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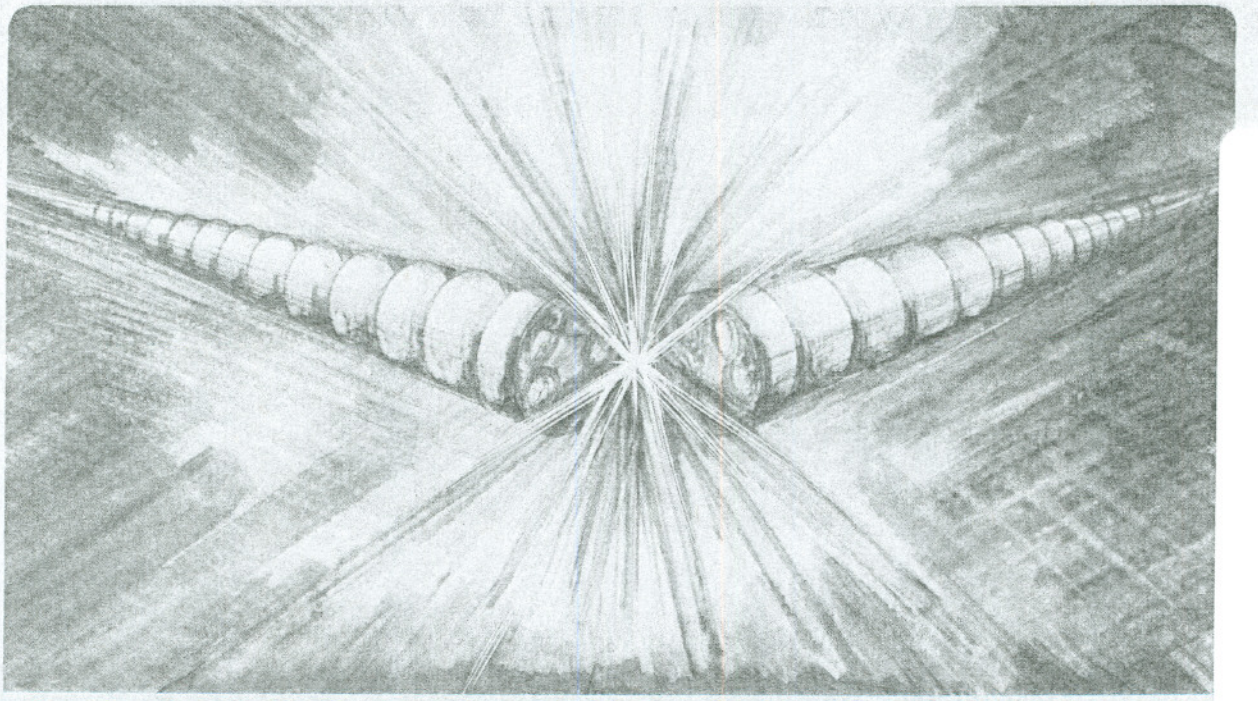
3D Field Harmonics

S. Caspi, M. Helm, and L.J. Laslett

March 1991



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3D Field Harmonics.*

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Abstract

We have developed an harmonic representation for the three dimensional field components within the windings of accelerator magnets. The form by which the field is presented is suitable for interfacing with other codes that make use of the 3D field components (particle tracking and stability). The field components can be calculated with high precision and reduced cpu time at any location (r,θ,z) inside the magnet bore. The same conductor geometry which is used to simulate line currents is also used in CAD with modifications more readily available. It is our hope that the format used here for magnetic fields can be used not only as a means of delivering fields but also as a way by which beam dynamics can suggest possible correction to the conductor geometry.

Introduction

In 2D problems we are accustomed to express the transverse components of a non-skew magnetic field in the form :

$$\begin{aligned} B_r &= \sum_{n=1} b_n r^{n-1} \sin n\theta \\ B_\theta &= \sum_{n=1} b_n r^{n-1} \cos n\theta \end{aligned} \tag{1}$$

It is of interest to exhibit similar means by which the components of a 3D field correspondingly might be usefully expressed in a non-skew situation.

We accordingly note that in the curl-free divergence-free region near the axis r=0 the field components may then be expressed as given by $\vec{B} = -\nabla V$ where V is a scalar potential function for which $\nabla^2 V = 0$. The proposed solution can be written in the form :

$$\begin{aligned} V &= \sum_{n=1} V_n(r,z) \sin n\theta \\ with\ : & \\ & \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_n}{\partial r}\right) + \frac{\partial^2 V_n}{\partial z^2} - \frac{n^2 V_n}{r^2} = 0 \end{aligned} \tag{2}$$

We note that if V_n were to be free of any z-dependence, the acceptable solution for V_n near the axis would be expressed by a single term proportional to r^n (i.e., involving r raised to the positive power n); more generally one would represent V_n by a power series involving factors r^{n+2k}, commencing with r^n, and employing z-dependent coefficients :

$$\begin{aligned} V_n &= \sum_{k=0} C_{n,k}(z) r^{n+2k} \\ & with\ C_{n,k}(z)\ satisfying\ the\ recursion\ relation \\ & C_{n,k}(z) = -\frac{1}{4k(n+k)}\frac{d^2 C_{n,k-1}}{dz^2} \quad ; \quad k = 1, \ldots \end{aligned} \tag{3}$$

in order that the series for V_n satisfy the differential equation written above. With introduction of $A_n(z)$ as a notation for $C_{n,0}(z)$, the solution of the recursion relation becomes expressed by

$$C_{n,k}(z) = (-1)^k \frac{n! \frac{d^{2k} A_n(z)}{dz^{2k}}}{2^{2k} k!(n+k)!} \tag{4}$$

and therefore :

$$V_n = -\left(\sum_{k=0} (-1)^{k+1} \frac{n!}{2^{2k} k!(n+k)!} A_n^{(2k)}(z) r^{2k} \right) r^n$$

or :

$$V_n = A_n(z) r^n - \frac{A_n''(z)}{4(n+1)} r^{n+2} + \frac{A_n''''(z)}{32(n+1)(n+2)} r^{n+4} - \frac{A_n''''''(z)}{384(n+1)(n+2)(n+3)} r^{n+6} + \ldots \tag{5}$$

With

$$\begin{aligned} B_{r,n} &= -\frac{\partial V_n}{\partial r} \sin n\theta = g_{rn} r^{n-1} \sin n\theta \\ B_{\theta,n} &= -\frac{n}{r} V_n \cos n\theta = g_{\theta n} r^{n-1} \cos n\theta \\ B_{z,n} &= -\frac{\partial V_n}{\partial z} \sin n\theta = g_{zn} r^n \sin n\theta \end{aligned} \tag{6}$$

2–Dimensional Field

The transverse components of a 2D non skewed magnetic field in a current-free region near the axis are conveniently expressible as[2]:

$$\begin{aligned} B_r &= \sum_{n=1} b_n r^{n-1} \sin n\theta \\ B_\theta &= \sum_{n=1} b_n r^{n-1} \cos n\theta \\ \text{and :} & \\ B_x &= \sum_{n=1} b_n r^{n-1} \sin (n-1)\theta \\ B_y &= \sum_{n=1} b_n r^{n-1} \cos (n-1)\theta \end{aligned} \tag{7}$$

The two dimensional Cartesian components can be written in complex notation as:

$$B^* = -i \sum_{n=1} b_n Z^{n-1} \qquad ; \quad B^* = B_x - iB_y \tag{8}$$

where $Z = r^{i\theta} = x + iy$. The above equation can be reduced to :

$$B^* = \sum_{n=1} C_n^* Z^{n-1} \tag{9}$$

where $C_n^* = a_n - ib_n$ *and* $a_n = 0$ in a non skewed field. For a dipole n=1,3,5,.., and for a quadrupole n=2,6,10...

[2] K. Halbach , "Fields And First Order Petrubation Effects In Two-Dimensional Conductor Dominated Magnets" , Nuclear Instruments And Methods 78 , 185–198 (1970) .

3–Dimensional Field

The transverse field components B_x and B_y

We attempt here to develop the transverse field components of a three—dimensional non skew field in the same spirit as was done for the two—dimensional case. The transverse components of the field can be put into a more general form than equation (7) :

$$B_r = \sum_{n=1} g_{rn} r^{n-1} \sin n\theta$$
$$B_\theta = \sum_{n=1} g_{\theta n} r^{n-1} \cos n\theta$$

and : (10)

$$B_x = \sum_{n=1} \frac{g_{rn} + g_{\theta n}}{2} r^{n-1} \sin(n-1)\theta + \sum_{n=1} \frac{g_{rn} - g_{\theta n}}{2} r^{n-1} \sin(n+1)\theta$$
$$B_y = \sum_{n=1} \frac{g_{rn} + g_{\theta n}}{2} r^{n-1} \cos(n-1)\theta - \sum_{n=1} \frac{g_{rn} - g_{\theta n}}{2} r^{n-1} \cos(n+1)\theta$$

Similarly to equation (8) the transverse Cartesian components can be expressed using complex notation :

$$B^* = -i \sum_{n=1} \frac{g_{rn} + g_{\theta n}}{2} Z^{n-1} + i \sum_{n=1} \frac{g_{rn} - g_{\theta n}}{2} \frac{1}{r^2} Z^{*n+1} \tag{11}$$

Note that in the limiting case where the field reduces to a two dimensional one $g_{rn} = g_{\theta n} = b_n$. Also , as will be shown later,

$$g_{rn} - g_{\theta n} = r \int \frac{\partial g_{zn}(r,z)}{\partial r} dz \tag{12}$$

The functions g_{rn} and $g_{\theta n}$

Whereas the harmonic coefficients b_n constitute a set of constants in the 2D case , their equivalent counterparts g_{rn} , $g_{\theta n}$ in the 3D case are in general functions of r and z (Eq. 5 and 6) as shown below for a current-free region (e.g. , near the axis) :

$$g_{rn}(r,z) = \sum_{k=0} (-1)^{k+1} \frac{n!(n+2k)}{2^{2k} k!(n+k)!} A_n^{(2k)}(z) r^{2k}$$
$$g_{\theta n}(r,z) = \sum_{k=0} (-1)^{k+1} \frac{n!n}{2^{2k} k!(n+k)!} A_n^{(2k)}(z) r^{2k} \tag{13}$$

Explicitly we can write the above as :

$$g_{rn}(r,z) = -nA_n(z) + \frac{n+2}{4(n+1)} A_n''(z) r^2 - \frac{n+4}{32(n+1)(n+2)} A_n''''(z) r^4 + \frac{n+6}{384(n+1)(n+2)(n+3)} A_n''''''(z) r^6 - \ldots$$
$$g_{\theta n}(r,z) = -nA_n(z) + \frac{n}{4(n+1)} A_n''(z) r^2 - \frac{n}{32(n+1)(n+2)} A_n''''(z) r^4 + \frac{n}{384(n+1)(n+2)(n+3)} A_n''''''(z) r^6 - \ldots \tag{14}$$

wherein the coefficients $A_n(z)$, $A_n''(z)$, etc. are coefficients of a scalar potential function as previously shown. Such formulas, if not truncated, lead to 3D field components that are both curl-free and divergence free, as is appropriate for a stationary magnetic field in a current-free region. If the summations over k are truncated so that the summations for g_{rn} , $g_{\theta n}$ are extended through values of this index that are greater by unity than the index limit for g_{zn}, then the corresponding magnetic field will still be seen to be divergence free (as may be appropriate for a Hamiltonian formulation of particle dynamics under the action of q[vxB] forces).

As shown above g_{rn} , $g_{\theta n}$ are power functions in r and functions of $A_n(z)$ and its derivatives. As will be shown we can evaluate $A_n(z)$ and set it in tables from which its derivatives can be calculated as well. Equation (11) now suggests the following form for the transverse component of the 3D field :

$$\begin{aligned} B^*(r,z) = B_x - iB_y = &-i\sum_{n=1}\left[\sum_{k=0}(-1)^{k+1}\frac{n!(n+k)}{2^{2k}k!(n+k)!}A_n^{(2k)}(z)r^{2k}\right]Z^{n-1} \\ &+i\sum_{n=1}\left[\sum_{k=0}(-1)^{k+1}\frac{n!k}{2^{2k}k!(n+k)!}A_n^{(2k)}(z)r^{2(k-1)}\right]Z^{*(n+1)} \end{aligned} \tag{15}$$

The z directed field component B_z

We can express the z directed field component as:

$$B_z = \sum_{n=1} g_{zn} r^n \sin n\theta = -\frac{\partial V}{\partial z} \tag{16}$$

(Note that we have here r^n and not r^{n-1} as written in equation 10). In analogy to equation 12 we have:

$$g_{zn}(r,z) = \sum_{k=0}(-1)^{k+1}\frac{n!}{2^{2k}k!(n+k)!}A_n^{(2k+1)}r^{2k} \tag{17}$$

or explicitly :

$$g_{zn}(r,z) = -A_n'(z) + \frac{1}{4(n+1)}A_n'''(z)r^2 - \frac{1}{32(n+1)(n+2)}A_n''''(z)r^4..... \tag{18}$$

Note also that :

$$g_{zn}(r,z) = \frac{1}{n}\frac{\partial g_{\theta n}(r,z)}{\partial z} \tag{19}$$

It will be noted from equation (10) – or, more explicitly, from equation (15)– that in truly 3D situations a single harmonic component (characterized by a single index n to describe the θ dependence of the r, θ, and z components of the field) will contain "pseudo-multipole"

components[3] of field in the sense that the transverse field components for such a harmonic no longer will exhibit a pure r^{n-1} dependence on the cylindrical-coordinate radius. Such additional pseudo-multipole terms arise not only from the presence of the factor Z^{*n+1} shown explicitly in equation (11), but also from the r-dependence of the many terms that form the (differing) expressions for $g_{rn}(r,z)$ and $g_{\theta n}(r,z)$ [as is evident from equations (12) , or (15)].Thus, as Krejcik and others (E.J.N. Wilson , G. Wustefeld) have indicated, a lens structure with quadrupole symmetry can lead in a 3D analysis of end fields to the presence of pseudo-octupole components.

The integrals of the transverse components of the field (integrated with respect to z at constant values of r and θ, completely through an end region of a lens element with a simple 2D design in the interior) of course will themselves have strictly 2D character , with the integrals of $g_{\theta n}$ and $g_{r n}$ becoming identical, and the "pseudo_multipole" elements of these integrated fields will be absent. [In simple cases in which a surrounding magnetic shield (if present) is of high permeability with an extended cylindrical interface, the 2D character of the integrated transverse components of field indeed can be directly related to the integral of the longitudinal component of current density.][4]

It may be regarded as desirable that the respective series for $g_{r n}$, $g_{\theta n}$, and g_{zn} be truncated so that the series for g_{zn} contains one fewer terms than either of the series for $g_{r n}$ and $g_{\theta n}$. It will be seen that, with such a termination, the divergence of B will vanish exactly (if the derivatives of $A_n(z)$ are accurately interrelated) , although the curl of B in general then will not do so exactly.

$$\nabla \cdot B = \frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0 \qquad (20)$$

The vanishing of the divergence, however, assures that this field is derivable from a vector-potential function (Appendix A) , as is required for a Hamiltonian representation of particle motion under the influence of q[vxB] forces. For symplectic computational work one of course must also take into consideration the need for symplectic integration algorithems.[5]

Example — dipole D19

As an example we have used the end region of dipole D19. Shown is the conductor geometry used for field calculation. We have computed both $g_{r n}$, $g_{\theta n}$ and g_{zn} from which the A(z) and its derivatives have been computed. In the following figures we show results for the A's derived from the conductor only the iron only and both iron and conductor together. For the conductor alone the A's have been computed up to $\frac{\partial^6 A(z)}{\partial z^6}$ for n=1,3,5,7 and 9. The quality of the iron contribution at the present time is limited and therefor the A's for the iron and for the iron + conductor have been computed only to $\frac{\partial^2 A(z)}{\partial z^2}$ for n=1,3,5,7 and 9

[3] This terminology has been suggested by P.Krejcik, CERN ps/87–21 (AA) –paper presented at the 1987 IEEE Particle Accelerator Conference , Washington, D.C. ; March 16–19, 1987 ; Proc. 1987 IEEE Particle Accel. Conf., V.2 , 1278–1280.

[4] For discussion of end windings of this character see, for example, Laslett, Caspi, and Helm, Particle Accelerator 22, 1–14 (1987) and 23, 149–150 (1988).

[5] See, for example, Etienne Forest and Ronald D. Ruth, "Fourth-Order Symplectic Integration", Physica D, 43 , 105–117 (North-Holland, 1990). We have been informed by Dr. Forest that he has also obtained similar results for sixth-order integration. Earlier third-order results were presented by Dr. Ruth at the 1983 Particle Accelerator Conference, "A Canonical Integration Technique", IEEE Trans. Nucl. Sci., NS-30 (No. 4, Part 1), 2669–2671 (1983).

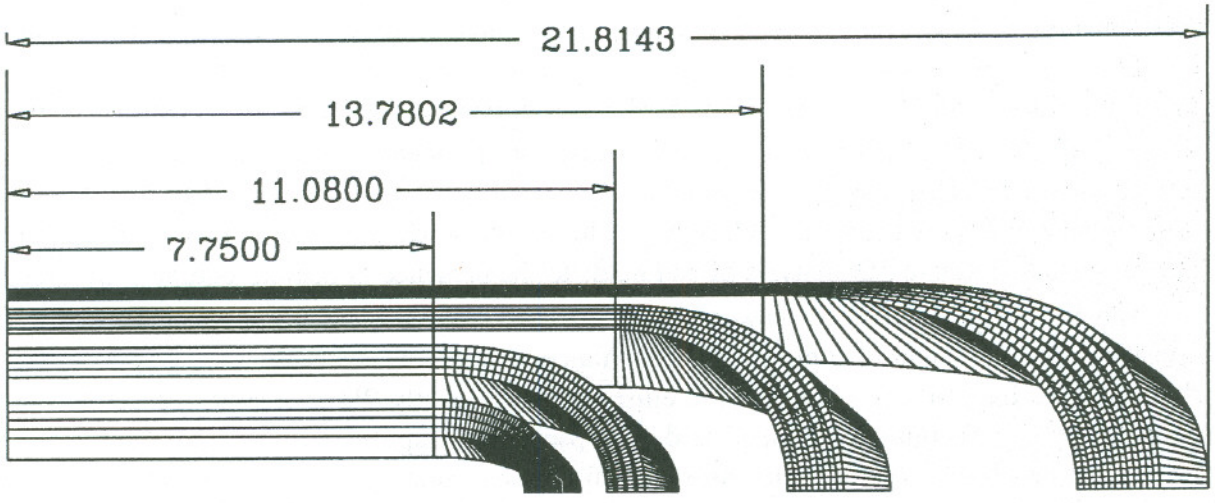

Figure 1 Conductor geometry in the end region of dipole magnet D19 — LAYER-1 TOP.

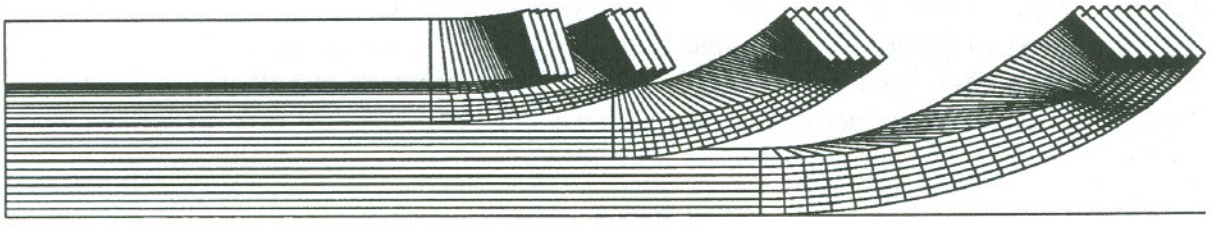

Figure 2 Conductor geometry in the end region of dipole magnet D19 — LAYER-1 SIDE.

