actual non-finite cases being governed by the same laws that govern the finite but were instead about a sequence of finite terms alone: “then the law of continuity can only be asserting a tautology: a sequence of standard entities consists of standard entities arranged in a sequence.”[139] Unfortunately, they are not clear on exactly why this consequence follows from a reductive reading of the Law of Continuity. The best reasoning I can come up with is that under

[137] Jesseph 2011, p. 203.
[138] Jesseph 2011, p. 197.
[139] Katz and Sherry 2013, p. 588.

a robust reading of the Law of Continuity, we are introducing mathematical objects that have the same properties as the finite portions of a continuous transformation that approaches a *terminus*, such as infinitely small lines that are in the proportion that is approached continuously during the finite portion of a given transition. But a syncategorematic reading would instead say that as the finite portions of the transition approach some *terminus* continuously, one will always be able to find a further finite stage in the transition in which the *terminus* is closer than before.

In a separate article, Katz and Sherry argue again that Leibniz's fictionalism is not reductive and say that Leibniz's approach to infinitesimals bears a close enough analogy with Hilbert's account of infinite sets to make the term "formalist" an appropriate description. Using a set of criteria given by Abraham Robinson, the architect of non-standard analysis, Katz and Sherry argue that Leibniz's use of infinitesimals meets the essential features of a formalist approach. First, non-finite quantities (as well as imaginary roots) are not themselves the objects of mathematical investigation. Second, mathematics is advanced by employing such ideal elements in calculations. And third, while some entities have the status of being merely fictional, there is a core of mathematical reasoning that is not fictional. For Hilbert, this core was Peano Arithmetic and a rudimentary logic, and Katz and Sherry argue that this core for Leibniz are the idealized geometric representations of sensible experience.[140] While this analogy may match some of the ways that Hilbert employed entities he deemed ideal, notably lacking is any mention of a finitistic consistency proof, a core desideratum of Hilbert's formalist program.

Katz and Sherry argue that what the Law of Continuity enables is a type of "concept-stretching," through which rules for existing mathematical concepts are applied to new species of concepts. An example of concept-stretching outside of the calculus that they give is the case of negative quantities. Despite differing from the properties of finite quantities, old axioms such

[140] Katz and Sherry 2012, p. 190.

as “equals added to equals gives equals” still apply.[141] The utility of these stretched concepts resides in their ability to systematize existing knowledge about the original un-stretched domain within mathematics. For example, Gerolamo Cardano’s 1545 text on algebra, *Ars Magna*, gives an equation for finding roots from cubics of the form $x^3 = px + q$:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

If we only allow real quantities in our algebraic manipulations, this formula ceases to apply when $\frac{p^3}{27} > \frac{q^2}{4}$, but allowing imaginary roots allows this rule to apply for all real values of p and q.[142] Additionally, in his own work on algebra, Leibniz says that introducing imaginary roots gives all equations of the same degree the same number of roots, which is useful in making general claims about algebraic formulas.[143] And just as imaginary roots provide unity to equations that would otherwise be grouped and treated separately, the infinitesimals allow a common method for solving the quadratures of both conic sections (which could already be achieved by the method of exhaustion), and “transcendental” curves (ones that cannot be expressed by polynomial equations, such as logarithms). For this reason, Katz and Sherry see Leibniz’s appeal to finite surrogates as superfluous. Just as imaginary roots are accepted without an explicit procedure for replacing them with ordinary real quantities, infinitesimals would have been justified even if Leibniz had not discovered a method for replacing them with series of finite proxy values.[144] While they disagree with reductive accounts that claim infinitesimals in logical actuality refer to certain processes of taking diminishing finite values, Katz and Sherry wind up arguing that Leibniz’s use of fictions is non-referential as well, concluding that “Leibniz's best

[141] Katz and Sherry 2012, p. 186.
[142] Katz and Sherry 2012, pp. 168 & 187.
[143] One place where this occurs is in the 1683 text *Elementa Nova Matheseos Universalis*, A.VI p. 520.
[144] Katz and Sherry (2012), p. 189.

thinking about mathematical fictions is closer to Robinson's idea that infinitary concepts are literally meaningless, in the sense that they ultimately lack referents."[145]

The debate about the independence of the Law of Continuity from the method of finite surrogates and the role that both play in Leibniz's fictionalism is an important one. But I think they detract from an alternate question that I will explore in the rest of this chapter: are infinitesimals in and of themselves consistent? In purely reductive accounts of fictionalism, the question of the consistency of non-finite quantities need not be decided. If the notion of an actual non-finite quantity entails a logical contradiction, there will be no damage done to Leibniz's calculus, since references to such entities are only illusory. In fact, if infinitesimals were inconsistent, Leibniz would have had all the more motivation to eliminate actual references to them in the calculus. Although this view is compatible with the inconsistency of non-finite quantities, it does not entail it. Perhaps non-finite quantities are consistent, and the motivation to replace them with finite proxies has a different origin, perhaps to satisfy an epistemic demand, such as some traditional conception of rigor.

One could also believe that the Law of Continuity serves as a separate, but consistent, foundation for the calculus while still believing that infinitesimals themselves are inconsistent. This is the approach that Katz and Sherry themselves take. In 2013, they present the view that "Leibniz's system for the calculus was free of contradiction."[146] But a year earlier, they argued that "These concepts [of imaginary roots and non-finite quantities], because they contain a contradiction, prevent us from imagining objects in accordance with their definition."[147] They argue that because we can represent such impossible quantities symbolically, we can still reason with them via symbolic manipulation. While they are not clear on how this process is

[145] Katz and Sherry (2012), p. 191.
[146] Katz and Sherry (2013), p. 572.
[147] Katz and Sherry (2012), p. 179.

supposed to work, it appears as if this symbolic representation mixed with their belief that such objects are not themselves the proper *objects* of mathematics stops the contradictory nature of these concepts from throwing the entirety of Leibniz's calculus into contradiction.

In the remainder of this chapter, I argue that contrary to Katz and Sherry, as well as other commentators, Leibniz did not consider non-finite quantities to be contradictory, even though he occasionally calls them impossible.[148] I begin by presenting a series of passages that appear to show Leibniz repeatedly altering his position back and forth on this issue over time in the next section. I then use Section 3.3 to advance my positive account for why infinitesimals should be seen as consistent, using analogies Leibniz makes between the calculus and algebra. I use these analogies to argue that when Leibniz says infinitesimals are fictions, he means that they have no place in the created world but can still be consistently employed as respectable entities within mathematics. I then spend Sections 3.4 and 3.5 showing how two seemingly troubling passages for this reading can be interpreted in ways that do not damage my reading of Leibniz's fictionalism.

3.2 Conflicting Comments

Looking directly at some Leibnizian texts to settle the question of the consistency of infinitesimals, we see a collection of opinions that are difficult to interpret consistently. I choose these texts specifically in part because of the fact that when placed in chronological order, they alternate between support for and against the position that infinitesimals are logically consistent. That is, after highlighting a text in which there is clear evidence that Leibniz argues for the consistency of infinitesimals, I pass over other chronologically-ordered texts that may also

[148] Rabouin 2015b claims that the notion of an infinitesimal is the same as a minimal quantity and hence implies a contradiction (p. 362-363). However, by 1676, Leibniz asserts that minima, i.e. points, are distinct from the idea of an infinitely small line, as this formed his rejection of the method of indivisibles in favor of his own conception of the calculus.

support the consistency of infinitesimals until reaching a text that appears to support the view that infinitesimals are logically inconsistent (and *vice versa*). By cultivating the collection of texts in this way, I hope to stave off an interpretation in which we resolve the tensions between the texts by claiming that Leibniz simply changed his mind about the consistency of infinitesimals, as this requires attributing a large number of changes to Leibniz's thought that he himself does not explicitly acknowledge.

Let us begin with a list of these four texts, with a (+) indicating one in which there is evidence for Leibniz believing that infinitesimals are consistent and (-) indicating one in which Leibniz seems to claim that infinitesimals are logically inconsistent. The first of these texts is 1683's *Elementa Nova Matheseos Universalis* (+), a treatise on technical and philosophical foundations of algebra.[149] Next, is Leibniz's November 18, 1698 letter to Johann Bernoulli (-), a correspondence that is frequently cited in connection with Leibniz's thoughts on the foundations of the infinitesimal calculus.[150] The next positive text is Leibniz's Feb 2, 1702 letter to Varignon (+), a letter frequently cited in connection with Leibniz's beliefs on the foundations of the calculus.[151] Next are passages from the *New Essays on Human Understanding* (-), Leibniz's response to Locke's work of a similar title. In the remainder of this section, I present the relevant evidence from each text, and devote the remaining sections to massaging these tensions into a consistent position that speaks in favor of the consistency of infinitesimals and their corresponding infinite lines within pure mathematics.

Let us begin with a very brief overview of how *Elementa Nova Matheseos Universalis* supports the position that infinitesimals are consistent, with a richer analysis appearing in the

[149] In A VI.4, pp. 513-524. A translation of page 521 appears in De Risi 2016, p. 135.
[150] The correspondence stretches across a few volumes of Series III of the Akademie Edition. This letter in particular can be found in A III.7 pp. 942-947. A partial translation of this letter appears in L, pp. 511-513.
[151] In GM IV, pp 91-95 and in translation in L, pp. 542-544. The letter itself will appear in A III.9, which has not yet been published, although pre-prints are available on the Akademie Edition website. (https://leibnizedition.de/)

next section. In one portion of this text, Leibniz discusses negative numbers and imaginary roots. He says that while these entities are “impossible by accident” (*per accidens impossibiles*), they differ from the absolutely impossible (*impossibiles absolute*). The defining feature of the absolutely impossible is containing a logical contradiction, such as a claim that a whole is equal to one of its parts.[152] Leibniz is very clear in this text that while some may consider concepts like imaginary roots to be contradictory, one who is knowledgeable about mathematics will recognize that this contradiction is only apparent. After clearly defending the view that these other types of mathematical concepts are consistent, Leibniz goes on to say that infinitesimals and infinite quantities arise in the same way, strongly implying that these non-finite quantities are also free from contradiction.

However, in 1698, Leibniz seems to deny the consistency of infinitesimals, telling Johann Bernoulli in a letter from November 18 of that year:

> As concerns infinitesimal terms, it seems to me not only that we cannot penetrate to them but that there are none in nature, that is, that they are not possible. Otherwise, as I have already said, I admit that if I could concede their possibility, I should concede their being.[153]

This appears to be a heavy blow to any interpretation in which infinitesimals are seen as consistent, for Leibniz frequently equates “possibility” with “freedom from contradiction.” Additionally, this is a very puzzling claim coming from Leibniz, for it appears to imply that the non-existence of infinitesimals implies their impossibility, rather than saying that their impossibility is what implies their absence from nature. As we will see in Section 3.4, Leibniz is very clear in other texts that it is not licit to infer impossibility from non-existence. But the fact

[152] To see that Leibniz considers violations of the part-whole axiom as logically contradictory, see 1689’s “*Principia Logico-Metaphysica*” where Leibniz shows how the definitions of “part” and “whole” can be analyzed to establish this axiom (A VI.4 pp 1643-1649 and in translation as “Primary Truths” in AG pp. 30-35).
[153] In L, p. 511.

remains that this exchange seems to be evidence that Leibniz thought infinitesimals were contradictory.

We see Leibniz return to making favorable claims about the consistency of infinitesimals in the Feb 2, 1702 letter to Varignon. There, Leibniz gives remarks that appear to contradict what he said to Bernoulli:

> So it can also be said that infinites and infinitesimals are grounded in such a way that everything in geometry, and even in nature, takes place as if they were perfect realities… And conversely the rules of the infinite apply to the finite, as if there were infinitely small metaphysical beings, although we have no need of them, and the division of matter never does proceed to infinitely small particles. This is because everything is governed by reason; otherwise, there could be no science and no rule, and this would not at all conform with the nature of the sovereign principle.[154]

The language in the first portion of this quote is highly suggestive of the view that non-finite quantities are consistent, for it is difficult to make sense of the claim that nature proceeds "as if" non-finite quantities were "perfect realities" if such entities were in fact contradictory. In addition to being able to act as if these concepts were "perfect realities," the second half of this quote implies that the principles we rely upon to rationally comprehend the world would fail if we were unable to reason "as if" infinitesimals existed. And it would be strange to say the least if the rationality that God instilled in us obliges us to operate in accordance with contradictory hypotheses.

But a reading in which infinitesimals are consistent reading runs into trouble yet again when we look at the following passage from Leibniz's *New Essays*, written in 1703-1705:

> But it would be a mistake to try to suppose an absolute space which is an infinite whole made up of parts. There is no such thing: it is a notion which implies a contradiction; and these infinite wholes, and their opposites the infinitesimals, have no place except in geometrical calculations, just like the imaginary roots of algebra.[155]

[154] In L, p. 544.
[155] *New Essays* II.vii.3

Here we explicitly see the phrase “notion which implies a contradiction” mentioned in connection with non-finite quantities.

