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One Modal Logic to Rule Them All?

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Abstract

In this paper, we introduce an extension of the modal language with what we call the *global quantificational modality* $[\forall p]$. In essence, this modality combines the propositional quantifier $\forall p$ with the global modality A: $[\forall p]$ plays the same role as the compound modality $\forall p\mathsf{A}$. Unlike the propositional quantifier by itself, the global quantificational modality can be straightforwardly interpreted in any Boolean Algebra Expansion (BAE). We present a logic GQM for this language and prove that it is complete with respect to the intended algebraic semantics. This logic enables a conceptual shift, as what have traditionally been called different "modal logics" now become $[\forall p]$-universal theories over the base logic GQM: instead of defining a new logic with an axiom schema such as $\Box\varphi \to \Box\Box\varphi$, one reasons in GQM about what follows from the globally quantified formula $[\forall p](\Box p \to \Box\Box p)$.

Keywords: global quantificational modalities, propositional quantifiers, Boolean algebra expansions, Boolean algebras with operators, modal consequence.

1 Introduction

In this paper, we investigate the effect of extending modal syntax with the *global quantificational modality* $[\forall p]$. This proposal arose from our efforts to tackle two foundational problems in modal logic, which under careful inspection turn out to be related to each other.

1.1 The proliferation of modal "logics"

According to a standard view, there is a striking contrast between first-order logic and modal logic: the former leads to a single system of First-Order Logic, while the latter leads to a vast landscape of different logics. In some contexts this has prompted the "suggestion. . . that the great proliferation of modal logics is an epidemy from which modal logic ought to be cured" [8, p. 25]. [1] One could object that first-order logic is not so monolithic, given the choice between classical, intuitionistic, superintuitionistic, or substructural bases. But even in

[1] The authors of [8] attribute this suggestion to others rather than endorsing it themselves.

the classical context, there are objections to the claimed contrast, now coming from the modal side. As van Benthem [4] writes:

> [T]hese systems are not "different modal logics", but different special theories of particular kinds of accessibility relation. We do not speak of "different first-order logics" when we vary the underlying model class. There is no good reason for that here, either. (p. 93)

Yet in modal logic there remains a distinction between theories and logics: the set of formulas satisfied at all points (or a point) in a model counts as a *theory*, but not a logic, while the set of formulas validated at all points (or a point) of a frame counts as a *logic*. Here we will not address the question in the philosophy of logic about what should count as a "logic". Instead, we answer the question: is there a mathematically appealing way in which what are traditionally called "modal logics" are special theories relative to one logical system?

1.2 The riddle of propositional quantification

The other problem we are concerned with is that of conservatively handling propositional quantifiers. Historically, propositional quantifiers were considered in modal logic from the very beginning. Most of the literature quotes references such as Kripke [23], Bull [7], Fine [13], and Kaplan [21]. In fact, however, propositional quantifiers were already present not only in Ruth Barcan Marcus's post-war papers [3], but even in a chapter about the "existence postulate" by C. I. Lewis in his famous 1932 monograph with Langford [24, § VI.6]. Lewis's postulate is classically equivalent to $\exists p(\Diamond p \wedge \Diamond\neg p)$, and he insisted that "it is only through such principles that the outlines of a logical system can be positively delineated" [24, p. 181].

The problem, however, is that the addition of propositional quantifiers is not necessarily conservative for a given logic. It appears most natural, for example, to interpret them using infinite meets and joins in (algebras dual to) a suitable semantics (see § 3). Unfortunately, there are logics that are not even weakly complete with respect to lattice-complete algebras [25,27,26,29,33]. Moreover, even for standard logics such algebras might well validate undesirable quantified principles, as shown by "Kaplan's paradox" [22] for possible world semantics (see § 4). Logics with propositional quantifiers also tend to display very bad computational behavior over the dual algebras of Kripke frames: even for logics as strong as $\mathsf{S4.2}$, propositional quantification over Kripke frames produces a system as complex as full second-order logic [13,20].

1.3 Our proposal

In this paper, we intend to solve both problems by showing, on the one hand, how "different modal logics" can indeed be seen as different theories over a single base logic, and on the other hand, how each and every modal logic can be conservatively extended with a form of propositional quantification. This is made possible by extending the language with the *global quantificational modality* $[\forall p]$, which combines the propositional quantifier $\forall p$ with the global modality A. Semi-formally, one can introduce it as $[\forall p]\varphi := \forall p\mathsf{A}\varphi$. One can then think of the global modality as definable by taking a fresh variable in $[\forall p]$ and introduce further global quantificational modalities (GQMs) as follows:

$$\begin{aligned}\langle \exists p\rangle\varphi &:= \neg[\forall p]\neg\varphi = \exists p\mathsf{E}\varphi\\ [\exists p]\varphi &:= \langle \exists p\rangle\mathsf{A}\varphi = \exists p\mathsf{A}\varphi\\ \langle \forall p\rangle\varphi &:= \neg[\exists p]\neg\varphi = \forall p\mathsf{E}\varphi.\end{aligned}$$

This language of global quantificational modalities, formally defined in § 2, will be our object of study. In § 3, we show that in a global sense, this language is as expressive as the standard language of second-order propositional modal logic over the lattice-complete algebras used to interpret that language; and in § 4, we show that the flexibility to interpret our language in incomplete algebras provides a response to "Kaplan's paradox" for possible world semantics. In § 5, we introduce our logic GQM, and in § 6 we show that GQM solves the twin problems of proliferation (§ 1.1) and nonconservativity (§ 1.2). Toward proving the completeness of GQM with respect to its intended semantics, we establish prenex normal form results in § 7 and mutual translations with the first-order theory of "discriminator BAEs" in § 8. The storyline culminates with the completeness theorem at the end of § 8. We conclude in § 9.

Several proofs are deferred to appendices. A number of proofs involving syntactic derivations are given in an extended technical report online [17]. An open-source git repository containing formalizations of our proofs in Coq by Michael Sammler is available at https://gitlab.cs.fau.de/lo22tobe/GQM-Coq.

2 Language and semantics

In § 1.3, we introduced the idea of $[\forall p]$ in terms of the propositional quantifier and the global modality. Officially, we take $[\forall p]$ as primitive.

Fix a countably infinite set Prop of propositional variables and define:

$$\mathcal{L}_{\mathrm{GQM}} \qquad \varphi ::= p \mid \neg\varphi \mid (\varphi\wedge\varphi) \mid \Box\varphi \mid [\forall p]\varphi,$$

where $p \in \mathsf{Prop}$. We treat $\vee$, $\to$, $\leftrightarrow$, and $\Diamond$ as abbreviations as usual and define:

- $\mathsf{A}\varphi := [\forall r]\varphi$ for an $r \in \mathsf{Prop}$ not free (in the usual sense) in φ; $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$;
- $\langle\exists p\rangle\varphi := \neg[\forall p]\neg\varphi$, $[\exists p]\varphi := \langle\exists p\rangle\mathsf{A}\varphi$, and $\langle\forall p\rangle\varphi := [\forall p]\mathsf{E}\varphi$.
- $\bot := (p\wedge\neg p)$ and $\top := \neg\bot$ for some $p \in \mathsf{Prop}$; [2]
- for each GQM $\{\!\!\{Qp\}\!\!\} \in \{[\forall p], [\exists p], \langle\exists p\rangle, \langle\forall p\rangle\}$, its dual $\overline{\{\!\!\{Qp\}\!\!\}}$ is defined in the obvious way, i.e., $[\forall p]$ is dual to $\langle\exists p\rangle$, and $[\exists p]$ is dual to $\langle\forall p\rangle$;
- for $* \in \{\wedge,\vee\}$, let G_* be A if $* = \wedge$ and E otherwise, and let us use plain G to stand for A or E (uniformly in a formula) in results that hold for both;
- for any formulas φ, ψ and propositional variable p, φ^p_ψ is the result of substituting ψ for all free occurrences of p in φ.

[2] Note that since $\bot$ can be defined as $[\forall p]p$, another elegant choice would be to have $\to$ as the *only* Boolean primitive.

Let $\mathcal{L}_{\Box}$ ($\mathcal{L}_{\Box\mathsf{A}}$) be the set of GQM formulas in which no global quantificational modalities (no global quantificational modalities other than A and E) appear.

Remark 2.1 The use of a single unary $\Box$ is for simplicity only. What follows could instead be developed in a polymodal language with polyadic modalities.

We now introduce the intended algebraic semantics for $\mathcal{L}_{\mathrm{GQM}}$.

Definition 2.2 A *Boolean algebra expansion* (BAE) is a tuple $\mathfrak{A} = \langle A, \neg, \wedge, \bot, \top, \Box\rangle$ where $\langle A, \neg, \wedge, \bot, \top\rangle$ is a Boolean algebra and $\Box : A \to A$.

Definition 2.3

(i) A $\mathcal{C}$-BAE (resp. $\mathcal{A}$-BAE) is a BAE whose Boolean reduct is complete (resp. atomic).