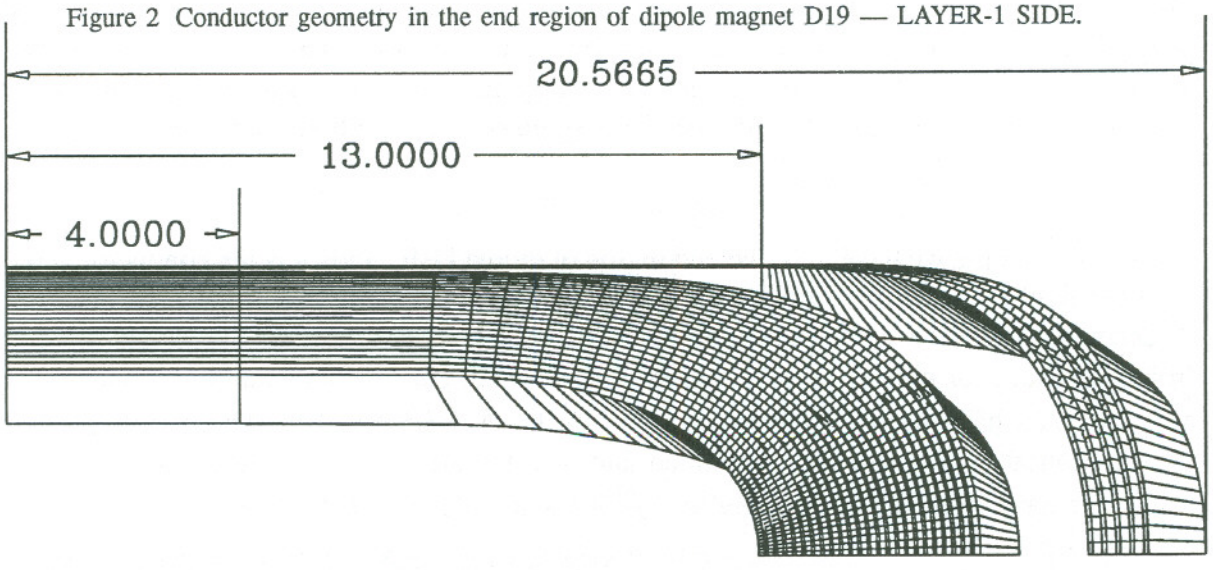

Figure 3 Conductor geometry in the end region of dipole magnet D19 — LAYER-2 TOP.

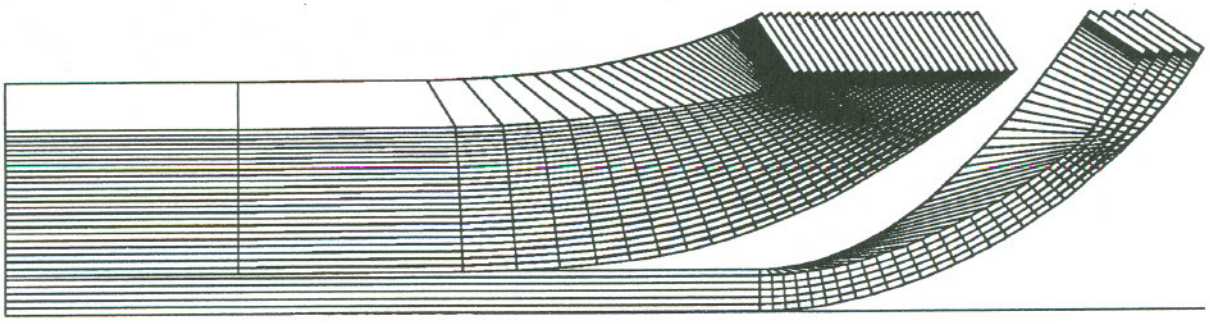

Figure 4 Conductor geometry in the end region of dipole magnet D19 — LAYER-2 SIDE.

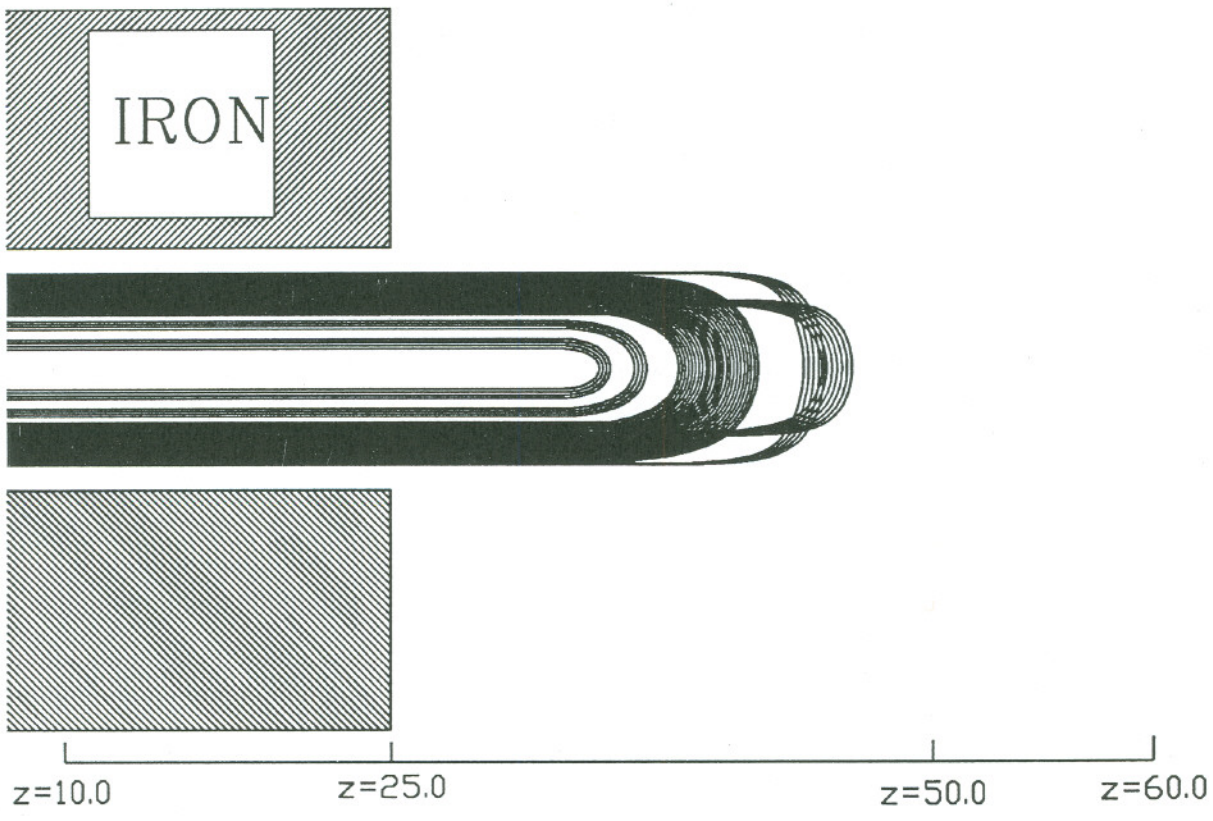

Figure 5 Coil schematic and IRON location in the end region.

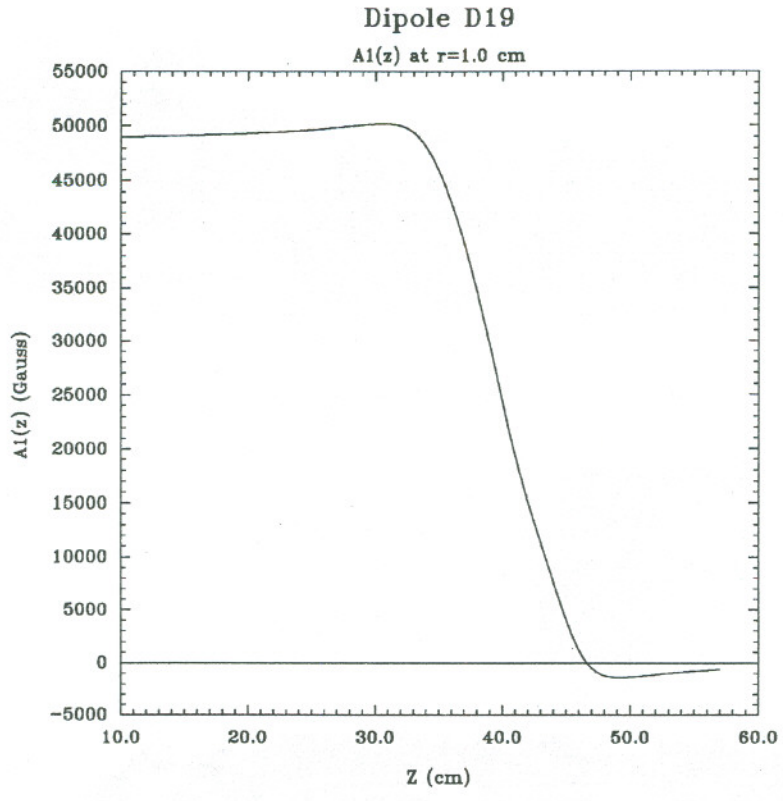

Figure 6 The dipole function A1(z) — CONDUCTOR ONLY.

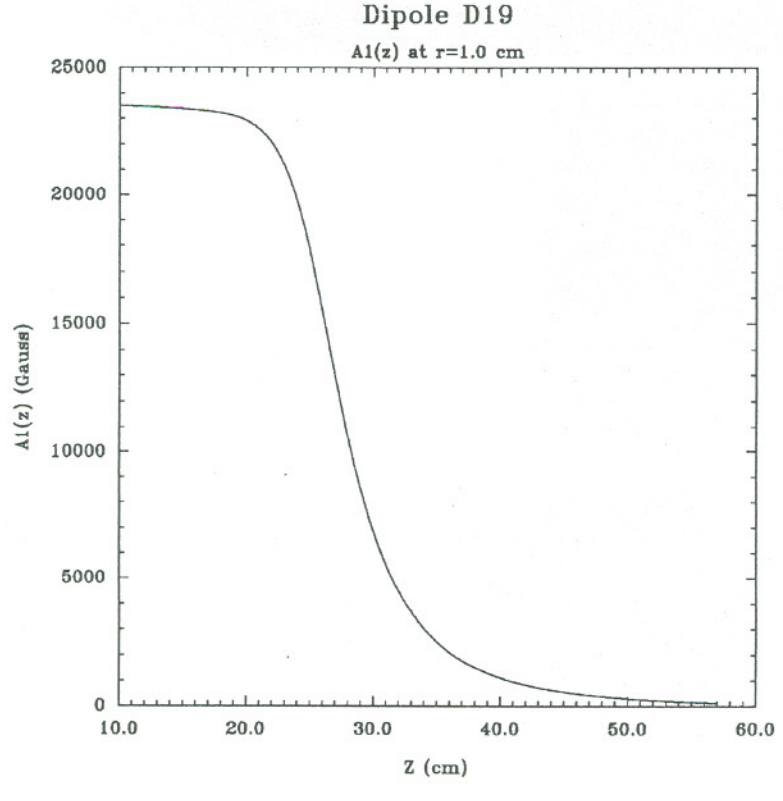

Figure 7 The dipole function A1(z) — IRON ONLY.

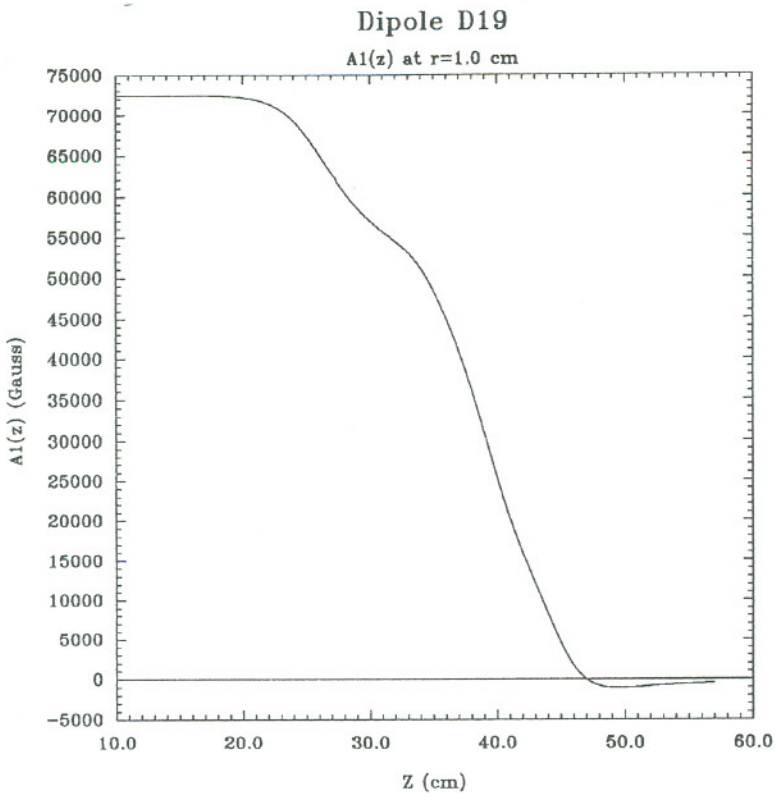

Figure 8 The dipole function A1(z) — CONDUCTOR and IRON.

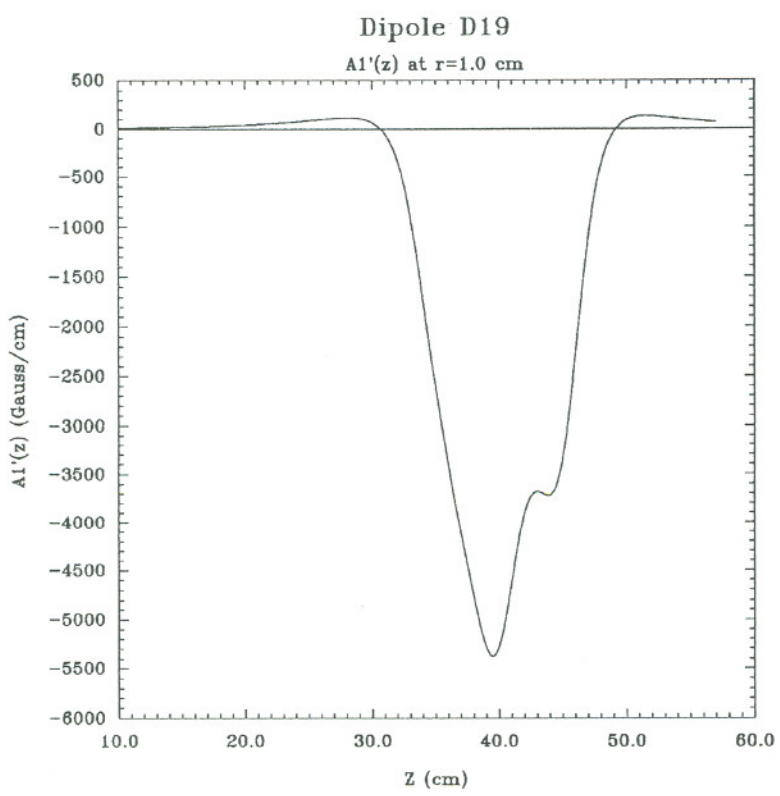

Figure 9 The first derivative function of n=1 — A1'(z) — CONDUCTOR ONLY.

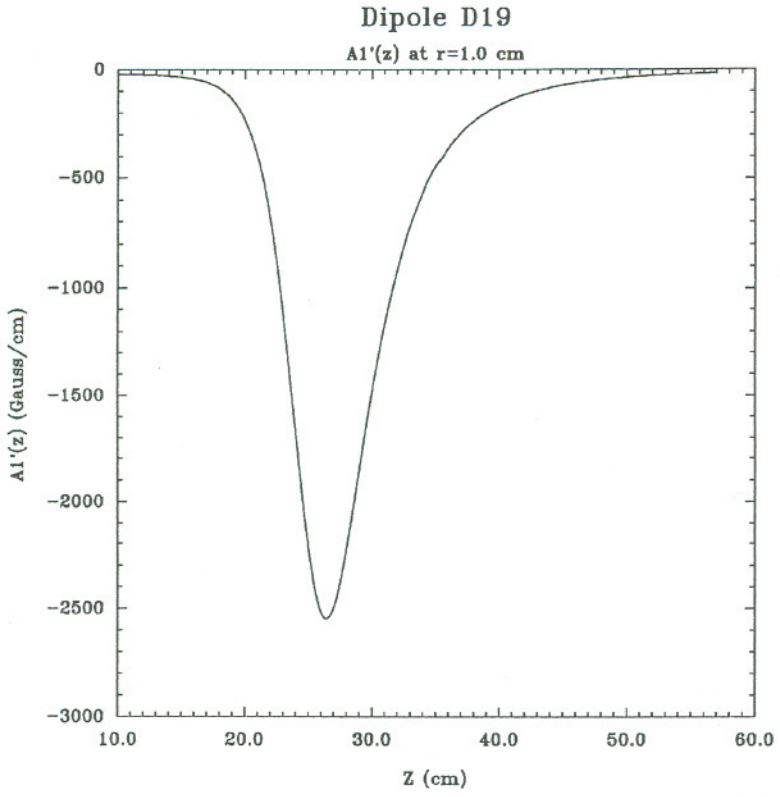

Figure 10 The first derivative function of n=1 — A1'(z) — IRON ONLY .

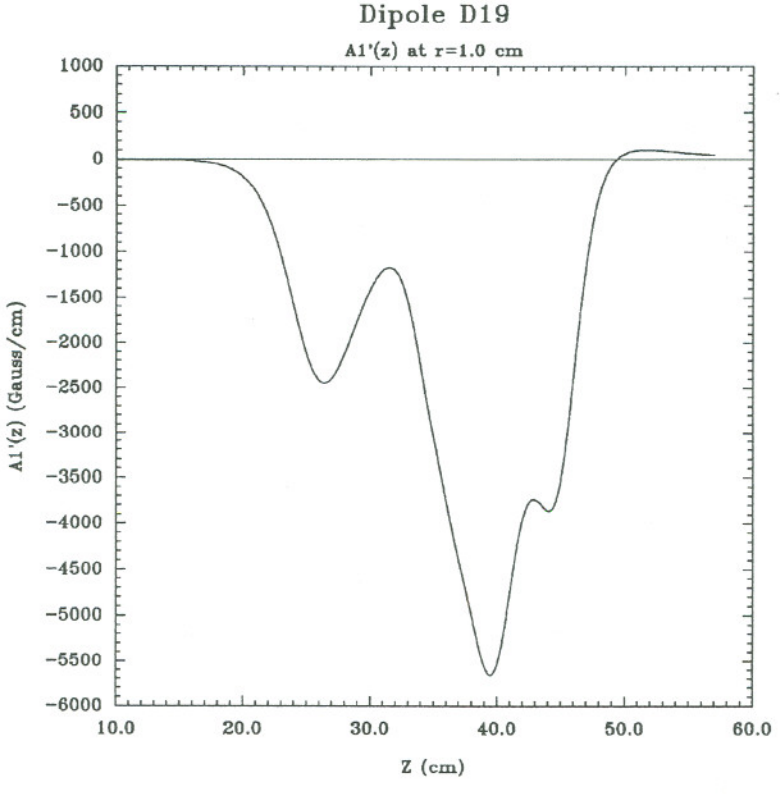

Figure 11 The first derivative function of n=1 — A1'(z) — CONDUCTOR and IRON.

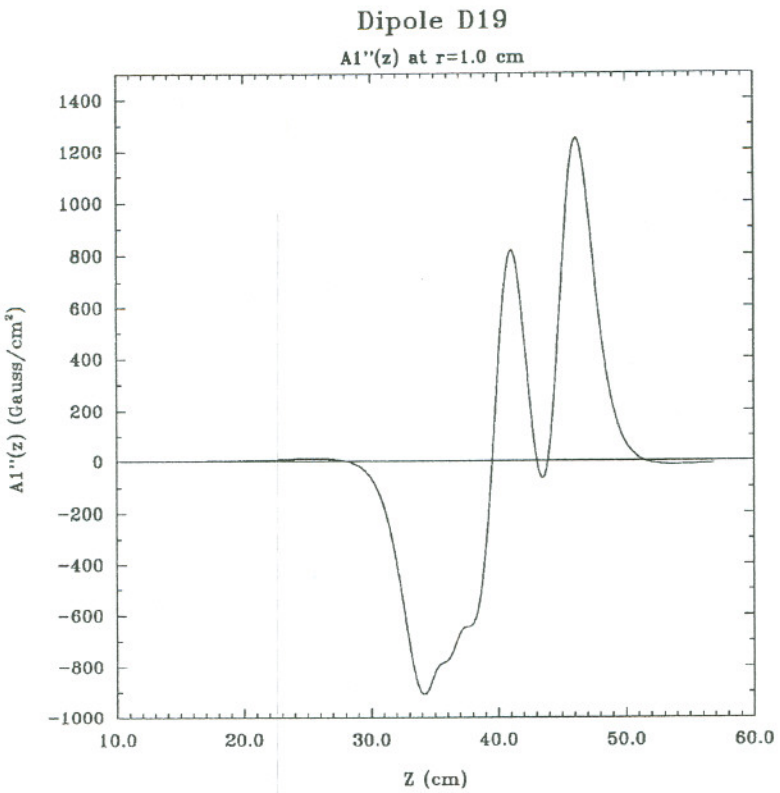

Figure 12 The second derivative function of n=1 — A1"(z) — CONDUCTOR ONLY.