One way to account for these different remarks is to say that Leibniz changed his mind about the status of infinitesimals. But as mentioned at the start of this section, the difficulty in this view is that it requires an explanation of not only why Leibniz would change his mind from 1683(+) to 1698(-), but why he then changed his mind again in 1702(+), only to revert once more to the view that infinitesimals are impossible in 1703-1705(-). The next section is devoted to drawing out the position Leibniz stakes in the “*Elementa Nova*” and the comparisons he makes between infinitesimals and imaginary roots in which infinitesimals fall into the category of concepts that are impossible but still consistent. In the section after that, I explain how to interpret Leibniz’s remarks to Bernoulli as consistent with the position identified in the *“Elementa Nova.*” I then briefly describe how distinguishing between mathematical and physical space allows us to read Leibniz’s remarks in the *New Essays* as consistent with my proposed reading of the consistency of infinitesimals.

3.3: A Consistent Reading on Consistency

In this section, I argue that infinitesimals and the kinds of infinite quantity that arise from taking their reciprocals were taken to be consistent by Leibniz. In order to do this, I take seriously the comparisons Leibniz makes between these non-finite quantities and other “imaginary” entities such as the square roots of negative numbers. In his discussion of these imaginary entities, Leibniz is clear that such entities are free of contradiction, and any supposed impossibility that arises from considering their properties is only apparent. Instead, such entities violate either metaphysical principles or conditions of phenomenal experience, rather than the logical principle of non-contradiction. To make sense of this distinction, I appeal to remarks Leibniz makes where he distinguishes between two different types of impossibility. This first is an absolute impossibility, and it reduces to logical contradiction. The other kind is a qualified notion, a notion which Leibniz refers to as *per accidens* impossibility. This kind of impossibility involves violations of metaphysical principles but can be employed safely in mathematical reasoning. I argue that this latter type of impossibility is what applies to infinitesimals. Such a reading can also resolve the tension between Leibniz's statements in favor of the consistency of infinitesimals and his remarks to Bernoulli mentioned in the previous section. Furthermore, I resolve the apparent conflict between my proposed reading and the remarks from the *New Essays* by distinguishing between the role of the infinite in accounts of mathematics and in the physical world. I argue that Leibniz's claim that parts are prior to their whole in physical objects underpins the remarks in the *New Essays*, and that the priority of the whole over its parts in ideal objects allow non-finite quantities to be consistent within pure mathematics. Before showing how to use Leibniz's comparisons between imaginary quantities in algebra with infinitesimals to interpret the remarks he makes elsewhere, we must turn to those comparisons themselves.

To make this argument, I present the distinction between the two types of impossibility that Leibniz presents in the text "*Elementa Nova Matheseos Universalis*." I then present an example from another Leibnizian text in algebra "*Mathesis Universalis*" in order to clarify an analogy Leibniz makes in the "*Elementa Nova*" regarding imaginary numbers. I then present passages from the text "On Freedom and Impossibility" to link remarks Leibniz makes about imaginary numbers with metaphysical considerations about possibility. The chapter ends with an attempt to trace how *per accidens* impossibility arises in the case of infinitesimals and bounded infinite lines.

The first text I identified in the previous section as supporting the view that infinitesimals are consistent is 1683's "*Elementa Nova Matheseos Universalis*." This treatise is devoted specifically to the foundational uses of algebra, and Leibniz believes his presentation of algebra will have a wider range of applicability than the algebra developed by his predecessors. The importance of this text for our purposes is that it presents the two types of impossibility and presents arguments that highly suggest that infinitesimals belong to the non-contradictory type of impossibility. Here Leibniz says that a truly universal mathematics ought to apply to all possible objects of the imagination, and when properly developed in this way, algebra will become a logic of the imagination.[156] Leibniz says that this implies that metaphysics concerning purely intelligible things such as thought, and action is to be excluded from the development of a universal mathematics. This accords with a classification of the sciences given in a brief note written in the same time period as the "*Elementa Nova*."[157] In the classificatory scheme of this other text, logic is listed as the general science, mathematics is presented as the science of imaginable things, metaphysics is referred to as the science of intellectual things, and morals is considered to be the science of affects. Leibniz frequently defends the independence of mathematics from metaphysical controversies, and this note explains why; the two sciences have different domains of inquiry.[158]

To further explain this independence afforded to mathematics, we must understand what it means to call something a science of the imagination. In the "*Elementa Nova*," Leibniz claims imagination is primarily concerned with two notions: quality and quantity. After discussing similarity, congruence, equality, correspondence, and different operations as notions proper to the domain of mathematics, Leibniz notes that some operations cannot actually be carried out under certain conditions. Despite these failures, it is sometimes possible to give an interpretation of these operations in the symbolism of a formal language, or even exhibit natural processes that can be interpreted as alternative representations of such operations. One

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156 "*Mathesis Universalis tradere debet Methodum aliquid exacte determinandi per ea quae sub imaginationem cadunt, sive ut ita dicam Logicam imaginationis*." A VI.4 p. 513.
157 "*De Artis Combinatoriae Usu in Scientia Generali*" A VI.4 pp. 510-512.
158 The start of the Feb 2, 1702 letter to Varignon is another prominent text in which the independence of mathematics from metaphysics is defended.

example Leibniz gives is subtraction from zero. He claims that this is impossible to subtract from nothing, yet this can be exhibited in nature when one owes more than they have.[159] Other such impossibilities are imaginary roots, as well as the division of prime numbers by another integer. But as noted in the previous section, these cases only constitute one type of impossibility. The first kind of impossibility is only a seeming impossibility (*in speciem impossibiles*), and Leibniz's language strongly implies should be read as being equivalent to what he calls in a subsequent sentence the *per accidens impossibiles*.[160] It is difficult to pin down exactly what causes something to be impossible in this way. In the case of imaginary roots, Leibniz says that the impossibility is due to "a lack of the sufficient constitution that is necessary for an intersection," presumably a reference to a claim in the previous paragraph that imaginary roots occur when there is no intersection between a given line and circle.[161] In this passage Leibniz is referring to one particular way of representing roots that can be found in a later text on algebra with a similar name to "*Elementa Nova Matheseos Universalis*": "*Mathesis Universalis*."[162] It is worth presenting this procedure of representing root extraction, for the details of this method provide a greater insight into the frequent comparisons Leibniz makes between imaginary roots and non-finite quantities.

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[159] "*Ita impossibile est subtrahi cum nihil adest, et tamen hoc in natura repraesentatur, cum quis plus debet, quam habet in bonis*." A VI.4 p. 520.

[160] One paragraph ends with the following characterization of *in speciem impossibiles*: "*Et has quantitates appello* in speciem impossibiles*; cum reapse sint reales, praeceptaque trado, quibus id agnosci possit*" A VI. 4 p. 520. Emphasis in original. The next paragraph begins with "*Multum autem interest inter quantitates imaginarias, seu impossibiles per accidens, et impossibiles absolute quae involvunt contradictionem*." Given that *in speciem impossibles* quantities are also called *imaginariae* in former paragraph, it seems that *in speciem* and *per accidens* are supposed to be equivalent characterizations of the same kind of impossibility.

[161] Translation from De Risi 2016, p. 135. The full quote is: "*Imaginariae vero seu per accidens impossibiles, quae scilicet non possunt exhiberi ob defectum sufficientis constitutionis ad intersectionem necessariae, possunt comparari cum Quantitatibus infinitis et infinite parvis, quae eodem modo oriuntur*." VI. 4 p. 521.

[162] "Mathesis Universalis," GM vii pp. 53-76. Although undated in the Gerhardt edition, the date of 1695 is given to this manuscript in De Risi 2008.

The method of representing roots of quadratic equations as explained in "*Mathesis Universalis*" is as follows. Consider a circle with center K and radius KA [Figure 3.1]. Produce a line segment tangent to the circle with one endpoint being the point of contact between the line segment and the circle and call this line $A(F)$. Produce a line segment perpendicular to AF at point Φ, call it ΦG, and call its points of intersection with the circle points H and L (it is crucial at this stage that the point Φ is chosen such that $A\Phi$ is smaller than the radius KA). Leibniz then uses variables to specify the lengths of these line segments: $r = AK$,$x = A\Phi$, and $y = \Phi H$ (or $y = \Phi L$, depending on which root of the quadratic equation/point of intersection one wishes to find). Next, he draws the radius $K\lambda$ parallel to AF, and the point of intersection between this radius and the line ΦG is called M. Point Mdivides the line segment HL into equal halves $HM = ML$. Since KH and KL are both radii, each is equal to *r*. Additionally, one can see that $\Phi M = AK = r$ and $A\Phi = KM = x$. The triangle HKM is a right triangle with hypotenuse HM, so by the Pythagorean Theorem, $HM = \sqrt{KH^2 - KM^2}$, and one can substitute the previously-specified variables to arrive at the equation $HM = \sqrt{r^2 - x^2}$. Since $\Phi H = \Phi M - HM$, and $y = \Phi H$, $y = r - \sqrt{r^2 - x^2}$. For the other value of y, i.e., ΦL, one has that $y = r + \sqrt{r^2 - x^2}$, using essentially the same reasoning that identified the value of ΦH. Even though Leibniz does not write out the quadratic equation for which these values are roots, one can work backwards and see that this figure represents the solutions for y in the quadratic equation $0 = \frac{1}{2}y^2 - ry + \frac{1}{2}x^2$, where r and x are positive constant quantities.

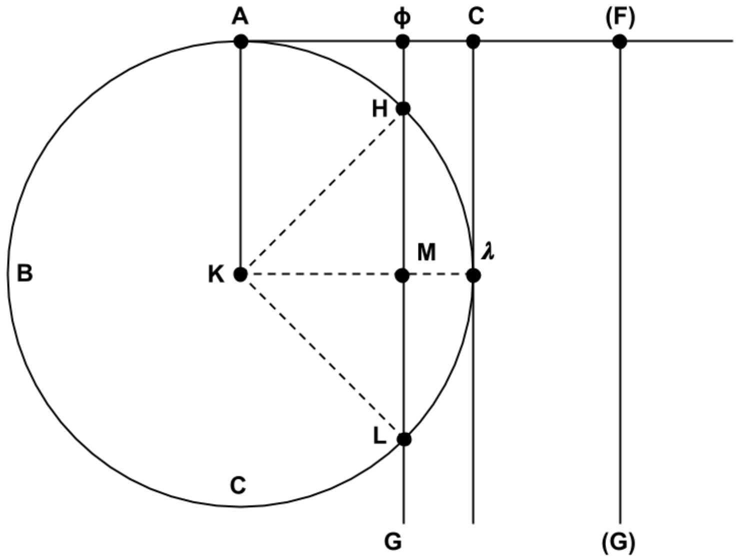

*Figure 3.1: Imaginary Roots in "*Mathesis Universalis*"*[163]

Leibniz notes that under most conditions, the equation $y = r \pm \sqrt{r^2 - x^2}$ ambiguously represents two values, corresponding to the two roots of a quadratic equation. But when $r = x$, we have the situation represented in the diagram in which the normal produced from point C intersects the circle in the single point λ rather than the two points H and L (where the distance AC is equal to the circle's radius). Leibniz further notes that when $x > r$ (indicated in the diagram by letting x equal the line segment $A(F)$, where $A(F)$ is greater than the circle's radius), there will be no points of intersection between the circle and the normal produced from the relevant endpoint. Leibniz pairs the geometric impossibility of the line and circle intersecting with the impossibility of finding real values for the term $\sqrt{r^2 - x^2}$ when $x^2 > r^2$. In cases where the line segment from which the normal is produced exceeds the length of the circle's radius, Leibniz claims "Whence we learn that this question is not well constituted. And either circle ABC

[163] Diagram from the appendix to GM VII. Modified to replace the C from this diagram with the Φ that is mentioned in the body of Leibniz's proof.

whose radius is r ought to be assumed greater, or with the same circle remaining, x or AFought to be assumed less, so that what is sought for is able to be obtained."[164] Presumably, what is "sought-for" here are the lengths between the endpoint of the normal and the points intersection with the circle: H and L. This remark is best read as referring to the problem of specifying points of intersection between a line and a circle under the problem's conditions, rather than extracting the roots of a quadratic equation. For if the problem is conceived of as extracting roots from an equation, altering the values of r or x would mean finding roots to an equation other than the one initially specified. And the claim that the problem is not "well constituted" (*non esse bene constitutam*) can be read as equivalent to the claim from the "*Elementa Nova*" that the *per accidens* impossibility of imaginary numbers is a consequence of a "lack of constitution necessary for an intersection" (*ob defectum sufficientis constitutionis ad intersectionem necessariae*). Additionally, the fact that Leibniz refers to this problem as poorly constituted in "*Mathesis Universalis*" means that the "lack of constitution" has to do with the way the problem is constituted, rather than some kind of deficit that resides in our mathematical or perceptual abilities.

