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Missing Symmetries of the Standard Model

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Missing Symmetries of the Standard Model

Robust Solutions to the Strong CP Problem

DISSERTATION

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by

Benjamin Lillard

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In memory of

Ethan Robert Lillard

and

Robert Morgan Guion

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Missing Symmetries of the Standard Model

Robust Solutions to the Strong CP Problem

Benjamin Lillard

Abstract of the Dissertation

The strong CP problem is a compelling motivation for the existence of as-yet-undiscovered additions to the Standard Model of particle physics. An extraordinary cancellation between two apparently unrelated parameters in the Standard Model endows the neutron with an essentially symmetric distribution of electric charge, implying that quantum chromodynamics (QCD) conserves parity and time reversal symmetries P and CP, despite the fact that both are broken by electroweak interactions.

Axion models provide a popular explanation to this puzzle of the Standard Model, by dynamically restoring CP as a symmetry of the QCD vacuum. Yet in the context of a high-energy theory with broken global symmetries, which encodes for example the expected effects from quantum gravity, simple axion models require their own severe form of fine-tuned cancellations to prevent unacceptably large violations of CP symmetry in the vacuum.

Constructing a model that safeguards the axion against these catastrophic effects is highly nontrivial, and has been an active area of research from around 1990 to the present. Typical solutions in the literature invoke intricate structures of new symmetries and particles, leading an ongoing search for simpler and more aesthetically pleasing models.

This thesis explores some supersymmetric models proposed by the author [1–3] as new, robust solutions to the strong CP problem. In particular, the composite axion model of [3] provides a compellingly simple extension to the MSSM, with built-in $B - L$ symmetry, a naturally $\mathcal{O}(\text{TeV})$ scale for electroweak physics, and sufficient protection from symmetry-violating effects for the axion model in the preferred window of parameter space, where the axion is a viable candidate for dark matter.

Chapter 1

Introduction

In terms of its ability to describe and predict the behavior of matter at its smallest observed scales, the Standard Model of particle physics has been remarkably successful. Even so, it has several unresolved and well documented shortcomings which indicate that it cannot be the complete description of Nature. It does not address the particle nature of the dark matter which, by mass, is evidently the dominant form of matter in the universe. The Standard Model does not explain the asymmetric abundance of baryons over antibaryons in the early universe which prevented the complete annihilation of matter, allowing the formation of complex structures that provided the necessary conditions for life. Reflecting the broader difficulty in reconciling gravitation with quantum mechanics, the Standard Model is also incapable of describing gravity.

Other puzzling aspects of the Standard Model include the hierarchy between the electroweak and Planck scales; the observed pattern of fermion masses; and the strong CP problem, which is the focus of this thesis.

In the case of the strong CP problem, two apparently unrelated parameters in the Standard Model conspire to cancel each other to one part in 10^{10} or more. This extreme cancellation may even be exact: state of the art measurements of this physical parameter through the neutron electric dipole moment find only upper limits on its value. As a result, the strong interaction appears to respect the symmetries of parity (P) and the combination of charge conjugation and parity (CP) to at least a very high degree, despite the fact that both P and CP are violated by the electroweak interactions.

Two terms in the Standard Model can in principle induce P and CP violation in the strong sector: the first,

$$\mathcal{L} = \frac{g^2}{64\pi^2}\theta\epsilon^{\alpha\beta\mu\nu}G_{\alpha\beta}G_{\mu\nu} = \frac{g^2}{32\pi^2}\theta G_{\mu\nu}\widetilde{G}^{\mu\nu}, \tag{1.0.1}$$

is odd under the action of P or CP, which interchanges $\widetilde{G}_{\mu\nu} \to -\widetilde{G}_{\mu\nu}$ and leaves $G_{\mu\nu}$ invariant. Complex phases in the quark mass matrix also violate CP. By performing subsequent chiral rotations on the quark fields, the degree of CP violation in the Standard Model is encoded by a single physical parameter:

$$\bar{\theta} = \theta + \arg\det M_q, \tag{1.0.2}$$

where $\arg\det M_q$ is the phase of the determinant of the quark mass matrix. The physical parameter $\bar{\theta}$ is defined on the interval $[0, 2\pi)$: yet, measurements of the neutron electric dipole moment [4,5] show that it is

$$\left|\bar{\theta}\right| < 6 \times 10^{-11}, \tag{1.0.3}$$

suggesting an extraordinary cancellation between two apparently unrelated quantities.

A number of explanations for this tiny value have been proposed. For example, if CP is invoked as a fundamental symmetry of the Standard Model, then $|\bar{\theta}| = 0$ automatically. However, to reproduce the observed violation of CP in the electroweak interactions, manifested for example by the CP-violating mixing and decay of mesons, CP must be spontaneously broken in the low-energy theory in such a way that $\bar{\theta}$ remains suitably small. Nelson [6] and Barr [7] proposed models in which this is achieved at tree level. Preventing higher dimensional operators in these models from shifting the $\bar{\theta}$ parameter away from zero remains a significant challenge: as outlined for example in [8], the Nelson-Barr mechanism may need to be augmented by supersymmetry and additional structure in order for it to produce $\bar{\theta} \lesssim 10^{-10}$ without requiring additional fine tuning.

Other possible explanations for the lack of CP violation in the strong sector invoke a new axial $U(1)$ symmetry, which if unbroken would render $\bar{\theta}$ unphysical. If for example the up quark were massless, $m_u = 0$, then the determinant of the quark mass matrix would be zero, and its phase would no longer be a physical quantity. In the absence of the mass term $\mathcal{L} \sim m_u u_L \bar{u}_R$, a global $U(1)_A$ symmetry emerges, under which $u_L \to e^{i\alpha} u_L$ and $\bar{u}_R \to e^{i\alpha} \bar{u}_R$. The action of the $U(1)_A$ symmetry, $\alpha \to \alpha + \Delta\alpha$ induces an analogous shift in $\bar{\theta} \to \bar{\theta} + 2\Delta\alpha$, rendering $\bar{\theta}$ unphysical.

The massless up quark scenario is no longer considered to be a serious solution to the strong CP problem: lattice results [9] suggest $m_u/m_d \approx 0.5$, which is roughly ten orders of magnitude too large to explain the value of $\bar{\theta}$. If any such axial symmetry does provide the explanation to the strong CP problem, then, it must be spontaneously broken. This axial $U(1)$ is referred to as a Peccei-Quinn symmetry, after [10, 11], and is the foundation of QCD axion models.

1.1 Axion Solution to the Strong CP Problem

This section is devoted to unpacking the following statement:[1]

> The QCD axion is the pseudo-Nambu–Goldstone boson of a spontaneously broken global $U(1)_{\rm PQ}$ symmetry; its vacuum expectation value dynamically sets $\bar{\theta} = 0$, restoring CP as a symmetry of the low energy theory.

A simple model that achieves this goal can be constructed by adding a complex scalar, ϕ; left-handed color (anti)-triplet fermions Q and $\overline{Q}$; and the interactions

$$\mathcal{L} \supset V(\phi) + \phi Q\overline{Q} + h.c. \tag{1.1.1}$$

where $V(\phi)$ is arranged such that ϕ acquires a nonzero vacuum expectation value. We use Weyl notation, where Q and $\overline{Q}$ are two-component left-handed fermions, and $Q^\dagger$ and $\overline{Q}^\dagger$ are their right-handed anti-particles. By design, the mass term $m_q Q\overline{Q} + m_q^\star \overline{Q}^\dagger Q^\dagger$ is forbidden, so that the Lagrangian respects the following $U(1)_{\rm PQ}$ symmetry:

$$\phi \to e^{i\alpha}\phi \qquad Q \to e^{-i\alpha/2} Q \qquad \overline{Q} \to e^{-i\alpha/2}\overline{Q}. \tag{1.1.2}$$

Similarly, $V(\phi)$ is taken to be a function only of $(\phi^\star\phi)$: just like $m_q \neq 0$, terms such as $(\phi + \phi^\star)$ in $V(\phi)$ would violate the Peccei-Quinn symmetry.

The potential $V(\phi)$ is minimized by some value of $\langle\phi\rangle \neq 0$: expanding about this vacuum solution, the axion is identified as the phase of ϕ:

$$\phi = \left(\langle\phi\rangle + \frac{\sigma}{\sqrt{2}}\right) \exp\left(i\frac{a}{f_a}\right), \tag{1.1.3}$$

[1]For a more complete review, see for example [12, 13].

where σ and a are real scalar and pseudoscalar fields, respectively, and where $f_a = \sqrt{2}\langle\phi\rangle$. Upon replacing ϕ with σ and a in the Lagrangian, one finds that σ has a mass on the order of $m_\sigma \sim f_a$, while the axion a remains massless. This is a result of the Peccei-Quinn symmetry: a $U(1)_{\rm PQ}$ rotation by some phase α, as in Eq. (1.1.2), corresponds to a linear shift in the field $a(x)$:

$$a(x) \to a(x) + \alpha f_a. \tag{1.1.4}$$

A mass term of the form $m_a^2 a^2$ is inconsistent with the shift symmetry, and is therefore forbidden if $U(1)_{\rm PQ}$ is an exact symmetry.

In the case of the chiral theory described above, however, $U(1)_{\rm PQ}$ is not exactly conserved. Even though it is a symmetry of the Lagrangian, it is explicitly broken in the quantized theory by the chiral anomaly. As described by Adler [14], Bell and Jackiw [15], the axial vector current is violated by the divergence of some triangle diagrams, specifically those with a closed fermionic loop with gauge bosons as the external states. These triangle diagrams were originally invoked to explain the $\pi^0 \to \gamma\gamma$ decay of the neutral pion. For the axion, the effect of the chiral anomaly is to induce a coupling between the axion and the gluon field strength tensor,

$$\mathcal{L} = \frac{g^2}{32\pi^2}\left(\bar{\theta} + \frac{a}{f_a}\right) G_{\mu\nu}\widetilde{G}^{\mu\nu}. \tag{1.1.5}$$

This can be seen more simply by repeating the procedure that led us to combine θ and $\arg\det M_q$ into one physically relevant parameter, $\bar{\theta} = \theta + \arg\det M_q$. Once ϕ acquires a vacuum expectation value, the $\phi Q\overline{Q}$ interaction endows the quarks with an effective mass term of order f_a:

$$\phi Q\overline{Q} \to \langle\phi\rangle e^{ia/f_a} Q\overline{Q} \qquad\qquad M_Q = e^{ia/f_a}\frac{f_a}{\sqrt{2}}. \tag{1.1.6}$$

Notice that the phase of the quark mass is determined by the value of a, making it a dynamical quantity. In the same way that a chiral rotation of the Standard Model quarks allows the phase of the determinant of the quark mass matrix to be shifted into the value of $\bar{\theta}$, an analogous rotation on Q and $\overline{Q}$ shifts this phase into $\bar{\theta}$ as well, so that

$$\bar{\theta}' = \theta + \arg\det M_q + \arg\det M_Q = \bar{\theta} + \frac{a}{f_a} \tag{1.1.7}$$

is the physical quantity that determines the size of the neutron electric dipole moment.

Nonperturbative effects from QCD induce a periodic potential for the axion, which can be heuristically described by the one-instanton potential

$$V(a) \approx m_\pi^2 f_\pi^2 \left(1 - \cos\left(\frac{a}{f_a} + \bar{\theta}\right)\right), \tag{1.1.8}$$

inducing a mass for the axion $m_a \sim m_\pi f_\pi / f_a$. A more precise expression can be derived from the chiral Lagrangian, to include the effect of broken chiral symmetry on the axion potential [16]:

$$V(a) = m_\pi^2 f_\pi^2 \left(2 - \sqrt{1 + \frac{2 m_u m_d}{(m_u + m_d)^2}\cos\left(\frac{a}{f_a} + \bar{\theta}\right)}\right), \tag{1.1.9}$$

where m_u and m_d are the up and down quark masses, respectively. In this case the axion mass would be

$$m_a^2 = \frac{m_u m_d}{(m_u + m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} = \frac{z}{(1+z)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2}, \tag{1.1.10}$$

where $z = m_u/m_d$. As a tangential observation, note that in the limit $m_u \to 0$, the shift symmetry $a \to a + \alpha f_a$ is restored in the vacuum.

Whether we use Eq. (1.1.8) or Eq. (1.1.9), the result is effectively the same: both potentials are minimized by the vacuum solution

$$\langle a \rangle = -\bar{\theta} f_a, \tag{1.1.11}$$

setting the physically relevant combination of CP violating phases to zero:

$$\bar{\theta} + \frac{a}{f_a} = 0. \tag{1.1.12}$$

Thus, CP is restored as a symmetry of the QCD vacuum, in perfect agreement with measurements of the neutron electric dipole moment.

1.2 QCD Axion Phenomenology

Originally, it was typically assumed that the axion decay constant f_a would be related to the electroweak scale [10,11,17,18], but this was quickly found to be phenomenologically unviable [19]. Two types of "invisible axion" models were soon formulated, both with $f_a \gg v_w$. The KSVZ-type "hadronic" axion [20,21] is the prototype for the axion model outlined in Section 1.1: new quarks carrying PQ charge gain $\mathcal{O}(f_a)$ masses when the complex scalar ϕ acquires an expectation value. In the KSVZ model, the quarks and leptons of the Standard Model are neutral under $U(1)_{\rm PQ}$, so that the new superheavy quarks are the only fermions with PQ charge. A second class of invisible axion, DFSZ [22,23] avoids adding any new fermions by giving the Standard Model matter fields PQ charges: this model requires two Higgs doublets ϕ_u and ϕ_d with hypercharge ± 1, and an electroweak singlet ϕ, with Peccei-Quinn charges such that the combination $(\phi_u \phi_d \phi^2)$ is neutral under $U(1)_{\rm PQ}$. The expectation value of ϕ produces a light axion as the pseudo-Nambu–Goldstone boson for the spontaneously broken $U(1)_{\rm PQ}$ symmetry, just as in the KSVZ model.

Both the KSVZ and DFSZ models feature a high scale for f_a and a correspondingly light mass for the axion. As the interactions between the Standard Model fields and the axion are suppressed by powers of f_a, the axion is also very weakly coupled: hence the nickname, "invisible axion". Astrophysical observations, especially of stellar cooling, constrain the value of f_a: if f_a is too low, then production of axions through the Primakov process [24] allows stars to cool at an enhanced rate. Observations of hydrogen- and helium-burning stars sets a lower bound $f_a \gtrsim 10^9$ GeV, or an upper bound on the axion mass $m_a < 10^{-2}$ eV [25,26].

A notable difference between the DFSZ and KSVZ models arises when considering the electromagnetic anomaly of the $U(1)_{\rm PQ}$ current. The mixed $SU(2)^2$-$U(1)_{\rm PQ}$ and $U(1)_Y^2$-$U(1)_{\rm PQ}$ anomalies induce an axion-photon coupling, $\mathcal{L} = \frac{1}{4} G_{a\gamma\gamma} a F_{\mu\nu} \widetilde{F}^{\mu\nu}$, where

$$G_{a\gamma\gamma} = \frac{\alpha}{2\pi f_a} \left(\frac{E}{N} - \frac{2}{3} \frac{4+z}{1+z} \right), \tag{1.2.1}$$

where $z = m_u/m_d$, and E and N represent the electromagnetic and color anomaly coefficients of the Peccei-Quinn current [27]. Because the DFSZ model utilizes only the Standard Model fermions, the value of E/N is unambiguously $E/N = 8/3$. In the KSVZ model, the value of E depends on the electroweak charges of Q and $\overline{Q}$: these are typically taken to be electroweak singlets, implying $E = 0$, but $E \neq 0$ can be achieved simply by giving some hypercharge to the superheavy quarks.

Bounds on f_a from Primakov cooling, then, are model-dependent, but do not become weaker at $E = 0$: on the contrary, with $z \approx 0.5$ the smallest values for $G_{a\gamma\gamma}$ are derived from models with $E/N \approx 2$.[2]

Part of the enduring popularity of axion models is due to the fact that they provide an excellent candidate for dark matter. Not only is the invisible axion suitably weakly coupled, but they can be produced with sufficient abundance in the early universe through the misalignment mechanism [29–31]. At temperatures above the electroweak scale ($T > 100$ GeV), the initial vacuum expectation value of the field ϕ is set when the electroweak symmetry is still unbroken: because the nonperturbative effects which generate the potential $V(a)$ for the axion turn off in the massless quark limit, the initial phase of $\langle\phi\rangle$ is essentially random:

$$\langle\phi\rangle = \frac{f_a}{\sqrt{2}} e^{i\theta_0}. \tag{1.2.2}$$

As the temperature decreases to $T \sim \sqrt{m_\pi f_\pi} \approx 100$ MeV, the QCD instanton effects serve to realign the vacuum towards the true minimum of the potential $V(a)$ from Eq. (1.1.9), from $\langle a\rangle/f_a = \theta_0$ to $\langle a\rangle/f_a = -\bar{\theta}$. The oscillation of the axion about its true minimum is described by the equations of motion:

$$\frac{d^2a}{dt^2} + (3H + \Gamma_a)\frac{da}{dt} + \frac{\partial V(a,T)}{\partial a} = 0, \tag{1.2.3}$$

where $H \sim T^2/M_{\rm P}$ is the Hubble rate, Γ_a represents the axion decay rate to photons, $a \to \gamma\gamma$, and $V(a,T)$ is the temperature-dependent potential for the axion. Considering that $\Gamma_a \sim m_a^3/f_a^2$ well exceeds the lifetime of the universe, we treat the axion as fundamentally stable with $\Gamma_a \approx 0$. Approximating $V(a,T) \approx m_\pi^2 f_\pi^2 + \frac{1}{2} m_a^2 a^2$ for the temperature-dependent mass $m_a(T)$, the equation of motion for the axion reduces to

$$\frac{d^2a}{dt^2} + 3H\frac{da}{dt} + m_a^2 a = 0, \tag{1.2.4}$$

which represents the coherent production of nonrelativistic axions. As the universe continues to cool, the energy density of the axion oscillations decreases as T^{-3}, while the energy densities of the relativistic degrees of freedom decrease more rapidly as T^{-4}: thus, the axion can soon become the dominant component of the universe's energy density.

Overly robust production of axions in the early universe can create a catastrophically overdense environment, where the energy density of axions ρ_a exceeds the critical density ρ_c, "overclosing" the universe and causing it to collapse. By solving Eq. (1.2.4) with the appropriate temperature dependence of the axion mass, the ratio ρ_a/ρ_c can be written in terms of f_a and the initial misalignment angle θ_0 [32]:

$$\Omega_a h^2 = \frac{\rho_a}{\rho_c} h^2 = 0.7 \left(\frac{f_a}{10^{12}\ \text{GeV}}\right)^{7/6} \left(\frac{\theta_0}{\pi}\right)^2, \tag{1.2.5}$$

where the requirement that axions not overclose the universe for $\mathcal{O}(1)$ values of θ_0 provides the constraint $f_a \lesssim 10^{12}$ GeV [29–31, 33].

[2] Taking for example $z = 0.56$ [28], it is in principle possible to tune the fractional hypercharge assignment in the KSVZ model with $E/N \approx 1.95$ so as to make $G_{a\gamma\gamma}$ effectively zero. The exercise in model-building is perhaps a poor way to make friends.

If axions are to be the dark matter, Eq. (1.2.5) suggests a lower bound on f_a: taking a large initial misalignment $\theta_0 \approx \pi$ and matching Ω_a to the measured density of cold dark matter, $\Omega_{CDM}h^2 \simeq 0.12$, we find

$$f_a^{(CDM)} \gtrsim 2.2 \times 10^{11} \text{ GeV}. \quad (1.2.6)$$

For smaller values of f_a, the misalignment mechanism no longer produces axions with the correct abundance to compose 100% of the dark matter. Larger values of f_a are made possible by somewhat smaller $\theta_0 < \pi$, though values of f_a much larger than 10^{12} GeV begin to require fine-tuned initial conditions in the early universe.

The density Eq. (1.2.5) assumes that the spontaneous breaking of $U(1)_{\text{PQ}}$ occurs before inflation, so that θ_0 is a uniform feature of the observable universe. If the order is reversed, then θ_0 varies across different parts of the early universe, and the average contribution to Ω_a is [32]:

$$\Omega_a h^2 = 0.3 \left(\frac{f_a}{10^{12} \text{ GeV}} \right)^{7/6}, \quad (1.2.7)$$

suggesting that QCD axion dark matter in this scenario calls for

$$f_a^{(CDM)} \approx 4.5 \times 10^{11} \text{ GeV}. \quad (1.2.8)$$

Bounds on axion models from the misalignment mechanism do depend on an initial temperature well above $T \sim 100$ MeV. Although a high reheating temperature T_R after the decay of the inflaton is a standard feature of many cosmological scenarios, measurements of the primordial abundances of light nuclei suggest only that $T_R \gtrsim 1$ MeV to match the successful predictions of Big Bang Nucleosynthesis (BBN). At such small initial temperatures the explicit Peccei-Quinn violating effects from QCD instantons are felt strongly, the assumption that the initial phase of $\langle \phi \rangle$ is randomly determined no longer applies, and the upper bound on f_a from the misalignment mechanism are relaxed. While many models of baryogenesis require high reheating temperatures, a number of low-temperature baryogenesis models with $T_R < 100$ GeV have been developed [34–36] that would be consistent with a higher value of f_a.

Many of the constraints on the QCD axion rely on its relation between f_a and m_a, $m_a f_a \sim m_\pi f_\pi$, which is the result of $U(1)_{\text{PQ}}$ being explicitly broken by QCD instantons. By generalizing the QCD axion to "axion-like particles", or ALPs, and dropping the requirement that the subsequent model is a solution to the strong CP problem, a wider field of phenomenological possibilities emerges, providing for example a more generic class of dark matter models. Though the production rate of the invisible QCD axion at particle colliders is negligible, and its lifetime very long, a less weakly coupled ALP could be produced and detected by collider searches.

1.3 The Axion Quality Problem

As discussed in Section 1.1, axion solutions to the strong CP problem have a generic common property: a spontaneously broken global $U(1)_{\text{PQ}}$, which is a symmetry of the classical action, but is explicitly violated by the QCD chiral anomaly. This simplicity, together with the predictive power of the theory and its viability as a dark matter candidate, has ensured the lasting popularity of the QCD axion as a possible extension to the Standard Model. In this section we explore the theoretical problems associated with simple axion models, namely a hierarchy problem and a fine-tuning problem.[3]

[3]Parts of this introduction quote the description of the axion quality problem found in [3], by the author and Tim M.P. Tait.

Simple axion models are plagued by the theoretical inconsistencies endemic to theories containing fundamental scalar fields. The scale f_a, set by the expectation value of a scalar field, is sensitive to potentially large additive threshold corrections based on details of the high-energy theory, for example at the GUT or Planck scales. This generic property of scalar fields is especially well-studied in the more severe case of the electroweak hierarchy problem. Solutions such as supersymmetry or compositeness which can render the electroweak scale technically natural work equally well to stabilize $f_a \ll M_{\mathrm{P}}$ in the axion model.

The requirement that $U(1)_{\mathrm{PQ}}$ is an exact symmetry of the Lagrangian turns out to be a significantly more serious liability than the hierarchy between f_a and M_{P}, introducing a degree of fine-tuning that is much more severe than the factor of 10^{-10} it was designed to explain. Arguments from general relativity [37–44] suggest that non-perturbative quantum gravitational effects do not respect global symmetries such as baryon number or $U(1)_{\mathrm{PQ}}$. If additional PQ violating operators representing the short distance influence of quantum gravity such as

$$\Delta V(\phi) = \frac{|\phi|^{k+3}}{M_{\mathrm{P}}^k} \left(\lambda_k \phi + \lambda_k^\star \phi^\star\right) \tag{1.3.1}$$

are present, the corresponding perturbation in $V(a)$ can shift $\langle a \rangle$ far away from the CP-conserving value of Eq. (1.1.12):

$$\delta V(a) \sim \lambda_k f_a^4 \left(\frac{f_a}{M_{\mathrm{P}}}\right)^k \cos\left(\Delta_{\mathrm{PQ}} \frac{a}{f_a} - \varphi\right), \tag{1.3.2}$$

where the phase φ is determined by λ_k, and Δ_{PQ} is the $U(1)_{\mathrm{PQ}}$ charge of the operator ϕ. It is convenient to describe such perturbations by defining a "quality factor" Q:

$$\delta V(a) = Q f_a^4 \cos\left(\frac{a}{f_a} - \varphi\right). \tag{1.3.3}$$

If we assume φ is not tuned to match the value of $\bar{\theta}$, the measured value of $\left|\bar{\theta}\right| \lesssim 10^{-10}$ is achieved only if $\delta V(a)$ satisfies

$$Q \lesssim 10^{-62} \left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)^4. \tag{1.3.4}$$

Satisfying this bound requires that the theory of quantum gravity somehow produce a severe fine-tuning in the λ_k, such that even the dimension-12 operators in Eq. (1.3.2) must have $\lambda_k \ll 1$. The derivation for Eq. (1.3.4) is left to Appendix A.1.