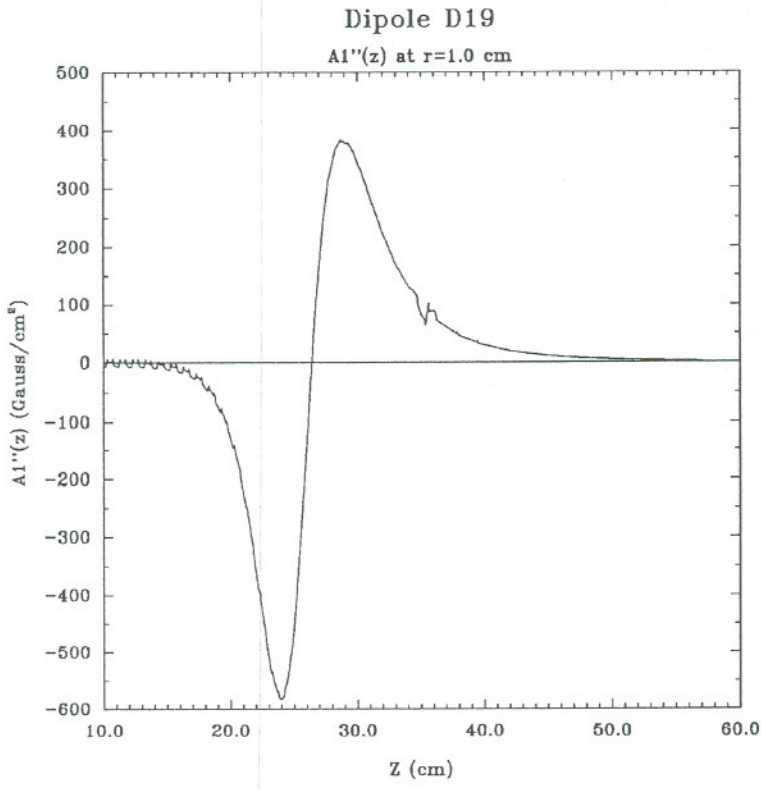

Figure 13 The second derivative function of n=1 — A1"(z) — IRON ONLY.

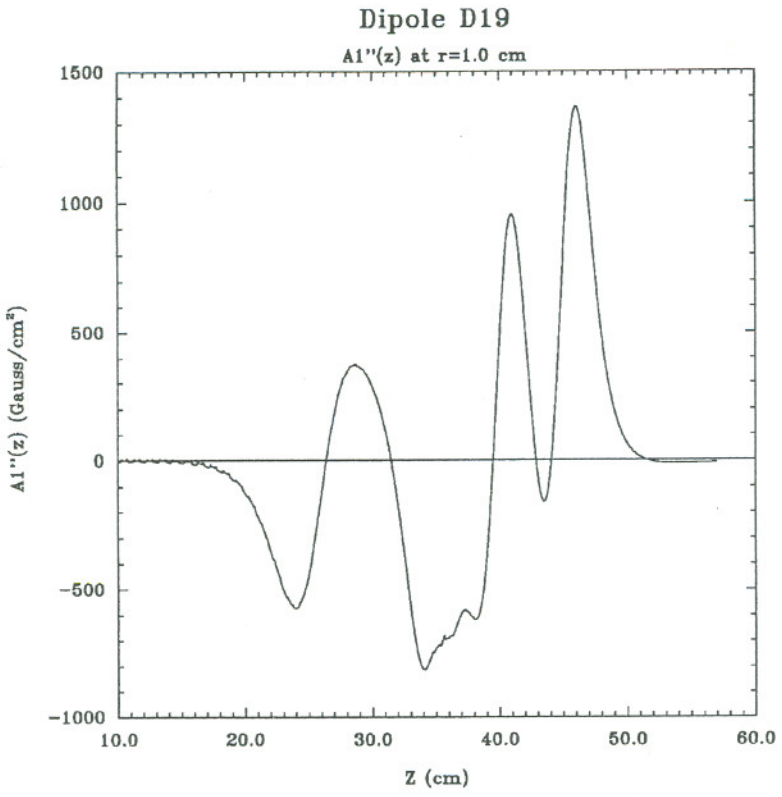

Figure 14 The second derivative function of n=1 — A1"(z) — CONDUCTOR and IRON.

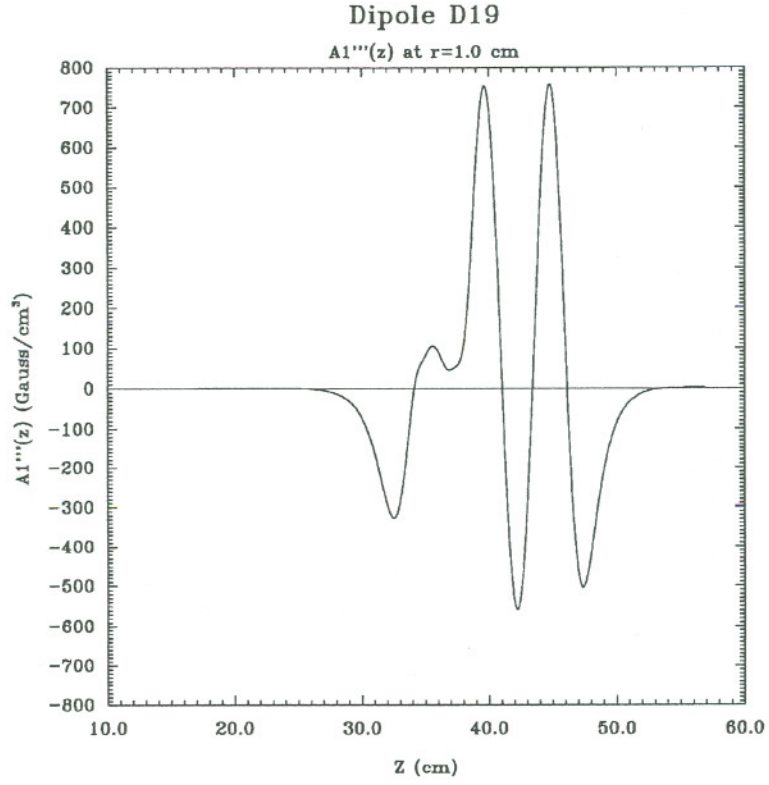

Figure 15 The 3–rd derivative function of n=1 — A1'"(z) — CONDUCTOR ONLY.

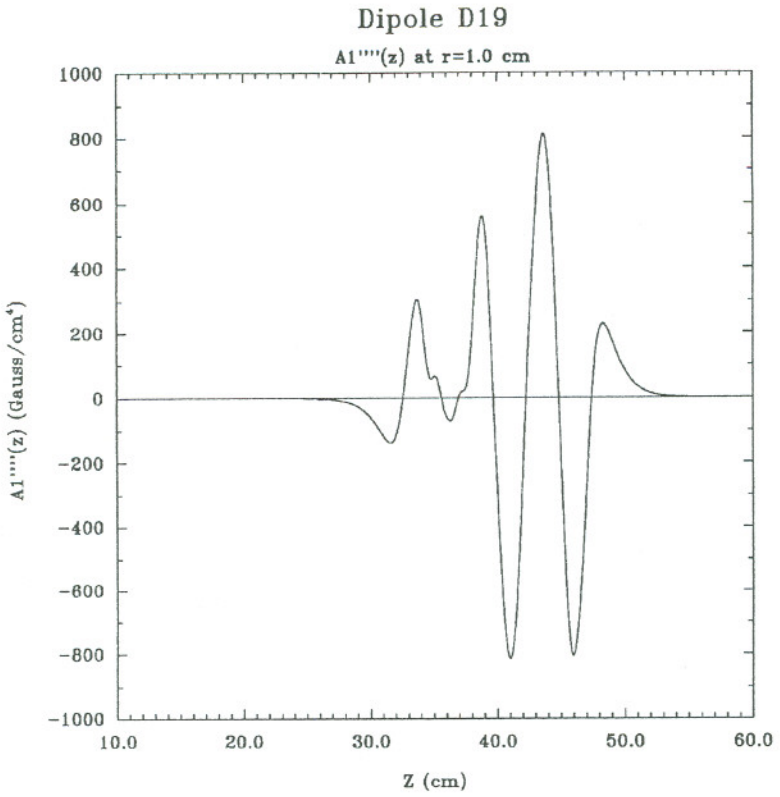

Figure 16 The fourth derivative function of n=1 — A1''''(z) — CONDUCTOR ONLY.

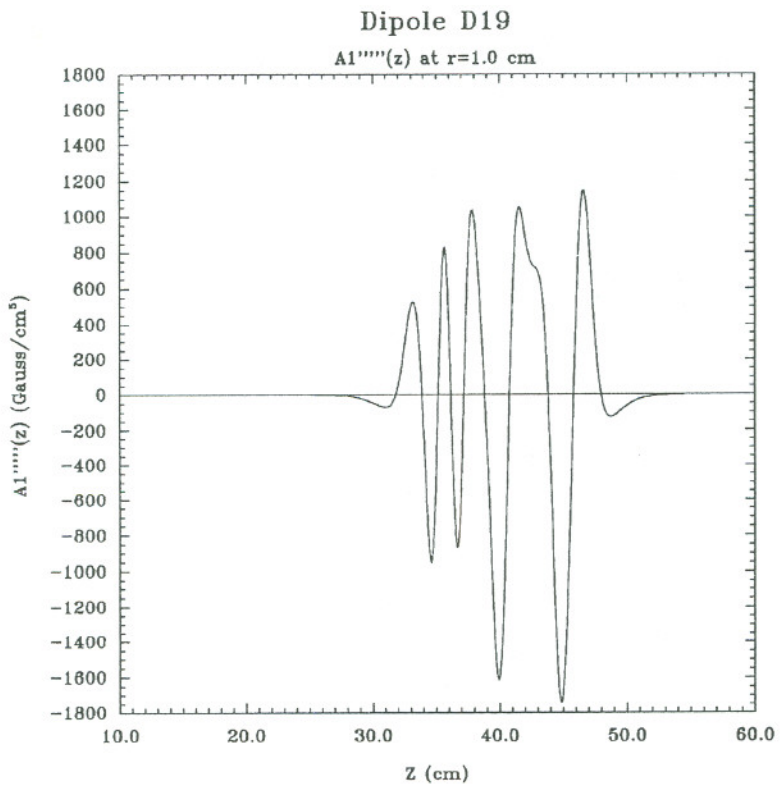

Figure 17 The fifth derivative function of n=1 — A1'''''(z) — CONDUCTOR ONLY.

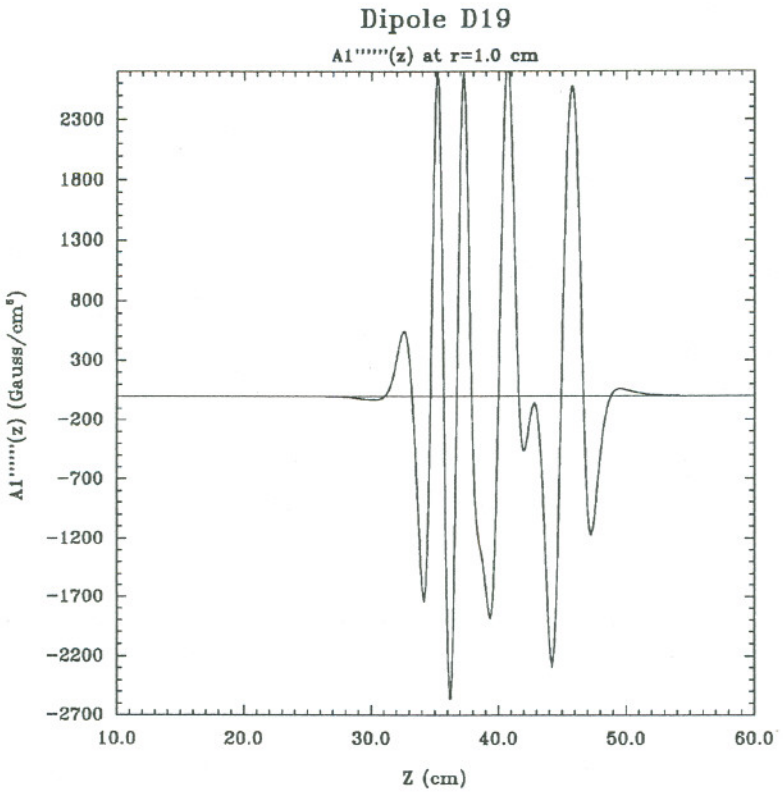

Figure 18 The sixth derivative function of n=1 — A1''''''(z) — CONDUCTOR ONLY.

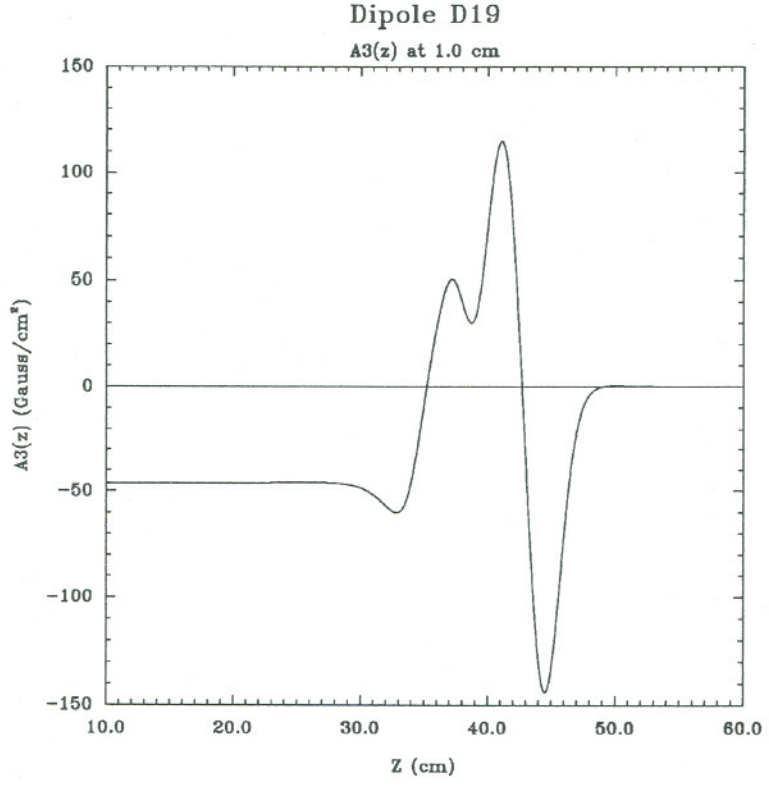

Figure 19 The sextupole function A3(z) — CONDUCTOR ONLY.

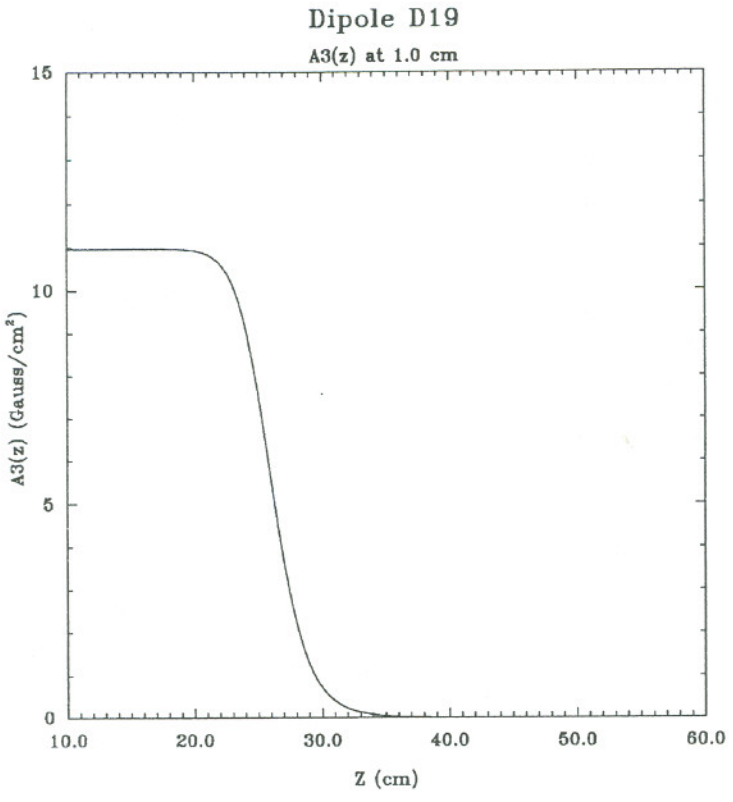

Figure 20 The sextupole function A3(z) — IRON ONLY.

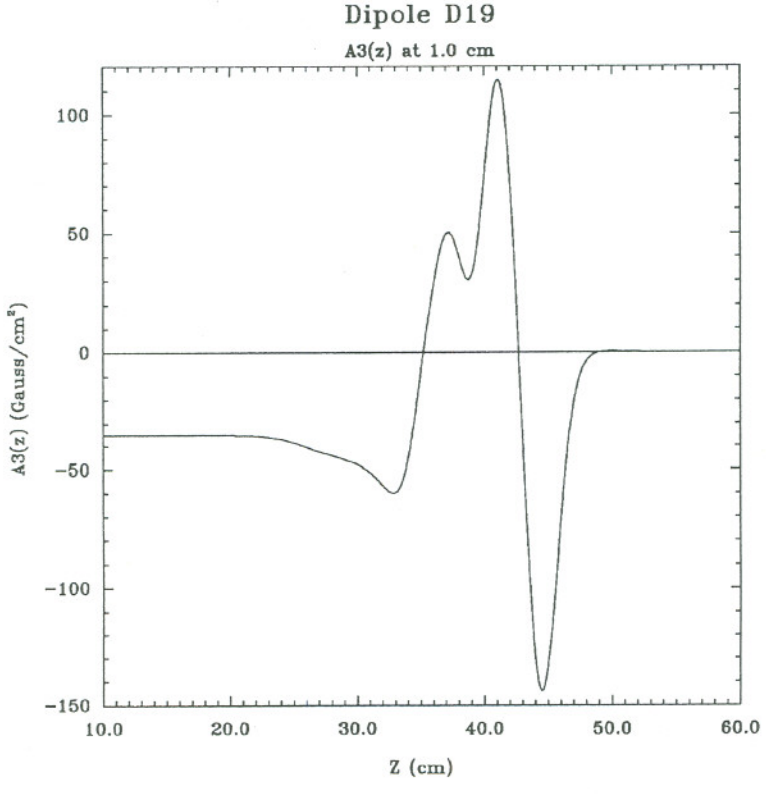

Figure 21 The sextupole function A3(z) — CONDUCTOR and IRON.

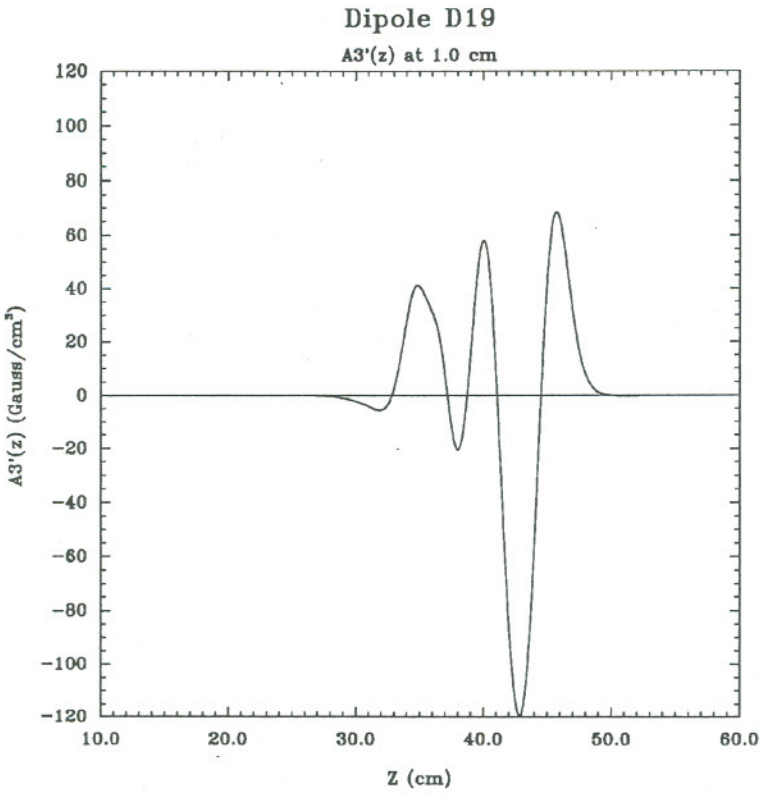

Figure 22 The first derivative function of n=3 — A3'(z) — CONDUCTOR ONLY.