Immediately following the remark about modifying the variables to cause the problem to yield real points of intersection, Leibniz remarks upon the importance of allowing such imaginary quantities in cases where the initial conditions do not allow for real solutions: "And if such imaginary quantities were not given in the calculus, it would be impossible for general calculations to be instituted, or common values to be found by possible and impossible [values], which differ only through the explication of the letters."[165] That is, if one is given an equation

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[164] "*Unde discimus, quaestionem non esse bene constitutam, et vel debere circulum ABC sive radium ejus r assumi majorem, vel eodem manente circulo, ipsam x vel AF assumendam minorem, ut quaesitum obtineri possit*." GM vii p. 74

[165] "*Et nisi darentur tales quantitates imaginariae in calculo, impossibile foret institui calculos generales, seu valores reperiri possibilibus et impossibilibus communes, qui sola differunt explicatione literarum*." GM vii. pp. 74-75. "*Calculos*" in this quote should be read as a generic term for a symbolic system, rather than the infinitesimal calculus in particular. The paragraph after this quote is an explanation of how non-

such as $0 = \frac{1}{2}y^2 - ry + \frac{1}{2}x^2$, then the equation $y = r \pm \sqrt{r^2 - x^2}$ will represent the roots regardless of whether r is "explicated" as a value less than x.

Now that we know the technical details of the connection between imaginary roots and non-existent points of intersection that Leibniz makes in *"Elementa Nova*," we return to that text with a greater understanding of the analogy Leibniz presents between such imaginary quantities and non-finite quantities. Although he only mentions explicitly mentions *per accidens* impossibility as arising from the lack of an intersection, we should not read Leibniz as saying this is the origin of *all* cases of this type of impossibility, for this mode of representation through intersections of lines and circles is not relevant to Leibniz's above-mentioned example of negative numbers as being represented by one's debt exceeding their wealth.[166] Instead, we should read Leibniz's remark that a lack of an intersection is the source of impossibility *per accidens* as applying only to square roots of negative numbers, rather than all entities that are impossible in this limited sense. However, it is easy to imagine how Leibniz would treat other cases of impossible representations. If addition and subtraction of quantities are interpreted as adjoining or removing undirected line segments of specified lengths, then one cannot represent the subtraction of a larger quantity from a smaller one. But by introducing negative quantities through the properties that they have in symbolic calculations, equations such as $a - b$ retain significance in the cases where $a < b$. In this way, quantities that are impossible *per accidens* can be used to systematically "fill in" gaps where an equation would otherwise fail to apply when

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finite quantities can be introduced using similar symbolic methods in the infinitesimal calculus, highlighting the close connection between such quantities and imaginary numbers that also appears in the "*Elementa Nova*."

[166] Recall that in the *Elementa Nova*, negative numbers are considered to be impossible *per accidens* as well.

certain conditions are no longer met, even if such quantities can only be represented through the symbolism of mathematical notation.[167]

The second kind of impossibility mentioned in the "*Elementa Nova*" is the more familiar notion: that which involves a contradiction. This text is not the only place where Leibniz makes the distinction between these two types of impossibility, for in another work from the early 1680s, "On Freedom and Possibility," Leibniz again mentions the impossibility of representing imaginary roots as the intersection of a line and circle and distinguishes this kind of impossibility from the that which involves an absolute absurdity.[168] Notably, Leibniz uses the difference between these types of impossibility to draw a contrast between possible but nonexistent entities in the physical world and entities whose existence is absolutely impossible. In the "*Elementa Nova*," Leibniz simply gives the equation $3 = 4$ as an example of an absolutely impossible claim, but "On Freedom and Possibility" elaborates on this example and how it highlights the difference between the two types of impossibility. If one is asked to solve for x where $x^2 = 9$ and $x + 5 = 9$, one must claim both that $x = 3$ and $x = 4$ (Leibniz omits the possibility that $x = -3$ in the first case). And in this text, Leibniz explains how the contradiction between equating 3 and 4 arises; if 3 and 4 were equal, then a whole would equal one if its parts, the same principle that Leibniz used to rule out the possibility of infinite numbers in the 1670s. Presumably, this stems from treating a number *n* as just a collection of *n* units, for then the number 4 would contain the number 3 as a proper part. Continuing, Leibniz says that this type of contradiction is distinct from the case where one is to solve for *x* in the equation $x^2 - 3x + 9 \ = 0$. The roots of this equation are $\frac{3 \pm 3\sqrt{-3}}{2}$, a solution which contains the imaginary

[167] For more on the importance Leibniz sees in allowing equations to apply even in situations where they traditionally would not, see Grosholz 2008.
[168] "On Freedom and Possibility" appears in translation in AG pp. 19-23, and in Latin in A VI.4 pp. 1444-1449. AG dates it between 1680-1682, while the Akademie editors date it between 1680-1684. Regardless of the exact dating, this piece is from the same general period as 1683's "*Elementa Nova Matheseos Universalis*."

quantity $\sqrt{-3}$. Acknowledging that one cannot "designate" (*non posse designari*) a number that satisfies this equation, Leibniz maintains that even if one were to admit this number, one would not be able to show that a part is greater than its whole or establish any other mathematical contradiction. While both cases involve finding the solution to equations that cannot have real numbers as their solutions, it is only the case where it is claimed that $x = 3 = 4$ that leads to contradiction.

In "On Freedom and Impossibility," the distinction between equations that are absolutely impossible to satisfy and ones that are only impossible to satisfy when restricting ourselves to real numbers is presented as an analogy for the difference between necessary and contingent truths, respectively. Some things may not exist due to their incompatibility with other events in the actual world, yet this does not mean that they are in and of themselves impossible. The example Leibniz gives to illustrate this point in this text is a pentagon. Even if no perfect pentagon were to exist in nature, it would still remain a possible notion, for there is no contradiction within the mere concept of a pentagon. So, if there were no pentagons to be found in nature, the reason for this pentagon-less world would have to be some incompatibility between pentagons and the series of objects and events present in the best of all possible worlds. I take the analogy between the examples of the different equations and the possibility of a pentagon-less world to be that just as the reason for the supposed non-existence of a pentagon would have to be located in a principle other than non-contradiction, so to the reason for the non-existence of imaginary roots must be located outside of non-contradiction, such as the inability to spatially locate such roots.[169] While he does not use the terminology of the impossible *per accidens* and the absolutely impossible in "On Freedom and Possibility," Leibniz

[169] Two years after the "*Elementa Nova,*" John Wallis gives a geometric interpretation of imaginary roots in his *Algebra*, but the historian of mathematics Morris Kline claims his method was "not a useful representation" for most purposes (Kline, p. 595). The contemporary representation using the complex plane was not developed until the early 19th Century.

is clear that there is a stark difference between equations whose solutions involve imaginary roots and those that lead to outright logical contradiction.

The contrast between equations with only imaginary roots and those that have no solution whatsoever is used in “On Freedom and Possibility” as an analogy between non-existent possible physical objects and absolutely impossible ones, respectively. However, in the “*Elementa Nova*,” Leibniz uses the difference between these two types of impossibility to make a point about infinitesimals and their reciprocal infinites, saying such entities arise “in the same way” as the other mathematical entities that are specified as impossible *per accidens*. As an example of how such quantities can arise, Leibniz gives the following example (it also occurs almost two decades later in 1701’s important text on the infinitesimal calculus, *Cum Prodiisset*.) Let AB be a straight line segment in which the location of point B is allowed to vary, and let line AC be a line perpendicular to AB [Figure 3.2]. The precise location of point B is determined as the intersection of a straight line that passes through point C and the line AB. In the diagram, these positions for the different determinate locations of the point B are represented by the points B_1, B_2 etc. Leibniz notes that the closer angle CBA is to a right angle, the smaller the line ABbecomes. Leibniz then says that in the case where CBA is actually a right angle, the line AB_1becomes infinitely small. Additionally, as the angle CBA becomes smaller and smaller, the point of intersection, B, moves farther and farther away from the point A. In the ultimate case, where line $CB$is parallel to line AB, the common point B becomes imaginary, a point standing infinitely distant. Furthermore, the straight line AB (and presumably CB as well) become infinite.[170] While the Law of Continuity is not expressly used to justify these considerations in “*Elementa Nova*,” this principle can be reasonably said to be implicitly invoked. For, just as there is always a straight line between points A and B as the point B approaches point A, there continues to be a line when the points are on the verge of coinciding– albeit an infinitely small

[170] A VI.4 p. 521.

one. And just as there always exists a point of intersection that recedes farther from point A as the angle of intersection decreases, in the case where the lines are parallel, the point of intersection becomes located infinitely far away.[171] Returning to the question of possibility, Leibniz says that those lacking in mathematical prowess may believe that these quantities lead to absurdity, i.e. a contradiction, but those who are skilled know that this “apparent impossibility” (*apparentem illam impossibilitatem*) is illusory. Instead, Leibniz says that those who have an adequate grasp on this issue see that the apparent impossibility “only means that a parallel is drawn making a sought angle to a straight line; and thus parallelism is that sought angle, or better a quasi-angle.”[172] Presumably the reference to the “sought angle” means that one can think of this example as reflecting some construction problem, such as: given the line AB and line AC perpendicular to it, find a point B_n, such that angle AB_nC is equal to some specified value. For 0° < AB_nC < 90°, one draws straight line CB_n so that the AB_nC is equal to the angle specified in the problem. Moreover, there is a continuous movement of point B_n away from point A as the sought angle approaches 0°, and as this occurs, line CB_n approaches parallelism towards line AB. So, if one were to set the “sought angle” in the problem to 0°, one could obtain the construction by drawing a parallel line. However, such a solution is imaginary, for parallel lines do not make an angle, just as there is no actual intersection between the circle and the straight line from the example concerning square roots of negative numbers. Similarly, when the sought-for angle is specified as a right angle, the straight line AB will be infinitesimal.

[171] As further evidence that some form of the Law of Continuity is being invoked to introduce the infinitesimal and infinite lines in *Elementa Nova*, see 1701’s *Cum Prodiisset* (Gerhardt 1846 p. 40-41) where a substantially similar version of this example is given as an explicit illustration of the Law of Continuity.
[172] Translation quoted from De Risi p. 135. Original Latin is: “*ut loco rectae angulum quaesitum facientis ducatur parallela; hunc parallelismum esse angulum illum seu quasi angulum quaesitum*.” In A VI.4 p. 521.

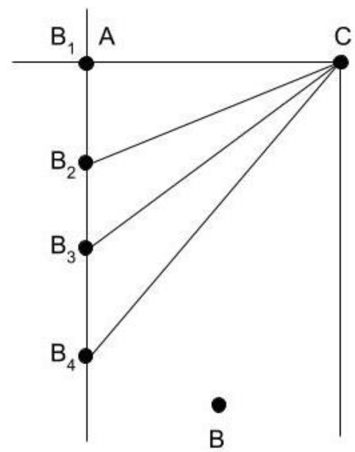

Figure 3.2: Passing from finite to infinite points of intersection[173]

Although the only remarks Leibniz makes about the origin of impossibility *per accidens* in the "*Elementa Nova*" refers specifically to imaginary roots of quadratic equations and the inability to represent them as real points of intersection between a line and circle, it is clear that mathematical objects that are impossible *per accidens* differ from the absolutely impossible, and that the absolutely impossible is the only of these two types of impossibility that involves a direct contradiction. In his explanation of the distinction between these two types of impossibility, Vincenzo De Risi says that perspectival geometry is the only type of non-Euclidean geometry that Leibniz would have been able to conceive of at this stage in history. But for Leibniz, this geometry would not strictly be "true" of the physical world, for it "does not obey the Principle of Determinant Reason and thus cannot exist."[174] At the same time, citing the distinction between the absolutely impossible and the impossible *per accidens* from, De Risi claims that an alternative geometry of this form would still be seen as non-contradictory by Leibniz, even if it

[173] Figure modified from the one present in A VI.6 p. 521, such that the subscripted indices come after each letter, rather than before.
[174] De Risi 2016, p. 101.

was not physically instantiated. This explanation helps us to determine more about the space between what is absolutely impossible and what is impossible *per accidens*.

