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Authors
Jitomirskaya, Svetlana
Mavi, Rajinder

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Continuity of the measure of the spectrum for quasiperiodic Schrödinger operators with rough potentials

Svetlana Jitomirskaya and Rajinder Mavi*

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Abstract

We study discrete quasiperiodic Schrödinger operators on $\ell^2(\mathbb{Z})$ with potentials defined by γ-Hölder functions. We prove a general statement that for $\gamma > 1/2$ and under the condition of positive Lyapunov exponents, measure of the spectrum at irrational frequencies is the limit of measures of spectra of periodic approximants. An important ingredient in our analysis is a general result on uniformity of the upper Lyapunov exponent of strictly ergodic cocycles.

1 Introduction

Consider quasiperiodic operators acting on $l^2(\mathbb{Z})$ and given by:

$$(H_{\omega,\theta}\psi)(n) = \psi(n-1) + \psi(n+1) + f(\omega n + \theta)\psi(n), \qquad n = \dots, -1, 0, 1, \dots, \tag{1.1}$$

where $f(x)$ is a real-valued sampling function of period 1. Denote by $S(\omega,\theta)$ the spectrum of $H_{\omega,\theta}$. For rational $\alpha = p/q$ the spectrum consists of at most q intervals. Let $S(\omega) = \bigcup_{\theta\in\mathbb{R}} S(\omega,\theta)$. Note that for irrational ω the spectrum of H (as a set) is independent of θ (see, e.g., [13]), and therefore $S(\omega,\theta) = S(\omega)$. In this paper we study continuity of $S(\omega)$ and its measure upon rational approximation of ω, for rough sampling functions f.

The last decade has seen an explosion of general results for operators (1.1) with analytic f, see e.g. [9, 18] and references therein, and by now even the global theory of such operators is well developed [2, 3]. There are very few complete results, however, beyond the analytic category. Indeed, not only the methods of the mentioned papers intrinsically require analyticity or at least Gevrey regularity (e.g. the large deviation theorems), but it is essential for some results too. For example, continuity of the Lyapunov exponent [10], an important ingredient of many later developments, may not hold in the case of even C^∞ regularity [35] (see also [21]). The surprising counterexample in [35] has made it natural to conjecture that (near) analyticity is essential for many other general properties of quasiperiodic potentials: positive Lyapunov exponents at high couplings, localization in the regime of positive Lyapunov exponents, finiteness of transitions between supercritical and subcritical regimes, almost reducibility conjecture, etc. This paper presents a result in the opposite direction. We show that, under certain conditions, for the fundamental question of continuity in ω (previously established under the analyticity condition) not only analyticity is not essential, but such continuity always holds even at surprisingly weak regularity. Namely, in the regime of positive Lyapunov exponents, spectra of rational approximants converge a.e. to $S(\omega)$ for all f, with Hölder-1/2+ continuity. To our knowledge, other than the very basic facts that require, at most, continuity of f, there are no other results that do not require exclusion of potentially relevant parameteres or additional assumptions (e.g. transversality) and work for potentials that rough, and the fact that one can even go beyond the Lipshitz condition has been a surprise to the authors. Moreover, we have reasons to expect that our condition is optimal as far as Hölder regularity go.

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The fact that various quantities could be easier to analyze and sometimes are even computable for periodic operators, $H_{p/q,\theta}$, makes results on continuity in ω particularly important. For example, the famous Hofstadter butterfly [16] is a plot of the almost Mathieu spectra for 50 rational values of ω and visually based inferences about the spectrum for irrational ω implicitly assume continuity. It is therefore an important and natural question if and in what sense the spectral properties of such rational approximants relate to those of the quasi-periodic operator $H_{\omega,\theta}$.

The history of this question was centered around the Aubry conjecture on the measure of the spectrum [1], popularized by B. Simon [32, 33] : that for the almost Mathieu operator given by (1.1) with $f(\theta) = 2\lambda \cos 2\pi\theta$, for irrational ω and all real λ, θ there is an equality

$$|S_\lambda(\omega, \theta)| = 4|1 - |\lambda||. \tag{1.2}$$

Here, for sets, we use $|\cdot|$ to denote the Lebesgue measure. Avron,van Mouche, Simon [8] proved that, for $|\lambda| \neq 1$, $|S_\lambda(p_n/q_n)| \to 4|1 - |\lambda||$ as $q_n \to \infty$, and Last [30] established this fact for $|\lambda| = 1$. Given these theorems, the proof of the Aubry-Andre conjecture was reduced to a continuity result.

The continuity in ω of $S(\omega)$ in Hausdorff metric was proven in [7, 14], requiring only the continuity of f. Continuity of the measure of the spectrum is a more delicate issue, since, in particular, $|S(\omega)|$ can be (and is, for the almost Mathieu operator) discontinuous at rational ω. We will actually use an even stronger notion of a.e. setwise continuity. Namely, we say $\lim_{n\to\infty} B_n = B$ if and only if

$$\limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n = B \Longleftrightarrow \lim_{n\to\infty} \chi_{B_n} = \chi_B \text{ Lebesgue a.e.} \tag{1.3}$$

.

Establishing continuity at irrational ω requires quantitative estimates on the Hausdorff continuity of the spectrum. The first such result, namely the Hölder-$\frac{1}{3}$ continuity was proved in [11], where it was used to establish a zero-measure spectrum (and therefore the Aubry-Andre conjecture) for the almost Mathieu operator with Liouville frequencies ω at the critical coupling $\lambda = 1$. That argument was improved to the Hölder-1/2 continuity (for arbitrary $f \in C^1$) in [8] and subsequently used in [29, 30] to establish (1.2) for the almost Mathieu operator for ω with unbounded continuous fraction expansion, therefore proving the Aubry-Andre conjecture for a.e. (but not all) ω. The extension to all irrational ω is due to [19, 6][1].

It was argued in [8] that Hölder continuity of any order larger than 1/2 would imply the desired continuity property of the measure of the spectrum for all ω and $f \in C^1$. It was first noted in [23] that in the regime of semi-uniform localization, the appropriate cut-offs of the exponentially localized eigenfunctions provide good enough approximate eigenvectors for a perturbed operator to establish almost Lipshitz continuity (thus establishing the Aubry-Andre conjecture in the localization regime available at that time). The idea of [19] was that for Diophantine ω and analytic f one can extract such eigenvectors (and thus establish almost Lipshitz continuity of S) by finding the cut-off places at distance L from each other where the generalized eigenfunction is exponentially small in L, simply as a corollary of positive Lyapunov exponents, without establishing localization. This led to establishing that, in the regime of positive Lyapunov exponents, for any analytic f, $|S(\frac{p_n}{q_n})| \to |S(\omega)|$ for every Diophantine ω and its approximants $\frac{p_n}{q_n}$. Recently, it was shown in [22] that positivity of the Lyapunov exponent is not needed for this result, in particular, for analytic f, and all irrational ω, $S(\frac{p_n}{q_n}) \to S(\omega)$.