In the "worst-case scenario" for the axion model, threshold corrections at the Planck scale induce the $k = -3$ operator $\mathcal{L} \sim \lambda_{-3} M_{\mathrm{P}}^3 (\phi + \phi^\star)$, which spoils the axion solution to the strong CP problem unless

$$\lambda_{-3} \lesssim 10^{-83} \tag{1.3.5}$$

for typical values of $f_a \sim 10^{12}$ GeV. Considering that the axion is introduced to explain fine-tuning of $\mathcal{O}(10^{-10})$, this calls its motivation into serious question.

1.4 Robust Solutions to the Strong CP Problem

In a truly compelling axion model, the $U(1)_{\mathrm{PQ}}$ symmetry should emerge as a consequence of some other underlying structure which forbids the problematic operators. For example, a gauged discrete $\mathbb{Z}_n$ symmetry [45] for some $n \gtrsim 13$ can forbid all PQ violating operators smaller than

($\phi^n + c.c.$). More sophisticated models can employ discrete groups as small as $\mathbb{Z}_4$ while forbidding the problematic operators [46, 47].

Solutions without gauged discrete symmetries also exist: for example, a composite model [48] with a gauged $SU(N) \times SU(m) \times SU(3)_c$ protects $U(1)_{\rm PQ}$ to arbitrarily high order, with the added benefit that the axion scale f_a can be generated dynamically. Qualitatively different recent models [49, 50] have also been shown to provide the appropriate protection from Planck scale corrections.

Other constructions protect $U(1)_{\rm PQ}$ by gauging a related Abelian group. In one model [51] with a compact extra dimension, a gauged $U(1)$ symmetry is spontaneously broken by fields localized on two separated four-dimensional branes. One combination of the fields is eaten by the gauge field, while the other acts as the QCD axion and is protected from gravitational corrections. A related model [52] gauges a product group of the form $U(1)^k$ with $k \geq 14$, which can also be interpreted as a k site deconstruction of a compact fifth dimension. In a different class of models [42,53], the fields are assigned large and relatively prime $U(1)$ charges, so that an accidental $U(1)_{\rm PQ}$ is protected from low-dimensional operators. Many of these constructions are intricate and also rather delicate in the sense that the axion quality is easily ruined in extensions of the model.

Some of these models, while successful at forbidding low-dimensional $U(1)_{\rm PQ}$-breaking operators, still suffer from a hierarchy problem. One resolution is supersymmetry (SUSY), which protects f_a from loop-level corrections, so that the theory is technically natural if the SUSY-breaking scale is not much larger than f_a. Another compelling direction is composite models, which can suppress dangerous gravitational contributions to the axion potential while allowing the scale of $U(1)_{\rm PQ}$ breaking to be determined from the confining dynamics. For asymptotically free gauge theories the confinement scale is expected to be exponentially suppressed compared to $M_{\rm P}$, so the hierarchy between f_a and $M_{\rm P}$ can be naturally generated dynamically.

In this thesis, we explore the solutions to the axion quality problem based on confining supersymmetric theories which have been proposed in two recent papers, [2] and [3]. Chapter 2 features a brief introduction to supersymmetry, focusing in particular on the use of Seiberg dualities to describe the low-energy behavior of strongly coupled gauge theories. While the behavior of product gauge groups is generally less well understood, a class of models investigated by the author [1] is shown to confine without breaking chiral symmetry, and we devote the remainder of Chapter 2 to these results.

In Chapter 4 we describe how a product gauge theory of this type can be employed as a solution of the strong CP problem, following [2]. To ensure spontaneous breaking of the $U(1)_{\rm PQ}$ symmetry at the appropriate scale, some modification to the original model is required: either a deformation of the tree level superpotential, or an expansion of the gauge structure. While we show that the resulting model does suppress PQ violating operators to the appropriate degree in the superpotential, its intricate nature and large gauge groups lead us to suggest that the model is not more realistic than the previously studied solutions to the axion quality problem [45,48,50,51,53].

A simpler and more promising model is examined in Chapter 5, based on the recently published results of [3]. Here we show that a more concise group structure with strongly coupled $SU(5)$ gauge groups achieves sufficiently small values of $\bar{\theta}$ for $f_a \lesssim 3 \times 10^{11}$ GeV, in the preferred region of parameter space where the axion is a compelling dark matter candidate. In addition to satisfactorily addressing the stated goal of solving the strong CP problem, the model is well suited to provide a solution to the μ problem of the MSSM, as well as messenger candidates for gauge-mediated supersymmetry breaking.

In this respect the model of [3] fulfills many of the criteria for a "truly compelling axion model": not only does it solve the strong CP problem in a robust way while providing a dark matter candidate, but it also supplies a naturally TeV scale for the μ term in the MSSM, all without additional model-building effort.

In Chapter 6, we conclude with some remarks about possible future directions of research.

Chapter 2

Supersymmetry and Confinement

Quantum field theory, as the union of special relativity and quantum mechanics, is designed around the symmetries intrinsic to flat spacetime: the translations, rotations and boosts that comprise the Poincaré group. This framework greatly restricts the range of possible additional symmetries that can be built into a model: a no-go theorem by Coleman and Mandula [54] based on the analyticity of the S-matrix shows that the only symmetries consistent with the Poincaré group are those which are internal, such as the interchange of two particles of identical spin, mass and interactions with other particles.

Supersymmetry (SUSY) developed as an extension to the internal symmetries known to be consistent with the Poincaré group. A supersymmetric transformation interchanges bosons and fermions, and is described by a graded Lie algebra [55, 56]. It was soon shown [57] that supersymmetry, combined with the types of internal symmetries previously discussed, formed the most general set of symmetries consistent with the Poincaré group.

2.1 Supersymmetry and the Electroweak Hierarchy

A number of observations about supersymmetric theories led to a dramatic increase in their popularity. Notably, supersymmetric theories are not perturbatively renormalized [58]: this property makes supersymmetry an appealing possible solution to the electroweak hierarchy problem.

In the Standard Model, the electroweak scale depends on the quadratic term in the Higgs potential, $V \sim -\mu^2 |H|^2 + \lambda |H|^4$, with $\mu^2 \ll M_{\text{P}}^2$. A small value of μ^2 is not natural in the 't Hooft sense [59]; no symmetry emerges in the $\mu^2 \to 0$ limit, and the electroweak scale is very sensitive to deformations of the high-energy theory. The presence of new heavy particles at some high energy scale $M_\star$, such as the GUT or Planck scale, can introduce large threshold corrections to the Higgs mass that tend to raise it to the high energy scale, $\mu^2 \sim \mathcal{O}(M_\star^2)$. Achieving $\mu \sim 100$ GeV in the low energy effective theory from such a framework requires the parameters of the high energy theory to be tuned to precisely cancel any such threshold corrections.[1] This sensitivity of the electroweak scale to the details of the high-energy theory is referred to as the electroweak hierarchy problem.

Supersymmetric theories provide an exception to this rule. Nonrenormalization of the superpotential stabilizes the Higgs mass parameter, as we discuss more generally in Section 2.2. In the MSSM superpotential, a $U(1)_R$ symmetry is restored in the limit where the supersymmetric Higgs mass vanishes: thus, if supersymmetry is manifest below the scales M_{P} and M_{GUT}, then the observed hierarchy between M_{P} and μ is consistent with the 't Hooft definition of naturalness.

[1]For a recent comprehensive review of effective field theory, see for example [60].

2.2 Superfields and Superpotentials

In this section we address some of the mechanics of supersymmetric theories that will be relevant for the present discussion. As a canonical introduction to supersymmetry, we recommend [61] to readers wishing for more detail.

The minimal realization of supersymmetry in four dimensions invokes four spinor generators Q_α and $Q^\dagger_{\dot\alpha}$ which interchange fermionic and bosonic degrees of freedom. These generators of the supersymmetric transformation satisfy anticommutation relations,

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu \qquad \{Q_\alpha, Q_\beta\} = 0 \qquad \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0; \tag{2.2.1}$$

commute with the generators of spacetime translations,

$$[Q_\alpha, P^\mu] = 0 \qquad [Q^\dagger_{\dot\alpha}, P^\mu] = 0, \tag{2.2.2}$$

and therefore also with $P^2 = P_\mu P^\mu$; are charged under a global $U(1)_R$ symmetry,

$$[Q_\alpha, R] = Q_\alpha \qquad [Q^\dagger_{\dot\alpha}, R] = -Q^\dagger_{\dot\alpha}; \tag{2.2.3}$$

and carry mass dimension $\frac{1}{2}$, as can be seen from the appearance of momentum P^μ in Eq. (2.2.1). The dotted and undotted spinor indices $\dot\alpha = 1,2$ and $\alpha = 1,2$ refer respectively to left and right handed spinors. Extended supersymmetric algebras $\mathcal{N} > 1$ can be constructed by introducing multiple copies of Q and $Q^\dagger$, but we focus exclusively on the $\mathcal{N} = 1$ case.

Generically, the irreducible representations of the SUSY algebra are *supermultiplets* composed of both bosons and fermions. In $\mathcal{N} = 1$ SUSY, the simplest examples pair a chiral fermion either with a spin-0 complex scalar field or a spin-1 vector boson; the resulting supermultiplets are referred to as *chiral* (matter) or *vector* (gauge) supermultiplets, respectively, and comprise the building blocks of renormalizable $\mathcal{N} = 1$ theories [61]. Note that the left- and right-handed superpartners of the gauge bosons, "gauginos", transform in the adjoint representation of the gauge group and are Majorana in nature, while the fermionic components of chiral supermultiplets are generally free to take complex representations.

The construction of an interacting supersymmetric theory is simplified by the introduction of *superfield* and *superpotential* notation. A generic superfield is written as an expansion in terms of four Grassmannian variables θ_α and $\theta^\dagger_{\dot\alpha}$ [61],

$$S = a + \theta\xi + \theta^\dagger\chi^\dagger + \theta\theta b + \theta^\dagger\theta^\dagger c + \theta^\dagger\bar\sigma^\mu\theta v_\mu + \theta^\dagger\theta^\dagger\eta + \theta\theta\theta^\dagger\zeta^\dagger + \theta\theta\theta^\dagger\theta^\dagger d, \tag{2.2.4}$$

where spinor indices have been suppressed: repeated variables $\theta\theta$ and $\theta^\dagger\theta^\dagger$ implicitly refer to the nonzero products of Grassmann variables $\theta_1\theta_2$ and $\theta^\dagger_1\theta^\dagger_2$. The generic superfield Eq. (2.2.4) includes several spin-0 fields, a, b, c, d; left- and right- handed fermions ξ, η, $\chi^\dagger$ and $\zeta^\dagger$; and a spin-1 boson, v_μ, and corresponds to a sum of irreducible representations of the SUSY algebra.

A description of the chiral and anti-chiral covariant derivatives used to define the irreducible representations can be found in [61]. The result is quoted below: chiral and anti-chiral superfields can generically be written as

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F \qquad \Phi^\star = \phi^\star + \sqrt{2}\theta^\dagger\psi^\dagger + \theta^\dagger\theta^\dagger F^\star, \tag{2.2.5}$$

while the expansion of the vector superfield in the Wess-Zumino gauge [56] is

$$V = \theta^\dagger\bar\sigma^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\star + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D. \tag{2.2.6}$$

The scalar fields F, $F^\star$ and D have no kinetic terms in the Lagrangian, and are referred to as auxiliary scalars. Their presence is needed to match the number of off-shell bosonic and fermionic degrees of freedom.

In the superfield notation, the simple supersymmetric theory of a single free chiral supermultiplet Φ from Wess and Zumino [56] can be written concisely as [61]:

$$\mathcal{L}_{\text{free}} = \int d^2\theta d^2\theta^\dagger \Phi^\star\Phi = \partial^\mu\phi\partial_\mu\phi^\star + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^\star F, \tag{2.2.7}$$

up to a total derivative. Supersymmetric masses and interactions can be added as well, for example by

$$\mathcal{L}_{\text{int}} = \int d^2\theta \left(\mu\Phi^2 + \lambda\Phi^3\right) + h.c. \tag{2.2.8}$$

More generically, the full Lagrangian of a theory with multiple chiral superfields Φ_i can be written in the form

$$\mathcal{L} = \int d^2\theta d^2\theta^\dagger K(\Phi_i, \Phi_i^\dagger) + \int d^2\theta W(\Phi_i) + \int d^2\theta^\dagger W^\star(\Phi_i^\star), \tag{2.2.9}$$

where the *superpotential* W is a holomorphic function of the chiral superfields Φ_i, and the *Kähler potential* $K(\Phi_i, \Phi_i^\dagger)$ is a real function of the chiral and anti-chiral superfields.

If we restrict our attention to theories of k chiral superfields $\Phi_{1\ldots k}$ with renormalizable interactions, the canonically normalized Kähler potential is simply

$$K = \sum_{i=1}^{k} \Phi_i^\star\Phi_i, \tag{2.2.10}$$

while the superpotential has the form

$$W = b_i\Phi_i + \mu_{ij}\Phi_i\Phi_j + \lambda_{ijk}\Phi_i\Phi_j\Phi_k, \tag{2.2.11}$$

where repeated indices imply summation. Note that the mass dimensions of θ and $\theta^\dagger$ are $-\frac{1}{2}$, so that the integration rules $\int d^2\theta\,\theta\theta = 1$ and $\int d^2\theta^\dagger\,\theta^\dagger\theta^\dagger = 1$ imply that K and W have mass dimensions $+2$ and $+3$, respectively. Nonrenormalizable interactions can be easily accommodated by the Kähler and superpotentials as well: for example, by $K \sim \frac{1}{\Lambda^2}(\Phi^\star\Phi)^2$ or $W \sim \frac{1}{\Lambda}\Phi^4$ for some mass scale Λ.

Recall that each Φ_i superfield contains a scalar F_i with mass dimension $+2$, which appears in the renormalizable Kähler potential only as $\int d^4\theta\,K \to \mathcal{L} \sim F^\star F$, with no derivative interactions. These auxiliary scalars represent off-shell bosonic degrees of freedom, rather than on-shell propagating states. As F_i is multiplied by θ^2 in the supermultiplet, the integral $\int d^2\theta\,W$ can be reformulated as a partial derivative, so that the F_i dependence of the Lagrangian can be written:

$$\mathcal{L}_F = F_i^\star F_i + \frac{\partial W}{\partial\Phi_i}F_i + \frac{\partial W^\star}{\partial\Phi_i^\star}F_i^\star. \tag{2.2.12}$$

In cases where we are concerned primarily with the vacuum of the theory, we can integrate out the auxiliary scalars to derive the scalar potential for the associated ϕ_i, using the equations of motion from Eq. (2.2.12):

$$F_i = -\left.\frac{\partial W}{\partial\Phi_i}\right|_{\Phi\to\phi} \qquad\qquad F_i^\star = -\left.\frac{\partial W^\star}{\partial\Phi_i^\star}\right|_{\Phi^\star\to\phi^\star}, \tag{2.2.13}$$

where $\Phi \to \phi$ indicates that after taking the partial derivative of W with respect to the superfield Φ, the equations of motion are obtained by replacing Φ with its scalar component ϕ. It is a standard abuse of notation to use Φ to refer both to the superfield and its non-auxiliary scalar, which we adopt throughout this thesis. With this convention, the scalar potential $V(\phi_i)$ is written:

$$V(\phi_i) = -\mathcal{L}_F = \left|\frac{\partial W}{\partial \Phi_i}\right|^2 \geq 0. \tag{2.2.14}$$

Supersymmetric theories often feature a continuous space of degenerate vacua satisfying $V = 0$, known as a *moduli space*, parameterized by the expectation values of the scalar fields. The subject of spontaneous symmetry breaking in supersymmetric theories is thus intrinsically linked to the study of moduli spaces.

Thus far we have considered only supersymmetric theories with chiral superfields. Gauge theories introduce vector superfields of the form Eq. (2.2.6). It is convenient to use the chiral covariant derivatives to define the field strength chiral superfields:

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{DD}\left(e^{-V} D_\alpha e^V\right), \tag{2.2.15}$$

which in the Wess-Zumino gauge becomes:

$$\mathcal{W}^a_\alpha = \lambda^a_\alpha + \theta_\alpha D^a + \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + i\theta\theta(\sigma^\mu \nabla_\mu \lambda^{\dagger a})_\alpha \tag{2.2.16}$$

following the conventions of [61] where ∇_μ is used to refer to the standard gauge covariant derivative,

$$\nabla_\mu \lambda = \partial_\mu \lambda + igA^a_\mu T^a \lambda \tag{2.2.17}$$

and where $\mathcal{W}_\alpha = 2gT^a \mathcal{W}^a_\alpha$, for generators T^a in the adjoint representation. The coupling g is typically combined with the CP violating $\bar{\theta}$ term to define a holomorphic gauge coupling

$$\tau = \frac{1}{g^2} - i\frac{\bar{\theta}}{8\pi^2}. \tag{2.2.18}$$

Supposing that there are some chiral superfields Φ_i which transform as some nontrivial representation r of the gauge group, the gauge-invariant supersymmetric Lagrangian for the theory can be written [61]:

$$\mathcal{L} = \frac{g^2}{4}\left(\int d^2\theta\, \tau \mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha + c.c.\right) + \int d^4\theta\, \Phi^{\star i}_i (e^{2gT^a_r V^a})_i{}^j \Phi_j + \int d^2\theta\, W(\Phi) + c.c. \tag{2.2.19}$$

2.3 The Minimal Supersymmetric Standard Model

The minimal supersymmetric extension to the Standard Model, the MSSM, introduces a scalar superpartner for each quark and lepton, *squarks* and *sleptons*, and fermionic gaugino superpartners for each of the gauge bosons. It also replaces the Standard Model Higgs boson with not one but two chiral superfields, H_u and H_d. A full list of the chiral superfields and their $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge charges is given below:

$$Q_i : (\mathbf{3}, \mathbf{2})_{\frac{1}{6}}, \quad \bar{u}_i : (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}}, \quad \bar{d}_i : (\overline{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}, \quad L_i : (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}, \quad \bar{e}_i : (\mathbf{1}, \mathbf{1})_1, \quad H_u : (\mathbf{1}, \mathbf{2})_{\frac{1}{2}}, \quad H_d : (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}, \tag{2.3.1}$$

where the index $i = 1, 2, 3$ refers to the three generations of matter.

The need for two Higgs supermultiplets arises by requiring the cancellation of all anomaly coefficients for the electroweak gauge symmetries. Adding a single fermionic partner to the Standard Model Higgs would create a model with an odd number of $SU(2)_L$ fermion doublets, introducing a Witten anomaly [62] and several other nonvanishing anomaly coefficients involving hypercharge, any of which would render the theory inconsistent.

Interactions between the superfields are encoded in the superpotential

$$W = \mu H_u H_d + (Y_u)_{ij}\bar{u}_i Q_j H_u - (Y_d)_{ij}\bar{d}_i Q_j H_d - (Y_e)_{ij}\bar{e}_i L_j H_d, \tag{2.3.2}$$

in addition to the standard field strength terms for the gauge bosons. Crucially, several terms allowed by gauge invariance are absent from this superpotential: for example, those derived by interchanging L_i and H_d in Eq. (5.2.28), or the gauge-invariant operator $\bar{u}\bar{d}\bar{d}$. These renormalizable operators violate baryon and lepton number in contradiction with observation, for example by inducing rapid proton decay.

An additional discrete symmetry is imposed on the MSSM to avoid these disastrous consequences: "matter parity", a discrete Z_2 subgroup of $U(1)_{B-L}$,

$$P_M = (-1)^{3(B-L)}. \tag{2.3.3}$$

Equivalently, by combining matter parity with fermion number F, this symmetry can be recast as the "R parity",

$$P_R = (-1)^{3(B-L)+F}, \tag{2.3.4}$$

under which all the Standard Model fields are even, and their superpartners are odd. Including P_M as an exact discrete symmetry of the MSSM is natural in contexts where $B - L$ is locally conserved, and spontaneously broken to a discrete subgroup at some high energy scale.

In general, $B - L$ symmetry does not guarantee the absolute stability of the proton, as the initial and final states of the decay process $p \to \pi^0 + e^+$ have the same $B - L$ number. However, by forbidding the renormalizable operators in the MSSM superpotential, $B - L$ forces proton decay to be mediated by irrelevant operators at signficantly suppressed rates.

Imposing R parity on the MSSM has significant phenomenological implications: the lightest supersymmetric particle (LSP) is absolutely stable, providing a dark matter candidate if the LSP is electrically neutral.

As the mechanics of supersymmetry breaking are unknown, the MSSM superpotential is supplemented by a wide variety of Lagrangian interactions that explicitly violate supersymmetry, but which could in principle be calculated once a precise mechanism of spontaneous supersymmetry breaking (SSB) is chosen. This SUSY-breaking Lagrangian $\mathcal{L}_{\text{soft}}$ is not taken to be entirely generic: instead, we impose the requirement of "soft" supersymmetry breaking, where $\mathcal{L}_{\text{soft}}$ contains only relevant operators (mass terms and other couplings with positive mass dimension), so that supersymmetry is restored in the limit where all these mass scales are taken to be vanishingly small, $m_{\text{soft}} \to 0$. Such a result stands in contrast to the counterexample in which marginal operators that explicitly violate SUSY are included in $\mathcal{L}$. A $\mathcal{L}_{\text{hard}}$ of this type reintroduces the dangerous sensitivity of the Higgs mass parameter to the high-energy limit of the theory, spoiling supersymmetry as a solution to the electroweak hierarchy problem.

Even with the requirement of soft supersymmetry breaking, the generic form for $\mathcal{L}_{\text{soft}}$ includes a great number of terms. This is the origin of the masses for the squarks, the sleptons and the gauginos which make these heavier than their superpartners. Gauge-invariant trilinear couplings between the scalars are also included, in what is often referred to as the "A-term potential". Notably, the supersymmetric mass term $W = \mu H_u H_d$ for the Higgs doublets is supplemented by a non-supersymmetric "b" term, $\mathcal{L}_{\text{soft}} = -bH_u H_d + c.c.$, as well as their individual SUSY-breaking masses $m^2_{H_u}$ and $m^2_{H_d}$.

Generating the appropriate potential for electroweak symmetry breaking requires μ^2 and b to match each other fairly well, $\mu^2 \sim b$. In principle the scales $b \sim m^2_{\text{soft}}$ and μ^2 need not have any relation to each other, making their coincidence highly suspicious. This is variously referred to as the μ problem, the μ/b problem, or the μ/B_μ problem of the MSSM. Various models have been proposed which would explain the proximity of their values, typically by generating μ by the same mechanism that spontaneously breaks supersymmetry [63–66].

In Chapter 5, we show how the search for a simpler composite solution to the axion quality problem led to the development of a model which accidentally resolves the μ problem of the MSSM, while at the same time supplying the $B - L$ symmetry needed to forbid the problematic operators in its superpotential.

First, we review some aspects of confining SUSY gauge theories in the following Sections 2.4 and 2.4.1. Section 2.4.2 discusses more specific properties of the $A + 4Q + N\overline{Q}$ model with an $SU(N)$ gauge group, including the coefficients in its dynamically generated superpotential. The derivation is presented in Appendix B.1. All of the material in Sections 2.4, 2.4.1 and 2.4.2 has been previously published by the author in [1].

2.4 Seiberg Dualities

It is generally difficult to analyze the infrared behavior of strongly coupled theories, due to the failure of perturbation theory in this limit. Seiberg, Intriligator and others have made this problem more tractable by exploiting some of the remarkable properties of supersymmetry, allowing some infrared properties of SUSY gauge theories to be calculated exactly [67, 68]. Seiberg's infrared dualities between different phases of gauge theories were central to these developments. We summarize some of the results in this section; a more detailed review is given in [69].

Seiberg found that in $SU(N)$ gauge groups with F flavors of quarks and antiquarks, also known as SUSY QCD, the infrared behavior of the $F = N$ and $F = N+1$ cases can be completely described by a set of gauge invariant operators, $M = Q\overline{Q}$, $B = Q^N$, and $\overline{B} = \overline{Q}^N$. This dual theory has no gauge interactions, so the $F = N$ and $F = N + 1$ theories are said to confine: every test charge can be "screened" by creating quark-antiquark pairs from the vacuum, and a gauge-invariant Wilson loop obeys a perimeter law.

Classically, the gauge invariant operators obey particular constraints, following from the Bose symmetry of the superfields and the definitions of M, B, and $\overline{B}$. For $F = N + 1$,

$$\begin{aligned} B_i M^i_j &= 0 \\ M^i_j \overline{B}^j &= 0 \\ (M^i_j)^{-1} \det M &= B_i \overline{B}^j , \end{aligned} \tag{2.4.1}$$

while for $F = N$

$$\det M - B\overline{B} = 0, \tag{2.4.2}$$

where the indices i and j refer to the family $SU(F)$ symmetries of the Q and $\overline{Q}$. It has been shown [70–72] that Eq. (2.4.2) is modified quantum mechanically:

$$\det M - B\overline{B} = \Lambda^b, \tag{2.4.3}$$

where Λ^b is the holomorphic scale

$$\Lambda^b = \mu^b \exp\left\{-8\pi^2/g^2 + i\theta_{\rm YM}\right\}. \tag{2.4.4}$$

Here $\theta_{\rm YM}$ is the CP-violating θ-term of the $SU(N)$ gauge group, g is the gauge coupling, and $b = 3N - F = 2N$ is derived from the β function for the gauge coupling. The quantum-modified constraint Eq. (2.4.3) can be enforced by a superpotential

$$W = \lambda\left(\det M - B\overline{B} - \Lambda^{2N}\right) \tag{2.4.5}$$

if we introduce a Lagrange multiplier superfield λ. At the origin of the classical moduli space, $M = B = \overline{B} = 0$, the UV family symmetry $SU(F)_L \times SU(F)_R \times U(1)_B$ is conserved. However, this point is not on the quantum-deformed moduli space given by Eq. (2.4.3), so the chiral symmetry is broken in the vacuum.