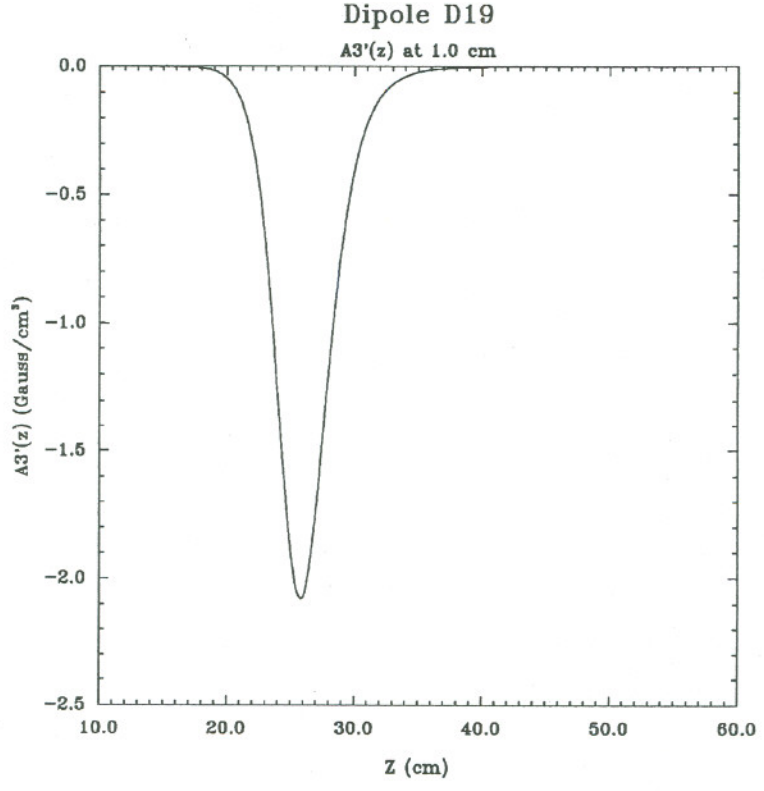

Figure 23 The first derivative function of n=3 — A3'(z) — IRON ONLY.

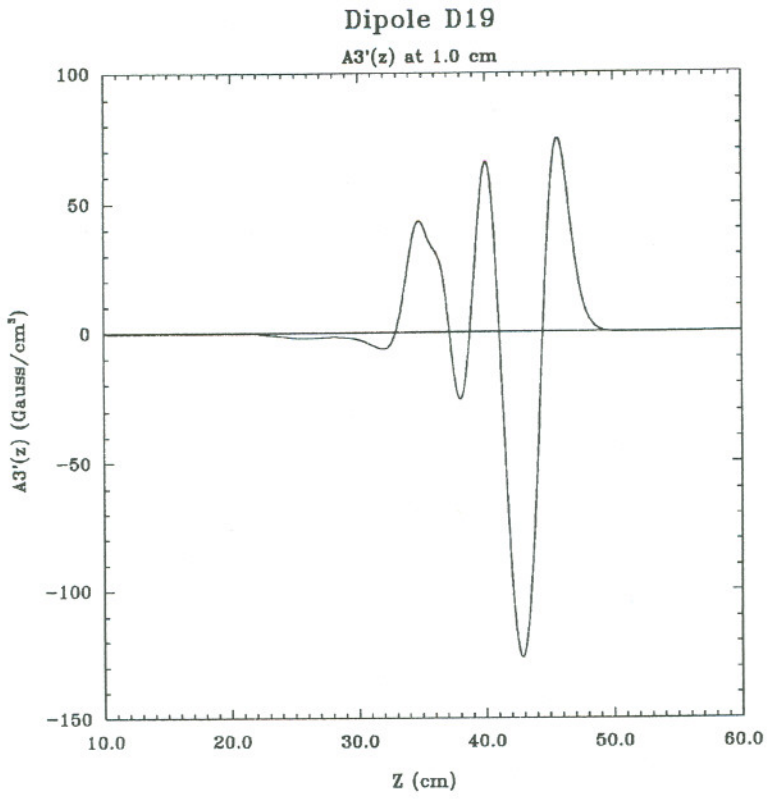

Figure 24 The first derivative function of n=3 — A3'(z) — CONDUCTOR and IRON.

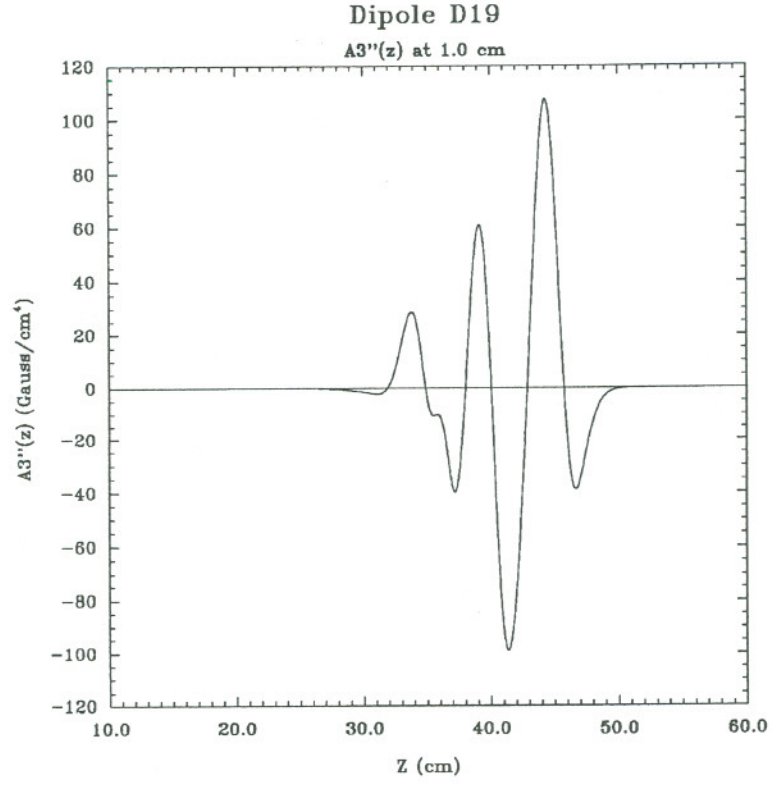

Figure 25 The second derivative function of n=3 — A3"(z) — CONDUCTOR ONLY.

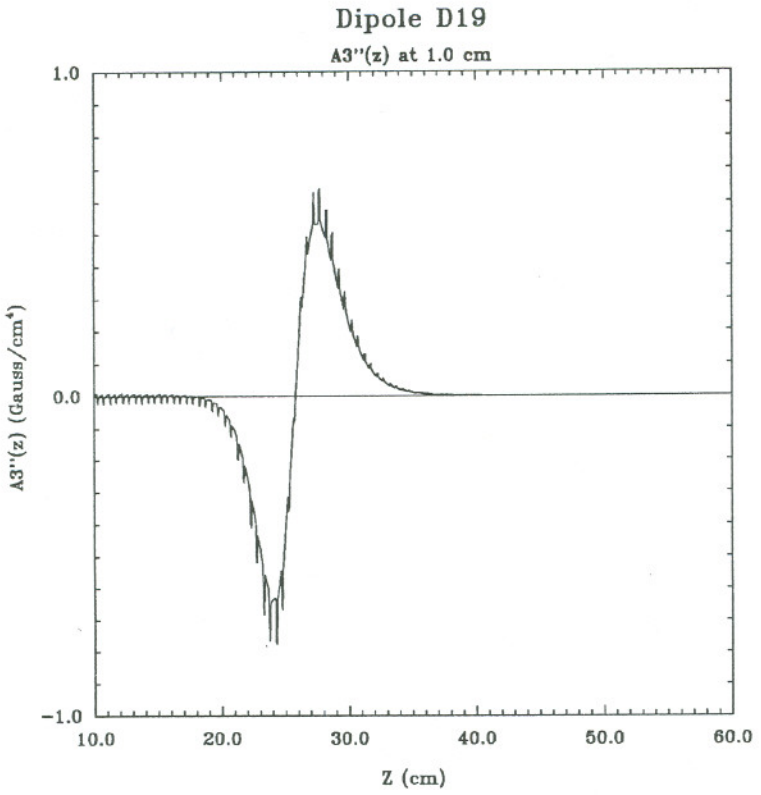

Figure 26 The second derivative function of n=3 — A3"(z) — IRON ONLY.

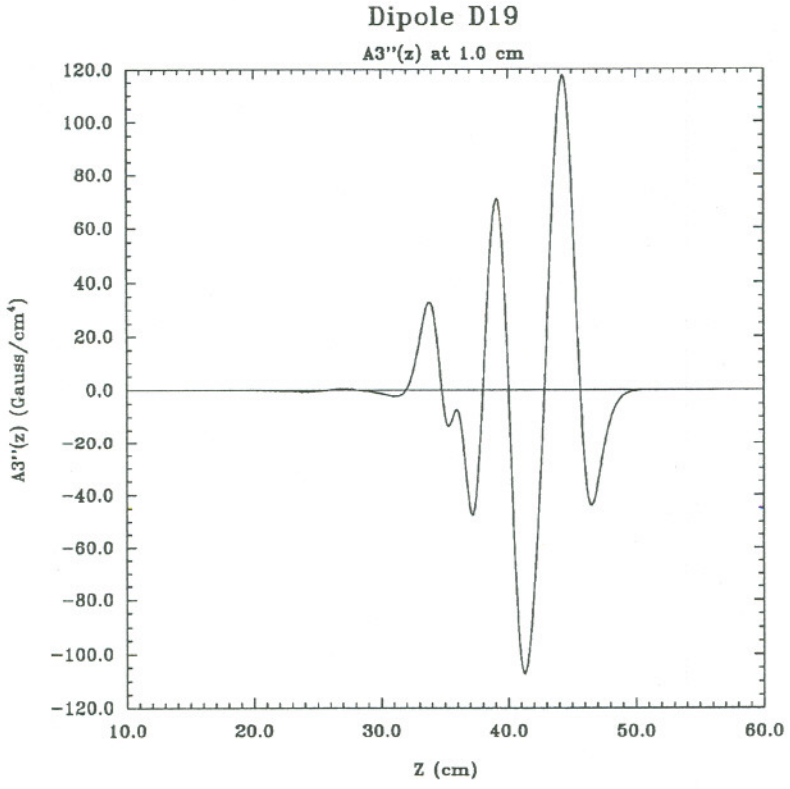

Figure 27 The second derivative function of n=3 — A3"(z) — CONDUCTOR and IRON.

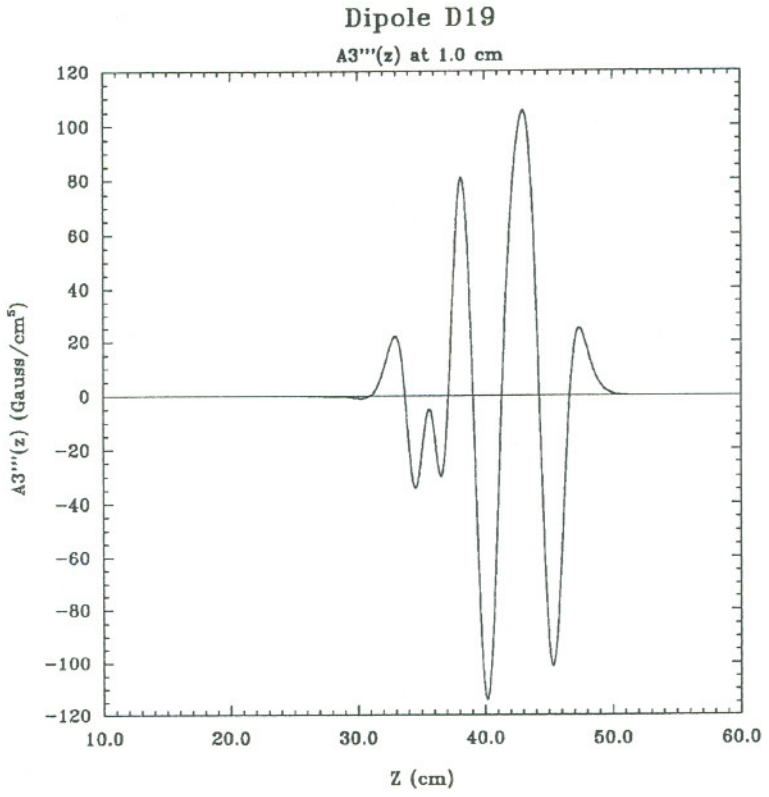

Figure 28 The 3–rd derivative function of n=3 — A3'''(z) — CONDUCTOR ONLY.

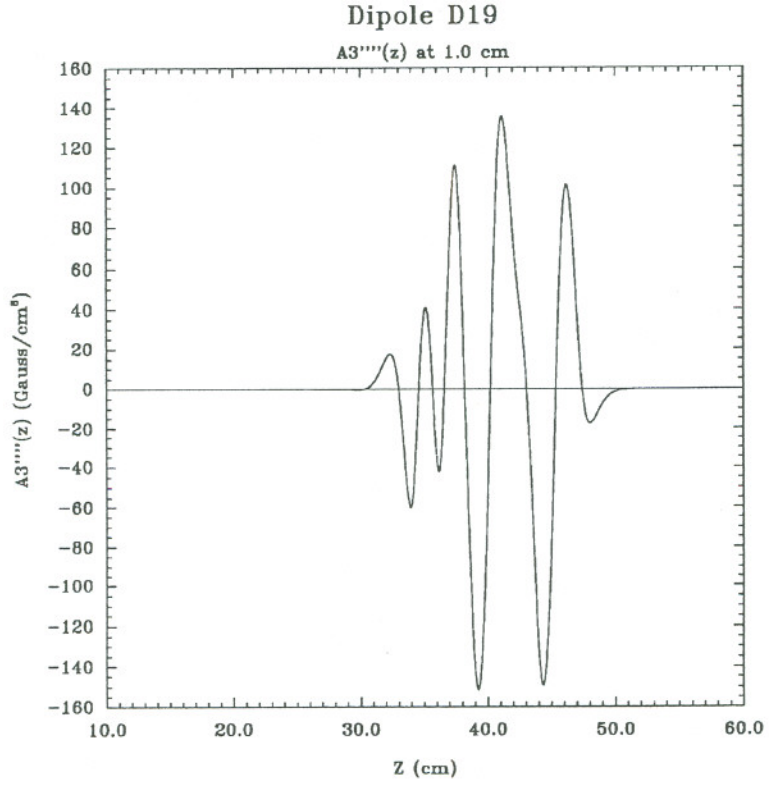

Figure 29 The fourth derivative function of n=3 — A3''''(z) — CONDUCTOR ONLY.

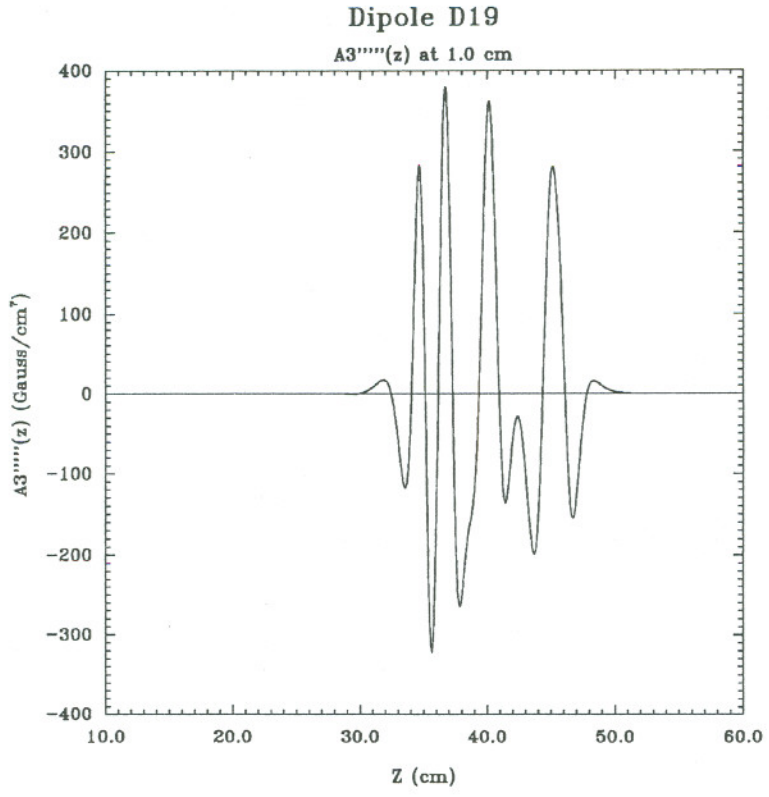

Figure 30 The fifth derivative function of n=3 — A3''''''(z) — CONDUCTOR ONLY.

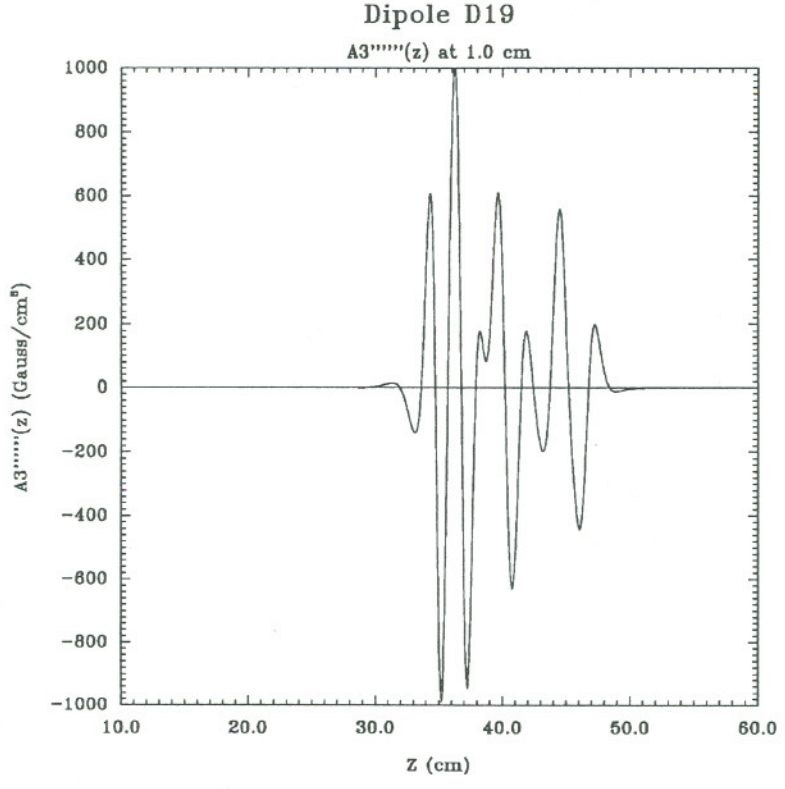

Figure 31 The sixth derivative function of n=3 — A3''''''(z) — CONDUCTOR ONLY.