Our goal is to show that under the condition of positivity of the Lyapunov exponent, one can significantly relax the required regularity of f.

For a given energy $E \in \mathbb{R}$, a formal solution u of

$$Hu = Eu \tag{1.4}$$

with operator H given by (1.1) can be reconstructed from its values at two consecutive points with the transfer matrix

$$A^E(\theta) = \begin{pmatrix} E - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}; \qquad A^E : \mathbb{T} \to \mathbf{SL}_2(\mathbb{R}) \tag{1.5}$$

[1] It should be noted that the argument of [6] that, in particular, completed the result for the critical value of λ, did not involve continuity in frequency

via the equation

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A^E(\theta + n\omega) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}. \tag{1.6}$$

Setting $R : \mathbb{T} \to \mathbb{T},\ Rx := x + \omega$, the pair (ω, A^E) viewed as a linear skew-product $(x, v) \to (Rx, A(x)v),\ x \in \mathbb{T},\ v \in \mathbb{R}^2$, is called the corresponding Schrödinger cocycle. The iterations of the cocycle (ω, A^E) for $k \geq 0$ are given by

$$A_k^E(\theta) = A^E(R^{(k-1)}\theta) \cdots A^E(R^1\theta)A^E(\theta), \qquad A_0^E = I \tag{1.7}$$

and

$$A_k^E(\theta) = \left(A_{-k}^E(R^{k+1}\theta)\right)^{-1}; \quad k < 0. \tag{1.8}$$

Therefore, it can be seen from (1.6) that a solution to (1.4) for chosen initial conditions $(u(0), u(-1))$ for all $k \in \mathbb{Z}$ is given by,

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_k^E(\theta) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}. \tag{1.9}$$

From general properties of subadditive ergodic cocycles, we can define the Lyapunov exponent

$$\mathcal{L}(E) = \lim_k \frac{1}{k} \int \ln \|A_k^E(\theta)\| \mathrm{d}\theta = \inf_k \frac{1}{k} \int \ln \|A_k^E(\theta)\| \mathrm{d}\theta, \tag{1.10}$$

furthermore, $\mathcal{L}(E) = \lim_k \frac{1}{k} \ln \|A_k^E(\theta)\|$ for almost all $\theta \in \mathbb{T}$.

As mentioned above, $\mathcal{L}$ may be discontinuous in the non-analytic category. Set $L_+(\omega) := \{E : \mathcal{L}(E) > 0\}$.

Our main result is

Theorem 1.1 *For every irrational ω, there exists a sequence of rationals $\frac{p_n}{q_n} \to \omega$ such that for any $f \in \mathcal{C}^\gamma(\mathbb{T})$ with $\gamma > \frac{1}{2}$*

$$S\left(\frac{p_n}{q_n}\right) \cap L_+(\omega) \to S(\omega) \cap L_+(\omega). \tag{1.11}$$

Remark

1. The convergence holds in the strong sense of (1.3).
2. The sequence $\frac{p_n}{q_n}$ will be the full sequence of continued fraction approximants of ω in the Diophantine case, and an appropriate subsequence of it otherwise. For practical purposes of making conclusions about $S(\omega)$ based on the information on $S\left(\frac{p_n}{q_n}\right)$ it is sufficient to have convergence along a subsequence.
3. It is an interesting question whether $\gamma = 1/2$ represents a sharp regularity threshold for this result for a.e. ω.
4. Lower regularity is sufficient for a measure zero set of non-Diophantine ω, see Theorem 2.2
5. It is also interesting to find out what is the lowest regularity requirement for the convergence of full union spectra, without condition of positive $\mathcal{L}$, and for the related Last's intersection spectrum conjecture. Both are more delicate and currently established only for analytic f ([22], see also [31]). We expect that higher than $1/2$ regularity should be required for those results, but that analytic condition is improvable.
6. If we replace f in (1.1) with λf, then $\mathcal{L}$ is expected to be positive for most f and large λ through most of the spectrum, creating a wide range of applicability for Theorem 1.1. For analytic f this is known to hold uniformly in (E, ω) for large λ. For the rough case, the relevant results are [34, 24](reviewer requests Sinai)

Theorem 1.1 certainly has an immediate corollary:

Corollary 1.2 *For every irrational ω, there exists a sequence of rationals $\frac{p_n}{q_n} \to \omega$ such that for any $f \in \mathcal{C}^\gamma(\mathbb{T})$ with $\gamma > \frac{1}{2}$*

$$|S\left(\frac{p_n}{q_n}\right) \cap L_+(\omega)| \to |S(\omega) \cap L_+(\omega)|. \tag{1.12}$$

This corollary for analytic f was the main result of [19] and our proof borrows some important ingredients from that work. The main idea of the current paper is to show that, for Diophantine frequencies, γ-Hölder continuity of f is sufficient to find the cut-off places at distance L from each other where the generalized eigenfunction is *polynomially* small in L, thus establishing β-Hölder continuity of the spectrum with $\beta < \gamma$. The requirement $\gamma > 1/2$ comes from the application of the original argument in [8]. For non-Diophantine frequencies we obtain the statement by extending the Hölder continuity theorem of [8] in the following way: for γ-Hölder functions f the spectrum is $\frac{\gamma}{1+\gamma}$-Hölder continuous, which is sufficient, under an appropriate anti-Diophantine condition, even without positivity of the Lyapunov exponent, see Theorem 5.2.

The proof requires very tight control on the perturbations of cocycles, in absence of continuity of the Lyapunov exponent. To this end, we show that generally, for cocycles over uniquely ergodic dynamics, upper bound is uniform in phases and neighborhoods (Theorem 3.2).

The main part of the proof of Theorem 1.1 follows from Hölder continuity properties of the spectrum in the Hausdorff metric which are stated in section 5. The argument for the positive Lyapunov exponent regime uses tight bounds on matrix cocycle approximation covered in Section 4 which in turn depend on a general result on uniform upper-semicontinuity of Lyapunov exponents for cocycles over uniquely ergodic dynamics, and is proven in Section 3. Section 6 completes the proof of Theorem 1.1.