2.4.1 S-Confinement

In the $F = N + 1$ case, the classical constraint equations are not modified. Instead, they are enforced by a dynamically generated superpotential [73].

$$W_d = \frac{1}{\Lambda^{2N-1}}\left[BM\overline{B} - \det M\right], \tag{2.4.6}$$

which has $\langle M\rangle = \langle B\rangle = \langle\overline{B}\rangle = 0$ as a solution to the equations of motion. This vacuum corresponds to confinement without chiral symmetry breaking, which we refer to as s-confinement. More precisely, a theory is s-confining if [74]:

- All infrared degrees of freedom are gauge invariant composite fields;
- The infrared physics is described by a smooth effective theory, which is valid everywhere on the moduli space (including the origin);
- There is a dynamically generated superpotential.

For the effective theory to be smooth, there should be no gauge invariant order parameter that can distinguish the Higgs and confined phases of the theory. The infrared degrees of freedom must also satisfy the anomaly matching conditions.

Generally, the dynamically generated superpotential is determined up to an overall factor based on symmetry arguments, and by matching its equations of motion to the classical constraints. Its dependence on the holomorphic scale Λ^b can be found either on dimensional grounds, or by requiring that W_d is neutral under the anomalous $U(1)$ symmetry.

The requirement that a superpotential is dynamically generated adds a powerful constraint on the matter content of any s-confining theory. An $\mathcal{N} = 1$ SUSY theory with f massless matter superfields has a classical family symmetry of rank $f+1$ including the R symmetry, but the $G^2U(1)$ anomaly removes one linear combination of the $U(1)$ family symmetries. This allows us to define a $U(1)_R$ symmetry such that exactly one of the matter superfields ϕ_i has R charge, q_i, with all other fields neutral. Using the normalization in which the gauginos have R charge $+1$, cancellation of the $G^2U(1)_R$ anomaly requires that

$$q_i = \frac{1}{\mu_i}\left[\sum_j \mu_j - \mu_G\right], \tag{2.4.7}$$

where μ_j and μ_G are the Dynkin indices of the matter fields ϕ_j and the gluinos, respectively, with the normalization $\mu(\Box) = 1$. For the dynamically generated superpotential to have R charge $+2$ under any of the possible anomaly-free R symmetries, it must have the form

$$W \sim \prod_i \left[\phi_i^{2/q_i}\right] = \prod_i \left(\phi_i^{\mu_i}\right)^{2/\left[\sum_j \mu_j - \mu_G\right]}. \tag{2.4.8}$$

The matter content must therefore satisfy the index constraint of Csaki *et al.* [74]:

$$\sum_j \mu_j - \mu_G = 2. \tag{2.4.9}$$

In [75] this index constraint is used to find all $\mathcal{N} = 1$ s-confining theories with one gauge group and no tree-level superpotential. Both $F = N + 1$ SUSY QCD and the $A + 4Q + N\overline{Q}$ model are included.

In theories with a product gauge group this constraint is relaxed: the number of fields exceeds the rank of the family symmetry, and it is no longer possible to identify a unique R symmetry for each field.

2.4.2 $SU(N)$ with antisymmetric tensor

Properties of the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + F\,\Box + (N + F - 4)\,\overline{\Box}$ model have been studied by several authors [76–79]. In the $F = 2$ case there is a superpotential generated by a one-instanton effect; for $F = 3$ the theory confines, with a quantum-deformed moduli space that induces dynamical symmetry breaking; and for $F = 4$, the theory is s-confining. The quantum modified constraints have been derived in [77] for $F = 3$, but the classical constraints for the $A + 4Q + N\overline{Q}$ model do not appear in the literature. We derive the relative coefficients of the dynamically generated superpotential in Appendix B.1, and quote the results in this section.

Infrared operators: In the $A + 4Q + N\overline{Q}$ model, the set of gauge invariant operators changes based on whether N is even or odd. This is due to the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ representation: if $N = 2m$ is even, then the gauge invariants include the antisymmetrized products (A^m), $(A^{m-1}Q^2)$, and $(A^{m-2}Q^4)$, while for odd $N = 2m + 1$ the gauge invariants include $(A^m Q)$ and $(A^{m-1}Q^3)$.

Below, we define the simplest gauge invariant operators for the $N = 2m$ and $N = 2m + 1$ models. Both cases include the operators $(Q\overline{Q})$, $(A\overline{Q}^2)$, and $(\overline{Q}^N)$:

$$J^i_j = Q^i_\alpha \overline{Q}^\alpha_j, \tag{2.4.10}$$

$$K_{j_1 j_2} = A_{\alpha\beta} \overline{Q}^\alpha_{j_1} \overline{Q}^\beta_{j_2}, \tag{2.4.11}$$

$$Z = \det \overline{Q} = \frac{\epsilon_{\alpha_1 \ldots \alpha_N} \epsilon^{j_1 \ldots j_N}}{N!} \left(\overline{Q}^{\alpha_1}_{j_1} \overline{Q}^{\alpha_2}_{j_2} \ldots \overline{Q}^{\alpha_N}_{j_N}\right). \tag{2.4.12}$$

For even $N \geq 4$, we also add the gauge invariants

$$U = \mathrm{Pf}\, A = \frac{\epsilon^{a_1 a_2 \ldots a_N}}{2^m m!} \left(A_{a_1 a_2} A_{a_3 a_4} \ldots A_{a_{N-1} a_N}\right), \tag{2.4.13}$$

$$V_{i_1 i_2} = \frac{\epsilon^{a_1 a_2 \ldots a_N}}{2^{m-1}(m-1)!2!} \left(A_{a_1 a_2} A_{a_3 a_4} \ldots A_{a_{N-3} a_{N-2}}\right) Q^{i_1}_{a_{N-1}} Q^{i_2}_{a_N}, \tag{2.4.14}$$

$$\mathcal{W} = \frac{\epsilon^{a_1 a_2 \ldots a_N}}{2^{m-2}(m-2)!} \frac{\epsilon_{j_1 j_2 j_3 j_4}}{4!} \left(A_{a_1 a_2} A_{a_3 a_4} \ldots A_{a_{N-5} a_{N-4}}\right) Q^{j_1}_{a_{N-3}} Q^{j_2}_{a_{N-2}} Q^{j_3}_{a_{N-1}} Q^{j_4}_{a_N}, \tag{2.4.15}$$

	G	$SU(4)_L$	$SU(N)_R$	U_A	U_B	U_R	U_1
A	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$			-4	-1	0	0
Q	$\Box$	$\Box$		$N-2$	$-1/2$	$1/2$	0
$\overline{Q}$	$\overline{\Box}$		$\Box$	0	1	0	1
Λ^b				0	0	0	N
J		$\Box$	$\Box$	$N-2$	$1/2$	$1/2$	1
K			$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	-4	1	0	2
Z				0	N	0	N
U				$-2N$	$-N/2$	0	0
V		$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$		0	$-N/2$	1	0
$\mathcal{W}$				$2N$	$-N/2$	2	0
X		$\Box$		$-N$	$-N/2$	$1/2$	0
Y		$\overline{\Box}$		N	$-N/2$	$3/2$	0

Table 2.1: The transformation properties of the UV and IR fields under the family $SU(4)_L \times SU(N)_R \times U(1)_A \times U(1)_B \times U(1)_R$ symmetry for the $F = 4$ model are shown, along with the charges under the spurious $U(1)_1$. The operators J, K, and Z are defined whether N is even or odd; the fields U, V and $\mathcal{W}$ are specific to the even N case, while the fields X and Y correspond to the odd N case. The $U(1)_R$ charges listed refer to the scalar component of each superfield.

whereas for odd $N \geq 5$ we include

$$X^j = \frac{\epsilon^{a_1 a_2 \ldots a_N}}{2^m m!} \left(A_{a_1 a_2} A_{a_3 a_4} \ldots A_{a_{N-2} a_{N-1}} \right) Q^j_{a_N}, \tag{2.4.16}$$

$$Y_j = \frac{\epsilon^{a_1 a_2 \ldots a_N}}{2^{m-1}(m-1)!} \frac{\epsilon_{j j_2 j_3 j_4}}{3!} \left(A_{a_1 a_2} A_{a_3 a_4} \ldots A_{a_{N-4} a_{N-3}} \right) Q^{j_2}_{a_{N-2}} Q^{j_3}_{a_{N-1}} Q^{j_4}_{a_N}. \tag{2.4.17}$$

The numeric coefficients absorb the combinatoric factors from the ϵ tensors, with the convention $\epsilon_{123\ldots N} = +1$. In general, we reserve the indices a, b, α, β for gauge groups, and use the indices i, j to refer to family symmetries. Superscripts and subscripts are chosen for visual clarity, and do not signify any particular group representation.

It is useful to classify the $\{U, V, \mathcal{W}, X, Y, Z\}$ fields as "baryons" and the J and K fields as "mesons," to separate the operators which scale with N from those which are independent of N. The transformation properties of these operators under the family symmetries are shown in Table 2.1. There is a continuous family of equivalent $U(1)_A \times U(1)_B \times U(1)_R$ charge assignments, but the choice shown in Table 2.1 is particularly convenient.

For $N = 4$, the theory contains four flavors of $Q + \overline{Q}$. This value of N is unique in that both $m_A \mathrm{Pf}\, A$ and $m^i_j Q^\alpha_i \overline{Q}^j_\alpha$ are gauge-invariant mass terms: if these masses are large compared to Λ, then every field can be integrated out above the confinement scale. This special case is discussed in Section 3.3.1. For $N = 3$ the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and $\overline{\Box}$ representations are equivalent, and the $A + 4Q + 3\overline{Q}$ model reduces to SUSY QCD with $F = 4$.

As discussed in Section 2.4, the form of the dynamically generated superpotential is determined by the representations of the matter fields. For the $A + 4Q + N\overline{Q}$ model,

$$W_d \sim \sum \frac{A^{N-2} Q^4 \overline{Q}^N}{\Lambda^b}. \tag{2.4.18}$$

The sum includes all possible gauge-invariant contractions of the group indices, with some relative coefficients:

$$W_{\text{odd N}} \sim \frac{1}{\Lambda^b}\left[XYZ + XK^{m-1}J^3 + YK^mJ\right], \tag{2.4.19}$$

$$W_{\text{even N}} \sim \frac{1}{\Lambda^b}\left[U\mathcal{W}Z + V^2Z + UK^{m-2}J^4 + VK^{m-1}J^2 + \mathcal{W}K^m\right]. \tag{2.4.20}$$

Both $\mathcal{F}_{\text{odd}} = \{J, K, X, Y, Z\}$ and $\mathcal{F}_{\text{even}} = \{J, K, U, V, \mathcal{W}, Z\}$ satisfy the t' Hooft anomaly matching conditions for the mixed $SU(4)^2U(1)$ and $SU(N)^2U(1)$ anomalies, the various $U(1)^3$ anomalies, and the mixed $U(1)$ gravitational anomalies, for all $U(1)$ symmetries listed in Table 2.1 except for $U(1)_1$. The $G_1^2U(1)_1$ anomaly breaks $U(1)_1$ explicitly at the scale Λ_1, so it is not a symmetry of the infrared theory.

Dynamically generated superpotential: The number of infrared operators, $\dim\mathcal{F}$, is larger than the dimension of the moduli space, $\dim M_0 = N(N-1)/2 + 4N + 1$. For $N = 2m+1$,

$$\dim\{J, K, X, Y, Z\} = \left(4N + \frac{N(N-1)}{2} + 4 + 4 + 1\right), \tag{2.4.21}$$

and for $N = 2m$,

$$\dim\{J, K, U, V, \mathcal{W}, Z\} = \left(4N + \frac{N(N-1)}{2} + 1 + \frac{4(3)}{2} + 1 + 1\right), \tag{2.4.22}$$

implying for both cases that the number of constraints is

$$N_{\text{con}} = \dim\mathcal{F} - \dim M_0 = 8. \tag{2.4.23}$$

For odd N, the eight constraints are

$$\begin{aligned} X^iZ &= \frac{\epsilon^{j_1j_2\ldots j_N}}{2^m m!}\left(K_{j_1j_2}K_{j_3j_4}\ldots K_{j_{N-2}j_{N-1}}\right)J^i_{j_N} \\ Y_iZ &= \frac{\epsilon^{j_1j_2\ldots j_N}\epsilon_{ii_2i_3i_4}}{2^{m-1}(m-1)!3!}\left(K_{j_1j_2}K_{j_3j_4}\ldots K_{j_{N-4}j_{N-3}}\right)J^{i_2}_{j_{N-2}}J^{i_3}_{j_{N-1}}J^{i_4}_{j_N}, \end{aligned} \tag{2.4.24}$$

while for even N

$$\begin{aligned} UZ &= \frac{\epsilon_{j_1\ldots j_N}}{2^m m!}K_{j_1j_2}K_{j_3j_4}\ldots K_{j_{N-1}j_N} = \operatorname{Pf}K, \\ V_{i_1i_2}Z &= \frac{\epsilon_{j_1\ldots j_N}}{2^{m-1}(m-1)!}\frac{\epsilon_{i_1i_2i_3i_4}}{2!}K_{j_1j_2}K_{j_3j_4}\ldots K_{j_{N-3}j_{N-2}}J^{i_3}_{j_{N-1}}J^{i_4}_{j_N}, \\ \mathcal{W}Z &= \frac{\epsilon_{j_1\ldots j_N}}{2^{m-2}(m-2)!}\frac{\epsilon_{i_1i_2i_3i_4}}{4!}K_{j_1j_2}K_{j_3j_4}\ldots K_{j_{N-5}j_{N-4}}J^{i_1}_{j_{N-3}}J^{i_2}_{j_{N-2}}J^{i_3}_{j_{N-1}}J^{i_4}_{j_N}. \end{aligned} \tag{2.4.25}$$

The index $i = 1\ldots4$ refers to the $SU(4)$ family symmetry.

By taking partial derivatives of Eq. (2.4.19) and Eq. (2.4.20) and matching the equations of motion to the classical constraints, one can determine the relative coefficient of each term in the dynamically generated superpotential. The results appear below:

$$
\begin{aligned}
W_{\mathrm{odd}} &= \frac{\alpha}{\Lambda^b}\Big\{X^i Y_i Z - \frac{\epsilon^{j_1 \ldots j_N}\epsilon_{i_1 \ldots i_4}}{2^{m-1}(m-1)!3!} X^{i_1}(K_{j_1 j_2} \ldots K_{j_{N-4} j_{N-3}}) J^{i_2}_{j_{N-2}} J^{i_3}_{j_{N-1}} J^{i_4}_{j_N} \\
&\quad - \frac{\epsilon^{j_1 \ldots j_N}}{2^m m!} Y_i (K_{j_1 j_2} \ldots K_{j_{N-2} j_{N-1}}) J^i_{j_N}\Big\}; \qquad (2.4.26) \\
W_{\mathrm{even}} &= \frac{\alpha}{\Lambda^b}\Big\{U\mathcal{W}Z - \frac{\epsilon_{i_1 \ldots i_4}}{2^2 2!} V^{i_1 i_2} V^{i_3 i_4} Z - \mathcal{W}\,\mathrm{Pf}\,K \\
&\quad - \frac{\epsilon_{j_1 \ldots j_N}}{2^{m-2}(m-2)!}\frac{\epsilon_{i_1 i_2 i_3 i_4}}{4!} U (K_{j_1 j_2} \ldots K_{j_{N-5} j_{N-4}})(J^{i_1}_{j_{N-3}} \ldots J^{i_4}_{j_N}) \\
&\quad + \frac{\epsilon_{j_1 \ldots j_N}\epsilon_{i_1 i_2 i_3 i_4}}{4 \cdot 2^{m-1}(m-1)!} V^{i_1 i_2}(K_{j_1 j_2} \ldots K_{j_{N-3} j_{N-2}}) J^{i_3}_{j_{N-1}} J^{i_4}_{j_N}\Big\}. \qquad (2.4.27)
\end{aligned}
$$

As in SUSY QCD, the overall factor α cannot be determined by symmetry arguments. In principle, it is possible to add heavy quark masses and integrate out two flavors of $(Q\overline{Q})$ so as to match the $F = 2$ model, whose superpotential can be calculated from a one-instanton calculation analogous to $F = N - 1$ SUSY QCD.

It is useful to consider the phases of α and Λ^b. As defined in Eq. (2.4.4), the phase of Λ^b is determined by the CP-violating θ_{YM} parameter. The phase of α is arbitrary: because W_d is charged under an unbroken $U(1)_R$ symmetry, α can be made real by a $U(1)_R$ rotation.

Chapter 3

An S-Confining Product Gauge Group

The following is based on a previously published paper by the author [1].

3.1 Introduction

Experimental evidence so far suggests that the Standard Model gauge group $G_{\text{SM}} = SU(3)_c \times SU(2)_L \times U(1)_Y$ well describes the universe. Attempts to expand the gauge sector beyond G_{SM} must therefore explain why the additional interactions have not yet presented any evidence for their existence.

There are several well-motivated ways to achieve this. The new gauge bosons and matter fields might form a "dark sector" and interact weakly (or not at all) with the particles described by the Standard Model. It is also possible for an extended gauge symmetry to be spontaneously broken to G_{SM} at some high-energy scale which we have not yet probed. In this chapter we consider the alternative in which the new dynamics are so strongly coupled that particles charged under the new interactions confine to form neutral bound states, with binding energies at the TeV scale or larger.

We focus on a particular class of $\mathcal{N} = 1$ supersymmetric (SUSY) gauge theories with product gauge groups of the form $SU(N)_1 \times SU(N)_2 \times \ldots \times SU(N)_k$. Our model includes one antisymmetric tensor $A_{\alpha\beta}$ and four quark fields Q^i_α charged under $SU(N)_1$, and a series of bifundamental fields $(\overline{Q}_i)^\alpha_\beta$ charged under adjacent gauge groups $SU(N)_i \times SU(N)_{i+1}$ as shown in Table 3.1. This theory is an extension of a model, $SU(N) : (\yng(1,1) + 4\Box + N\overline{\Box})$, which has been shown to confine [76, 78, 80].

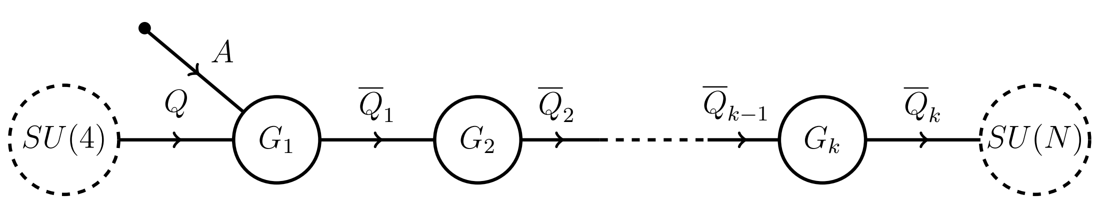

Figure 3.1: The matter content of the proposed s-confining theory is shown as a moose diagram. Each G_i represents a gauged $SU(N)$ group, while the dashed circles represent the $SU(4)_L \times SU(N)_R$ family symmetry.

We propose in the language of [74] that this $SU(N)^k$ model is "s-confining:" that is, the theory confines smoothly in the infrared without breaking chiral symmetry, and it generates a non-vanishing superpotential that describes the interactions between the gauge invariant composite fields. Although the $\mathcal{N} = 1$ s-confining theories with a simple gauge group are fully classified [75], very few examples of s-confinement in product gauge groups are known [81, 82].

Our $SU(N)^k$ product group model has two distinctive features which may be useful for model-building. First, there are no small gauge-singlet operators: the number of fields contained in every gauge invariant operator depends on k or N. Second, the various $SU(N)_i$ subgroups generally confine at different scales Λ_i, with hierarchies based on the coupling constants g_i.

Product groups of this form appear in studies of five-dimensional gauge theories [83–86]. The model shown in Table 3.1 can be interpreted as a k-site deconstruction of a 5d SUSY $SU(N)$ gauge theory with a $\mathbb{Z}_2$ orbifold. In the 5d theory the chiral fields $\{A, Q\}$ and $\overline{Q}_k$ exist on opposing 4d branes, while the bifundamental $\overline{Q}_i$ superfields correspond to a single bulk $\overline{Q}$ field. A natural hierarchy between the Λ_i arises if the extra dimension is warped: for example, the model with $\Lambda_1 > \ldots > \Lambda_k$ has A and Q on the ultraviolet brane and $\overline{Q}_k$ on the infrared brane.

3.2 Product Group Extension for an S-Confining Theory

Our interest in the product group model of Table 3.1 is motivated by an observation from the $G_1 \times G_2$ case, in which the family symmetry $G_2 = SU(N)_R$ of the $\overline{Q}$ is weakly gauged. In the confined phase of G_1, there are three types of operators charged under G_2: one antisymmetric $K = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$, four quarks $J = \Box$, and N antiquarks $\overline{Q}_2 = \overline{\Box}$. Remarkably, this is identical to the original s-confining model.

The model described in Section 2.4.2 can be extended indefinitely by adding more gauge groups G_i and bifundamental matter $\overline{Q}_i$. As long as $\Lambda_1 > \Lambda_2 \ldots > \Lambda_i > \Lambda_{i+1}$, confinement under G_i always produces mesons charged as $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4\Box$ under G_{i+1}. This is the model shown in Table 3.1, where the gauge group is $G_1 \times \ldots \times G_k$. In this section we devote our attention to the question: is this $SU(N)^k$ theory s-confining, or is s-confinement disrupted by the product group?

There are two obvious ways in which the $K + 4J + N\overline{Q}_2$ "k=2" model differs from the original ("k=1") s-confining theory. First, in the $k = 1$ model there is no tree-level superpotential, but in the $k = 2$ case there is a superpotential from G_1 confinement that may alter how $\{K, J, P\}$ confine under G_2. Luckily, inspection of the classical constraints shows that K, J, and $\overline{Q}_2$ may be varied freely, as long as the baryon products $\{UZ, VZ, \mathcal{W}Z\}$ or $\{XZ, YZ\}$ vary in accordance with Eqs. (2.4.24) and (2.4.25). The second main difference is that under G_2, the classical moduli space is modified quantum mechanically. For the $k \geq 2$ theory to be s-confining, we must determine whether or not the origin remains on the moduli space.

Of the existing literature regarding SUSY product groups, the work of Chang and Georgi [86] on $SU(N)^k$ extensions to $F = N$ SUSY QCD is particularly relevant to our present study. Our method also has some similarities to deconfinement [76, 87], particularly in Section 3.3 when we consider $Sp(2N)$ groups.

3.2.1 Infrared Operators

To understand the infrared behavior of the theory, we develop a basis of gauge invariant operators which describe the moduli space and obey anomaly matching conditions. Then in Sections 3.2.2 and 3.2.3, we find the dynamically generated superpotential and perform some consistency checks.

Let us define a basis for the anomalous $U(1)$ charges, $U(1)_{j=1\ldots k}$, such that the anomaly coefficient $\mathcal{A}(G_i^2 U(1)_j)$ is zero if and only if $i \neq j$, as shown in Table 3.1. Each $U(1)_i$ is explicitly broken at a scale associated with Λ_i, so that the approximate UV symmetry is broken to:

$$SU(4)_L \times SU(N)_R \times U(1)_R \times U(1)^{k+2} \longrightarrow SU(4)_L \times SU(N)_R \times U(1)_R \times U(1)_A \times U(1)_B. \quad (3.2.1)$$

The $U(1)_i$ charges of the Λ_i^b are determined by the $G^2U(1)$ anomaly coefficients. Note that $b = 2N-1$ for Λ_1^b, while $b = 2N$ for $\Lambda_{i\neq1}^b$.

