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Exploring Five-Dimensional Superconformal Field Theories with Holography and M-theory

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Exploring Five-Dimensional Superconformal Field Theories with Holography and M-theory

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

Justin Kamyar Kaidi

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ABSTRACT OF THE DISSERTATION

Exploring Five-Dimensional
Superconformal Field Theories
with Holography and M-theory

by

Justin Kamyar Kaidi
Doctor of Philosophy in Physics
University of California, Los Angeles, 2020
Professor Eric D'Hoker, Chair

In this dissertation, we discuss two results relevant to the study of five-dimensional superconformal field theories. In the first half of this work, we use six-dimensional Euclidean $F(4)$ gauged supergravity to construct a holographic renormalization group flow for a superconformal field theory on S^5. Numerical solutions to the BPS equations are obtained and the free energy of the theory is determined holographically by calculation of the renormalized on-shell supergravity action. A candidate field theory dual to these solutions is then proposed. This tentative dual is a supersymmetry-preserving deformation of the theory engineered via the D4-D8 system in string theory. In the infrared, this theory is a mass deformation of a $USp(2N)$ gauge theory. A localization calculation of the free energy is performed for this infrared theory, and is found to match the holographic free energy.

In the second half of this work, we establish a close relation between recently constructed AdS_6 solutions in Type IIB supergravity, which describe the near-horizon limit of (p, q) 5-brane junctions, and the curves wrapped by M5-branes in the M-theory realization of the 5-brane junctions. This provides a geometric interpretation of various objects appearing in the construction of the Type IIB solutions and a physical interpretation of the regularity conditions. Conversely, the Type IIB solutions can be used to

obtain explicit solutions to the equations defining the M-theory curves associated with (p, q) 5-brane junctions.

The dissertation of Justin Kamyar Kaidi is approved.

Terence Tao

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Michael Gutperle

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Chapter 2 of this thesis reports on joint work with Michael Gutperle and Himashu Raj [GKR18]. Chapter 3 of this thesis reports on joint work with Christoph Uhlemann [KU18].

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PUBLICATIONS

• J. Kaidi, J. Parra-Martinez, Y. Tachikawa, "Topological Superconductors on Superstring Worldsheets," arXiv:1911.11780 [hep-th].

• J. Kaidi, J. Parra-Martinez, Y. Tachikawa, "GSO Projections via SPT phases,"Phys.Rev.Lett. **124** (2020) 12, 121601 arXiv:1908.04805 [hep-th].

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• D. Corbino, E. D'Hoker, J. Kaidi and C. F. Uhlemann, "Global half-BPS $AdS_2 \times S^6$ solutions in Type IIB," JHEP **1903**, 039 (2019) arXiv:1812.10206 [hep-th].

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• K. Arisaka *e*t al., "Expected Sensitivity to Galactic/Solar Axions and Bosonic Super-WIMPs based on the Axio-electric Effect in Liquid Xenon Dark Matter Detectors," Astropart. Phys. **44**, 59 (2013) [arXiv:1209.3810 [astro-ph.CO]].

☐ Publications marked with a square are discussed in this thesis.

CHAPTER 1

Introduction

One of the many remarkable outcomes of string theory is evidence for the existence of interacting superconformal field theories (SCFTs) in five and six dimensions. Such theories were historically thought not to exist, due in part to the fact that they do not admit a conventional Lagrangian description. However, it is now known they can be realized as low-energy limits of string and M-theory, which allows one to study e.g. their moduli spaces and relevant deformations. In many cases, deformations can be found that *do* admit an effective Lagrangian description, allowing for a match to effective field theory analyses and providing further evidence for the stringy constructions.

Five-dimensional SCFTs, which are the main concern of this thesis, can be realized in a variety of ways. The first realizations were described in Type IIA, where they correspond to the worldvolume theories of D4-branes probing a stack of D8-branes and $O8^-$-planes [Sei96, BO99, BR12]. More general classes of theories can be realized in Type IIB on the intersection point of (p, q) five-brane junctions [AH97, AHK98, DHI99], and in M-theory either on Calabi-Yau threefolds [MS97, DKV97, IMS97] or by considering the worldvolume theory of an M5-brane wrapping a holomorphic curve with one compact direction [Kol99, BIS97, AHK98, KR98].

Though these stringy constructions provide evidence for the existence of five-dimensional theories, many of their properties remain difficult to study directly. Fortunately – at least in the large N limit – we may use the tools of holography to make some progress. This is one line of study which we pursue in this work. To begin the holographic analysis, one may first notice that, in contrast to theories in other dimensions, five-dimensional superconformal field theories (SCFTs) have a unique superalgebra $F(4)$ [Nah78, Kac77, Shn88], containing $SO(2, 5)$ conformal symmetry, $SU(2)_R$ R-symmetry, and sixteen supercharges (eight Poincare and eight conformal supercharges). Hence the dual supergravities are all expected to have geometries of the form $\mathrm{AdS}_6 \times S^2$, warped over some 2-manifold. We may separate our discussion into two broad classes of solutions:

D4-D8-O8$^-$ solutions: We begin by discussing solutions obtained from massive Type IIA with geometry $\mathrm{AdS}_6 \times S^4/\mathbb{Z}_2$. Such solutions were obtained in [BO99, BR12, Pas13], and describe the near-horizon limit of the aforementioned D4-D8-O8$^-$ system. The S^4 is subject to an antipodal identification, with the singular locus along the equator corresponding to the location of the O8$^-$-plane. As will be reviewed in Chapter 2, the non-Lagrangian SCFTs engineered by this system admit a deformation to $USp(2N)$ gauge theory with some number of hypermultiplets. Holographic study of these SCFTs was initiated in [JP14].

Instead of studying the full massive Type IIA solutions, which are generically rather involved, one can hope to make mileage by restricting to some consistent truncation of them. A particularly well-known truncation is six-dimensional $F(4)$ gauged supergravity [Rom86], which will be reviewed in Chapter 2. Restricting ourselves to this truncation allows for a number of simplifications. First, this theory can be easily coupled to any number of six-dimensional vector multiplets, with the resulting Lagrangian, supersymmetry transformations, and possible gaugings identified in [ADV01]. These theories admit supersymmetric AdS_6 vacua, and determining the spectrum of linearized supergravity fluctuations dual to primary operators is also straightforward [FKP98, DFV00, KL17]. There is a venerable history of work on the use of $F(4)$ gauged supergravity in holography, including [Kar13, Kar14, AFR14, ARS15, HNU14].

On the downside, it is not yet known how to lift generic solutions of six-dimensional gauged supergravity to ten dimensions, and hence a microscopic understanding of the CFT described by such solutions is often lacking. In Chapter 2, we will introduce a certain class of solutions which we claim *does* admit a ten-dimensional interpretation, as a deformation of the D4-D8-O8$^-$ system. From the field theory perspective, this corresponds to a certain supersymmetry-preserving mass-deformation, which we specify. By computing the on-shell effective action of the supergravity solutions and comparing it to the free energy of the deformed 5d SCFT, we obtain a convincing check of this ten-dimensional interpretation.

Brane web solutions: A much larger (and potentially all-encompassing) class of five-dimensional SCFTs can be engineered using so-called (p,q) five-brane webs [AH97, AHK98, DHI99]. This motivates the search for Type IIB supergravity solutions which could describe the near-horizon limit of brane webs. Recently,[1] a family of Type IIB supergravity solutions were found with the geometry $\mathrm{AdS}_6 \times S^2$ warped over a Riemann surface Σ_{IIB} [DGK16a, DGU17a, DGU17b, GMT17]. The solutions are given in terms of a pair of locally holomorphic functions $\mathcal{A}_\pm$ on Σ_{IIB}. For the solutions to be physically regular, Σ_{IIB} is required to have a boundary and the functions $\mathcal{A}_\pm$ are required to satisfy certain constraints, to be reviewed in Chapter 3. Along the boundary of Σ_{IIB}, the differentials $\partial\mathcal{A}_\pm$ have poles, from which the semi-infinite external five-branes of the associated 5-brane web emerge. The (p,q) charges of the emerging 5-brane are fixed by the residues of $\partial\mathcal{A}_\pm$. The solutions are completely specified by the choice of Riemann surface Σ_{IIB}, together with the number of poles and associated residues.

As will be explain in Chapter 3, the locations of the poles and their residues can be given a physical interpretation as capturing the data of semi-infinite (p,q) five-branes resulting from the conformal limit of (p,q) five-brane webs. Various aspects of the solutions and the dual SCFTs have since been studied holographically [GMT17, Kai17, GUV18], and comparisons to field theory calculations supporting the proposed dualities have been presented in [BRU18, FU18]. The solutions have also been extended to describe five-brane webs containing mutually local seven-branes [DGU17c, GTU18].

Though these supergravity solutions have already been used to great effect, there is a sense in which they are quite physically opaque. To remedy this, in Chapter 3 we reinterpret some of the objects appearing in the supergravity solutions in a more intuitive M-theory language. In particular, we will outline a relationship between Σ_{IIB} with the locally holomorphic functions $\mathcal{A}_\pm$ on the one hand, and Σ_{M5} with a holomorphic one-form λ on the other. Here Σ_{M5} is a holomorphic curve wrapped by an M5-brane, and λ is the Seiberg-Witten differential, to be reviewed latter. More precisely, we will argue that the locally holomorphic functions $\mathcal{A}_\pm$ provide an embedding of the doubled Type IIB Riemann surface $\hat{\Sigma}_{IIB}$ into the flat M-theory geometry, and that this embedded surface *is*

[1]For earlier work in this direction, see [LOR13, LOR14, KLM15, AFP14, KKS15, KK16].

the surface Σ_{M5} wrapped by the M5-brane. The Seiberg-Witten differential λ is identified with a locally holomorphic one-form $\mathcal{A}_+\partial\mathcal{A}_- - \mathcal{A}_-\partial\mathcal{A}_+$, which features prominently in the construction of the Type IIB solutions.

CHAPTER 2

Mass deformations of 5d SCFTs via holography

2.1 Introduction

In this chapter, we utilize Romans' $F(4)$ gauged supergravity to study deformations of 5d SCFTs. We will be primarily concerned with deformations of SCFTs by relevant operators which keep some Poincáre supersymmetries unbroken. Well-known cases of such deformations include the $\mathcal{N} = 2^*$ and $\mathcal{N} = 1^*$ theories obtained by mass deformations of $\mathcal{N} = 4$ super Yang-Mills. A systematic classification of operators which break superconformal symmetry but leave all Poincare supersymmetries unbroken was obtained in [CDI16].

In order to make use of localization results, we will furthermore be interested in deformed SCFTs on the Euclidean sphere S^5. Conformal field theories defined on $\mathbb{R}^d$ can be put on other conformally flat manifolds such as the d-dimensional sphere in a unique fashion. However, for non-conformal theories this is not the case, though for many theories it is possible turn on additional terms in the Lagrangian which preserve supersymmetry on the curved space. For $\mathcal{N} = 2^*$ these terms were found in [Pes12] and for gauge theories on S^5 such terms were given in [HST12, KQZ12].

In the context of the AdS/CFT correspondence, such deformations on spheres have been studied for $\mathcal{N} = 2^*$ [BEF14], $\mathcal{N} = 1^*$ [BEK16], and ABJM theories [FP14]. The method used to study these theories holographically is as follows. For a field theory in d-dimensions, one considers a gauged supergravity with an AdS$_{d+1}$ vacuum corresponding to the undeformed superconformal field theory. The ansatz for the metric corresponding to the deformed theory is given by a Euclidean RG-flow/domain wall, where a d-dimensional sphere is warped over a one-dimensional holographic direction. The scalars which are dual to the mass deformations, as well as the additional terms which are necessary for preserving supersymmetry on the sphere, are sourced in the UV. The preservation of supersymmetry in the supergravity demands the vanishing of fermionic supersymmetry variations and provides first-order flow equations for the scalars. The integrability conditions for the gravitino variation determine the metric. For generic scalar sources, the flow will lead to a singular solution, but demanding that the sphere closes off smoothly in the IR provides relations among the UV sources and leads to a nonsingular supersymmetric RG flow. Using holographic renormalization, the free energy of the theory on the sphere is determined by calculating the renormalized on-shell action of the supergravity solutions. The continuation of the supergravity theory from Lorentzian to Euclidean signature, the precise mapping of supergravity fields to field theory operators, and the choice of finite counterterms preserving supersymmetry are among the subtle issues which the papers [BEF14, BEK16, FP14] address in five- and four-dimensional gauged supergravity.

The goal of this chapter is to apply these techniques to matter-coupled six-dimensional gauged supergravity [Rom86, ADV01] in order to study mass deformations of a five-dimensional SCFT on S^5. The structure of this chapter is as follows. In Section 2.2, we review features of the Lorentzian matter-coupled $F(4)$ gauged supergravity theory. In Section 2.3, we discuss the continuation of the supergravity to Euclidean signature and construct the ansatz describing the RG flow on S^5. Vanishing of the fermionic variations leads to the Euclidean BPS equations. We solve these equations numerically and obtain a one parameter family of smooth solutions. In Section 2.4, we use holographic renormalization to evaluate the on-shell action as a function of the mass parameter. In the process, we deal with the subtle issue of identification of finite counterterms needed to preserve supersymmetry on S^5. In Section 2.5, we compare the holographic sphere free energy with the corresponding result obtained via localization in the large N limit of a $USp(2N)$ gauge theory with one massless hypermultiplet in the antisymmetric representation and one massive hypermultiplet in the fundamental representation of the gauge group. In Section 2.6, we close with a discussion.

2.2 Lorentzian matter-coupled $F(4)$ gauged supergravity

The theory of matter-coupled $F(4)$ gauged supergravity was first studied in [ADV01, DFV00], with some applications and extensions given in [KL17, Kar13, Kar14]. Below we present a short review of this theory, similar to that given in [GKR17].

2.2.1 The bosonic Lagrangian

We begin by recalling the field content of the 6-dimensional supergravity multiplet,

$$(e_\mu^a,\, \psi_\mu^A,\, A_\mu^\alpha,\, B_{\mu\nu},\, \chi^A,\, \sigma) \tag{2.2.1}$$

The field e_μ^a is the 6-dimensional frame field, with spacetime indices denoted by $\{\mu, \nu\}$ and local Lorentz indices denoted by $\{a, b\}$. The field ψ_μ^A is the gravitino with the index $A, B = 1, 2$ denoting the fundamental representation of the gauged $SU(2)_R$ group. The supergravity multiplet contains four vectors A_μ^α labelled by the index $\alpha = 0, \ldots 3$. It will often prove useful to split $\alpha = (0, r)$ with $r = 1, \ldots, 3$ an $SU(2)_R$ adjoint index. Finally, the remaining fields consist of a two-form $B_{\mu\nu}$, a spin-$\frac{1}{2}$ field χ^A, and the dilaton σ.

The only allowable matter in the $d = 6$, $\mathcal{N} = 2$ theory is the vector multiplet, which has the following field content

$$(A_\mu,\, \lambda_A,\, \phi^\alpha)^I \tag{2.2.2}$$

where $I = 1, \ldots, n$ labels the distinct matter multiplets included in the theory. The presence of the n new vector fields A_μ^I allows for the existence of a further gauge group G_+ of dimension $\dim G_+ = n$, in addition to the gauged $SU(2)_R$ R-symmetry. The presence of this new gauge group contributes an additional parameter to the theory, in the form of a coupling constant λ. Throughout this section, we will denote the structure constants of the additional gauge group G_+ by C_{IJK}. However, these will play no role in what follows, since we will be restricting to the case of only a single vector multiplet $n = 1$, in which case $G_+ = U(1)$.

In (half-)maximal supergravity, the dynamics of the $4n$ vector multiplet scalars $\phi^{\alpha I}$ is given by a non-linear sigma model with target space G/K; see e.g. [Sam08]. The group G is the global symmetry group of the theory, while K is the maximal compact subgroup of G. As such, in the Lorentzian case the target space is identified with the following coset space,

$$\mathcal{M} = \frac{SO(4,n)}{SO(4) \times SO(n)} \times SO(1,1) \tag{2.2.3}$$

where the second factor corresponds to the scalar σ which is already present in the gauged supergravity without added matter. In the particular case of $n = 1$, explored here and in [GKR17], the first factor is nothing but four-dimensional hyperbolic space $\mathbb{H}_4$. When we analytically continue to the Euclidean case, it will prove very important that we analytically continue the coset space as well, resulting in a dS_4 coset space. This will be discussed more in the following section.

In both the Lorentzian and Euclidean cases, a convenient way of formulating the coset space non-linear sigma model is to have the scalars $\phi^{\alpha I}$ parameterize an element L of G. The so-called coset representative L is an $(n+4) \times (n+4)$ matrix with matrix elements $L^{\Lambda}{}_{\Sigma}$, for $\Lambda, \Sigma = 1, \dots n+4$. Using this representative, one may construct a left-invariant 1-form,

$$L^{-1} dL \in \mathfrak{g} \tag{2.2.4}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$. To build a K-invariant kinetic term from the above, we decompose

$$L^{-1} dL = Q + P \tag{2.2.5}$$

where $Q \in \mathfrak{k} = \mathrm{Lie}(K)$ and P lies in the complement of $\mathfrak{k}$ in $\mathfrak{g}$. Explicitly, the coset vielbein forms are given by,

$$P^{I}{}_{\alpha} = \left(L^{-1}\right)^{I}{}_{\Lambda} \left(dL^{\Lambda}{}_{\alpha} + f^{\Lambda}{}_{\Gamma\Pi} A^{\Gamma} L^{\Pi}{}_{\alpha}\right) \tag{2.2.6}$$

where the $f_{\Lambda\Sigma}{}^{\Gamma}$ are structure constants of the gauge algebra, i.e.

$$[T_{\Lambda}, T_{\Sigma}] = f_{\Lambda\Sigma}{}^{\Gamma} T_{\Gamma} \tag{2.2.7}$$

We may then use P to build the kinetic term for the vector multiplet scalars as,

$$\mathcal{L}_{\text{coset}} = -\frac{1}{4} e P_{I\alpha\mu} P^{I\alpha\mu} \tag{2.2.8}$$

where $e = \sqrt{|\det g|}$ and we've defined $P^{I\alpha}_{\mu} = P^{I\alpha}_{i} \partial_{\mu} \phi^{i}$, for $i = 0, \dots, 4n-1$. With this formulation for the coset space non-linear sigma model, we may now write down the full bosonic Lagrangian of the theory. We will be interested in the case in which only the metric and the scalars are non-vanishing. In this case the Lorentzian theory is given by

$$e^{-1}\mathcal{L} = -\frac{1}{4}R + \partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}P_{I\alpha\mu}P^{I\alpha\mu} - V \tag{2.2.9}$$

with the scalar potential V given by

$$\begin{aligned} V &= -e^{2\sigma}\left[\frac{1}{36}A^2 + \frac{1}{4}B^iB_i + \frac{1}{4}(C^I_t C_{It} + 4D^I_t D_{It})\right] + m^2 e^{-6\sigma}\mathcal{N}_{00} \\ &\quad -me^{-2\sigma}\left[\frac{2}{3}AL_{00} - 2B^iL_{0i}\right] \end{aligned} \tag{2.2.10}$$

The scalar potential features the following quantities,

$$\begin{aligned} A &= \epsilon^{rst}K_{rst} & B^r &= \epsilon^{rst}K_{st0} \\ C^t_I &= \epsilon^{trs}K_{rIs} & D_{It} &= K_{0It} \end{aligned} \tag{2.2.11}$$

with the so-called “boosted structure constants” K given by,

$$\begin{aligned} K_{rs\alpha} &= g\,\epsilon_{\ell mn}L^\ell{}_r(L^{-1})_s{}^m L^n{}_\alpha + \lambda\, C_{IJK}L^I{}_r(L^{-1})_s{}^J L^K{}_\alpha \\ K_{\alpha It} &= g\,\epsilon_{\ell mn}L^\ell{}_\alpha(L^{-1})_I{}^m L^n{}_t + \lambda\, C_{MJK}L^M{}_\alpha(L^{-1})_I{}^J L^K{}_t \end{aligned} \tag{2.2.12}$$

We remind the reader that $r,s,t = 1,2,3$ are obtained from splitting the index α into a 0 index and an $SU(2)_R$ adjoint index. Also appearing in the Lagrangian is $\mathcal{N}_{00}$, which is the 00 component of the matrix

$$\mathcal{N}_{\Lambda\Sigma} = L_\Lambda{}^\alpha\left(L^{-1}\right)_{\alpha\Sigma} - L_\Lambda{}^I\left(L^{-1}\right)_{I\Sigma} \tag{2.2.13}$$

2.2.2 Supersymmetry variations

We now review the supersymmetry variations for the fermionic fields in the Lorentzian theory. In the following section, we will discuss the continuation of this theory to Euclidean signature, which is complicated by the necessary modification of the symplectic Majorana condition imposed on the spinor fields.

In order to write the fermionic variations, it is first necessary to introduce a matrix γ^7 defined as

$$\gamma^7 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 \tag{2.2.14}$$

and satisfying $(\gamma^7)^2 = -\mathbb{1}$. With this, the supersymmetry transformations of the fermions in the Lorentzian case can be given as

$$\begin{aligned}
\delta\chi_A &= \frac{i}{2}\gamma^\mu\partial_\mu\sigma\varepsilon_A + N_{AB}\varepsilon^B \\
\delta\psi_{A\mu} &= \mathcal{D}_\mu\varepsilon_A + S_{AB}\gamma_\mu\varepsilon^B \\
\delta\lambda^I_A &= i\hat{P}^I_{ri}\sigma^r_{AB}\partial_\mu\phi^i\gamma^\mu\varepsilon^B - i\hat{P}^I_{0i}\epsilon_{AB}\partial_\mu\phi^i\gamma^7\gamma^\mu\varepsilon^B + M^I_{AB}\varepsilon^B
\end{aligned} \tag{2.2.15}$$

where we have defined

$$\begin{aligned}
S_{AB} &= \frac{i}{24}[Ae^\sigma + 6me^{-3\sigma}(L^{-1})_{00}]\varepsilon_{AB} - \frac{i}{8}[B_te^\sigma - 2me^{-3\sigma}(L^{-1})_{t0}]\gamma^7\sigma^t_{AB} \\
N_{AB} &= \frac{1}{24}[Ae^\sigma - 18me^{-3\sigma}(L^{-1})_{00}]\varepsilon_{AB} + \frac{1}{8}[B_te^\sigma + 6me^{-3\sigma}(L^{-1})_{t0}]\gamma^7\sigma^t_{AB} \\
M^I_{AB} &= (-C^I{}_t + 2i\gamma^7D^I{}_t)e^\sigma\sigma^t_{AB} - 2me^{-3\sigma}(L^{-1})^I{}_0\gamma^7\varepsilon_{AB},
\end{aligned} \tag{2.2.16}$$

In the above, the matrix σ^r_{AB} defined as $\sigma^r_{AB} \equiv \sigma^{rC}{}_B\varepsilon_{CA}$ is symmetric in A, B. For more details, see our previous paper [GKR17].