2 Continued fraction approximants

For $\kappa \geq 0$, $\omega \in \mathbb{R}$ is said to be κ-Diophantine if there exists some $C_\omega > 0$ so that

$$\|n\omega\| > \frac{C_\omega}{n^{1+\kappa}} \tag{2.1}$$

for all $n \in \mathbb{Z}$, where $\|\cdot\|$ denotes distance to the integers. For $\kappa > 0$ a.e. $\omega \in \mathbb{T}$ is κ-Diophantine. For $\kappa = 0$ this condition is equivalent to having bounded type, so a.e. $\omega \in \mathbb{T}$ is not 0-Diophantine.

Writing ω in continued fraction expansion,

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots],$$

the truncations $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ are known as the continued fraction approximants. From the theory of continued fractions [27], for κ-Diophantine ω and $n > n_\omega$ we have for some $C_\omega > 0$,

$$\frac{C_\omega}{q_n^{2+\kappa}} < \left|\omega - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{2.2}$$

We will also need the following fact:

Lemma 2.1 *(e.g. [20]) For an interval $I \subset \mathbb{T}$, if n is such that $|I| > \frac{1}{q_n}$ then for any $\theta \in \mathbb{T}$ there is $0 \leq j \leq q_n + q_{n-1} - 1$ so that $\theta + j\omega \in I$.*

We are now ready to formulate a more detailed version of the main Theorem

Theorem 2.2 *Assume $f \in \mathcal{C}^\gamma(\mathbb{T})$ with $1 \geq \gamma > 0$. Then*

1. *If ω is κ-Diophantine, $\kappa > 0$, and $\gamma > \frac{1}{2}$, then*

$$S\left(\frac{p_n}{q_n}\right) \cap L_+(\omega) \to S(\omega) \cap L_+(\omega). \tag{2.3}$$

for p_n/q_n the sequence of continued fraction approximants of ω.

2. *If ω is not κ-Diophantine with $\kappa = \gamma^{-1} - 1$, then*

$$S\left(\frac{p_n}{q_n}\right) \to S(\omega) \tag{2.4}$$

for a subsequence of approximants

Remark 1. Thus, for Lipshitz f (2.4) holds for a.e. ω (all except possibly for the bounded type). This is already implicit in [29].

2. Theorem 2.2 certainly implies Theorem 1.1

3 Uniform upper semicontinuity of the upper Lyapunov exponent

This section is devoted to some fundamental properties of the Lyapunov exponent in the general setting. It is well known that the Lyapunov exponent of ergodic cocycles is upper semicontinuous. For a uniquely ergodic underlying dynamics, Furman [15] has shown, by a subadditivity argument originally used by Katznelson and Weiss [26] to prove Kingman's ergodic theorem, that rate of convergence of a cocycle from above can be bounded uniformly in the phase. Now we investigate the coincidence of these properties.

Assume (X, T, μ) is an ergodic Borel probability space. We use the notation $\{f\}$ for a sequence $(f_n) \in \mathcal{C}(X, \mathbb{R}) \cap L^1(X, \mu)$ which is a continuous subadditive cocycle with respect to T, that is

$$f_{n+m}(x) \leq f_n(x) + f_m(T^n x).$$

The category of continuous subadditive cocycles will be denoted $\Gamma(X)$. We define the Lyapunov exponent as,

$$\Lambda(f) = \lim_n \frac{1}{n} \int_X f_n(x) \mu(\mathrm{d}x).$$

By Kingman's subadditive ergodic theorem we have, for almost all $x \in X$,

$$\lim_n \frac{1}{n} f_n(x) = \inf_n \frac{1}{n} \int_X f_n(x) \mu(\mathrm{d}x) = \Lambda(f). \tag{3.1}$$

To proceed it will be useful to introduce a metric on $\Gamma(X)$. For two continuous cocycles $\{g\}, \{f\} \in \Gamma(X)$ define

$$\mathrm{d}(\{g\}, \{f\}) = \sum_{n \geq 1} \frac{1}{2^n} \frac{\|g_n - f_n\|_0}{1 + \|g_n - f_n\|_0}, \tag{3.2}$$

where the norm $\|f\|_0 = \max_\theta |f(\theta)|$. Then $(\Gamma(X), \mathrm{d})$ is a metric space. Since for any n the map $\{f\} \to \frac{1}{n} \int_X f_n$ is continuous in $(\Gamma(X), d)$, it follows that the infimum

$$\{f\} \to \inf_n \frac{1}{n} \int_X f_n \mu(\mathrm{d}x) = \Lambda(f)$$

is upper semicontinuous in $(\Gamma(X), \mathrm{d})$.[2] On the other hand, for a fixed cocycle over uniquely ergodic dynamics the convergence is uniform in the phase.

[2] This is true for general L^1 cocycles, with no continuity required.

Theorem 3.1 (Furman [15]) *Let $\{f\}$ be a continuous subadditive cocycle on a compact uniquely ergodic space. Given $\epsilon > 0$, there exists n_ϵ so that for $n > n_\epsilon$ for any $x \in X$ we have*

$$\frac{1}{n} f_n(x) < \Lambda(f) + \epsilon.$$

In the following theorem we combine these properties to obtain uniform uppersemicontinuity in both the cocycle and phase. Note that our simple proof is self-contained, and except for a basic result in [12] (requiring unique ergodicity) it only uses compactness,continuity and subadditivity. In particular, it gives a significantly streamlined proof of Theorem 3.1.[3]

Theorem 3.2 *Let (X, T, μ) be a compact uniquely ergodic dynamical system. Then $\Lambda : \Gamma(X) \to \mathbb{R}$ is uniformly uppersemicontinuous with respect to* d, *meaning that given $\epsilon > 0$ there exist $\delta_\epsilon, n_\epsilon$ such that for g with $\mathrm{d}(f, g) < \delta_\epsilon$ and $n > n_\epsilon$, for all $x \in X$,*

$$\frac{1}{n} g_n(x) \leq \Lambda(f) + \epsilon.$$

Proof By [12] for any x and $\epsilon > 0$ one finds $m(x) > 0$ such that $\frac{1}{m(x)} f_{m(x)}(x) < \Lambda(f) + \epsilon$. By continuity and compactness, we find $M < \infty$ so that for all x, $m(x) \leq M$ and $\delta_\epsilon > 0$ such that for $\mathrm{d}(f, g) < \delta_\epsilon$, $\frac{1}{M} g_{m(x)}(x) < \Lambda(f) + 2\epsilon$. By subadditivity, for k large enough and any $r = 0, \ldots, M$ and all x one has

$$\frac{1}{km(x) + r} g_{km(x)+r}(x) \leq \frac{1}{km(x) + r}(km(x)(\Lambda(f) + 2\epsilon) + Cr) < \Lambda(f) + 3\epsilon \ . \ \blacksquare$$