	G_1	G_2	G_3	$\ldots$	G_k	$SU(4)$	$SU(N)$	U_A	U_B	U_R	U_1	U_2	U_3	$\ldots$	U_k
Q	$\square$					$\square$		$N-2$	$-1/2$	$1/2$	0	0	0		0
A	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$							-4	-1	0	0	0	0		0
$\overline{Q}_1$	$\overline{\square}$	$\square$						0	1	0	1	0	0		0
$\overline{Q}_2$		$\overline{\square}$	$\square$					0	-1	0	-1	1	0	$\ldots$	0
$\overline{Q}_3$			$\overline{\square}$					0	1	0	1	-1	1		0
$\vdots$				$\ddots$	$\square$			0	$\vdots$	0	$\vdots$		$\vdots$		0
$\overline{Q}_k$					$\overline{\square}$		$\square$	0	±1	0	±1	∓1	±1	$\ldots$	1
Λ_1^b								0	0	0	N	0	0		0
Λ_2^b								0	0	0	0	N	0		0
Λ_3^b								0	0	0	0	0	N		0
$\vdots$								$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\ddots$	0
Λ_k^b								0	0	0	0	0	0		N

Table 3.1: Matter content of the proposed s-confining theory, showing the transformation properties under the gauged $SU(N)^k$ and the $SU(4)_L \times SU(N)_R \times U(1)_A \times U(1)_B \times U(1)_R$ family symmetry. The spurious $U(1)_{i=1\ldots k}$ charges are also shown. The alternating ($\pm$) factors in the $\overline{Q}_k$ charges depend on whether k is odd or even: the upper choice corresponds to odd k.

From Table 3.1, it is clear that combinations of the form

$$\left(\frac{\overline{Q}_1^N \overline{Q}_2^N}{\Lambda_2^b}\right), \left(\frac{\overline{Q}_2^N \overline{Q}_3^N}{\Lambda_3^b}\right), \ldots \left(\frac{\overline{Q}_{k-1}^N \overline{Q}_k^N}{\Lambda_k^b}\right)$$

are neutral under all of the symmetries, including the spurious $U(1)_i$. Therefore, the dynamically generated superpotential has the form

$$W_d \sim \sum_{p_2\cdots p_k} \left\{ \left(\frac{A^{N-2}Q^4\overline{Q}_1^N}{\Lambda_1^b}\right) \left(\frac{\overline{Q}_1^N \overline{Q}_2^N}{\Lambda_2^b}\right)^{p_2} \left(\frac{\overline{Q}_2^N \overline{Q}_3^N}{\Lambda_3^b}\right)^{p_3} \ldots \left(\frac{\overline{Q}_{k-1}^N \overline{Q}_k^N}{\Lambda_k^b}\right)^{p_k} \right\} \quad (3.2.2)$$

for some powers $p_i = 0, 1, \ldots$ for each $i = 2, 3, \ldots k$. Any such superpotential has an R charge of $+2$ under all of the possible $U(1)_R$ symmetries. Before we can find the individual terms that appear in W_d, it is necessary to understand the equations of motion between the infrared operators.

To find a set of gauge invariant operators in the far infrared, let us consider the ordered case $\Lambda_1 \gg \Lambda_2 \gg \ldots \gg \Lambda_k$. As discussed in Section 2.4.2, G_1 confinement produces the operators

$$J_1 = (Q\overline{Q}_1), \qquad K_1 = (A\overline{Q}_1^2), \qquad Z_1 = (\overline{Q}_1^N), \quad (3.2.3)$$

$$U_1 = (A^m), \quad V_1 = (A^{m-1}Q^2), \quad \mathcal{W}_1 = (A^{m-2}Q^4); \quad X_1 = (A^mQ), \quad Y_1 = (A^{m-1}Q^3), \quad (3.2.4)$$

where J_1 and K_1 are charged under G_2. Although $U(1)_1$ is broken, the $U(1)_2 \times \ldots \times U(1)_k$ symmetry is approximately preserved above the scale Λ_2, adding $\mathcal{O}(k^3)$ anomaly coefficients that must be calculated.

This is the benefit of the strategically-defined $U(1)_i$ charges shown in Table 3.1: the fields $\{Q, A, \overline{Q}_1\}$ are neutral under $U(1)_2 \ldots U(1)_k$, and all of these anomaly matching conditions are trivially satisfied. The fields J_1 and K_1 transform similarly to Q and A under the non-Abelian symmetries, but their $U(1)_B$ charges are different, as shown in Table 3.2.

	G_2	G_3	$\ldots$	G_k	$SU(4)$	$SU(N)$	U_A	U_B	U_R	U_2	U_3	$\ldots$	U_k
J_1	$\square$				$\square$		$N-2$	$+1/2$	$1/2$	0	0		0
K_1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$						-4	$+1$	0	0	0		0
$\overline{Q}_2$	$\overline{\square}$	$\square$					0	-1	0	1	0		0
$\overline{Q}_3$		$\overline{\square}$					0	$+1$	0	-1	1	$\ldots$	0
$\vdots$			$\ddots$	$\square$			0	$\vdots$	0	$\vdots$	$\vdots$		0
$\overline{Q}_k$				$\overline{\square}$		$\square$	0	± 1	0	∓ 1	± 1	$\ldots$	1
U_1							$-2N$	$-N/2$	0	0	0		0
V_1					$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		0	$-N/2$	1	0	0		0
$\mathcal{W}_1$							$2N$	$-N/2$	2	0	0		0
X_1					$\square$		$-N$	$-N/2$	$1/2$	0	0		0
Y_1					$\overline{\square}$		N	$-N/2$	$3/2$	0	0		0
Z_1							0	N	0	0	0		0

Table 3.2: Transformation properties of the composite fields in the confined phase of G_1, in the limit where $G_2 \times \ldots \times G_k$ is weakly gauged. The composite fields U, V, and $\mathcal{W}$ exist only if N is even; if N is odd, then they are replaced by X and Y.

At the scale $\Lambda_2 < \Lambda_1$, the G_2 fields confine to form the following $G_1 \times G_2$ singlets:

$$J_2 = (J_1 \overline{Q}_2) \qquad K_2 = (K_1 \overline{Q}_2^2) \qquad X_2 = (K_1^m J_1) \qquad Y_2 = (K_1^{m-1} J_1^3) \tag{3.2.5}$$

$$U_2 = (K_1^m) \qquad V_2 = (K_1^{m-1} J_1^2) \qquad \mathcal{W}_2 = (K_1^{m-2} J_1^4) \qquad Z_2 = (\overline{Q}_2^N). \tag{3.2.6}$$

The fields J_2 and K_2 transform under G_3 as $\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ respectively.

It is convenient to define the shorthand notation B_i, where $B_i = \{U_i, V_i, \mathcal{W}_i\}$ for even $N = 2m$, and $B_i = \{X_i, Y_i\}$ for odd $N = 2m+1$. At scales below Λ_2 and above Λ_3, the intermediate degrees of freedom are $\{J_2, K_2, B_1, B_2, Z_1, Z_2, \overline{Q}_3, \ldots, \overline{Q}_k\}$. This set of fields satisfies the anomaly matching conditions for $SU(4)_L \times SU(N)_R \times U(1)_A \times U(1)_B \times U(1)_R \times U(1)_3 \times \ldots \times U(1)_k$.

It is straightforward to continue this procedure until all groups including G_k have confined, using the following recursive operator definition:

$$J_i = (J_{i-1} \overline{Q}_i) \qquad K_i = (K_{i-1} \overline{Q}_i^2) \qquad X_i = (K_{i-1}^m J_{i-1}) \qquad Y_i = (K_{i-1}^{m-1} J_{i-1}^3) \tag{3.2.7}$$

$$U_i = (K_{i-1}^m) \qquad V_i = (K_{i-1}^{m-1} J_{i-1}^2) \qquad \mathcal{W}_i = (K_{i-1}^{m-2} J_{i-1}^4) \qquad Z_i = (\overline{Q}_i^N). \tag{3.2.8}$$

This definition can be applied to $i = 1$ as well if we define $J_0 = Q$ and $K_0 = A$. Below the scale Λ_k, all of the gauge groups have confined, and the approximate $U(1)_{i=1\ldots k}$ symmetries are broken

	$SU(4)_L$	$SU(N)_R$	U_A	U_B	U_R
J_k	$\square$	$\square$	$N-2$	$\pm 1/2$	$1/2$
K_k		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	-4	± 1	0
U_{odd}			$-2N$	$-N/2$	0
V_{odd}	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		0	$-N/2$	1
$\mathcal{W}_{\text{odd}}$			$2N$	$-N/2$	2
U_{even}			$-2N$	$+N/2$	0
V_{even}	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		0	$+N/2$	1
$\mathcal{W}_{\text{even}}$			$2N$	$+N/2$	2
X_{odd}	$\square$		$-N$	$-N/2$	$1/2$
Y_{odd}	$\overline{\square}$		N	$-N/2$	$3/2$
X_{even}	$\square$		$-N$	$+N/2$	$1/2$
Y_{even}	$\overline{\square}$		N	$+N/2$	$3/2$
Z_{odd}			0	N	0
Z_{even}			0	$-N$	0

Table 3.3: The transformation properties of the composite fields in the fully confined phase of $SU(N)^k$ are shown. The subscript $B_{\text{odd,even}}$ refers to $i = 1 \dots k$, whereas the baryon content $B_i = \{U_i, V_i, \mathcal{W}_i\}$ or $B_i = \{X_i, Y_i\}$ depends on N. The $U(1)_B$ charges of J_k and K_k are positive if k is odd, and negative if k is even.

to discrete $\mathbb{Z}_N$ groups. The charges under the remaining continuous family symmetries are shown in Table 3.3.

It must be shown that the basis of infrared operators is large enough to cover the moduli space. For the $SU(N)^k$ gauge group with fields $\{A, Q, \overline{Q}_1, \dots, \overline{Q}_k\}$, the dimension of the moduli space is

$$\dim M_0(k) = \frac{N(N-1)}{2} + 4N + kN^2 - k(N^2-1) = 4N + \frac{N(N-1)}{2} + k, \tag{3.2.9}$$

while the operator basis $\{J_k, K_k; B_1, \dots, B_k; Z_1, \dots, Z_k\}$ has dimension

$$N_{\text{ops}} = 4N + \frac{1}{2}N(N-1) + 9k, \tag{3.2.10}$$

implying that there are $8k$ complex constraints. By rearranging Eq. (3.2.7) as follows, we can find $8(k-1)$ of the constraint equations:

$$\begin{aligned} X_i &= (K_{i-1}^m J_{i-1}) = (K_{i-2}\overline{Q}_{i-1}^2)^m (J_{i-2}\overline{Q}_{i-1}) = (K_{i-2}^m J_{i-2})(\overline{Q}_{i-1}^{2m+1}) = X_{i-1}Z_{i-1} \\ Y_i &= (K_{i-1}^{m-1} J_{i-1}^3) = (K_{i-2}\overline{Q}_{i-1}^2)^{m-1} (J_{i-2}\overline{Q}_{i-1})^3 = (K_{i-2}^{m-1} J_{i-2}^3)(\overline{Q}_{i-1}^{2m+1}) = Y_{i-1}Z_{i-1}, \end{aligned} \tag{3.2.11}$$

for $i = 2, 3 \dots k$. Similarly,

$$U_i = U_{i-1}Z_{i-1} \qquad V_i = V_{i-1}Z_{i-1} \qquad \mathcal{W}_i = \mathcal{U}_{i-1}Z_{i-1}. \tag{3.2.12}$$

The eight remaining constraints are provided by

$$X_k Z_k = K_k^m J_k \qquad Y_k Z_k = K_k^{m-1} J_k^3, \tag{3.2.13}$$

or

$$U_k Z_k = \text{Pf}\,(K_k) \qquad V_k Z_k = K_k^{m-1} J_k^2 \qquad \mathcal{W}_k Z_k = K_k^{m-2} J_k^4. \tag{3.2.14}$$

It is possible that these classical constraints may be quantum-modified.

Reduced operator basis: The classical constraints for $B_{i>1}$ are mildly problematic, because Eqs. (3.2.11) and (3.2.12) imply that these operators are redundant: that is, they can be written as products from a smaller operator basis, $\{B_1, Z_1, Z_2, \ldots Z_k\}$, and are therefore not independent degrees of freedom. Excitations of the B_i fields above the vacuum acquire $\mathcal{O}(\Lambda_i)$ masses if they do not obey the classical constraints. These massive modes decouple at the scale Λ_k, leaving only the degrees of freedom consistent with the classical (or quantum-modified) constraints. Unfortunately, anomaly cancelation depended on the fields $B_{i=2\ldots k}$: if these are not true degrees of freedom, then the anomaly matching conditions might not be satisfied.

A solution to this problem can be seen by studying the X_{odd} and Y_{even} charges in Table 3.3. Their fermionic components have opposite charges under each of $U(1)_A$, $U(1)_B$, and $U(1)_R$. When we calculate the anomaly coefficients for each of the mixed and pure $U(1)$ anomalies, the contributions from each X_{odd} cancel those from a Y_{even} field. This is also true for the $SU(4)^2U(1)$ and $SU(4)^3$ anomalies. Therefore, we refer to X_{odd} and Y_{even} as an "anomaly neutral pair," indicating that they can be removed without changing any of the anomaly coefficients. Similarly, X_{even} and Y_{odd} also form an anomaly neutral pair.

If k is odd, then all of the operators $\{X_2, Y_2, \ldots, X_k, Y_k\}$ can be removed in neutral pairs. Substituting X_k and Y_k with their equations of motion, Eq. (3.2.13) becomes

$$(X_1 Z_1 Z_2 \ldots Z_{k-1})\, Z_k = K_k^m J_k \qquad\qquad (Y_1 Z_1 Z_2 \ldots Z_{k-1})\, Z_k = K_k^{m-1} J_k^3 \tag{3.2.15}$$

This is not possible if k is even. To remove all the redundant operators, we must also remove a pair $\{X_1, Y_{\text{even}}\}$ or $\{X_{\text{even}}, Y_1\}$, and this is inconsistent: both X_1 and Y_1 are necessary to describe the moduli space.

This can be seen if we move away from the origin along the flat direction parameterized by $(A^m Q)$, while keeping $\overline{Q}_1 = 0$. Along this flat direction X_1 increases, but $X_{\text{even}} = 0$. Therefore, X_1 describes directions on the moduli space that cannot be described by X_{even}. Similarly, by increasing $(A^{m-1}Q^3)$ and fixing $\overline{Q}_1 = 0$, we can see that Y_1 is just as necessary.

Quantum modification to Eq. (3.2.15) could explain why the odd k and even k situations are different. If $U(1)_B$ is broken in the vacuum, then $\{X_i, Y_i\}$ become an anomaly-neutral pair under the remaining symmetries, for any value of $i = 1\ldots k$. Based on $F = N$ SUSY QCD, one would expect the classical relationships involving $\overline{Q}_i$ and $\overline{Q}_{i+1}$ to be quantum-modified. Specifically, the combination $(Z_{i-1}Z_i)$ has the same spurious $U(1)_i$ charge as $\Lambda_i^{b=2N}$, allowing modifications to equations such as Eq. (3.2.15). For example, the classical $k = 4$ constraint for $X_4 Z_4$ might become

$$X_1 \left(Z_1 Z_2 Z_3 Z_4 + \beta_1 \Lambda_2^b Z_3 Z_4 + \beta_2 Z_1 \Lambda_3^b Z_4 + \beta_3 Z_1 Z_2 \Lambda_4^b + \beta_4 \Lambda_2^b \Lambda_4^b \right) = K_4^m J_4, \tag{3.2.16}$$

with some as-yet-unknown coefficients β_i. As long as the coefficients are not zero, then the flat direction corresponding to $(A^m Q) \neq 0$ with $\overline{Q}_1 = 0$ now requires some of the $Z_{i\neq 1}$ to have nonzero expectation values. In this $Z_1 = 0$, $X_1 \neq 0$ example, Eq. (3.2.16) implies that $\Lambda_2^b(Z_3 Z_4 + \Lambda_4^b) = 0$, spontaneously breaking $U(1)_B$ even in the limit where $\langle X_1 \rangle \gg \Lambda_k$. Once $U(1)_B$ is broken in the vacuum, the operators $\{J_4, K_4, X_1, Y_1, Z_{i=1\ldots 4}\}$ obey the anomaly matching conditions.

A quantum-modified constraint like Eq. (3.2.16) also explains why $\{J_k, K_k, X_1, Y_1, Z_{i=1\ldots k}\}$ is consistent at the origin of moduli space if k is odd. In this case the $Z_i = 0$ solution remains valid far away from the origin, because every Λ^b term multiplies at least one Z field. Consider Eq. (3.2.16) with $k = 5$:

$$\begin{aligned} K_5^m J_5 \;\; = \;\; & X_1 \big(Z_1 Z_2 Z_3 Z_4 Z_5 + \beta_1 \Lambda_2^b Z_3 Z_4 Z_5 + \beta_2 Z_1 \Lambda_3^b Z_4 Z_5 + \beta_3 Z_1 Z_2 \Lambda_4^b Z_5 + \beta_4 Z_1 Z_2 Z_3 \Lambda_5^b \\ & + \beta_5 Z_1 \Lambda_3^b \Lambda_5^b + \beta_6 \Lambda_2^b Z_3 \Lambda_5^b + \beta_7 \Lambda_2^b \Lambda_4^b Z_5 \big). \end{aligned} \tag{3.2.17}$$

In this case, the $(A^m Q) \neq 0$, $\overline{Q}^N_{i=1\ldots k} = 0$ flat direction remains on the moduli space for arbitrarily large values of $(A^m Q)$.

This does not mean that $U(1)_B$ is necessarily broken in the vacuum if k is even. Let us fix $Z_i = 0$ for all $i = 1\ldots k$ to ensure that $U(1)_B$ is not broken at the scale Λ_i. After imposing this constraint, Eq. (3.2.16) becomes

$$X_1 = \frac{K_4^m J_4}{\Lambda_2^b \Lambda_4^b}, \tag{3.2.18}$$

implying that X_1 is not an IR degree of freedom when $U(1)_B$ is conserved. The same is true for $Y_1 \Lambda_2^b \Lambda_4^b = K_4^{m-1} J_4^3$. In this particular vacuum X_1 and Y_1 are redundant operators, and after they are removed from the calculation the $U(1)_B$ anomaly coefficients match the ultraviolet theory.

Theories with even N behave in essentially the same way. Under the exact family symmetries, the operator pairs $\{U_{\rm odd}, \mathcal{W}_{\rm even}\}$, $\{U_{\rm even}, \mathcal{W}_{\rm odd}\}$, and $\{V_{\rm odd}, V_{\rm even}\}$ are anomaly-neutral. As in the odd N case, if k is even then it is not possible to remove all the redundant $\{U_i, V_i, \mathcal{W}_i\}$ operators while preserving the anomaly matching. This leads us to expect that the classical constraint equations

$$U_k = U_1\,(Z_1 Z_2 \ldots Z_{k-1}), \qquad V_k = V_1\,(Z_1 Z_2 \ldots Z_{k-1}), \qquad \mathcal{W}_k = \mathcal{W}_1\,(Z_1 Z_2 \ldots Z_{k-1}) \tag{3.2.19}$$

receive quantum modifications of the form

$$\mathrm{Pf}\, K_k = U_1 \left(Z_1 Z_2 \ldots Z_{k-1} + \ldots + (\Lambda_2^b \Lambda_4^b \ldots \Lambda_{k-2}^b) Z_{k-1} Z_k + (\Lambda_2^b \Lambda_4^b \ldots \Lambda_k^b) \right). \tag{3.2.20}$$

if k is even. Either $U(1)_B$ is broken in the vacuum, or the operators $\{U_1, V_1, \mathcal{W}_1\}$ are not degrees of freedom: in both cases, the IR theory satisfies t' Hooft anomaly matching. Thus, the reduced operator basis describes all infrared degrees of freedom, for both even and odd N.

3.2.2 Dynamically generated superpotential

In this section we find a dynamically generated superpotential in the region of parameter space with $\Lambda_1 \gg \Lambda_2 \gg \ldots \gg \Lambda_k$. We begin by considering how the W_d of Eq. (2.4.26) and Eq. (2.4.27) becomes modified at the G_2 confinement scale. Ignoring the precise relative coefficients between terms,

$$W_{\rm odd}^{(1)} = \frac{1}{\Lambda_1^b} \left(X_1 Y_1 Z_1 - X_1 K_1^{m-1} J_1^3 - Y_1 K_1^m J_1 \right) \tag{3.2.21}$$

$$W_{\rm even}^{(1)} = \frac{1}{\Lambda_1^b} \left(U_1 \mathcal{W}_1 Z_1 - V_1^2 Z_1 - U_1 K_1^{m-2} J_1^4 + V_1 K_1^{m-1} J_1^2 - \mathcal{W}_1 K_1^m \right) \tag{3.2.22}$$

At the scale Λ_2, we expect J_1 and K_1 to confine to form the B_2 baryons. If we make these replacements in $W^{(1)}$, it becomes

$$W_{\rm odd}^{(1)} = \frac{1}{\Lambda_1^b} \left(X_1 Y_1 Z_1 - X_1 Y_2 - Y_1 X_2 \right) \tag{3.2.23}$$

$$W_{\rm even}^{(1)} = \frac{1}{\Lambda_1^b} \left(U_1 \mathcal{W}_1 Z_1 - V_1^2 Z_1 - \mathcal{W}_1 U_2 - U_1 \mathcal{W}_2 + V_1 V_2 \right) \tag{3.2.24}$$

It is likely that G_1 confinement changes the holomorphic scale Λ_2 to some new $\widetilde{\Lambda}_2$. To find the relationship between Λ_2 and $\widetilde{\Lambda}_2$, let us normalize the hadrons to have mass dimension +1:[1]

$$\widetilde{J}_1 = \frac{J_1}{\Lambda_1} \qquad \widetilde{K}_1 = \frac{K_1}{\Lambda_1^2} \qquad \widetilde{Z}_1 = \frac{Z_1}{\Lambda_1^{N-1}}, \tag{3.2.25}$$

and similarly for the baryon operators B_1. The dynamically generated superpotential W_2 has the form

$$W^{(2)} = \sum_{\text{contr.}} \left(\frac{\widetilde{K}_1^{N-2} \widetilde{J}_1^4 \overline{Q}_2^N}{\widetilde{\Lambda}_2^b} \right) = \sum_{\text{contr.}} \left(\frac{K_1^{N-2} J_1^4 \overline{Q}_2^N}{\Lambda_1^{2N} \widetilde{\Lambda}_2^b} \right). \tag{3.2.26}$$

From Eq. (3.2.2), symmetry requirements ensure that the superpotential has the form

$$W^{(2)} \sim \frac{A^{N-2} Q^4 \overline{Q}_1^N}{\Lambda_1^b} \frac{\overline{Q}_1^N \overline{Q}_2^N}{\Lambda_2^b} \longrightarrow \frac{K_1^{N-2} J_1^4 \overline{Q}_2^N}{\Lambda_1^b \Lambda_2^b}, \tag{3.2.27}$$

allowing $\widetilde{\Lambda}_2^b$ to be expressed as

$$\widetilde{\Lambda}_2^{2N-1} = \frac{1}{\Lambda_1} \Lambda_2^{2N}. \tag{3.2.28}$$

This expression can also be derived with the same result by matching the gauge couplings at the mass threshold Λ_1. Based on this agreement, we do not expect the superpotential W_2 to receive modifications of the form

$$W^{(2)} \to \left(1 + \frac{Z_1 Z_2}{\Lambda_2^b} + \dots \right) W^{(2)}, \tag{3.2.29}$$

even though such terms are consistent with the family symmetries.

As confinement continues, the products of intermediate mesons J_2 and K_2 can be replaced with G_3 baryons. Each $i = 1 \dots k$ superpotential $W^{(i)}$ becomes

$$W_{\text{odd}}^{(i<k)} = \left(\prod_{j=1}^{i} \Lambda_j^b \right)^{-1} (X_i Y_i Z_i - X_i Y_{i+1} - Y_i X_{i+1}) \tag{3.2.30}$$

$$W_{\text{odd}}^{(k)} = \left(\prod_{j=1}^{k} \Lambda_j^b \right)^{-1} \left(X_k Y_k Z_k - X_k K_k^{m-1} J_k^3 - Y_k K_k^m J_k \right), \tag{3.2.31}$$

$$W_{\text{even}}^{(i<k)} = \left(\prod_{j=1}^{i} \Lambda_j^b \right)^{-1} \left(U_i \mathcal{W}_i Z_i - V_i^2 Z_i - \mathcal{W}_i U_{i+1} - U_i \mathcal{W}_{i+1} + V_i V_{i+1} \right) \tag{3.2.32}$$

$$W_{\text{even}}^{(k)} = \left(\prod_{j=1}^{k} \Lambda_j^b \right)^{-1} \left(U_k \mathcal{W}_k Z_k - V_k^2 Z_k - \mathcal{W}_k K_k^m - U_k K_k^{m-2} J_k^4 + V_k K_k^{m-1} J_k^2 \right). \tag{3.2.33}$$

The full superpotential is the sum

$$W_d = \sum_{i=1}^{k} W^{(i)}. \tag{3.2.34}$$

[1] Even after dividing by these powers of Λ, it is not necessarily true that the fields are canonically normalized. Corrections in the Kähler potential are likely to require additional normalization.