2.2.3 Mass deformations

In the following, we consider the coset (2.2.3) with $n = 1$, i.e. a single vector multiplet. The coset representative is expressed in terms of four scalars $\phi^i, i = 0, 1, 2, 3$ via

$$L = \prod_{i=0}^{3} e^{\phi^i K^i} \tag{2.2.17}$$

where K^i are the non compact generators of $SO(4, 1)$; see [GKR17] for details. Note that ϕ^0 is an $SU(2)_R$ singlet, while the other three scalars ϕ^r form an $SU(2)_R$ triplet. The scalar potential for this specific case can be obtained from (2.2.10) and takes the following form

$$\begin{aligned}
V(\sigma, \phi^i) =& - g^2e^{2\sigma} + \frac{1}{8}me^{-6\sigma}\bigg[- 32ge^{4\sigma}\cosh\phi^0\cosh\phi^1\cosh\phi^2\cosh\phi^3 + 8m\cosh^2\phi^0 \\
&+ m\sinh^2\phi^0\Big(- 6 + 8\cosh^2\phi^1\cosh^2\phi^2\cosh(2\phi^3) + \cosh(2(\phi^1 - \phi^2)) \\
&+ \cosh(2(\phi^1 + \phi^2)) + 2\cosh(2\phi^1) + 2\cosh(2\phi^2)\Big)\bigg]
\end{aligned} \tag{2.2.18}$$

The supersymmetric AdS$_6$ vacuum is given by setting $g = 3m$ and setting all scalars to vanish. The masses of the linearized scalar fluctuation around the AdS vacuum determine the dimensions of the dual scalar operators in the SCFT via

$$m^2 l^2 = \Delta(\Delta - 5) \tag{2.2.19}$$

where l is the curvature radius of the AdS$_6$ vacuum. For the scalars at hand, one finds

$$m_\sigma^2 l^2 = -6 \qquad m_{\phi^0}^2 l^2 = -4 \qquad m_{\phi^r}^2 l^2 = -6\ , \quad r = 1, 2, 3 \tag{2.2.20}$$

Hence the dimensions of the dual operators are

$$\Delta_{\mathcal{O}_\sigma} = 3, \qquad \Delta_{\mathcal{O}_{\phi^0}} = 4, \qquad \Delta_{\mathcal{O}_{\phi^r}} = 3\ , \quad r = 1, 2, 3 \tag{2.2.21}$$

In [FKP98] these CFT operators were expressed in terms of free hypermultiplets (i.e. the singleton sector). The case of $n = 1$ corresponds to having a single free hypermultiplet, consisting of four real scalars q_A^I and two symplectic Majorana spinors ψ^I. Here $I = 1, 2$ is the $SU(2)_R$ R-symmetry index and $A = 1, 2$ is the $SU(2)$ flavor symmetry index. The gauge invariant operators appearing in (2.2.21) are related to these fundamental fields as follows,

$$\mathcal{O}_\sigma = (q^*)^A{}_I q^I{}_A, \qquad \mathcal{O}_{\phi^0} = \bar{\psi}_I \psi^I, \qquad \mathcal{O}_{\phi^r} = (q^*)^A{}_I (\sigma^r)_A{}^B q^I{}_B\ , \quad r = 1, 2, 3 \tag{2.2.22}$$

Note that the first two operators correspond to mass terms for the scalars and fermions, respectively, in the hypermultiplet. The third operator is a triplet with respect to the $SU(2)_R$ R-symmetry. As argued in [FKP98], the field ϕ^0 is the top component of the global current supermultiplet. Therefore a deformation by $\mathcal{O}_{\phi^0}$ will break superconformal symmetry but preserve all Poincare supersymmetry [CDI16]. However, deformation by $\mathcal{O}_{\phi^0}$ alone is inconsistent. Poincare supersymmetry demands that we also turn on the scalar masses $\mathcal{O}_\sigma$. Moreover, supersymmetry on S^5 requires an additional operator in the action that breaks the superconformal $SU(2)_R$ symmetry to $U(1)_R$ symmetry [HST12]. Without loss of generality, we may choose this operator to be $\mathcal{O}_{\phi^3}$.

2.3 Euclidean theory and BPS solutions

In this section we will obtain the six-dimensional holographic dual of a mass deformation of a 5D SCFT on S^5. Such a dual is given by S^5-sliced domain wall solutions of matter-coupled Euclidean $F(4)$ gauged supergravity. In order to obtain such solutions, we must first continue the Lorentzian signature gauged supergravity outlined above to Euclidean signature, which has subtleties for both the scalar and fermionic sectors. Once the Euclidean theory is obtained, we turn on relevant scalars necessary to support the domain wall. As discussed in the previous section, at least three scalars must be turned on to obtain supersymmetric solutions. The ansatz for the domain wall solutions takes the following form

$$ds^2 = du^2 + e^{2f(u)} ds^2_{S^5}, \qquad \sigma = \sigma(u), \qquad \phi^i = \phi^i(u), \quad i = 0, 3 \tag{2.3.1}$$

with the remaining fields set to zero. Next we will obtain a consistent set of BPS equations on the above ansatz, and then solve them numerically. When solving them, we will demand as an initial condition that for some finite u the metric factor e^{2f} vanishes, so that the geometry closes off smoothly.

2.3.1 Euclidean action

The Euclidean action may be obtained from the Lorentzian one by first performing a simple Wick rotation of Lorentzian time $t \to -ix^6$. This makes the spacetime metric negative definite, since the metric in the Lorentzian theory was taken to be of mostly negative signature. However, we will choose to work with the Euclidean theory with positive definite metric. Making this modification involves a change in the sign of the Ricci scalar. Then noting that the Euclidean action is related to the Lorentzian action by $\exp\left(iS^{Lor}\right) = \exp\left(-S^{Euc}\right)$, the final result of the Wick rotation is the following Euclidean action,

$$S_{6D} = \frac{1}{4\pi G_6}\int d^6x\,\sqrt{G}\mathcal{L}\ , \qquad \mathcal{L} = \left(-\frac{1}{4}R + \partial_\mu\sigma\partial^\mu\sigma + \frac{1}{4}G_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + V(\sigma,\phi^i)\right) \tag{2.3.2}$$

where the spacetime metric G is positive definite and G_6 is the six-dimensional Newton's constant. By abuse of notation, $G_{ij}(\phi)$ with indices refers to the metric on the scalar manifold, which for the coset representative (2.2.17) is given by

$$G_{ij} = \mathrm{diag}\left(\cosh^2\phi^1\cosh^2\phi^2\cosh^2\phi^3, \cosh^2\phi^2\cosh^2\phi^3, \cosh^2\phi^3, 1\right) \tag{2.3.3}$$

In addition to performing the above Wick rotation, we also perform a Wick rotation on the sigma model [BCP09, HTV17, RTV18]

$$\frac{SO(4,1)}{SO(4)} \to \frac{SO(4,1)}{SO(3,1)} \simeq dS_4 \tag{2.3.4}$$

The metric on the sigma model is now that of dS_4, as opposed to the $\mathbb{H}_4$ that we had in the Lorentzian case [GKR17]. This can be obtained by making the following change to the $\mathbb{H}_4$ coset,

$$\phi_r \to i\phi_r \qquad r = 1, 2, 3 \tag{2.3.5}$$

It would be interesting to understand this analytic continuation from first principles and its relation to Euclidean supersymmetry, possibly along the lines of [GGP96, CMM04]. For now, we just note that such a Wick rotated model seems necessary to obtain regular, supersymmetric solutions.

2.3.2 Euclidean supersymmetry

The next task is to identify the form of the Euclidean supersymmetry variations. Motivation for the form of these variations may be obtained by analysis of the free differential algebra (FDA) of the $F(4)$ gauged supergravity theory with $\mathbb{H}_6$ vacuum, as discussed in Appendix B. The final result for this FDA is given in (B.0.6), and is noted to be of the same form as the FDA for the theory with dS_6 background (identified in [DV02]), with two differences. The first obvious difference is that the metrics differ - the space considered in [DV02] was dS_6 with mostly minus signature, whereas we are currently focused on positive definite $\mathbb{H}_6$. However, both of these spaces have $R_{\mu\nu} = -20m^2 g_{\mu\nu}$. The second difference is in the definition of Dirac conjugate spinors. However, once the difference in definition of the gamma matrices is accounted for, the only difference is a factor of i, i.e.

$$\bar{\psi}_A^{(\mathbb{H}_6)} = i\bar{\psi}_A^{(dS_6)} \tag{2.3.6}$$

Because of these similarities, the supersymmetry variations in the current case are expected to be of a similar form to that of [DV02]. In particular, the variations of the fermions are expected to be of the form

$$
\begin{aligned}
\delta\chi_A &= -\frac{1}{2}\gamma^\mu \partial_\mu \sigma \varepsilon_A + N_{AB}\varepsilon^B + \ldots \\
\delta\psi_{A\mu} &= \mathcal{D}_\mu \varepsilon_A + i S_{AB}\gamma_\mu \varepsilon^B + \ldots \\
\delta\lambda^I_A &= -\hat{P}^I_{ri}\sigma^r_{AB}\partial_\mu \phi^i \gamma^\mu \varepsilon^B + \hat{P}^I_{0i}\epsilon_{AB}\partial_\mu\phi^i\gamma^7\gamma^\mu\varepsilon^B + M^I_{AB}\varepsilon^B + \ldots
\end{aligned}
\tag{2.3.7}
$$

where N_{AB}, S_{AB}, and M^I_{AB} are again given by (2.2.16), but now with the appropriate redefinition of the coset representative as per (2.3.5). It should be noted that while the FDA analysis presented in Appendix B is a strong motivation for the form of the supersymmetry variations presented above, it is not a proof. To actually derive the form of these variations, one must first introduce curvature terms representing deviations from zero of each line in the free differential algebra. An application of the exterior derivative to the resulting expressions then gives rise to Bianchi identities, which must be solved before obtaining the explicit form of the fermion variations. This is a rather involved process, and so for the moment we will content ourselves with the motivating comments provided by the FDA. We will take the eventual presence of smooth supersymmetric solutions consistent with the equations of motion as *a posteriori* evidence for the legitimacy of these variations.

A nice property of the variations above is the fact that they are consistent with the following $SO(6)$-invariant symplectic Majorana condition,

$$
\bar{\psi}_A = \epsilon^{AB}\psi^T_B \mathcal{C} \tag{2.3.8}
$$

The consistency of such a condition allows us to work with symplectic Majorana spinors just as in the Lorentzian case, though the symplectic Majorana condition utilized here is different than that of the Lorentzian case.[2]

As mentioned before, we will be concerned with only the simplest case of a single non-zero $SU(2)_R$-charged vector multiplet scalar ϕ^3, i.e. we take $\phi^1 = \phi^2 = 0$. It can be

[2]The fact that the symplectic Majorana condition must be different in the current case follows from $SO(6)$ invariance. The condition used in the Lorentzian case [GKR17] was expressed in terms of γ_0, which explicitly breaks $SO(6)$ symmetry.

easily verified that this is a consistent truncation, and is in fact the most general choice of non-vanishing fields that can preserve $SO(4,2)\times U(1)_R$. With this consistent truncation, the functions N_{AB}, S_{AB}, and M^I_{AB} appearing in the supersymmetry variations reduce to

$$\begin{aligned} S_{AB} &= iS_0\epsilon_{AB} + iS_3\gamma^7\sigma^3_{AB} \\ N_{AB} &= -N_0\epsilon_{AB} - N_3\gamma^7\sigma^3_{AB} \\ M^I_{AB} &= M_0\gamma^7\epsilon_{AB} + M_3\sigma^3_{AB} \end{aligned} \tag{2.3.9}$$

where we have defined

$$\begin{aligned} S_0 &= \frac{1}{4}\left(g\cos\phi^3 e^\sigma + me^{-3\sigma}\cosh\phi^0\right) \\ S_3 &= \frac{1}{4} i\, m\; e^{-3\sigma}\sinh\phi^0\sin\phi^3 \\ N_0 &= -\frac{1}{4}\left(g\cos\phi^3 e^\sigma - 3me^{-3\sigma}\cosh\phi^0\right) \\ N_3 &= -\frac{3}{4} i\, me^{-3\sigma}\sinh\phi^0\sin\phi^3 \\ M_0 &= 2m\; e^{-3\sigma}\cos\phi^3\sinh\phi^0 \\ M_3 &= -2i\, g\; e^\sigma\sin\phi^3 \end{aligned} \tag{2.3.10}$$

Importantly, note that S_3, N_3, and M_3 are now purely imaginary, in contrast to the Lorentzian case [GKR17]. In all that follows we will set $m = -1/2\,\eta$ such that the radius of AdS_6 is one.

2.3.3 BPS Equations

We now use the vanishing of the fermionic variations (2.3.7) to obtain BPS equations for the warp factor and the three non-zero scalars.

2.3.3.1 Dilatino equation and projector

We begin by imposing the vanishing of the dilatino variation, $\delta\chi_A = 0$, which implies

$$\frac{1}{2}\gamma^5\sigma'\varepsilon_A = N_0\varepsilon_A + N_3\gamma^7(\sigma^3)^B{}_A\varepsilon_B \tag{2.3.11}$$

This equation can be interpreted as a projection condition on the spinors ε_A. Consistency of this projection condition then requires that

$$\sigma' = 2\eta\sqrt{N_0^2 + N_3^2} \tag{2.3.12}$$

where $\eta = \pm 1$. Plugging this BPS equation back into (2.3.11) then yields a second form of the projection condition,

$$\gamma^5 \varepsilon_A = G_0 \varepsilon_A - G_3 \gamma^7 (\sigma^3)^B{}_A \varepsilon_B \tag{2.3.13}$$

which is more useful in the derivation of the other BPS equations. In the above, we have defined

$$G_0 = \eta \frac{N_0}{\sqrt{N_0^2 + N_3^2}} \qquad\qquad G_3 = -\eta \frac{N_3}{\sqrt{N_0^2 + N_3^2}} \tag{2.3.14}$$

2.3.3.2 Gravitino equation

The analysis of the gravitino equation $\delta\psi_{A\mu} = 0$ proceeds in exactly the same way as for the Lorentzian case studied in [GKR17]. The procedure gives rise to a first-order equation for the warp factor f and an algebraic constraint. To avoid excessive overlap with that paper, we simply cite the result,

$$f' = 2(G_0 S_0 + G_3 S_3) \qquad\qquad e^{-2f} = 4(G_0 S_0 + G_3 S_3)^2 - 4(S_0^2 + S_3^2) \tag{2.3.15}$$

2.3.3.3 Gaugino equations

Finally, we turn toward the gaugino equation $\delta\lambda_A^I = 0$. Again the analysis of this equation proceeds in an exactly analogous manner to the Lorentzian case [GKR17]. The result is

$$\cos\phi^3 (\phi^0)' = -(G_0 M_0 + G_3 M_3) \qquad\qquad (\phi^3)' = i(G_3 M_0 - G_0 M_3) \tag{2.3.16}$$

The right-hand sides of both equations are real, and thus give rise to real solutions when appropriate initial conditions are imposed.

2.3.3.4 Summary of first-order equations

To summarize, the first-order equations for the warp factor f and the scalars σ, ϕ^0, ϕ^3 are found to be

$$\begin{aligned} f' &= 2\,(G_0 S_0 + G_3 S_3) \\ \sigma' &= 2\eta\sqrt{N_0^2 + N_3^2} \\ \cos\phi^3\,\left(\phi^0\right)' &= -\,(G_0 M_0 + G_3 M_3) \\ \left(\phi^3\right)' &= i\,(G_3 M_0 - G_0 M_3) \end{aligned} \tag{2.3.17}$$

Furthermore, for consistency these were required to satisfy the algebraic constraint

$$e^{-2f} = 4\left(G_0 S_0 + G_3 S_3\right)^2 - 4\left(S_0^2 + S_3^2\right) \tag{2.3.18}$$

The various functions featured in these equations were defined in (2.3.10) and (2.3.14).

2.3.4 Numeric solutions

In order to get acceptable numerical solutions from these equations, we must choose appropriate initial conditions. It is easy to check that the following initial conditions ensure smoothness of all three scalars, as well as the vanishing of e^{2f} at the origin,

$$\begin{aligned} \phi_0^3 &= \sin^{-1}\left[\frac{1}{8\tanh\phi_0^0}\left(-3+\sqrt{9+16\tanh^2\phi_0^0}\right)\right] \\ \sigma_0 &= \frac{1}{4}\log\left[\frac{\cosh\phi_0^0\left(5+\sqrt{9+16\tanh^2\phi_0^0}\right)}{\sqrt{6}\sqrt{8+\coth^2\phi_0^0\left(-3+\sqrt{9+16\tanh^2\phi_0^0}\right)}}\right] \end{aligned} \tag{2.3.19}$$

We have defined for notational convenience $\phi_0^\alpha \equiv \phi^\alpha(0)$ and $\sigma_0 \equiv \sigma(0)$. For these initial conditions to be real, we must ensure that

$$|f(\phi_0^0)| \leq 1 \qquad\qquad f(\phi_0^0) \equiv \frac{1}{8\tanh\phi_0^0}\left(-3+\sqrt{9+16\tanh^2\phi_0^0}\right) \tag{2.3.20}$$

Noting that

$$\lim_{\phi_0^0\to-\infty} f(\phi_0^0) = -\frac{1}{4} \qquad\qquad \lim_{\phi_0^0\to+\infty} f(\phi_0^0) = \frac{1}{4} \tag{2.3.21}$$

and also that $f(\phi_0^0)$ is monotonically increasing, i.e.

$$\frac{df}{d\phi_0^0} > 0 \qquad \forall\phi_0^0 \in \mathbb{R} \tag{2.3.22}$$

allows us to conclude that this is always the case for real initial conditions ϕ_0^0. Thus we have a one parameter family of real smooth solutions, labeled by the IR parameter ϕ_0^0.

With this in mind, we may choose any value of ϕ_0^0 and solve the BPS equations in (2.3.17) numerically. In Figure 2.1, we plot the solutions obtained for the following choices of initial condition: $\phi_0^0 = \{0.25, 0.5, 1, 1.5, 2\}$. In order to get smooth solutions for $u > 0$, we must take $\eta = -1$. It is straighforward to verify that the resulting solutions are completely smooth and have the expected vanishing of e^{2f} at the origin, implying that the spacetime smoothly pinches off. Furthermore, e^{2f}/e^{2u} is seen to asymptote to a constant, which we denote by e^{2f_k}.

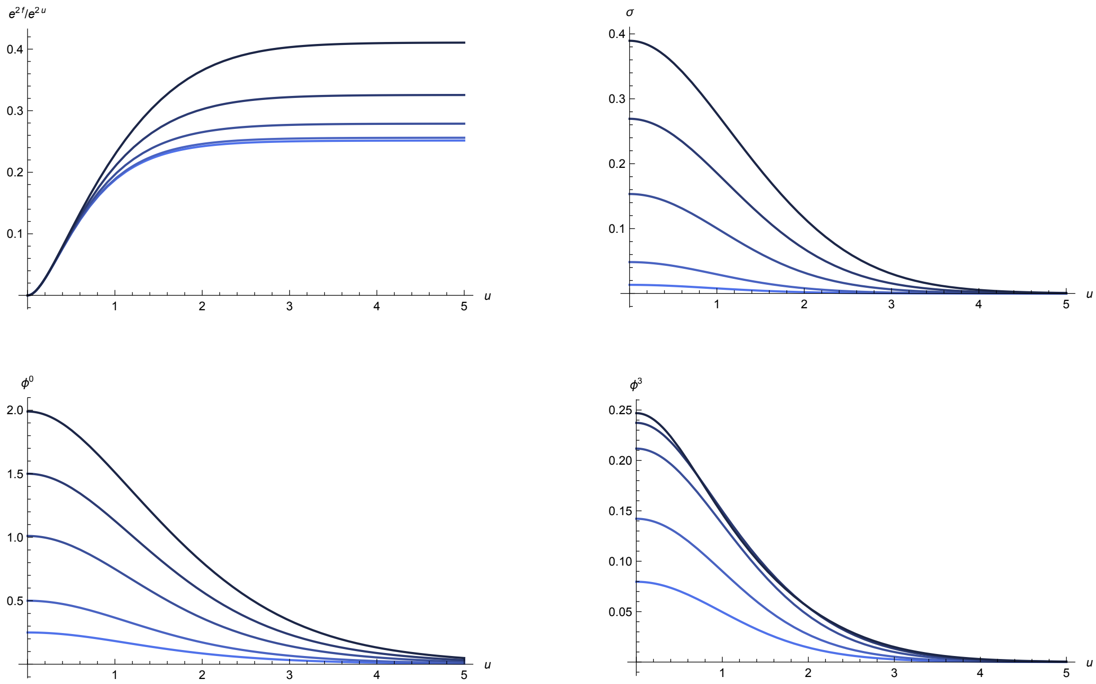


Figure 2.1: Smooth solutions for the four scalar fields in the Euclidean theory. We take $\eta = -1$ and have chosen the following values for the initial conditions: $\phi_0^0 = \{0.25, 0.5, 1, 1.5, 2\}$ (light to dark blue). Importantly, we see that e^{2f} vanishes at the origin - signaling a smooth closing off of the spacetime - and asymptotes to a constant e^{2f_k}.