(ii) A BAO is a BAE with a *normal* $\Box$, i.e., $\Box$ distributes over all finite meets.

(iii) A $\mathcal{V}$-BAO is a BAO in which $\Box$ distributes over all existing meets.

We may concatenate '$\mathcal{C}$', '$\mathcal{A}$', and '$\mathcal{V}$' to indicate multiple properties; e.g., a $\mathcal{CA}$-BAE is a BAE whose Boolean reduct is both complete and atomic. This is a convention used in our earlier papers [27,26,29,16,15].

Definition 2.4 A *valuation* on a BAE $\mathfrak{A}$ is a function $\theta : \mathsf{Prop} \to \mathfrak{A}$ that extends to a function $\tilde{\theta} : \mathcal{L}_{\mathrm{GQM}} \to \mathfrak{A}$ as follows:

$$\tilde{\theta}(p) := \theta(p) \qquad \tilde{\theta}(\neg\varphi) := \neg\tilde{\theta}(\varphi)$$
$$\tilde{\theta}(\varphi \wedge \psi) := \tilde{\theta}(\varphi) \wedge \tilde{\theta}(\psi) \qquad \tilde{\theta}(\Box\varphi) := \Box\tilde{\theta}(\varphi)$$
$$\tilde{\theta}([\forall p]\varphi) := \begin{cases} \top & \text{if } \tilde{\gamma}(\varphi) = \top \text{ for all valuations } \gamma \sim_p \theta \\ \bot & \text{otherwise} \end{cases}$$

where $\gamma \sim_p \theta$ iff γ and θ disagree at most at p.

A formula φ is *valid in* $\mathfrak{A}$ iff for every valuation θ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \top$. Let $\vDash_{\mathsf{GQM}} \varphi$ iff φ is valid in all BAEs, in which case φ is simply *valid*.

Lemma 2.5 *For any valuation θ on a* BAE $\mathfrak{A}$*:*

$$\tilde{\theta}(\mathsf{A}\varphi) = \begin{cases} \top & \textit{if } \tilde{\theta}(\varphi) = \top \\ \bot & \textit{otherwise} \end{cases} \qquad \tilde{\theta}(\mathsf{E}\varphi) = \begin{cases} \top & \textit{if } \tilde{\theta}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases}$$
$$\tilde{\theta}(\langle\exists p\rangle\varphi) = \begin{cases} \top & \textit{if } \exists\gamma \sim_p \theta.\tilde{\gamma}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases} \qquad \tilde{\theta}([\exists p]\varphi) = \begin{cases} \top & \textit{if } \exists\gamma \sim_p \theta.\tilde{\gamma}(\varphi) = \top \\ \bot & \textit{otherwise} \end{cases}$$
$$\tilde{\theta}(\langle\forall p\rangle\varphi) = \begin{cases} \top & \textit{if } \forall\gamma \sim_p \theta.\tilde{\gamma}(\varphi) \neq \bot \\ \bot & \textit{otherwise} \end{cases}.$$

Several definitions of semantic consequence are available, but as our default we pick the algebraic analogue of global model consequence [5, § 1.5].

Definition 2.6 Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, let $\Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi$ iff for any BAE $\mathfrak{A}$ and $\theta : \mathsf{Prop} \to \mathfrak{A}$, if $\tilde{\theta}(\gamma) = \top$ for each $\gamma \in \Gamma$, then $\tilde{\theta}(\varphi) = \top$.

One of our main goals is to find a proof system complete with respect to $\vDash^{\mathsf{A}}_{\mathrm{GQM}}$. For this relation we have the following semantic deduction theorem.

Lemma 2.7 *For any formulas $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}_{\mathsf{GQM}}$, $\{\varphi_1, \dots, \varphi_n\} \vDash^{\mathsf{A}}_{\mathrm{GQM}} \psi$ iff $\vDash_{\mathsf{GQM}} \mathsf{A}(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \mathsf{A}\psi$.*

Proof. Immediate from Definition 2.6 and Lemma 2.5. □

We also distinguish two senses in which formulas may be equivalent.

Definition 2.8 For any $\varphi, \psi \in \mathcal{L}_{\mathrm{GQM}}$ and class $\mathcal{K}$ of BAEs:

(i) φ and ψ are *equivalent over* $\mathcal{K}$ iff for every $\mathfrak{A} \in \mathcal{K}$ and valuation θ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \tilde{\theta}(\psi)$ (or equivalently, $\varphi \leftrightarrow \psi$ is valid in $\mathfrak{A}$);

(ii) φ and ψ are *globally equivalent over* $\mathcal{K}$ iff for every $\mathfrak{A} \in \mathcal{K}$ and valuation θ on $\mathfrak{A}$, $\tilde{\theta}(\varphi) = \top$ iff $\tilde{\theta}(\psi) = \top$ (or equivalently, $\mathsf{A}\varphi \leftrightarrow \mathsf{A}\psi$ is valid in $\mathfrak{A}$).

(iii) φ and ψ are *equivalent* (resp. *globally equivalent*) iff they are equivalent (resp. globally equivalent) over the class of all BAEs.

Remark 2.9 Since $\mathcal{L}_{\mathrm{GQM}}$ can be interpreted in arbitrary BAEs, it can be interpreted in any frames that give rise to BAEs, e.g.: Kripke frames (corresponding to $\mathcal{CAV}$-BAOs); relational possibility frames [15] (corresponding to $\mathcal{CV}$-BAOs); neighborhood frames (corresponding to $\mathcal{CA}$-BAEs); neighborhood possibility frames [15] (corresponding to $\mathcal{C}$-BAEs); discrete general frames [10] (corresponding to $\mathcal{AV}$-BAOs); discrete general neighborhood frames (corresponding to $\mathcal{A}$-BAEs); general neighborhood frames (corresponding to BAEs).

Remark 2.10 Given that the "predicate lifting" approach in coalgebraic logic reduces any set-based coalgebra to a neigborhood frame, it would be interesting to investigate coalgebraic applications of GQMs.

3 Reduction of SOPML to GQM

The standard language $\mathcal{L}_{\mathrm{SOPML}}$ of second-order propositional modal logic replaces $[\forall p]$ by the propositional quantifier $\forall p$. At first one might expect that the implicit global modality in $[\forall p]$ reduces the expressivity of $\mathcal{L}_{\mathrm{GQM}}$ relative to the language $\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}}$ of second-order propositional modal logic plus the global modality. In fact, we will show that every $\mathrm{SOPML}_\mathsf{A}$ formula is globally equivalent to a GQM formula over standard semantics. For convenience, in this section we regard GQM formulas as $\mathrm{SOPML}_\mathsf{A}$ formulas with $[\forall p]$ as $\forall p \mathsf{A}$.

First, let us recall the algebraic semantics for $\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}}$ that interprets $\forall p$ using the meets in a $\mathcal{C}$-BAE, as in, e.g., [16].

Definition 3.1 We extend a valuation θ on a $\mathcal{C}$-BAE $\mathfrak{A}$ to a valuation $\tilde{\theta} : \mathcal{L}_{\mathrm{SOPML}_\mathsf{A}} \to \mathfrak{A}$ using the clauses for $\neg$, $\wedge$, and $\Box$ from Definition 2.4 plus:

$$\tilde{\theta}(\forall p\varphi) = \bigwedge\{\tilde{\gamma}(\varphi) \mid \gamma \sim_p \theta\} \qquad \tilde{\theta}(\mathsf{A}\varphi) = \begin{cases} \top & \text{if } \tilde{\theta}(\varphi) = \top \\ \bot & \text{otherwise.} \end{cases}$$

Dually, $\exists p\varphi$ is interpreted using the join. The definitions of local and global equivalence from Definition 2.8 transfer in the obvious way to $\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}}$.

We will reduce $\mathcal{L}_{\mathrm{SOPML}_\mathsf{A}}$ to $\mathcal{L}_{\mathrm{GQM}}$ over $\mathcal{C}$-BAEs using a prenex form result. In [9] it was shown that over $\mathcal{CAV}$-BAOs, every $\mathcal{L}_{\mathrm{SOPML}}$ formula is equivalent to a prenex one, i.e., a formula of the form $Q_1 p_1 \ldots Q_n p_n \varphi$ where $Q_i \in \{\forall, \exists\}$ and φ is quantifier-free. In fact, the following more general result holds.