4 Rate of convergence for matrix cocycles

The first application of Theorem 3.2 is to approximations of matrix cocycles. Consider a continuous matrix $A \in \mathcal{C}(X, \mathbf{GL}_n(\mathbb{C}))$ defined on a compact uniquely ergodic space (X, T, μ). Let the metric on $\mathcal{C}(X, \mathbf{GL}_n(\mathbb{C}))$ be defined by the norm $\|A\|_0 = \max_\theta \|A(\theta)\|$. Then

$$\ln \|A_n(\theta)\| := \ln \|A(T^{n-1}x) \cdots A(x)\|, \ A_0 = I,$$

is a subadditive cocycle and its Lyapunov exponent is defined by

$$\mathcal{L}(A) = \inf_n \frac{1}{n} \int_X \ln \|A(T^{n-1}x) \cdots A(x)\| \mu(\mathrm{d}x).$$

Immediately, an application of Theorem 3.2 results in uniform uppersemicontinuity of the Lyapunov exponent: given ϵ, for D near A and large k, we have

$$\|D_k(x)\| \leq \exp\{k(\mathcal{L}(A) + \epsilon)\} \tag{4.1}$$

uniformly in x, since D in a small C_0 neighborhood of A implies $\{\ln \|D_n\|\}$ in a small d neighborhood of $\{\ln \|A_n\|\}$. This observation leads to the following

Corollary 4.1 *Let $\epsilon > 0$, and $A \in \mathcal{C}(X, \mathbf{GL}_n(\mathbb{C}))$. For small enough δ and large k_ϵ, if*

$$\|D - A\|_0 < \delta$$

and $k \geq k_\epsilon$, then

$$\|A_k - D_k\|_0 \leq \delta e^{\{k(\mathcal{L}(A)+\epsilon)\}}. \tag{4.2}$$

[3] a similar idea has been used in more specialized settings in [4, 5]

Proof To bound the left hand side of (4.2) we will break it into terms composed of iterates of cocycles A and D. We obtain this by a standard trick

$$A_k(\theta) \;=\; A(T^{k-1}\theta)\circ A_{k-1}(\theta) = (A-D)\,(T^{k-1}\theta)\circ A_{k-1}(\theta) + D(T^{k-1}\theta)\circ A_{k-1}(\theta)$$

and iterate on the last term to retrieve the iterates of D

$$\begin{aligned}\|A_k(\theta)-D_k(\theta)\| &\le \sum_{0\le\ell\le k-1}\|D_\ell(T^{k-\ell}\theta)(D-A)(T^{k-1-\ell}\theta)A_{k-1-\ell}(\theta)\|\\ &\le \sum_{0\le\ell\le k-1}\|D_\ell\|_0\|D-A\|_0\|A_{k-1-\ell}\|_0.\end{aligned}$$

For small enough $\delta>0$ we may apply Theorem 3.2 to both the A and D iterates in the last term. Particularly, setting $f(x)=\ln\|A(x)\|$ we obtain that for $0<\epsilon'<\epsilon$ there is $\|D-A\|_0<\delta$ and $k(\epsilon')$ large so for $\ell\ge k(\epsilon')$,

$$\|D_\ell(x)\|\le e^{k\mathcal{L}(A)+\epsilon'} \tag{4.3}$$

We partition the sum accordingly, for $k>2k(\epsilon')$

$$\|A_k(\theta)-D_k(\theta)\|\le\left(\sum_{0\le\ell\le k(\epsilon')-1}+\sum_{k(\epsilon')\le\ell\le k-k(\epsilon')-2}+\sum_{k-k(\epsilon')-1\le\ell\le k-1}\right)\|D_\ell\|_0\|D-A\|_0\|A_{k-1-\ell}\|_0 \tag{4.4}$$

Applying (4.3) to all iterates A_j, D_j for $j>k(\epsilon')$, noting that the first and last summands consist of $k(\epsilon')$ terms each and that for $\ell<k(\epsilon')$ we can bound $\|D_\ell\|_0\le(\|A\|_0+\epsilon)^\ell$, $\|A_{k-1-\ell}\|_0\le(\|A\|_0+\epsilon)^{k-1-\ell}$, we obtain

$$\begin{aligned}\|A_k(\theta)-D_k(\theta)\| &\le \delta\sum_{k_{\epsilon'}\le\ell\le k-k_{\epsilon'}-2}\|D_\ell\|_0\|A_{k-1-\ell}\|_0+2\delta e^{\{(k-1)(\mathcal{L}+\epsilon')\}}\sum_{0\le\ell\le k_\epsilon-1}(\|A\|+\delta)^\ell\\ &\le \delta e^{\{(k-1)(\mathcal{L}+\epsilon')\}}\left(k+2\sum_{0\le\ell\le k_{\epsilon'}-1}(\|A\|+\delta)^\ell\right)\\ &\le \delta e^{\{k(\mathcal{L}+\epsilon)\}}\end{aligned}$$

for large enough $k>k_\epsilon$. ∎

Remark A standard argument would easily obtain (4.2) with $\exp\{k(\mathcal{L}+\epsilon)\}$ replaced by $C\|A\|_0^k$. The issue here is tight control on the exponential rate of growth of the error, without assuming continuity of $\mathcal{L}$.