2.3.5 UV asymptotic expansions

As in the holographic Janus solutions in Lorentzian signature [GKR17], the BPS equations may also be used to obtain the UV asymptotic behavior of the solutions. To do so, we begin by defining an asymptotic coordinate $z = e^{-u}$, where the asymptotic S^5 boundary is reached by taking $u \to \infty$. Consequently, an asymptotic expansion is an expansion around $z = 0$. The coefficients in the UV expansions of the non-zero fields may now be solved for order-by-order using the BPS equations. One finds explicitly that all coefficients are determined in terms of only three independent parameters α, β, and f_k, in accord with the fact that there are three independent first-order differential equations. The first few terms in the expansions are

$$
\begin{aligned}
f(z) &= -\log z + f_k - \left(\frac{1}{4}e^{-2f_k} + \frac{1}{16}\alpha^2\right) z^2 + O(z^4) \\
\sigma(z) &= \frac{3}{8}\alpha^2 z^2 + \frac{1}{4}e^{f_k}\alpha\beta\, z^3 + O(z^4) \\
\phi^0(z) &= \alpha\, z - \left(\frac{5}{4}\alpha\, e^{-2f_k} + \frac{23}{48}\alpha^3\right) z^3 + O(z^4) \\
\phi^3(z) &= e^{-f_k}\alpha z^2 + \beta\, z^3 + O(z^4)
\end{aligned} \tag{2.3.23}
$$

We have obtained the expansions up to $O(z^8)$, but we display only the first few terms here.

2.4 Holographic sphere free energy

The goal of this section is to obtain the holographic free energy, i.e. the renormalized on-shell action. We begin by writing the full action,

$$
S = S_{\mathrm{6D}} + S_{\mathrm{GH}}
$$

$$
S_{\mathrm{6D}} = \int du\; d^5x\; \sqrt{G}\,\mathcal{L} \qquad\qquad S_{\mathrm{GH}} = -\frac{1}{2}\int d^5x \sqrt{\gamma}\,\mathcal{K} \tag{2.4.1}
$$

where S_{6D} is the six-dimensional Euclidean action given in (2.3.2) and S_{GH} is the Gibbons-Hawking term.[3] The γ appearing in S_{GH} is the determinant of the induced metric on the boundary (located at some cutoff distance $u = \Lambda$), while $\mathcal{K}$ is the trace of the extrinsic curvature $\mathcal{K}_{ij}$ of the radial S^5 slices. The latter is defined as

$$
\mathcal{K}_{ij} = \frac{1}{2}\frac{d}{du}\gamma_{ij} \tag{2.4.2}
$$

In general, the on-shell action is divergent and requires renormalization. The addition of infinite counterterms is standard in holographic renormalization [BFS02, Ske02, PS04], but in the current case we must also add finite counterterms in order to preserve supersymmetry [BFS01]. We will begin our exploration of counterterms in this section by first considering the finite counterterms in the limit of a flat domain wall, after which we move

[3] We have set $4\pi G_6 = 1$ to avoid clutter in the formulas. We will restore this factor in the final expression for the free energy.

onto infinite counterterms in the more general case of a curved domain wall. Finally, appropriate curved space finite counterterms will be fixed by demanding finiteness of the one-point functions of the dual operators.

2.4.1 Finite counterterms

In order to obtain finite counterterms, we will make use of the Bogomolnyi trick [BEF14, BEK16, FP14]. To do so, we will first need to identify a superpotential W. Though we will find that no exact superpotential can be found for our solutions - in the sense that there is no superpotential which can recast all of the BPS equations in gradient flow form - we will be able to identify an *approximate* superpotential. By "approximate" here, we mean that it does yield gradient flow equations up to terms of order $O(z^5)$, where the asymptotic coordinate z was defined earlier as $z = e^{-u}$. This is useful since, as we will see later, we will only need terms up to $O(z^5)$ to obtain all divergent and finite counterterms. Terms of higher order will all vanish in the $\epsilon \to 0$ limit, i.e. when the UV cutoff is removed. Thus the approximate superpotential will yield all finite counterterms.

2.4.1.1 Approximate superpotential

In order to identify a candidate superpotential, we begin by recalling the form of the scalar potential V. With the choice of coset representative and consistent truncation outlined in Section 2.3, one finds that

$$V(\sigma, \phi^i) = -9m^2 e^{2\sigma} - 12m^2 e^{-2\sigma} \cosh\phi^0 \cos\phi^3 + m^2 e^{-6\sigma} \cosh^2\phi^0 + m^2 e^{-6\sigma} \cos 2\phi^3 \sinh^2\phi^0$$

This scalar potential can in fact be rewritten as

$$V = 4(N_0^2 + N_3^2) + \frac{1}{4}(M_0^2 + M_3^2) - 20(S_0^2 + S_3^2) \tag{2.4.3}$$

Then for BPS solutions, (2.3.17) implies that

$$V = (\sigma')^2 + \frac{1}{4}\left(-(\phi^{3\prime})^2 + \cos^2\phi^3 (\phi^{0\prime})^2\right) - 20(S_0^2 + S_3^2) \tag{2.4.4}$$

This motivates us to define a superpotential W as

$$W = \sqrt{S_0^2 + S_3^2} \tag{2.4.5}$$

Unfortunately, this superpotential does *not* allow one to write the BPS equations for both ϕ^0 and ϕ^3 as gradient flow equations. The reason for this failure is that the integrability condition required to convert the BPS equation into a gradient flow form is not satisfied; see e.g. Appendix C.2.1 of [BEF14].[4] We thus follow the strategy of [BEF14] to construct an approximate superpotential. Our model consists of two consistent truncations that admit flat domain walls and an exact superpotential. These are the $\phi^3 = 0, \phi^0 \neq 0$ truncation and the $\phi^0 = 0, \phi^3 \neq 0$ truncation. The corresponding flow equations are (we set $\eta = -1$ henceforth)

$$\phi^{0\prime} = -8\,\partial_{\phi^0} W|_{\phi^3=0} \qquad\qquad \phi^{3\prime} = 8\,\partial_{\phi^3} W|_{\phi^0=0} \tag{2.4.6}$$

respectively. In either truncation, the BPS equations for the warp factor and dilaton σ can be put in the following form,

$$f' = 2\,W \qquad\qquad \sigma' = 2\,\partial_\sigma W \tag{2.4.7}$$

An important fact is that, though the gradient flow equations of (2.4.6) do not hold exactly in the full model with $\phi^0 \neq 0, \phi^3 \neq 0$, they *do* hold up to and including $O(z^5)$. Looking at the form of the UV asymptotics of the scalar fields, one may expand the superpotential of (2.4.5) keeping only terms contributing up to this order. This gives

$$W = \frac{1}{2} + \frac{3}{4}\sigma^2 + \frac{1}{16}(\phi^0)^2 - \frac{3}{16}(\phi^3)^2 + \frac{1}{192}(\phi^0)^4 - \frac{3}{16}(\phi^0)^2\sigma + \ldots \tag{2.4.8}$$

where the dots represent terms of order $O(z^6)$. This is the approximate superpotential we will use in what follows.

2.4.1.2 Bogomolnyi trick

We now use the Bogomolnyi trick [BEF14, BEK16, FP14] to get the finite counterterms needed to preserve supersymmetry in the case of a flat domain wall. The central idea of the Bogomolnyi trick is that for a BPS solution, the renormalized on-shell action must vanish. In order to make use of this fact, we will first want to recast the on-shell action in a simpler form.

[4] See however [LPT15, CKP18] where an effective superpotential involving the warp factor was derived, in terms of which the first-order equations take the form of a gradient flow.

To do so, we begin by inserting (2.4.4) into (2.2.9). We find that

$$\mathcal{L} = -\frac{1}{4}R - 20W^2 + 2\mathcal{L}_{\text{kin}} \tag{2.4.9}$$

where we've defined

$$\mathcal{L}_{\text{kin}} = (\sigma')^2 + \frac{1}{4}\left[-(\phi^{3\prime})^2 + \cos^2\phi^3(\phi^{0\prime})^2\right] \tag{2.4.10}$$

The non-zero components of the Ricci tensor are

$$R_{uu} = -5\left(f'' + (f')^2\right) \qquad\qquad R_{mn} = -g_{mn}\left(f'' + 5(f')^2\right) \tag{2.4.11}$$

while the Ricci scalar is given by

$$R = -10f'' - 30(f')^2 \tag{2.4.12}$$

Furthermore, we have that $\sqrt{G} = e^{5f}\sqrt{g}$, where g is the determinant of the unit S^5 metric. Upon integration by parts, part of the Einstein-Hilbert term cancels with the Gibbons-Hawking term to give the following simple expression

$$S = \int du \int d^5x\sqrt{g}\, e^{5f}\left[-5\left((f')^2 + 4W^2\right) + 2\mathcal{L}_{\text{kin}}\right] \tag{2.4.13}$$

The restriction to the flat case was not strictly necessary so far, but it will be crucial in the next step. The gradient flow equations (2.4.6) and (2.4.7), together with the chain-rule, allows us to rewrite

$$\mathcal{L}_{\text{kin}} = -2\,W' \tag{2.4.14}$$

Plugging this into (2.4.13) and using the BPS equation of the warp factor, we find

$$S = -4\int d^5x\sqrt{g}\, e^{5f}\,W\Big|_0^\Lambda \tag{2.4.15}$$

where Λ is the UV cutoff. Only the Λ part of the action contributes, since $e^{5f}W|_0$ vanishes due to the close-off of the geometry.

Removing the UV cutoff $\Lambda \to \infty$ is equivalent to removing the cutoff ε on our asymptotic coordinate z, i.e. $\varepsilon \to 0$. From the UV asymptotics (2.3.23) we find that in this limit the factor e^{5f} diverges like

$$e^{5f} \sim \frac{1}{\varepsilon^5} \tag{2.4.16}$$

This is the reason for the previous claims that only the terms up to $O(z^5)$ in the superpotential are relevant for obtaining counterterms. All the higher-order terms vanish as the cutoff is removed. We may thus legitimately insert the approximate superpotential (2.4.8) into (2.4.15) to get the counterterms,

$$S_{\rm ct}^{(W)} = 4\int d^5x\sqrt{\gamma}\left[\frac{1}{2}+\frac{3}{4}\sigma^2+\frac{1}{16}(\phi^0)^2-\frac{3}{16}(\phi^3)^2+\frac{1}{192}(\phi^0)^4-\frac{3}{16}(\phi^0)^2\sigma\right] \quad (2.4.17)$$

where γ is the induced metric on the $z=\varepsilon$ boundary. All fields are evaluated at $z=\varepsilon$. This gives all finite and infinite counterterms for the flat domain wall solutions.

2.4.2 Infinite counterterms

We now turn towards the identification of the infinite counterterms in the more general curved domain wall case. We may first solve for all of the infinite counterterms via the usual holographic renormalization procedure. Once we have these, we will

1. Check that in the flat limit, they reduce to the divergent pieces of the flat counterterms (2.4.17) found above.

2. Add to them the finite pieces found in (2.4.17) but missing in the holographic renormalization procedure.

For simplicity, we will perform holographic renormalization on supersymmetric solutions only, and thus the infinite counterterms we obtain are universal for supersymmetric solutions only.

We begin by using the expression for the on-shell Ricci scalar,

$$R = 4(\sigma')^2+\left[-(\phi^{3\prime})^2+\cos^2\phi^3(\phi^{0\prime})^2\right]+6V \quad (2.4.18)$$

to rewrite the action (2.4.1) as

$$S_{\rm 6D} \;=\; -\frac{1}{2}\int du\, d^5x\sqrt{g}\, e^{5f}V \quad (2.4.19)$$

We have not included the Gibbons-Hawking term yet, but will do so later. The first step of holographic renormalization is to isolate the divergent terms. We may do so by expanding all fields using their UV asymptotics, then integrating over small z and evaluating on the cutoff ϵ. Doing so, we find

$$
\begin{aligned}
S_{\rm 6D} &= -\frac{1}{2}\int d^5x\sqrt{g}e^{5f_k}\left[\frac{1}{\epsilon^5}+\frac{1}{3\epsilon^3}\left(25f_2+\left(\phi_1^0\right)^2\right)\right.\\
&\qquad +\frac{1}{24\epsilon}\left(1500f_2^2+600f_4+120f_2\left(\phi_1^0\right)^2-\left(\phi_1^0\right)^4\right.\\
&\qquad\qquad \left.\left.+48\,\phi_1^0\phi_3^0+36\left(-\left(\phi_2^3\right)^2+4\sigma_2^2\right)\right)\right]
\end{aligned} \tag{2.4.20}
$$

where we've thrown out all non-divergent contributions. Note that the integration would naively give a $\log\epsilon$, but this vanishes on the BPS equations since they constrain the UV asymptotic expansion coefficients in the following way,[5]

$$
25f_5+2\,\phi_1^0\phi_4^0-3\,\phi_2^3\phi_3^3+12\sigma_2\sigma_3=0 \tag{2.4.21}
$$

The absence of the logarithmic term is to be expected, since any dual five-dimensional field theory is anomaly-free. The Gibbons-Hawking term is

$$
S_{\rm GH}=-\frac{5}{2}\int d^5x\sqrt{g}\,e^{5f}f' \tag{2.4.22}
$$

We again use the asymptotic expansions to write

$$
S_{\rm GH}=-\frac{5}{2}\int d^5x\sqrt{g}e^{5f_k}\left[\frac{1}{\epsilon^5}+\frac{3f_2}{\epsilon^3}+\frac{1}{2\epsilon}\left(5f_2^2+2f_4\right)\right] \tag{2.4.23}
$$

Adding the two together, we find in total that

$$
\begin{aligned}
S_{\rm 6D}+S_{\rm GH} &= -\int d^5x\sqrt{g}e^{5f_k}\left[\frac{2}{\epsilon^5}+\frac{1}{6\epsilon^3}\left(20f_2-\left(\phi_1^0\right)^2\right)-\frac{1}{48\epsilon}\left(1200f_2^2+480f_4\right.\right.\\
&\qquad \left.\left.+120f_2\left(\phi_1^0\right)^2-\left(\phi_1^0\right)^4+48\,\phi_1^0\phi_3^0-36(\phi_2^3)^2+144\sigma_2^2\right)\right]
\end{aligned} \tag{2.4.24}
$$

We must now undergo the task of inverting all of the UV modes to rewrite the action in terms of induced fields at the cut-off surface (since it is the latter which transform nicely under bulk diffeomorphism). Before quoting the result, we note that at the cut-off $z=\epsilon$, the induced metric γ_{ij} is given by

$$
\gamma_{ij}=e^{2f}\big|_{z=\epsilon}\,g_{ij}^{(S^5)} \tag{2.4.25}
$$

The Ricci tensor and Ricci scalar are given by

$$
R_{ij}[\gamma]=4e^{-2f}\gamma_{ij}\big|_{z=\epsilon} \qquad\qquad R[\gamma]=20\,e^{-2f}\big|_{z=\epsilon} \tag{2.4.26}
$$

[5]We have shown this using the solutions of the BPS equations, but it must hold for general solutions of the equations of motion as well.

In terms of these quantities, we find that the inverted form of the divergent part of the on-shell action is

$$S = -\int d^5x\sqrt{\gamma}\left[2+\frac{1}{4}\left(\phi^0\right)^2+\frac{3}{4}\left(\phi^3\right)^2-3\sigma^2+\frac{7}{12}\left(\phi^0\right)^4 \right. \\ \left. +\frac{1}{12}R[\gamma]-\frac{1}{320}R[\gamma]^2-\frac{3}{32}R[\gamma]\left(\phi^0\right)^2\right] \quad (2.4.27)$$

We may now address the two points mentioned at the start of this subsection. To begin, we check that in the flat limit, we reproduce the divergent terms obtained in (2.4.17). In particular, we expect that the first line of (2.4.27) should be equal to $-S_{ct}^{(W)}$ up to and including order $O(z^4)$. Though the expressions look different at first sight, it can be checked via the relationships between expansion coefficients in (2.3.23) (along with their higher order counterparts) that in the limit $e^{-2f} \to 0$ the two expressions indeed *are* equivalent up to $O(z^4)$. Thus all of their divergent contributions are the same in the flat limit. However, even in this limit the two differ at order $O(z^5)$, which means that they have different finite contributions. As mentioned earlier, the finite terms we must work with are those coming from (2.4.17). An action which has both the required finite and infinite counterterms is[6]

$$S_{\text{ct}} = \int d^5x\sqrt{\gamma}\left[2+\frac{1}{4}\left(\phi^0\right)^2+\frac{3}{4}\left(\phi^3\right)^2+3\sigma^2+\frac{1}{48}\left(\phi^0\right)^4-\frac{3}{4}\left(\phi^0\right)^2\sigma \right. \\ \left. +\frac{1}{12}R[\gamma]-\frac{1}{320}R[\gamma]^2-\frac{3}{32}R[\gamma]\left(\phi^0\right)^2\right] \quad (2.4.28)$$

The three gravitational counterterms 2, $R[\gamma]$, and $R[\gamma]^2$ match with the ones obtained in [EJM99, AFG14]. On our S^5 domain-wall ansatz, the term proportional to the square of the Ricci tensor simplifies in terms of the square of the Ricci scalar $R_{ij}[\gamma]R[\gamma]^{ij} = \frac{1}{5}R[\gamma]^2$.

Note that there is still a question of curved space finite counterterms, which we have not yet fixed. If we insist on including only terms even under

$$\varphi^0 \to -\varphi^0 \qquad \text{and} \qquad \varphi^3 \to -\varphi^3 \quad (2.4.29)$$

(which is a symmetry of the action) it can be shown that the only way to add terms which change the curved space finite counterterms but leave the other counterterms unchanged

[6] Note the sign of the $(\phi^3)^2$ term, which is different than the sign in (2.4.17).

is to add a combination of the form

$$(\phi^3)^2 - \frac{1}{20}R[\gamma](\phi^0)^2 = 2\,e^{-f_k}\beta\alpha\,z^5 + O(z^6) \tag{2.4.30}$$

This freedom is fixed by demanding that the vevs of the dual operators stay finite. We will simply quote the result here,

$$\begin{aligned} S_{\text{ct}} \;\; = \;\; & \int d^5x\sqrt{\gamma}\,\Big[2 + \frac{1}{4}\left(\phi^0\right)^2 - \frac{1}{2}\left(\phi^3\right)^2 + 3\sigma^2 + \frac{1}{48}\left(\phi^0\right)^4 - \frac{3}{4}\left(\phi^0\right)^2\sigma \\ & \qquad\qquad + \frac{1}{12}R[\gamma] - \frac{1}{320}R[\gamma]^2 - \frac{1}{32}R[\gamma]\left(\phi^0\right)^2\Big] \end{aligned} \tag{2.4.31}$$

and postpone showing that this gives finite vacuum expectation values to the next subsection.

At this level, everything has seemed unique. However, when thinking in terms of the induced fields instead of the modes appearing in asymptotic expansions, the counterterms of (2.4.31) are just one of many possible sets of counterterms that can be written down. In particular, since on-shell we have the relationship

$$I_0 \equiv 5\sigma^2 + \frac{45}{64}(\varphi^0)^4 - \frac{15}{4}(\varphi^0)^2\sigma = O(z^6) \tag{2.4.32}$$

we may add I_0 freely to (2.4.31) without changing either finite or infinite contributions. However, the inclusion of this term will have an impact on some of the one-point functions, which we calculate next.

2.4.3 Vevs and free energy

The renormalized on-shell action is given by

$$S_{\text{ren}} = S_{\text{6D}} + S_{\text{GH}} + S_{\text{ct}} + \Omega\int d^5x\,\sqrt{\gamma}\,I_0 \tag{2.4.33}$$

where the counterterm action S_{ct} is given by (2.4.31), Ω is a constant parameterizing choice of scheme, and I_0 is given in (2.4.32). Note that the free energy is independent of the choice of Ω, since I_0 is $O(z^6)$ and hence vanishes in the $\epsilon \to 0$ limit. However, some of the one-point functions *will* depend on Ω. It may be the case that only certain choices of Ω correspond to supersymmetric schemes, but since the final free energy will be independent of Ω we will not worry about this choice.

While in principle (2.4.33) gives us the free energy, its evaluation on our numerical solutions is complicated by the integration over u in S_{6D}. As such, we will take a slightly roundabout approach to the calculation of the free energy, first calculating its derivative $dF/d\alpha$ and then integrating over the UV parameter α. This will allow us to circumvent the integration over u. In order to get $dF/d\alpha$, it will first be necessary to calculate the one-point functions of the dual field theory operators. This is the topic of the following subsection.