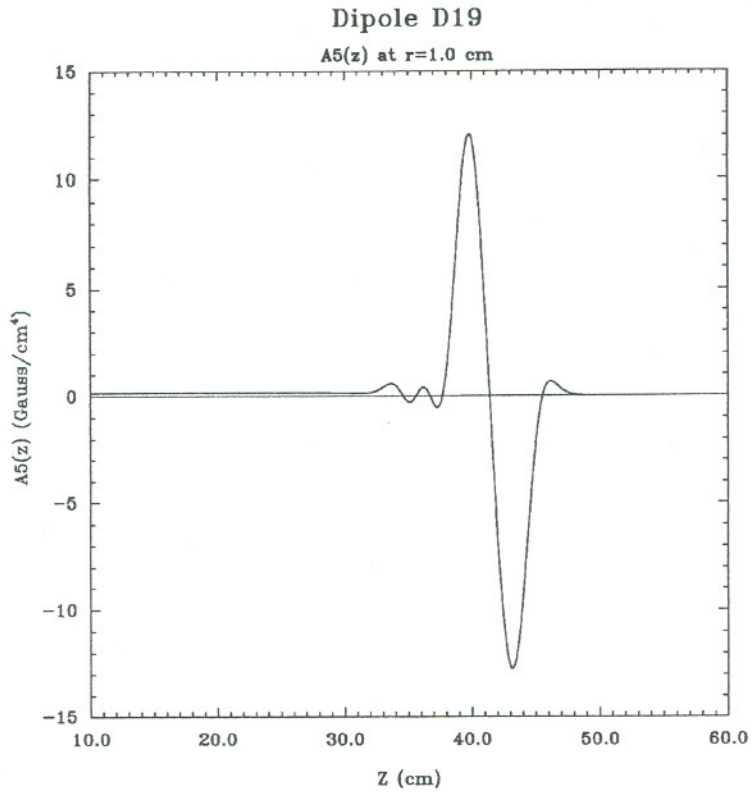

Figure 32 The decapole function A5(z) — CONDUCTOR ONLY.

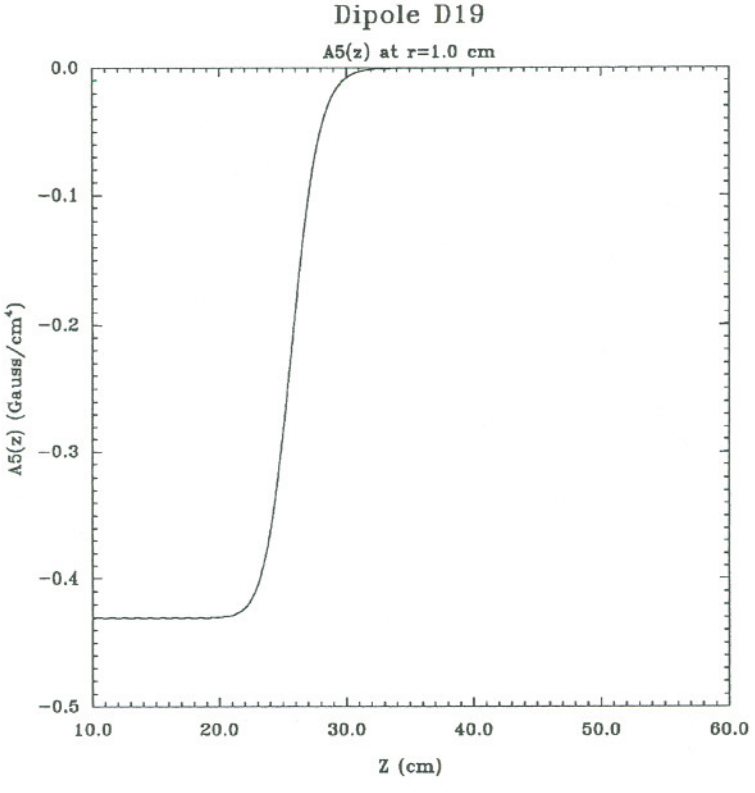

Figure 33 The decapole function A5(z) — IRON ONLY.

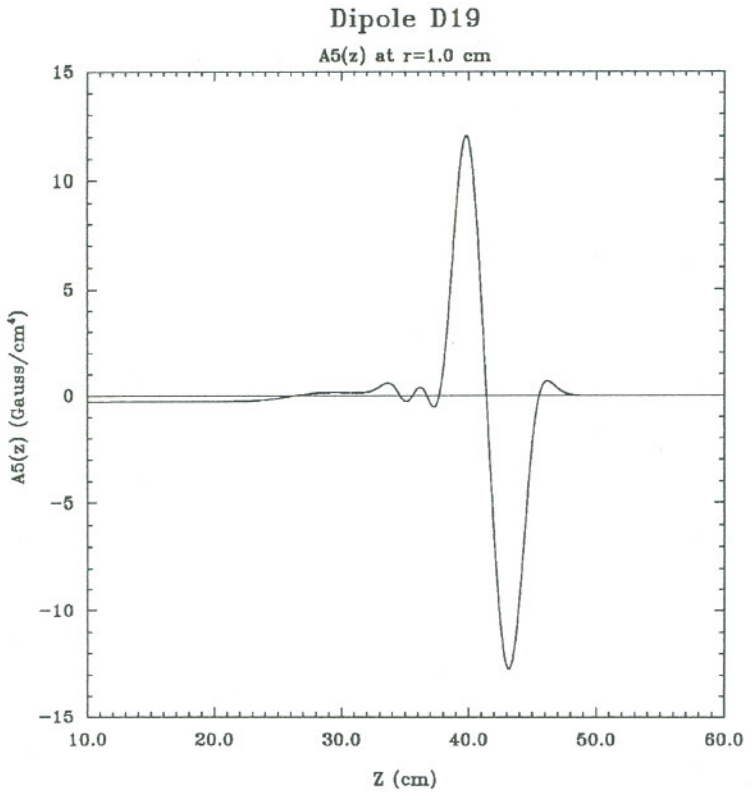

Figure 34 The decapole function A5(z) — CONDUCTOR and IRON.

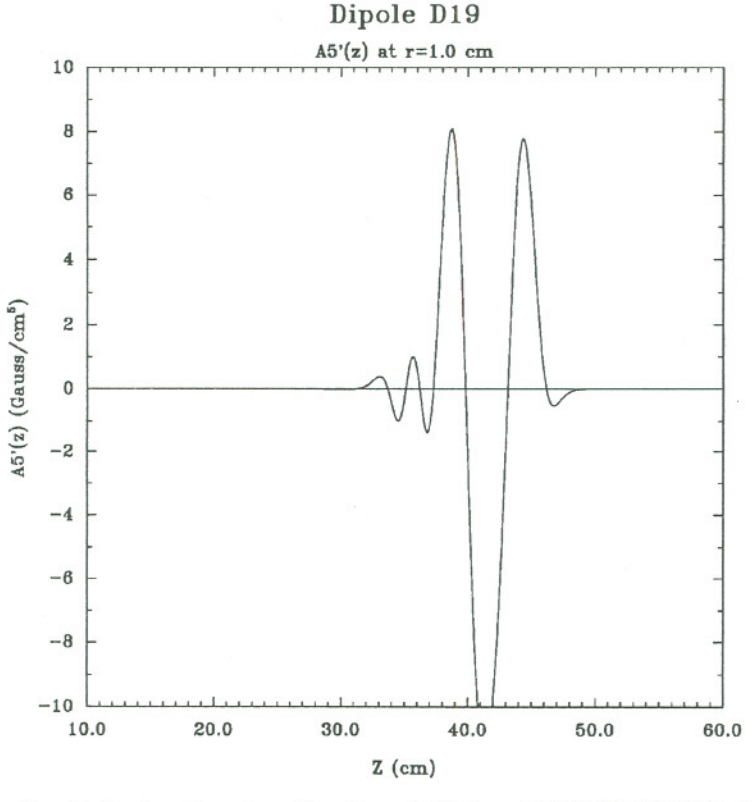

Figure 35 The first derivative function of n=5 — A5'(z) — CONDUCTOR ONLY.

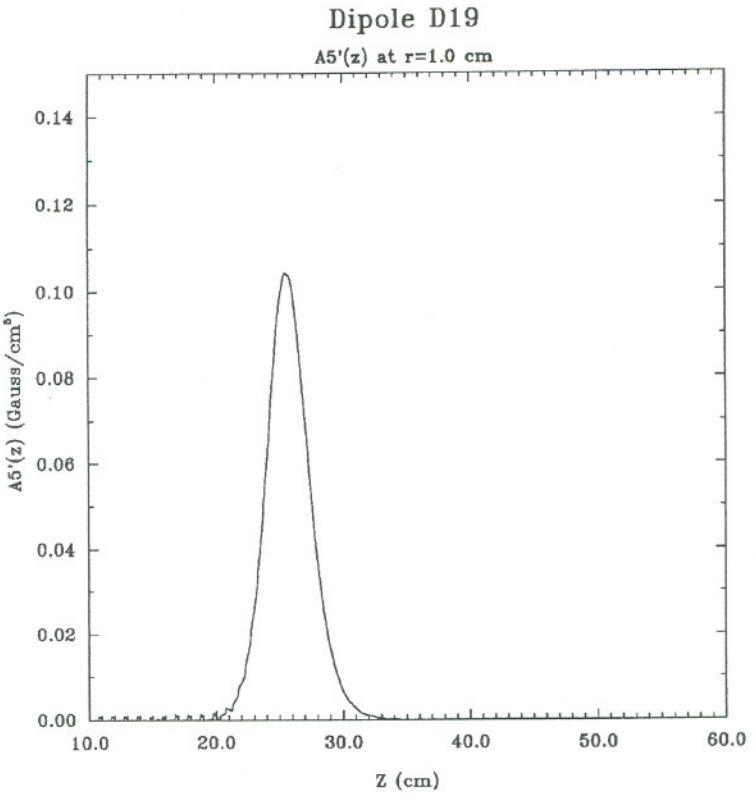

Figure 36 The first derivative function of n=5 — A5'(z) — IRON ONLY.

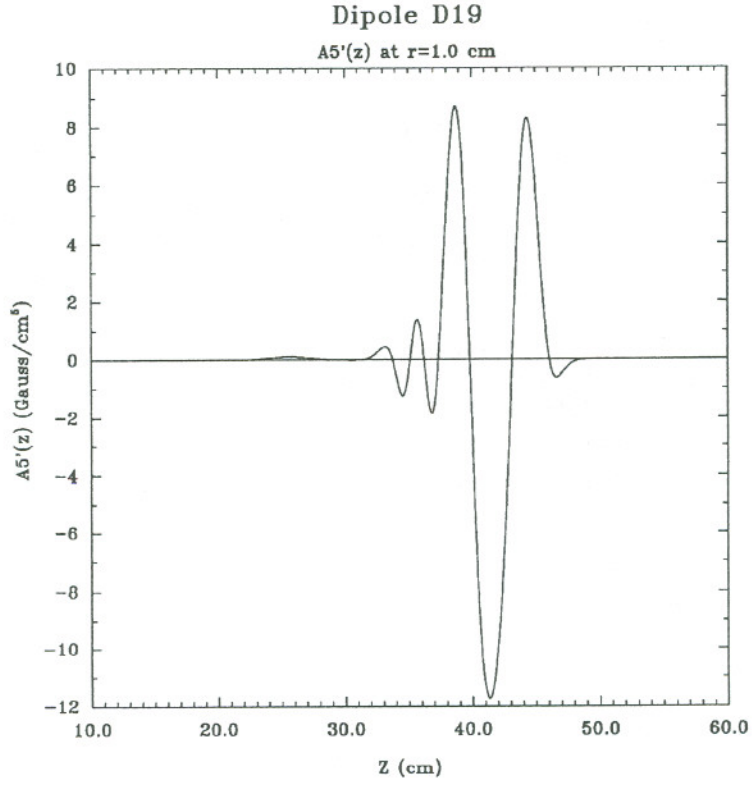

Figure 37 The first derivative function of n=5 — A5'(z) — CONDUCTOR and IRON.

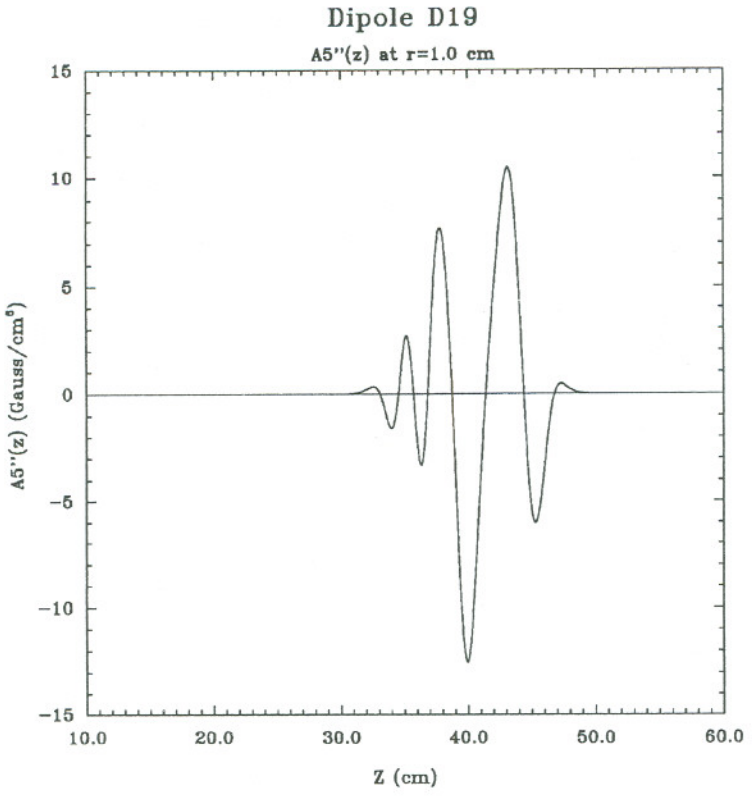

Figure 38 The second derivative function of n=5 — A5"(z) — CONDUCTOR ONLY.

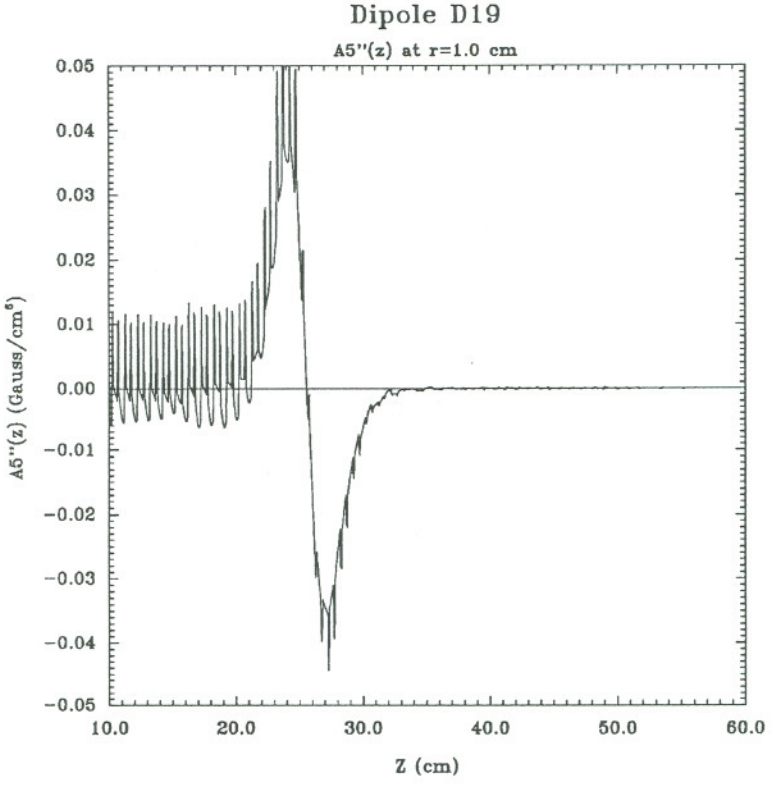

Figure 39 The second derivative function of n=5 — A5"(z) — IRON ONLY.

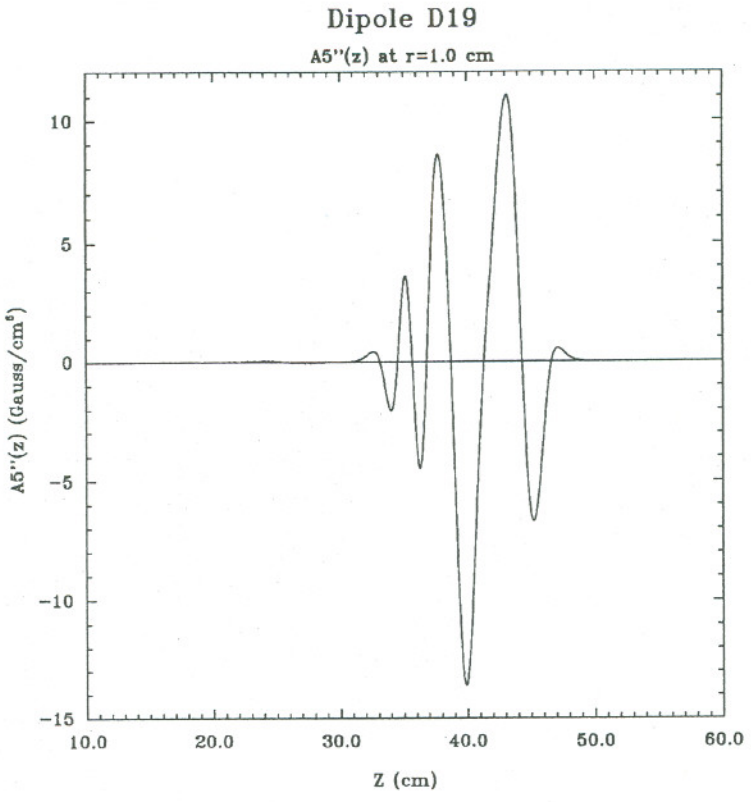

Figure 40 The second derivative function of n=5 — A5"(z) — CONDUCTOR and IRON.

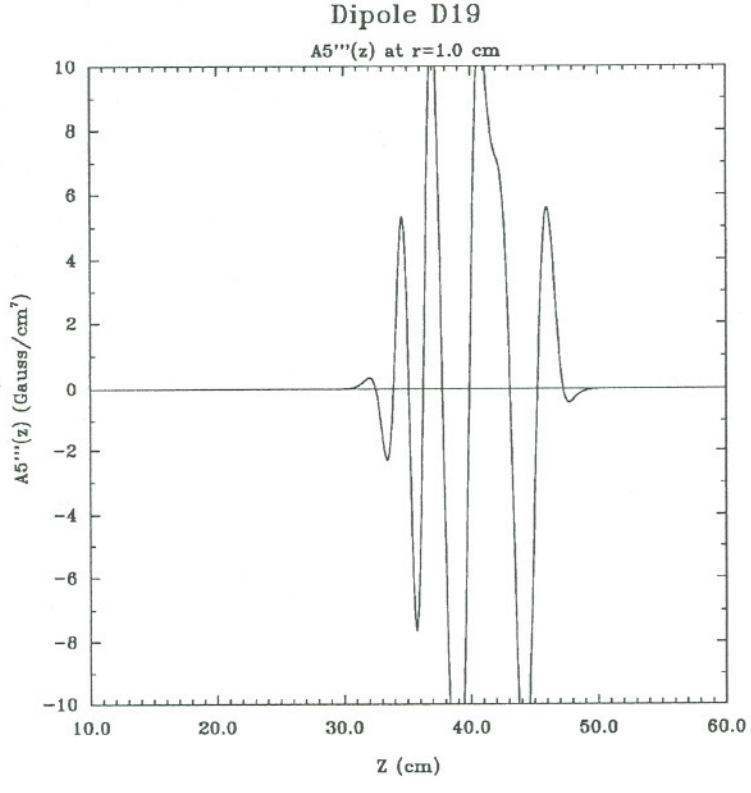

Figure 41 The 3–rd derivative function of n=5 — A5"'(z) — CONDUCTOR ONLY.

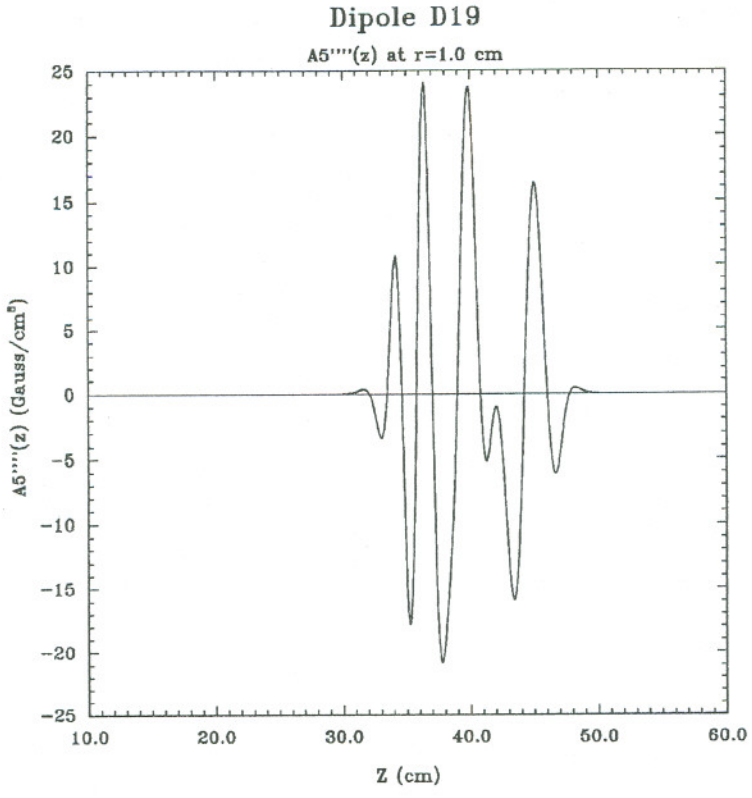

Figure 42 The fourth derivative function of n=5 — A5''''(z) — CONDUCTOR ONLY.