5 Hölder Continuity in Frequency

If $I=[u,v]\subset\mathbb{Z}$ we write

$$H_{I;\theta}=H_{[u,v],\theta}:=R_I H_\theta R_I$$

where R_I projects onto the subspace of coordinates restricted to I. The Green's function for the interval is the inverse of the restriction $G_I(i,j)=\delta_i^T H_I^{-1}\delta_j$. The determinants of the truncated matrix will be labeled $P_k^E(\theta):=\det(H_{[0,k-1];\theta}-E)$. The truncated Hamiltonian relates to the cocycle matrices by the equation

$$A_k^E(\theta)=\begin{bmatrix} P_k^E(\theta) & -P_{k-1}^E(\theta+\omega)\\ P_{k-1}^E(\theta) & -P_{k-1}^E(\theta+\omega)\end{bmatrix}. \tag{5.1}$$

The following simple lemma allows to bound $|P_k|$ from above uniformly in θ and for a large measure subset of the spectrum

Lemma 5.1 *For any $\zeta, \eta > 0$ there exists a set $F(\zeta, \eta) \subset S(\omega)$, $|F(\zeta, \eta)| < \zeta$, and $k(\omega, \zeta, \eta) = k_F$ so that $E \in S(\omega) \backslash F(\zeta, \eta)$ and $k > k_F$ implies*

$$|P_k^E(\theta)| < e^{k(\mathcal{L}(E)+\eta)}. \tag{5.2}$$

Furthermore there is some $\zeta_F > 0$ so that uniform upper convergence in the sense of Corollary 4.1 holds. Thus, $E \in S(\omega) \backslash F(\zeta, \eta)$ implies if $\|D - A^E\| < \zeta_F$ and $k > k_F$ then (4.2) holds.

Proof For all E there exists $k_{E,\eta}$ and ζ_E so that Corollary 4.1 holds. Thus,

$$|\{E : k_{E,\eta} > k\}| \to 0 \text{ as } k \to \infty$$

and

$$|\{E : \zeta_E < \delta\}| \to 0 \text{ as } \zeta \to 0.$$

Therefore,

$$F(\zeta, \eta) = \{E : \zeta_E < \delta\} \cup \{E : k_{E,\eta} > k\}$$

for small enough ζ and large enough $k = k_F$ so that $|F(\zeta, \eta)| < \zeta$. ■

5.1 The general case

Here we observe that a result of Avron, Mouche and Simon on 1/2-Hölder continuity of the spectrum easily generalizes from $f \in \mathcal{C}^1$ to γ-Hölder case.

Theorem 5.2 *Suppose $f \in \mathcal{C}^\gamma(\mathbb{T})$, $1 \geq \gamma > 0$. Then for $E \in S(\omega)$ and for small enough $|\omega - \omega'|$, there exists an $E' \in S(\omega')$ so that $|E - E'| < C_f |\omega - \omega'|^{\frac{\gamma}{1+\gamma}}$, for some constant $C_f > 0$ not depending on ω or ω'.*

Note that by C_f we mean a constant that depends only on f. Different such constants are denoted by the same C_f in the proofs below. The proof is very similar to that of [8]. Starting with an approximate eigenfunction for $H_{\omega,\theta} - E$ and using the same test function as in [8], upon a cutoff at a distance L we obtain an approximate eigenfunction for $H_{\omega',\theta'}$ with an error in the kinetic energy of order L^{-1}. The main difference is that the potential energy error is now bounded by $CL|\omega - \omega'|^\gamma$, so the choice of L is optimized by $L = C_f |\omega - \omega'|^{-\frac{\gamma}{1+\gamma}}$.

More precisely, given $\epsilon > 0$ and $E \in S(\omega)$, there exists an approximate eigenfunction $\phi_\epsilon \in \ell^2(\mathbb{Z})$ so that $\|(H_{\omega,\theta} - E)\phi_\epsilon\| < \epsilon \|\phi_\epsilon\|$. Set $g_{j,L}(n) = \left(1 - \frac{|j-n|}{L}\right)^+$, where $g^+(n) = g(n), n \geq 0$ and $g^+(n) = 0$ otherwise.

Avron-van Mouche-Simon [8] prove that for sufficiently large L for *any* bounded $f : \mathbb{T} \to \mathbb{R}$ there exists j such that $g_{j,L}\phi_\epsilon \neq 0$,and for any $\epsilon > 0$,

$$\|(H_{\omega,\theta} - E) g_{j,L}\phi_\epsilon\|^2 \leq C\left(\epsilon^2 + L^{-2}\right) \|g_{j,L}\phi_\epsilon\|^2, \tag{5.3}$$

where C is universal. Now let θ' be given by $\omega j + \theta = \omega' j + \theta'$. By the Hölder assumption on f and $j - L \leq n \leq j + L$, observe that

$$|f(\theta + n\omega) - f(\theta' + n\omega')| \leq C_f \left(L|\omega - \omega'|\right)^\gamma$$

Thus,

$$\begin{aligned} \|(H_{\omega',\theta'} - E) g_{j,L}\phi_\epsilon\| &\leq \|(H_{\omega',\theta'} - H_{\omega,\theta}) g_{j,L}\phi_\epsilon\| + \|(H_{\omega,\theta} - E) g_{j,L}\phi_\epsilon\| \\ &\leq \left(C_f \left(L|\omega - \omega'|\right)^\gamma + C\left(\epsilon^2 + L^{-2}\right)^{1/2}\right) \|g_{j,L}\phi_\epsilon\|. \end{aligned} \tag{5.4}$$

Since ϵ can be arbitrarily small, choosing $L = C_f |\omega - \omega'|^{-\frac{\gamma}{1+\gamma}}$, to make both addends on the right-hand side of (5.4) equal, we obtain the statement of Theorem 5.2 by the variational principle. ■

5.2 Diophantine case

As discussed in detail in [8] (the last section), for Diophantine rotations 1/2-Hölder continuity of the spectrum (the best that can be obtained from Theorem 5.2) is not sufficient, so that is what we aim to improve.

Theorem 5.3 *Suppose $H_{\omega,\theta}$ is an operator of the form (1.1) where $f \in \mathcal{C}^{\gamma}$, $1 \geq \gamma > 0$, $\omega \in [0,1]$ is κ-Diophantine, $\kappa > 0$. Fix $0 < \beta < \gamma$. Given $\zeta > 0$ there is a B_ζ, $0 < |B_\zeta| < \zeta$ so that for $E \in S(\omega) \cap L_+(\omega) \backslash B_\zeta$ and any ω' near ω, there exists $E' \in S(\omega')$ such that*

$$|E - E'| < C_f |\omega - \omega'|^{\beta}.$$

Remark The theorem holds for $\gamma > \beta > 0$, but the application we are interested in will require $\gamma > \beta > \frac{1}{2}$.