2.4.3.1 One-point functions

By the usual AdS/CFT dictionary, the one-point functions of the operators dual to the three scalar fields and the metric are given by

$$\begin{aligned} \langle \mathcal{O}_\sigma \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{ren}}{\delta \sigma} & \qquad \langle \mathcal{O}_{\phi^0} \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^4} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{ren}}{\delta \phi^0} \\ \langle \mathcal{O}_{\phi^3} \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{ren}}{\delta \phi^3} & \qquad \langle T^i{}_j \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^5} \frac{1}{\sqrt{\gamma}} \gamma_{jk} \frac{\delta S_{ren}}{\delta \gamma_{ik}} \end{aligned} \tag{2.4.34}$$

We may obtain the explicit values of these vacuum expectation values by varying the on-shell action (2.4.33). The variation of the counterterm action S_{ct} is straightforward. The variation of S_{6D} gives rise to one piece which vanishes on the equations of motion, as well as a boundary term which must be accounted for. We find,

$$\begin{aligned} \langle \mathcal{O}_\sigma \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \left[-2z\partial_z \sigma + 6\sigma - \frac{3}{4}(\varphi^0)^2 + \Omega \left(10\sigma - \frac{15}{4} \left(\phi^0\right)^2 \right) \right] \\ \langle \mathcal{O}_{\phi^0} \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^4} \Bigg[-\frac{1}{2} \cos^2 \phi^3 z \partial_z \phi^0 + \frac{1}{2}\phi^0 + \frac{1}{12} \left(\phi^0\right)^3 - \frac{3}{2}\phi^0 \sigma - \frac{1}{16} R \phi^0 \\ &\qquad\qquad + \Omega \left(\frac{45}{16} \left(\phi^0\right)^3 - \frac{15}{2} \phi^0 \sigma \right) \Bigg] \\ \langle \mathcal{O}_{\phi^3} \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \left[\frac{1}{2} z \partial_z \phi^3 - \phi^3 \right] \\ \langle T^i{}_j \rangle &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^5} \left[\frac{1}{2} \left(\mathcal{K} \gamma^{ij} - \mathcal{K}^{ij} \right) + \frac{2}{\sqrt{\gamma}} \frac{\delta S_{ct}}{\delta \gamma_{ij}} \right] \end{aligned} \tag{2.4.35}$$

Evaluating the limits, we get the following one-point functions

$$\begin{aligned} \langle \mathcal{O}_\sigma \rangle &= \frac{5}{2} e^{f_k} \alpha \beta \, \Omega & \qquad \langle \mathcal{O}_{\phi^0} \rangle &= \frac{3}{2} e^{-f_k} \beta - \frac{15}{8} e^{f_k} \alpha^2 \beta \, \Omega \\ \langle \mathcal{O}_{\phi^3} \rangle &= \frac{1}{2} \beta & \qquad \langle T^i{}_i \rangle &= -\frac{5}{2} e^{-f_k} \alpha \beta \end{aligned} \tag{2.4.36}$$

The expectation values of the operator $\mathcal{O}_{\phi^3}$ and the trace of the energy-momentum tensor are independent of Ω. As a check, we note that the four one-point functions satisfy the following operator relation, which is associated to the violation of conformal invariance by non-zero classical beta functions,

$$\langle T^i{}_i \rangle = -\sum_{\mathcal{O}} (d - \Delta_{\mathcal{O}})\, \phi_{\mathcal{O}}\, \langle \mathcal{O} \rangle \tag{2.4.37}$$

Here $\phi_{\mathcal{O}}$ is the source for the operator $\mathcal{O}$ and is obtained from the asymptotic solutions given in (2.3.23).

2.4.3.2 Derivative of the free energy

Following [BEF14], we may now compute the derivative of F with respect to α as follows. First we note that

$$\frac{dF}{d\alpha} = \frac{dS_{\text{ren}}}{d\alpha} = \lim_{\epsilon\to 0} \int d^5x \sum_{\text{fields } \Phi} \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\text{ren}}\right)}{\delta\Phi} \frac{d\Phi}{d\alpha}\bigg|_{z=\epsilon} \tag{2.4.38}$$

In our case, the terms appearing in the sum over fields are

$$\begin{aligned}
\frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\text{ren}}\right)}{\delta\sigma} &= \sqrt{\gamma}\, \langle O_\sigma \rangle \epsilon^3 + \ldots & \qquad \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\text{ren}}\right)}{\delta\phi^0} &= \sqrt{\gamma}\, \langle O^0_\phi \rangle \epsilon^4 + \ldots \\
\frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\text{ren}}\right)}{\delta\phi^3} &= \sqrt{\gamma}\, \langle O^3_\phi \rangle \epsilon^3 + \ldots & \qquad \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\text{ren}}\right)}{\delta\gamma^{ij}} &= \frac{1}{2}\sqrt{\gamma}\, \langle T_{ij} \rangle \epsilon^5 + \ldots
\end{aligned} \tag{2.4.39}$$

The dots represent terms of strictly lower order in ϵ. Furthermore, from the form of the UV asymptotic expansions (2.3.23), we have

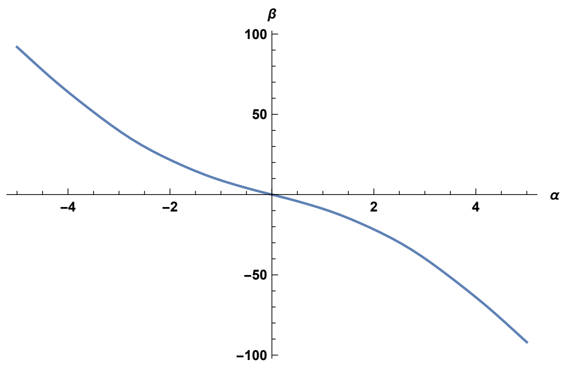
$$\begin{aligned}
\frac{d\sigma}{d\alpha} &= \frac{3}{4}\alpha\epsilon^2 + O(\epsilon^3) & \qquad \frac{d\phi^0}{d\alpha} &= \epsilon + O(\epsilon^3) \\
\frac{d\phi^3}{d\alpha} &= \left(1 - \alpha\frac{df_k}{d\alpha}\right) e^{-f_k}\epsilon^2 + O(\epsilon^3) & \qquad \frac{d\gamma^{ij}}{d\alpha} &= -2\frac{df_k}{d\alpha}e^{-2f_k}\epsilon^2 + O(\epsilon^2)
\end{aligned} \tag{2.4.40}$$

Combining the pieces (2.4.39),(2.4.40) with the results for the one-point functions in (2.4.36), we find that the contribution of the metric in (2.4.38) is suppressed by ϵ^2 compared to other terms. The derivative of the free energy is then

$$\begin{aligned}
\frac{dF}{d\alpha} &= \lim_{\epsilon\to 0} \int d^5x \sqrt{\gamma}\, \epsilon^5 \left[\frac{3}{2}\beta e^{-f_k} + \frac{1}{2}\beta e^{-f_k}\left(1 - \alpha\frac{df_k}{d\alpha}\right) + O(\epsilon)\right] \\
&= \text{vol}_0\left(S^5\right)\, \frac{1}{2}\beta\, e^{4f_k}\left(4 - \alpha\frac{df_k}{d\alpha}\right)
\end{aligned} \tag{2.4.41}$$

where $\text{vol}_0(S^5) = \pi^3$ is the volume of a unit S^5. The Ω dependence in the one-point functions cancels out, consistent with the fact that F itself is independent of Ω. We thus obtain the final result

$$\frac{dF}{d\alpha} = \frac{\pi^2}{8\,G_6}\beta\, e^{4f_k}\left(4 - \alpha\frac{df_k}{d\alpha}\right) \tag{2.4.42}$$



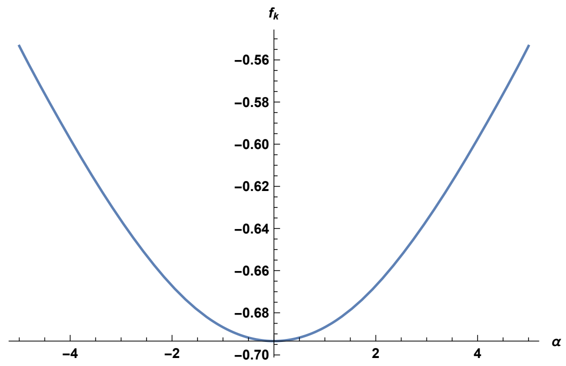


Figure 2.2: Plots of β vs. α and f_k vs. α. The relationships between the three parameters α, β, and f_k may be used to express (2.4.42) in terms of only a single parameter α.

Note that we've reintroduced the six-dimensional Newton's constant G_6, which had been previously set to $4\pi G_6 = 1$. This factor is important for the identification with the free energy on the field theory side.

Treating $\beta(\alpha)$ and $f_k(\alpha)$ as functions of α, this gives us an expression which may be numerically integrated to obtain the free energy $F(\alpha) - F(0)$ of the domain wall. The functional forms of $\beta(\alpha), f_k(\alpha)$ are obtained by fitting curves to the numerical data, as shown in Figure 2.2. Integrating to obtain $F(\alpha) - F(0)$ gives the result shown in Figure 2.3.

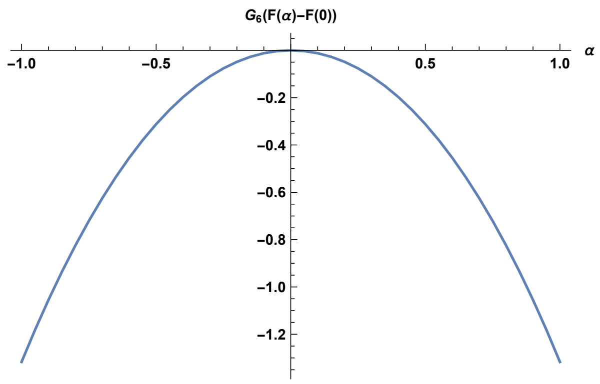


Figure 2.3: Plot of $G_6(F(\alpha)-F(0))$ obtained by numerical integration of the holographic result (2.4.42) in the range $|\alpha| \leq 1$.

2.5 Field theory calculation

Localization [Pes12] is a powerful tool used to obtain exact results in supersymmetric quantum field theories. In the large N limit, results obtained via localization calculations can be compared with results obtained via holography. The goal of this section is to calculate the sphere free energy for a five-dimensional mass-deformed SCFT using localization, and then to compare it to the holographic result obtained in the previous section.

A potential complication is that the five-dimensional field theory dual to the matter-coupled six-dimensional gauged supergravity described in section 2.2 has not been fully identified. This is because the full gauged supergravity has not been shown to arise as a consistent truncation of any ten-dimensional theory. In the following, the tentative field theory dual we will use for the localization calculation in the IR is a $USp(2N)$ gauge theory coupled to N_f fundamental representation hypermultiplets, and a single hypermultiplet in the anti-symmetric representation. As we will review below, this theory is believed to be obtained from the D4-D8 system [BO99] in type I' string theory/massive type IIA supergravity.

One fundamental limitation in our comparison between field theory and holographic results is that our holographic RG flow is completely numerical, and there is no analytic formula for the free energy that can be derived from it. Nevertheless, we will find qualitative similarities between the holographic free energy and the localization result for the free energy of the aforementioned $USp(2N)$ gauge theory with mass deformation. For completeness, we will review the origin of the field theory from the brane system before presenting the localization calculation.

2.5.1 The D4-D8 system

The original D4-D8 system [BO99] is a brane configuration in type I' string theory involving N D4 branes on $\mathbb{R}^{1,8} \times S^1/\mathbb{Z}^2$. The D4 branes have their worldvolume along $\mathbb{R}^{1,8}$ and sit at points along the interval $S^1/\mathbb{Z}^2$. There is an O8$^-$ plane living at each of the two ends of the interval. These orientifold planes carry -16 units of D8 brane charge, and thus require the inclusion of 16 D8 branes at points along the interval for tadpole cancellation. The usual construction is to stack N_f D8 branes atop one of the O8$^-$ planes and to stack the remaining $(16 - N_f)$ D8 branes atop the other O8$^-$ plane. One then considers the case in which the N D4 branes are very near to the former stack, in which case the second boundary may be neglected. We are thus left with a consistent string theory configuration involving N D4 branes probing N_f D8 branes and a single O8$^-$ plane.

This string theory setup allows for an AdS/CFT interpretation. On the closed string side of the correspondence, the near-horizon geometry of the brane configuration is found to be $\mathrm{AdS}_6 \times S^4$ with N units of 4-form flux passing through the S^4 [BO99]. This is a background of massive type IIA supergravity. While ten-dimensional uplifts of general solutions to $F(4)$ gauged supergravity are not known, pure Roman's supergravity *does* have a known uplift to massive type IIA supergravity [CLP99]. In that case, the $\mathrm{AdS}_6 \times S^4$ background may be interpreted as an AdS_6 background of the six-dimensional pure Roman's theory.[7] With this as motivation, we will be optimistic and assume that the solution of the six-dimensional $F(4)$ gauged supergravity theory being studied in the present case also has some uplift to massive type IIA, even though the details have not been worked out.

On the open string side of the correspondence, the worldvolume theory of the N D4 branes (together with their images) is a strongly-coupled 5D SCFT which does not admit a Lagrangian description. However, one may deform this theory by a relevant operator to flow to a 5D $\mathcal{N} = 1$ Yang-Mills-matter theory in the IR [Sei96]. In the setup described above, the resulting flow is to a 5D $\mathcal{N} = 1$ $USp(2N)$ gauge theory, where the relevant deformation has an interpretation as the gauge theory kinetic operator TrF^2. The gauge theory is also accompanied by N_f hypermultiplets in the fundamental representation and a single hypermultiplet in the antisymmetric representation. The fundamental hypermultiplets arise from D4-D8 strings, while the antisymmetric hypermultiplet arises from strings stretched between the D4 branes and their images.

The UV SCFT has a moduli space of vacua, and this maps in the IR to the Coulomb branch of the Yang-Mills theory. The Coulomb branch is parameterized by vevs of the vector multiplet scalars, which correspond in the string theory picture to the location of the D4 branes along the interval. The locations of the D8 branes along the interval tune the masses of the fundamental hypermultiplets, while leaving the mass of the

[7] The reduction to six dimensions is done in two steps. One first integrates over one of the coordinates of the sphere, leaving a nine-dimensional space of the form $\mathrm{AdS}_6 \times S^3$. Then one reduces on the S^3 to six dimensions, while gauging an $SU(2)$ subgroup of the sphere's $SO(4)$ isometry group [BO99].

antisymmetric hypermultiplet unchanged.

From the two points of view outlined above, one is led to conjecture a duality between the fluctuations around the $\mathrm{AdS}_6 \times S^4$ background of massive type IIA supergravity on one hand, and the non-Lagrangian worldvolume theory of the N D4 branes on the other. Though the non-Lagrangian nature of the field theory would naively make checking the duality extremely difficult, the fact that the UV SCFT admits a deformation to a 5D $\mathcal{N}=1$ Yang-Mills theory coupled to matter allows for the following crucial simplification. Given the Lagrangian description of the IR gauge theory, we may add an infinite number of gauge-invariant, supersymmetric irrelevant operators to deform the theory back to the UV fixed point with arbitrary precision. If one assumes these irrelevant operators to be Q-exact, then their coefficients can be tuned freely without changing the path integral on S^5. Thus the sphere partition function, and hence the free energy, calculated in the IR Yang-Mills theory is expected to be equivalent to that calculated in the original non-Lagrangian theory, allowing one to test the conjectured duality. This reasoning was used in [JP14] to calculate the free energy on both sides of the above duality. Comparison of the two results showed a perfect match.

We may now offer a microscopic description of the supergravity solutions described in this paper. Under the previous assumption that the solutions of the $F(4)$ gauged supergravity theory being studied here can be uplifted to an $\mathrm{AdS}_6 \times S^4$ background of massive type IIA, our solutions should be captured by the D4-D8 brane framework. To identify the details of the relevant brane configuration, we first recall from section 2.2.1 that the group which is gauged in the supergravity theory is $SU(2)_R \times G_+$, where G_+ is the additional gauge group arising from the presence of vector multiplets. Indeed, the presence of n vector fields A^I_μ allows for the existence of a gauge group G_+ of dimension $\dim G_+ = n$. The gauge group G_+ in the bulk corresponds to a flavor symmetry group E_{N_f+1} of the boundary SCFT [FKP98]. The RG-flow triggered by the gauge coupling breaks this symmetry group to $SO(2N_f) \times U(1)$ in the IR. Deformation by the relevant mass parameters will generically break $SO(2N_f)$ further. For the solution studied in this paper, an $SO(2)$ symmetry survives, which suggests that a minimal choice for the dual field theory would be one with $N_f = 1$ (i.e. a single D8 brane).

However, even in this minimal case the enhanced gauge group $E_2 \cong SU(2) \times U(1)$ of the conformal fixed point is found to have dimension $\dim E_2 = 4$, which suggests that the holographic dual to such a theory should contain at least four bulk vector multiplets. Fortunately, it is possible to embed our $n = 1$ solution in a theory with $n = 4$, which can accommodate the extended flavor symmetry in the UV. Setting the fields of the three additional vector multiplets to vanish then reproduces exactly the solutions explored in this paper. In fact, such an embedding is possible for any value of $n > 1$. This suggests that our holographic solutions are generic enough to capture the behavior of all single-mass deformations of E_{N_f+1} theories for any N_f. As such, we will carry out the localization calculation in section 2.5.3 for generic N_f. We will find that for every choice of $1 \leq N_f \leq 7$, one obtains a good match between the analytic field theory expression and our previous numerical results.

Having addressed the identification of flavor symmetries, it is natural to interpret the holographic solutions of this paper as dual to RG flows emanating from the same UV SCFTs that were found to be the duals of pure Roman's supergravity. The flow is driven by three relevant operators of dimension $\Delta = 3, 4, 3$, in addition to the gauge coupling deformation which brings the non-Lagrangian UV SCFT to an IR Yang-Mills-matter theory. In the IR, the three relevant deformations are interpreted respectively as a mass term for the hypermultiplet scalars, a mass term for the hypermultiplet fermions, and a dimension three operator needed to preserve supersymmetry on the five-sphere [HST12, KQZ12]. The explicit form of these deformations is shown in (2.2.22).

To support this interpretation, we now calculate the free energy of the mass-deformed $USp(2N)$ gauge theory and compare it to the holographic result displayed in Figure 2.3. For the unfamiliar reader, we will first reproduce the results of [JP14], where the $USp(2N)$ theory without mass deformation was studied. The techniques used for the mass-deformed theory will be the same, and the new calculation is presented in section 2.5.3.

2.5.2 Undeformed $USp(2N)$ gauge theory

In [KQZ12], localization techniques were used to find the perturbative partition function of $\mathcal{N}=1$ five-dimensional Yang-Mills theory with matter in a representation R on S^5, with the result given by

$$\begin{aligned} Z &= \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}}[d\sigma]\, e^{-\frac{8\pi^3 r}{g_{YM}^2}\text{Tr}(\sigma^2)} det_{\,\text{Ad}}\left(\sin(i\pi\sigma)e^{\frac{1}{2}f(i\sigma)}\right) \\ &\qquad \times\prod_I det_{\,R_I}\left((\cos(i\pi\sigma))^{\frac{1}{4}}e^{-\frac{1}{4}f(\frac{1}{2}-i\sigma)-\frac{1}{4}f(\frac{1}{2}+i\sigma)}\right)+O\left(e^{\frac{-16\pi^3 r}{g_{YM}^2}}\right) \end{aligned} \tag{2.5.1}$$

where r is the radius of S^5, σ is a dimensionless matrix, and f is defined as

$$f(y)=\frac{i\pi y^3}{3}+y^2\log\left(1-e^{-2\pi iy}\right)+\frac{iy}{\pi}\text{Li}_2\left(e^{-2\pi iy}\right)+\frac{1}{2\pi^2}\text{Li}_3\left(e^{-2\pi iy}\right)-\frac{\zeta(3)}{2\pi^2} \tag{2.5.2}$$

The quotient by the Weyl group in (2.5.1) amounts to division by a simple numerical factor $|\mathcal{W}|=2^N N!$. The integral over σ is not restricted to a Weyl chamber. Though this localization result was obtained in the IR theory, it is expected to hold in the UV due to the assumed Q-exactness of the irrelevant UV completion terms.

One may rewrite the partition function in terms of the free energy as

$$\begin{aligned} Z &= \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}}[d\sigma]\, e^{-F(\sigma)}+O\left(e^{\frac{-16\pi^3 r}{g_{YM}^2}}\right) \\ F(\sigma) &= \frac{4\pi^3 r}{g_{YM}^2}\text{Tr}\,\sigma^2+\text{Tr}_{\text{Ad}}F_V(\sigma)+\sum_I \text{Tr}_{R_I}F_H(\sigma) \end{aligned} \tag{2.5.3}$$

The definitions of $F_V(\sigma)$ and $F_H(\sigma)$ follow simply from (2.5.1), and using (2.5.2) one may obtain the following large argument expansions

$$F_V(\sigma)\approx\frac{\pi}{6}|\sigma|^3-\pi|\sigma| \qquad\qquad F_H(\sigma)\approx-\frac{\pi}{6}|\sigma|^3-\frac{\pi}{8}|\sigma| \tag{2.5.4}$$

It was argued in [JP14] that in the large N limit, the perturbative Yang-Mills term - i.e. the first term in the expression for $F(\sigma)$ in (2.5.3) - can be neglected, as can be the instanton contributions. Thus in our evaluation of the free energy, we will only concern ourselves with the contributions coming from $F_V(\sigma)$ and $F_H(\sigma)$.

The first step in the evaluation of (2.5.3) is recasting the matrix integral in a simpler form. The integral over σ in (2.5.3) is an integration over the Coulomb branch, which is parameterized by the non-zero vevs of σ. One may write

$$\sigma=\text{diag}\{\lambda_1,\ldots,\lambda_N,-\lambda_1,\ldots,-\lambda_N\} \tag{2.5.5}$$

since $USp(2N)$ has N elements in its Cartan. The integration variables are these N λ_i. Normalizing the weights of the fundamental representation of $USp(2N)$ to be $\pm e_i$ with e_i forming a basis of unit vectors for $\mathbb{R}^N$, it follows that the adjoint representation has weights $\pm 2e_i$ and $e_i \pm e_j$ for all $i \neq j$, whereas the anti-symmetric representation has only weights $e_i \pm e_j$ for all $i \neq j$. The free energy in the specific case of a vector multiplet in the adjoint, a single antisymmetric hypermultiplet, and N_f fundamental hypermultiplets then is

$$\begin{aligned} F(\lambda_i) &= \sum_{i \neq j} \left[F_V(\lambda_i - \lambda_j) + F_V(\lambda_i + \lambda_j) + F_H(\lambda_i - \lambda_j) + F_H(\lambda_i + \lambda_j)\right] \\ &\quad + \sum_i \left[F_V(2\lambda_i) + F_V(-2\lambda_i) + N_f F_H(\lambda_i) + N_f F_H(-\lambda_i)\right] \end{aligned} \tag{2.5.6}$$

The next step is to look for extrema of this function in the specific case of $\lambda_i \geq 0$ for all i. Extrema in the case of non-positive λ_i can be obtained from these through action of the Weyl group.