In saying that perspectival geometry does not obey the Principle of Determinant Reason (also known as the Principle of Sufficient Reason), yet still remains consistent, De Risi is making a claim that is different from the example of the non-existent pentagon that Leibniz mentioned earlier. In that example, the pentagon's existence or non-existence is determined by the series of events in the world with the greatest possible perfection, *i.e.,* the world God chooses to create. If the pentagon were included in that infinite series of events, it would exist. If not, it would not. But even if the pentagon's existence were ruled out by its incompatibility with the best of all possible worlds, there would still be suboptimal possible worlds that obey the principle of Determinate Reason in which that pentagon would exist.[175] That is, there will be possible worlds in which every effect has a traceable and determinate cause (satisfying the Principle of Sufficient Reason), but when all of these causes and effects are taken together, the degree of perfection is less than that of other possible worlds (which is why God would not choose to actualize these suboptimal worlds). Because comparisons of the total perfection of a possible world involves analyzing an infinitude of causes and effects, there is no *a priori* way for our finite minds to know that some existing being will lead to more or less perfection for the world as a whole, and thus we cannot make *a priori* determinations about the existence of such objects.[176]

Simply put, there is no way for us to make existence claims between competing possible worlds and their constituent objects/events when both satisfy the Principle of Sufficient Reason. On the other hand, Leibniz appears to believe that human minds are able to know *a priori* that

[175] Here I am setting aside concerns Leibniz believes no precise geometric objects exist in the physical world due to the ever-shifting nature of bodies that are constantly in motion. See Garber 2015 for more on Leibniz's denial of precise geometric objects within the physical world.
[176] "On Contingency," a 1686 text translated in AG pp. 28-30 is just one of many places where this claim occurs. The belief that contingent truths cannot be known *a priori* to finite minds due to them requiring an infinite analysis is a recurrent theme in Leibniz's thought.

worlds that contain some violation of the Principle of Sufficient Reason will fail to exist, even if such worlds are free from logical contradiction. That is, we are able to confidently assert the non-existence of events/objects that come into being for no reason whatsoever, even if such events/events are non-contradictory (and hence possible). One of the most famous instances of this kind of reasoning comes from Leibniz's correspondence with Clarke concerning the nature of space. In the Postscript to his Fourth Letter, Leibniz says should there be portions of space that contain a vacuum, there would have to be a reason for why some specific proportion of unoccupied to occupied space obtains in the world. But Leibniz says, "It is impossible that there should be any principle to determine what proportion of matter there ought to be, out of all the possible degrees from a plenum to a vacuum, or from a vacuum to a plenum."[177] Leibniz then argues that a 50/50 split between occupied and empty space is too arbitrary, and the proportion should then be equivalent to the degrees of perfection present in matter and vacuum. But since by definition, there are no objects in a vacuum, there would be nothing in this unoccupied space, which is capable of perfection, and hence God would not create a world in which vacua exist. One would not arrive at a *logical* contradiction by positing a vacuum, but one would have to say that God acted without any possible reason by creating a vacuum. Since the existence of vacua would violate the Principle of Sufficient Reason, we know that the world must be filled with matter in all places. The specifics of how this space is filled remains unknowable *a priori* because it would require an infinite analysis, but we are able to confidently assert that there is no portion of matter, for there would be a violation of the Principle of Sufficient Reason if not.[178]

In my reading of these texts, we see that an equivalence between "impossible *per accidens*" and "that which violates the Principle of Sufficient Reason" provides a tidy explanation of why it is that some things are possible in mathematics, but not in the created physical world.

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[177] *LC* p. 28.

[178] Similar arguments rule out the existence of indivisible atoms in favor of matter that is divided into smaller and smaller parts, each endowed with their own motions.

We already saw that Leibniz distinguishes mathematics from metaphysics based on the subject matter of imaginary things or purely intelligible things, respectively. And in his Fourth Letter to Clarke, Leibniz lists the Principle of Sufficient Reason and the Identity of Indiscernibles as great principles of metaphysics, as opposed to some other structured domain of knowledge.[179] Additionally, in his Fifth Letter to Clarke, Leibniz says that indivisible bodies and bodies that are perfectly similar are both consequences of a lazy philosophy that is overly reliant on the imagination, as opposed to true metaphysical principles. He again notes that while such entities are not absolute impossibilities, their existence would be contrary to the divine wisdom, and hence would not exist.[180] A few paragraphs later speaking of the concept of motion relative to absolute space alone, Leibniz declares "Mere mathematicians who are only taken up with the conceits of imagination are apt to forge such notions, but they are destroyed by superior reasons."[181] Thus we see that even in his late years, Leibniz distinguishes between what is impossible in virtue of contradiction, and what we know cannot happen due to our knowledge of metaphysical principles. And these additional principles are not reducible to the Law of Contradiction alone. Additionally, the link between mathematics and the imagination as well as metaphysics and understanding mirrors the taxonomy of the sciences from 1683 discussed above. Hence, some things can be possible within imagination, such as two indiscernible but distinct bodies, a vacuum in the physical world, and non-finite quantities. But when one reasons about God's creation, one must account for metaphysical principles, such as the Principle of Sufficient Reason, that further winnow the candidates for the actually existent world beyond a consideration of non-contradiction alone.

In this section, we saw that Leibniz draws a distinction between two kinds of impossibility: *per accidens* and absolute. *Per accidens* impossibility involves no contradiction

[179] Letter Four, Paragraph 5. In *LC* p. 22.
[180] Letter Five, Pagraphs 21-25. In *LC* pp. 40-41.
[181] Letter Five, Paragraph 29. In *LC* p. 42.

and is compared with imaginary roots. Absolute impossibility is that which involves a contradiction, and Leibniz offers “$3 = 4$” as the paradigm example of this type of impossibility. In some cases, the contrast between these two mathematical examples is used to illuminate the difference between non-existent possibles and absolutely impossible entities, as seen in “On Freedom and Possibility.” In the *Elementa Nova*, the difference between these two equations is used to claim that non-finite quantities are as consistent as these imaginary roots. Developing the line that some kinds of geometry may be consistent yet fail to be physically instantiated because they violate metaphysical principles, we saw similar remarks that Leibniz makes about voids and atoms. I believe that this status that infinitesimals possess as legitimate entities to use in mathematics (as opposed to contradictory notions like infinite number) while simultaneously being barred from the physical world explains why Leibniz refers to them as “fictions.” Armed with this distinction and the conceptual space between claims that violate the Law of Contradiction, the Principle of Sufficient Reason, and the Principle of Perfection, we can address the tension between holding infinitesimals to be consistent and Leibniz’s remarks to Johann Bernoulli that seem to indicate otherwise.

3.4 Accounting for the Bernoulli Correspondence

Armed with a fuller understanding of the distinction between the two different types of impossibility that Leibniz identifies, we can return to the correspondence with Johann Bernoulli. In this section, I argue that although Leibniz does not explicitly draw the distinction between the two types of impossibility in his correspondence with Bernoulli, his remarks on the impossibility of infinitesimals are best read as referring to *per accidens* impossibility rather than absolute impossibility. Under this reading, there is no tension between these remarks and the consistency of infinitesimals.

Recall the problematic passage for the reading where infinitesimals are consistent is the following passage from Leibniz's November 18, 1698 letter:

> As concerns infinitesimal terms, it seems to me not only that we cannot penetrate to them but that there are none in nature, that is, that they are not possible. Otherwise, as I have already said [in a letter dated July 29 of the same year], I admit that if I could concede their possibility, I should concede their being.[182]

The first notable feature of this argument is its conflict with what Leibniz says elsewhere about possibility. Here Leibniz starts by saying that there are no infinitesimals in nature and uses this fact to justify their impossibility. But elsewhere, including his correspondence with Johann Bernoulli, Leibniz is very careful in emphasizing that just because something does not exist does not mean that it must be impossible.[183] In fact, this was the whole point of the pentagon example described above was that non-existence is not sufficient to indicate impossibility. Despite his emphatic defense of this position, this quote in his letter to Bernoulli seems to be an argument from the non-existence of infinitesimals to their impossibility.

However, the second sentence of this quote shows that the argument rests on the premise that if infinitesimals were possible, they would be actual. This is a premise that is lacking from the pentagon example and other discussions of non-actual possibles. The letter to Bernoulli from July 29, 1698 contains the line where Leibniz gives this uncharacteristic link between possibility and existence referred to in the November 18 letter, but he does not give evidence for his claim, other than simply saying that should non-finite quantities be possible,

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[182] Translation from Loemker, p. 511. Original Latin is: "*Quod terminos infinitesimos attinet, videtur mihi non tantum ad eos non posse a nobis perveniri, sed etiam eos non esse in natura id est non esse possibiles alioqui fateor ut jam dixi si concederem esse posse, concederem esse.*" In A III.7 p. 943.

[183] One place where this occurs is in the January 13, 1699 letter to Bernoulli, where Leibniz denies the existence of atoms and vacua, but does not say they are therefore impossible, only contrary to divine wisdom ("*non esse divinae sapientiae consentanea*") (in A III.8 p. 38). And in another letter in this correspondence, from February 24 of the same year, Leibniz says that actual things are just the best possible things, and there are hence many non-existent possibles (in A III.8 p. 64).

they would exist.[184] And one month later, Leibniz tells Bernoulli that this earlier hypothetical was not a declaration of the impossibility of infinitesimals, but simply a way to leave the matter in the middle.[185]

This justification for the additional premise that if infinitesimals were possible, they would actually exist becomes even harder to understand when one considers that Leibniz frequently says that God is the only entity whose existence follows from its possibility.[186] However, this argument becomes easier to square with Leibniz's statements about this unique property of God if one reads it as analogous to his arguments against empty space and invisible atoms rather than reading them as on par with his statements about God's necessary existence. Prior to delivering the line in the July 29, 1698 letter that if infinitesimals were possible, they would be actual, Leibniz repeats claims that we have seen him make elsewhere about infinitesimals being similar to imaginary roots in algebra and replaceable with a series of continually diminishing finite quantities. Leibniz then states another doctrine that appears in numerous other texts; there is no portion of matter that is not divided into other portions of matter. But this does not mean that matter is divided into infinitely small parts; the divisions are always into finite parts that continually become smaller. Switching from bare matter itself to organic bodies, Leibniz says that even conceding that every animal is always composed of smaller animals does not force one into accepting infinitely small animals, let alone animals that are the ultimate constituents of other animals. In a letter to Bernoulli from a year later, Leibniz explicitly draws an analogy between vacua and atoms, saying that while there is no contradiction in a world with empty space or undivided bodies, such worlds are less than perfect: "However it is manifest that a

[184] "*Si talia de quibus inter nos agitur infinita et infinite parva, possibilia esse concederem, etiam crederem esse*." A III.7 p. 858.
[185] "*Cum dixi si infinite parva et possibilia crederem, me concessurum ea esse; non ideo dixi ea esse impossibilia, sed rem in medio adhuc reliqui*." August 22, 1698. A III.7 p. 884.
[186] "On Freedom and Possibility" is one place in which this unique aspect of God is listed (AG p. 19). Bassler 1998 argues that there is a structural analogy between arguments for God's existence and the existence of infinitesimals here (p. 860, footnote 34).

vacuum (and equally atoms) leave places sterile and uncultivated, in which another thing could have been produced, yet with all remaining things preserved."[187] Thus, we can use the Principle of Sufficient Reason to confidently know that there will be no undivided portions of matter, even if we cannot *a priori* determine which possible divisions will lead to the greatest possible perfection, and hence be actual.

This later message to Bernoulli concerning the infinitely-nested division of matter tells us why Leibniz believes that the possibility of infinitesimals entails their existence. If it were possible to divide matter into infinitely small bodies with differing motions, then there would be a layer of diversity that would not be present in a world where there are only finitely small beings. To use Leibniz's phrase, God would not allow such a "sterile and uncultivated" realm to remain without infinitely small bodies, although allowing them to remain would only violate the Principle of Sufficient Reason, not the Law of Contradiction. In this way, God's existence and the existence of infinitesimals are not on even footing, for Leibniz believes that if a definition of God were possible, God would exist by necessity. Whereas the existence of infinitesimals would only follow from their possibility with the additional consideration that God decides to only create the world with maximal diversity/perfection. And it is crucial to Leibniz's account of God's that this choice is made freely, rather than by necessity.[188] Hence, when Leibniz says of infinitesimals "if I could concede their possibility, I should concede their being," this claim only holds under additional suppositions about the nature of God's creative act. This is a strange conclusion since other objects cannot be inferred based on their possibility alone, and we are unable to say

[187] "*Vacuum autem (perinde ac Atomos) relinquere loca sterilia atque inculta manifestum est, in quibus tamen salvis caeteris omnibus aliquid adhuc produci potuisset. Talia vero relinqui cum sapientia pugnat.*" From January 13, 1699. A III.8 p. 38.

[188] This is an early belief of Leibniz that persists throughout his thought. One can find it in early on in 1678 notes on Spinoza's Ethics, where he says that Spinoza's claim that God necessarily created the world in the only possible way it could be created is only true under the hypothesis that God chooses the best, rather than necessary in an unqualified sense. (Note to Proposition 33, in L p. 204). As evidence that it remains a stable element of Leibniz's thought, one can find the claim in 1714 in "The Principles of Nature and Grace, Based on Reason," §11 (translated in L p. 639-640).

that any particular object would bring more good into the world than another because we need to take account of how each would or would not be compatible with infinitely many other possible events that could lead to a world with greater perfection. Part of the strangeness of this reading is the fact that rejecting something's possibility on account of its non-existence is a strange claim for Leibniz to be making in the first place. However, we can reduce this tension by realizing that Leibniz is not saying if some individual infinitesimal body is possible, it would be actual. Instead, this is a claim about the existence of infinitesimal-sized bodies in general, rather than the existence of some determinate configuration of such bodies in particular. And for whatever reason, Leibniz seems to believe that infinitesimals are impossible, but the reason for their non-existence could come from metaphysical, rather than logical, principles. Thus, we can read the reference to infinitesimals as not being possible as a claim that they are impossible *per accidens*, rather than absolutely impossible.