Equations of motion: Let us consider equations of motion of the form $\partial W/\partial B_1$, where $B_1 = \{U_1, V_1, \mathcal{W}_1, X_1, Y_1\}$ is any of the G_1 baryons. It is easy to show that these equations are

$$Y_2 = Y_1 Z_1 \qquad X_2 = X_1 Z_1 \qquad X_1 Y_1 = 0 \tag{3.2.35}$$

for odd N, and

$$\mathcal{W}_2 = \mathcal{W}_1 Z_1 \qquad V_2 = V_1 Z_1 \qquad U_2 = U_1 Z_1 \qquad U_1 \mathcal{W}_1 = V_1^2 \tag{3.2.36}$$

for even N. The $\partial W/\partial B_2$ equations yield more surprising results: for example,

$$\frac{\partial W_d}{\partial X_2} = -\frac{Y_1}{\Lambda_1^b} + \frac{Y_2 Z_2}{\Lambda_1^b \Lambda_2^b} = 0 \quad \longrightarrow \quad Y_2 Z_2 - Y_3 = Y_1 \Lambda_2^b. \tag{3.2.37}$$

The classical constraint $Y_2 Z_2 = Y_3$ is modified, due to the appearance of X_2 in both $W^{(1)}$ and $W^{(2)}$. For $i = 2, 3 \ldots (k-1)$, we find

$$B_i Z_i = B_{i+1} + \Lambda_i^b B_{i-1}. \tag{3.2.38}$$

The equations of motion $\partial W_d/\partial Z_i$ are not modified, so that

$$X_i Y_i = 0, \qquad U_i \mathcal{W}_i = V_i^2 \tag{3.2.39}$$

for all i. Finally, the B_k equations of motion are

$$X_k Z_k = K_k^m J_k + \Lambda_k^b X_{k-1}, \qquad Y_k Z_k = K_k^{m-1} J_k^3 + \Lambda_k^b Y_{k-1} \tag{3.2.40}$$

for odd N, and

$$U_k Z_k = K_k^m + \Lambda_k^b U_{k-1}, \quad V_k Z_k = K_k^{m-1} J_k^2 + \Lambda_k^b V_{k-1}, \quad \mathcal{W}_k Z_k = K_k^{m-2} J_k^4 + \Lambda_k^b \mathcal{W}_{k-1} \tag{3.2.41}$$

for even N.

Recall from Section 2.4.2 that each gauge group $SU(N)_i$ has a related CP parameter θ_i, which determines the phase of the holomorphic scale Λ_i^b. Although Λ^b did not appear in the $k = 1$ equations of motion, the phases of Λ_i^b do affect the equations of motion in the product group case. The overall phase of W_d can still be removed by performing a $U(1)_R$ rotation; however, the relative phases between the Λ_i may have physical effects.

Armed with these iterative equations of motion, we can rewrite the larger baryons $B_{i>1}$ in terms of $\{B_1\}$ and the Z_i fields. For example,

$$B_2 = B_1 Z_1 \tag{3.2.42}$$
$$B_3 = B_1 (Z_1 Z_2 - \Lambda_2^b) \tag{3.2.43}$$
$$B_4 = B_1 (Z_1 Z_2 Z_3 - \Lambda_2^b Z_3 - Z_1 \Lambda_3^b) \tag{3.2.44}$$
$$B_5 = B_1 (Z_1 Z_2 Z_3 Z_4 - \Lambda_2^b Z_3 Z_4 - Z_1 \Lambda_3^b Z_4 - Z_1 Z_2 \Lambda_4^b + \Lambda_2^b \Lambda_4^b). \tag{3.2.45}$$

Our guesses in Eqs. 3.2.16 and 3.2.17 as to the form of the quantum modification are correct, with $\beta_i = \pm 1$ for each coefficient. This process is extended to arbitrary B_i in the following way: each classical constraint involving products of the form $(Z_1 Z_2 \ldots Z_j)$ is modified by replacing adjacent pairs $(Z_{i-1}Z_i)$ by $(-\Lambda_i^b)$, and each possible term is added to the product $(Z_1 \ldots Z_j)$. After making these adjustments, the k^{th} equations of motion return the following constraints if k is odd:

$$\begin{aligned}
K_k^m J_k &= X_1\big\{(Z_1 \ldots Z_k) - \Lambda_2^b(Z_3 \ldots Z_k) + \ldots + (-1)^{(k-1)/2}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_{k-1}^b)Z_k\big\} \\
K_k^{m-1} J_k^3 &= Y_1\big\{(Z_1 \ldots Z_k) - \Lambda_2^b(Z_3 \ldots Z_k) + \ldots + (-1)^{(k-1)/2}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_{k-1}^b)Z_k\big\}, \\
K_k^m &= U_1\big\{(Z_1 \ldots Z_k) - \Lambda_2^b(Z_3 \ldots Z_k) + \ldots + (-1)^{(k-1)/2}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_{k-1}^b)Z_k\big\} \\
K_k^{m-1} J_k^2 &= V_1\big\{(Z_1 \ldots Z_k) - \Lambda_2^b(Z_3 \ldots Z_k) + \ldots + (-1)^{(k-1)/2}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_{k-1}^b)Z_k\big\} \\
K_k^{m-2} J_k^4 &= \mathcal{W}_1\big\{(Z_1 \ldots Z_k) - \Lambda_2^b(Z_3 \ldots Z_k) + \ldots + (-1)^{(k-1)/2}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_{k-1}^b)Z_k\big\},
\end{aligned} \tag{3.2.46}$$

or if k is even:

$$\begin{aligned}
K_k^m J_k &= X_1\big\{(Z_1 \ldots Z_k) + \ldots - (-1)^{\frac{k}{2}}(\Lambda_2^b \ldots \Lambda_{k-2}^b)Z_{k-1}Z_k + (-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)\big\} \\
K_k^{m-1} J_k^3 &= Y_1\big\{(Z_1 \ldots Z_k) + \ldots - (-1)^{\frac{k}{2}}(\Lambda_2^b \ldots \Lambda_{k-2}^b)Z_{k-1}Z_k + (-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)\big\}, \\
K_k^m &= U_1\big\{(Z_1 \ldots Z_k) + \ldots - (-1)^{\frac{k}{2}}(\Lambda_2^b \ldots \Lambda_{k-2}^b)Z_{k-1}Z_k + (-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)\big\} \\
K_k^{m-1} J_k^2 &= V_1\big\{(Z_1 \ldots Z_k) + \ldots - (-1)^{\frac{k}{2}}(\Lambda_2^b \ldots \Lambda_{k-2}^b)Z_{k-1}Z_k + (-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)\big\} \\
K_k^{m-2} J_k^4 &= \mathcal{W}_1\big\{(Z_1 \ldots Z_k) + \ldots - (-1)^{\frac{k}{2}}(\Lambda_2^b \ldots \Lambda_{k-2}^b)Z_{k-1}Z_k + (-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)\big\}.
\end{aligned} \tag{3.2.47}$$

In both cases, the origin of moduli space is a solution to the equations of motion.

As we suggested in Section 3.2.1, if k is even then the B_1 fields are not independent degrees of freedom when $Z_{i=1\ldots k} = 0$:

$$\begin{aligned}
K_k^m &= U_1(-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b) \\
K_k^{m-1} J_k^2 &= V_1(-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b) \\
K_k^{m-2} J_k^4 &= \mathcal{W}_1(-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b)
\end{aligned}
\quad ; \quad
\begin{aligned}
K_k^m J_k &= X_1(-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b) \\
K_k^{m-1} J_k^3 &= Y_1(-1)^{\frac{k}{2}}(\Lambda_2^b\Lambda_4^b \ldots \Lambda_k^b).
\end{aligned} \tag{3.2.48}$$

Therefore, if $U(1)_B$ is a symmetry of the vacuum and k is even, then the B_1 fields are completely determined by J_k and K_k. After removing the B_1 fields, the t' Hooft anomaly matching conditions are satisfied. Elsewhere on the moduli space the B_1 fields may vary independently from K_k and J_k, $U(1)_B$ is spontaneously broken by $\langle Z_i \rangle \neq 0$, and the anomaly coefficients for the infrared symmetries match the values calculated in the ultraviolet theory.

3.2.3 Additional tests

So far we have restricted our attention to the ordered $\Lambda_1 > \ldots > \Lambda_k$ case to find the dynamically generated superpotential. Due to the holomorphy of the superpotential, changes in the Λ_i hierarchy should not alter the form of the superpotential. In this section we test this supposition by considering the $\Lambda_1 \ll \Lambda_{i\neq 1}$ case. In this limit the $SU(N)^k$ model reduces to an $SU(N)^{k-1}$ extension to $F = N$ SUSY QCD which has been studied by Chang and Georgi [86].

As $\Lambda_1 \to 0$, the A and Q fields decouple from the strongly coupled $\overline{Q}_i$. Chang and Georgi find that the infrared operators involving only $\overline{Q}_i$ obey the following constraints:

$$\det(\overline{Q}_1\overline{Q}_2) = Z_1Z_2 - \Lambda_2^b \tag{3.2.49}$$
$$\det(\overline{Q}_1\overline{Q}_2\overline{Q}_3) = Z_1Z_2Z_3 - \Lambda_2^bZ_3 - Z_1\Lambda_3^b \tag{3.2.50}$$
$$\det(\overline{Q}_1\overline{Q}_2\overline{Q}_3\overline{Q}_4) = Z_1Z_2Z_3Z_4 - \Lambda_2^bZ_3Z_4 - Z_1\Lambda_3^bZ_4 - Z_1Z_2\Lambda_4^b + \Lambda_2^b\Lambda_4^b, \tag{3.2.51}$$

and so on. This is exactly the same form we derived for $B_{i\geq 2}$ in Section 3.2.2. At scales above $\mathcal{O}(\Lambda_1)$ but below $\Lambda_{i>1}$, the G_1 charged degrees of freedom include A, Q, and $M = (\overline{Q}_1\overline{Q}_2 \ldots \overline{Q}_k)$.

Let us define the mass-normalized field $\mathcal{M}$,

$$\mathcal{M} = \frac{(\overline{Q}_1\overline{Q}_2 \dots \overline{Q}_k)}{\Lambda_2\Lambda_3 \dots \Lambda_k}, \tag{3.2.52}$$

and let the fields $\{A, Q, \mathcal{M}\}$ confine under G_1, producing

$$J_k = Q\mathcal{M}, \qquad K_k = A\mathcal{M}^2, \qquad Z_M = \det(\mathcal{M}), \tag{3.2.53}$$

and the baryons $B_1 = \{U_1, V_1, \mathcal{W}_1; X_1, Y_1\}$ as defined in Section 3.2.1. The dynamically generated superpotential is

$$W_{\text{odd}} = \frac{X_1Y_1Z_M - X_1K_k^{m-1}J_k^3 - Y_1K_k^mJ_k}{\widetilde{\Lambda}_1^b} \tag{3.2.54}$$

$$W_{\text{even}} = \frac{(U_1\mathcal{W}_1 - V_1^2)Z_M - U_1K_k^{m-2}J_k^4 + V_1K_k^{m-1}J_k^2 - \mathcal{W}_1K_k^m}{\widetilde{\Lambda}_1^b}. \tag{3.2.55}$$

The effective scale $\widetilde{\Lambda}_1^b$ contains a product of $(\overline{Q}_1^N \dots \overline{Q}_k^N)$ and $\Lambda_2^b \dots \Lambda_k^b$, so that the superpotential is invariant under the spurious symmetries.

There is also a quantum modified constraint

$$Z_M = \det \mathcal{M} = (Z_1 \dots Z_k) - \Lambda_2^b(Z_3 \dots Z_k) + \{\text{all other contractions}\}. \tag{3.2.56}$$

If we use a Lagrange multiplier λ, Eq. (3.2.56) follows from the superpotential

$$W_d' = \lambda \left\{Z_M - (Z_1 \dots Z_k) + (\text{all contractions})\right\}. \tag{3.2.57}$$

After replacing Z_M with $\{Z_i\}$, the equations of motion are identical to Eqs. (3.2.46) and (3.2.47), suggesting that there is no phase transition in the parameter space.

Notice that the equations of motion from Z_M also determine a vacuum solution for λ:

$$\frac{\partial W_{\text{odd}}}{\partial Z_M} = \frac{X_1Y_1}{\widetilde{\Lambda}_1^b} + \lambda = 0 \tag{3.2.58}$$

$$\frac{\partial W_{\text{even}}}{\partial Z_M} = \frac{U_1\mathcal{W}_1 - V_1^2}{\widetilde{\Lambda}_1^b} + \lambda = 0 \tag{3.2.59}$$

(3.2.60)

Thus, the Lagrange multiplier can be treated as a new redundant baryon operator, which should be integrated out along with the other redundant fields.

Finally, let us consider regions of parameter space in which Λ_1 is neither the largest nor the smallest confinement scale. In these cases the redundant operators include a mix of B_i and Z_{ij}, all of which produce the same equations of motion in the reduced operator basis. For any arrangement, at the last confinement scale Λ_f there is a dynamically generated superpotential of the form

$$W^{(f)} \sim \frac{\widetilde{K}_f^{N-2}\widetilde{J}_f^4\mathcal{M}^N}{\widetilde{\Lambda}_f^b}, \tag{3.2.61}$$

where J_f, K_f, and M are such that

$$(J_fM) = (Q\overline{Q}_1 \dots \overline{Q}_f)(\overline{Q}_{f+1} \dots \overline{Q}_k) = J_k, \quad (K_fM^2) = \left(A\overline{Q}_1^2 \dots \overline{Q}_f^2\right)\left(\overline{Q}_{f+1} \dots \overline{Q}_k^2\right)^2 = K_k, \tag{3.2.62}$$

and where $\{\widetilde{J}_f, \widetilde{K}_f, \mathcal{M}\}$ are normalized to have mass dimension +1. Under the remaining gauged G_f, these fields satisfy the index condition for s-confinement, $\sum_j \mu_j - \mu_G = 2$, and there is a dynamically generated superpotential. Lagrange multipliers λ_i enforce the constraint between the operators $\det(\overline{Q}_i \ldots \overline{Q}_j)$ and $\{Z_i \ldots Z_j\}$, and the equations of motion provide a relationship between λ_i and the other hadrons. After replacing the redundant operators with their equations of motion, we find that the constraints relating $\{J_k, K_k\}$ to $\{B_1, Z_i\}$ are unchanged.

Flow: It is a necessary condition for s-confining theories that their description in terms of gauge-invariants is valid in the Higgs phase, when some fields acquire large expectation values and spontaneously break the gauge group to a subgroup. If the low-energy theory does not s-confine, then the original theory cannot be s-confining either. This is the "flow requirement" of [74], which we use in this section to test the $SU(N)^k$ theory.

In the $\langle J_k \rangle^i_j \gg \Lambda$ vacuum with $\langle A_{\alpha\beta} \rangle = 0$, the $SU(N)^k$ group is broken to $SU(N-1)^k$ in the classical limit. This requires a nonzero $(\overline{Q}_i)^\alpha_\beta$ for every $\overline{Q}_i$, which break each gauged $SU(N)_i$ to $SU(N-1)_i$. The $SU(N)_i \times SU(N)_{i+1}$ bifundamentals $\overline{Q}_i$ decompose into $SU(N-1) \times SU(N-1)$ representations as follows:

$$SU(N) \times SU(N) \to SU(N-1) \times SU(N-1): \quad (\overline{\Box}, \Box) \longrightarrow (\overline{\Box}, \Box) \oplus (\overline{\Box}, \mathbf{1}) \oplus (\mathbf{1}, \Box) \oplus (\mathbf{1}, \mathbf{1}). \tag{3.2.63}$$

The $(2N-1)$ broken generators of each gauge group $G_{i \neq 1}$ "eat" the combination $\Box + \overline{\Box} + \mathbf{1}$ from $\overline{Q}_{i-1}$ and $\overline{Q}_i$ to create $(2N-1)$ massive gauge superfields, leaving behind the $(\overline{\Box}, \Box)$ bifundamental fields.

The G_1 group behaves somewhat differently: its broken generators "eat" the $(\overline{\Box}, \mathbf{1})$ part of $\overline{Q}_1$ and a linear combination of the $\Box$ superfields $Q_{i=1\ldots4}$. Under $SU(N-1)_1$ the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ field decomposes as $(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \oplus \Box)$, so that the "eaten" Q field is replaced by a component of A. After removing the massive superfields, the $SU(N-1)_1$ charged matter is $A' + 4Q' + (N-1)\overline{Q}'_1$. The overall effect of $\langle J_k \rangle \gg \Lambda$ on the $SU(N)^k$ model is to replace N with $N-1$.

Now let us consider the limit where $\langle A_{\alpha\beta} \rangle \gg \Lambda$ and $\langle J \rangle = 0$. In the even $N = 2m$ case with $\langle U_1 = \mathrm{Pf}\, A \rangle \gg \Lambda_1$, $SU(2m)_1$ is broken to $Sp(2m)_1$ and $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ decomposes into $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{Sp} \oplus \mathbf{1}$. Here $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{Sp}$ is the $(2m^2 - m - 1)$ dimensional representation of $Sp(2m)$. There are also $(2m^2 - m - 1)$ broken $SU(2m)$ generators, so the superfield $A' = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{Sp}$ is eaten.

The fields Q and $\overline{Q}_i$ are not directly affected by $\langle \mathrm{Pf}\, A \rangle$: however, as $Sp(2m)$ has no complex representations, Q and $\overline{Q}_1$ are effectively $(2m+4)$ quarks charged in the $\Box$ representation of $Sp(2m)$. This theory is known to s-confine [88]. It is likely that the $Sp(2m) \times SU(2m)^{k-1}$ product group theory is also s-confining: we explore this possibility in Section 3.3.2.

In the case where N is odd, an expectation value $\langle X_1 \rangle = \langle A^m Q \rangle \gg \Lambda$ breaks $SU(2m+1)$ to $Sp(2m)$ instead. Aside from a few extra singlets and massive gauge bosons, there is little difference between the odd N and even N cases: the infrared theory is $Sp(2m) \times SU(2m)^{k-1}$.

Conclusion: Our product group extension to the $A + 4Q + N\overline{Q}$ model exhibits the behavior required for an s-confining theory. The set of gauge invariant operators $\{J_k, K_k, B_1, Z_{1\ldots k}\}$ satisfies the t' Hooft anomaly matching conditions; the origin remains on the quantum moduli space, so the theory can confine without breaking chiral symmetry; and there is a dynamically generated superpotential. Furthermore, the operators $\{J_k, K_k, B_1, Z_{1\ldots k}\}$ provide a smooth description of the entire moduli space: there is no gauge invariant order parameter to distinguish the confined and

Higgs phases. By considering the flow along flat directions, we have also found another product group extension to an s-confining theory, $Sp(2m) \times SU(N)^{k-1}$.

3.3 Other S-Confining Theories

In the previous section we find strong evidence that the product group extension to the $A+4Q+N\overline{Q}$ model is s-confining. In this section we consider the follow-up question: how many other s-confining models can be extended into product groups? We have already suggested that $Sp(2m)$ with $(2m+4)\Box$ can be extended into an $Sp(2m) \times SU(N)^{k-1}$ product group model. If this theory is not s-confining, then the $SU(N)^k$ $A+4Q+N\overline{Q}$ model is not s-confining either. We discuss the behavior of this theory in Section 3.3.2.

There are also additional possibilities for the $A+4Q+N\overline{Q}$ model in the case where $N=4$. In this special case the entire $SU(4)_L \times SU(N)_R$ family symmetry can be gauged: we consider whether or not such theories are s-confining in Section 3.3.1. In Sections 3.3.3 and 3.3.4 we discuss the other s-confining theories in [75] with family symmetries large enough to accommodate a gauged $SU(N)$ subgroup. This includes SUSY QCD with $F = N+1$ flavors, and $Sp(2m)$ with $(\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 6\Box)$ matter for $m=2$ and $m=3$. We show that some of these theories are not s-confining.

Due to the lack of an index constraint on the matter content, it is difficult to conduct a systematic search for new s-confining product groups. We have seen in the $A+4Q+N\overline{Q}$ model that G_1 confinement increases the index sum of the G_2 charged matter by $+2$, but other confining theories tend to change the index sum by varying amounts. Therefore, the list of theories considered in this section is presumably incomplete.

We restrict our attention to s-confining models which can be extended by gauging a subgroup of the family symmetries and adding bifundamental fields. Our goal is to determine whether product group s-confinement is possible in each model, based on the index constraint after confinement. This is sufficient to show which of the product group extensions are obviously not s-confining. A more detailed analysis is appropriate for the theories which pass this test.

3.3.1 Special case: $SU(4)$

In this section, we extend the $N=4$ $A+4Q+N\overline{Q}$ model by gauging $SU(4)_L^\ell \times G_0 \times SU(4)_R^r$ for some ℓ and r. Here G_0 is the $SU(4)$ gauge group containing the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 4(\Box + \overline{\Box})$ matter, and every other gauged $SU(4)$ contains four flavors of $(\Box + \overline{\Box})$. It is convenient to relabel the hadrons to reflect the $Q \leftrightarrow \overline{Q}$ symmetry of the matter content of the $A+4Q+4\overline{Q}$ model:

$$M = Q\overline{Q}, \qquad \overline{K} = A\overline{Q}^2, \qquad K = AQ^2, \qquad U = A^2, \qquad Z = Q^4, \qquad \overline{Z} = \overline{Q}^4. \tag{3.3.1}$$

A convenient redefinition of the $U(1)_A \times U(1)_B \times U(1)_R$ charges is shown in Table 3.4, for $\ell = r = 2$.

After extending the model in this way, the model has a "left-right" symmetry which simplifies many of the calculations in this section:

$$\ell \leftrightarrow r, \quad G_i \leftrightarrow \widetilde{G}_i, \quad \Lambda_i \leftrightarrow \overline{\Lambda}_i, \quad SU(4)_L \leftrightarrow SU(4)_R, \quad U(1)_A \leftrightarrow U(1)_B, \quad Q_i \leftrightarrow \overline{Q}_i. \tag{3.3.2}$$

Above, Λ_i corresponds to the group G_i, while $\overline{\Lambda}_i$ is the confinement scale of the group $\widetilde{G}_i$. The group $G_0 \times U(1)_R$ and the field A are invariant under the discrete transformation.

	$SU(4)_L$	G_2	G_1	G_0	$\widetilde{G}_1$	$\widetilde{G}_2$	$SU(4)_R$	U_A	U_B	U_R
Q_2	$\overline{\square}$	$\square$						1	0	0
Q_1		$\overline{\square}$	$\square$					-1	0	0
Q_0			$\overline{\square}$	$\square$				1	0	0
A				$\begin{smallmatrix}\square\\\square\end{smallmatrix}$				-2	-2	1
$\overline{Q}_0$				$\overline{\square}$	$\square$			0	1	0
$\overline{Q}_1$					$\overline{\square}$	$\square$		0	-1	0
$\overline{Q}_2$						$\overline{\square}$	$\square$	0	1	0

Table 3.4: Above, the original s-confining theory $A + 4(Q_0 + \overline{Q}_0)$ is extended on the left and right by gauging $G_L^2 \times \widetilde{G}_R^2$ and adding the Q_i and $\overline{Q}_i$ fields to cancel the anomalies. To extend the model beyond $\ell = r = 2$, more quarks Q_i and $\overline{Q}_j$ can be added with alternating $U(1)_A$ and $U(1)_B$ charges.

Infrared operators: Based on our understanding of the $(\ell = 0, r = k-1)$ models developed in the previous section and the vectorlike nature of the G_0-charged fields, we can guess the form of the gauge-invariant operators which describe the moduli space:

$$\mathcal{F} \equiv \left\{ \begin{array}{rclrcl} U_1 &=& A^2 & M_{\ell r} &=& (Q_\ell \ldots Q_1 Q_0 \overline{Q}_0 \overline{Q}_1 \ldots \overline{Q}_r) \\ Z_i &=& Q_i^4 & K_\ell &=& (Q_\ell^2 \ldots Q_0^2 A) \\ \overline{Z}_j &=& \overline{Q}_j^4, & \overline{K}_r &=& (A \overline{Q}_0^2 \ldots \overline{Q}_r^2) \end{array} \right\}, \tag{3.3.3}$$

for $i = 0, 1, \ldots, \ell$ and $j = 0, 1, \ldots, r$.

Only under certain conditions do we expect the basis $\mathcal{F}$ to obey the anomaly matching conditions for the family symmetries listed in Table 3.4. We have already seen that in the $(\ell = 0, r = k-1)$ models with even k, some of the operators in $\mathcal{F}$ become redundant in the $U(1)_B$ preserving vacuum. If this pattern continues in the (ℓ, r) models with $\ell \neq 0$ and $r \neq 0$, then we would expect that the set $\mathcal{F}$ obeys the anomaly matching conditions only if ℓ and r are even. If either ℓ or r is odd, we expect that some operators in $\mathcal{F}$ become redundant if $U(1)_A \times U(1)_B$ is preserved in the vacuum.

For a given (ℓ, r), the number of infrared operators is given by

$$\dim \mathcal{F} = 1 + (\ell + 1) + (r + 1) + 4^2 + \frac{4(3)}{2} + \frac{4(3)}{2} = \ell + r + 31, \tag{3.3.4}$$

while the dimension of the classical moduli space is

$$\dim M_0 = (\ell + 1)4^2 + \frac{4(3)}{2} + (r + 1)4^2 - (\ell + 1 + r)(4^2 - 1) = \ell + r + 23. \tag{3.3.5}$$

This implies that there should exist $N_{\text{con}} = 8$ constraint equations.