To calculate the extrema, one first assumes that as $N \to \infty$, the vevs scale as $\lambda_i = N^\alpha x_i$ for $\alpha > 0$ and x_i of order $O(N^0)$. One then introduces a density function

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \tag{2.5.7}$$

which in the continuum limit should approach an L^1 function normalized as

$$\int dx\, \rho(x) = 1 \tag{2.5.8}$$

In terms of this density function, one finds that

$$F \approx -\frac{9\pi}{8} N^{2+\alpha} \int dx dy\, \rho(x)\rho(y)\left(|x-y| + |x+y|\right) + \frac{\pi(8-N_f)}{3} N^{1+3\alpha} \int dx\, \rho(x)\, |x|^3 \tag{2.5.9}$$

where the large argument expansions (2.5.4) have been used, and terms subleading in N have been dropped. This only has non-trivial saddle points when both terms scale the same with N, which demands that $\alpha = 1/2$ and gives the famous result that $F \propto N^{5/2}$. Extremizing the free energy over normalized density functions then gives

$$F \approx -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}} \tag{2.5.10}$$

This value of the free energy is to be identified with the renormalized on-shell action of the supersymmetric AdS_6 solution. This identification yields the following relation between the six-dimensional Newton's constant G_6 and the parameters N and N_f of the dual SCFT,

$$G_6 = \frac{5\pi\sqrt{8-N_f}}{27\sqrt{2}}\, N^{-5/2} \tag{2.5.11}$$

2.5.3 Mass-deformed $USp(2N)$ gauge theory

As discussed previously, we now give a mass to a single hypermultiplet in the fundamental representation. This amounts to making a shift $\sigma \to \sigma + m$ in the relevant functional determinant. The result of this shift may be accounted for in (2.5.6) by writing

$$\begin{aligned} F(\lambda_i, m) \;=\; & \sum_{i\neq j} \left[F_V(\lambda_i-\lambda_j) + F_V(\lambda_i+\lambda_j) + F_H(\lambda_i-\lambda_j) + F_H(\lambda_i+\lambda_j)\right] \\ & + \sum_i \left[F_V(2\lambda_i) + F_V(-2\lambda_i) + F_H(\lambda_i+m) + F_H(-\lambda_i+m)\right. \\ & \left. + (N_f-1)F_H(\lambda_i) + (N_f-1)F_H(-\lambda_i)\right] \end{aligned} \tag{2.5.12}$$

As before, we assume that $\lambda_i = N^\alpha x_i$ for $\alpha > 0$ and introduce a density $\rho(x)$ satisfying (2.5.8). Using the expansions (2.5.4), we find the analog of (2.5.9) to be

$$\begin{aligned} F(\mu) \;\approx\; & -\frac{9\pi}{8}N^{2+\alpha}\int dxdy\,\rho(x)\rho(y)\,(|x-y|+|x+y|) + \frac{\pi}{3}(9-N_f)N^{1+3\alpha}\int dx\,\rho(x)\,|x|^3 \\ & -\frac{\pi}{6}N^{1+3\alpha}\int dx\,\rho(x)\left[|x+\mu|^3 + |x-\mu|^3\right] \end{aligned} \tag{2.5.13}$$

where for convenience we have defined $\mu \equiv m/N^\alpha$. As in the undeformed case, there is a non-trivial saddle point only when $\alpha = 1/2$. A normalized density function which extremizes the free energy is

$$\rho(x) = \frac{1}{(8-N_f)x_*^2 - \mu^2}\left(2(9-N_f)|x| - |x+\mu| - |x-\mu|\right) \qquad x_* = \sqrt{\frac{9+2\mu^2}{2(8-N_f)}} \tag{2.5.14}$$

with $\rho(x)$ having support only on the interval $x \in [0, x_*]$. Inserting this result back into (2.5.13) then gives our final result,[8]

$$F(\mu) = \frac{\pi}{135}\left((N_f-1)|\mu|^5 - \sqrt{\frac{2}{8-N_f}}\,(9+2\mu^2)^{5/2}\right)N^{5/2} \tag{2.5.15}$$

---

[8]The first term in the large N expansion of this result agrees with Eq. (3.22) of [CFL18], up to a factor of N_f. This difference is due to the fact that we give mass to only a single fundamental hypermultiplet.

We may check that when $\mu = 0$, we reobtain the result of the undeformed case (2.5.10).

With this result and G_6 given by (2.5.11), we may now try to compare $G_6(F(\mu)-F(0))$ to the same result calculated holographically in Figure 2.3. Importantly, since μ scales as $N^{-1/2}$, we see that in the large N limit the first term of (2.5.15) is subleading and may be neglected. Thus to leading order in N, the combination $G_6 F(\mu)$ is in fact independent of N_f. Since comparison with the holographic result requires taking the large N limit, our supergravity solutions will be unable to capture information about the precise flavor content of the SCFT dual. This agrees with the previous comments that, from the point of view of six-dimensional supergravity, the $n = 1$ solutions we are considering can be consistently embedded into theories with any number of bulk vector multiplets.

To proceed with the comparison between field theory and holographic results, we require a relation between the holographic deformation parameter α and the field theory mass parameter μ, i.e. $\alpha = A^{-1}\mu$ for some A, whose numerical value can be obtained by fitting the the two results. The result of this one parameter fit is given by the red curve in Figure 2.4.

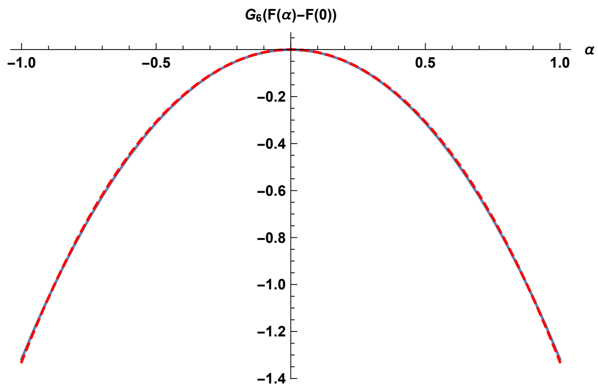


Figure 2.4: The free energy obtained by a holographic computation (solid blue), together with the free energy obtained by a field theory localization calculation (dashed red).

To the numerical accuracy of the holographic result, we see that the behavior of the holographic free energy as a function of the deformation parameter agrees with the field theory result obtained via localization. The value of A furnishing the fit in the range $|\alpha| \leq 1$ is found to be $A \approx 0.81$.

2.6 Discussion

In this chapter, we used the simple setup of six-dimensional gauged supergravity coupled to a single vector multiplet to study supersymmetric mass deformations of strongly coupled five-dimensional CFTs on a five-sphere. The numerical integration of the Euclidean BPS equations and the careful treatment of holographic renormalization allowed us to obtain the holographic free energy of the theory by calculating the on-shell action for the supergravity solutions. Due to the regularity of the solutions, the free energy depends on only one parameter, which can be interpreted as the supersymmetric mass deformation in the boundary RG flow.

We were able to find good numerical agreement between the holographic result and a localization calculation for a free $USp(2N)$ field theory in the IR, at least in the case of reasonably small deformation parameter. To proceed, one could next consider cosets with $n > 1$ and gaugings which realize larger flavor symmetries at the UV fixed points. It would also be interesting to see whether the six-dimensional solutions found here could be lifted to ten dimensions, both in the context of massive type IIA supergravity [BO99] as well as type IIB supergravity [DGK16a, DGU17a].

Furthermore, in obtaining our solutions we demanded that the five-sphere smoothly closes off in the IR. It should also be possible to impose a different boundary condition where at finite radius one side of the RG flow is glued to a second one, resulting in a Euclidean wormhole configuration in AdS [GS02, MM04]. It is likely that such a solution would be related to the holographic defect solutions found in [GKR17].

CHAPTER 3

M-theory curves from warped AdS_6 in Type IIB

3.1 Introduction

As mentioned in Chapter 1, in [DGK16b, DGU17a, DGU17b] Type IIB supergravity solutions of the form $AdS_6 \times S^2$ warped over a Riemann surface Σ_{IIB} were obtained, and were written in terms of a pair of locally holomorphic functions $\mathcal{A}_\pm$ on Σ_{IIB}. As will be reviewed below, for the solutions to be physically regular, Σ_{IIB} is required to have a boundary and the functions $\mathcal{A}_\pm$ are required to satisfy certain constraints. Along the boundary of Σ_{IIB}, the differentials $\partial\mathcal{A}_\pm$ have poles, at which the semi-infinite external five-branes of the associated 5-brane web emerge. The (p, q) charges of the emerging 5-brane are fixed by the residues of $\partial\mathcal{A}_\pm$. The solutions are completely specified by the choice of Riemann surface Σ_{IIB}, together with the number of poles and associated residues.

The prominent role of a Riemann surface and holomorphic functions in specifying the Type IIB supergravity solutions may seem reminiscent of the data used by Seiberg and Witten to specify 4d $\mathcal{N} = 2$ theories [SW94, Wit97]. Indeed, the same data can be used to specify 5d $\mathcal{N} = 1$ theories engineered by (p, q) 5-brane webs in Type IIB – that is, such theories may be defined by a holomorphic curve Σ_{M5}, which contains one compact direction, together with a holomorphic one-form λ on that curve [Kol99, BIS97, AHK98, KR98]. The physical interpretation is that the 5d $\mathcal{N} = 1$ theory is the worldvolume theory of an M5-brane wrapped on Σ_{M5}. This suggests that the Riemann surface and holomorphic data characterizing the Type IIB supergravity solutions may be related to the Riemann surface wrapped by the M5-brane in M-theory.

In this chapter, we show that this expectation is indeed realized, and explicate the relationship between Σ_{IIB} with the locally holomorphic functions $\mathcal{A}_\pm$ on the one hand,

and Σ_{M5} with a holomorphic one-form λ on the other. More precisely, we will argue that the locally holomorphic functions $\mathcal{A}_\pm$ provide an embedding of the doubled Type IIB Riemann surface $\hat{\Sigma}_{IIB}$ into the flat M-theory geometry, and that this embedded surface *is* the surface Σ_{M5} wrapped by the M5-brane. The Seiberg-Witten differential λ is identified with a locally holomorphic one-form $\mathcal{A}_+\partial\mathcal{A}_- - \mathcal{A}_-\partial\mathcal{A}_+$, which features prominently in the construction of the Type IIB solutions.

This identification between the data defining the Type IIB supergravity solutions and the data used to construct 5d SCFTs in M-theory is useful in a variety of ways. For the Type IIB solutions, it provides a geometric and physical understanding of certain aspects of the construction that are not directly apparent in Type IIB. For example, the physical meaning of the regularity conditions is not immediately apparent in the original formulation. In the M-theory picture, on the other hand, they become the simple condition that the BPS masses associated with the punctures of Σ_{M5} vanish - i.e. they enforce conformality of the dual 5d theory. This gives a physical reason for the absence of Type IIB AdS_6 solutions with Σ_{IIB} being an annulus, or more generally a Riemann surface with multiple boundary components or higher genus. Such solutions would map to M-theory curves describing mass deformations of 5d SCFTs, and are thus not expected to have the full AdS_6 isometries. For the solutions with Σ_{IIB} being a disc, the identification with the M-theory curve provides independent support for the identification of the solutions with the near-horizon limit of (p, q) 5-brane junctions.

For the M-theory side, the AdS_6 solutions provide explicit solutions to the polynomial equations defining the M-theory curves. We discuss this for a number of explicit classes, where the AdS_6 solutions provide simple generating functions for the polynomials defining the curves. This gives a more direct understanding of the pattern of "binomial edge coefficients," discussed in the separate context of brane tilings and their relations to dimer models in [HK05], and provides a simple way to compute certain multiplicities. We also discuss an interesting relation between the polynomial defining the T_N theory curve and a seemingly unrelated quantity in the field of combinatorics and number theory - namely, the Wendt determinant [Wen94, Hel97]. We show that the polynomial defining the M-theory curve for the 5d T_N theories [BBT09], evaluated for unit arguments, coincides with the Wendt determinant. We leave further exploration to the future, where we certainly expect the connection between Type IIB solutions and M-theory curves to be mutually beneficial. For example, the M-theory perspective may help identify operators in the SCFTs dual to the Type IIB solutions [HY98, Mik98]. It may also be useful for generalizing the construction of Type IIB AdS$_6$ solutions with 7-branes [DGU17c] to incorporate non-commuting monodromies.

The rest of this chapter is organized as follows. In Section 3.2, we review the relevant aspects of the Type IIB AdS$_6$ solutions as well as of the M-theory curves. In Section 3.3, we expand upon the relation between the two pictures and formulate the concrete identification. In Section 3.4, we verify the proposed identification for five families of supergravity solutions and M-theory curves.

3.2 Review: Type IIB AdS$_6$ and M-theory curves

This section contains a review of relevant aspects of the AdS$_6$ solutions, as well as of the relation between Type IIB 5-brane webs and M5-branes wrapping holomorphic curves in M-theory.

3.2.1 Warped AdS$_6$ in Type IIB

The geometry of the Type IIB AdS$_6$ solutions constructed in [DGK16b] is a warped product

$$AdS_6 \times S^2 \times \Sigma_{IIB} \tag{3.2.1}$$

of AdS$_6$ and S^2 over a Riemann surface Σ_{IIB}. The general solution to the BPS equations is parametrized by two locally holomorphic functions $\mathcal{A}_\pm$ on Σ_{IIB}. From these functions a locally holomorphic one-form $d\mathcal{B}$ on Σ_{IIB} is defined,

$$d\mathcal{B} = \mathcal{A}_+ d\mathcal{A}_- - \mathcal{A}_- d\mathcal{A}_+ \ . \tag{3.2.2}$$

The $SL(2,\mathbb{R})$ transformations of Type IIB supergravity are induced by a linear action of $SU(1,1) \times \mathbb{C}$ on the differentials (Section 5.3 of [DGK16b]),

$$\mathcal{A}_+ \to u\mathcal{A}_+ + v\mathcal{A}_- + c \ , \qquad\qquad \mathcal{A}_- \to \bar{v}\mathcal{A}_+ + \bar{u}\mathcal{A}_- + \bar{c} \ , \tag{3.2.3}$$

with $|u|^2 - |v|^2 = 1$ and $c \in \mathbb{C}$. The one-form $d\mathcal{B}$ is invariant under these transformations. The shifts parametrized by c leave the supergravity fields invariant, except for a gauge transformation of the two-form field. The supergravity fields are expressed in terms of $\mathcal{A}_\pm$, $\mathcal{B}$, and the composite functions [DGK16b]

$$\kappa^2 = -|\partial_w \mathcal{A}_+|^2 + |\partial_w \mathcal{A}_-|^2 \,, \qquad \mathcal{G} = |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}} \,, \tag{3.2.4}$$

where w is a local coordinate on Σ. Their explicit expressions will not be needed here.

Imposing global regularity conditions constrains the $\mathcal{A}_\pm$ and requires that Σ_{IIB} have non-empty boundary. Physically regular solutions without monodromy were constructed in [DGU17a, DGU17b] for the case in which Σ_{IIB} is a disc, or equivalently the upper half-plane. At the boundary of the Riemann surface, $\partial\Sigma_{IIB}$, the spacetime S^2 collapses, closing off the ten-dimensional geometry smoothly. With a complex coordinate w on the upper half-plane, the $\mathcal{A}_\pm$ are given by

$$\mathcal{A}_\pm = \mathcal{A}^0_\pm + \sum_{\ell=1}^{L} Z^\ell_\pm \ln(w - r_\ell) \,, \tag{3.2.5}$$

with $\bar{Z}^\ell_\pm = -Z^\ell_\mp$ and $\bar{\mathcal{A}}^0_\pm = -\mathcal{A}^0_\mp$. The differentials $\partial_w \mathcal{A}_\pm$ have $L \geq 3$ poles at $w = r_\ell$ on the real line, with residues $Z^\ell_\pm$. The residues are constructed in terms of a distribution of auxiliary charges and sum to zero by construction. The locations of the poles are fixed by a set of regularity conditions

$$\mathcal{A}^0_+ Z^k_- - \mathcal{A}^0_- Z^k_+ + \sum_{\ell \neq k} (Z^\ell_+ Z^k_- - Z^k_+ Z^\ell_-) \ln |r_\ell - r_k| = 0 \,, \qquad k = 1, \cdots, L \,. \tag{3.2.6}$$

These physically regular solutions admit a natural identification with (p,q) 5-brane junctions in Type IIB string theory, involving L 5-branes whose charges we denote by (p_ℓ, q_ℓ) for $\ell = 1, .., L$. At the poles r_ℓ, the external (p,q) 5-branes of the associated 5-brane junction emerge, with the charges given in terms of the residues by

$$Z^\ell_\pm = \frac{3}{4} \alpha' (\pm q_\ell + i p_\ell) \,, \tag{3.2.7}$$

where a D5-brane corresponds to charge $(\pm 1, 0)$ and an NS5-brane to $(0, \pm 1)$ [BRU18].

3.2.2 M5-branes on holomorphic curves

Consider a (p,q) 5-brane web in Type IIB in the (x^5, x^6) plane. All 5-branes extend in the field theory directions $x^0, \ldots, x^4$. Compactifying x^4 on a circle with radius R_4 and

T-dualizing leads to Type IIA compactified on the T-dual circle with radius $\tilde{R}_4 = \alpha'/R_4$ and $g_{IIA} = \sqrt{\alpha'} g_{IIB}/R_4$. This is equivalent to M-theory compactified on a torus with coordinates (x^4, x^{10}) and $R_{10} = \sqrt{\alpha'} g_{IIA} = g_{IIB}\tilde{R}_4$. Decompactified Type IIB corresponds to the limit of vanishing volume, $\tilde{R}_4 R_{10} \to 0$, with fixed $R_{10}/\tilde{R}_4$.

In M-theory, the 5-brane web corresponds to a single M5-brane wrapping $x^0, \ldots, x^3$ and a complex curve $\Sigma_{M5} \subset \mathcal{M}_4$, where $\mathcal{M}_4 = \mathbb{R}^2 \times T^2$ is the space spanned by (x^5, x^6, x^4, x^{10}). Using complex coordinates s, t, defined by

$$s = \exp\left(\frac{x^5 + ix^4}{\tilde{R}_4}\right) , \qquad t = \exp\left(\frac{x^6 + ix^{10}}{R_{10}}\right) , \tag{3.2.8}$$

the curve is an algebraic variety defined by

$$\Sigma_{M5} : \qquad P(s,t) = 0 . \tag{3.2.9}$$

The polynomial $P(s,t)$ can be constructed in an algorithmic way from the brane web, as will be reviewed shortly, and Σ_{M5} is directly related to the Seiberg-Witten curve of the 4d theory obtained by compactifying x^4 [Wit97]. Supersymmetry requires Σ_{M5} to be a calibrated submanifold. The calibration is given by

$$d\lambda = d\ln t \wedge d\ln s , \tag{3.2.10}$$

and the primitive yields the Seiberg-Witten differential, e.g.

$$\lambda = \frac{dt}{2t}\ln s - \frac{ds}{2s}\ln t . \tag{3.2.11}$$

The Type IIB $SL(2,\mathbb{Z})$ duality is realized in M-theory as the $SL(2,\mathbb{Z})$ acting on the (x^4, x^{10}) torus via

$$s \to s^a t^b , \qquad t \to s^c t^d , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) . \tag{3.2.12}$$

The Seiberg-Witten differential in (3.2.11) is invariant under these $SL(2,\mathbb{Z})$ transformations.

3.2.2.1 M-theory curves and grid diagrams

The polynomial $P(s,t)$ defining Σ_{M5} is obtained from the grid diagram associated with a given 5-brane web [AHK98]. The grid diagram is constructed by placing one vertex in each face of the web and connecting vertices in adjacent faces by a line that crosses the intermediate 5-brane perpendicularly. This gives a convex polygon $\Delta(P) \subset \mathbb{Z}^2$.[9] One may read off the polynomial $P(s,t)$ from $\Delta(P)$ as follows: for each point in $\Delta(P)$ with coordinates $(\alpha_i, \beta_i) \in \mathbb{Z}^2$, one adds a monomial $s^{\alpha_i} t^{\beta_i}$ with an arbitrary coefficient, resulting in

$$P(s,t) = \sum_i c_i s^{\alpha_i} t^{\beta_i} \ . \tag{3.2.13}$$

Explicit examples will be shown in section 3.4.

Now consider one of the asymptotic 5-branes with charges (p,q), in all-ingoing convention. Supersymmetry requires the slope of this brane in the (x^5, x^6)-plane to be

$$\frac{\Delta x^6}{\Delta x^5} = \frac{Im(\tau)q}{p + Re(\tau)q} \ . \tag{3.2.14}$$

This is the condition that there be zero force at the vertices of the web. In M-theory, holomorphicity demands that this constraint be completed to an analogous constraint on s and t. The imaginary part of the holomorphic constraint is

$$\frac{\Delta x^{10}}{\Delta x^4} = \frac{Im(\tau)q}{p + Re(\tau)q} \ . \tag{3.2.15}$$

Interpreting τ as the modular parameter of the M-theory torus, this fixes the M5-brane to be oriented along the (p,q) cycle of T^2.

Without loss of generality, we set the asymptotic value of the axio-dilaton scalar to $\tau_\infty = i$.[10] The embedding of the (p,q) 5-brane into the (x^5, x^6)-plane is then given by

$$m + (-qx^5 + px^6)T_s = 0 \ , \tag{3.2.16}$$

where m corresponds to a mass parameter. The projection of the M5-brane curve onto the (x^5, x^6)-plane should approach this embedding asymptotically. In the s,t coordinates, (3.2.16) becomes $\exp(m/\tilde{R}_4 T_s)|s|^{-q}|t|^p = 1$, while the asymptotic region corresponds to

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[9] In the math literature, the grid diagram is also referred to as the "Newton polygon."

[10] In M-theory this corresponds to $\tilde{R}_4 = R_{10}$. Expressions for generic values of τ_∞ are obtained by replacing $x^5 \to \tilde{x}^5 = x^5 - Re(\tau_\infty)/Im(\tau_\infty)x^6$, $x^6 \to \tilde{x}^6 = x^6/Im(\tau_\infty)$ [AHK98].