Proposition 3.2

(i) *Over* $\mathcal{CV}$-BAOs, *every* SOPML *formula is equivalent to a prenex* SOPML *formula.*

(ii) *Over* $\mathcal{C}$-BAEs, *every* $\mathrm{SOPML}_{\mathsf{A}}$ *formula is equivalent to a prenex* $\mathrm{SOPML}_{\mathsf{A}}$ *formula.*

Proof. The proof of part (i) is the same as in [9, Prop. 3] except that we give a different argument for pulling the quantifier out of $\Diamond\forall p\varphi$, which does not assume $\mathcal{A}$. For consistency with [9], we work with $\Diamond$, though everything dualizes easily. For any normal $\Box$, we have the following equivalence:

$$\Diamond\psi \Leftrightarrow \exists q(\Diamond q \wedge \Box(q \to \psi))$$

for a variable q that does not occur in ψ. Thus, we have the following equivalences where $q \neq p$ and q does not occur in φ:

$$\begin{aligned}
\Diamond\forall p\varphi &\Leftrightarrow \exists q(\Diamond q \wedge \Box(q \to \forall p\varphi)) \text{ setting } \psi := \forall p\varphi \\
&\Leftrightarrow \exists q(\Diamond q \wedge \Box\forall p(q \to \varphi)) \text{ because } q \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\Box(q \to \varphi)) \text{ by } \mathcal{V} \text{ for } \Box \\
&\Leftrightarrow \exists q\forall p(\Diamond q \wedge \Box(q \to \varphi)) \text{ because } q \neq p.
\end{aligned}$$

For part (ii), we use the fact that A distributes over arbitrary meets, so the reasoning for part (i) shows that $\mathsf{E}\forall p\varphi$ is equivalent to $\exists q\forall p(\mathsf{E}q \wedge \mathsf{A}(q \to \varphi))$, which implies ($\star$): $\mathsf{A}\exists p\psi$ is equivalent to $\forall q\exists p(\mathsf{E}q \to \mathsf{E}(q \wedge \psi))$.

Now for any $\Box$ in a $\mathcal{C}$-BAE, we have the following equivalence:

$$\Diamond\psi \Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(q \leftrightarrow \psi))$$

for a variable q that does not occur in ψ. Thus, we have the following equivalences where $q \neq p$ and q does not occur in φ:

$$\begin{aligned}
\Diamond\forall p\varphi &\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(q \leftrightarrow \forall p\varphi)) \text{ setting } \psi := \forall p\varphi \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(\forall p(q \to \varphi) \wedge \exists p(\varphi \to q))) \text{ because } q \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}(\forall p(q \to \varphi) \wedge \exists r(\varphi^p_r \to q))) \text{ for a fresh } r \\
&\Leftrightarrow \exists q(\Diamond q \wedge \mathsf{A}\forall p\exists r\underbrace{((q \to \varphi) \wedge (\varphi^p_r \to q))}_{\alpha}) \text{ because } r \neq p \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\mathsf{A}\exists r\alpha) \text{ by } \mathcal{V} \text{ for } \mathsf{A} \\
&\Leftrightarrow \exists q(\Diamond q \wedge \forall p\forall q'\exists r(\mathsf{E}q' \to \mathsf{E}(q' \wedge \alpha))) \text{ by } (\star) \text{ where } q' \text{ is fresh} \\
&\Leftrightarrow \exists q\forall p\forall q'\exists r(\Diamond q \wedge (\mathsf{E}q' \to \mathsf{E}(q' \wedge \alpha))) \text{ because } q \neq p, q \neq q', q \neq r.
\end{aligned}$$

The rest of the proof is as in [9]. $\Box$

Proposition 3.3 *If* α *is a prenex* $\mathrm{SOPML}_{\mathsf{A}}$ *formula, then* $\mathsf{A}\alpha$ *is equivalent over* $\mathcal{C}$-BAEs *to a* GQM *formula.*

Proof. Proved in Appendix A as Proposition A.2. □

Theorem 3.4 *Every* SOPML$_\mathsf{A}$ *formula is globally equivalent over* $\mathcal{C}$-BAEs *to a* GQM *formula.*

Proof. By Proposition 3.2.(ii), φ is globally equivalent over $\mathcal{C}$-BAEs to a prenex SOPML$_\mathsf{A}$ formula ψ. Then since ψ is globally equivalent to $\mathsf{A}\psi$, it follows by Proposition 3.3 that ψ is globally equivalent to a GQM formula. □

Theorem 3.5 *The set of* GQM *formulas valid over any class of* $\mathcal{C}$-BAEs *containing the class of* $\mathcal{CAV}$-BAOs *validating* S4.2 *is not recursively enumerable.*

Proof. [Sketch] Fine [13, Prop. 6] (cf. [20]) showed that the set of SOPML sentences valid in $\mathcal{CAV}$-BAOs validating S4.2 is not recursively enumerable. The property of a BAE being an $\mathcal{AV}$-BAO is expressible in $\mathcal{L}_{\mathrm{GQM}}$; we leave this to the reader as an exercise (cf. § 8, [18, § 9]). Let $\chi_{\mathcal{AV}}$ be the corresponding sentence. The validity of an SOPML sentence φ over S4.2 $\mathcal{CAV}$-BAOs is equivalent to the validity of the GQM sentence $(\chi_{\mathcal{AV}} \wedge [\forall]\mathsf{S4.2}) \to \varphi^*$ over $\mathcal{C}$-BAEs, where φ^* is obtained from $\mathsf{A}\varphi$ by Theorem 3.4 and $[\forall]\mathsf{S4.2}$ is the GQM statement of the S4.2 axioms. Thus, the existence of a semi-decision procedure contradicting the statement would yield a semi-decision procedure contradicting [13,20]. □

Another route would be to use results of Thomason [32]. Both [32] and [20] deal with a stronger property: reducibility of full second-order consequence. We postpone the details to a sequel paper.

4 Interlude: "Kaplan's paradox"

In a festschrift for Ruth Barcan Marcus, Kaplan [22] posed a problem for possible world semantics involving propositional quantification. In brief, Kaplan claimed that the following should be consistent for a non-monotonic $\Box$:

- $\kappa := \forall p \mathsf{E} \forall q(\Box q \leftrightarrow \mathsf{A}(p \leftrightarrow q))$.

For example, if $\Box$ means "it is entertained at time t that...", then κ says that for all propositions p, it could have been that p was the unique proposition entertained at time t. Kaplan argued that *logic* should not rule this out. Yet he noted that κ is unsatisfiable in possible world semantics with $\forall$ quantifying over the powerset. For the truth of κ would yield an injection from the powerset of the set of worlds to the set of worlds. In fact, as Yifeng Ding (p.c.) observed, it is unsatisfiable in any $\mathcal{C}$-BAE. The truth of κ would yield (a) an injection from the algebra to an antichain of elements. But since the algebra is complete, (b) every subset of the antichain has a join, and all such joins are distinct. Together (a) and (b) contradict Cantor's theorem.

We will show there is a GQM formula φ, regarded as an SOPML$_\mathsf{A}$ formula as in § 3, such that (i) in any logic that derives some plausible equivalences, φ is provably equivalent to (the A-necessitation of) Kaplan's formula, so intuitively the truth of φ implies the truth of Kaplan's formula, and (ii) φ can be made true in an incomplete BAE. First, in any modal logic with propositional quantifiers in which the equivalences in the proof of Proposition 3.2.(ii) are provable, (the A-necessitation of) Kaplan's formula is provably equivalent to

- $\mathsf{A}\forall p\exists r(\mathsf{E}r \wedge \forall q\mathsf{A}(r \to (\Box q \leftrightarrow \mathsf{A}(p \leftrightarrow q))))$. [3]

Then using Barcan's equivalence $\mathsf{A}\forall p\psi \leftrightarrow \forall p\mathsf{A}\psi$ and S5 reasoning with E and A, the preceding formula is provably equivalent to

- $\forall p\mathsf{A}\exists r\mathsf{A}(\mathsf{E}r \wedge \forall q\mathsf{A}(r \to (\Box q \leftrightarrow \mathsf{A}(p \leftrightarrow q))))$,

which becomes the GQM formula

- $[\forall p][\exists r](\mathsf{E}r \wedge [\forall q](r \to (\Box q \leftrightarrow \mathsf{A}(p \leftrightarrow q))))$.

Now this formula can be made true in a BAE. Pick any infinite set X, and let $\mathfrak{A}$ be the Boolean algebra of its finite and cofinite subsets. Clearly, not only is X identifiable with the set $At(\mathfrak{A})$ of atoms of $\mathfrak{A}$, but also there is an injective (and hence bijective) map $\Box : \mathfrak{A} \to At(\mathfrak{A})$. It is easy to see that the formula above evaluates to $\top$ in $\mathfrak{A}$ with this interpretation of $\Box$. Thus, according to the logic GQM, the GQM translation of Kaplan's formula is consistent. [4]

5 Axiomatization

Let us now turn to formulating a complete proof system for $\mathcal{L}_{\mathrm{GQM}}$.