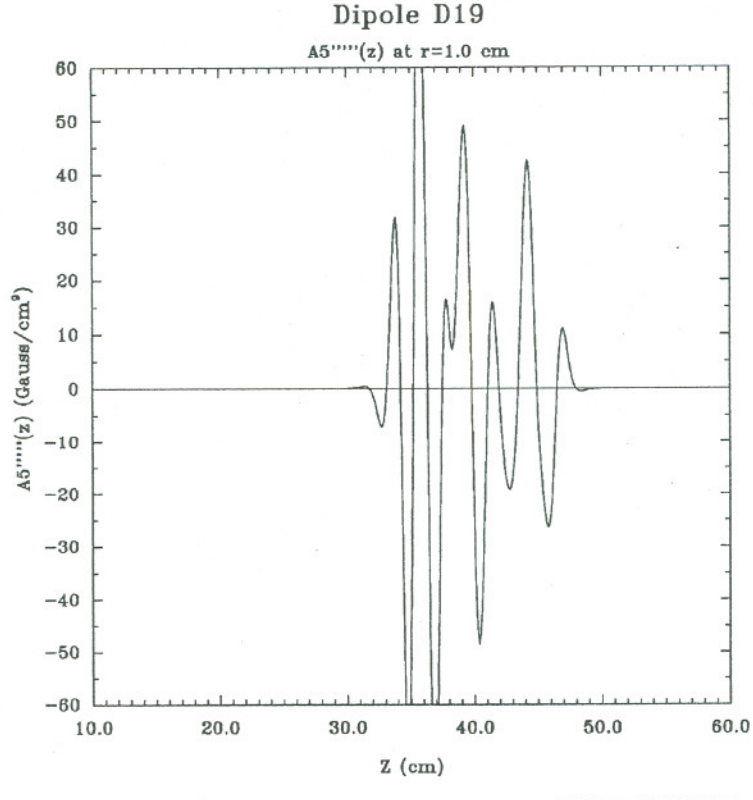

Figure 43 The fifth derivative function of n=5 — A5'''''(z) — CONDUCTOR ONLY.

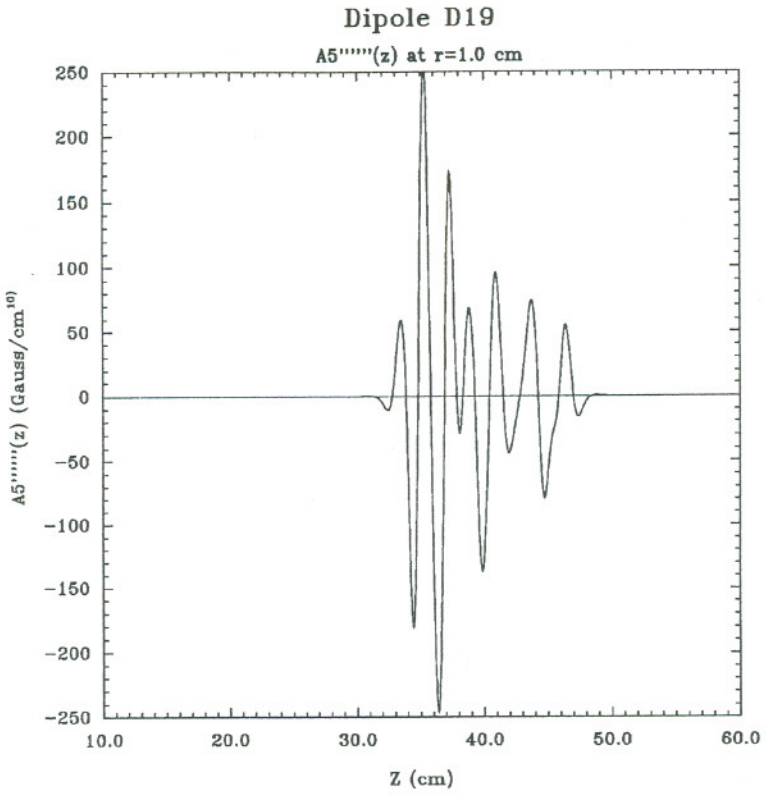

Figure 44 The sixth derivative function of n=5 — A5''''''(z) — CONDUCTOR ONLY.

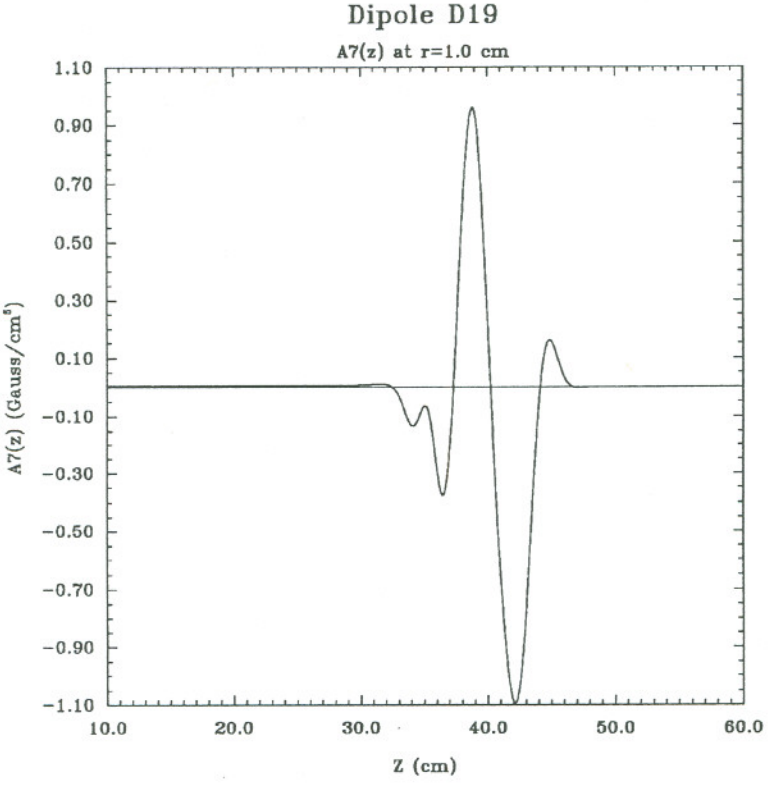

Figure 45 The 14 pole function A7(z) — CONDUCTOR ONLY.

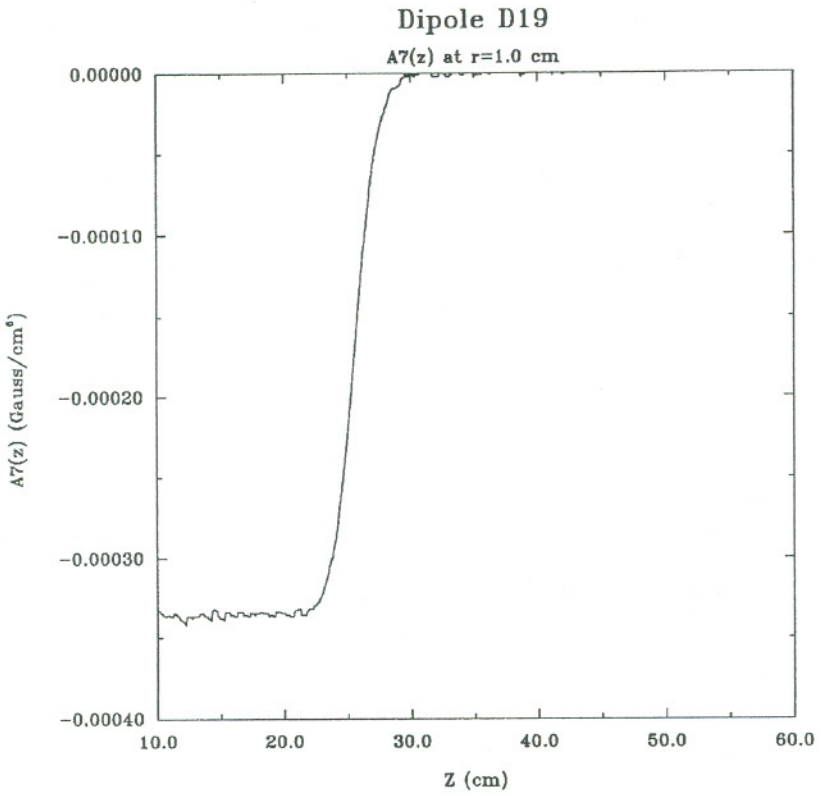

Figure 46 The 14 pole function A7(z) — IRON ONLY.

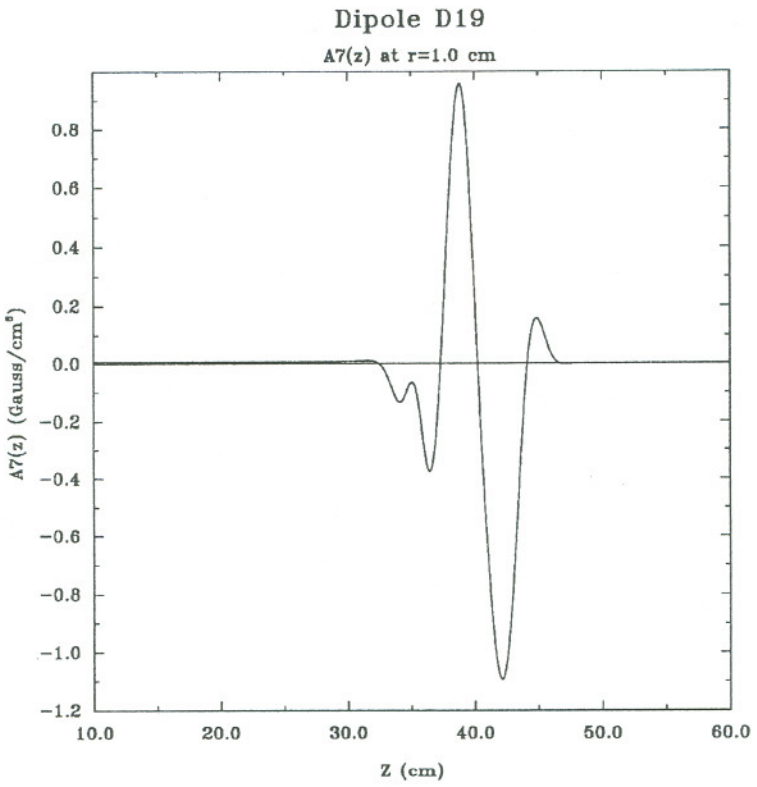

Figure 47 The 14 pole function A7(z) — CONDUCTOR and IRON.

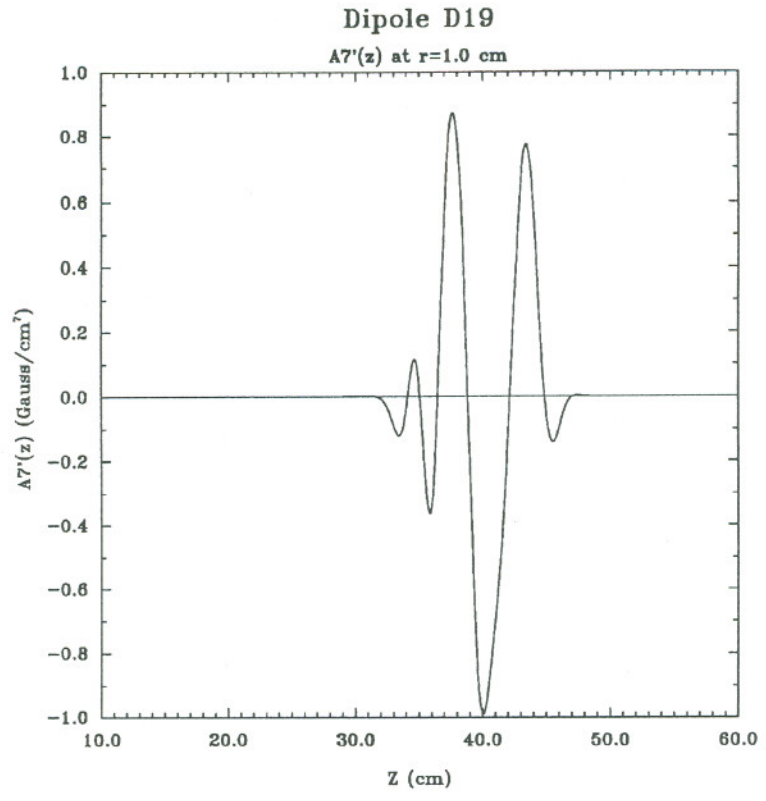

Figure 48 The first derivative function of n=7 — A7'(z) — CONDUCTOR ONLY.

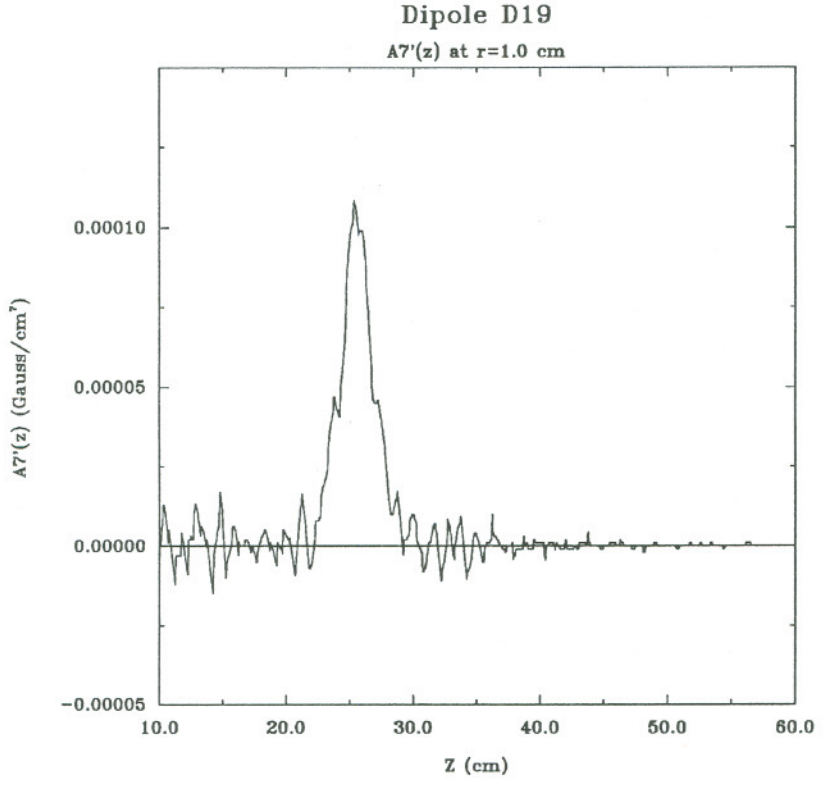

Figure 49 The first derivative function of n=7 — A7'(z) — IRON ONLY.

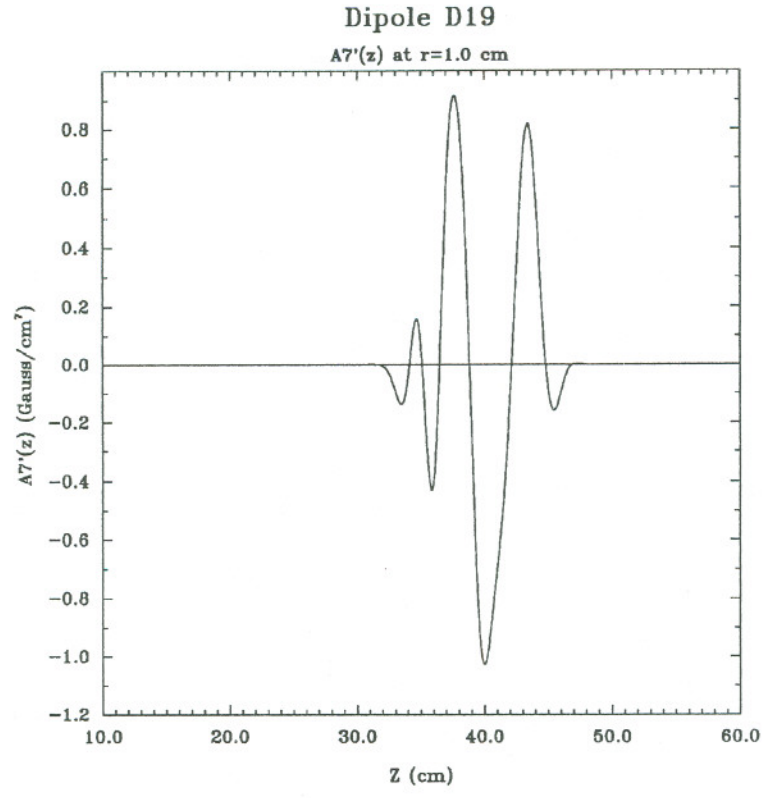

Figure 50 The first derivative function of n=7 — A7'(z) — CONDUCTOR and IRON.

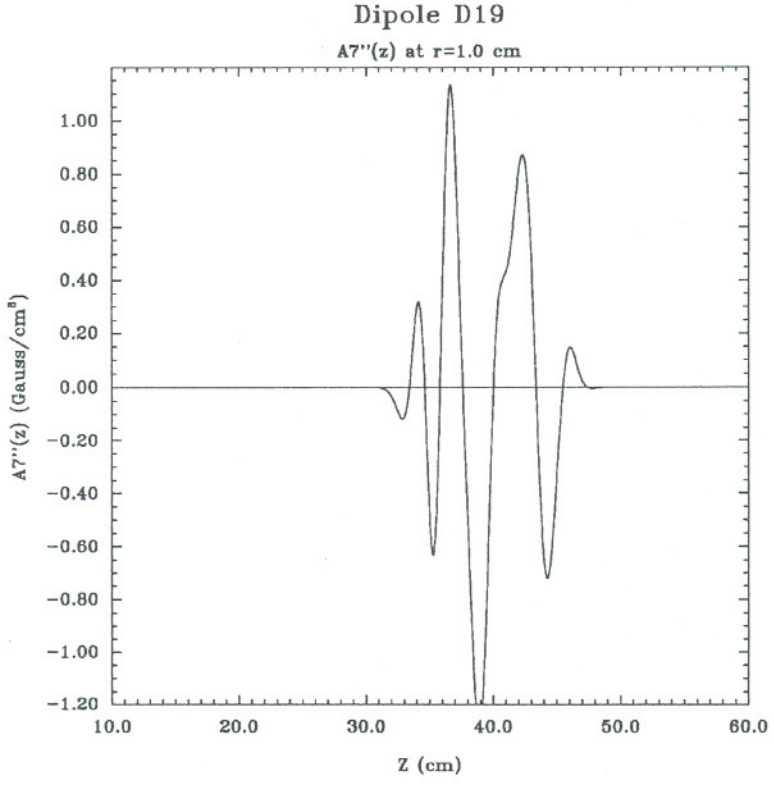

Figure 51 The second derivative function of n=7 — A7''(z) — CONDUCTOR ONLY.