$-px^5, -qx^6 \to \infty$, or $|s|^{-p}, |t|^{-q} \to \infty$. In summary, the M-theory curve should behave as

$$As^{-q}t^p \sim 1 \;, \qquad\qquad \text{for} \quad |s|^{-p}, |t|^{-q} \to \infty \;, \tag{3.2.17}$$

with $|A| = \exp(m/\tilde{R}_4 T_s)$. Requiring that $P(s,t) = 0$ exhibits this behavior puts constraints on the coefficients c_i. For a group of N external 5-branes with charges (p,q), the constraint is

$$P(s,t) \sim \prod_{i=1}^{N}(A_i t^p - s^q) \qquad\qquad \text{for} \quad |s|^{-p}, |t|^{-q} \to \infty \;. \tag{3.2.18}$$

In the conformal limit, these 5-branes are coincident, and the M5-brane curve is expected to approach this stack of coincident branes. The boundary condition then becomes

$$P(s,t) \sim (\alpha t^p - s^q)^N \;, \qquad\qquad \text{for} \quad |s|^{-p}, |t|^{-q} \to \infty \;, \tag{3.2.19}$$

where α is a phase, i.e. $|\alpha| = 1$, which encodes the asymptotic behavior of the M-theory curve in the (x^4, x^{10}) directions.

3.3 M-theory curves from Type IIB AdS$_6$

In this section we discuss the connection between AdS$_6$ solutions in Type IIB and the holomorphic curves wrapped by M5-branes in M-theory. Our main result is a relation between the Riemann surface Σ_{IIB} appearing in the supergravity solution and the M-theory curve Σ_{M5}. Detailed evidence for the proposed relation will be presented in Section 3.4.

3.3.1 $\mathcal{A}_\pm$ and algebraic equations

Before discussing the identification in detail, we rewrite the locally holomorphic functions $\mathcal{A}_\pm$ in (3.2.5) in a more suggestive way. Using the relation between residues and 5-brane charges (3.2.7), as well as the conjugation relations spelled out below (3.2.5), we have

$$\mathcal{A}_\pm = \frac{3}{4}\alpha' \left(i \ln \sigma \pm \ln \mathfrak{t}\right) \;, \tag{3.3.20}$$

where the combinations σ and $\mathfrak{t}$ are defined as

$$\sigma = e^{Im\, a} \prod_{\ell=1}^{L}(w - r_\ell)^{p_\ell} \;, \qquad\qquad \mathfrak{t} = e^{Re\, a} \prod_{\ell=1}^{L}(w - r_\ell)^{q_\ell} \;, \tag{3.3.21}$$

and we have introduced a constant a defined by $\mathcal{A}_+^0 \equiv \frac{3}{4}\alpha' a$. With these definitions, the locally holomorphic one-form $d\mathcal{B}$ defined in (3.2.2) takes the form

$$d\mathcal{B} = \frac{9}{8} i\alpha'^2 \left(\frac{d\sigma}{\sigma} \ln \mathfrak{t} - \frac{d\mathfrak{t}}{\mathfrak{t}} \ln \sigma \right) \ , \tag{3.3.22}$$

while κ^2 and $\mathcal{G}$ of (3.2.4) are given by

$$\kappa^2 dw \wedge d\bar{w} = \frac{9}{8i}\alpha'^2 \left(d\ln\sigma \wedge \overline{d\ln\mathfrak{t}} - d\ln\mathfrak{t} \wedge \overline{d\ln\sigma} \right) \ , \tag{3.3.23}$$

$$\mathcal{G} = \frac{9}{8i}\alpha'^2 \left(\overline{\ln\sigma}\, \ln\mathfrak{t} - \ln\sigma\, \overline{\ln\mathfrak{t}} \right) + \mathcal{B} + \bar{\mathcal{B}} \ . \tag{3.3.24}$$

The first claim, which we will verify for a number of explicit examples in section 3.4, is that the Riemann surface Σ_{IIB} with the locally holomorphic functions $\mathcal{A}_\pm$ provides a solution to equation (3.2.9) defining the associated M-theory curve, via the identification

$$s = \sigma \ , \qquad\qquad t = \mathfrak{t} \ . \tag{3.3.25}$$

Note that we could in principle allow for arbitrary rescalings of s, t in this identification, corresponding to translations of the web - cf. (3.2.8). As a first consistency check, we note that the $SL(2,\mathbb{Z})$ transformations of s, t in (3.2.12) induce the corresponding transformations of $\mathcal{A}_\pm$ in (3.2.3) via (3.3.20) and (3.3.25). Moreover, the constant shifts by $c, \bar{c}$ in (3.2.3) correspond to translations in (x^4, x^{10}) via (3.2.8).

An immediate consequence of this identification is that the holomorphic one-form $d\mathcal{B}$ in (3.3.22) is directly related to the Seiberg-Witten differential λ in (3.2.11), via

$$d\mathcal{B} = -\frac{9}{4} i\alpha'^2 \lambda \ . \tag{3.3.26}$$

3.3.2 Global structure

We have claimed that the functions $\mathcal{A}_\pm$ on the Riemann surface Σ_{IIB} provide a solution to the equation defining the M-theory curve, Σ_{M5}. We now address this identification at the global level. The relation (3.3.20) with (3.3.25) and (3.2.8) in fact suggests a more direct identification of $\mathcal{A}_\pm$ with the coordinates in M-theory as follows,

$$\frac{x^5 + ix^4}{\tilde{R}_4} = -\frac{2i}{3\alpha'} \left(\mathcal{A}_+ + \mathcal{A}_- \right) \ , \qquad \frac{x^6 + ix^{10}}{R_{10}} = \frac{2}{3\alpha'} \left(\mathcal{A}_+ - \mathcal{A}_- \right) \ . \tag{3.3.27}$$

That is, the functions $\mathcal{A}_\pm$ provide an embedding of Σ_{IIB} into the four-dimensional space $\mathcal{M}_4 = \mathbb{R}^2 \times T^2$ spanned by the M-theory coordinates (x^5, x^6, x^4, x^{10}). An apparent challenge to a direct identification of Σ_{IIB} and Σ_{M5} is the fact that, being a disc or the upper half-plane, Σ_{IIB} has a boundary, while Σ_{M5} does not. We note that

$$\bar{\mathcal{A}}_\pm - \mathcal{A}_\mp = 2\pi i \sum_{k=1}^{L} \Theta(r_k - w) Z_\mp^k \ , \qquad\qquad w \in \partial\Sigma_{IIB} \ , \tag{3.3.28}$$

with Θ the Heaviside function. Consequently, for integer charges p_k, q_k,

$$\frac{x^{10}}{R_{10}} = \frac{x^4}{\tilde{R}_4} = 0 \mod \pi \qquad \forall w \in \partial\Sigma_{IIB} \ . \tag{3.3.29}$$

Thus, the segments of the boundary of Σ_{IIB} in between poles are mapped to curves in planes of constant x^4 and x^{10}. The embedding of Σ_{IIB} into $\mathcal{M}_4$ is illustrated in Figures 3.2 and 3.5 for the T_1 and $+_{1,1}$ solutions, respectively.

A natural interpretation for the boundary in Σ_{IIB} can be obtained as follows. We recall that the regularity conditions in Type IIB supergravity have two branches of solutions (Section 5.4 of [DGK16b]),

$$\mathcal{R}_+ : \quad \{\kappa^2 > 0 \ , \quad \mathcal{G} > 0\} \ , \qquad\qquad \mathcal{R}_- : \quad \{\kappa^2 < 0 \ , \quad \mathcal{G} < 0\} \ . \tag{3.3.30}$$

These two branches are mapped into one another by complex conjugation. The regular solutions discussed above with Σ_{IIB} being the upper half-plane realize the branch $\mathcal{R}_+$. For any such regular solution in the upper half-plane, the extension of the $\mathcal{A}_\pm$ into the lower half-plane provides an equivalent regular solution, realizing the second branch of regularity conditions $\mathcal{R}_-$. The two solutions are separated at the boundary of Σ_{IIB}, where $\kappa^2 = \mathcal{G} = 0$.

Since the 10d spacetime in Type IIB is closed off smoothly at $\partial\Sigma_{IIB}$ by the collapsing S^2, the solutions in the upper and lower half-planes are two realizations of equivalent Type IIB solutions. But for the identification of Σ_{IIB} with the M-theory curve, it is natural to consider the full, doubled, Riemann surface $\hat{\Sigma}_{IIB}$.[11] The precise relation we propose is then

$$\Sigma_{M5} : \qquad \hat{\Sigma}_{IIB} \xrightarrow{\mathcal{A}_\pm} \mathcal{M}_4 = \mathbb{R}^2 \times T^2 \ . \tag{3.3.31}$$

[11] In fact, the construction of regular solutions in [DGU17b] employed an auxiliary electrostatics potential, in which the doubled Riemann surface $\hat{\Sigma}_{IIB}$ already played a crucial role.

That is, the embedding of the doubled Type IIB Riemann surface $\hat{\Sigma}_{IIB}$ into the four-dimensional part of the M-theory geometry, with the embedding functions given by $\mathcal{A}_\pm$ via (3.3.27), *is* the M-theory curve Σ_{M5}.

The doubled Type IIB Riemann surface $\hat{\Sigma}_{IIB}$ is a closed surface with punctures at the poles r_ℓ. Suppose we encircle one of the poles r_ℓ. Then $\ln(w - r_\ell) \to \ln(w - r_\ell) + 2\pi i$, and consequently

$$\mathcal{A}_+ \pm \mathcal{A}_- \to \mathcal{A}_+ \pm \mathcal{A}_- + 2\pi i \left(Z_+^\ell \pm Z_-^\ell \right) \ . \tag{3.3.32}$$

With the identifications (3.2.7) and (3.3.27), this means that

$$x^4 \to x^4 + 2\pi \tilde{R}_4 q_\ell \ , \qquad\qquad x^{10} \to x^{10} + 2\pi R_{10} p_\ell \ . \tag{3.3.33}$$

This is indeed the desired behavior: the (p, q) 5-brane charges become the winding numbers of the M5-brane, with the winding on the M-theory circle x^{10} encoding the D5 charge and the winding on x^4 encoding the NS5 charge. This furthermore implies that the curve defined by the embedding (3.3.27) is smooth across the boundary of Σ_{IIB}, despite the fact that the $\mathcal{A}_\pm$ are not single-valued in the doubled Riemann surface $\hat{\Sigma}_{IIB}$ (noting that the differentials $\partial_w \mathcal{A}_\pm$ are single-valued on $\hat{\Sigma}_{IIB}$). That is, since

$$\overline{\mathcal{A}_\pm(\bar{w})} = -\mathcal{A}_\mp(w) + \frac{3}{2}\alpha' i\pi k \ , \qquad\qquad k \in \mathbb{Z} \ , \tag{3.3.34}$$

mapping from the upper half-plane Σ_{IIB} into the lower half-plane of $\hat{\Sigma}_{IIB}$ induces the following map on the M-theory curve,

$$\begin{aligned} w \mapsto \bar{w} \ : \qquad\qquad & x^4 \mapsto -x^4 \quad \mod 2\pi \tilde{R}_4 \ , \\ & x^{10} \mapsto -x^{10} \quad \mod 2\pi R_{10} \ . \end{aligned} \tag{3.3.35}$$

Then due to (3.3.29), the boundary of Σ_{IIB} is mapped to fixed points of this action on the torus.

3.3.3 Type IIB regularity conditions

The asymptotic behavior of the M5-brane curve is constrained by the conditions (3.2.17). We will now discuss how this behavior is realized by the identification (3.3.25), and obtain a geometric perspective on the Type IIB regularity conditions (3.2.6). Consider the limit in which

$$w \to r_k \ . \tag{3.3.36}$$

With the explicit expressions in (3.3.21), we find that in this limit

$$|\sigma|^{-p_k}, |\mathfrak{t}|^{-q_k} \to \infty \ , \tag{3.3.37}$$

corresponding to the asymptotic region where 5-branes with charges (p_k, q_k) are, as expected. Furthermore, in this limit the explicit expressions in (3.3.21) give

$$\sigma^{-q_k} \mathfrak{t}^{p_k} = e^{p_k \, Re(a) - q_k \, Im(a)} \prod_{\ell \neq k} (r_k - r_\ell)^{q_\ell p_k - p_\ell q_k} \ , \tag{3.3.38}$$

which is finite, as required by (3.2.17). As seen from (3.2.17), the mass parameter associated with the external 5-branes is given by

$$\begin{aligned} -m_k^2 &= \ln \left| \sigma^{-q_k} \mathfrak{t}^{p_k} \right|^2 \\ &= 2p_k \, Re(a) - 2q_k \, Im(a) + \sum_{\ell \neq k} (q_\ell p_k - p_\ell q_k) \ln |r_k - r_\ell|^2 \ . \end{aligned} \tag{3.3.39}$$

Using the identification of the residues with the 5-brane charges (3.2.7), as well as the definition of the constant a below (3.3.21), the Type IIB regularity conditions in (3.2.6) are precisely the statement that $m_k^2 = 0$ for all k. The Type IIB regularity conditions are therefore interpreted from the M-theory perspective as the requirement that the 5-branes within each group of like-charged external 5-branes are coincident, with the associated mass parameter vanishing.

The identification of $d\mathcal{B}$ with the Seiberg-Witten differential allows for an additional physical interpretation of the regularity conditions (3.2.6) from the 4d perspective. Of the L conditions in (3.2.6) only $L-1$ are independent, due to the fact that the $Z_\pm^\ell$ sum to zero by construction, implementing charge conservation at the 5-brane junction. These conditions may be formulated more concisely in the upper half-plane as

$$\int_{C_k} d\mathcal{B} + c.c. = 0 \ , \qquad\qquad k = 1, \dots, L \ , \tag{3.3.40}$$

where C_k denotes a curve connecting two points on the boundary $\partial\Sigma_{IIB}$ to either side of the pole r_k. In this formulation, charge conservation amounts to the fact that the sum of the cycles C_k is contractible.

In the doubled surface $\hat{\Sigma}_{IIB}$, the addition of the complex conjugate on the left hand side in (3.3.40) can be implemented by closing the contour C_k in the lower half-plane, such that the pole is encircled completely. Denoting by $\hat{C}_k$ a closed contour around the pole p_k in $\hat{\Sigma}_{IIB}$, the regularity conditions become

$$\int_{\hat{C}_k} d\mathcal{B} = 0 \,, \qquad\qquad k = 1, \dots, L \,. \tag{3.3.41}$$

With the identification of $\hat{\Sigma}_{IIB}$ as the Seiberg-Witten curve of the 5d theory compactified on x^4, and of $d\mathcal{B}$ as the Seiberg-Witten differential via (3.3.26), the regularity conditions (3.3.41) again become the statement that the BPS masses associated with the punctures vanish.

3.3.4 Σ_{IIB} of general topology

The identification of $\hat{\Sigma}_{IIB}$ with the M-theory curve Σ_{M5} gives an interesting perspective on potential AdS_6 solutions in Type IIB where Σ_{IIB} is a Riemann surface with multiple boundary components or higher genus. From the Type IIB perspective, it is not *a priori* clear whether such solutions should exist. The construction used in [DGU17b] of imposing the global regularity conditions on the general local solution to the BPS equations and reducing them to a finite number of constraints in principle works for Riemann surfaces of arbitrary topology. This was spelled out explicitly in Section 6 of [DGU17b]. But solutions to these constraints were only found for the upper half-plane. For the annulus, an explicit search was conducted, but no solutions were found.

From the perspective of the associated M-theory curve, assuming that the identification of $\hat{\Sigma}_{IIB}$ with Σ_{M5} extends to Σ_{IIB} of more general topology, Σ_{IIB} with multiple boundaries or higher genus would correspond to M-theory curves Σ_{M5} of higher genus. Such curves are associated to 5-brane webs with open faces, i.e. mass deformations. These webs describe renormalization group flows, as opposed to renormalization group fixed points, and are therefore not expected to have an AdS_6 dual. This gives a physical interpretation for the absence of annulus solutions in Type IIB, and suggests more generally the absence of AdS_6 solutions for Riemann surfaces with multiple boundary components or higher genus.

3.4 Case studies

In this section, we verify the relation between the Type IIB AdS$_6$ solutions and M-theory curves discussed in Section 3.3 for a number of explicit examples.

3.4.1 T_N solutions

As a first example we discuss the 5d T_N theories [BBT09]. These are realized by triple junctions of N D5, N NS5, and N $(1,1)$ 5-branes (fig. 3.1(a)). The polynomial $P(s,t)$, obtained from the grid diagram (fig. 3.1(b)), is given by

$$P(s,t) = \sum_{i=0}^{N} \sum_{j=0}^{N-i} c_{i,j} s^i t^j \ . \tag{3.4.42}$$

The boundary conditions, in the conformal limit, are

$$\begin{aligned}
s,t \to \infty : \qquad & P(s,t) \sim \sum_{k=0}^{N} c_{k,N-k} s^k t^{N-k} && \overset{!}{\sim}\ (s-\alpha_1 t)^N \ , \\
s \text{ finite, } t \to 0 : \qquad & P(s,t) \sim \sum_{k=0}^{N} c_{k,0} s^k && \overset{!}{\sim}\ (s-\alpha_2)^N \ , \\
t \text{ finite, } s \to 0 : \qquad & P(s,t) \sim \sum_{k=0}^{N} c_{0,k} t^k && \overset{!}{\sim}\ (1-\alpha_3 t)^N \ ,
\end{aligned} \tag{3.4.43}$$

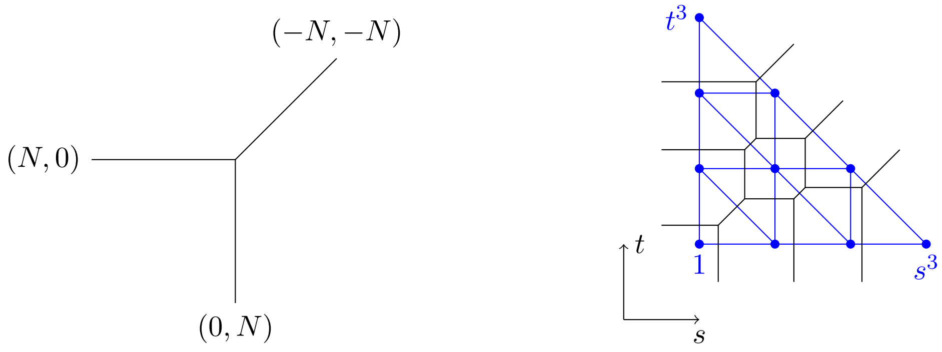
with $|\alpha_i| = 1$. This fixes the coefficients $c_{k,N-k}$, $c_{k,0}$ and $c_{0,k}$ for $k = 0,\dots,N$ to be binomial. The remaining coefficients encode Coulomb branch parameters. Without loss of generality, we fix $c_{0,0} = 1$. Then for $N = 1$, one finds

$$P_{T_1}(s,t) = 1 - \alpha_2^{-1} s - \alpha_3 t \ . \tag{3.4.44}$$

Consistency of the boundary conditions requires $\alpha_1 = \alpha_2\alpha_3$. The remaining freedom in α_2, α_3 corresponds to translations in the compact directions.

The Type IIB supergravity solutions corresponding to triple junctions of D5, NS5, and $(1,1)$ 5-branes were discussed in detail in [BRU18, FU18], including comparisons of holographic results to field theory computations. The functions $\mathcal{A}_\pm$ are given by (Section 4.3 of [BRU18])

$$\mathcal{A}_\pm = \frac{3}{4}\alpha' N \left[\pm \ln(w-1) + i\ln(2w) - (i \pm 1)\ln(w+1)\right] \ . \tag{3.4.45}$$



(a) (b)

Figure 3.1: Left: the 5-brane junction describing the T_N SCFTs with charge assignments in ingoing convention. Right: brane web and grid diagram for a mass deformation of the T_3 theory. Some examples of the monomials associated to the grid points are shown.

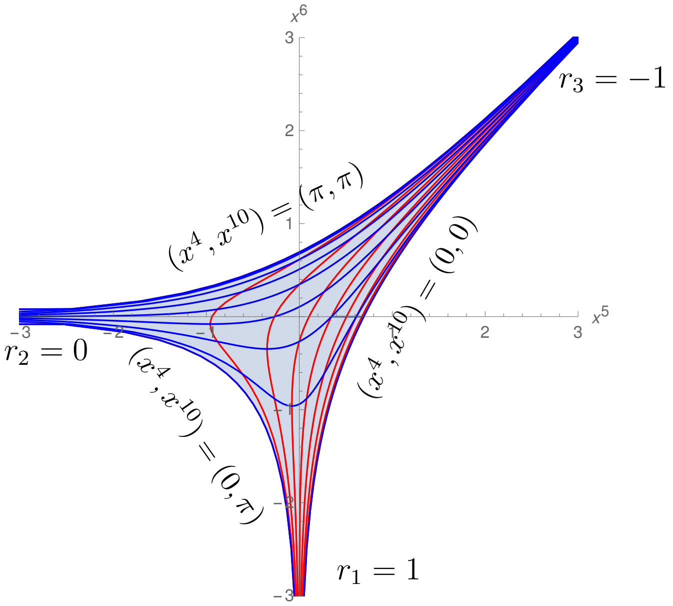


Figure 3.2: T_1 curve with $\tilde{R}_4 = R_{10} = 1$ obtained by embedding Σ_{IIB} into $\mathcal{M}_4$ via (3.3.27). The poles r_ℓ on Σ_{IIB} correspond to the external 5-branes in the asymptotic regions as indicated. The segments of the boundary $\partial\Sigma_{IIB}$ in between poles are mapped to the outer curves connecting the asymptotic regions, with values of x^4, x^{10} as indicated. The blue curves correspond to constant x^4, the red curves to constant x^{10}. Both are positive for $w \in \Sigma_{IIB}$. The embedding of the second half of $\hat{\Sigma}_{IIB}$, with w in the lower half-plane, is obtained by reversing the signs of x^4 and x^{10} (3.3.35).