Definition 5.1 The logic GQM is the smallest set of formulas containing the following axioms and closed under the following rules.

propositional axioms

- all classical propositional tautologies.

axioms for $[\forall p]$

- distribution: $[\forall p](\varphi \to \psi) \to ([\forall p]\varphi \to [\forall p]\psi)$;
- instantiation: $[\forall p]\varphi \to \varphi^p_\psi$ where ψ is substitutable for p in φ; [5]
- global instantiation: $[\forall p]\varphi \to [\forall r]\varphi^p_\psi$ where ψ is substitutable for p in φ and r is not free in φ^p_ψ;
- quantificational 5 axiom: $\neg[\forall p]\varphi \to [\forall r]\neg[\forall p]\varphi$ where r is not free in $[\forall p]\varphi$.

axioms linking $[\forall p]$ and $\Box$

- $\Box$-congruence: $[\forall p](\varphi \leftrightarrow \psi) \to (\Box\varphi \leftrightarrow \Box\psi)$.

rules

- modus ponens: if $\vdash_{\mathsf{GQM}} \varphi$ and $\vdash_{\mathsf{GQM}} \varphi \to \psi$, then $\vdash_{\mathsf{GQM}} \psi$;
- $[\forall p]$-necessitation: if $\vdash_{\mathsf{GQM}} \varphi$, then $\vdash_{\mathsf{GQM}} [\forall p]\varphi$;
- universal generalization: if $\vdash_{\mathsf{GQM}} \alpha \to [\forall p]\varphi$ and q is not free in α, then

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[3] Of course the quantifier $\forall q$ can be pulled to the front for a prenex form, but here we opt for better human readability.

[4] After submitting this paper, we were informed by John Hawthorne of the paper [2] in which the finite-cofinite algebra has also been used in response to Kaplan's paradox.

[5] The definition of ψ being substitutable for p in φ is the obvious analogue of the definition of a term t being substitutable for a variable x in a first-order formula [12, p. 113]: no propositional variable in ψ should be captured by a quantifier in φ upon substituting ψ for p.

$\vdash_{\mathsf{GQM}} \ \alpha \to \ [\forall q][\forall p]\varphi.$

Here '$\vdash_{\mathsf{GQM}} \varphi$' means $\varphi \in \mathsf{GQM}$. We write '$\vdash \varphi$' when no confusion will arise.

Let us now record some useful theorems and metatheorems.

Lemma 5.2 *If q is substitutable for p in φ, and q is not free in φ, then* $\vdash [\forall p]\varphi \leftrightarrow [\forall q]\varphi^p_q$.

Proof. See the extended technical report [17]. □

Lemma 5.3 *If* $\vdash \varphi \to \psi$, *then* $\vdash \{Qp\}\varphi \to \{Qp\}\psi$.

Proof. See the extended technical report [17]. □

Lemma 5.4

(i) $\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$;	(vii) $\vdash \mathsf{EA}\varphi \leftrightarrow \mathsf{A}\varphi$;
(ii) $\vdash \mathsf{G}_*(\varphi * \psi) \leftrightarrow (\mathsf{G}_*\varphi * \mathsf{G}_*\psi)$;	(viii) $\vdash \mathsf{GG}\varphi \leftrightarrow \mathsf{G}\varphi$;
(iii) *if* $\vdash \varphi \to \psi$, *then* $\vdash \mathsf{G}\varphi \to \mathsf{G}\psi$;	(ix) $\vdash \{Qp\}\mathsf{A}\psi \leftrightarrow [Qp]\psi$;
(iv) $\vdash \mathsf{A}\varphi \to \varphi$;	(x) $\vdash \{Qp\}\mathsf{E}\psi \leftrightarrow \langle Qp\rangle\psi$;
(v) $\vdash \varphi \to \mathsf{E}\varphi$;	(xi) $\vdash \{Qp\}\psi \leftrightarrow \mathsf{A}\{Qp\}\psi$;
(vi) $\vdash \mathsf{E}\varphi \leftrightarrow \mathsf{AE}\varphi$;	(xii) $\vdash \{Qp\}\psi \leftrightarrow \mathsf{E}\{Qp\}\psi$.

Proof. See the extended technical report [17]. □

Let us now introduce a relation of syntactic consequence. In the following definition, Γ may be regarded as a set of globally true premises.

Definition 5.5 Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, let $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ iff φ belongs to the smallest set Λ of GQM formulas that includes $\Gamma \cup \mathsf{GQM}$ and is closed under modus ponens and A-necessitation: if $\psi \in \Lambda$, then $\mathsf{A}\psi \in \Lambda$.

Now we obtain a syntactic deduction theorem parallel to Lemma 2.7.

Lemma 5.6 *For any formulas* $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}_{\mathsf{GQM}}$, $\{\varphi_1, \dots, \varphi_n\} \vdash^{\mathsf{A}}_{\mathsf{GQM}} \psi$ *iff* $\vdash_{\mathsf{GQM}} \mathsf{A}(\varphi_1 \wedge \dots \wedge \varphi_n) \to \mathsf{A}\psi$.

Proof. By application of Lemma 5.4. □

By design, $\vdash^{\mathsf{A}}_{\mathsf{GQM}}$ is sound with respect to $\vDash^{\mathsf{A}}_{\mathrm{GQM}}$.

Lemma 5.7 *For* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ *implies* $\Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi$.

Proof. Straightforward induction. □

It follows from Lemma 5.7 and the example in § 4 (or Theorem 3.5) that GQM is incomplete with respect to validity over the class of $\mathcal{C}$-BAEs. However, we will see in § 8 that GQM is complete with respect to validity over all BAEs.

6 Conservativity and modal logics as GQM theories

Before proving completeness, we show that GQM solves our two problems from § 1: the proliferation problem and the nonconservativity problem.

A *congruential modal logic* is a set $\mathsf{L} \subseteq \mathcal{L}_\Box$ containing all propositional tautologies and closed under uniform substitution, modus ponens, and the rule that if $\varphi \leftrightarrow \psi \in \mathsf{L}$, then $\Box\varphi \leftrightarrow \Box\psi \in \mathsf{L}$. Let GQM-L be the smallest set of formulas that includes $\mathsf{GQM} \cup \mathsf{L}$ and is closed under all three rules of GQM.

Proposition 6.1 (Conservativity) *For any $\varphi \in \mathcal{L}_\Box$, $\varphi \in \mathsf{GQM}$-L iff $\varphi \in \mathsf{L}$.*

Proof. The Lindenbaum-Tarski algebra for L is a BAE in which every $\varphi \in \mathsf{GQM}$-L is valid and in which any $\mathcal{L}_\Box$ formula not in L can be refuted. □

A set $\Sigma \subseteq \mathcal{L}_\Box$ *axiomatizes* a congruential modal logic L iff L is the smallest congruential modal logic such that $\Sigma \subseteq \mathsf{L}$.

Theorem 6.2 *If Σ axiomatizes L, then we have the following equivalence: $\varphi \in \mathsf{L}$ iff there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\vdash_{\mathsf{GQM}} [\forall \boldsymbol{p}](\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$, where $\boldsymbol{p}$ is the tuple of variables occurring in $\psi_1, \ldots, \psi_n$.*

Proof. From right to left, we have:

$$\begin{aligned} & \vdash_{\mathsf{GQM}} [\forall \boldsymbol{p}](\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi \\ \Rightarrow\ & \varphi \in \mathsf{GQM}\text{-}\mathsf{L} \text{ by } [\forall]\textsf{-necessitation} \text{ to } \psi_1 \wedge \cdots \wedge \psi_n \in \mathsf{L} \text{ and } \textsf{modus ponens} \\ \Rightarrow\ & \varphi \in \mathsf{L} \text{ by Proposition 6.1.} \end{aligned}$$

From left to right, the proof proceeds by induction on the length of derivations. Details are in the extended technical report [17]. Also see Remark 8.7. □

We can easily rephrase Theorem 6.2 in the language of "theories."

Definition 6.3 A $\vdash_{\mathsf{GQM}}$-*theory* is a set of GQM formulas that includes GQM and is closed under modus ponens.

Corollary 6.4 *If $\Sigma \subseteq \mathcal{L}_\Box$ axiomatizes a congruential modal logic L, then we have the following equivalence: $\varphi \in \mathsf{L}$ iff φ belongs to the smallest $\vdash_{\mathsf{GQM}}$-theory that includes $[\forall]\Sigma = \{[\forall \boldsymbol{p}]\varphi \mid \varphi \in \Sigma$ and $\boldsymbol{p}$ are the variables in $\varphi\}$.*

Given this reduction of modal logics to $\vdash_{\mathsf{GQM}}$-theories, we have the following.

Corollary 6.5 GQM *theoremhood is undecidable.*

Proof. In light of Theorem 6.2, a decision procedure for GQM would yield a decision procedure for every finitely axiomatizable modal logic. But there are undecidable logics with finite axiomatizations [11, § 16.4]. □

Theorem 3.5 showed that the set of GQM formulas valid over $\mathcal{C}$-BAEs is not recursively enumerable. Our completeness result in Theorem 8.6 will yield that the situation is better over general algebraic semantics.

7 Prenex forms

Our path to completeness begins with suitable normal and prenex forms.