If we interpret this line of argument with Bernoulli as claiming that non-finite quantities are impossible in nature, but not absolutely impossible, we have a way to preserve the logical consistency of infinitesimals. But we still need a reason for their metaphysical impossibility. Unfortunately, Leibniz never explicitly gives such a proof.[189] In the remainder of this section, I construct an argument for exactly why non-finite quantities would violate metaphysical principles, in particular the Identity of Indiscernibles. My reading of Leibniz's remarks to Bernoulli requires there to be a reason that rules out non-finite quantities that does not reduce to the Law of Noncontradiction. I am not committed to this being Leibniz's actual argument, but merely one that he could have given. While such speculation is always fraught with danger, Leibniz's failure to actually provide his reasons for ruling out infinitesimals in nature puts my

[189] Levey, 2008 notes the point that no such proof has yet surfaced in Leibniz's extant writings (p. 116). Although he appears to be looking for a proof of absolute impossibility rather than impossibility *per accidens*, the point remains that there appears to be no clearly labeled proof in support of the claim to Bernoulli that non-finite quantities are not possible.

reading in no worse a position than one that seeks to ascribe absolute impossibility to infinitesimals, for he does not give a definitive proof of absolute impossibility either. The upshot of the argument I develop here is that if organisms of an infinitesimal size were to exist, there would be no way to distinguish between them and finite bodies.

The following argument shows that if infinitesimals or bounded infinite bodies were real, we would be unable to tell whether our bodies were finite quantities, infinitesimal ones of any given degree, or a bounded infinite corresponding to reciprocal of an infinitesimal quantity. This supposition is not a *reductio ad absurdum* argument *per se*, for no logical contradiction results for the supposition of infinitesimal quantities. Instead, we will see that a world in which our bodies are finitely sized would be indistinguishable from ones where they were on some level of non-finite quantity, a case in which things go haywire, but not because of absolute impossibility. Suppose our own bodies were finite in extension, and let x stand for our height. Then if we consider some other quantity y, it is possible to deduce whether or not $y$is finite by using the Archimedean property: if $x < y$, then there should be some natural number n such that $nx > y$, and if $x > y$, there should be some natural number n such that $x < ny$. If there is no such natural number n for either of these possibilities, then y is either infinitely long in the case where $x < y$, or y is infinitesimal in the case where $x > y$. But if there were no such natural number n, any conclusion that y is either infinitely long or infinitesimal only holds under the supposition that we already know that our own bodies are finite. For suppose our own bodies were actually at the infinitesimal level, with a length of dx. Then if we consider a quantity y that is actually finite, there will be no finite number n such that $ndx > y$, so finite quantities will appear to us as if they were infinite. Additionally, any quantity dy that does satisfy Archimedes' Axiom with respect to dx will also be infinitesimal. In this case, infinitesimals of the second order, such as ddx and ddy would appear as infinitesimal. And if our bodies were actually infinitely extended, finite quantities would appear as infinitesimal, and only infinities of a higher order would be considered as infinite. Thus, using the Archimedean property, we can only tell if a quantity is infinitely smaller or larger relative to another quantity.

One other possible attempt to discern finite quantities from non-finite ones is by considering a quantity's reciprocal. If the quantity x is finite, then x^{-1}will also be finite. However, should x be infinitely small, then x^{-1}would be infinitely large. And x^{-1}would be infinitely small should x be infinitely large.[190] Given this assumption, we can simply classify finite lengths as those whose inverses can be compared using the Archimedean property; there will exist some number n such that $nx^{-1} > x$ or $nx > x^{-1}$, in cases where $x^{-1} < x$ and $x < x^{-1}$, respectively. While this may seem promising, any attempt to implement it to bodies in the world would fall into trouble. Taking the inverse of x means finding some value y such that $xy = 1$. But since quantities in the world do not come with a pre-established metric attached to them, we must stipulate a quantity that serves as a unit length. A traditional way to construct this would be to use the construction in Euclid VI.8. This proposition tells us how to construct the fourth proportional between three given quantities, *i.e.,* is to find x such that $\frac{a}{b} = \frac{c}{x}$ (where a, b, and c are given quantities). If one lets $b = c = 1$, then the proportion becomes $a/1 = 1/x$, equivalent to $1 = ax$, making x the inverse of a. The general method of finding a fourth proportional is to represent the original three quantities as lines $A,\ B,$ and C [Figure 3.3]. Then one draws two lines DE and EF at any angle (other than $180°$) that meet at some point D. Then, one specifies points G, E, and H such that $DG = A$, $GE = B$, and $DH = C$. One then draws a straight line segment GH, and the line segment EF parallel to GH. Since line GH is parallel to the base of triangle DEF, the proportion $\frac{DG}{GE} = \frac{DH}{HF}$ holds. And by the earlier equalities, $\frac{A}{B} = \frac{C}{HF}$, meaning HF is the fourth proportional we originally wished to construct. Stipulating some element as our unit

[190] According to Rules 10 and 11 of Knboloch's reconstruction of Leibniz's arithmetic of the infinite in the early 1760's text *De Quadratura Arithmetica*. Knobloch 2002, p. 67.

length and setting the two middle terms in the proportion equal to that unit gives us a way to represent the algebraic concept of taking a reciprocal quantity.[191]

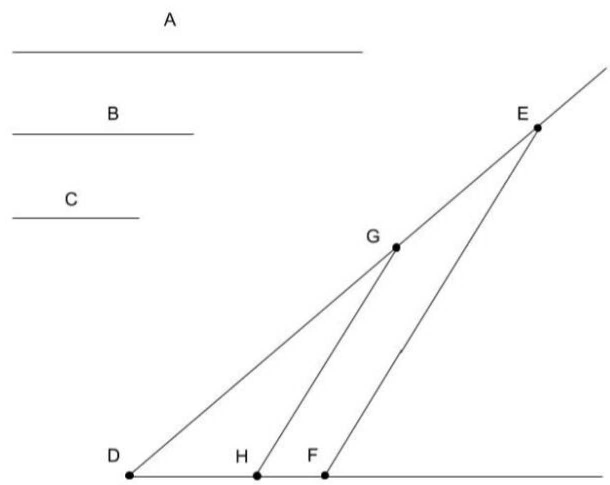

Figure 3.3: The construction of a fourth proportional

However, this inverse construction procedure also fails to allow us to distinguish the finite from the infinite in the physical world. For we must first stipulate some line to serve as 1. If we stipulate a finite line, then we do get the desired result that a quantity x is finite if and only if x and $x^{-1}$are comparable by the Archimedean property. But if we allowed some infinitesimal line dy to serve as our stand-in for the unit length, then the proportion would be $\frac{x}{dy} = \frac{dy}{z}$ (where $z$is written in place of the earlier-used x^{-1}, for as we will see, there are reasons to be wary of calling this z the "actual" inverse of x). In this case, if x is finite, then $\frac{x}{dy}$ will be infinitely large, since dy is infinitely small.[192] And if $z$is finite, $\frac{dy}{z}$ will be an infinitely small quantity divided by a finite quantity, which is another infinitely small quantity. Since $\frac{x}{dy}$ is infinitely large and $\frac{dy}{z}$ is

[191] Note that this is similar to the construction Descartes used to construct the product of two lines without representing it as a rectangle. There, Descartes picks an arbitrary line to serve as 1, and gets the product *ab* by the proportion 1:a :: b:ab. Both techniques involve using Euclid's construction with a privileged unit element, the only difference is where in the equation that element occurs.
[192] By Rule 10 in Knobloch 2002. p. 67.

infinitely small, the two patently cannot be equal. Hence the supposition that z is finite must be false. The only option is for z to be an infinitesimal of a lower order that is infinitely small with respect to dy, which would allow $\frac{dy}{z}$ to be equal to $\frac{x}{dy}$. And under the supposition that we chose infinitesimal dyas our unit element, if we take an infinitesimal quantity of the same order, say dx, and construct its inverse, the ratio $\frac{dx}{dy}$ will be finite, and so will $\frac{dy}{z}$. This means z will be an infinitesimal quantity of the same rank as the initial dx. Similar problems will occur if we choose an infinitely long line as our initial unit length. If we select a finite line as our unit element, the procedure for constructing inverses will tell us that the inverse of a finite quantity is another finite quantity. But if we originally pick a non-finite quantity, the quantities whose inverses are comparable via the Archimedean property will be quantities of the same non-finite rank as the unit element. And an attempted definition of finite quantities as those whose inverse is also finite only works under the condition that the line, we use to construct inverses is itself finite. But assuming that we can select such a line simply begs the question.

Hence, it seems that there is no way to determine whether or not a quantity is finite or infinite. This means that if there were actually infinitesimal beings, they would be unable to tell that they were infinitely small. From their perspective, they would be finite, and we would be infinitely large. And given Leibniz's earlier-mentioned thoughts on infinite divisibility, these infinitely small bodies would themselves be divided into smaller and smaller portions. And since division of our finite matter proceeds into infinitely small parts, there seems to be nothing stopping the possibility of the matter in this infinitesimal realm as being divided into parts infinitely small with respect to it. And in this new doubly infinitely small realm, everything will seem finite, and it will also be subdivided into parts infinitely smaller than it. And I cannot think of any reason why this process should stop at any particular level, and so there will be an infinitely descending chain of infinitely smaller and infinitely smaller active levels of the universe. In this originally stipulated infinitely smaller world, our own world would be infinitely large with respect to it. For this reason, there seems to be nothing stopping there from being a world infinitely larger than our own, of which our bodies are the divisions into infinitely small parts. And if this is possible, what possible reason could prevent God from creating organic bodies infinitely greater than those, and so on? This supposition leads to a problem, however. What if God shifted these levels? So instead of being finite, we become the first infinitely small level, and what was formerly the first infinitely large level becomes finite. There would be no way for us to discern that we are no longer the finite level, for there was no way to tell whether or not we had this property to begin with. Since this would mean a change without a difference, God would have no reason to designate one level as finite over another, and Leibniz firmly believes that changes in name only are no changes at all. Hence, we can be metaphysically certain that there will be no infinitesimals, since their existence would force God into a choice (about which quantities will really be finite) without any possible reason for the basis of that choice. And the Principle of Sufficient Reason guarantees that this will never happen. Hence, while infinitesimals are not themselves contradictory, their existence would be a violation of the Principle of Sufficient Reason and can be deemed "impossible" in the *per accidens* sense of the term.

Given these considerations, we can construct a hierarchy of possibilities [Table 1]. First, there is what necessarily exists. For Leibniz, this is God alone, and all else is contingent.[193] Next, there is the actual world and all of the individual objects that compose it. The existence of our world is contingent, and it is selected because according to Leibniz, famously and infamously, it is the best of all possible worlds. We are unable to *a priori* specify the existence of any particular individual within this world because we would have to do an infinite comparison between all of the worlds and their constituent individuals (save for atoms, voids, and other violations of the PSR, which fall within the penultimate category). The next category is the unactualized possible worlds that contain fully determinate series of events but are not selected

[193] In "On Freedom and Possibility" AG p. 19.

for creation by God due to their inferiority in comparison to the actual world. Their non-existence is contingent in the sense that God has the power to actualize them should he be so inclined (but the nature of God's wisdom and benevolence means he never will have such an inclination). Next is the category of worlds and objects that are still possible in the sense that God could actualize them but violate the PSR. Unfortunately, Leibniz never schematically expounds upon, or even explicitly recognizes, this category. But his remarks about atoms and voids make it clear that such worlds are possible in the sense that they are free from contradiction yet are distinct from the previous class of possible worlds because we can rule out their existence *a priori*. I believe that worlds with infinitesimal bodies fall into this category, for infinitesimals are possible in and of themselves, and Leibniz's banishment of them from the real world must have a source in a principle beyond logic alone. Finally, is that which by necessity does not and cannot exist. In this category falls everything that contains logical contradiction, and not even God could actualize the denizens of this category.

Although the argument from the impossibility of distinguishing the finite from the infinite is a speculative reconstruction of a possible reason for why Leibniz believed in the metaphysical impossibility of infinitesimals, the lack of a smoking gun is an evidentiary gap that all interpretations of Leibniz's fictionalism must reckon with. Building any case that accounts for Leibniz's statements about the impossibility of non-finite quantities with evidence that he regards them as free from contradiction requires a good deal of circumstantial evidence to explain either how the two views fit together, or why Leibniz changes his mind in a relatively short time span. For this reason, while I can only offer a tentative source for Leibniz's denial of infinitesimals, I am not in a uniquely problematic position in doing so.

Necessarily existent objects.	**The actual world and its inhabitants.**	**Possible worlds that uphold PSR.**	**Possible worlds and objects in which the PSR fails.**	**Necessarily non-existent / contradictory objects.**
God is the only being Leibniz believes exists by necessity.	The best of all possible worlds. Obeys the PSR and when taken as a whole, as the most diversity and perfection in comparison to other possible worlds.	Possible worlds in which every event has a cause. Not actualized by God due to being suboptimal relative to the actual world.	Possible worlds that are not only suboptimal, but also violate the PSR. Includes worlds with vacuums and atoms, as well as worlds with infinitesimals and concepts impossible *per accidens*.	Square circles, wholes equal to some of their parts, etc. Logically contradictory and not even God has the power to create them.
Existence is necessary.	Existence is contingent. (Depends on Principle of Perfection)	Existence is contingent. (Depends on Principle of Perfection)	Existence is contingent (Depends on Principle of Sufficient Reason)	Non-existence is necessary.
Existence can be argued for *a priori*.	No *a priori* way to specify what particular individuals are included / excluded.	No *a priori* way to specify what particular individuals are included / excluded.	Can *a priori* know that worlds with these violations of the PSR will not be created.	Can *a priori* know that such contradictory concepts have no place in the created world.