Equations of Motion: It is easiest to derive the equations of motion in the case where G_0 confines last. The groups $G_1 \times \ldots \times G_\ell$ and $\widetilde{G}_1 \times \ldots \times \widetilde{G}_r$ confine separately to form the mesons $M_L = (Q_0 \ldots Q_\ell)$ and $M_R = (\overline{Q}_0 \ldots \overline{Q}_r)$, the baryons $Z_{i=0\ldots\ell}$ and $\overline{Z}_{j=0\ldots r}$, and some larger baryon operators with quantum-modified constraints. The charges of M_L and M_R are shown in Table 3.5. In the limit where Λ_0 is small, the theory reduces to two copies of $F = N$ SUSY QCD with product group extensions. According to [86], the fields obey the following constraints:

$$\det M_L = (Z_0 Z_1 \ldots Z_\ell) - \Lambda_1^b (Z_2 \ldots Z_\ell) - \ldots - (Z_0 \ldots Z_{\ell-2}) \Lambda_\ell^b + \ldots \tag{3.3.6}$$

$$\det M_R = (\overline{Z}_0 \overline{Z}_1 \ldots \overline{Z}_r) - \overline{\Lambda}_1^b (\overline{Z}_2 \ldots \overline{Z}_r) - \ldots - (\overline{Z}_0 \ldots \overline{Z}_{r-2}) \overline{\Lambda}_r^b + \ldots \tag{3.3.7}$$

If ℓ is odd-valued, then the sum of neighbor contractions includes a constant term, $(\Lambda_1^b\Lambda_3^b\ldots\Lambda_\ell^b)$; if ℓ is even, then all terms include some power of Z_i. The same relationship holds for r and $\det M_R$. As in the $SU(N)^k$ models, we expect that the distinction between even and odd ℓ and r determines which of the operators in $\mathcal{F}$ are redundant when $U(1)_A$ and $U(1)_B$ are conserved in the vacuum.

	$SU(4)_L$	G_0	$SU(4)_R$	U_A	U_B	U_R
M_L	$\overline{\square}$	$\square$		$\{0,1\}$	0	0
A		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		-2	-2	1
M_R		$\overline{\square}$	$\square$	0	$\{0,1\}$	0

Table 3.5: All gauge groups except G_0 have confined, leaving M_L and M_R. The $\{0,1\}$ charges of M_L and M_R correspond to the cases where ℓ and r are odd or even, respectively. Not shown are the baryons Z_i and $\overline{Z}_j$, which do not transform under the non-Abelian symmetries.

When G_0 confines, $\{M_L, A, M_R\}$ form the following hadrons:

$$\begin{aligned} U_1 &= A^2 & M_{\ell r} &= (M_L M_R) \\ Z_L &= \det M_L & K_\ell &= (A M_L^2) \\ Z_R &= \det M_R & \overline{K}_r &= (A M_R^2), \end{aligned} \tag{3.3.8}$$

with the dynamically-generated superpotential

$$W_d \sim \frac{A^2\mathcal{M}_L^4\mathcal{M}_R^4}{\widetilde{\Lambda}_0^b} \sim \frac{U_1 Z_L Z_R - Z_R K_\ell^2 - Z_L \overline{K}_r^2 - U_1 M_{\ell r}^4 + K_\ell M_{\ell r}^2 \overline{K}_r}{\widetilde{\Lambda}_0^b(\Lambda_1\ldots\Lambda_\ell)^4(\overline{\Lambda}_1\ldots\overline{\Lambda}_r)^4}, \tag{3.3.9}$$

for some $\widetilde{\Lambda}_0^b$ consistent with the anomalous symmetries. We show the charges of the composite fields in Table 3.6.

The equations of motion from U_1, K_ℓ, and Z_L produce the following constraints:

$$\begin{aligned} \det M_{\ell r} &= Z_L Z_R & K_\ell Z_R &= M_{\ell r}^2 \overline{K}_r & \operatorname{Pf}\overline{K}_r &= U_1 Z_R \\ U_1 M^3 &= K_\ell M \overline{K}_r, & \overline{K}_r Z_L &= K_\ell M_{\ell r}^2, & \operatorname{Pf} K_\ell &= U_1 Z_L. \end{aligned} \tag{3.3.10}$$

These equations are not all independent, but contain $N_{\text{cons}} = 8$ independent constraints.

	$SU(4)_L$	$SU(4)_R$	$U_A^{\text{odd }\ell}$	$U_A^{\text{even }\ell}$	$U_B^{\text{odd }r}$	$U_B^{\text{even }r}$	U_R
K_ℓ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		-2	0	-2	-2	1
$M_{\ell r}$	$\overline{\square}$	$\square$	0	1	0	1	0
$\overline{K}_r$		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	-2	-2	-2	0	1
U_1			-4	-4	-4	-4	2
$Z_{\text{even } i}$			+4	+4	0	0	0
$Z_{\text{odd } i}$			-4	-4	0	0	0
$\overline{Z}_{\text{even } j}$			0	0	+4	+4	0
$\overline{Z}_{\text{odd } j}$			0	0	-4	-4	0

Table 3.6: After all of the gauge groups confine, the infrared degrees of freedom are described by the hadrons shown above. Their $U(1)_A$ and $U(1)_B$ charges depend on ℓ and r, respectively.

If we introduce Lagrange superfields λ_L and λ_R, the quantum modified constraints relating $\{Z_L, Z_R\}$ to $\{Z_i, \overline{Z}_j\}$ as a superpotential:

$$W_L = \lambda_L\Big(Z_L - (Z_0 Z_1 \dots Z_\ell) + \Lambda_1^b(Z_2 \dots Z_\ell) + \dots + (Z_0 \dots Z_{\ell-2})\Lambda_\ell^b + \dots\Big) \quad (3.3.11)$$

$$W_R = \lambda_R\left(Z_R - (\overline{Z}_0 \overline{Z}_1 \dots \overline{Z}_r) + \overline{\Lambda}_1^b(\overline{Z}_2 \dots \overline{Z}_r) + \dots + (\overline{Z}_0 \dots \overline{Z}_{r-2})\overline{\Lambda}_r^b + \dots\right). \quad (3.3.12)$$

Redundant Operators: In this section we use the equations of motion to study the operator basis $\mathcal{F}$. In the $U(1)_A$ preserving vacuum with $\langle Z_i \rangle = 0$, the expectation value of Z_L depends heavily on whether ℓ is even or odd. If ℓ is even, then $Z_L \approx 0$; if ℓ is odd, then $Z_L \approx (\Lambda_1^b \Lambda_3^b \dots \Lambda_\ell^b) \gg 0$. The same pattern holds for r and $\overline{Z}_j$ when $U(1)_B$ is preserved.

It is simplest to consider the case in which both ℓ and r are even. Expanding about the $Z_i = \overline{Z}_j = 0$ vacuum to first order in Z_i and $\overline{Z}_j$, we find that every term in Eq. (3.3.10) contains a product of at least two fields, so that none of the operators in the set $\mathcal{F}$ are redundant. This is consistent with the fact that all of the anomaly coefficients from $SU(4)_L \times SU(4)_R \times U(1)_A \times U(1)_B \times U(1)_R$ match the ultraviolet theory when r and ℓ are even.

This is not true if ℓ is odd. In this case the equations of motion for $\overline{K}_r Z_L$ and $U_1 Z_L$ can be rewritten as

$$\overline{K}_r = \frac{K_\ell M_{\ell r}^2}{(\Lambda_1^b \Lambda_3^b \dots \Lambda_\ell^b)}, \qquad U_1 = \frac{\operatorname{Pf} K_\ell}{(\Lambda_1^b \Lambda_3^b \dots \Lambda_\ell^b)}. \quad (3.3.13)$$

near the $U(1)_A \times U(1)_B$ preserving vacuum. Similarly, the equation of motion for $\det M_{\ell r}$ becomes

$$\overline{Z}_0(\overline{\Lambda}_2^b \overline{\Lambda}_4^b \dots \overline{\Lambda}_r^b) + \overline{\Lambda}_1^b \overline{Z}_2(\overline{\Lambda}_4^b \dots \overline{\Lambda}_r^b) + \dots + (\overline{\Lambda}_1^b \overline{\Lambda}_3^b \dots \overline{\Lambda}_{r-1}^b)\overline{Z}_r = \frac{\det M_{\ell r}}{(\Lambda_1^b \Lambda_3^b \dots \Lambda_\ell^b)}, \quad (3.3.14)$$

which can be recast into a linear constraint equation for any one of the $\overline{Z}_{\text{even}}$ fields. Taken together, Eqs. (3.3.13) and (3.3.14) imply that the operators $\{\overline{K}_r, U_1, \overline{Z}_{\text{even}}\}$ should be removed in the $U(1)_A \times U(1)_B$ preserving vacuum if ℓ is odd and r is even. In the even ℓ, odd r case it is the operators $\{K_\ell, U_1, Z_{\text{even}}\}$ which become redundant, and Z_R rather than Z_L remains large in the $\overline{Z}_j = 0$ vacuum.

If both ℓ and r are odd, then the origin of moduli space is no longer a solution to the equations of motion:

$$\begin{aligned}\det M_{\ell r} &= (\Lambda_1^b \Lambda_3^b \dots \Lambda_\ell^b)(\overline{\Lambda}_1^b \overline{\Lambda}_3^b \dots \overline{\Lambda}_r^b) - \Big(Z_0 Z_1 \Lambda_3^b \dots \Lambda_\ell^b + Z_0 \Lambda_2^b Z_3 \dots \Lambda_\ell^b + \dots\Big)(\overline{\Lambda}_1^b \dots \overline{\Lambda}_r^b) \\ &\quad -(\Lambda_1^b \dots \Lambda_\ell^b)\left(\overline{Z}_0 \overline{Z}_1 \overline{\Lambda}_3^b \dots \overline{\Lambda}_r^b + \overline{Z}_0 \overline{\Lambda}_2^b \overline{Z}_3 \dots \overline{\Lambda}_r^b + \dots\right) + \dots \end{aligned} \quad (3.3.15)$$

To satisfy this constraint, either $\langle M \rangle \neq 0$, $\langle Z_{\text{even}} Z_{\text{odd}} \rangle \neq 0$, or $\langle \overline{Z}_{\text{even}} \overline{Z}_{\text{odd}} \rangle \neq 0$. Different family symmetries are broken in each case, leaving different sets of independent operators.

In the $\langle M \rangle \neq 0$ vacuum where M_j^i is proportional to δ_j^i, $SU(4)_L \times SU(4)_R$ is broken to its diagonal subgroup $SU(4)_d$. The fields Q_ℓ and $\overline{Q}_r$ transform under $SU(4)_d$ as $\overline{\Box}$ and $\Box$, respectively, while the meson M decomposes as

$$\overline{\Box} \otimes \Box = \mathbf{1} \oplus \mathbf{Adj}: \qquad M_{\ell r} \longrightarrow (\operatorname{Tr} M_{\ell r}) \oplus (M_{\ell r} - \operatorname{Tr} M_{\ell r}). \quad (3.3.16)$$

In the $U(1)_A \times U(1)_B$ preserving vacuum with $Z_i = \overline{Z}_j = 0$, it is possible to write $\overline{K}_r$ and U_1 either in terms of K_ℓ and $M_{\ell r}$, or K_ℓ and U_1 in terms of $\overline{K}_r$ and $M_{\ell r}$. Therefore, we can either remove the set $\{K_\ell, U_1, \mathrm{Tr}\, M\}$ or $\{\overline{K}_r, U_1, \mathrm{Tr}\, M\}$. This degeneracy is related to the fact that K_ℓ and $\overline{K}_r$ have the same transformation properties under $SU(4)_d \times U(1)_A \times U(1)_B \times U(1)_R$.

If instead $\langle M \rangle = 0$ and $\langle Z_{\rm even} Z_{\rm odd} \rangle \neq 0$, only $U(1)_A$ is broken in the vacuum. One "$(Z_{\rm even} + Z_{\rm odd})$" linear combination determined by the ratio of the expectation values becomes massive, and all sixteen M^i_j degrees of freedom remain independent. The operator $\overline{K}_r$ is not redundant in this vacuum: the $Z_L \overline{K}_r$ equation of motion includes a term $Z_{\rm even} Z_{\rm odd} \overline{K}_r$ which is not small. The set of redundant operators is $\{K_\ell, U_1, (Z_{\rm even} + Z_{\rm odd})\}$.

Finally, if the nonzero expectation value is $\langle \overline{Z}_{\rm even} \overline{Z}_{\rm odd} \rangle$, then $U(1)_B$ is broken. As we would expect from the left-right symmetry, the redundant operators are $\{\overline{K}_r, U_1, (\overline{Z}_{\rm even} + \overline{Z}_{\rm odd})\}$ in this vacuum. It is also possible to break a linear combination of $U(1)_A$ and $U(1)_B$ if $\langle Z_{\rm even} Z_{\rm odd} \rangle \neq 0$ and $\langle \overline{Z}_{\rm even} \overline{Z}_{\rm odd} \rangle \neq 0$.

Anomaly Matching: We have discussed six distinct cases with maximal symmetry in the vacuum, based on ℓ and r. Below, we show a summary of our results for each case:

(ℓ, r)	Broken symmetry	Redundant operators
(even, even)	None	None
(odd, even)	None	$\{\overline{K}_r, U_1, \overline{Z}_{\rm even}\}$
(even, odd)	None	$\{K_\ell, U_1, Z_{\rm even}\}$
(odd, odd)	$SU(4)_L \times SU(4)_R$	$\{K_\ell$ or $\overline{K}_r, U_1, \mathrm{Tr}\, M_{\ell r}\}$
	$U(1)_A$	$\{K_\ell, U_1, (Z_{\rm even} + Z_{\rm odd})\}$
	$U(1)_B$	$\{\overline{K}_r, U_1, (\overline{Z}_{\rm even} + \overline{Z}_{\rm odd})\}$

For the remaining symmetries and operators in each case, we have verified that the anomaly coefficients match the UV theory. There are 21 matching conditions for each of the first three cases, 17 for the fourth case, and 12 each for the final two cases. Although some of these coefficients are related to each other via the left-right symmetry, the explicit calculation is lengthy and not very illuminating.

Let us also consider points on the moduli space with nonzero $\langle Z_i \rangle$ or $\langle \overline{Z}_j \rangle$, where none of the operators in the set $\mathcal{F}$ are redundant. In these vacua $U(1)_A \times U(1)_B$ is spontaneously broken, and the infrared operators should obey anomaly matching conditions for the remaining symmetries.

For the odd ℓ, even r case, $U(1)_A$ is broken by $\langle Z_i \rangle \neq 0$ for some Z_i. After $U(1)_A$ is broken, $\{U_1, \overline{Z}_{\rm even}\}$ form an anomaly-neutral pair: their $U(1)_{B,R}$ charges are opposite, so all of the $U(1)^3$ and gravitational $U(1)$ anomalies cancel. The fermionic part of $\overline{K}_r$ is neutral under $U(1)_B \times U(1)_R$, and it is in a real representation of $SU(4)_R$: therefore, $\overline{K}_r$ contributes nothing to the remaining anomaly coefficients. Thus, the t' Hooft anomaly matching conditions are also satisfied in the $\langle Z_i \rangle \neq 0$ vacuum where the operators $\{\overline{K}_r, U_1, \overline{Z}_{\rm even}\}$ are independent degrees of freedom.

In the even-ℓ, odd-r models, the operators $\{K_\ell, U_1, Z_{\rm even}\}$ are restored as independent degrees of freedom when $\langle \overline{Z}_j \rangle \neq 0$ and $U(1)_B$ is spontaneously broken. Applying the left-right transformation to the above results, the introduction of $\{K_\ell, U_1, Z_{\rm even}\}$ has no net effect on the anomaly coefficients once $U(1)_B$ is removed. Finally, when $\langle Z_i \rangle \neq 0$ and $\langle \overline{Z}_j \rangle \neq 0$ in the odd-ℓ, odd-r models, the operators $\{K_\ell, U_1, Z_{\rm even}\}$ are restored as independent degrees of freedom without contributing to the anomaly coefficients of the remaining symmetries. Both $U(1)_A$ and $U(1)_B$ are broken in this case.

Flows: Our proposed s-confining extensions to the $SU(4)$ model pass several consistency checks. As a final test, let us spontaneously break the gauge group by giving large expectation values to the gauge invariant operators. For example, $\langle M_{\ell r}\rangle \gg \Lambda$ breaks $SU(4)^{\ell+r+1}$ to $SU(3)^{\ell+r+1}$, leaving $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix} + 4\Box + 3\overline{\Box}$ matter charged under $SU(3)_0$. Three of the $\Box$ fields come from the $G_0 \times G_1$ bifundamental Q_0, while the fourth comes from

$$SU(4) \to SU(3): \qquad \begin{smallmatrix}\Box\\ \Box\end{smallmatrix} \longrightarrow \begin{smallmatrix}\Box\\ \Box\end{smallmatrix} \oplus \Box. \tag{3.3.17}$$

Note that $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix} = \overline{\Box}$ for $SU(3)$, so that there are effectively $(3+1)$ flavors of $(\Box + \overline{\Box})$ charged under $SU(3)_0$. The low-energy theory is a left-right extension of $F = 4$, $N = 3$ SUSY QCD, where an $SU(3)_L \times SU(3)_R$ subgroup of the family $SU(4)_L \times SU(4)_R$ is gauged. In Section 3.3.3 we consider such models in more detail.

Along flat directions with $\langle \mathrm{Pf}\, A\rangle \gg \Lambda_0$, $SU(4)_0$ is broken to $Sp(4)$, leaving an (ℓ, r) product group extension of the s-confining $Sp(4) : (4 + 4)\Box$ model. In this theory an $SU(4)_L \times SU(4)_R$ subgroup of the $SU(8)$ family symmetry is gauged. We discuss models of this type in Section 3.3.2.

Summary: In every (ℓ, r) model with $(\ell, r) \neq (0, 0)$, there are quantum deformations to the classical moduli space. The origin remains on the moduli space unless both ℓ and r are odd. In the mixed case where only one of $\{\ell, r\}$ is odd, eight of the fields become redundant in the vacua which conserve $U(1)_A \times U(1)_B$. If ℓ and r are both even, all of the infrared operators in Eq. (3.3.3) are independent, interacting degrees of freedom even at the origin of moduli space. Due to the existence of a dynamically generated superpotential and the possibility of confinement without chiral symmetry breaking, we conclude that the (ℓ, r) models are s-confining if ℓ and r are not both odd.

$SU(4)$ **Ring Extension:** Before moving on to consider other types of models, let us extend the (ℓ, r) model even further by gauging a diagonal subgroup G_d of the family $SU(4)_L \times SU(4)_R$ symmetry. This connects the left and right ends of the (ℓ, r) extension as shown in Table 3.7, so that different models are labelled by the sum $(\ell + r)$. Models of this type appear in deconstructions of 5d gauge theories, as in [83].

	G_ℓ	$G_{\ell-1}$	$\ldots$	G_1	G_0
Q_ℓ	$\Box$				$\overline{\Box}$
$Q_{\ell-1}$	$\overline{\Box}$	$\Box$			
$\vdots$		$\overline{\Box}$			
Q_1			$\ddots$	$\Box$	
Q_0				$\overline{\Box}$	$\Box$
A					$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$

Table 3.7: Above, we show the matter fields of the $SU(4)$ ring extension to the $A + 4Q + 4\overline{Q}$ model.

Although the baryon operators $\mathrm{Pf}\, A$ and $\det Q_i$ are unaffected by the ringlike nature of the product gauge group, there is now only one gauge-invariant meson operator: $\mathrm{Tr}\, M = \mathrm{Tr}\,(Q_0 Q_1 \ldots Q_\ell)$. For any group G_i, the adjoint operator

$$(\hat{M}_i)^\alpha_\beta = (Q_i Q_{i+1} \ldots Q_\ell Q_0 \ldots Q_{i-1})^\alpha_\beta - \frac{1}{4}(\mathrm{Tr}\, M)\delta^\alpha_\beta \tag{3.3.18}$$

is a degree of freedom in the limit where G_i is weakly gauged, and can be used to create gauge-invariant operators of the type $\mathrm{Tr}\,(\hat{M}_i \hat{M}_i)$ and $\mathrm{Tr}\,(\hat{M}_i^3)$. In this notation, $Q_{-1} = Q_\ell$ for the $i = 0$ case.

Even when these operators have large expectation values, the gauge group is not completely broken. It has been shown [89] in the $SU(N)^k$ extension to $F = N$ SUSY QCD that at an arbitrary point on the moduli space has a remaining $U(1)^3$ gauge group. In the $A + 4Q + 4\overline{Q}$ model it is also possible to set $\langle \text{Pf}\, A\rangle \gg \Lambda_0$, so that $SU(4)_0$ is broken to $Sp(4)$. This reduces the rank of the group by one, but is not sufficient to break $U(1)^3$ completely. Therefore, the $SU(4)$ ring extension has a Coulomb branch, and is not s-confining.

3.3.2 $Sp(2m)$ with $(2m+4)$ quarks

In Section 3.2.3, we found that the $SU(N)^k$ extension of the $A + 4Q + N\overline{Q}$ model flows to an $Sp(2m) \times SU(2m)^{k-1}$ theory. In the limit where $Sp(2m)$ is much more strongly coupled than the $SU(2m)$ groups, the $(2m+4)$ quarks confine to produce the operator $M = (Q^2)$, which transforms in the $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$ representation under the approximate $SU(2m+4)$ family symmetry.

The fields Q and M have the following charges:

	$Sp(2m)$	$SU(2m+4)$	$U(1)_R$
Q	$\Box$	$\Box$	$1/(m+2)$
M		$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$	$2/(m+2)$

A dynamically generated superpotential

$$W_d = \frac{\text{Pf}\, M}{\Lambda^{2m+1}} \tag{3.3.19}$$

reproduces the classical constraints on the Q_i fields.

	$SU(4)_L$	$Sp(2m)$	$SU(2m)_1$	$\ldots$	$SU(2m)_k$	$SU(2m)_R$
Q_L	$\Box$	$\Box$				
$\overline{Q}_0$		$\Box$	$\Box$			
$\overline{Q}_1$			$\overline{\Box}$			
$\vdots$				$\ddots$		
$\overline{Q}_{k-1}$					$\Box$	
$\overline{Q}_k$					$\overline{\Box}$	$\Box$
(Q_L^2)	$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$					
$(Q_L\overline{Q}_0)$	$\Box$		$\Box$			
$(\overline{Q}_0^2)$			$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$			
$\overline{Q}_1$			$\overline{\Box}$			
$\vdots$				$\ddots$		
$\overline{Q}_k$					$\overline{\Box}$	$\Box$

Table 3.8: An $Sp(2m) \times SU(2m)^k$ model is shown, which is expected to s-confine. At the bottom of the table, we list the degrees of freedom in the confined phase of $Sp(2m)$. Subsequent confinement follows the pattern of the $A + 4Q + N\overline{Q}$ model.

In the product gauge group model shown in Table 3.8, an $SU(2m)$ subgroup of the family symmetry is gauged and new bifundamental fields are added to cancel the anomalies. The family

$SU(2m+4)$ is explicitly broken to $SU(2m)\times SU(4)\times U(1)$, under which the meson M decomposes as

$$\yng(1,1) \longrightarrow (\yng(1,1), \mathbf{1}; -4) \oplus (\Box, \Box; m-2) \oplus (\mathbf{1}, \yng(1,1); 2m) : \qquad M \longrightarrow M_A \oplus M_Q \oplus M_0, \tag{3.3.20}$$

and the dynamically generated superpotential becomes

$$W_d \longrightarrow \frac{M_A^{m-1} M_Q^2 M_0}{\Lambda^{2m+1}}. \tag{3.3.21}$$

Including the bifundamental field $\overline{Q}_1$, the $SU(2m)_1$ charged matter in the confined phase of $Sp(2m)$ is $M_A + 4M_Q + 2m\overline{Q}_1$, which is expected to s-confine.

This model can also be derived using the deconfinement technique of Berkooz [76], by treating the matter field A as a bound state of two quarks transforming in the fundamental representation of a new $Sp(N)$.

3.3.3 SUSY QCD

A product group extension to $F = N+1$ SUSY QCD can be derived from the $N = 3$ case of $A + 4Q + N\overline{Q}$. In $SU(3)$, the $\yng(1,1)$ representation is the same as $\overline{\Box}$, so that the G_1 matter is effectively $4\Box + 4\overline{\Box}$. By gauging the $SU(3)$ family symmetry of the $\overline{Q}$ and adding a sequence of bifundamental fields $\overline{Q}_i$, we have found a product group extension to SUSY QCD.