7.1 Weak prenex forms

Definition 7.1

(i) A *nontrivial weak prenex* (NWP) *formula* is a formula of the form $\boldsymbol{\{Qp\}}\,\varphi$, where $\boldsymbol{\{Qp\}}$ is a nonempty sequence of GQMs and φ is a $\mathcal{L}_{\Box\mathsf{A}}$-formula.

(ii) A *normal clause* is a disjunction each disjunct of which is either (a) a literal, (b) of the form $\Box\psi$ or $\Diamond\psi$ with ψ quantifier free, or (c) a formula in NWP form.

(iii) A *conjunctive normal form weak prenex* (CNFWP) *formula* is a conjunction of normal clauses.

The following is a key lemma for the purposes of showing that formulas can be transformed into equivalent CNFWP formulas.

Lemma 7.2

(i) $\vdash (\mathsf{G}_*\alpha * \{Qp\}\beta) \leftrightarrow \{Qp\}(\mathsf{G}_*\alpha * \beta)$ *where p is not free in α;*

(ii) $\vdash \mathsf{A}(\varphi \vee \boldsymbol{\{Qp\}}\psi) \leftrightarrow (\mathsf{A}\varphi \vee \boldsymbol{\{Qp\}}\psi)$*;*

(iii) $\vdash \Box(\alpha \wedge (\varphi \vee \{Qp\}\psi)) \leftrightarrow ((\{Qp\}\psi \wedge \Box\alpha) \vee (\neg\{Qp\}\psi \wedge \Box(\alpha \wedge \varphi)))$.

Proof. See the extended technical report [17]. $\Box$

Theorem 7.3 *For every $\varphi \in \mathcal{L}_{\mathrm{GQM}}$:*

(i) *φ is provably equivalent to a CNFWP formula;*

(ii) *$\mathsf{A}\varphi$ is provably equivalent to an NWP formula.*

Proof. Proved in Appendix B as Theorem B.2. $\Box$

7.2 Pure weak prenex forms

The following special case of NWP form will be essential in relating $\mathcal{L}_{\mathrm{GQM}}$ to the first-order language in § 8.

Definition 7.4 A formula is in *pure weak prenex form* (PWP) iff it is of the form $\boldsymbol{\{Qp\}}\mathsf{G}\varphi$ where $\boldsymbol{\{Qp\}}$ is a sequence of $[\forall p_i]$ and $\langle\exists p_i\rangle$ GQMs only, G is either A or E, and φ is a $\mathcal{L}_{\Box\mathsf{A}}$-formula.

Theorem 7.5 *Every NWP formula is provably equivalent to a PWP formula.*

Proof. By induction on the length of the quantifier prefix. Assuming φ is a PWP formula, we must show that $\{Qp\}\varphi$ is equivalent to a PWP formula. If $\{Qp\} \in \{[\forall p], \langle\exists p\rangle\}$, there is nothing to do. Case 1: $\{Qp\} := \langle\forall p\rangle$. By Lemma 5.4.(ix)-(x), where r is not free in φ, $\langle\forall p\rangle\varphi$ is equivalent to $[\forall p]\langle\exists r\rangle\varphi$, which is a PWP formula. Case 2: $\{Qp\} := [\exists p]$. By Lemma 5.4.(ix)-(x), where r is not free in φ, $[\exists p]\varphi$ is equivalent to $\langle\exists p\rangle[\forall r]\varphi$, which is a PWP formula. $\Box$

8 Completeness via FO-theory of discriminator BAEs

Using the prenex results of § 7, we will now prove the completeness of GQM via mutual translations with the first-order theory of discriminator BAEs.

Definition 8.1 A *Boolean algebra expansion with a discriminator* ($\mathrm{BAE}_{\mathsf{A}}$) is a tuple $\mathfrak{A} = \langle A, \neg, \wedge, \bot, \top, \Box, \mathsf{A}\rangle$ where $\langle A, \neg, \wedge, \bot, \top, \Box\rangle$ is a BAE and A is the dual form of the *unary discriminator term* [19], i.e., an algebraic counterpart of the global modality: $\mathsf{A}a = \top$ if $a = \top$, and $\mathsf{A}a = \bot$ otherwise.

Let $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ (resp. $\mathrm{FO}_{\mathrm{BAE}}$) be the set of first-order formulas in the $\mathrm{BAE}_{\mathsf{A}}$ (resp. BAE) signature (recycling Prop for our set of first-order variables).

The class of all $\mathrm{BAE}_{\mathsf{A}}$s is elementary, although it is not exactly a variety (an equationally definable class): rather, it is the class of all *simple* members of the corresponding variety [19, Thm. 3]. BAEs and $\mathrm{BAE}_{\mathsf{A}}$s are in 1-1 correspondence: $\mathrm{BAE}_{\mathsf{A}}$s have BAEs as reducts; every BAE $\mathfrak{A}$ can be trivially extended to a $\mathrm{BAE}_{\mathsf{A}}$ $\mathfrak{A}_{\mathsf{A}}$; and both operations are mutual inverses.

In a similar way, we can assign to every formula of $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ a formula equivalent to a PWP formula (where $\sim$ and & are the negation and conjunction connectives in the first-order language, whereas $\neg$ and $\wedge$ in the first-order language are function symbols for the Boolean algebraic operations):

$$(\varphi \approx \psi)_* := \mathsf{A}(\varphi \leftrightarrow \psi) \qquad (\sim\alpha)_* := \neg(\alpha)_*$$
$$(\alpha \,\&\, \beta)_* := ((\alpha)_* \wedge (\beta)_*) \qquad (\forall p\alpha)_* := [\forall p](\alpha)_*.$$

Note that the *terms* in the $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ formula become formulas of $\mathcal{L}_{\mathrm{GQM}}$, with the Boolean function symbols becoming propositional connectives.

In the reverse direction, define for each PWP formula:

$$(\mathsf{A}\varphi)^* := \varphi \approx \top \qquad (\mathsf{E}\varphi)^* := \varphi \not\approx \bot$$
$$([\forall p]\varphi)^* := \forall p(\varphi)^* \qquad (\langle\exists p\rangle\varphi)^* := \exists p(\varphi)^*.$$

Any A or E GQMs inside φ become function symbols in the $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$ translation.

Lemma 8.2 *For any nontrivial*[6] BAE $\mathfrak{A}$, $\theta : \mathsf{Prop} \to \mathfrak{A}$, *and* $\alpha \in \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$*:*

$$\mathfrak{A}, \theta \vDash \alpha \text{ iff } \tilde{\theta}((\alpha)_*) = \top \qquad \text{and} \qquad \mathfrak{A}, \theta \nvDash \alpha \text{ iff } \tilde{\theta}((\alpha)_*) = \bot.$$

Proof. By induction on the complexity of α. The atomic case follows directly from properties of the connective $\leftrightarrow$, Lemma 2.5, and the fact that in a nontrivial Boolean algebra, $\top$ and $\bot$ are distinct. The Boolean cases follow from the first-order satisfaction definition, the inductive hypothesis, and the algebraic behavior of $\top$ and $\bot$. The GQM case is by Definition 2.4. □

Corollary 8.3 *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$, $\Delta \vDash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} \alpha$ *iff* $(\Delta)_* \vDash^{\mathsf{A}}_{\mathrm{GQM}} (\alpha)_*$.

Proof. Immediate from Lemma 8.2 and the definitions of consequence. □

Theorem 8.4

(i) *For any PWP formula* $\varphi \in \mathcal{L}_{\mathrm{GQM}}$, $\varphi \vdash^{\mathsf{A}}_{\mathsf{GQM}} ((\varphi)^*)_*$ *and* $((\varphi)^*)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$.

(ii) *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$, $\Delta \vdash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} \alpha$ *implies* $(\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\alpha)_*$.

(iii) *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$, $\Delta \vdash_{\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}} \alpha$ *iff* $(\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\alpha)_*$.

Proof. For part (i), given $\langle\exists p\rangle\psi := \neg[\forall p]\neg\psi$, we have (for a fresh q):

[6] By a nontrivial BAE, we mean a BAE in which $\top \neq \bot$.

$$(\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{A}\varphi)^* = \boldsymbol{Qp}\forall q(\varphi \approx \top) \qquad ([\boldsymbol{Qp}]\mathsf{E}\varphi)^* = \boldsymbol{Qp}\exists q(\varphi \not\approx \bot)$$
$$((\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{A}\varphi)^*)_* = \{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{A}(\varphi \leftrightarrow \top) \quad ((\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{E}\varphi)^*)_* = \{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{E}\neg(\varphi \leftrightarrow \bot).$$

It is an easy exercise using Lemmas 5.3 and 5.4 to show that $\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{A}\varphi$ is GQM-equivalent to $\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{A}(\varphi \leftrightarrow \top)$ and $\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{E}\varphi$ to $\{\!\![\boldsymbol{Qp}]\!\!\}\mathsf{E}\neg(\varphi \leftrightarrow \bot)$.