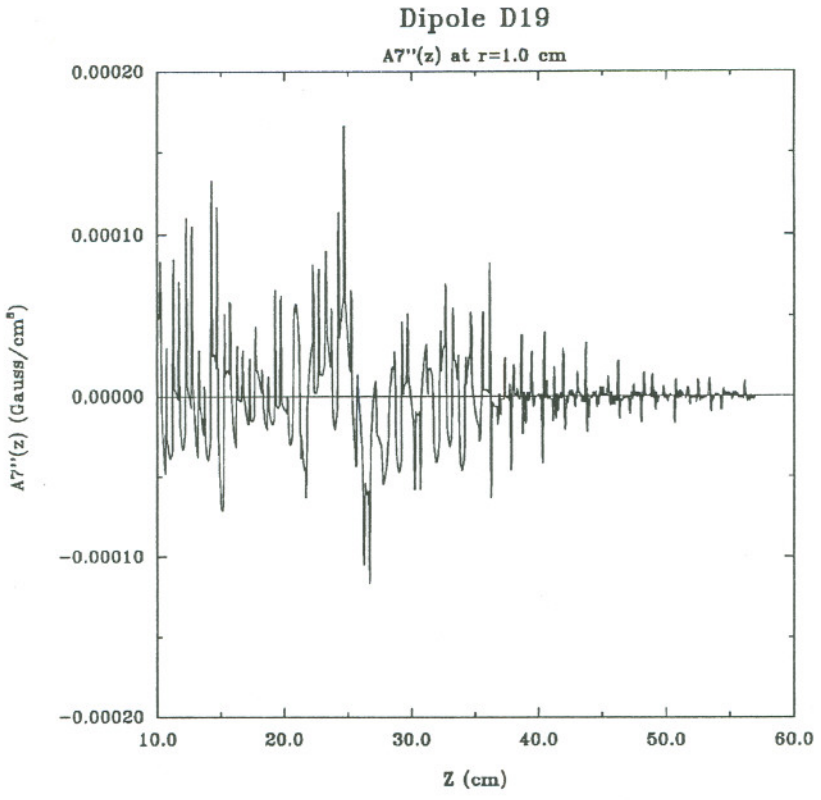

Figure 52 The second derivative function of n=7 — A7"(z) — IRON ONLY.

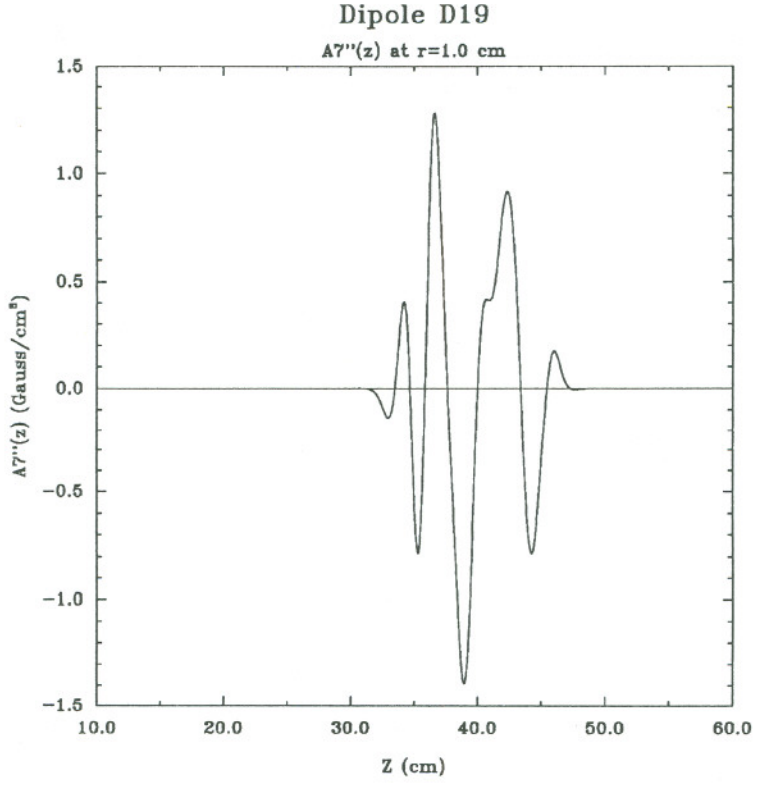

Figure 53 The second derivative function of n=7 — A7"(z) — CONDUCTOR and IRON.

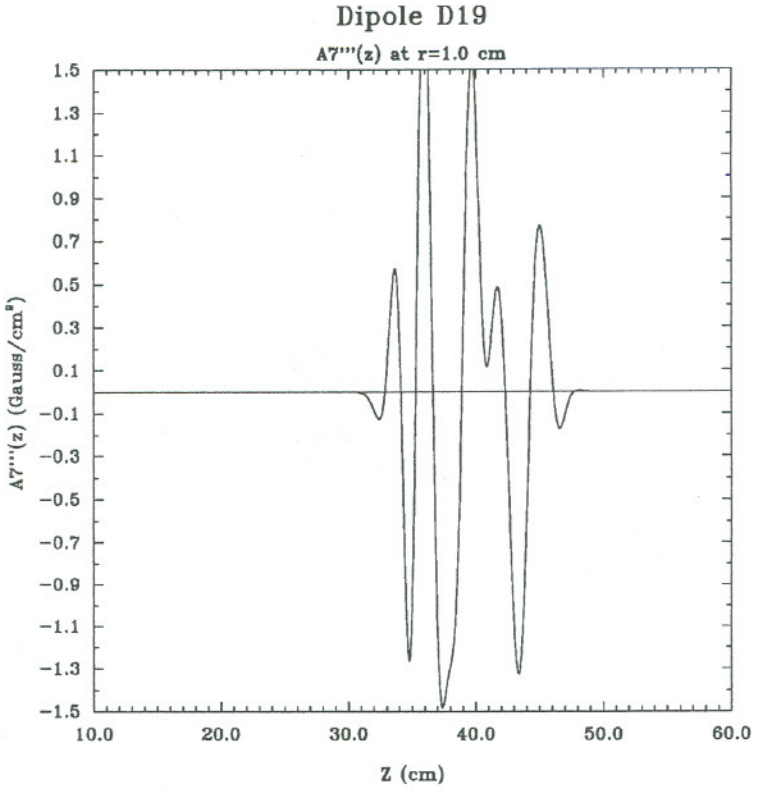

Figure 54 The 3–rd derivative function of n=7 — A7'''(z) — CONDUCTOR ONLY.

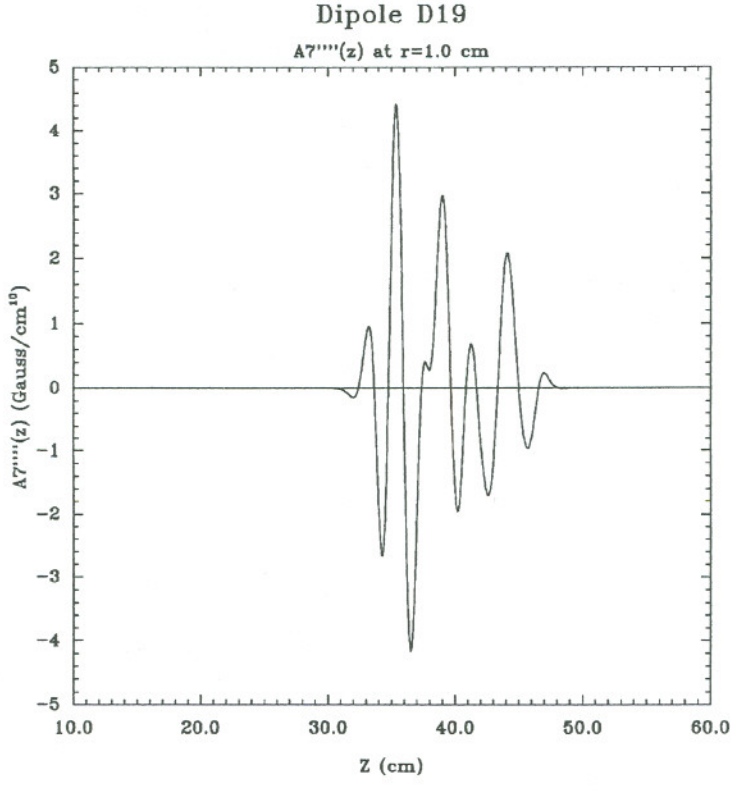

Figure 55 The fourth derivative function of n=7 — A7''''(z) — CONDUCTOR ONLY.

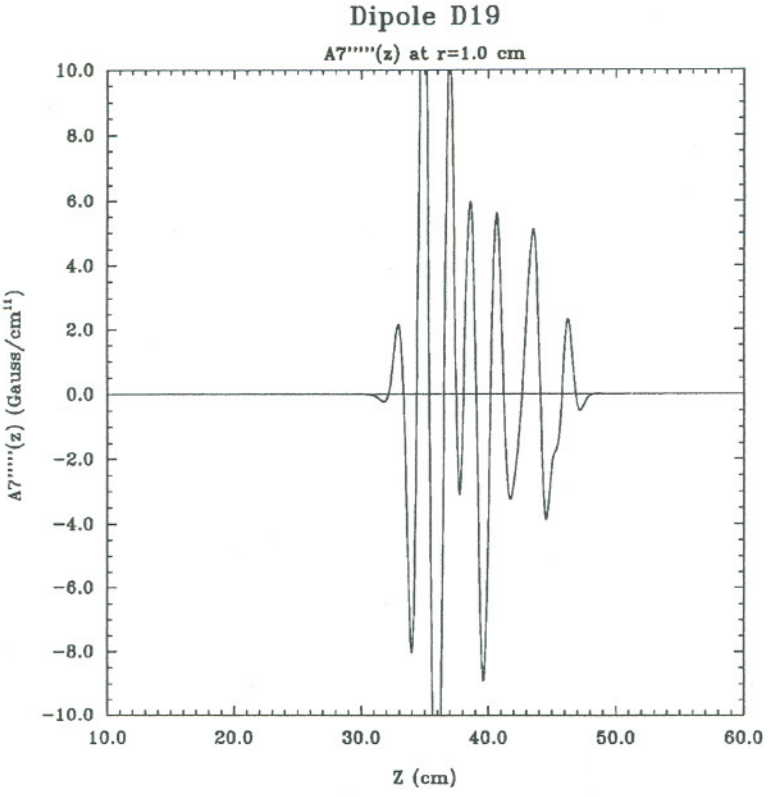

Figure 56 The fifth derivative function of n=7 — A7''''''(z) — CONDUCTOR ONLY.

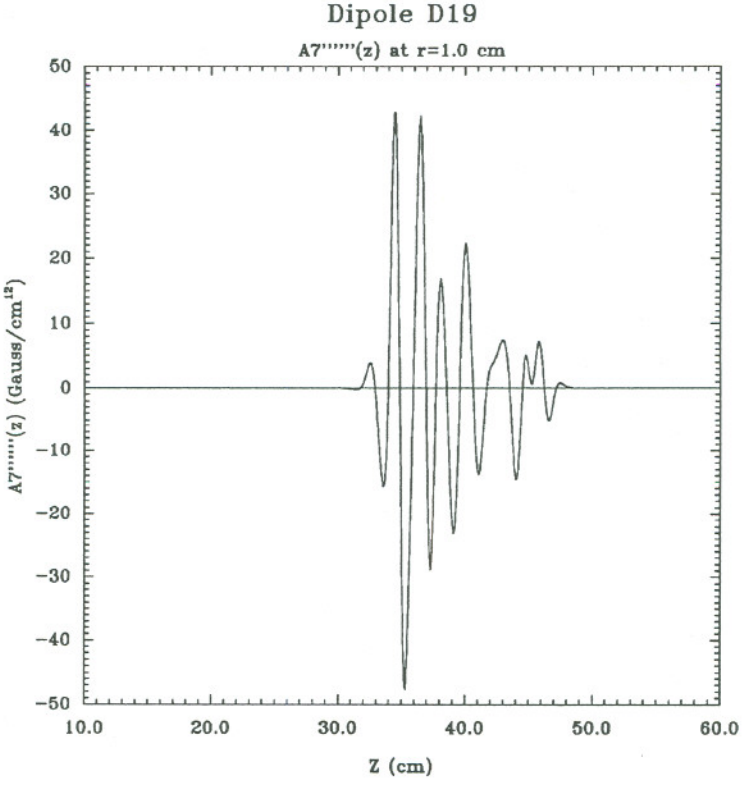

Figure 57 The sixth derivative function of n=7 — A7''''''(z) — CONDUCTOR ONLY.

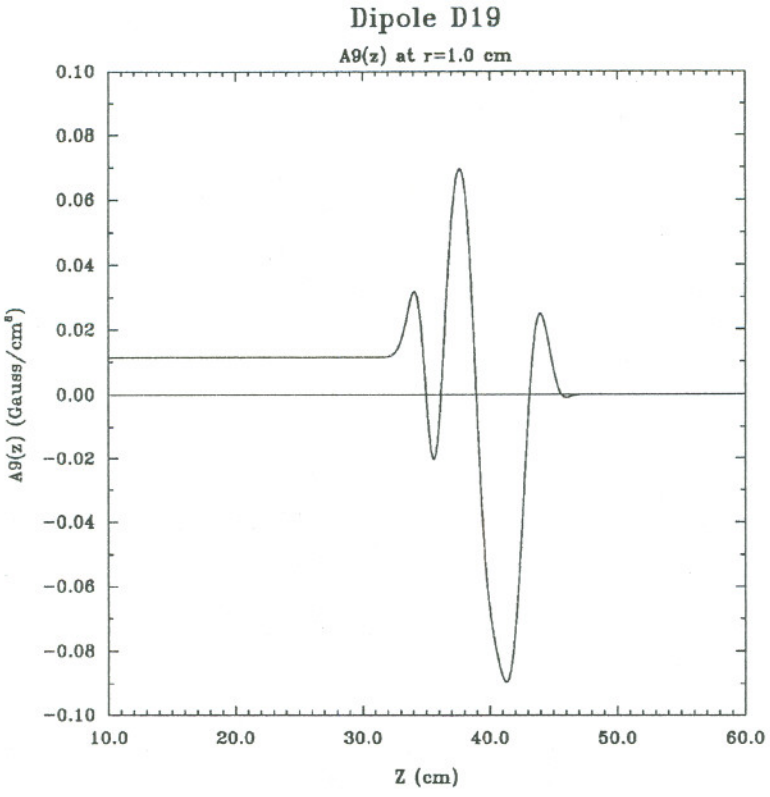

Figure 58 The 18 pole function A9(z) — CONDUCTOR ONLY.

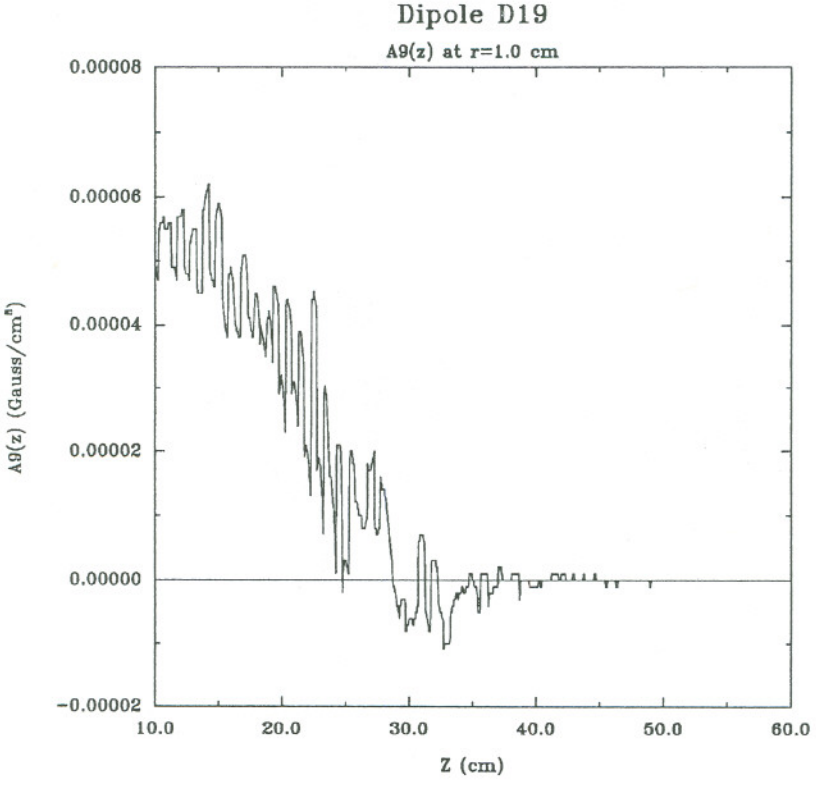

Figure 59 The 18 pole function A9(z) — IRON ONLY.

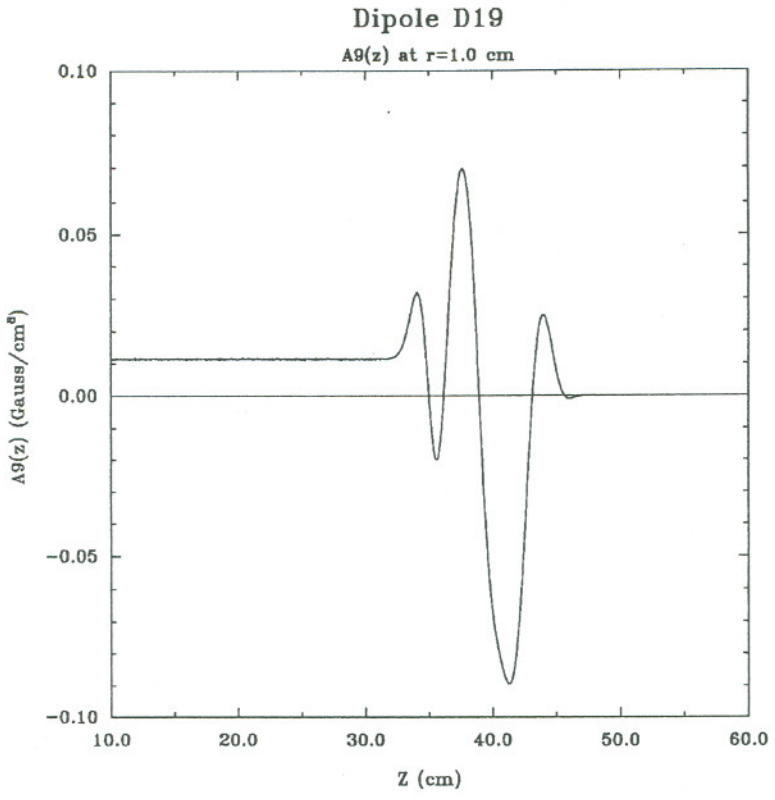

Figure 60 The 18 pole function A9(z) — CONDUCTOR and IRON.

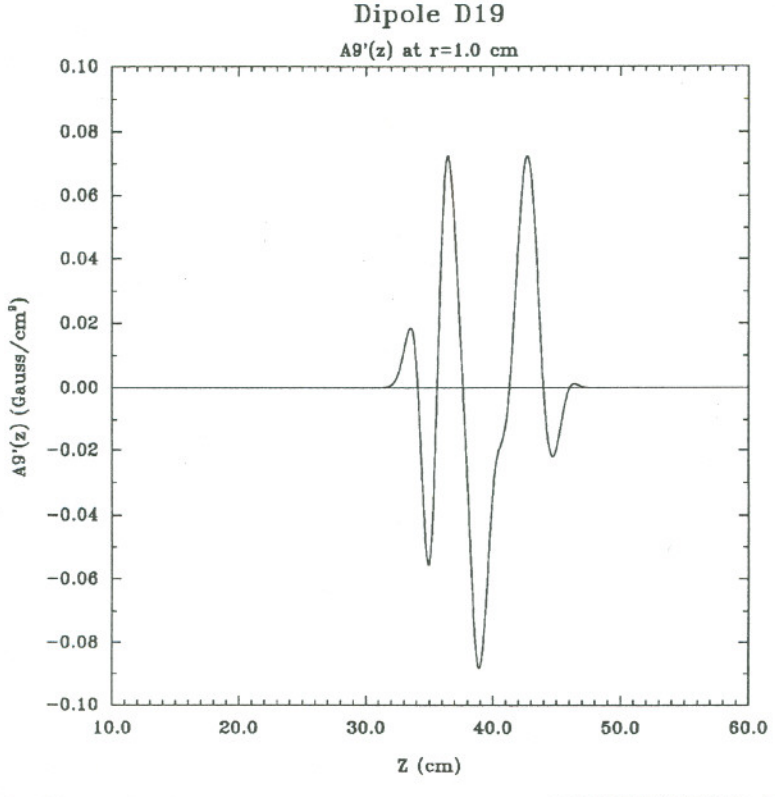

Figure 61 The first derivative function of n=9 — A9'(z) — CONDUCTOR ONLY.