This realizes the T_N charges in all-ingoing convention. Via (3.3.20) this yields

$$\sigma = \left(\frac{2w}{1+w}\right)^N , \qquad\qquad \mathfrak{t} = \left(\frac{w-1}{w+1}\right)^N . \tag{3.4.46}$$

For $N = 1$, these solve (3.4.44) with $\alpha_2 = -\alpha_3 = 1$ via $s = \sigma$ and $t = \mathfrak{t}$. More generally, σ and $\mathfrak{t}$ satisfy

$$0 = \mathcal{P}_{T_N}(\sigma, \mathfrak{t}) , \qquad\qquad \mathcal{P}_{T_N}(\sigma, \mathfrak{t}) \equiv 1 - \sigma^{1/N} + \mathfrak{t}^{1/N} . \tag{3.4.47}$$

Solving this equation for either σ in terms of $\mathfrak{t}$ or $\mathfrak{t}$ in terms of σ yields N branches of solutions. These are realized in (3.4.46) by the fact that solving for w in terms of σ or $\mathfrak{t}$ yields N branches of solutions. Evaluating the expression for the remaining one of σ or $\mathfrak{t}$ for these w gives N branches for σ in terms of $\mathfrak{t}$ and $\mathfrak{t}$ in terms of σ.

Eq. (3.4.47) can be converted to a polynomial equation $\tilde{P}_{T_N}(\sigma, \mathfrak{t}) = 0$ with the same roots. The result is

$$0 = \tilde{P}_{T_N}(\sigma, \mathfrak{t}) , \qquad \tilde{P}_{T_N}(\sigma, \mathfrak{t}) \equiv \prod_{n=0}^{N-1} \prod_{m=0}^{N-1} \mathcal{P}_{T_1}\left(e^{\frac{2\pi i n}{N}} \sigma^{\frac{1}{N}}, e^{\frac{2\pi i m}{N}} \mathfrak{t}^{\frac{1}{N}}\right) . \tag{3.4.48}$$

This is indeed a polynomial in σ and $\mathfrak{t}$ for each N, where each term has combined degree at most N, as in (3.4.42); all fractional powers of σ and $\mathfrak{t}$ drop out. This shows that the subspace in $\mathcal{M}_4$ defined by (3.3.27) is indeed an algebraic variety. That the polynomial satisfies the boundary conditions spelled out in (3.4.43) for general N can be verified directly by inspecting $\mathcal{P}_{T_N}$ in (3.4.47). It also follows from the general discussion in Section 3.3.3, which showed that σ and $\mathfrak{t}$ extracted from regular supergravity solutions automatically realize the appropriate asymptotic behavior. Some explicit forms of the coefficients $\tilde{c}_{ij}$ of $\tilde{P}_{T_N}(\sigma, \mathfrak{t}) = \sum_{ij} \tilde{c}_{ij} \sigma^i \mathfrak{t}^j$ for small N are

$$\tilde{c}_{ij}^{T_2} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & -2 & \\ 1 & & \end{pmatrix} \qquad\qquad \tilde{c}_{ij}^{T_3} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ -3 & 21 & -3 & \\ 3 & 3 & & \\ -1 & & & \end{pmatrix} . \tag{3.4.49}$$

The coefficients which are not fixed by the boundary conditions (3.4.43) are tuned to specific values, corresponding to the origin of the Coulomb branch. This is the expected result for the curve extracted from a Type IIB supergravity solution with an AdS_6 factor, describing the conformally invariant vacuum state.

We now discuss the mapping of the Type IIB Riemann surface Σ_{IIB} to the M-theory curve. With the identification of σ, $\mathfrak{t}$ given in (3.4.46) with s, t and their relation (3.2.8) to the M-theory coordinates (x^5, x^6, x^4, x^{10}) on $\mathcal{M}_4 = \mathbb{R}^2 \times T^2$, we obtain the embedding of $\hat{\Sigma}_{IIB}$ into $\mathcal{M}_4$ as

$$x^5 + ix^4 = \tilde{R}_4 N \ln\left(\frac{2w}{1+w}\right) , \qquad x^6 + ix^{10} = R_{10} N \ln\left(\frac{w-1}{w+1}\right) . \tag{3.4.50}$$

The poles at r_1, r_2, r_3 correspond to the NS5, D5, and $(1,1)$ 5-branes, respectively. The geometry of the curve for $N = 1$ is illustrated in fig. 3.2. The curve for generic N is obtained by a simple rescaling.

We note that eq. (3.4.48) is precisely the formula quoted in (3.13) of [HK05], which made use of earlier results in [KOS03]. The context of that result was a proposed correspondence between brane tilings and dimer models. Though we have not been considering brane tilings in the current work, the curves wrapped by the NS5-branes in the brane tiling construction are of the same form as the curves being wrapped by the M5-brane here. In the current context, the formula of [HK05] appears more naturally in the form (3.4.42), coming directly from the warped AdS_6 solutions. The pattern of binomial coefficients on the edges (cf. (3.4.49)), which was traced back in [HK05] to the expression (3.4.48), implements the boundary conditions on the curve as discussed in Section 3.3.3.

We also note an interesting relation between the polynomial defining the T_N theory curve and a seemingly unrelated quantity in the field of combinatorics and number theory. Namely, this is the Wendt determinant [Wen94, Hel97], given by

$$W_n = \prod_{j=0}^{m-1} \left(\left(1 + \zeta_m^j\right)^m - 1 \right) , \tag{3.4.51}$$

where ζ_m is a primitive m-th root of unity. To make the relation to the polynomial $\tilde{P}_{T_N}(\sigma, \mathfrak{t})$ transparent, we note the alternative expression

$$\tilde{P}_{T_N}(\sigma, \mathfrak{t}) = \prod_{n=0}^{N-1} \left(\left(1 + e^{\frac{2\pi i n}{N}} \mathfrak{t}^{1/N}\right)^N - \sigma \right) . \tag{3.4.52}$$

This expression shows that the Wendt determinant W_n is obtained by evaluating the polynomial for $\sigma = \mathfrak{t} = 1$,

$$W_n = \tilde{P}_{T_n}(1, 1) . \tag{3.4.53}$$

The first terms in the sequence are given by

$$W_1 = 1\ , \quad W_2 = -3\ , \quad W_3 = 28\ , \quad W_4 = -375\ , \quad W_5 = 3751\ , \quad W_6 = 0\ . \tag{3.4.54}$$

The relation of the Wendt determinant to circulant matrices with all binomial coefficients may provide an interesting perspective on the conformal invariance of the curve. We leave further investigation of this relation to the future.

For each theory obtained by wrapping an M5-brane on a holomorphic curve, there is an alternative interpretation as M-theory on a (singular) Calabi-Yau threefold. In the particular case of rank 1 SCFTs with toric realizations (i.e. theories with grid diagrams with a single internal dot), this threefold is a complex cone over $\mathbb{F}^0$ or a del Pezzo surface dP_n, $n \leq 3$ [MS97, DKV97, IMS97]. This may be seen by interpreting the brane web as the toric skeleton defining the geometry [LV98]. In the case of the T_1 theory, the corresponding Calabi-Yau threefold is simply $\mathbb{C}^3$. The higher rank T_N theories correspond to orbifolds of $\mathbb{C}^3$, i.e. $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)$ with the orbifold action given by [HK05]

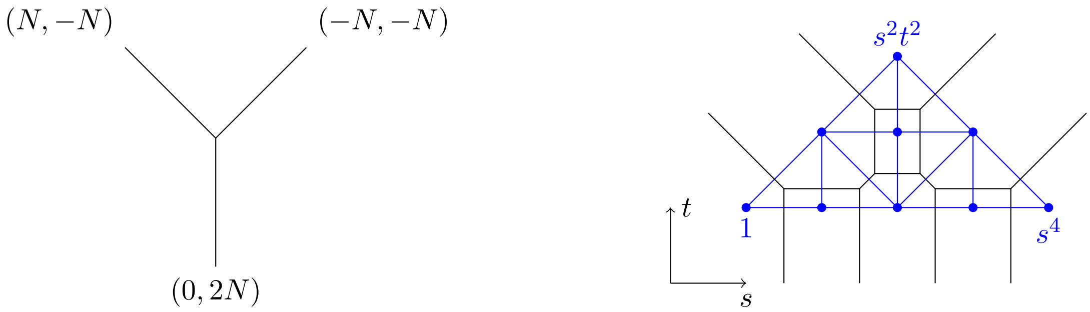
$$\begin{aligned}
&(z_1, z_2, z_3) \mapsto (\lambda z_1, z_2, \lambda^{-1} z_3)\ , && \lambda^N = 1\ , \\
&(z_1, z_2, z_3) \mapsto (z_1, \nu z_2, \nu^{-1} z_3)\ , && \nu^N = 1\ .
\end{aligned} \tag{3.4.55}$$

3.4.2 Y_N solutions

As a next example we discuss the closely related Y_N junctions, which are triple junctions of N $(1,1)$ 5-branes, N $(-1,1)$ 5-branes, and $2N$ D5-branes (fig. 3.3(a)). Although generally different from the T_N junctions, at the level of supergravity the solutions corresponding to the Y_N theories are related to the T_N solutions by an $SL(2,\mathbb{R})$ transformation combined with a rescaling of the charges (Section 4.3 of [BRU18]). This leads to simple relations between the large-N limits of the two theories. The curves are likewise closely related, as we will discuss now.

We start with the supergravity picture in this case, and compare to the construction of the curve via the grid diagram associated with the brane web at the end. The functions $\mathcal{A}_\pm$ are given by

$$\mathcal{A}_\pm = \frac{3}{4}\alpha' N \left[(i \mp 1)\ln(w+1) \pm 2\ln(4w) - (i \pm 1)\ln(w-1)\right]\ , \tag{3.4.56}$$



(a) (b)

Figure 3.3: Left: the 5-brane junction describing the Y_N SCFTs. Right: brane web and grid diagram for a mass deformation of the Y_2 theory.

from which we extract, via (3.3.20),

$$\sigma = \left(\frac{w+1}{w-1}\right)^N , \qquad \mathfrak{t} = \left(\frac{4w^2}{w^2-1}\right)^N . \tag{3.4.57}$$

They satisfy

$$0 = \mathcal{P}_{Y_N}(\sigma, \mathfrak{t}) , \qquad \mathcal{P}_{Y_N}(\sigma, \mathfrak{t}) = 1 + \sigma^{1/N} - (\sigma \mathfrak{t})^{1/(2N)} . \tag{3.4.58}$$

This can be understood from the result for the T_N solution as follows. We first note that σ, $\mathfrak{t}$ for the Y_N solution are related to σ, $\mathfrak{t}$ for the T_N solution by

$$\sigma_{Y_N} = \mathfrak{t}_{T_N}^{-1} , \qquad \mathfrak{t}_{Y_N} = \sigma_{T_N}^2 \mathfrak{t}_{T_N}^{-1} . \tag{3.4.59}$$

This may be interpreted as the Y_N solution being obtained from the T_N solution by an $SL(2,\mathbb{R})$ transformation with $a = 0$, $c = -1/b = -1/d = \sqrt{2}$, acting as in (3.2.12), combined with a charge rescaling $N \to \sqrt{2}N$. As a consequence of (3.4.59), we have

$$\mathcal{P}_{Y_N}(\sigma, \mathfrak{t}) = \sigma^{1/N} \mathcal{P}_{T_N}\left(\sqrt{\frac{\mathfrak{t}}{\sigma}}, \frac{1}{\sigma}\right) . \tag{3.4.60}$$

We now compare to the polynomial equation obtained from the grid diagram of the Y_N junctions. A sample grid diagram is shown in fig. 3.3(b), and the resulting polynomial takes the form

$$P(s,t) = \sum_{i=0}^{2N} \sum_{j=0}^{N-|N-i|} c_{i,j} s^i t^j . \tag{3.4.61}$$

The boundary conditions in the conformal limit demand that the coefficients on the edges be binomial. More precisely, the requirements are

$$P(s,t)\big|_{s,t\to\infty} \overset{!}{\sim} s^N(s-\alpha_1 t)^N \ , \qquad\qquad P(s,t)\big|_{s\to 0,t\to\infty} \overset{!}{\sim} s^N\left(t-\frac{\alpha_2}{s}\right)^N \ ,$$
$$P(s,t)\big|_{s\ \text{finite},t\to 0} \overset{!}{\sim} (s-\alpha_3)^{2N} \ . \tag{3.4.62}$$

Consistency of the boundary conditions requires $\alpha_1\alpha_2=\alpha_3^2$.

Eq. (3.4.58), which is satisfied by σ and $\mathfrak{t}$ obtained from the supergravity solution, may again be converted to a polynomial equation, $0=\tilde{P}_{Y_N}(\sigma,\mathfrak{t})$, as follows. Eq. (3.4.58) for $N=1$ is equivalent to

$$0=\tilde{P}_{Y_1}(\sigma,\mathfrak{t}) \qquad\qquad \tilde{P}_{Y_1}(\sigma,\mathfrak{t})=(\sigma+1)^2-\sigma\mathfrak{t} \ . \tag{3.4.63}$$

For higher $N\geq 2$,

$$\tilde{P}_{Y_N}(\sigma,\mathfrak{t})\equiv\prod_{n=0}^{N-1}\prod_{m=0}^{N-1}\tilde{P}_{Y_1}\left(e^{\frac{2\pi i n}{N}}\sigma^{\frac{1}{N}},e^{\frac{2\pi i m}{N}}\mathfrak{t}^{\frac{1}{N}}\right) \ . \tag{3.4.64}$$

This is again a polynomial in σ and $\mathfrak{t}$, and takes precisely the form in (3.4.61). Moreover, the edge coefficients are binomial, reflecting the fact that the curve obtained from the supergravity solution automatically satisfies the correct boundary conditions. Some explicit forms for small N are

$$\tilde{c}_{ij}^{Y_2}=\begin{pmatrix}1 & & \\ -4 & -2 & \\ 6 & -12 & 1 \\ -4 & -2 & \\ 1 & & \end{pmatrix} \ , \qquad\qquad \tilde{c}_{ij}^{Y_3}=\begin{pmatrix}1 & & & \\ 6 & -3 & & \\ 15 & 150 & 3 & \\ 20 & -423 & 60 & -1 \\ 15 & 150 & 3 & \\ 6 & -3 & & \\ 1 & & & \end{pmatrix} \ . \tag{3.4.65}$$

As before, the coefficients corresponding to Coulomb branch deformations are tuned to particular values for the conformally invariant vacuum state. The supergravity solution again provides an explicit solution to the equation defining the M-theory curve, with $\mathcal{A}_\pm$ providing the embedding as discussed in Section 3.3.

The Y_1 theory may also be obtained by considering M-theory on $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$. The Y_N theories are obtained via orbifolds thereof.

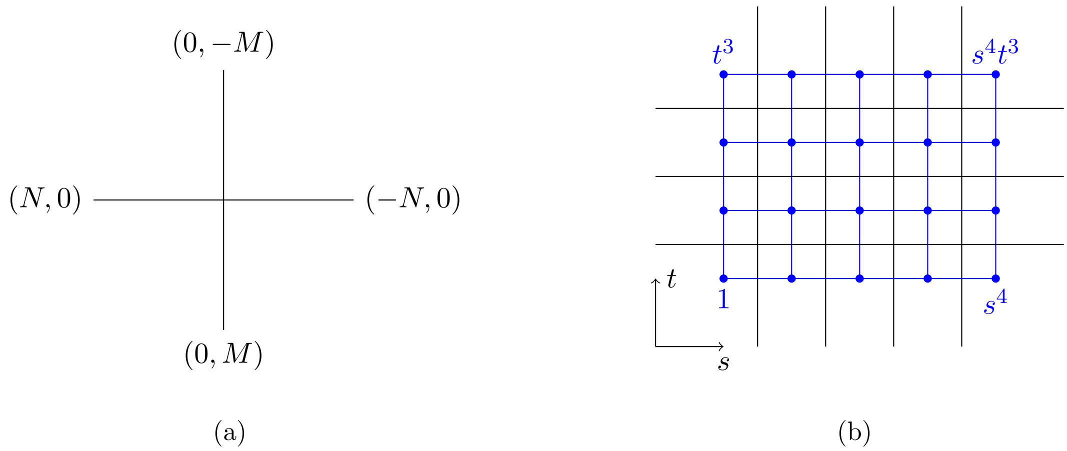


Figure 3.4: Left: the 5-brane junction describing the $+_{N,M}$ SCFT. Right: brane web and grid diagram for a mass deformation of the $+_{3,4}$ theory (a complete triangulation of the grid diagram can be obtained by resolving the remaining brane intersections).

3.4.3 $+_{N,M}$ solutions

The next example is a quartic junction of N D5-branes and M NS5-branes, as shown in fig. 3.4(a). This configuration has been discussed already in [AHK98]. An example for the associated grid diagram is shown in fig. 3.4(b). The polynomial $P(s,t)$ defining the M-theory curve is given by

$$P(s,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j} s^i t^j \ . \tag{3.4.66}$$

The boundary conditions in the conformal limit are,

$$\begin{aligned} P(s,t)\big|_{s\to\infty,t\text{ finite}} &\stackrel{!}{\sim} s^M(t-\alpha_1)^N \ , & P(s,t)\big|_{s\text{ finite},t\to\infty} &\stackrel{!}{\sim} t^N(s-\alpha_2)^M \ , \\ P(s,t)\big|_{s\text{ finite},t\to 0} &\stackrel{!}{\sim} (s-\alpha_3)^M \ , & P(s,t)\big|_{s\to 0,t\text{ finite}} &\stackrel{!}{\sim} (t-\alpha_4)^N \ , \end{aligned} \tag{3.4.67}$$

with $|\alpha_i| = 1$. Consistency of the boundary conditions requires $\alpha_1\alpha_3 = \alpha_2\alpha_4$.

We again show that the functions $\mathcal{A}_\pm$ of the corresponding supergravity solution provide an explicit parametrization of the curve. They are given by (Section 4.2 of [BRU18])

$$\mathcal{A}_\pm = \frac{3}{4}\alpha' \left[\pm M(\ln(3w-2) - \ln w) + iN(\ln(2w-1) - \ln(w-1))\right] \ . \tag{3.4.68}$$

From (3.3.20), σ and $\mathfrak{t}$ are obtained as

$$\sigma = \left(\frac{2w-1}{w-1}\right)^N \ , \qquad\qquad \mathfrak{t} = \left(\frac{3w-2}{w}\right)^M \ . \tag{3.4.69}$$

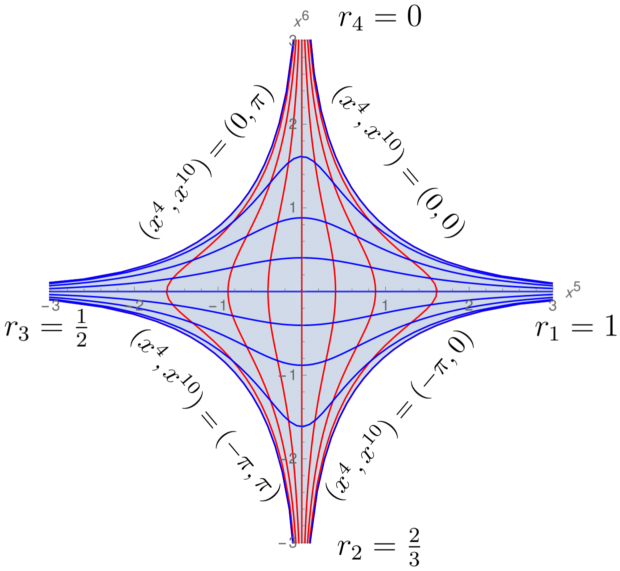


Figure 3.5: $+_{1,1}$ curve with $\tilde{R}_4 = R_{10} = 1$ obtained by embedding Σ_{IIB} into $\mathcal{M}_4$ via (3.3.27). The poles r_ℓ on Σ_{IIB} correspond to the external 5-branes as indicated. The segments of $\partial\Sigma_{IIB}$ in between poles are mapped to the outer curves connecting the asymptotic regions, with x^4, x^{10} as indicated. The blue and red curves correspond to constant x^4 and x^{10}, respectively. The embedding of the second half of $\hat{\Sigma}_{IIB}$, with w in the lower half-plane, is obtained via (3.3.35).

They satisfy

$$0 = \mathcal{P}_{+_{N,M}}(\sigma, \mathfrak{t}) \ , \qquad \mathcal{P}_{+_{N,M}}(\sigma, \mathfrak{t}) \equiv 1 + \sigma^{1/N} + \mathfrak{t}^{1/M} - \sigma^{1/N}\mathfrak{t}^{1/M} \ . \tag{3.4.70}$$

To compare to the definition of the curve via (3.4.66), this equation can again be recast in terms of a polynomial $\tilde{P}_{+_{N,M}}(\sigma, \mathfrak{t})$. Namely,

$$0 = \tilde{P}_{+_{N,M}}(\sigma, \mathfrak{t}) \ , \qquad \tilde{P}_{+_{N,M}}(\sigma, \mathfrak{t}) \equiv \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} \mathcal{P}_{+_{1,1}}\left(e^{\frac{2\pi i n}{N}}\sigma^{\frac{1}{N}}, e^{\frac{2\pi i m}{M}}\mathfrak{t}^{\frac{1}{M}}\right) \ . \tag{3.4.71}$$

This indeed yields polynomials of the form (3.4.66) satisfying the boundary conditions in (3.4.67). Some explicit examples for small N are

$$\tilde{c}_{ij}^{+_{1,4}} = \begin{pmatrix} 1 & -1 \\ 4 & 4 \\ 6 & -6 \\ 4 & 4 \\ 1 & -1 \end{pmatrix} \ , \qquad \tilde{c}_{ij}^{+_{5,3}} = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 3 & -495 & 3390 & -3390 & 495 & -3 \\ 3 & 495 & 3390 & 3390 & 495 & 3 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix} \ . \tag{3.4.72}$$

The binomial form of the edge coefficients again implies that the correct boundary conditions are satisfied. The curve obtained from the supergravity solution is shown in fig. 3.5.

The $+_{1,1}$ theory may also be obtained by considering M-theory on the conifold $\mathcal{C}$. The $+_{N,M}$ theories are obtained by considering M-theory on $\mathcal{C}/(\mathbb{Z}_N \times \mathbb{Z}_M)$, with the orbifold action given in (3.4.55), but with $\nu^M = 1$.