For part (ii), see Appendix C. Part (iii) is obtained from (ii) by noting that the opposite direction follows from the soundness of GQM (Lemma 5.7), Corollary 8.3, and the completeness of $\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$. □

An astute reader will note here that even though $(\cdot)_*$ and $(\cdot)^*$ are mutual inverses up to equivalence, the matrix of $((\alpha)_*)^*$ consists of a single equation or its negation, including for those α whose matrix is a nontrivial conjunction of disjunctions. This is in keeping with general discriminator theory [34].

Corollary 8.5

(i) *For any* $\Delta \cup \{\alpha\} \subseteq \mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$, $\Delta \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} \alpha$ *iff* $(\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\alpha)_*$.

(ii) *For any set of PWP fomulas* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$, $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ *iff* $(\Gamma)^* \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} (\varphi)^*$.

Proof. For part (i), we proceed as follows:

$$\begin{aligned}\Delta \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} \alpha &\Leftrightarrow \Delta \vdash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} \alpha && \text{by completeness of } \mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}\\ &\Leftrightarrow (\Delta)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} (\varphi)_* && \text{by Theorem 8.4.(iii).}\end{aligned}$$

For part (ii), we have:

$$\begin{aligned}(\Gamma)^* \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} (\varphi)^* &\Leftrightarrow ((\Gamma)^*)_* \vdash^{\mathsf{A}}_{\mathsf{GQM}} ((\varphi)^*)_* && \text{by part (i)}\\ &\Leftrightarrow \Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi && \text{by Theorem 8.4.(i).}\end{aligned}$$ □

Theorem 8.6 (Completeness) *For any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\mathrm{GQM}}$,

$$\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi \textit{ iff } \Gamma \vDash^{\mathsf{A}}_{\mathrm{GQM}} \varphi.$$

Proof. First, as far as the consequence relation $\vdash^{\mathsf{A}}_{\mathsf{GQM}}$ is concerned, we can prefix all formulas in $\Gamma \cup \{\varphi\}$ by A (by Lemma 5.6) and then transform them into equivalent PWP formulas (Theorems 7.3.(ii) and 7.5). Corollary 8.5 established that $\Gamma \vdash^{\mathsf{A}}_{\mathsf{GQM}} \varphi$ iff $(\Gamma)^* \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} (\varphi)^*$. By Corollary 8.3, this is equivalent to $((\Gamma)^*)_* \vDash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} ((\varphi)^*)_*$. The result then follows by Theorem 8.4.(i). □

Remark 8.7 The use of Theorem 8.4 in this section should be compared with [6, Thm. 3.7] and [28, Lem. 19]. Our success in establishing the equivalence between the (global) GQM-consequence relation and that of $\mathrm{FO}_{\mathrm{BAE}}$ means that we can internalize the metatheory of modal logics concisely and in a generic way using "bridge theorems" of abstract algebraic logic [6,1,14]. For lack of space, we are not pursuing this option further in this paper, but a good illustration of how GQM can be used in such a generalization can be found in our recent paper [18, § 9], which in fact led us to the invention of this formalism. Bases of admissible rules (see, e.g., [31]) seem to provide another promising candidate. Details and more examples will be provided in a sequel paper.

9 Conclusions

We have seen that GQM provides the sought-after way of viewing "modal logics" as theories relative to one logical system, while also offering a generic and conservative way to enrich any modal logic with propositional quantifiers. This study led us to new perspectives on the first-order correspondence language for BAEs (§ 8) on the one hand and SOPML on the other hand (§ 3). In the first case, the equivalence between $\mathrm{FO}_{\mathrm{BAE}_{\mathsf{A}}}$-consequence and $\vdash^{\mathsf{A}}_{\mathsf{GQM}}$-consequence illustrates a curious use of techniques from abstract algebraic logic (cf. Remark 8.7) beyond their usual scope. In the second case, we were led to new prenex normal form results. We also believe that focusing on the syntax of GQM and its algebraic semantics can lead to a clarification of philosophical problems concerning propositional quantification, such as Kaplan's paradox (§ 4).

Along the way, a number of issues have been postponed to a follow-up paper. In particular, we mentioned that over dual, set-based semantics GQM-consequence may be intractable, as it is over Kripke frames. On the other hand, given that modal logics can be identified with (fragments of) universal GQM-theories, the existence of a rich modal completeness apparatus indicates that for suitable fragments of GQM and formulas of specific syntactic shapes, developing GQM model theory is not hopeless and may yield additional insights in modal logic. This will be a subject of future investigation, as will the systematic internalization of "bridge theorems" mentioned in Remark 8.7.

Another issue we have not touched on is that of Gentzen-style proof theory for (well-behaved fragments of) GQM. Since the difficulty of developing Gentzen systems for many modal logics was behind the idea "that the great proliferation of modal logics is an epidemy from which modal logic ought to be cured" [8, p. 25], it would be of interest to see if GQM could help here as well.

A further intriguing possibility is that of weakening the classical base of GQM to an intuitionistic one. Just as modal logics are [∀]-universal theories in classical GQM, intermediate (modal) logics could be [∀]-universal theories in intuitionistic GQM. There is a connection here with the origins of modal logic: not only was C. I. Lewis a proponent of propositional quantification in modal logic, as well as perhaps the earliest opponent of modal proliferation, but also he seemed interested in the idea of strict implication on an intuitionistic base (see [30] for discussion). It is argued in [30] that moving Lewis's strict implication to an intuitionistic base is indeed a conceptually fruitful step. The enrichment of that system with GQM may be the ultimate Lewisian logic.

One need not stop at intuitionistic logic. With the power of the global quantificational modality, there is the possibility that even vaster swaths of "logics" could become special theories over a generalized version of GQM. Alternatively, the classical base of GQM could be retained, while the connectives of different "logics" are treated as modal operators in BAEs, whose behavior is governed by [∀]-universal GQM formulas. Under one of these approaches, a version of GQM could bring us closer to the idea of "one logic to rule them all."

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A Proof of Proposition 3.3

To prove Proposition 3.3, it is convenient to have another equivalence for $\mathsf{A}\exists p\psi$ (other than $(\star)$ in the proof of Proposition 3.2) in part (ii) of the following.

Lemma A.1 *The following are valid in all* $\mathcal{C}$-BAEs*:*

(i) $\mathsf{A}\forall p\psi \leftrightarrow \forall p\mathsf{A}\psi$*;*

(ii) $\mathsf{A}\exists p\psi \leftrightarrow \forall q\mathsf{A}(\mathsf{E}q \to \exists r\mathsf{A}(\mathsf{E}r \wedge (r \to q) \wedge \exists p\mathsf{A}(r \to \psi)))$ *where q and r do not occur in ψ.*

Proof. Part (i) again follows from the distribution of A over arbitrary meets.

Part (ii) follows from the Boolean algebraic fact that for any $\mathcal{C}$-BA $\mathfrak{A}$ and $Y \subseteq \mathfrak{A}$, we have $\bigvee Y = \top$ (take $Y = \{\tilde{\gamma}(\psi) \mid \gamma \sim_p \theta\}$) iff for all $x \in \mathfrak{A}$ (take x as the semantic value of q), if $x \neq 0$, then there exists a $y \in Y$ such that $x \wedge y \neq 0$, which is equivalent to there being a $z \in \mathfrak{A}$ (take z to be the semantic value of r) such that $z \neq 0$, $z \leq x$, and z is under some element of Y. $\square$

Proposition A.2 *If α is a prenex* $\mathrm{SOPML}_{\mathsf{A}}$ *formula, then $\mathsf{A}\alpha$ is equivalent over* $\mathcal{C}$-BAEs *to a* GQM *formula.*

Proof. We continue to regard GQM formulas as $\mathrm{SOPML}_{\mathsf{A}}$ formulas as in § 3.

The proof is by induction on the number of quantifiers in the prenex formula $\alpha := Q_1p_1 \dots Q_np_n\chi$. Let $\alpha' := Q_2p_2 \dots Q_np_n\chi$.

Case 1: $Q_1 = \forall$. Then by Lemma A.1.(i), $\mathsf{A}\alpha$ is equivalent to $\forall p_1\mathsf{A}\alpha'$, which is equivalent to $\forall p_1\mathsf{A}\mathsf{A}\alpha'$. Since α' has fewer quantifiers than α, by the inductive hypothesis $\mathsf{A}\alpha'$ is equivalent to a GQM formula β. Hence $\forall p_1\mathsf{A}\mathsf{A}\alpha'$ is equivalent to the GQM formula $\forall p_1\mathsf{A}\beta$.

Case 2: $Q_1 = \exists$. Then by Lemma A.1.(ii), $\mathsf{A}\alpha$ is equivalent to

(1) $\forall q \mathsf{A}(\mathsf{E}q \to \exists r \mathsf{A}(\mathsf{E}r \wedge (r \to q) \wedge \exists p_1 \mathsf{A}(r \to \alpha')))$

and hence to

(2) $\forall q \mathsf{A}(\mathsf{E}q \to \exists r \mathsf{A}(\mathsf{E}r \wedge (r \to q) \wedge \exists p_1 \mathsf{A}\mathsf{A}(r \to \alpha')))$.