Proof We assume $L_+(\omega) \cap S(\omega) \neq \emptyset$ otherwise the Theorem holds vacuously. Suppose f is γ-Hölder. Let $0 < \beta < \gamma$. Let $\mathcal{E}_\chi = \{E \in S(\omega) \cap L_+(\omega) : \mathcal{L}(E) < \chi\}$, with $\chi > 0$ so small that $|\mathcal{E}_\chi| < \frac{\zeta}{2}$. By upper semicontinuity of the Lyapunov exponent, the Lyapunov exponent is bounded on compact sets. Let $\bar{\chi} > 0$ be an upper bound of the Lyapunov exponent on $S(\omega)$. Let $1 > c > \frac{3}{4}$. Choose d so that $c - \frac{1}{2} > d > \frac{1}{4}$. Choose

$$0 < \tau < \varsigma < \frac{\gamma - \beta}{\beta + 1 - \frac{\beta}{\gamma}} \frac{d}{(1 + 2\kappa)}; \ 1 > b > \max(1 - \frac{\chi}{\bar{\chi}}\tau, c) \text{ and } b < a < 1. \tag{5.5}$$

Finally, let $\eta > 0$ be such that

$$0 < \eta < \min\{\chi\tau - \bar{\chi}(1 - b), \chi(1 - a), \chi(c - d - \frac{1}{2})\}. \tag{5.6}$$

Define $B_\zeta = \mathcal{E}_\chi \cup F(\zeta/2, \eta)$ with $F(\cdot, \cdot)$ from Lemma 5.1 with associated k_F and δ_F. Take $E \in S(\omega) \cap L_+(\omega) \backslash B_\zeta$. We now find an Nth degree trigonometric polynomial f_N that approximates f. Namely, for γ-Hölder functions f, we have

$$\|f_N - f\| < C_f N^{-\gamma}$$

where

$$f_N(\theta) := K_N * f(\theta) = \sum_{-N \leq j \leq N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j) e^{ij\theta},$$

K_N being Fejer's summability kernel, see for example [25].

Set

$$N = \exp\left\{\chi k \frac{\tau}{\gamma}\right\}$$

and let $A^{(N),E}$ be the cocycle matrix defined by the potential determined by the sampling function f_N.

For a map $B : \mathbb{T} \to SL_2(\mathbb{R})$ and associated cocycle,

$$V_k(t, B) = \left\{\theta \in \mathbb{T} : \frac{1}{k} \ln \|B_k(\theta)\| > t\right\} \subset \mathbb{T}. \tag{5.7}$$

The measure of this set, for large k, can be bounded below by use of Corollary 4.1. Indeed, for $k > k_F$ we have for all θ, $\frac{1}{k} \ln \|A_k^E(\theta)\| < \mathcal{L}(E) + \eta$, thus using (1.10) and (4.1),

$$\mathcal{L}(E) \leq \int_{\mathbb{T}} \frac{1}{k} \ln \|A_k^E(\theta)\| \mathrm{d}\theta \leq |V_k\left(a\mathcal{L}(E), A^E\right)|(\mathcal{L}(E) + \eta) + \left(1 - \left|V_k\left(a\mathcal{L}(E), A^E\right)\right|\right) a\mathcal{L}(E), \tag{5.8}$$

the lower bound on the measure of V_k follows immediately,

$$\frac{(1-a)\mathcal{L}(E)}{(1-a)\mathcal{L}(E) + \eta} \leq \left|V_k\left(a\mathcal{L}(E), A^E\right)\right|. \tag{5.9}$$

Furthermore, we make the following claim regarding the sets $V_k(\cdot,\cdot)$ for $k > k_G = \max\{k_F, k_{a,b,c}\}$, and $|E-\bar{E}|, |E-\bar{\bar{E}}| < \exp\{-\chi\tau k\}$,

$$V_k(a\mathcal{L}(E), A^E) \subset V_k(b\mathcal{L}(E), A^{(N),\bar{E}}) \subset V_k(c\mathcal{L}(E), A^{\bar{\bar{E}}}). \tag{5.10}$$

The left inclusion of (5.10) follows from the approximation,

$$\begin{aligned} \theta &\in V_k\left(a\mathcal{L}(E), A^E\right) \implies \\ \left\|A_k^{(N),\bar{E}}(\theta)\right\| &\geq \left\|A_k^E(\theta)\right\| - \left\|A_k^E(\theta) - A_k^{(N),\bar{E}}(\theta)\right\| \\ &> e^{ak\mathcal{L}(E)} - \left(\left|E-\bar{E}\right| + C_f N^{-\gamma}\right) e^{k(\mathcal{L}(E)+\eta)} > e^{ak\mathcal{L}(E)} - Ce^{k(\mathcal{L}(E)+\eta-\chi\tau)} > e^{bk\mathcal{L}(E)}. \end{aligned}$$

The second inequality follows from the definition of 5.7 and an application of Corollary 4.1, the next inequality is immediate from the choice of $\bar{E}$ and N, finally the by the choice of parameters in (5.6) we have $\mathcal{L}(E) + \eta - \tau\chi < b\mathcal{L}(E)$ so that the final inequality holds. The right inclusion of (5.10) is similar, with comparisons (applications of Corollary 4.1) made to A^E,

$$\begin{aligned} \theta &\in V_k\left(b\mathcal{L}(E), A^{(N),\bar{E}}\right) \implies \\ \left\|A_k^{\bar{\bar{E}}}(\theta)\right\| &\geq \left\|A_k^{(N),\bar{E}}(\theta)\right\| - \left\|A_k^{(N),\bar{E}}(\theta) - A_k^E(\theta)\right\| - \left\|A_k^E(\theta) - A_k^{\bar{\bar{E}}}(\theta)\right\| \\ &> e^{bk\mathcal{L}(E)} - (C_f N^{-\gamma} + \left|\bar{E}-E\right| + \left|E-\bar{\bar{E}}\right|)e^{k(\mathcal{L}(E)+\eta)} \\ &> e^{bk\mathcal{L}(E)} - Ce^{k(\mathcal{L}(E)+\eta-\chi\tau)} > e^{ck\mathcal{L}(E)}, \end{aligned}$$

again using (5.6) to obtain the final inequality. Using the inclusion (5.10) and the lower bound on measure (5.9) we have

$$\left|V_k(b\mathcal{L}(E), A^{(N),\bar{E}})\right| \geq \frac{\chi}{\chi + \eta/(1-a)} \geq \frac{1}{2}, \tag{5.11}$$

with the final inequality following from (5.6). Thus $V_k(b\mathcal{L}(E), A^{(N),E})$, being defined by a polynomial of order $4k\exp\{\chi k\tau/\gamma\}$, contains an interval of length $\exp\{-\chi k\varsigma/\gamma\}$, for sufficiently large k. It follows from (5.10) that $V_k(c\mathcal{L}(E), A^{\bar{E}})$ also contains an interval I of length $\exp\{-\chi k\varsigma/\gamma\}$.