Table 1: Ranges of (im)possibility.

It is also worth mentioning an additional difficulty in the argument for this position.[194] In Section 3.3, I used the comparison between imaginary roots and infinitesimals to show how infinitely small lines and their infinitely large reciprocals are only impossible *per accidens*, meaning there is no contradiction involved in positing them as objects in the course of mathematics. In this chapter, I used the fact that infinitesimals were only impossible *per accidens* to read Leibniz's remarks to Bernoulli where Leibniz reasons from infinitesimals not being actual to their being impossible as an indicator of *metaphysical* impossibility rather than logical impossibility. This metaphysical impossibility was then said to be rooted in violations of the Principle of Sufficient Reason. This puts infinitesimals in the same category as voids and atoms, objects that God could create without contradiction, although we can *a priori* rule them out of the order of existing things because of their violations of divine wisdom. The problem is that reading impossible *per accidens* as "something God could create in virtue of His power yet is deficient in ways that allow us to *a priori* deny their existence from metaphysical considerations" works much better for infinitesimals than for imaginary numbers. But can we say that God could create a world where imaginary roots are real? Part of the difficulty in answering this question is that while we could be able to give at least a vague description of what it would look like if infinitesimals were real, it is much harder to give a story about what a world where imaginary numbers are actually instantiated even means. Any answer that mentions the complex plane would only be of use to anyone who was defending this position on their own behalf rather than actually thinking about how Leibniz would have answered this question, for the complex plane was not used as a representation for imaginary numbers until long after Leibniz's death.

Despite this difficulty in imagining what such a world would look like, I will do my best to explain how it would violate principles of intelligibility that would rule out the existence of

[194] My thanks to Jeremy Heis for pressing me on this topic.

imaginary roots, while trying to avoid using mathematical tools that would have been unavailable to Leibniz. Recall how Leibniz motivated imaginary numbers as an intelligible concept: he used geometric reasoning to extract an algebraic formula that represented the relationship between a circle, a straight line tangent to the circle, and the points of intersection between the circle and lines normal to the tangent line: $y = r \pm \sqrt{r^2 - x^2}$ (the above discussion of Figure 3.1 was where this formula was derived). Leibniz remarks that even though there are no real points of intersection when $x > r$, there is no contradiction with using symbolic and algebraic reasoning to act as if there were points of intersection that are not present on the normal geometric plane. The trouble with saying what it would be like if such points of intersection were created by God in our real world is that Leibniz only reasons with such entities via their symbolic representation in algebraic reasoning; he does not have a fully-developed geometric theory of such objects in the same way as he does with infinitesimals. Faced with the question of what it would mean to physically reify imaginary quantities, the answer that strikes me as involving the least amount of speculation is to say that it would mean bodies that do not intersect phenomenally could nevertheless have common borders that were in principle unobservable phenomenally, for this seems to capture the essence of Leibniz's introduction of imaginary quantities as representing points of intersection of otherwise non-intersecting objects. One meaning of these imaginary points of intersection being real is that bodies could exert force on one another without touching phenomenally, for the imaginary point of intersection is the "place" where these two objects touch outside of phenomenal space. Leibniz was notoriously harsh on the Newtonian conception of gravity being a revision of inexplicable occult forces, reflecting his belief that Pre-Established Harmony ensures that God's divine wisdom will ensure that we live in a world explicable through mechanistic laws that involve contact action alone. If Clarke had responded to Leibniz that gravity was not action at a distance but was in fact the result of contact between bodies that only *seemed* to be distant because the points of intersection were real but phenomenally inaccessible, it is far more likely than not that Leibniz would have ridiculed this view rather than entertain it as a viable explanation of natural phenomena. The problem with worlds that have phenomena that cannot be explained using the principles of mechanism alone is that they violate intelligibility criteria imposed by God, rather than being worlds that are absolutely beyond his power. Therefore, if we can think of a world where imaginary points of intersection are real as one in which bodies are able to act on each other in ways that appear to be at a distance, then Leibniz would reject such a world because it violates metaphysical criteria imposed by God's wisdom. Obviously, I cannot claim that Leibniz would have viewed the reification of imaginary points of intersection as being represented by apparent action at a distance, or even that descriptions of a world with imaginary points would have been recognized as an intelligible question in the first place. But this is one way to see how Leibniz would have *a priori* rejected such a possibility without declaring it an absolute impossibility, meaning it would be in the same ontological category as atoms and voids. And this category is where I argued infinitesimals and their reciprocals belong.

While the interpretation that helps us see how Leibniz could assert that infinitesimals are impossible in nature while still believing that they are free from contradiction (by being impossible *per accidens*), the passages from the *New Essays* still must be accounted for, which leads us to the final section of this chapter. This evidence is especially concerning because the reading given here still requires them to be free from contradiction, yet contradiction is explicitly mentioned in the *New Essays*.

3.5 "A space … made up of parts"

I now take up another possible piece of evidence against my reading of infinitesimals as consistent fictions. As I have repeatedly emphasized throughout this chapter, Leibniz's descriptions of infinitesimals make them more possible than the logical contradictory but are still barred from existence in the created world. Perhaps the most damning piece of evidence against this view is the line from the *New Essays* that was quoted in Section 3.2 of this chapter. There, Leibniz says that a space that is thought of as an infinite whole made up of parts is contradictory, groups this concept with infinitesimals, and says that such quantities are only useful in mathematics. In this section, I explain how the qualifier "*made up of parts*" is essential to the point of this passage in the *New Essays*, allowing us to say that infinitesimals in a mathematical context remain free of contradiction. The distinction between the ideal space of geometry and the real space of the created world is introduced in this section but receives a more detailed treatment in the next chapter on the ways infinitude arises in Leibniz's account of the created world.

Book 2, Chapter xvii of the *New Essays* is titled "Of Infinity." Leibniz's overall aim in this chapter is focused on arguing that, contrary to Locke, we have a positive idea of the infinite, and that this idea is innate and is not given to us by sense experience, a consideration that again goes against Locke's account of human understanding. Leibniz uses the first section of this chapter to mention the difference between a categorematic and syncategorematic understanding of the infinite, and only the latter is actual. He says that it is correct to say that there exists an infinity of things but denies that one can collect these into a genuine whole. He says that any true infinite is that which is absolute and precedes any composition from parts. He goes on to disagree with Locke about whether the infinite is a "modification of expansion and duration" or any notion with "magnitude or multiplicity" (with Leibniz settling on the latter) and whether or not it is the infinite or the finite that is a modification (with Leibniz again opting for the latter).[195]

[195] *New Essays* p. 158.

Next comes Section 3, in which Leibniz gives the quote that is worrisome for the interpretation of infinitesimals I have been advancing. Due to its importance, I quote it in full:

> *Philalethes* [Locke's spokesman]: It has been our belief that the mind gets its idea of infinite space from the fact that no change occurs in its power to go on enlarging its idea of space by further additions.
>
> *Theophilus* [Leibniz's spokesman]: It is worth adding that it is because the same principle can be seen to apply at every stage. Let us take a straight line, and extend it to double its original length. It is clear that the second line, being perfectly similar to the first, can be doubled in its turn to yield a third line which is also similar to the preceding ones; and since the same principle is always applicable, it is impossible that we should ever be brought to a halt; and so the line can be lengthened to infinity. Accordingly, the thought of the infinite comes from the thought of likeness, or of the same principle, and it has the same origin as do universal necessary truths. That shows how our ability to carry through the conception of this idea comes from something within us, and could not come from sense-experience; just as necessary truths could not be proved by induction or through the senses. The idea of the absolute is internal to us, as is that of being: these absolutes are nothing but the attributes of God; and they may be said to be as much the source of ideas as God himself is the principle of beings. The idea of the absolute, with reference to space, is just the idea of the immensity of God and thus of other things. But it would be a mistake to try and suppose an absolute space which is an infinite whole made up of parts. There is no such thing: it is a notion which implies a contradiction; and these infinite wholes, and their opposites the infinitesimals have no place except in geometrical calculations, just like the use of imaginary roots in algebra.[196]

The first portion of Theophilus's reply mimics the reasoning Locke sets out in the corresponding passage of the "Old" *Essay Concerning Human Understanding*, in which he says that our power to continually add or multiply any given length without having a reason to stop is the origin of our idea of infinite space. In his recasting, however, Leibniz emphasizes the importance of continuing the *same* operation, rather than continuing with any arbitrary chain of different operations that increase the magnitude in question. It seems that under Locke's presentation, one could gain the idea of infinite extension by taking a quantity x, doubling it, adding x again to

[196] *New Essays* p. 158.

this quantity, doubling the result, adding another arbitrary quantity, quintupling that result, and continuing these operations without rhyme or reason. Instead, Leibniz thinks it is important that we apply one and the same operation without limit (with the tacit assumption that the operation is one that will increase the quantity without limit, rather than continually applying an operation that results in a convergent infinite series). However, it is also possible that this disagreement is superficial, as "increase this line segment by an arbitrary length" would be a principle that also yields a line segment similar to the original, and adding any other arbitrary line segment to the resulting one would create yet another similar segment. If this were the case, Leibniz would not so much be disagreeing with Locke in the first portion of Theophilus's response but drawing out something he sees as implicit in Locke's argument.

Regardless of whether or not this part of Theophilus's reply is a genuine disagreement with Locke, it is clear that the remainder of his response is a critique of Locke's viewpoint. Leibniz compares our idea of the infinite with necessary truths. This point of comparison revolves around neither being something that Leibniz sees as obtainable through empirical sources. In the preface to the *New Essays*, Leibniz says that our understanding of necessary truths are the result of innate ideas and principles without our own mind. Leibniz agrees with Locke that sensory experience is necessary for all of our knowledge, but he points to the truths of mathematics and logic as showing that such empirical sensations are not sufficient for all instances of knowledge, for "we must contribute something from our side."[197] The necessity of sense experience comes from the fact that these innate ideas are not immediately transparent to us but arise from our careful introspection and prompting from sensory experience. Hence, the process of repeating one and the same operation does not generate the concept of the infinite, but instead illuminates an already existing idea within us. Leibniz makes this point earlier in his discussion of eternity, where he says, "the idea of the *absolute* is, in the nature of

[197] *New Essays,* p. 49. Emphases in original.

things, prior to that of the *limits* which we contribute, but we come to notice the former only by starting with whatever is limited and strikes our senses."[198]

We now arrive at the problematic passage: Leibniz's declaration that an infinite space is contradictory, and so are infinitesimals, by extension. The way out of this is to note that Leibniz here is specifically warning us to not dabble in an infinite space that is "composed of parts." This naturally raises the question of whether we can think of an infinite space that is not composed of parts, and answer that has a resounding "yes!" As Vincenzo De Risi makes clear in his work on Leibniz's *analysis situs*, Leibniz's mature position is that space is *constituted* by points, but is not *composed* of them.[199] In the case of the ideal space of geometry, this is because no parts exist unless specified, a reflection of Leibniz's principle that the mark of an ideal object is the priority of the whole to its parts.[200] Thus, while we can specify an infinitude of points within geometric space, or an infinitude of finite shapes, we can think of an infinite space without thinking of it as being composed of parts. This is an incomplete notion, for such a space leaves open the possibility of an infinitude of divisions. And it is precisely because it is an ideal and incomplete notion that does not already presuppose the existence of infinitely many parts that we are able to think of it as one whole without contradiction. So long as we remain in the realm of the geometric where parts are merely possible, we remain free to reason with notions that would suddenly become contradictory if we were forced to think of them as actuals composed from an infinitude of parts. And since we have seen how we can have a concept of space that is infinite, ideal, not composed of parts, and free of contradiction, we can say that the consistency of infinitesimals within pure geometry is not a target of this quote from the *New Essays*.

[198] *New Essays*, p. 154.
[199] De Risi 2007, pp. 173*ff.*
[200] This is a claim that resurfaces through Leibniz's career. One sees it in 1676's "Infinite Number" (in *DLC* p. 97), 1695's "Note on Foucher's Objection" (*AG* p. 147) and the July 31, 1709 letter to Des Bosses.

Admittedly, this reading of the passage is somewhat contorted, but it is the best way to tie this passage together with other themes of Leibniz's thought that we have already seen. When we look at the passage in question, there may be some equivocation on the meaning of "these infinite wholes":

> But it would be a mistake to try and suppose an absolute space which is an infinite whole made up of parts. There is no such thing: it is a notion which implies a contradiction; and these infinite wholes, and their opposites the infinitesimals have no place except in geometrical calculations, just like the use of imaginary roots in algebra.