Since r is not among $p_2, \dots, p_n$, (2) is equivalent to

(3) $\forall q \mathsf{A}(\mathsf{E}q \to \exists r \mathsf{A}(\mathsf{E}r \wedge (r \to q) \wedge \exists p_1 \mathsf{A}\mathsf{A}Q_2 p_2 \dots Q_n p_n (r \to \chi)))$.

Since $Q_2 p_2 \dots Q_n p_n (r \to \chi)$ has fewer quantifiers than α, by the inductive hypothesis $\mathsf{A}Q_2 p_2 \dots Q_n p_n (r \to \chi)$ is equivalent to a GQM formula γ. Hence (3) is equivalent to the GQM formula

(4) $\forall q \mathsf{A}(\mathsf{E}q \to \exists r \mathsf{A}(\mathsf{E}r \wedge (r \to q) \wedge \exists p_1 \mathsf{A}\gamma))$. □

B Proof of Theorem 7.3

In order to prove Theorem 7.3 (Theorem B.2), we first need the following lemma (in the proof, 'PL' stands for propositional logic).

Lemma B.1

(i) *If $\alpha_1, \dots, \alpha_m$ are each NWP formulas, then $\alpha_1 * \dots * \alpha_m$ is provably equivalent to $\mathsf{G}_*(\alpha_1 * \dots * \alpha_m)$.*

(ii) *If $\alpha_1, \dots, \alpha_n$ are each NWP formulas, then $\alpha_1 * \dots * \alpha_n$ is provably equivalent to an NWP formula.*

(iii) *If α is a normal clause, then $\mathsf{A}\alpha$ is provably equivalent to an NWP formula.*

(iv) *If φ is a CNFWP formula, then $\mathsf{A}\varphi$ is provably equivalent to an NWP formula.*

Proof. (i) We have:

(1) $\vdash (\alpha_1 * \dots * \alpha_n) \leftrightarrow (\mathsf{G}_*\alpha_1 * \dots * \mathsf{G}_*\alpha_n)$ by Lemma 5.4.(xi)-(xii) since each α_i is an NWP formula

(2) $\vdash (\mathsf{G}_*\alpha_1 * \dots * \mathsf{G}_*\alpha_n) \leftrightarrow \mathsf{G}_*(\alpha_1 * \dots * \alpha_n)$ by Lemma 5.4.(ii)

(3) $\vdash (\alpha_1 * \dots * \alpha_n) \leftrightarrow \mathsf{G}_*(\alpha_1 * \dots * \alpha_n)$ from (1) and (2) by PL.

(ii) By Lemma 5.2, we may assume without loss of generality that (a) no propositional variable occurs both free and bound in $\alpha_1, \dots, \alpha_n$. The proof is by induction on the number of nonvacuous GQMs (i.e., GQMs binding variables, unlike A and E) occurring in $\alpha_1, \dots, \alpha_n$. First, by Lemma 5.4.(xi)-(xii), we may replace each α_i with an equivalent NWP formula α_i' containing no more GQMs and in which no vacuous GQM occurs before a nonvacuous GQM. Thus, if no α_i' begins with a nonvacuous GQM, then $\alpha_1' * \dots * \alpha_n'$ is already an NWP formula, so we are done. Now suppose that some α_i', say α_n', is of the form $\{Qp\}\varphi$ where $\{Qp\}$ is nonvacuous. Since $\alpha_1', \dots, \alpha_{n-1}'$ are each NWP formulas, $\alpha := (\alpha_1' * \dots * \alpha_{n-1}')$ is equivalent to $\mathsf{G}_*\alpha$ by part (i). Then we have:

(4) $\vdash \alpha \leftrightarrow \mathsf{G}_*\alpha$

(5) $\vdash (\alpha_1' * \dots * \alpha_n') \leftrightarrow (\mathsf{G}_*\alpha * \{Qp\}\varphi)$ by (4) and PL

(6) $\vdash (\mathsf{G}_*\alpha * \{Qp\}\varphi) \leftrightarrow (\mathsf{G}_*\alpha * \{Qp\}\{Qr\}\varphi)$ where r is not free in φ or α, by Lemma 5.4.(ix)-(x)

(7) $\vdash (\mathsf{G}_*\alpha * \{Qp\}\{\forall r\}\varphi) \leftrightarrow \{Qp\}(\mathsf{G}_*\alpha * \{Qr\}\varphi)$ by Lemma 7.2.(i), since p is not free in α by (a) above

(8) $\vdash \{Qp\}(\mathsf{G}_*\alpha * \{\forall r\}\varphi) \leftrightarrow \{Qp\}(\alpha * \{Qr\}\varphi)$ by (4), PL, and Lemma 5.3

(9) $\vdash (\alpha'_1 * \cdots * \alpha'_n) \leftrightarrow \{Qp\}(\alpha'_1 * \cdots * \alpha'_{n-1} * \{Qr\}\varphi)$ by (5)–(8) by PL.

Since $\alpha'_1, \ldots, \alpha'_{n-1}, \{Qr\}\varphi$ are each NWP formulas, and there is one fewer non-vacuous GQM in $\alpha'_1, \ldots, \alpha'_{n-1}, \{Qr\}\varphi$ than in $\alpha'_1, \ldots, \alpha'_n$, the inductive hypothesis implies that $\alpha'_1 * \cdots * \alpha'_{n-1} * \{Qr\}\varphi$ is equivalent to an NWP formula. It follows by Lemma 5.3 that $\{Qp\}(\alpha'_1 * \cdots * \alpha'_{n-1} * \{Qr\}\varphi)$ is equivalent to an NWP formula, so by (9), $\alpha'_1 * \cdots * \alpha'_n$ is equivalent to an NWP formula, which means that $\alpha_1 * \cdots * \alpha_n$ is equivalent to an NWP formula.

(iii) The proof is by induction on the number of disjuncts in a normal clause. Suppose α is $\beta_1 \vee \cdots \vee \beta_m$. If no β_k is an NWP formula, then $\mathsf{A}\alpha$ is already an NWP formula. Suppose $\beta_m := \boldsymbol{\{Qp\}}\gamma$ is an NWP formula, and let $\beta := \beta_1 \vee \cdots \vee \beta_{m-1}$. Then β is a normal clause, so by the inductive hypothesis, $\mathsf{A}\beta$ is equivalent to an NWP formula δ. Now we have:

(10) $\vdash \alpha \leftrightarrow (\beta \vee \boldsymbol{\{Qp\}}\gamma)$ by our assumption of what α is

(11) $\vdash \mathsf{A}\alpha \leftrightarrow \mathsf{A}(\beta \vee \boldsymbol{\{Qp\}}\gamma)$ from (10) by Lemma 5.4.(iii)

(12) $\vdash \mathsf{A}(\beta \vee \boldsymbol{\{Qp\}}\gamma) \leftrightarrow (\mathsf{A}\beta \vee \boldsymbol{\{Qp\}}\gamma)$ by Lemma 7.2.(ii)

(13) $\vdash \mathsf{A}\beta \leftrightarrow \delta$ by the inductive hypothesis

(14) $\vdash (\mathsf{A}\beta \vee \boldsymbol{\{Qp\}}\gamma) \leftrightarrow (\delta \vee \boldsymbol{\{Qp\}}\gamma)$ from (12) and (13) by PL

(15) $\vdash \mathsf{A}\alpha \leftrightarrow (\delta \vee \boldsymbol{\{Qp\}}\gamma)$ from (11), (12), and (14) by PL.

Since δ is an NWP formula, $\delta \vee \boldsymbol{\{Qp\}}\gamma$ is a disjunction of NWP formulas. Thus, by part (ii), $\delta \vee \boldsymbol{\{Qp\}}\gamma$ is equivalent to an NWP formula, and hence by (15), $\mathsf{A}\alpha$ is equivalent to an NWP formula.

(iv) Suppose φ is a CNFWP formula $\alpha_1 \wedge \cdots \wedge \alpha_n$. Hence $\mathsf{A}\varphi$ is equivalent to $\mathsf{A}\alpha_1 \wedge \cdots \wedge \mathsf{A}\alpha_n$ by Lemma 5.4.(ii). Since each α_i is a normal clause, part (iii) implies that each $\mathsf{A}\alpha_i$ is equivalent to an NWP formula χ_i. Hence $\mathsf{A}\alpha_1 \wedge \cdots \wedge \mathsf{A}\alpha_n$ is equivalent to $\chi_1 \wedge \cdots \wedge \chi_n$, which by part (ii) is equivalent to an NWP formula. Thus, φ is equivalent to an NWP formula. □

Theorem B.2 *For every $\varphi \in \mathcal{L}_{\mathrm{GQM}}$:*

(i) *φ is provably equivalent to a CNFWP formula;*

(ii) *$\mathsf{A}\varphi$ is provably equivalent to an NWP formula.*

Proof. We prove part (i) by induction on φ. The base case for propositional variables is immediate. Suppose φ is $\neg\psi$. By the inductive hypothesis, ψ is equivalent to a CNFWP formula. One then uses de Morgan and distributive laws to show that $\neg\psi$ is also equivalent to a CNFWP formula. Suppose φ is $\psi_1 \wedge \psi_2$. By the inductive hypothesis, ψ_1 and ψ_2 are both equivalent to CNFWP formulas ψ'_1 and ψ'_2. Then $\psi_1 \wedge \psi_2$ is equivalent to the CNFWP formula $\psi'_1 \wedge \psi'_2$.