For larger values of N, let us gauge an $SU(N)$ subgroup of the $SU(N+1)_R$ family symmetry as shown below:

	$SU(N+1)_L$	$SU(N)_1$	$SU(N)_2$	$SU(N)_R$
Q	$\Box$	$\Box$		
$\bar{q}$		$\overline{\Box}$		
$\overline{Q}_1$		$\overline{\Box}$	$\Box$	
$\overline{Q}_2$			$\overline{\Box}$	$\Box$

After $SU(N)_1$ confinement, the hadrons are $(Q\bar{q})$, $(Q\overline{Q}_1)$, (Q^N), $(\overline{Q}_1^N)$, and $(\bar{q}\overline{Q}_1^{N-1})$, which transform under $SU(N)_2$ and the family symmetries as:

	$SU(N+1)_L$	$SU(N)_2$	$SU(N)_R$
$(Q\bar{q})$	$\Box$		
(Q^N)	$\overline{\Box}$		
$(\overline{Q}_1^N)$			
$(Q\overline{Q}_1)$	$\Box$	$\Box$	
$(\bar{q}\overline{Q}_1^{N-1})$		$\overline{\Box}$	
$\overline{Q}_2$		$\overline{\Box}$	$\Box$

Under $SU(N)_2$ there are $(N+1)(\Box + \overline{\Box})$ matter fields, which is consistent with the index constraint for s-confinement.

For this theory to be s-confining, it must be shown that the dynamically generated superpotential from $SU(N)_1$ does not prevent the operators $(Q\overline{Q}_1)$ and $(\bar{q}\overline{Q}_1^{N-1})$ from varying independently; that the infrared operators obey the appropriate anomaly matching conditions; and that the origin is on the moduli space. The additional gauge groups are likely to introduce quantum-modified constraints between some of the operators, which may induce chiral symmetry breaking in some cases.

This theory can also be extended by gauging an $SU(N)$ subgroup of $SU(N+1)_L$, so that the most general product group extension is $SU(N)^\ell \times SU(N)_0 \times SU(N)^r$. Based on the behavior of the (ℓ, r) $A + 4Q + 4\overline{Q}$ model for odd ℓ and r, we expect that some of the (ℓ, r) SUSY QCD models also break chiral symmetry.

Alternating Gauge Groups: The $F = N+1$ model can also be extended by gauging the entire $SU(N+1)$ family symmetry. In this case, the gauge group has the alternating form $SU(N) \times SU(N+1) \times SU(N) \times SU(N+1) \times \ldots$, with a series of bifundamental fields:

	$SU(N+1)_L$	$SU(N)_1$	$SU(N+1)_2$	$SU(N)_3$	$SU(N+1)_R$
Q	$\overline{\square}$	$\square$			
$\overline{Q}_1$		$\overline{\square}$	$\square$		
$\overline{Q}_2$			$\overline{\square}$	$\square$	
$\overline{Q}_3$				$\overline{\square}$	$\square$

The matter content is simpler in this case, as all of the fields are $SU(N+1) \times SU(N)$ bifundamentals. When $SU(N)_1$ confines, we are left with

	$SU(N+1)_L$	$SU(N+1)_2$	$SU(N)_3$	$SU(N+1)_R$
(Q^N)	$\square$			
$(Q\overline{Q}_1)$	$\overline{\square}$	$\square$		
$(\overline{Q}_1^N)$		$\overline{\square}$		
$\overline{Q}_2$		$\overline{\square}$	$\square$	
$\overline{Q}_3$			$\overline{\square}$	$\square$

Under $SU(N+1)_2$, there are $(N+1)$ flavors of $\square + \overline{\square}$ which is expected to confine with chiral symmetry breaking. Many of the G_2 singlets we would naïvely construct, such as $(Q\overline{Q}_1)(\overline{Q}_1^N)$, are set to zero by the equations of motion, so G_2 confinement leaves the following charged fields:

	$SU(N+1)_L$	$SU(N)_3$	$SU(N+1)_R$
(Q^N)	$\square$		
$(Q\overline{Q}_1\overline{Q}_2)$	$\overline{\square}$	$\square$	
$\overline{Q}_3$		$\overline{\square}$	$\square$

After $G_1 \times G_2$ confinement, the low energy theory is simply $F = N+1$ SUSY QCD with some gauge singlet fields.

Both product group models based on SUSY QCD have the potential to be s-confining, and may be promising directions for future study.

3.3.4 Other Models

Of the s-confining theories listed in [75], there are only a few models possessing non-Abelian family symmetries larger than the gauge group. We have already discussed the $SU(N)$ models with $A + 4Q + N\overline{Q}$ and $(N+1)(Q + \overline{Q})$, as well as the $Sp(2m)$ model with $(2m+4)Q$. There are two remaining cases based on $Sp(2m)$ with $A + 6Q$ [90, 91]. If $m = 2$ or $m = 3$, an $SU(4)$ or $SU(6)$ subgroup of the family symmetry can be gauged. In this section, we show that the product group extensions do not exhibit s-confinement.

$Sp(6)$ **with** $A + 6Q$**:** Consider the $m = 3$ case with just one extra product group. Below, we show the matter fields above and below the $Sp(6)$ confinement scale:

	$Sp(6)$	$SU(6)$	$SU(6)_R$
A	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		
Q	$\square$	$\square$	
$\overline{Q}$		$\overline{\square}$	$\square$
(A^2)			
(A^3)			
(Q^2)		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	
(QAQ)		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	
(QA^2Q)		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	
$\overline{Q}$		$\overline{\square}$	$\square$

In the confined phase of $Sp(6)$, the $SU(6)$ index sum becomes

$$\sum_j \mu_j - \mu_G = 3 \cdot (6 - 2) + 6 \cdot 1 - 2 \cdot 6 = +6, \tag{3.3.22}$$

so the product group does not s-confine. It may be possible to remove some of the degrees of freedom by adding a nonzero tree-level superpotential, but this is outside the scope of the current study.

$Sp(4)$ **with** $A + 6Q$**:** In the $Sp(4)$ case, an $SU(4)$ subgroup of the $SU(6)$ family symmetry is gauged.

	$SU(2)_L$	$Sp(4)$	$SU(4)$	$SU(6)_R$
Q_L	$\square$	$\square$		
A		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$		
Q_R		$\square$	$\square$	
$\overline{Q}$			$\overline{\square}$	$\square$

The set of $Sp(4)$ invariants is

$$\mathcal{F} = \{(A^2); (Q_L^2), (Q_L Q_R), (Q_R^2); (Q_L A Q_L), (Q_L A Q_R), (Q_R A Q_R)\}. \tag{3.3.23}$$

The operators $(Q_L Q_R)$ and $(Q_L A Q_R)$ are bifundamentals of $SU(2) \times SU(4)$, while (Q_R^2) and $(Q_R A Q_R)$ transform as $(\mathbf{1}, \begin{smallmatrix}\square\\\square\end{smallmatrix})$. The other hadrons are gauge singlets. Together with $\overline{Q}$, the $SU(4)$ charged matter is $2\begin{smallmatrix}\square\\\square\end{smallmatrix} + 4\square + 4\overline{\square}$, with the index sum

$$\sum_j \mu_j - \mu_G = 2(2) + 4(1) + 4(1) - 2 \cdot 4 = +4. \tag{3.3.24}$$

Therefore, the $Sp(4)$ product group extension to $Sp(4) : (A + 6Q)$ is also not s-confining.

3.4 Conclusion

For several s-confining theories, we find product gauge group models with the following properties:

- All infrared degrees of freedom are gauge invariant composite fields;
- The infrared physics is described by a smooth effective theory, which is valid everywhere on the moduli space (including the origin);
- There is a dynamically generated superpotential.

This allows confinement without symmetry breaking, even when the quantum and classical moduli spaces are different. In particular, this behavior may be found in the following models:

$$SU(N) : A + 4Q + N\overline{Q} \qquad Sp(2m) : (2m+4)Q \qquad SU(N) : (N+1)(Q+\overline{Q}).$$

In this chapter we argue that the $A + 4Q + N\overline{Q}$ and $Sp(2m) : (2m+4)Q$ product group models s-confine. Based on less rigorous arguments we suggest two product group extensions of SUSY QCD which may also be s-confining, but a more detailed analysis is required. It is also entirely possible that there are many other s-confining product group theories unrelated to the models considered here.

In the $A + 4Q + N\overline{Q}$ model with $N = 4$, we consider a set of product group extensions of the form $G_L^\ell \times G_0 \times G_R^r$. When ℓ and r are both odd, the chiral symmetry is necessarily broken in the vacuum, so the theory is not s-confining. If instead the sum $(\ell + r)$ is odd, then the origin remains on the quantum-deformed moduli space, and some of the infrared operators become redundant in the symmetry-enhanced vacua. Finally, if ℓ and r are both even, we find that all of the operators are interacting degrees of freedom in the neighborhood of the origin. In each case, there is a dynamically generated superpotential.

A promising direction for future study is to treat the product gauge groups as k site decompositions of 5d SUSY theories. Exact calculations in $\mathcal{N} = 2$ SUSY may provide us with a better understanding of the 4d $\mathcal{N} = 1$ models considered here.

One feature of the product group models is the lack of small gauge-invariant operators, which has a promising phenomenological application to composite axion models. After lifting some of the flat directions, a Peccei-Quinn $U(1)$ symmetry may be dynamically broken when the gauge group confines, producing a light composite axion. If the product gauge group is suitably large, the Peccei-Quinn symmetry is protected against the explicit symmetry breaking effects which would otherwise be induced by higher-dimensional operators. Chapter 4 is devoted to the study of this scenario.

Chapter 4

Composite Axion Model from S-Confinement

The following is based on a previously published paper by the author and Tim M.P. Tait [2].

4.1 Introduction

Compositeness is invoked in many of the solutions to the axion quality problem discussed in Section 1.4. A properly constructed model can protect the axion potential from the PQ violating effects that induce CP violation in the vacuum, while also generating the scale f_a dynamically.

If the axion is a strongly bound state of multiple fundamental fields, $\phi \sim \psi^n$, the problematic Lagrangian operators of the form $\mathcal{L} = m^3(\phi + \phi^\star)$ originate from nonrenormalizable interactions in the high-energy theory, suppressing the magnitude of the coupling m^3. Taking Λ as the scale at which the fields ψ confine to form the composite axion, $\phi \sim \psi^n/\Lambda^{n-1}$, Planck scale violation of the Peccei-Quinn symmetry induces the effective operator

$$\mathcal{L}_{\text{UV}} \sim \frac{\psi^n + \psi^{\star n}}{M_{\text{P}}^{n-4}} = \frac{\Lambda^{n-1}}{M_{\text{P}}^{n-4}} \left(\frac{\psi^n}{\Lambda^{n-1}} + \frac{\psi^{\star n}}{\Lambda^{n-1}} \right), \tag{4.1.1}$$

implying that the coupling m^3 is of order $\Lambda^{n-1} M_{\text{P}}^{4-n}$. In terms of the quality factor defined in Eq. (1.3.4), a satisfactory solution to the strong CP problem would require

$$\Lambda^{n-1} M_{\text{P}}^{4-n} f_a \lesssim 10^{-62} (10^{12}\ \text{GeV})^4. \tag{4.1.2}$$

With the simplifying choice $\Lambda \sim f_a \sim 10^{12}$ GeV, an appropriately small value for m^3 is attained if $n > 12$, just as in the case where $U(1)_{\text{PQ}}$ is protected by a $\mathbb{Z}_n$ discrete symmetry [45].

A complete solution to the axion quality problem must also satisfy several additional criteria. First, the $U(1)_{\text{PQ}}$ symmetry must have no chiral anomaly with the strongly coupled group: otherwise, instanton effects will explicitly break $U(1)_{\text{PQ}}$ at the scale Λ, completely invalidating the axion model. Second, *all* low dimensional PQ violating operators should be forbidden, even those not directly involving the scalar field ϕ. In particular, tree level masses for $SU(3)_c$ charged fermions or other types of $U(1)_{\text{PQ}}$ violation, $\mathcal{L} \sim m_q q\bar{q} + \frac{1}{M_\star}(q\bar{q})^2 + \dots$ introduce loop-level perturbations to the axion potential $V(a)$, easily resulting in effective values of $\bar{\theta}$ larger than 10^{-10}. This criteria applies even to supersymmetric theories, due to loop effects below the scale of spontaneous supersymmetry breaking.

Simultaneously satisfying these various requirements is a highly nontrivial exercise, and this is reflected in the relatively small set of models in the literature. Typical solutions, whether composite [48–50] or not [42,52,53], rely on some sort of product gauge group to protect the axion potential from all the possible loop-induced PQ violating effects.

In this regard, the structure of the s-confining theory outlined in Chapter 3 makes it a nearly ideal candidate for QCD axion model-building. The "meson-like" gauge invariant operators $J_k = Q\overline{Q}_1\overline{Q}_2 \ldots \overline{Q}_k$ and $K_k = A\overline{Q}_1^2\overline{Q}_2^2 \ldots \overline{Q}_k^2$ grow in dimension with the length of the product gauge group, $SU(N)_1 \times SU(N)_2 \times \ldots SU(N)_k$, while the dimensions of the "baryon-like" operators scale with N. By increasing N and k, Planck scale Peccei-Quinn violation can be made arbitrarily irrelevant at low energies, ameliorating the axion quality problem.

A number of practical questions must be addressed, in order to convert the s-confining theory of Chapter 3 into a working QCD axion model. To solve the strong CP problem, the $SU(3)_c$ gauge group must be present somewhere in the theory, embedded for example within the $SU(4)$ or $SU(N)_R$ global symmetries shown in Table 3.3. Ensuring anomaly cancellation will necessitate the addition of new, presumably non-composite fields. Sections 4.2 and 4.2.1 in this chapter develop a model based on one particular set of choices, corresponding to the moose diagram shown in Figure 5.1. Assuming that all of the anomaly-free global symmetries are broken to some extent by the Planck-scale perturbations to the theory, we identify the unique combination of the global $U(1)$ symmetries that corresponds to $U(1)_{\text{PQ}}$.

An s-confining theory by definition includes the origin on the moduli space: this is the vacuum in which the expectation value of every scalar field is zero, and every global symmetry of the high-scale theory is manifest in the infrared limit. This "confinement without chiral symmetry breaking" is not the right behavior for an axion model, because $U(1)_{\text{PQ}}$ must be spontaneously broken in order to solve the strong CP problem without new, massless fields. Engineering the right vacuum thus requires an external superpotential, which when combined with the dynamically generated superpotential successfully excises the origin from the moduli space, spontaneously breaking $U(1)_{\text{PQ}}$ with some f_a determined jointly by the confinement scale and parameters in the external superpotential. In Section 4.2.2 we show how this is accomplished. Section 4.2.3 estimates the size of the leading gravitational corrections, and determines parameters such that the axion quality problem is ameliorated to a sufficient degree. In Section 4.3, we show how a simple extension of the basic model can dynamically generate superpotential terms on which the basic module relies, resulting in a theory in which all of the essential mass scales are generated from strong dynamics.

As we shall see, solving the quality problem can imply that a theory whose low energy limit looks like a rather standard invisible axion model may blossom at high energies into a rich interlocking structure of gauge dynamics.

4.2 Axion from a Supersymmetric Product Group

The construction of our axion model begins with a gauge group $SU(N)_{(1)}$, with one matter field A transforming in the antisymmetric ($\boxminus$) representation; four quarks, Q; and N antiquarks $\overline{Q}_1$. This theory is known to s-confine [76–78,88]: that is, a set of gauge-invariant operators provides a smooth description of the moduli space which is valid at the origin, and a dynamically generated superpotential enforces the classical constraints between operators [74,75]. When supplemented by an appropriately chosen external superpotential, $U(1)_{\text{PQ}}$ is spontaneously broken when $SU(N)_{(1)}$ confines.

High axion quality is enforced by expanding $SU(N)_{(1)}$ into a product gauge group, $SU(N)^r =$

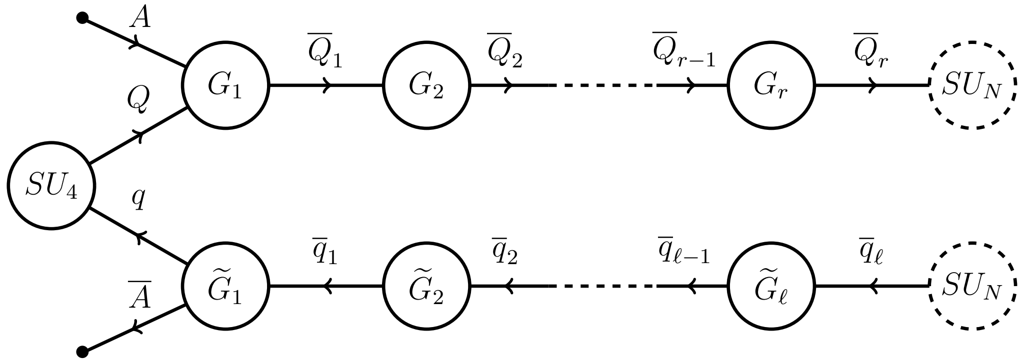

Figure 4.1: Moose diagram indicating the matter content and gauge interactions of the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite axion model. Each G_i and $\widetilde{G}_i$ corresponds to a gauged $SU(N)$, whereas $SU(N)$ flavor symmetries are represented by dashed circles. The bifundamental fields Q, $\overline{Q}_i$, q, and $\overline{q}_i$ are depicted as directed line segments connecting adjacent groups, while the field A ($\overline{A}$) transforms under G_1 ($\widetilde{G}_1$) in the antisymmetric two-tensor representation.

$SU(N)_{(1)} \times SU(N)_{(2)} \times \ldots \times SU(N)_{(r)}$. In addition to the $SU(N)_{(1)}$-charged $A + 4Q$, the matter fields include a set of bifundamentals $\overline{Q}_i$ which transform under $SU(N)_{(i)} \times SU(N)_{(i+1)}$, and N antiquarks $\overline{Q}_r$ charged only under $SU(N)_{(r)}$. It has recently been demonstrated that this product group model s-confines [1], and that the gauge-invariant operators include "mesons" of the form $(Q\overline{Q}_1\overline{Q}_2 \ldots \overline{Q}_r)$ and $(A\overline{Q}_1^2 \ldots \overline{Q}_r^2)$; "baryons" $(\overline{Q}_i^N)$ for each $i = 1 \ldots r$; and special baryons $(A^{\frac{N-p}{2}} Q^p)$ for $0 \leq p \leq 4$, subject to the condition that $(N - p)$ is even. An axion living in a combination of these fields enjoys the feature that increasing r and N results in increasingly suppressed gravitational corrections.

To accommodate QCD within the model, we introduce a second copy of the matter fields $\overline{A} + 4q + \overline{q}_1 + \ldots + \overline{q}_{\ell-1} + N\overline{q}_\ell$ charged under a new s-confining $SU(N)^\ell$ gauge group, and we let Q and q transform in the fundamental ($\square$) and antifundamental ($\overline{\square}$) representations under a weakly gauged $SU(4)$ which contains $SU(3)_c$ as a subgroup. The full matter content of our theory is thus $\{A, Q, \overline{Q}_1 \ldots \overline{Q}_r; \overline{A}, q, \overline{q}_1 \ldots \overline{q}_\ell\}$, with the gauge group $SU(N)^r \times SU(4) \times SU(N)^\ell$. The gauge structure and matter assignments is represented as a moose diagram in Figure 5.1, and is vaguely reminiscent of a deconstructed extra dimension with a bulk $SU(N)$ broken to $SU(4)$ on a defect. As we show in Section 4.2.3, this structure permits smaller values of N for a given axion qualty.

For convenience, we introduce the notation $SU(N)^\ell = \widetilde{G}_1 \times \widetilde{G}_2 \times \ldots \times \widetilde{G}_\ell$ and $SU(N)^r = G_1 \times G_2 \times \ldots \times G_r$, where $\widetilde{G}_i$ and G_i represent $SU(N)$ groups that confine at scales $\widetilde{\Lambda}_i$ and Λ_i respectively. Up to a constant, the holomorphic scales $\widetilde{\Lambda}_i$ and Λ_i are defined as

$$\widetilde{\Lambda}_i^b \equiv \mu^b \exp\{-8\pi^2/\widetilde{g}_i^2 + i\widetilde{\theta}_i\}, \qquad \Lambda_i^b \equiv \mu^b \exp\{-8\pi^2/g_i^2 + i\theta_i\}, \tag{4.2.1}$$

where $\widetilde{g}_i$ and g_i are the coupling constants of the gauge groups $\widetilde{G}_i$ and G_i. In the dynamically generated superpotential for each group there is an overall constant that is not determined by symmetry arguments; to simplify the notation, we absorb these constants into $\widetilde{\Lambda}_i^b$ and Λ_i^b.

In the absence of an external superpotential, there is a conserved $U(1)_A \times U(1)_B \times U(1)_C \times U(1)_R \times SU(N)_L \times SU(N)_R$ global symmetry, and an approximate $U(1)_{\rm PQ}$ that is broken by the $SU(4)^2$-$U(1)$ anomaly. Charges are shown in Table 5.1, where for convenience, we have taken the $U(1)_R$ charges of Q and A to be equal to q and $\overline{A}$, respectively, with $q_Q = \frac{N-4}{N}$ and $q_A = \frac{16-2N}{N(N-2)}$. By defining $U(1)_{\rm PQ}$ as in Table 5.1, we assume that the operator $(A\overline{Q}_1^2 \ldots \overline{Q}_r^2)$ is more suppressed than

	$SU(N)_L$	$\widetilde{G}_\ell$	$\dots$	$\widetilde{G}_1$	$SU(4)$	G_1	$\dots$	G_r	$SU(N)_R$	U_A	U_B	U_C	U_R	$U(1)_{\text{PQ}}$
$\overline{q}_\ell$	$\Box$	$\Box$								0	0	±1	0	0
$\overline{q}_{\ell-1}$		$\overline{\Box}$	$\Box$							0	0	∓1	0	0
$\vdots$			$\ddots$							$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{q}_1$			$\overline{\Box}$	$\Box$						0	0	1	0	0
$\overline{A}$				$\overline{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}$						-4	0	$\frac{-N}{N-2}$	q_A	0
q				$\overline{\Box}$	$\overline{\Box}$					$N-2$	0	0	q_Q	0
Q					$\Box$	$\Box$				$2-N$	0	0	q_Q	$\frac{2-N}{N}$
A						$\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$				4	$\frac{-N}{N-2}$	0	q_A	$4/N$
$\overline{Q}_1$						$\overline{\Box}$	$\Box$			0	1	0	0	0
$\vdots$							$\ddots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{Q}_{r-1}$							$\overline{\Box}$	$\Box$		0	∓1	0	0	0
$\overline{Q}_r$								$\overline{\Box}$	$\Box$	0	±1	0	0	0

Table 4.1: Representations of the matter fields under the gauged $SU(N)^\ell \times SU(4) \times SU(N)^r$ symmetries, the flavor symmetries $SU(N)_L \times SU(N)_R \times U(1)^4$, and the approximate $U(1)_{\text{PQ}}$ symmetry.

$(\overline{A}\overline{q}_1^2 \dots \overline{q}_\ell^2)$, so that $U(1)_{\text{PQ}}$ is expected to be a better symmetry than $U(1)_A$. Appropriate $U(1)_{\text{PQ}}$ charges in the opposite limit can be recovered by performing the following outer automorphism on the moose diagram:

$$\ell \leftrightarrow r, \qquad G_i \leftrightarrow \widetilde{G}_i, \qquad \Lambda_i \leftrightarrow \widetilde{\Lambda}_i, \qquad A \leftrightarrow \overline{A}, \qquad Q \leftrightarrow q, \qquad \overline{Q}_i \leftrightarrow \overline{q}_i. \tag{4.2.2}$$

At a generic point on the moduli space the full global symmetry is spontaneously broken, producing a number of Nambu-Goldstone bosons. Although the explicit symmetry breaking from gravity would supply masses for the pNGBs, a tree-level external superpotential

$$W_{\text{tree}} = \frac{(\overline{A}\overline{q}_1^2\overline{q}_2^2 \dots \overline{q}_\ell^2)}{M_A^{2\ell-2}} + \frac{(\overline{Q}_1^N)}{M_B^{N-3}} + \frac{(\overline{q}_1^N)}{M_C^{N-3}} + \frac{(A^m Q)(A^{m-1}Q^3)}{M_R^{N-1}} + \frac{(\overline{A}^m q)(\overline{A}^{m-1}q^3)}{M_r^{N-1}} \tag{4.2.3}$$

increases the pNGB masses by breaking the global symmetries more severely. This is essential in the case of the second (M_B) term, which as we shall see below determines the PQ symmetry breaking scale f_a after confinement. The remaining M_i could be safely taken to be M_{P} without harm. In addition, to avoid deforming the G_1 confinement, we choose them to satisfy $\Lambda_1 \lesssim M_i$.

In Section 4.3 we discuss the possibility that some of the terms in Eq. (4.2.3) are generated dynamically through the s-confinement of a strongly coupled $Sp(2n)$ gauge group, providing a natural and completely dynamical origin for the scale f_a.