3.4.4 $X_{N,M}$ solutions

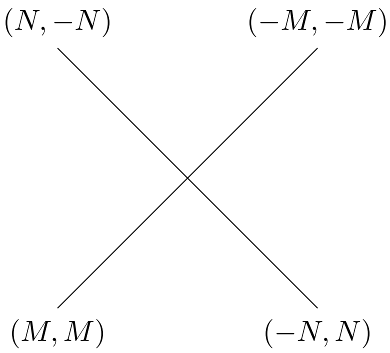
The $X_{N,M}$ theories are defined by quartic junctions of N $(1,-1)$ 5-branes and M (1,1) 5-branes, as in fig. 3.6(a). They are closely related to the $+_{N,M}$ theories, in a very similar way to how the Y_N theories are related to the T_N theories.

The quantities σ and $\mathfrak{t}$ extracted from the supergravity solution (as discussed in Section 4.2.2 of [BRU18]) via (3.3.20) are

$$\sigma = \left(\frac{2w-1}{w-1}\right)^N \left(\frac{3w-2}{w}\right)^M \ , \qquad \mathfrak{t} = \left(\frac{3w-2}{w}\right)^M \left(\frac{w-1}{2w-1}\right)^N \ . \tag{3.4.73}$$

These are related to the complex coordinates of the $+_{N,M}$ theory by

$$\sigma_{+_{N,M}} = \sqrt{\frac{\sigma}{\mathfrak{t}}}\Big|_{X_{N,M}} \ , \qquad \mathfrak{t}_{+_{N,M}} = \sqrt{\sigma\,\mathfrak{t}}\ \Big|_{X_{N,M}} \ . \tag{3.4.74}$$



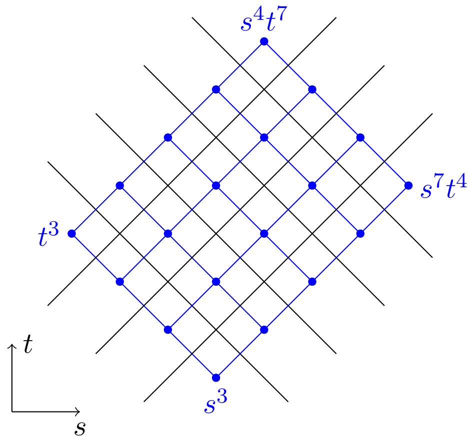


Figure 3.6: Left: the 5-brane junction describing the $X_{N,M}$ SCFT. Right: brane web and grid diagram for a mass deformation of the $X_{4,3}$ theory (a complete triangulation of the grid diagram can be obtained by resolving the remaining brane intersections).

In Type IIB, the two configurations are related by an $SL(2,\mathbb{R})$ rotation, together with a rescaling of charges. However, the two configurations are not related by $SL(2,\mathbb{Z})$ in the full string theory description, as can be seen by comparing (3.4.74) to (3.2.12). Using (3.4.74) and (3.4.70), the M-theory curve for the $X_{N,M}$ theory is

$$0 = \mathcal{P}_{X_{N,M}}(\sigma,\mathfrak{t}) \ , \qquad \mathcal{P}_{X_{N,M}}(\sigma,\mathfrak{t}) \equiv \sigma^{\frac{1}{2N}} + \mathfrak{t}^{\frac{1}{2N}} + (\sigma\mathfrak{t})^{\frac{1}{2M}}\left(\mathfrak{t}^{\frac{1}{2N}} - \sigma^{\frac{1}{2N}}\right) \ . \tag{3.4.75}$$

The factors of $1/2$ in the exponents imply that the $X_{N,M}$ M5-brane has twice the winding along the torus as the $+_{N,M}$ M5-brane.

One can once again convert (3.4.75) to polynomial form, $\tilde{P}_{X_{N,M}} = 0$. However, unlike for the T_N, Y_N, and $+_{N,M}$ curves, the grid diagram is not obtained by simply subdividing the lattice in the horizontal and vertical directions. Consequently, the polynomial for the general $X_{N,M}$ solutions does not follow the pattern in (3.4.48), (3.4.64), (3.4.71). Some examples for small N, M are

$$\tilde{c}_{ij}^{X_{1,2}} = \begin{pmatrix} & & 1 & \\ & -2 & 8 & -1 \\ 1 & 8 & 2 & \\ & -1 & & \end{pmatrix} , \quad \tilde{c}_{ij}^{X_{4,2}} = \begin{pmatrix} & & 1 & & & & \\ & -2 & -128 & -4 & & & \\ 1 & -128 & 2568 & -1920 & 6 & & \\ & -4 & -1920 & -13324 & -1920 & -4 & \\ & & 6 & -1920 & 2568 & -128 & 1 \\ & & & -4 & -128 & -2 & \\ & & & & 1 & & \end{pmatrix} . \tag{3.4.76}$$

These are generally polynomials of precisely the form implied by the grid diagram (fig. 3.6(b)), with binomial edge coefficients implementing the boundary conditions.

The $X_{1,1}$ theory may be described as M-theory on the cone over $\mathbb{F}^0 = \mathbb{P}^1 \times \mathbb{P}^1$.

3.4.5 $\not{+}_N$ solutions

As a final example we consider the $\not{+}_N$ theories, which are realized by sextic junctions of NS5, D5, and (1,1) 5-branes as shown in fig. 3.7(a). The polynomial $P(s,t)$ obtained from the grid diagram takes the form

$$P(s,t) = \sum_{\substack{0 \le i,j \le 2N \\ N \le i+j \le 3N}} c_{i,j} s^i t^j \ . \tag{3.4.77}$$

The boundary conditions are

$$\begin{aligned} P(s,t)\big|_{s,t\to\infty} &\stackrel{!}{\sim} s^N t^N (s-\alpha_1 t)^N \ , & P(s,t)\big|_{s,t\to 0} &\stackrel{!}{\sim} (s-\alpha_4 t)^N \ , \\ P(s,t)\big|_{t\text{ finite},\, s\to 0} &\stackrel{!}{\sim} t^N (t-\alpha_3)^N \ , & P(s,t)\big|_{s\text{ finite},\, t\to\infty} &\stackrel{!}{\sim} t^{2N}(s-\alpha_2)^N \ , \\ P(s,t)\big|_{s\text{ finite},\, t\to 0} &\stackrel{!}{\sim} s^N (s-\alpha_5)^N \ , & P(s,t)\big|_{t\text{ finite},\, s\to\infty} &\stackrel{!}{\sim} s^{2N}(t-\alpha_6)^N \ , \end{aligned} \tag{3.4.78}$$

with $|\alpha_i| = 1$. For consistency, we require that $\alpha_1\alpha_2\alpha_3 = \alpha_4\alpha_5\alpha_6$.

The supergravity solution has been discussed in Section 4.5 of [BRU18]. Via (3.3.20), σ and $\mathfrak{t}$ are found to be

$$\sigma = \left(\frac{1}{\sqrt{7+4\sqrt{3}}} \frac{(w-r_5)(w-r_6)}{(w-r_2)(w-r_3)} \right)^N , \qquad \mathfrak{t} = \left(\sqrt{7+4\sqrt{3}}\, \frac{(w-r_1)(w-r_6)}{(w-r_3)(w-r_4)} \right)^N , \tag{3.4.79}$$

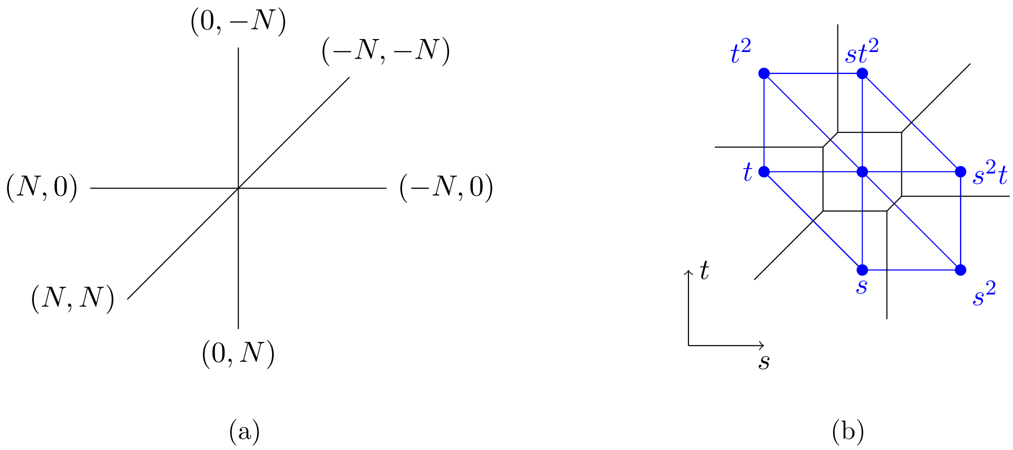


Figure 3.7: Left: the sextic junction describing the $\not{X}_N$ theory. Right: brane web and grid diagram for a deformation of the $\not{X}_1$ theory.

where

$$r_1 = -r_2 = -2+\sqrt{3}\,, \qquad r_4 = -r_5 = 2+\sqrt{3}\,, \qquad r_3 = -r_6 = 1\,. \tag{3.4.80}$$

They satisfy $\mathcal{P}_{\not{X}_N}(\sigma,\mathfrak{t}) = 0$ with

$$\mathcal{P}_{\not{X}_N}(\sigma,\mathfrak{t}) = \left(\sigma^{1/N}+\mathfrak{t}^{1/N}\right)\left(1+(\sigma\mathfrak{t})^{1/N}\right) - \sigma^{2/N} - \mathfrak{t}^{2/N} + 6(\sigma\mathfrak{t})^{1/N}\,. \tag{3.4.81}$$

For $N=1$ this is a polynomial. Converting the equation for generic N to polynomial form yields

$$\tilde{P}_{\not{X}_N}(\sigma,\mathfrak{t}) \equiv \prod_{n=0}^{N-1}\prod_{m=0}^{N-1} \mathcal{P}_{\not{X}_1}\left(e^{\frac{2\pi i n}{N}}\sigma^{\frac{1}{N}}, e^{\frac{2\pi i m}{N}}\mathfrak{t}^{\frac{1}{N}}\right)\,. \tag{3.4.82}$$

These are polynomials of the form (3.4.77), satisfying the constraints spelled out in (3.4.78). This establishes the identification of Type IIB supergravity solutions with M-theory curves, (3.3.25), also for this class of solutions. An example polynomial is

$$\tilde{c}_{ij}^{\not{X}_3} = \begin{pmatrix} & & & 1 & -3 & 3 & -1 \\ & & 3 & 2172 & 9474 & 2172 & 3 \\ & 3 & -9474 & 400119 & -400119 & 9474 & -3 \\ 1 & 2172 & 400119 & 2444568 & 400119 & 2172 & 1 \\ -3 & 9474 & -400119 & 400119 & -9474 & 3 & \\ 3 & 2172 & 9474 & 2172 & 3 & & \\ -1 & 3 & -3 & 1 & & & \end{pmatrix}\,. \tag{3.4.83}$$

The $\not{X}_1$ theory may be obtained from M-theory on the cone over dP_3. The $\not{X}_N$ theory is obtained by a $\mathbb{Z}_N\times\mathbb{Z}_N$ orbifold of this geometry.

APPENDIX A

Gamma matrix conventions

For concreteness, we take the following basis of six-dimensional gamma matrices,

$$\begin{aligned}
\gamma_1 &= \sigma_2 \otimes \mathbb{1}_2 \otimes \sigma_3 \\
\gamma_2 &= \sigma_2 \otimes \mathbb{1}_2 \otimes \sigma_1 \\
\gamma_3 &= \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_2 \\
\gamma_4 &= \mathbb{1}_2 \otimes \sigma_3 \otimes \sigma_2 \\
\gamma_5 &= \sigma_1 \otimes \sigma_2 \otimes \mathbb{1}_2 \\
\gamma_6 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_2
\end{aligned} \tag{A.0.1}$$

These gamma matrices satisfy the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{A.0.2}$$

as appropriate for a positive definite Euclidean spacetime. All matrices are purely imaginary and satisfy

$$(\gamma_\mu)^\dagger = \gamma_\mu \qquad\qquad (\gamma_\mu)^2 = \mathbb{1} \tag{A.0.3}$$

We will now be interested in a seven-dimensional Clifford algebra, which will require the introduction of a new matrix γ_7. The reason we are interested in this is that we would like to represent hyperbolic space $\mathbb{H}_6$ as a hypersurface in a seven-dimensional ambient space. This allows us to determine properties of the Dirac spinors in the Euclidean-continued $F(4)$ gauged supergravity theory with $\mathbb{H}_6$ background by first considering Dirac spinors in seven dimensions and then performing a timelike reduction. In particular, we will choose a 7D metric of signature $(+,+,+,+,+,+,-)$ for the ambient space. Then hyperbolic space $\mathbb{H}_6$ is given by the following quadratic form

$$x_1^2 + \cdots + x_6^2 - x_7^2 = -L^2 \tag{A.0.4}$$

The seven-dimensional Clifford algebra is made up of the set of matrices $\{\gamma_1, \ldots, \gamma_6, \gamma_7\}$, with γ_7 satisfying

$$(\gamma_7)^2 = -\mathbb{1} \qquad\qquad \{\gamma_\mu, \gamma_7\} = 0 \ \ \forall \mu \neq 7 \tag{A.0.5}$$

As usual, we use the notation $\gamma^7 = (\gamma_7)^{-1}$, so that by the above we have $\gamma^7 = -\gamma_7$.

We now discuss Dirac spinors in $d = 7$. We define the Dirac conjugate of ψ_A to be

$$\bar{\psi}_A = \psi_A^\dagger G^{-1} \tag{A.0.6}$$

for some matrix G. There are two possible choices for G [DV02], which in the particular case of the ambient space above are

$$G_1 = \gamma^7 \qquad\qquad G_2 = \gamma^1 \ldots \gamma^6 \tag{A.0.7}$$

These will turn out to be the same, so we just work with the former. Thus we have that

$$\bar{\psi}_A = \psi_A^\dagger \gamma_7 \tag{A.0.8}$$

If we choose γ_7 such that

$$(\gamma_7)^\dagger = -\gamma_7 \tag{A.0.9}$$

we can express the Hermitian conjugates of our gamma matrices as[12]

$$\gamma_\mu^\dagger = \eta\, G^{-1} \gamma_\mu G \tag{A.0.10}$$

Importantly, with $G = G_1$ in (A.0.7), we have

$$\eta = -1 \tag{A.0.11}$$

This will be important in Appendix B when the consistency of the symplectic Majorana condition is analyzed. For now, we just recall that the symplectic Majorana condition must take the form

$$\bar{\psi}_A = \epsilon^{AB} \psi_B^T \mathcal{C} \tag{A.0.12}$$

[12]Note that the η used in this Appendix has nothing to do with the η defined in (2.3.12), though they both end up being given the value -1 in this paper.

where

$$\mathcal{C}^2 = 1 \qquad\qquad \mathcal{C}^T = \mathcal{C} \qquad\qquad \gamma_\mu^T = -\mathcal{C}^{-1}\gamma_\mu\mathcal{C} \tag{A.0.13}$$

We now want to reduce from $d = 7$ to $d = 6$. In particular, we reduce on the time-like direction x_7. This entails finding a Euclidean induced metric on the six-dimensional surface (A.0.4). From the point of view of the Clifford algebra, we must remove the matrix γ_7 to get a six-dimensional Clifford algebra. However, the properties of the matrix γ^7 remain the same. In fact, we may choose

$$\gamma_7 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 \tag{A.0.14}$$

which satisfies all of the properties (A.0.5),(A.0.9).

APPENDIX B

Free differential algebra

In this appendix, we will construct the free differential algebra (FDA) of a supergravity theory with $\mathbb{H}_6$ background in order to motivate the form of the supersymmetry variations given in (2.3.7).

The first step of constructing the FDA is to write down the Maurer-Cartan equations (MCEs), which may be thought of as the geometrization of the (anti-)commutation relations of the superalgebra. In short, instead of defining the algebra via the (anti-)commutators of its generators, the MCEs encode the algebraic structure in integrability conditions. In the supergravity context, a nice introduction to the MCEs, as well as to the free differential algebras to be introduced shortly, may be found in [CP96]. In the current case, the MCEs are

$$\begin{aligned}
0 &= \mathcal{D}V^a + \frac{1}{2}\bar{\psi}_A\gamma^a\gamma^7\psi^A \\
0 &= R^{ab} - 4m^2V^aV^b + m\bar{\psi}_A\gamma^{ab}\psi^A \\
0 &= dA^r - \frac{1}{2}g\epsilon^{rst}A_sA_t - i\bar{\psi}_A\psi_B\sigma^{r\ AB} \\
0 &= D\psi_a + m\gamma_a\gamma_7\psi_AV^a
\end{aligned} \qquad \text{(B.0.1)}$$

Here $a = 1, \ldots, 6$ and V^a are the six-dimensional frame fields, given in terms of the seven-dimensional spin-connection as $V^a = \frac{1}{2m}\omega^{a7}$. These may be compared to the analogous expressions in the dS/AdS cases of [DV02].

As a simple check, the second equation of (B.0.1) tells us that when $\psi^A = 0$,

$$R_{\mu\nu} = -20m^2g_{\mu\nu} \qquad \text{(B.0.2)}$$

which is precisely as expected for an $\mathbb{H}_6$ background.

The next step is to enlarge the MCEs to a free differential algebra (FDA) by adding the following equations for the additional vector and 2-form fields of the full $d = 6$ $F(4)$

supergravity theory,

$$dA - mB + \alpha\bar{\psi}_A\gamma_7\psi^A = 0 \qquad\qquad dB + \beta\bar{\psi}_A\gamma_a\psi^A V^a = 0 \tag{B.0.3}$$

Above, α and β are two coefficients, which can be shown [DV02] to satisfy

$$\beta = -2\alpha \tag{B.0.4}$$

for our metric conventions. For the ambient space signature $(t, s) = (1, 6)$, it is furthermore found that $\beta = 2i$, and thus we have $\alpha = -i$.

We would now like to compare the FDA above to the results of [ADV01, DFV00, DV02]. To do so, we must first shift our notations by shifting

$$\gamma^a \to \gamma^7\gamma^a \qquad\qquad \gamma_a \to -\gamma_7\gamma_a \tag{B.0.5}$$

This preserves the square of the gamma matrices, and hence the signature of the metric. The definition of the Dirac conjugate spinor (A.0.8) remains the same under this change. So the FDA for the $\mathbb{H}_6$ theory in these conventions is,

$$\begin{aligned}
0 &= \mathcal{D}V^a + \frac{1}{2}\bar{\psi}_A\gamma^a\psi^A \\
0 &= R^{ab} - 4m^2V^aV^b + m\bar{\psi}_A\gamma^{ab}\psi^A \\
0 &= dA^r - \frac{1}{2}g\epsilon^{rst}A_sA_t - i\bar{\psi}_A\psi_B\sigma^{r\ AB} \\
0 &= D\psi_a - m\gamma_a\psi_AV^a \\
0 &= dA - mB - i\bar{\psi}_A\gamma_7\psi^A \\
0 &= dB - 2i\bar{\psi}_A\gamma_7\gamma_a\psi^AV^a
\end{aligned} \tag{B.0.6}$$

We may now compare the FDA written above to that obtained in the AdS$_6$ case, which for convenience we reproduce below,

$$\begin{aligned}
0 &= \mathcal{D}V^a - \frac{i}{2}\bar{\psi}_A\gamma^a\psi^A \\
0 &= R^{ab} + 4m^2V^aV^b + m\bar{\psi}_A\gamma^{ab}\psi^A \\
0 &= dA^r - \frac{1}{2}g\epsilon^{rst}A_sA_t - i\bar{\psi}_A\psi_B\sigma^{r\ AB} \\
0 &= D\psi_a - im\gamma_a\psi_AV^a \\
0 &= dA - mB - i\bar{\psi}_A\gamma_7\psi^A \\
0 &= dB + 2\bar{\psi}_A\gamma_7\gamma_a\psi^AV^a
\end{aligned} \tag{B.0.7}$$

We see that formally, we may obtain the $\mathbb{H}_6$ FDA from the AdS$_6$ FDA by exchanging

$$m \to -im \qquad \psi_A \to \psi_A \qquad \bar{\psi}_A \to i\bar{\psi}_A \qquad A^r \to iA^r \qquad g \to -ig \qquad B \to -B \qquad A \to iA$$

These exchanges are compatible with the relation $g = 3m$.

Finally, we will check that the $\mathbb{H}_6$ FDA is compatible with the symplectic Majorana condition. This is a statement about the fourth equation of (B.0.6). We begin by defining

$$\nabla\psi_A \equiv D\psi_A - q\gamma_a\psi_A V^a \tag{B.0.8}$$

where $q = m$ for $\mathbb{H}_6$ and $q = im$ for AdS$_6$. We then find that

$$\begin{aligned} \overline{\nabla\psi_A} &= D\psi_A^\dagger G^{-1} - q^*\psi_A^\dagger G^{-1} G\gamma_a^\dagger G^{-1} V^a = D\bar{\psi}_A - q^*\eta\,\bar{\psi}_A\gamma_a V^a \\ \epsilon^{AB}\nabla\psi_B^T\mathcal{C} &= \epsilon^{AB} D\psi_B^T\mathcal{C} - q\epsilon^{AB}\psi_B^T\mathcal{C}\mathcal{C}^{-1}\gamma_a^T\mathcal{C}V^a = D\bar{\psi}_A + q\bar{\psi}_A\gamma_a V^a \end{aligned} \tag{B.0.9}$$

where η is defined implicitly in (A.0.10). We thus find that the symplectic Majorana condition is consistent only when

$$-q^*\eta = q \tag{B.0.10}$$

For $\mathbb{H}_6$, the consistency of the symplectic Majorana condition thus requires $\eta = -1$, which we have already seen to be the case in (A.0.11). On the other hand, in the AdS$_6$ case, one would instead have required $\eta = 1$. Checking the results of [ADV01, DFV00] confirms that this was so.

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