On the one hand, it seems that "these infinite wholes" mentioned in the last sentence must be those that are "made up of parts," given the first sentence. On the other hand, we saw in Section 2.1 that infinite wholes made up of parts are not admissible in mathematical contexts. So, if they have a place in "geometrical calculations," they must be infinite wholes whose parts are not actual and do not compose them. And the fact that they are employed in geometry as imaginary roots are in algebra is also evidence that these must be infinite wholes without parts. Imaginary roots are consistent, infinite wholes composed of actual distinct parts are not, therefore the infinite wholes must be those that are not composed of parts, like the ideal space of geometry in which the whole precedes its potential parts.

3.6 Conclusion

In this chapter, we have built upon the groundwork of the previous chapters to develop a reading of infinitesimals in which they are logically possible but barred from existence for metaphysical principles. In Chapter 1, we saw how the relationship that parts have to their whole becomes central to Leibniz's classifications of the infinite, being at the forefront of Leibniz's 1706 taxonomy of the infinite. This involved a strong denial against the existence of any infinite wholes that are composed of actual distinct and separate parts. Or to use Leibniz's technical language, he denies the existence of a categorematic infinite in which there are parts that are contained formally and actually. While the 1676 taxonomy involved a distinction between lines that were “merely” infinite and unbounded lines that were maximally infinite, these two types of infinite lines would no longer be placed in different categories in the 1706 taxonomy, for they do not differ with regards to the relationship that the whole of each kind of line would bear to its possible parts.

In Chapter 2, we saw a variety of non-finite entities that Leibniz treats within his mathematics. There we again witnessed part-whole relations playing a central role. When some mathematical concepts, such as number, are extended into the infinite, they lead to conflict with the axiom that a whole is greater than any of its parts. Unlike a modern set theorist, Leibniz is not willing to sacrifice this principle and promptly rejects the very coherence of infinite number. Instead, his conception of the infinitude of numbers means that one cannot speak of all the numbers as a completed totality. This is a syncategorematic account because it involves the possibility of continually specifying new numbers without there being some further whole containing every single one of those indefinitely many numbers as parts (Section 2.1). Abandoning the concept of an infinite totality also raises problems with viewing the limit of a convergent series as a completed sum of infinitely many parts (Section 2.3). Unlike infinite number, Leibniz is inclined to say that one convergent series is twice another and also add, subtract, and multiply different series by one another. Leibniz speaks of the sum of convergent infinite series as a limit the series approaches as one considers more and more terms of the series because his block against the very logical coherence of the categorematic infinite prevents him from viewing the series as a completed summation of infinitely many terms. We again see a concern with wholes consisting of infinitely many parts in Leibniz’s treatment of the continuum. Leibniz’s ban on wholes consisting of infinitely many parts did not jeopardize the relationship between lines and points because Leibniz holds that the continuous objects of geometry are posited as wholes first, with the potential to be divided in infinitely many ways

(Section 2.2). Points are the minimum elements of the continuum and their inhomogeneity with lines means that Leibniz never speaks of a geometric proportion between points and lines. Opposite points as minima, are the unbounded lines that are treated as maxima. Here we see Leibniz try and compare the sizes of infinite lines by using geometric part-whole relations but wound up finding that he was unable to treat such lines as having a determinate length while still being able to freely rotate them or superpose them onto other lines while maintaining the same quantity in length (Section 2.5). For this reason, Leibniz says that bounded quantities are the median between points and the unbounded infinite by analogy only, for the latter are incapable of being subjected to the algebraic operations that proportions between homogenous quantities can undergo.

In Section 2.4 and its subsections, we saw some of the mathematical properties of infinitesimals. Many of these properties hinged on them being subjected to the same operations that are used on finite quantities, including the ability to put a finite line in geometric proportion to an infinitely small line and an infinitely long (but bounded) one. Unlike points, infinitesimals are still lines, allowing them to be further divided into smaller parts, even infinitely small ones. Infinitesimals and their infinitely bounded counterparts have an advantage over numbers and the terms of discrete series because like other continuous geometric objects, they are postulated as wholes whose parts can later be freely specified. And unlike points and unbounded lines, we can use the Law of Continuity to extend our reasoning about finite quantities to infinitesimals and bounded infinite lines.

In this chapter, I have argued that contrary to what some commentators have said, Leibniz's fictionalism does not imply logical contradiction in the concept of an infinitesimal, and references to infinitesimals do not have a hidden logical structure that reduces to a syncategorematic process. While references to infinitesimals may be *replaced* by a series of quantities that become arbitrarily small, they remain perfectly consistent if interpreted at face value. Leibniz's remarks about imaginary roots in algebra show his belief that something can be logically and mathematically consistent, while still having no place in the realm of created beings. And his frequent comparison between imaginary roots and infinitesimals show that he believes that infinitesimals and their reciprocal infinities fall into this category of *per accidens* impossibility. And this simultaneous freedom from contradiction and impossibility of physical instantiation is how we should interpret Leibniz's remarks about the fictionality of infinitesimals.

We thus see that Leibniz's fictionalism about certain types of mathematical entities reflects his professed attitude that mathematics and metaphysics are separate enterprises. While within the realm of pure mathematics, there is no need to worry about the ontological counterparts of the entities being symbolically manipulated and reasoned about. Instead, it is freedom from contradiction that underpins the mathematician's concerns when introducing new types of objects into one's reasoning.

Afterward: Leibnizian Infinities Outside of Mathematics

In the preceding pages, we saw how highly nuanced Leibniz's treatment of the infinite is within his pure mathematics. Infinitesimals have a place within the calculations performed by pure mathematics, even though their utility in mathematics should be given no weight when it comes to what kinds of beings exist in the created physical world. Additionally, unlike infinitesimals, any supposed whole that is composed from infinitely many distinct parts can only enter into mathematics for the purposes of proofs by contradiction. And as seen in Chapter 2, there are numerous other uses of the infinite in Leibniz's mathematics that are not reducible to each other. Understanding the nuanced ways that these distinct types of infinity are used in mathematics requires a careful approach that attends to the differences in their properties. In this afterward, I examine locations in which Leibniz uses the infinite that reaches beyond pure mathematics. Unfortunately, we do not have the space to fully treat each of these subjects with the depth that they require, which is why they are presented here as sketches for possible avenues that one can take when studying how the infinite functions in these diverse areas. In this afterward, I explain some of the ways that the infinite arises in Leibniz's physics and his logic.

The infinite in Leibniz's physics

In Leibniz's physics, one way that the infinite arises is in so-called "dead force." Leibniz tends to represent the effect of this force as infinitesimal in comparison to "living force." One place where Leibniz does this is in his *Specimen Dynamicum*, Suppose that a tube $AC$is rotating clockwise around the stationary point Cat a uniform rate [Figure 4.1]. Let B represent a ball within the tube that was held in position by a string during the early portion of the tube's rotation about point C, but has just been cut when the tube is in the position on the left-hand side of the diagram. Had the ball remained attached to the string, it would have been in position (D) as the tube rotated from A to (A). However, once the ball is no longer held in place by the string, it will recede from the center of rotation (i.e., point (C)) due to centrifugal force. Thus, the ball will be in position (B), rather than (D) when the tube is in position (A). Leibniz then says that the ball's impetus (the term he uses for an object's speed[201] multiplied by its mass) can be decomposed into the movement from D to (D) and (D) to (B), and one can "compare" (that is, form a ratio between) these two impetuses when the tube is in position $(A)C$. But Leibniz claims that at the initial moment in which the ball was released from the string, its tendency to recede away from the center of rotation is infinitely small in comparison to the rotational impetus, for its motion away from point C has not yet begun. Leibniz calls these infinitely small tendencies to move in a given direction "solicitations," and these solicitations are best understood as referring to what we now call acceleration.

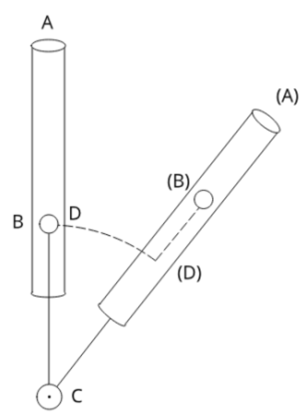

Figure 4.1 A diagram from Specimen Dynamicum

Leibniz uses the distinction between an infinitely small solicitation and a finite impetus to describe the effects of two different types of force. At first, Leibniz simply defines "Living force"

201 Leibniz uses the term "*velocitas*" in defining impetus (GM VI p. 237), which is rendered as "velocity" in the translation given in AG. This quantity only refers to the amount of space an object can cover over a given time, without taking direction into account. Instead, he uses the term "*conatus*" to refer to *velocitas* in conjunction with a direction. In contemporary physics, "speed" correlates to Leibniz's "*velocitas*," and "velocity" with his "*conatus*." For the purposes of this afterward, I have chosen to use the contemporary names for these quantities.

(*vis viva*) as “an ordinary force, joined with actual motion.”[202] While this description alone is far from enlightening, Leibniz goes on to say that the living force between two separate bodies is proportional to their mass multiplied by their velocity squared.[203] In the rotating tube example, living forces are responsible for the rotational motion of the ball and tube along the center C, as well as the ball’s motion away from that center point once it is in motion. In addition to these living forces, Leibniz also says that this example shows the presence of a “dead force” (*vis mortua*). In dead force, “motion does not yet exist in it, but only a solicitation to motion,” yet motion can arise as the result of an infinite repetition of dead forces.[204] Although sometimes Leibniz’s wording appears to equate actual motion with living force and “solicitations" with dead force, the forces are metaphysical causes that are inferred, and the motions they cause are merely their observable effects.[205]

When discussing the connection between living and dead forces, Leibniz says that it takes an infinite repetition of dead force to produce a living force, and he mathematically represents the solicitation to motion that is produced by dead force as infinitely small in comparison to an already existing motion. He cautions against making any ontological inferences from these mathematical representations:

> From [the example of the rotating tube], it is obvious that the *nisus*[206] is twofold, that is, elementary or infinitely small, which I also call *solicitation*, and that which is formed from the continuation or repetition of elementary nisus, that is, *impetus* itself. Nevertheless, I wouldn’t want to claim on these grounds that these mathematical entities are really found in nature, but I only wish to advance them for making careful calculations through mental abstraction.[207]

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[202] From *Specimen Dynamicum*, in AG p. 121.
[203] In AG p. 128. This shows that Leibniz’s *vis viva* is equivalent what we now call kinetic energy, which is measured as $\frac{1}{2}mv^2$.
[204] In AG pp. 121-122.
[205] Garber 2009, p. 136.
[206] “*Nisus*” is a term Leibniz leaves undefined in the *Specimen Dynamicum*, but from context, it appears to mean the tendency of a body to move in a given direction.
[207] In AG p. 121. Emphases in the original.

This presents a level of agnosticism about the existence of infinitesimals, for Leibniz simply says that this example does not establish their existence in the natural order of things, but he does not explicitly rule out the existence of infinitely small objects in the *Specimen Dynamicum* either.

Despite the independence of metaphysics from *pure* mathematics, Leibniz says that a mathematical understanding of nature will require one to deduce laws from metaphysical principles. One of the arguments that Leibniz advances in *Specimen Dynamicum* is that Cartesian laws of motion are false, for they only consider bodies as following purely geometric laws.[208] Instead, Leibniz argues that one must add something above and beyond purely geometric determinations to bodies, namely the concept of force. With this concept, Leibniz deduces a few mechanical laws *a priori*, such as the law that the amount of force produced in an effect is equal to the amount of force expended by a given cause. Leibniz shows that if nature did not obey this law, one would have a false conclusion: perpetual motion. Perpetual motion is not an absolute impossibility, but one that contradicts the "order of things."[209] Laws such as these are not grounded in the Law of Contradiction, but considerations of "the divine wisdom," that is by the consideration that God decreed that everything happen for a determinate reason and chose to create only the best of all possible worlds.[210] The role that final causes (i.e. considerations stemming from God creating the most perfect universe) play in Leibniz's physics is subtle; we cannot use them to explain any particular cause or effect, presumably because elsewhere Leibniz says that such considerations would involve an infinite comparison of possibilities in order to determine which truly contributes to the best of all possible worlds, the subject of the next section of this afterward.

[208] The details of Leibniz's arguments against these laws of motion need not concern us. He essentially argues that Descartes was wrong to think that momentum (i.e. mv) is conserved in collisions, and instead argues that *vis viva* (i.e. mv^2) is the conserved quantity.
[209] From a 1691 note that contains preliminary demonstrations of the work presented in *Specimen Dynamicum*, in AG p. 106.
[210] In *Specimen Dynamicum*, p. 126.