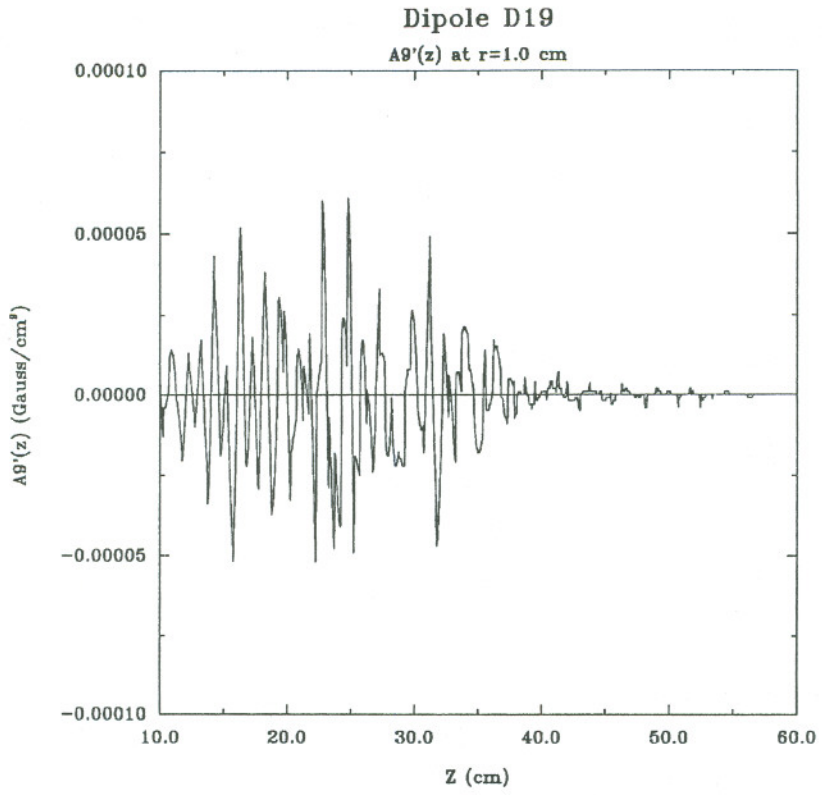

Figure 62 The first derivative function of n=9 — A9'(z) — IRON ONLY.

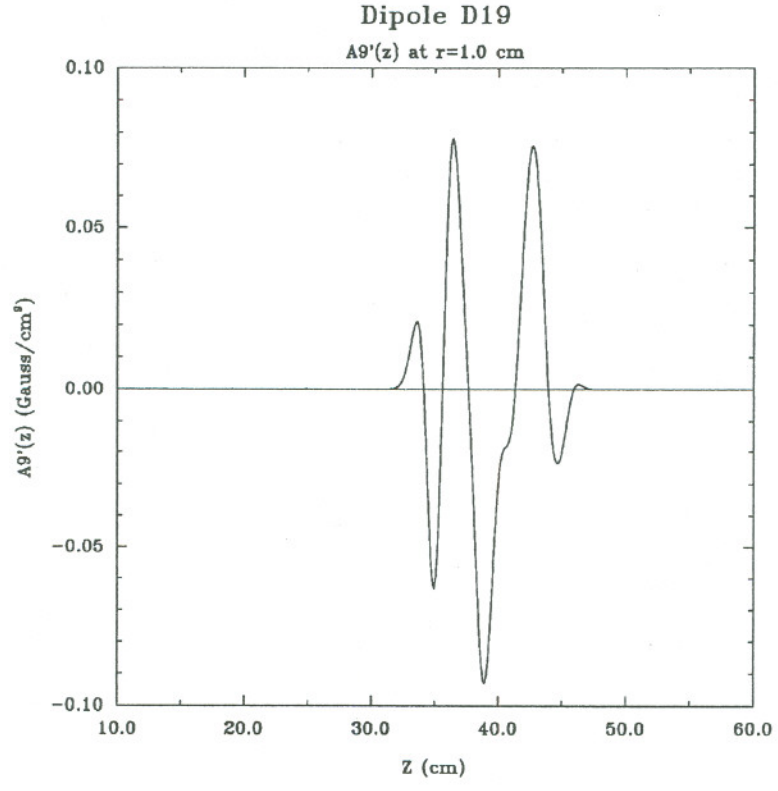

Figure 63 The first derivative function of n=9 — A9'(z) — CONDUCTOR and IRON.

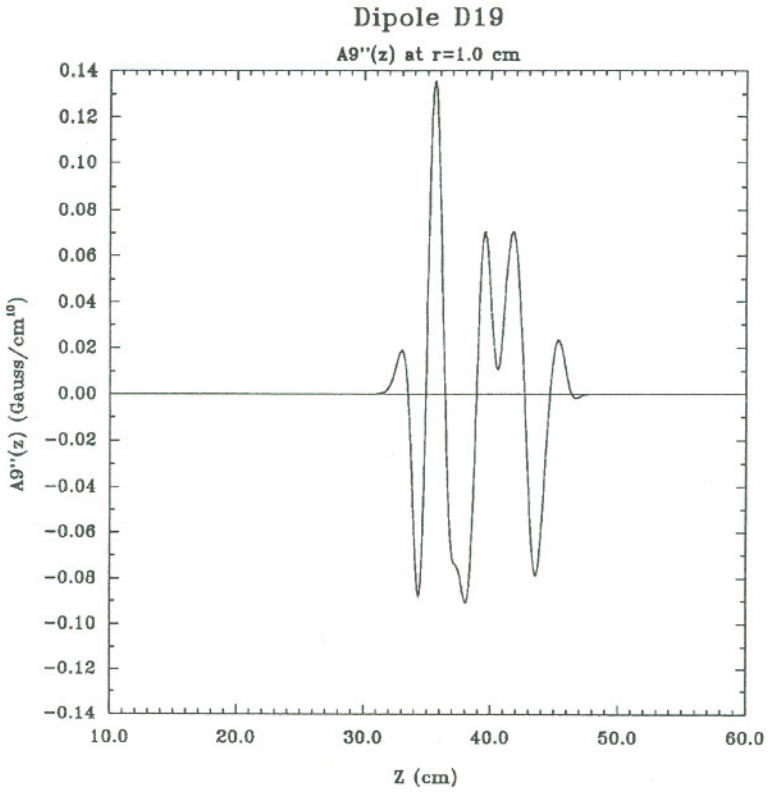

Figure 64 The second derivative function of n=9 — A9"(z) — CONDUCTOR ONLY.

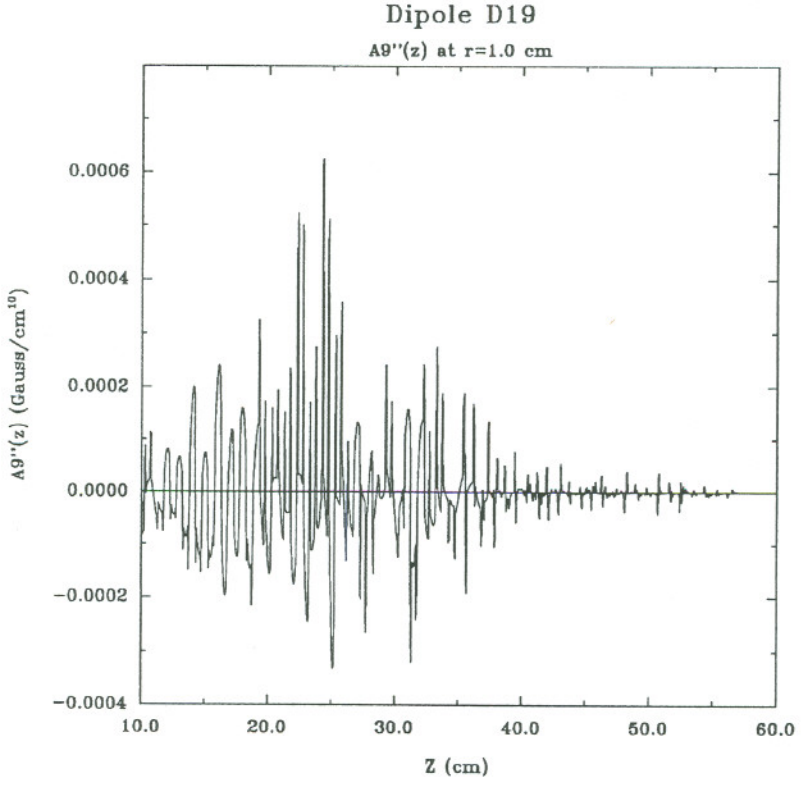

Figure 65 The second derivative function of n=9 — A9"(z) — IRON ONLY.

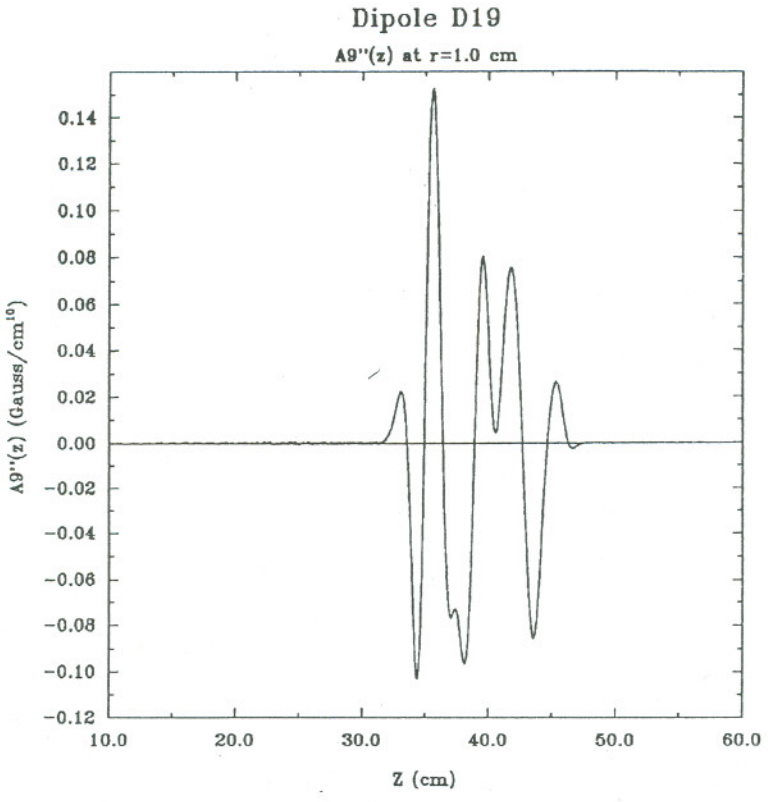

Figure 66 The second derivative function of n=9 — A9"(z) — CONDUCTOR and IRON.

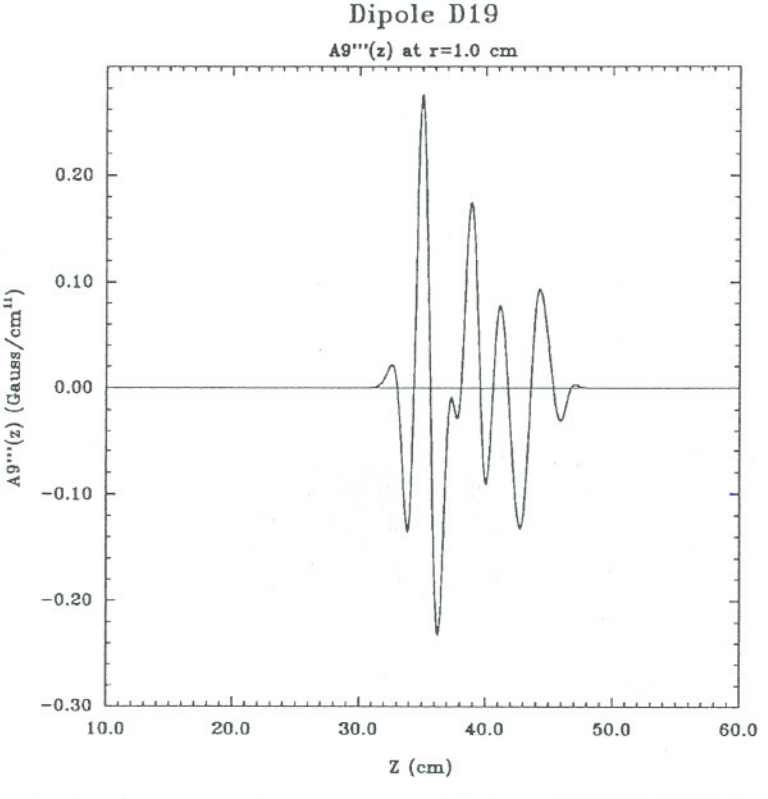

Figure 67 The 3–rd derivative function of n=9 — A9'"(z) — CONDUCTOR ONLY.

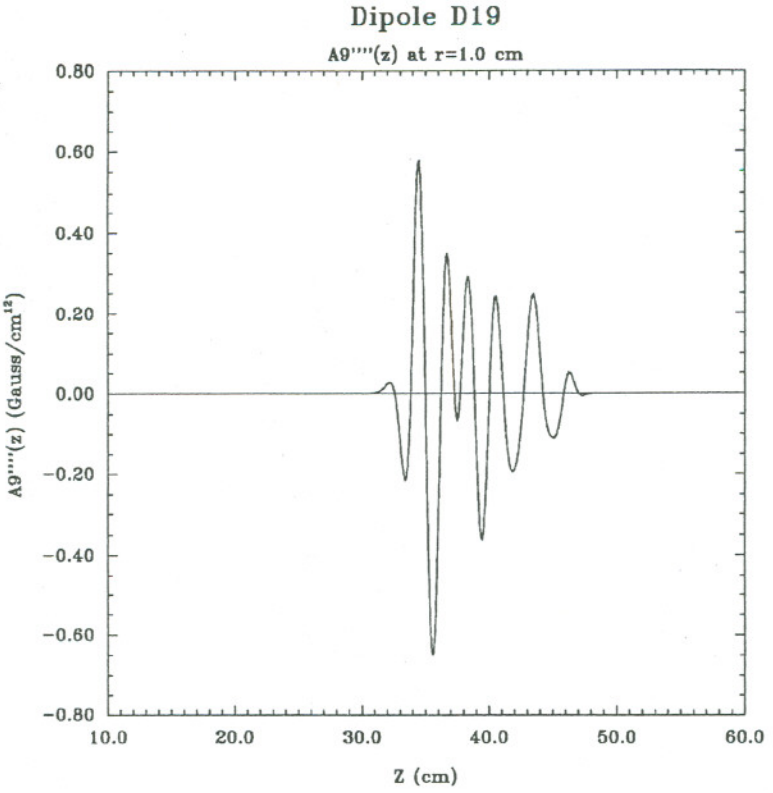

Figure 68 The fourth derivative function of n=9 — A9''''(z) — CONDUCTOR ONLY.

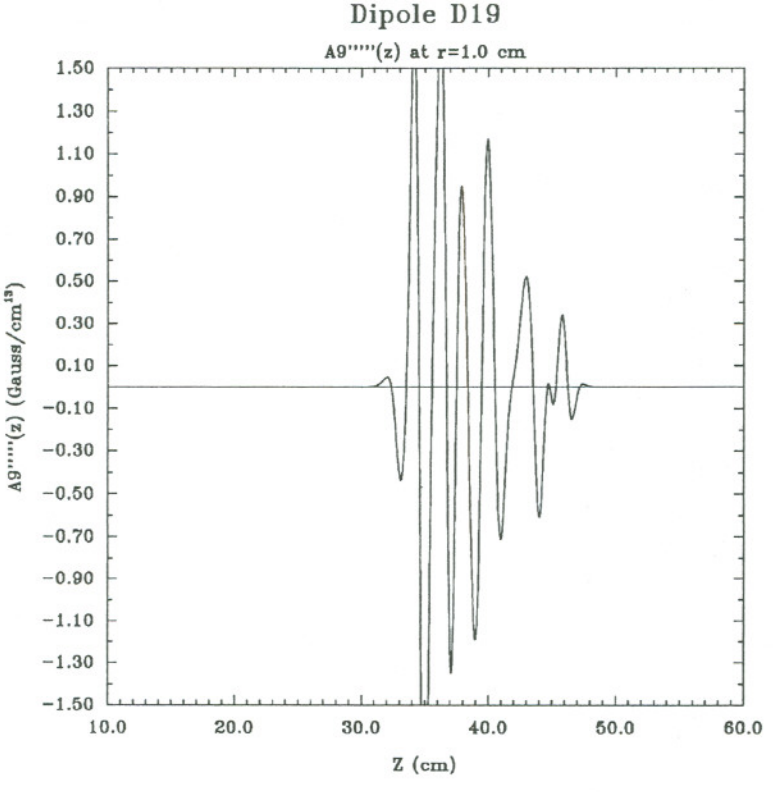

Figure 69 The fifth derivative function of n=9 — A9'''''(z) — CONDUCTOR ONLY.

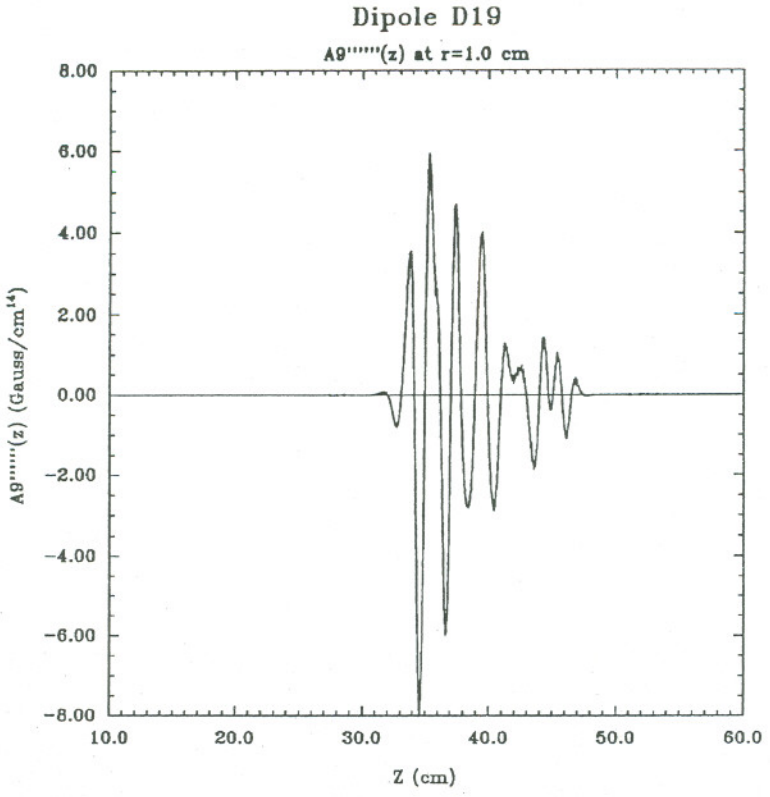

Figure 70 The sixth derivative function of n=9 — A9''''''(z) — CONDUCTOR ONLY.

Appendix A The Vector Potential

Consistent with the form of series proposed for expressing the field components we may provide in various ways forms for a vector-potential function whose curl will provide these field components. One such form is shown below and, if truncated in the manner suggested, will give a field that remains divergence free. We express the vector potential A as :

$$\vec{A} = \sum_n \left[A_{r,n}\hat{e_r} + A_{\theta,n}\hat{e_\theta} + A_{z,n}\hat{e_z} \right] \tag{1}$$

and write :

$$\begin{aligned}
A_{z,n} &= [A_n(z)r^n - \frac{A_n''(z)}{2^2 1!(n+1)}r^{n+2} + \frac{A_n''''(z)}{2^4 2!(n+1)(n+2)} \\
&\quad - \frac{A_n''''''(z)}{2^6 3!(n+1)(n+2)(n+3)}r^{n+4}r^{n+6} + ...]\cos n\theta \\
A_{r,n} &= [-\frac{A_n'(z)}{2(n+1)}r^{n+1} + \frac{A_n'''(z)}{2^3 1!(n+1)(n+2)}r^{n+3} \\
&\quad - \frac{A_n'''''(z)}{2^5 2!(n+1)(n+2)(n+3)}r^{n+5} + ...]\cos n\theta \\
A_{\theta,n} &= [-\frac{A_n'(z)}{2(n+1)}r^{n+1} + \frac{A_n'''(z)}{2^3 1!(n+1)(n+2)}r^{n+3} \\
&\quad - \frac{A_n'''''(z)}{2^5 2!(n+1)(n+2)(n+3)}r^{n+5} + ...]\sin n\theta
\end{aligned} \tag{2}$$