Suppose φ is $[\forall p]\psi$. By the inductive hypothesis, ψ is equivalent to a CNFWP formula χ, which implies that $\mathsf{A}\psi$ is equivalent to $\mathsf{A}\chi$ by Lemma 5.3. Hence $[\forall p]\psi$, which is equivalent to $[\forall p]\mathsf{A}\psi$ by Lemma 5.4.(ix), is equivalent to $[\forall p]\mathsf{A}\chi$ by Lemma 5.3. By Lemma B.1.(iv), $\mathsf{A}\chi$ is equivalent to an NWP formula, from which it follows by Lemma 5.3 that $[\forall p]\mathsf{A}\chi$ is equivalent to an NWP formula. Such a formula is in CNFWP.

Suppose φ is $\Box\psi$. By the inductive hypothesis, ψ is equivalent to a CNFWP formula $\alpha_1 \wedge \cdots \wedge \alpha_n$. We will prove that for any CNFWP formula $\sigma_1 \wedge \cdots \wedge \sigma_k$, $\Box(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is equivalent to a formula in CNFWP, by induction on the number of GQMs occurring in $\sigma_1 \wedge \cdots \wedge \sigma_k$. If no σ_i contains a disjunct in NWP, then no σ_i contains a GQM, which means $\Box(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is already in CNFWP. So suppose that some σ_i, say σ_k, contains as a disjunct an NWP formula $\{\!\!\{Qp\}\!\!\}\gamma$. Hence σ_k is equivalent to $\beta \vee \{\!\!\{Qp\}\!\!\}\gamma$ for a normal clause β. Let $\sigma := \sigma_1 \wedge \cdots \wedge \sigma_{k-1}$. Thus, $\Box(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is equivalent to $\Box(\sigma \wedge (\beta \vee \{\!\!\{Qp\}\!\!\}\gamma))$, which by Lemma 7.2.(iii) is equivalent to

(1) $(\{\!\!\{Qp\}\!\!\}\gamma \wedge \Box\sigma) \vee (\neg\{\!\!\{Qp\}\!\!\}\gamma \wedge \Box(\sigma \wedge \beta))$.

Now σ and $\sigma \wedge \beta$ are CNFWP formulas containing fewer GQMs than $\sigma_1 \wedge \cdots \wedge \sigma_k$. Hence by the inductive hypothesis, there are CNFWP formulas χ_1 and χ_2 such that $\Box\sigma$ is equivalent to χ_1 and $\Box(\sigma \wedge \beta)$ is equivalent to χ_2. Thus, $\Box(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is equivalent to

(2) $(\{\!\!\{Qp\}\!\!\}\gamma \wedge \chi_1) \vee (\neg\{\!\!\{Qp\}\!\!\}\gamma \wedge \chi_2)$.

Since $\{\!\!\{Qp\}\!\!\}\gamma$ is an NWP formula and χ_1 and χ_2 are CNFWP formulas, (2) can be transformed into an equivalent CNFWP using distributive laws and the fact that $\neg\{\!\!\{Qp\}\!\!\}\gamma$ is equivalent to the NWP formula $\overline{\{\!\!\{Qp\}\!\!\}}\neg\gamma$.

Part (ii) follows from part (i) and Lemma B.1.(iv). $\Box$

C Proof of Theorem 8.4.(ii)

In order to prove Theorem 8.4.(ii), we first recall the needed first-order apparatus, for which we follow Enderton [12]. A *generalization* of a first-order formula φ is any formula of the form $\forall p_1 \ldots \forall p_n \varphi$ for $n \geq 0$. Enderton takes as axioms all generalizations of the following:

- all substitution instances of propositional tautologies;
- $\forall p\varphi \to \varphi^p_t$ where the term t is substitutable for p in φ;
- $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$;
- $\varphi \to \forall p\varphi$ where p does not occur free in φ;
- $p \approx p$, and $p \approx q \to (\varphi \to \varphi')$ where φ is atomic and φ' is obtained from φ by replacing p in zero or more places by q.

In addition, we add all generalizations of the following axioms for the elementary theory of nontrivial discriminator BAEs:

- first-order axioms of Boolean algebras;
- $\forall p((p \approx \top \,\&\, \mathsf{A}p \approx \top) \text{ OR } (p \not\approx \top \,\&\, \mathsf{A}p \approx \bot))$ and $\top \not\approx \bot$.

Let $\Gamma \vdash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}} \varphi$ iff φ belongs to the smallest set of $\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$ formulas that includes all the axioms above, is closed under modus ponens, and includes Γ.

Lemma C.1 *For any $\varphi \in \mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$, term t and variable p:*

(i) *if p is not free in φ, then p is not free in $(\varphi)_*$;*

(ii) *if t is substitutable for p in φ, then t is substitutable for p in $(\varphi)_*$;*

(iii) $(\varphi^p_t)_* = ((\varphi)_*)^p_t$.

Lemma C.2 *For every $\varphi \in \mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}$, $\vdash_{\mathsf{GQM}} (\varphi)_* \leftrightarrow \mathsf{A}(\varphi)_*$.*

Proof. A straightforward induction using Lemma 5.4. □

We are now ready to prove Theorem 8.4.(ii).

Proof. The proof is by induction on the length of $\vdash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}}$ proofs. We first check that the translation of each axiom is a theorem of GQM. Since GQM has the $[\forall]$-necessitation rule that if $\vdash_{\mathsf{GQM}} \varphi$, then $\vdash_{\mathsf{GQM}} [\forall p]\varphi$, it suffices to check that each of the ungeneralized axioms translates to a theorem of GQM:

- The translation of any propositional tautology is clearly also a propositional tautology.
- By Lemma C.1.(iii), $(\forall p\varphi \to \varphi^p_t)_* = [\forall p](\varphi)_* \to ((\varphi)_*)^p_t$, which by Lemma C.1.(ii) is an instance of instantiation.
- $(\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi))_* = [\forall p]((\varphi)_* \to (\psi)_*) \to ([\forall p](\varphi)_* \to [\forall p](\psi)_*)$, which is an instance of distribution.
- $(\varphi \to \forall p\varphi)_* = (\varphi)_* \to [\forall p](\varphi)_*$, and we have $\vdash (\varphi)_* \to \mathsf{A}(\varphi)_*$ by Lemma C.2 and hence $\vdash (\varphi)_* \to [\forall p](\varphi)_*$ by Lemma 5.2 since p is not free in $(\varphi)_*$ (by Lemma C.1).
- $(p \approx p)_* := \mathsf{A}(p \leftrightarrow p)$, which is obtained from the tautology $p \leftrightarrow p$ by $[\forall]$-necessitation.
- $(p \approx q \to (\varphi \to \varphi'))_* = \mathsf{A}(p \leftrightarrow q) \to ((\varphi)_* \to ((\varphi)_*)')$ where $((\varphi)_*)'$ is obtained from $(\varphi)_*$ by replacing the appropriate occurrences of p by q. Proving that $\vdash_{\mathsf{GQM}} \mathsf{A}(p \leftrightarrow q) \to ((\varphi)_* \to ((\varphi)_*)')$ is routine.
- The translation of any axiom of Boolean algebra is clearly derivable in GQM using PL and $[\forall]$-necessitation.
- $(\forall p((p \approx \top \,\&\, \mathsf{A}p \approx \top) \,\mathrm{OR}\, (p \not\approx \top \,\&\, \mathsf{A}p \approx \bot)))_*$ is

  $$[\forall p]\big((\mathsf{A}(p \leftrightarrow \top) \wedge \mathsf{A}(\mathsf{A}p \leftrightarrow \top)) \vee (\neg\mathsf{A}(p \leftrightarrow \top) \wedge \mathsf{A}(\mathsf{A}p \leftrightarrow \bot))\big),$$

  which is straightforward to derive using Lemma 5.4 and $[\forall]$-necessitation.
- $(\top \neq \bot)_* = \neg\mathsf{A}(\top \leftrightarrow \bot)$, which is derivable by instantiation and PL.

Finally, any application of modus ponens for $\vdash_{\mathrm{FO}_{\mathrm{BAE}_\mathsf{A}}}$ can be matched—using the inductive hypothesis—by an application of modus ponens for $\vdash^{\mathsf{A}}_{\mathsf{GQM}}$. □