Despite our inability to appeal to final causes when treating particular causes and effects, Leibniz tells us in the *Specimen Dynamicum* we can, and indeed must, consider final causes when "establishing general and efficient principles" in an *a priori* way.[211] Once these laws are in place, we can return to a purely mathematical description of physical phenomena that includes the laws derived from metaphysical considerations as axioms. That is, we cannot explain a particular event like a glass falling to the ground by arguing that more good has been produced from it falling than a possible world where it remained on the table. But we can say that during its fall, the forces and motions involved will conform to the general laws of physics that are established through considerations of final causes. And although such causes are not necessary in the sense that a violation of them would entail a logical contradiction, they have what Leibniz refers to as a "moral certainty," for they follow from God's omnibenevolence.[212]

In addition to the laws that maintain that the equality of force between cause and effect, Leibniz refers to the Law of Continuity in the context of physics as an architectonic law, meaning it is knowable *a priori* from considerations of final causes, rather than logical necessity. Notably, the Law of Continuity is only architectonic when applied to physics; it is "absolutely necessary" in geometry.[213] The geometric use of the Law of Continuity is to treat infinitesimal cases using the same laws as finite cases; its physical interpretation is that rest can be treated as an infinitely small motion, and it allows one to reason mathematically about nature in the first place. Were the Law of Continuity false in physics, there would be discontinuous jumps in the motions of bodies, and there is nothing logically contradictory about this. But Leibniz believes that if bodies were to move from one position to another, then the only way to account for such motions would be

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[211] In *Specimen Dynamicum*, p. 126.

[212] This is a crucial point of dispute in the Leibniz-Clarke correspondence, and shows up specifically in Paragraphs 9, 73, and 76 of Leibniz's Fifth Letter in this exchange.

[213] From "Letter of Mr. Leibniz on a General Principle Useful in Explaining the Laws of Nature Through a Consideration of the Divine Wisdom…" in L. p. 352. This paper was published in 1687 in *Nouvelles de la republique des lettres* and is Leibniz's first public presentation of the Law of Continuity.

through the inscrutable actions of God, for the body would not move through its own power.[214] Leibniz does not elaborate on why a violation in the Law of Continuity would lead to this conclusion. However, he asserts that if nature did not obey the Law of Continuity, the very possibility of understanding it through mathematical laws would be demolished. This explains the puzzling remarks mentioned at the start of this paragraph in Leibniz's letter to Varignon, where he claims we could establish no science of motion if the hypothesis of infinitesimals contradicted the observed behavior of bodies. While infinitesimals may not exist in nature, we can be confident that nature will act as if there were such entities. And our ability to do this is guaranteed by the knowledge that God would not create a world in which we could not formulate a scientific understanding of bodies.[215]

In this dissertation, I have defended the view that the logical consistency of infinitesimals allows them to be freely deploying in mathematical reasoning without contradiction. I argued that the reasons that bar infinitesimally-sized bodies from existence in the real world is that they would violate the principles that guide, but do not force, God's decision in which possible world to create. These architectonic considerations that bar the real existence of infinitesimals at the same time justify the legitimacy of applying the Law of Continuity to nature and as reasoning as if there were infinitesimal relations between forces. There is nothing logically necessary about bodies moving in ways that comport to the Law of Continuity, so the reasons that give us *a priori* certainty that this law is applicable are distinct from the law of noncontradiction alone, just as the law of noncontradiction alone does not explain why infinitesimals are not present in our created world. This connection between metaphysical reasoning telling us that there cannot be infinitely small bodies or motions, while also telling us that we are warranted in behaving as if there were is a tension worthy of further analysis.

[214] From the Sep 11, 1699 letter to De Volder, in L, pp. 521-523.
[215] The letter to Sophia Charlotte, in L, p. 583.

The infinite in Leibniz's logic

One peculiar Leibnizian doctrine is the claim that all truths can be proven *a priori*, and as we will see in this section, Leibniz makes reference to the infinite when specifying how this doctrine works in the case of contingent truths. Leibniz held that a series of logical reasoning will reduce all truths to primary truths, defined as "those which assert the same thing of itself or deny the opposite of its opposite."[216] Primary truths are identities, broadly construed. In order to decompose a truth into a primary truth without the aid of experience, Leibniz relies on definitions. As an example of this process, Leibniz shows how to prove the axiom "a part is less than the whole," an axiom that surfaced frequently in the above section on infinite number. With assistance from to the definition of less, "The *less* is that which is equal to the part of another (the *greater*)," Leibniz gives the following analysis:

1. The part is less than the whole. (initial statement).
2. The part is equal to a part of the whole. (def of "less")
3. The part is equal to itself. (It is the "part of the whole" of step 2)

A large part of the theoretical apparatus that underlies this process of analysis is the doctrine of concept containment. All humans are rational" is a truth sourced in the containment of the predicate "rational" within "animal," the subject term. In Leibniz's own words:

> "Every true categorical proposition, [affirmative and universal],[217] signifies nothing but a certain connection between the predicate and the subject - in the direct case, that is, of which I am always speaking here. This connection is such that the predicate is said to be in the subject, [or to be contained in it, and this either absolutely and viewed in itself, or in some particular case.] Or in the same way, the subject is said to contain the predicate; that is, the concept of the subject, either in itself or with some addition, involves the concept of the predicate. And therefore, the subject and predicate are mutually related to

[216] From "Primary Truths" in AG, p. 30.
[217] "Elements of Calculus" in L, p. 236. Loemker's translation, used here, was based on Couturat's transcription. Couturat notes that "universalem" was a later addition by Leibniz, with "affirmativam" being an even later addition. Brackets added to note this feature of the manuscript.

> each other either as whole and part, or as whole and coinciding whole, or as part to whole"[218]

The containment of the subject within the predicate is not just restricted to logical truths; Leibniz believed that all truths whatsoever were grounded in concept containment and could thus be proven a priori by analysis. To take an example from Leibniz, "Julius Caesar crossed the Rubicon" is true, and hence must be provable through an *a priori* analysis.[219] Because analytic truths are grounded in concept containment, the concept "Caesar" must contain the concept "crossed the Rubicon" as one of its characteristic marks. But Caesar did a lot more in his life than cross the Rubicon, so "became dictator of Rome," "was killed on the Ides of March," "is not a number," and every other fact about the famous conqueror of Gaul are all contained in the concept "Julius Caesar." Thus, all individual concepts are fully determined with respect to all their properties. Analogous to maximally consistent sets, any marks added to the concept of an individual will either already be contained in the concept or will contradict an existing mark within the concept.[220] The level of detail of the concept of an individual would have to be massive for it to contain information about all the events that happen to the individual. But there is even more complexity than one might first think, for each concept contains the whole series of its world within it.[221] That is, the concept of Julius Caesar will contain a record of all of the events that happened directly to him during his life, as well as events in the distant parts of the universe during that time. Additionally, a restriction to just the events that happened during Caesar's lifespan is invalid, for when Leibniz says the *entire* series of the world is written within an

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[218] "Elements of Calculus," L p. 236
[219] *Discourse on Metaphysics* § 13, AG p. 45
[220] As Leibniz puts it: "If there is a term BA and B is an individual, A will be superfluous; or if BA = C, then B = C." Leibniz's marginal note to *General Inquiries* §72. In P, p. 65. Here it is assumed that "BA" is not a self-contradicting concept.
[221] This is a recurring theme: "Monadology" §56; "Primary Truths," AG p. 32; "Discourse on Metaphysics" §13; and even in the title of "Remarks on Arnuald's Letter about My Proposition That the Individual Notion of Each Person Includes Once and for All Everything That Will Even Happen to Him" AG p. 69.

individual concept, he means events both spatially and temporally remote from the actual individual.

This claim that all truths can be proven *a priori* is frequently presented as a reformulation of the Principle of Sufficient Reason. The Principle of Sufficient Reason states that "nothing happens without it being possible for someone who knows enough things to give a reason sufficient to determine why it is so and not otherwise."[222] The link between these two doctrines is revealed by noticing that Leibniz describes the process of a proof as an analysis "through reasons for reasons."[223] If every truth can be proven *a priori*, and proofs just are a chain of substituting reasons for reasons, then we arrive at the claim that every truth must have a sufficient reason. Conversely, if every truth has a sufficient reason for being one way rather than another, a proof just would be an appeal to those determining reasons.

Leibniz notes that his belief in the determination of all things by a sufficient reason led him to the brink of some dangerous conclusions:

> When I considered that nothing happens by chance or by accident (unless we are considering certain substances taken by themselves), that fortune distinguished from fate is an empty name, and that nothing exists unless its own particular conditions are present (conditions from whose joint presence it follows, in turn, that the thing exists), I was very close to the view of those who think that everything is absolutely necessary, who judge that it is enough for freedom that we be uncoerced, even though we might be subject to necessity, and close to the view of those who do not distinguish what is infallible or certainly known to be true, from that which is necessary.[224]

Leibniz goes on to say that considerations of self-consistent works of fiction drew him from the view this necessitarian view. For there one finds unactualized possibles. However, the predicate-in-subject account of truth pulled him back towards the position that all truths are necessary: "for if the notion of the predicate is in the notion of the subject at a given time, then

[222] "Principles of Nature and Grace Based on Reason" in AG p. 210.
[223] "On Contingency" in AG p. 28.
[224] "On Freedom" in AG p. 94

how could the subject lack the predicate without contradiction and impossibility, and without changing that notion?"[225] Leibniz claims that what released him from these competing tensions and allowed him to acknowledge the existence of genuine contingencies was an appeal to the role of infinity in logic.

While he maintains that all truths can be proven *a priori*, Leibniz claims that this proof can either be infinite or finite. For a necessary truth, the analysis from the truth to a primitive identity statement will be completed in a finite number of steps. Leibniz also says that necessary truths are those that depend on the principle of contradiction and the negation of a necessary truth implies a contradiction. Because a necessary statement can be turned into an identity statement, the negation of a necessary truth would be tantamount to asserting "A is not A."

In a contingent truth, any step in the analysis will contain some unanalyzed component that can be further decomposed before an identity statement is reached. The denial of a contingent statement is another contingent statement. This is because if the denial of a contingency implied an impossibility, then the denial of that impossibility would be a necessary statement, contrary to the supposition that the initial statement was a contingency.[226] Instead, these truths are based on the principle of perfection: the claim that God will choose to create the best of all possible worlds. While the negation of a contingent truth will not imply a contradiction, it will imply a sub-optimal configuration of the series of events within a world, and the sub-optimality of such a world gives a sufficient reason for its non-existence. But these assessments of the relative perfection of different possible worlds is not something that our finite minds are able to perform.

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[225] "On Freedom" in AG p. 95
[226] "On Freedom and Possibility" in AG p. 20

As finite minds, we cannot complete an analysis of contingent truths because we cannot fully exhaust the infinitude of reasons for such truths. God, according to Leibniz, does possess knowledge of such an infinite series through "knowledge by intuition [*scientia visionis*]." One might think that God is able to infinitely run through all the steps of this process and arrive at the final step: the resolution to an identity statement. But the way Leibniz frames God's perception of finite truths makes this interpretation unlikely:

> But in contingent truths, even though the predicate is in the subject, this can never be demonstrated, nor can a proposition ever be reduced to an equality or to an identity, but the resolution proceeds to infinity, God alone seeing, not the end of the resolution, of course, which does not exist, but the connection of the terms or the containment of the predicate in the subject, since he sees whatever is in the series.[227]

This seems to imply that God's faculty of "knowledge by intuition" is able to directly grasp the conclusion of an infinite analysis in a way that's distinct from the completion of the syncategorematic series of reasons generated by our finite minds. I need to do more research into the concept of this *scientia visionis*, as well as the "knowledge of simple understanding [*scientia simplicis intelligentiae*]" that it is contrasted with in order to say more about God's grasp of contingent truths. Regardless of how it is that these forms of knowing provide God with *a priori* knowledge about which contingent truths will lead to a better total sum of perfection in the world, our finite minds cannot possess such knowledge. We come to know things *a priori* by analyzing concepts. Given how Leibniz describes the process of analysis, it is clear that this process is through a sequential examination of reasons. If this sequence of reasons were infinite in scope, the infinitude could only be portrayed syncategorematically, owing to Leibniz's denial of the categorematic infinite. That is, we can always extend the analysis to represent more and more reasons for a fact to our minds, but there is never a final stage where all of the reasons for a fully determinate event are laid before our minds as a completed whole. And since

[227] "On Freedom." AG p. 96

a complete reckoning of a contingent truth is required for an *a priori* ruling on its truth, we can never fully analyze such contingent truths. However, God's comprehension of these truths cannot be a whole analysis consisting of infinitely many distinct concepts as its components, for early on in this dissertation, we saw Leibniz say that not even God could comprehend a whole made up of infinitely many parts. This therefore highlights that the difference between God's cognition of contingent truths and our own cannot be one of degree alone but must be a difference in kind.

One of the recurring themes of this dissertation is that Leibniz's treatment of the infinite is an exceptionally complex subject, prompting the lengthy analysis of how to understand Leibniz's attitude towards the infinite within pure mathematics. For this reason, I have avoided making any definitive claims about how the work of this dissertation affects our understanding of the role of the infinite in those areas. The same level of care should be made when comparing Leibniz's use of the infinite in different domains of inquiry as when comparing the differences in Leibniz's use of the infinite within and across the various taxonomies present in the main three chapters of this book. However, rigorously understanding Leibniz's use of the infinite within his pure mathematics will certainly equip one to better navigate the nuances of the infinite in Leibniz's other work.

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