Now we move on to constructing the approximate eigenfunction. Let E_0 be a generalized eigenvalue of $H_{\omega,\theta}$ so that $|E-E_0| < e^{-(\bar{\chi}+\eta)k}$, with generalized eigenvector ψ. For spectrally a.e. E, $|\psi(x)| = o((1+|x|)^{1/2+\epsilon})$ (known as Schnol's Theorem, see for example [28]), so we assume E_0 is such a value. Thus there exists an x_m so that

$$\frac{|\psi(x_m)|}{|x_m|+1} = \max_x \left(\frac{|\psi(x)|}{|x|+1}\right) \geq \frac{|\psi(x)|}{|x|+1}$$

for all $x \in \mathbb{Z}$. Let ψ be normalized so that,

$$\frac{|\psi(x_m)|}{|x_m|+1} = 1.$$

The sublinear growth property together with the convergence properties of cocycles we have discussed forces ψ to take on small values at controlled distances, allowing as to make an effective cutoff, as we will now show. Using the Diophantine property (2.2) for ω, we find a denominator of an approximant q_n such that

$$|I|^{-1} < \exp\{k\chi\varsigma/\gamma\} \leq q_n < \exp\{k\chi\varsigma(1+2\kappa)/\gamma\}. \tag{5.12}$$

where $I \subset V(c\mathcal{L}(E), A^E)$ is an interval discussed in the reasoning after (5.11). Using Lemma 2.1, applied to the interval I there exists an x_1', with $x_m - 2q_n - k \leq x_1' < x_m - k$, so that $T^{x_1'}\theta \in I \subset V(c\mathcal{L}(E), A^E)$. Similarly, there exists x_3', with $x_m < x_3' \leq x_m + 2q_n$, such that $T^{x_3'}\theta \in I$. The need for an upper bound on

q_n will arise later. The lower bound on the norm of A^E at $T^{x_1'}\theta$ (that follows from (5.7) implies by the form of the cocycles of A^E in (5.1) that for $x_1' = x_1$ or $x_1 = x_1' - 1$ and $k_\ell = k, k-1,$ or $k-2$, we have

$$|P_{k_\ell}^{E_0}(T^{x_1}\theta)| > \frac{1}{4}e^{c\mathcal{L}(E)k}.$$

Similarly for $x_3 = x_3'$ or $x_3 = x_3' - 1$ and $k_r = k, k-1,$ or $k-2$

$$|P_{k_r}^{E_0}(T^{x_3}\theta)| > \frac{1}{4}e^{c\mathcal{L}(E)k}.$$

Let

$$x_\ell = x_1 + \left[\frac{k_\ell}{2}\right]; \quad x_r = x_3 + \left[\frac{k_r}{2}\right].$$

Set also $x_2 = x_1 + k_\ell - 1$ and $x_4 = x_3 + k_r - 1$. Using Cramer's rule, as in [17]

$$|G_{[x_1,x_2]}^{E_0}(x_\ell, x_1)| < \frac{|P_{(x_2-x_\ell)}^{E_0}(T^{x_\ell+1}\theta)|}{|P_{k_\ell}^{E_0}(T^{x_1}\theta)|} < C\frac{(1+\exp\{-(\bar{\chi}+\eta)k_\ell\})\exp\left\{(\mathcal{L}(E)+\eta)\frac{k_\ell}{2}\right\}}{\exp\{c\mathcal{L}(E)k_\ell\}} < \exp\{-dk_\ell\mathcal{L}(E)\} \tag{5.13}$$

similarly,

$$|G_{[x_3,x_4]}^{E_0}(x_r, x_3)| < \exp\{-dk_r\mathcal{L}(E)\} \tag{5.14}$$

with the numerator in the second inequality bounded above with (4.2) and the last inequality following from (5.6) for sufficiently large k. For similar reasons (5.13) also holds with (x_ℓ, x_1) replaced with $(x_\ell - 1, x_1)$, $(x_\ell - 1, x_2)$ or (x_ℓ, x_2) and (5.14) holds with (x_r, x_3) replaced by $(x_r - 1, x_3)$, $(x_r - 1, x_4)$ or (x_r, x_4). Let $\Lambda = [x_\ell, x_r]$ and let ψ_Λ be the truncation of ψ to Λ or $\psi_\Lambda = R_\Lambda\psi$. We have $|\Lambda| \leq 4q_n + k + 2 < 5\exp\left\{k\chi\frac{\varsigma(1+2\kappa)}{\gamma}\right\}$ by the upper bound of q_n, (5.12). By choice of x_m, and with $x_a = x_r$ or x_ℓ,

$$\begin{aligned}\frac{|\psi(x_a)|}{|x_m|+1} &= \frac{|\psi(x_a)|}{|x_m|+1}\frac{|x_a|+1}{|x_a|+1} \leq \frac{|x_a|+1}{|x_m|+1} \\ &\leq \frac{|x_m|+|x_a-x_m|+1}{|x_m|+1} \leq 1 + |x_a - x_m| \leq 1 + 2q_n + k/2 < 3\exp\left\{k\chi\frac{\varsigma(1+2\kappa)}{\gamma}\right\}.\end{aligned} \tag{5.15}$$

As a formal eigenfunction, ψ satisfies, for $x_1 \leq x \leq x_2$,

$$\psi(x) = -G_{[x_1,x_2]}^{E_0}(x, x_1)\psi(x_1 - 1) - G_{[x_1,x_2]}^{E_0}(x, x_2)\psi(x_2 + 1), \tag{5.16}$$

and similarly for x_3, x_4. Applying both (5.15) and (5.13) to (5.16) we obtain bound at an end point of Λ,

$$\psi(x_\ell) \leq C(|x_m|+1)\exp\left\{k\chi\frac{\varsigma(1+2\kappa)}{\gamma}\right\}\exp\{-kd\mathcal{L}(E)\} \tag{5.17}$$

A similar bound follows for $\psi(x_\ell - 1)$, and following the same reasoning on $[x_3, x_4]$ and using (5.13) we have similar bounds for $\psi(x_r)$ and $\psi(x_r + 1)$. The cutoff function then satisfies,

$$\|(H_{\omega,\theta} - E_0)\psi_\Lambda\| \leq C(|x_m|+1)\exp\left\{-k\left(d\mathcal{L}(E) - \chi\frac{\varsigma(1+2\kappa)}{\gamma}\right)\right\}$$

Define $\phi_\Lambda = \psi_\Lambda/\|\psi_\Lambda\|$. By the normalization of ψ, we have $\|\psi_\Lambda\| \geq |x_m| + 1 \geq 1$ so that

$$\|(H_{\omega,\theta} - E_0)\phi_\Lambda\| \leq \frac{1}{|x_m|+1}\|(H_\omega - E_0)\psi_\Lambda\| \leq C\exp\left\{-k\left(d\mathcal{L}(E) - \chi\frac{\varsigma(1+2\kappa)}{\gamma}\right)\right\}.$$

For $\omega' \in \mathbb{T}$ set $\theta' = \theta - \frac{x_r + x_\ell}{2}(\omega - \omega')$. Then, perturbing the Hamiltonian's frequency,

$$\|(H_{\omega,\theta} - H_{\omega',\theta'})\,\phi_\Lambda\| \leq \max_{x_\ell \leq x \leq x_r} |f(\theta' + x\omega') - f(\theta + x\omega)| \leq C_f\,(|\Lambda| \cdot |\omega' - \omega|)^\gamma < C_f |\omega' - \omega|^\gamma \exp\{k\chi\varsigma(1+2\kappa)\} \tag{5.18}$$

Thus

$$\begin{aligned} \|(E - H_{\omega',\theta'})\,\phi_\Lambda\| &\leq |E - E_0| + \|(E_0 - H_{\omega,\theta})\phi_\Lambda\| + \|(H_{\omega,\theta} - H_{\omega',\theta'})\,\phi_\Lambda\| \qquad (5.19)\\ &\leq |E - E_0| + C\exp\left\{-k\left(d\mathcal{L}(E) - \chi\frac{\varsigma(1+2\kappa)}{\gamma}\right)\right\} + C_f|\omega' - \omega|^\gamma \exp\{k\chi\varsigma(1+2\kappa)\}\\ &\leq C\exp\left\{-k\left(d\mathcal{L}(E) - \chi\frac{\varsigma(1+2\kappa)}{\gamma}\right)\right\} + C_f|\omega' - \omega|^\gamma \exp\{k\chi\varsigma(1+2\kappa)\}. \qquad (5.20) \end{aligned}$$

Thus, by the variation principle, there exists an E' in $S(\omega')$ so that

$$|E' - E| \leq \|(E - H_{\omega',\theta'})\,\phi_\Lambda\|\,. \tag{5.21}$$

If we take $k > k_G$ such that

$$\frac{-\beta \ln|\omega - \omega'|}{\chi\left(d - \frac{\varsigma}{\gamma}(1+2\kappa)\right)} \leq k \leq \frac{-(\gamma - \beta)\ln|\omega - \omega'|}{\chi\varsigma(1+2\kappa)},$$

which we can do, by (5.5), for sufficiently small $|\omega - \omega'|$, we obtain $|E' - E| < |\omega - \omega'|^\beta$. The required smallness of $|\omega - \omega'|$ depends only on chosen parameters, therefore on ω (through its Diophantine parameters), β, ζ and f. ∎

6 The strong continuity. Proof of Theorem 2.2

This argument is very similar to that of [19] (which in turn is a modification of the proof in [29]). First, continuity of $S(\omega)$ in Hausdorff metric [7] implies

$$\limsup_{\frac{p}{q} \to \omega} S\left(\frac{p}{q}\right) \subseteq \Sigma(\omega)\ , \tag{6.1}$$

for any irrational $\omega \in \mathbb{T}$ (inclusion holds set-wise, not just a.e, for any continuous f and any sequence $\frac{p}{q} \to \omega$), which immediately implies the corresponding inclusion in Theorem 2.2. For the opposite inclusion we need to consider continued fraction approximants $\frac{p_n}{q_n}$. Note that because of continuity in θ, the set $S(p_n/q_n)$ consists of at most q_n disjoint intervals, say $S(p_n/q_n) = \cup_{i=1}^{q'_n}[a_{n,i}, b_{n,i}]$, $q'_n \leq q_n$.

We now treat Diophantine and non-Diophantine cases separately.

For a Diophantine ω, Theorem 5.3 implies that for $n > n(\omega, \beta, \zeta, f)$,

$$S(\omega) \cap L_+(\omega) \subset \cup_{i=1}^{q'_n}[a_{n,i} - C_f|\omega - \frac{p_n}{q_n}|^\beta, b_{n,i} + C_f|\omega - \frac{p_n}{q_n}|^\beta] \cup B_\zeta$$

thus

$$|(S(\omega) \cap L_+(\omega) \backslash B_\zeta) \backslash S(p_n/q_n)| < 2C_f q_n |\omega - \frac{p_n}{q_n}|^\beta \to 0$$

since $\beta > 1/2$.

Therefore, for every $\zeta > 0$, we have $|S(\omega) \cap L_+(\omega) \backslash B_\zeta) \backslash \liminf_{p_n/q_n \to \omega} S(p_n/q_n)| = 0$. Thus

$$|S(\omega) \cap L_+(\omega) \backslash \cap_{\zeta > 0} B_\zeta) \backslash \liminf_{p_n/q_n \to \omega} S(p_n/q_n)| = 0,$$

which gives the desired inclusion in Theorem 1.1.

Now, consider the irrational ω so that there exists a sequence of rational $\frac{p_n}{q_n}$ so that p_n and q_n are mutually prime and

$$q_n^{\frac{1+\gamma}{\gamma}} \left|\omega - \frac{p_n}{q_n}\right| \to 0, \tag{6.2}$$

so that, ω is not κ Diophantine for $\kappa > \frac{1}{\gamma} - 1$. Similar to the above calculation, we have, letting $S\left(\frac{p_n}{q_n}\right) = \cup_{1\le i\le q_n'}[a_{n,i}, b_{n,i}]$, and using Theorem 5.2 that

$$S(\omega) \subset \bigcup_{1\le i\le q_n'} \left[a_{n,i} - C_f \left|\omega - \frac{p_n}{q_n}\right|^{\frac{\gamma}{1+\gamma}}, b_{n,i} + C_f \left|\omega - \frac{p_n}{q_n}\right|^{\frac{\gamma}{1+\gamma}}\right]$$

Thus, by (6.2),

$$S(\omega) \subset \liminf_{p_n/q_n \to \omega} S\left(\frac{p_n}{q_n}\right). \ \blacksquare$$

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