4.2.1 Confinement

We choose the UV gauge couplings such that $SU(N)^\ell$ and $SU(N)^r$ confine at an intermediate scale where $SU(4)$ remains weakly coupled and supersymmetry is unbroken. For odd $N = 2m+1$, the groups $SU(N)^\ell$ and $SU(N)^r$ confine separately to produce the following hadrons:

$$J_L = (\overline{q}_\ell \overline{q}_{\ell-1} \ldots \overline{q}_1 q), \quad K_L = (\overline{q}_\ell^2 \overline{q}_{\ell-1}^2 \ldots \overline{q}_1^2 \overline{A}), \quad x_1 = (\overline{A}^m q), \quad y_1 = (\overline{A}^{m-1} q^3), \quad z_i = (\overline{q}_i)^N, \tag{4.2.4}$$

$$J_R = (Q\overline{Q}_1 \overline{Q}_2 \ldots \overline{Q}_r), \quad K_R = (A\overline{Q}_1^2 \overline{Q}_2^2 \ldots \overline{Q}_r^2), \quad X_1 = (A^m Q), \quad Y_1 = (A^{m-1} Q^3), \quad Z_i = (\overline{Q}_i)^N. \tag{4.2.5}$$

Their transformation properties under the global symmetries are summarized in Table 4.2. These operators obey quantum-modified equations of motion, for which we define the shorthand notation:

$$(\widetilde{\Pi}_1^\ell z) = \begin{cases} \text{even } \ell\text{:} & (z_1 z_2 z_3 \ldots z_\ell) - \widetilde{\Lambda}_2^b (z_3 z_4 \ldots z_\ell) - z_1 \widetilde{\Lambda}_3^b (z_4 \ldots z_\ell) + \widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b (z_5 \ldots z_\ell) + \ldots \\ & \quad + (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-2}^b) z_{\ell-1} z_\ell + (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-2}^b \widetilde{\Lambda}_\ell^b), \\ \text{odd } \ell\text{:} & (z_1 z_2 z_3 \ldots z_\ell) - \widetilde{\Lambda}_2^b (z_3 z_4 \ldots z_\ell) - z_1 \widetilde{\Lambda}_3^b (z_4 \ldots z_\ell) + \widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b (z_5 \ldots z_\ell) + \ldots \\ & \quad + z_1 (\widetilde{\Lambda}_3^b \widetilde{\Lambda}_5^b \widetilde{\Lambda}_7^b \ldots \widetilde{\Lambda}_\ell^b) + \ldots + (\widetilde{\Lambda}_2^b \widetilde{\Lambda}_4^b \widetilde{\Lambda}_6^b \ldots \widetilde{\Lambda}_{\ell-1}^b z_\ell); \end{cases} \tag{4.2.6}$$

$$(\widetilde{\Pi}_1^r Z) = \begin{cases} \text{even } r\text{:} & (Z_1 Z_2 Z_3 \ldots Z_r) - \Lambda_2^b (Z_3 Z_4 \ldots Z_r) - Z_1 \Lambda_3^b (Z_4 \ldots Z_r) + \Lambda_2^b \Lambda_4^b (Z_5 \ldots Z_r) + \ldots \\ & \quad + (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-2}^b) Z_{r-1} Z_r + (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-2}^b \Lambda_r^b), \\ \text{odd } r\text{:} & (Z_1 Z_2 Z_3 \ldots Z_r) - \Lambda_2^b (Z_3 Z_4 \ldots Z_r) - Z_1 \Lambda_3^b (Z_4 \ldots Z_r) + \Lambda_2^b \Lambda_4^b (Z_5 \ldots Z_r) + \ldots \\ & \quad + Z_1 (\Lambda_3^b \Lambda_5^b \Lambda_7^b \ldots \Lambda_r^b) + \ldots + (\Lambda_2^b \Lambda_4^b \Lambda_6^b \ldots \Lambda_{r-1}^b Z_r). \end{cases} \tag{4.2.7}$$

The constraint equations include:

$$\begin{array}{ccc} K_L^m J_L = x(\widetilde{\Pi}_1^\ell z) & K_L^{m-1} J_L^3 = y(\widetilde{\Pi}_1^\ell z) & xy = 0 \\ K_R^m J_R = X(\widetilde{\Pi}_1^r Z) & K_R^{m-1} J_R^3 = Y(\widetilde{\Pi}_1^r Z) & XY = 0. \end{array} \tag{4.2.8}$$

Not shown above, X, Y, x, and y each carry an $SU(4)$ gauge index, which is summed over in the expressions $x^\alpha y_\alpha = X_\alpha Y^\alpha = 0$. Each term in the equations above is invariant under the $SU(N)_L \times SU(N)_R$ family symmetry. Combinatoric coefficients have been suppressed for clarity.

The analysis is simplified by introducing spurion superfields $X_{i>1}$, $Y_{i>1}$, $x_{i>1}$ and $y_{i>1}$, such that the constraints between operators follow directly from the dynamically generated superpotential $W_d = W_L + W_R$, where

$$W_L = \frac{x_1 y_1 z_1 - x_1 y_2 - y_1 x_2}{\widetilde{\Lambda}_1^b} + \sum_{i=2}^{\ell-1} \frac{x_i y_i z_i - x_i y_{i+1} - y_i x_{i+1}}{\widetilde{\Lambda}_1^b \widetilde{\Lambda}_2^b \ldots \widetilde{\Lambda}_i^b} + \frac{x_\ell y_\ell z_\ell - x_\ell K_L^{m-1} J_L^3 - y_\ell K_L^m J_L}{\widetilde{\Lambda}_1^b \widetilde{\Lambda}_2^b \ldots \widetilde{\Lambda}_\ell^b} \tag{4.2.9}$$

$$W_R = \frac{X_1 Y_1 Z_1 - X_1 Y_2 - Y_1 X_2}{\Lambda_1^b} + \sum_{i=2}^{r-1} \frac{X_i Y_i Z_i - X_i Y_{i+1} - Y_i X_{i+1}}{\Lambda_1^b \Lambda_2^b \ldots \Lambda_i^b} + \frac{X_r Y_r Z_r - X_r K_R^{m-1} J_R^3 - Y_r K_R^m J_R}{\Lambda_1^b \Lambda_2^b \ldots \Lambda_r^b}. \tag{4.2.10}$$

Each of the fields $\{X_{i>1}, Y_{i>1}, x_{i>1}, y_{i>1}\}$ is a redundant operator: that is, the equations of motion determine the low-energy behavior of each superfield exactly, leaving no independent degrees of freedom. For example, the constraint $\partial W_d / \partial X_i = 0$ determines the value of Y_{i+1}:

$$Y_2 = Y_1 Z_1, \qquad Y_3 = Y_1 (Z_1 Z_2 - \Lambda_2^b), \qquad Y_{i+1} = Y_i Z_i - \Lambda_i^b Y_{i-1} = Y_1 (\widetilde{\Pi}_1^i Z). \tag{4.2.11}$$

	$SU(4)$	$SU(N)_L$	$SU(N)_R$	$U(1)_{\rm PQ}$
x_1	$\overline{\Box}$			0
y_1	$\Box$			0
z_i	1			0
J_L	$\overline{\Box}$	$\Box$		0
K_L	1	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$		0
X_1	$\Box$			1
Y_1	$\overline{\Box}$			-1
Z_i	1			0
J_R	$\Box$		$\Box$	$\frac{2-N}{N}$
K_R	1		$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$4/N$

Table 4.2: Operators describing infrared degrees of freedom in the confined phase of $SU(N)^\ell \times SU(N)^r$, and their transformation properties under the approximate $SU(N)_L \times SU(N)_R \times U(1)_{\rm PQ}$ flavor symmetries.

After confinement, the tree-level superpotential Eq. (4.2.3) leads to

$$W_{\rm tree} \to \frac{(K_L)_{i_1 i_2}}{M_A^{2\ell-2}} + \frac{Z_1}{M_B^{N-3}} + \frac{z_1}{M_C^{N-3}} + \frac{X_1^\alpha Y_1^\alpha}{M_R^{N-1}} + \frac{x_1^\alpha y_1^\alpha}{M_r^{N-1}}, \tag{4.2.12}$$

where the indices i and α refer to $SU(N)_L$ and $SU(4)$, respectively. In the discussion that follows, we assume that M_B is several orders of magnitude below $M_{\rm P}$, and that $M_B \lesssim M_{A,C,R,r} \lesssim M_{\rm P}$.

4.2.2 Symmetry Breaking

Each term in $W_{\rm tree}$ is introduced to break an undesired global symmetry: however, the Z_1 and z_1 tadpoles induced by $W_{\rm tree}$ also have a significant effect on the vacuum structure. Added to the full superpotential,

$$W = W_{\rm tree} + W_L + W_R, \tag{4.2.13}$$

the Z_1 and z_1 tadpole terms in $W_{\rm tree}$ shift the moduli space away from the origin: specifically, their equations of motion cause $\langle X_1 Y_1\rangle$ and $\langle x_1 y_1\rangle$ to be nonzero. In this section we consider the case $\langle X_1 Y_1\rangle \gg \langle x_1 y_1\rangle$ and show that $SU(4)\times U(1)_{\rm PQ}$ is spontaneously broken to $SU(3)_c$.

It is convenient to normalize the infrared operators by appropriate factors of Λ_i so as to give them canonical mass dimension $+1$:

$$\widetilde{J}_L \equiv \frac{J_L}{\Lambda_L^\ell}, \qquad \widetilde{K}_L \equiv \frac{K_L}{(\Lambda_L^\ell)^2}, \qquad \widetilde{x} \equiv \frac{x_1}{\widetilde{\Lambda}_1^m}, \qquad \widetilde{y} \equiv \frac{y_1}{\widetilde{\Lambda}_1^{m+1}}, \qquad \widetilde{z}_i \equiv \frac{z_i}{\widetilde{\Lambda}_i^{N-1}} \tag{4.2.14}$$

$$\widetilde{J}_R \equiv \frac{J_R}{\Lambda_R^r}, \qquad \widetilde{K}_R \equiv \frac{K_R}{(\Lambda_R^r)^2}, \qquad \widetilde{X} \equiv \frac{X_1}{\Lambda_1^m}, \qquad \widetilde{Y} \equiv \frac{Y_1}{\Lambda_1^{m+1}}, \qquad \widetilde{Z}_i \equiv \frac{Z_i}{\Lambda_i^{N-1}} \tag{4.2.15}$$

where

$$\Lambda_L^\ell \equiv (\widetilde{\Lambda}_1 \widetilde{\Lambda}_2 \ldots \widetilde{\Lambda}_\ell), \qquad \Lambda_R^r \equiv (\Lambda_1 \Lambda_2 \ldots \Lambda_r). \tag{4.2.16}$$

In terms of these operators, the tree-level superpotential Eq. (4.2.3) becomes

$$W_{\text{tree}} \;\to\; \Lambda_L^2\left(\frac{\Lambda_L}{M_A}\right)^{2\ell-2}(\widetilde{K}_L)_{i_1i_2}+\Lambda_1^2\left(\frac{\Lambda_1}{M_B}\right)^{N-3}\widetilde{Z}_1+\widetilde{\Lambda}_1^2\left(\frac{\widetilde{\Lambda}_1}{M_C}\right)^{N-3}\widetilde{z}_1 + \Lambda_1\left(\frac{\Lambda_1}{M_R}\right)^{N-1}\widetilde{X}\widetilde{Y}+\widetilde{\Lambda}_1\left(\frac{\widetilde{\Lambda}_1}{M_r}\right)^{N-1}\widetilde{x}\widetilde{y}, \tag{4.2.17}$$

and the dynamically generated superpotential includes the leading terms

$$W_L+W_R=\widetilde{x}\widetilde{y}\widetilde{z}_1+\widetilde{X}\widetilde{Y}\widetilde{Z}_1-\frac{x_1y_2+y_1x_2}{\widetilde{\Lambda}_1^b}-\frac{X_1Y_2+Y_1X_2}{\Lambda_1^b}+\dots \tag{4.2.18}$$

The equation of motion $\partial W/\partial\widetilde{Z}_1=0$ enforces:

$$\widetilde{X}_\alpha\widetilde{Y}^\alpha=-\frac{\Lambda_1^{N-1}}{M_B^{N-3}}\equiv\sigma^2. \tag{4.2.19}$$

By performing an $SU(4)$ gauge transformation, the nonzero expectation values can be rotated into the $\alpha=4$ component such that

$$\langle\widetilde{X}\rangle_{(4)}=\beta\sigma,\qquad \langle\widetilde{Y}\rangle_{(4)}=\frac{1}{\beta}\sigma,\qquad \langle\widetilde{X}\rangle_{\alpha=1,2,3}=\langle\widetilde{Y}\rangle_{\alpha=1,2,3}=0, \tag{4.2.20}$$

where β parametrizes a flat direction of the degenerate vacua, which is likely to be lifted in a particular model of SUSY breaking; we treat it as a free parameter. An $SU(3)_c$ subgroup of $SU(4)$ remains as an infrared symmetry, and the other $15-8=7$ generators of $SU(4)$ are broken. Through the super-Higgs mechanism, 7 of the 8 would-be NGBs are eaten by the $SU(4)$ superfields to make them massive, and a single NGB remains massless. The matter fields decompose into irreducible representations of $SU(3)_c$ as follows:

$$\begin{array}{cclcclccl}\Box&\longrightarrow&\Box\oplus\mathbf{1}, & \overline{\Box}&\longrightarrow&\overline{\Box}\oplus\mathbf{1}, & \mathbf{Adj}&\longrightarrow&\mathbf{Adj}\oplus\Box\oplus\overline{\Box}\oplus\mathbf{1},\\ \widetilde{X}_{\alpha'}&\longrightarrow&\widetilde{X}_\alpha\oplus\widetilde{X}_{(4)}, & \widetilde{Y}_{\alpha'}&\longrightarrow&\widetilde{Y}_\alpha\oplus\widetilde{Y}_{(4)}, & \lambda_a&\longrightarrow&\lambda'_a\oplus\lambda^+\oplus\lambda^-\oplus\lambda^0.\end{array} \tag{4.2.21}$$

A combination of the superfields $\widetilde{X}_{\alpha=1,2,3}$ and $\widetilde{Y}_{\alpha=1,2,3}$ are eaten by the massive $\lambda^\pm$ vector supermultiplets. Another linear combination of $\widetilde{X}$ and $\widetilde{Y}$ is eaten by the diagonal T^{15} generator of $SU(4)$, leaving exactly one massless superfield to play the role of the axion.

We introduce the real scalar fields ϕ_1, ϕ_2, a and η to describe the bosonic degrees of freedom:

$$\begin{aligned}\widetilde{X}_{(4)}&=\left(\tfrac{\phi_1}{\sqrt{2}}+\langle\widetilde{X}_{(4)}\rangle\right)\exp\left[\tfrac{i}{f_a}\left(a+\alpha\eta\right)\right]\\ \widetilde{Y}_{(4)}&=\left(\tfrac{\phi_2}{\sqrt{2}}+\langle\widetilde{Y}_{(4)}\rangle\right)\exp\left[\tfrac{i}{f_a}\left(-a+\tfrac{1}{\alpha}\eta\right)\right],\end{aligned} \tag{4.2.22}$$

where f_a is the axion decay constant, and α is a constant determined by requiring canonical normalization of the scalar kinetic terms. It is convenient to define $v_{1,2}$ such that

$$v_1=\sqrt{2}\left|\langle\widetilde{X}_{(4)}\rangle\right|=\sqrt{2}\,|\beta\sigma|\qquad v_2=\sqrt{2}\left|\langle\widetilde{Y}_{(4)}\rangle\right|=\sqrt{2}\left|\frac{\sigma}{\beta}\right|, \tag{4.2.23}$$

so that normalization of the scalar fields requires

$$f_a^2 = v_1^2 + v_2^2, \qquad\qquad \alpha = \frac{v_2}{v_1}. \tag{4.2.24}$$

In the discussion above we assume that $\widetilde{X}$ and $\widetilde{Y}$ are the only $U(1)_{\rm PQ}$-charged fields with nonzero expectation values. This is not necessarily true: for example, $\langle K_R \rangle$ may acquire an expectation value without breaking $SU(3)_c$. In the limit where $\langle K_R \rangle \ll \sigma$ its contribution to the axion potential is vanishingly small, and the physics remains approximately as discussed here. For completeness, in Appendix A.2 we derive the composition of the physical axion in the more general $\langle K_R \rangle \neq 0$ case.

To preserve $SU(3)_c$ in the vacuum, the QCD-charged components of the scalars $\widetilde{x}$, $\widetilde{y}$, $\widetilde{J}_L$ and $\widetilde{J}_R$ must not acquire expectation values, which places mild constraints on the unspecified nature of SUSY-breaking. Nonzero VEVs for the $i = 4$ components of the scalar fields are permitted.

4.2.3 Gravitational Corrections

Non-perturbative gravity produces $U(1)_{\rm PQ}$-violation, which at low energies are described by local gauge invariant operators in an effective superpotential. The leading (in $1/M_{\rm P}$) terms are:

$$W_g = \rho_1 \frac{(\overline{q}_\ell \overline{q}_{\ell-1} \ldots \overline{q}_1 q Q \overline{Q}_1 \overline{Q}_2 \ldots \overline{Q}_r)}{M_{\rm P}^{\ell+r-1}} + \rho_2 \frac{(\overline{q}_\ell \overline{q}_{\ell-1} \ldots \overline{q}_1 q)(A^m Q)}{M_{\rm P}^{\ell+m-1}} + \rho_3 \frac{(\overline{A}^m q)(A^m Q)}{M_{\rm P}^{2m-1}} + \rho_4 \frac{(A \overline{Q}_1^2 \overline{Q}_2^2 \ldots \overline{Q}_r^2)}{M_{\rm P}^{2r-2}}, \tag{4.2.25}$$

with coefficients ρ_i which encode the details of the unknown quantum gravitational physics. Naive power counting would argue for $\rho_i \sim \mathcal{O}(1)$, whereas computations based on wormhole configurations or stringy realizations of quantum gravity favor $\rho_i \sim \mathcal{O}\left(\exp\left[-S_{\rm wh}\right]\right)$ with $S_{\rm wh} \sim M_{\rm P}/f_a$. To capture the range of possibilities, we will consider a range of ρ_i (all taken to have roughly equal magnitudes) in our analysis below.

After confinement, W_g maps on to:

$$W_g \to \rho_1 \frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} (\widetilde{J}_L \widetilde{J}_R) + \rho_2 \frac{\Lambda_L^\ell \Lambda_1^m}{M_{\rm P}^{\ell+m-1}} (\widetilde{J}_L \widetilde{X}) + \rho_3 \frac{\widetilde{\Lambda}_1^m \Lambda_1^m}{M_{\rm P}^{2m-1}} (\widetilde{x} \widetilde{X}) + \rho_4 \frac{(\Lambda_R^r)^2}{M_{\rm P}^{2r-2}} (\widetilde{K}_R)_{j_1 j_2}, \tag{4.2.26}$$

where the index j refers to the $SU(N)_R$ family symmetry.

There are two types of tree-level corrections to the axion potential. In the supersymmetric limit, the equations of motion from $W_{\rm tree} + W_d + W_g$ produce operators in the Lagrangian of the form

$$\mathcal{L}_g \sim \left(\prod_{i,j} \phi_i \phi_j^\star \right) (\Phi + \Phi^\star), \tag{4.2.27}$$

where Φ has non-zero $U(1)_{\rm PQ}$ charge (and thus some of its phase is part of the axion), and ϕ_i and $\phi_j^\star$ are scalar fields as determined by the equations of motion. Replacing the fields with their expectation values, $\mathcal{L}_g$ corrects the axion potential by:

$$\delta V[a] \sim \left(\prod_{i,j} \langle \phi_i \rangle \langle \phi_j^\star \rangle \right) \langle \Phi \rangle \cos\left(\frac{q_\Phi a}{f_a} + \theta_0 \right). \tag{4.2.28}$$

Clearly this type of correction is only operative if all of the relevant fields $\phi_{i,j}$ have non-zero expectation values.

The second type of tree-level correction arises once SUSY is broken, and the low energy Lagrangian contains A-terms of the form

$$\mathcal{L}_g \sim m_s W_g + h.c. \tag{4.2.29}$$

(where W_g should be understood to have its super-fields replaced by their scalar components, and there is a separate SUSY-breaking coefficient of $\mathcal{O}(m_s)$ for each term in W_g). In the cases where the necessary scalar fields have zero expectation values, these terms can still correct the axion potential at loop level.

As can be seen from Eq. (4.2.8), the moduli space includes vacua with $\langle K_R \rangle = \langle J_R \rangle = 0$. These flat directions are lifted by SUSY-breaking, and thus model-dependent. Rather than getting bogged down in the details of a specific model, we make the pessimistic assumption that the resulting expectation values are large:

$$\langle \widetilde{J}^j_{(4)} \rangle, \langle \widetilde{K}^{j_1 j_2} \rangle \sim \mathcal{O}(m_s). \tag{4.2.30}$$

This assumption additionally simplifies the analysis in that for such large expectation values, the tree-level corrections to the axion potential are expected to dominate over any of the loop level corrections.

Generically, the leading contributions to the axion potential are expected to arise from SUSY-breaking rather than from the equations of motion. This is because the equations of motion from W_d involve high-dimensional operators, which are only important at tree level if all of the participating fields have relatively large expectation values. For example,

$$\left| \frac{\partial W}{\partial \widetilde{J}_R} \right|^2 = \left| \frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} (\widetilde{J}_L) - \frac{(\widetilde{X}_k \widetilde{J}_R^2) \widetilde{K}_R^{m-1}}{\Lambda_r^m} - \frac{(\widetilde{Y}_k) \widetilde{K}_R^m}{\Lambda_r^{m-1}} \right|^2 \tag{4.2.31}$$

reduces to

$$\mathcal{L}_g \sim \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} \frac{\langle \widetilde{K}_R^m \rangle}{\Lambda_r^{m-1}} \right) \langle \widetilde{J}_L^\star \rangle \widetilde{Y}_k + h.c. \tag{4.2.32}$$

In the product $\langle \widetilde{K}_R^m \rangle$, the $SU(N)_R$ indices are contracted antisymmetrically. If some of the expectation values are close to zero, the entire product vanishes. Only in the case where $\langle \widetilde{K} \rangle$ and $\langle \widetilde{J} \rangle$ are comparable to Λ_r does Eq. (4.2.32) contribute significantly.

Quality Factors: The SUSY-breaking A-term corresponding to the ρ_1 term in W_g is

$$\mathcal{L}_g \sim m_s \rho_1 \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} \right) (\widetilde{J}_L)_i^\alpha (\widetilde{J}_R)_j^\alpha + h.c., \tag{4.2.33}$$

where the indices i and j correspond to the $SU(N)_L \times SU(N)_R$ global symmetry. As $\widetilde{J}_R$ is charged under $U(1)_{\rm PQ}$ $\langle \widetilde{J}_L \widetilde{J}_R \rangle \neq 0$ shifts the axion potential by

$$\delta V[a] \sim \rho_1 m_s \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r-1}} \right) \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right| \cos\left(q_J \frac{a}{f_a} + \theta_0 \right), \tag{4.2.34}$$

with $q_J = \frac{2-N}{N} = \mathcal{O}(1)$. From Eq. (**??**), consistency with $\left|\bar{\theta}\right| < 10^{-10}$ requires

$$\rho_1 \frac{m_s M_{\rm P} \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right|}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\rm P}^{\ell+r}} \right) < 10^{-62}. \tag{4.2.35}$$

B1	(GeV)	**B2**	(GeV)	**B3**	(GeV)
f_a	10^{17}	f_a	10^{12}	f_a	10^{9}
Λ_1	10^{17}	Λ_1	10^{12}	Λ_1	10^{9}
$\Lambda_{i>1}$	10^{15}	$\Lambda_{i>1}$	10^{9}	$\Lambda_{i>1}$	10^{4}
$\widetilde{\Lambda}_i$	10^{15}	$\widetilde{\Lambda}_i$	10^{9}	$\widetilde{\Lambda}_i$	10^{4}
m_s	10^{6}	m_s	10^{4}	m_s	10^{4}

Table 4.3: Three benchmark points in the parameter space of Λ_i and $\widetilde{\Lambda}_i$. With the exception of $\langle\widetilde{X}\rangle$ and $\langle\widetilde{Y}\rangle$, the expectation values of the $SU(3)_c$ singlet fields are taken to be $\mathcal{O}(m_s)$.

A limit on r is set by the ρ_4 term:

$$\delta V[a] \sim \rho_4 m_s \frac{\Lambda_R^{2r}}{M_{\rm P}^{2r-2}} \left|\langle(\widetilde{K}_R)_{j_1 j_2}\rangle\right| \cos\left(q_K \frac{a}{f_a} + \theta_0\right), \tag{4.2.36}$$

where $q_K = 4/N$. Ignoring the $\mathcal{O}(1)$ number q_K,

$$\rho_4 \, \frac{m_s M_{\rm P}^2 \left|\langle\widetilde{K}_R\rangle\right|}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_R}{M_{\rm P}}\right)^{2r} < 10^{-62}. \tag{4.2.37}$$

From the ρ_3 term

$$\delta V[a] \sim m_s \rho_3 \frac{\widetilde{\Lambda}_1^m \Lambda_1^m}{M_{\rm P}^{2m-1}} \left|\langle\widetilde{x}_{(4)}\rangle\langle\widetilde{X}_{(4)}\rangle\right| \cos\left(\frac{a}{f_a} + \theta_0\right), \tag{4.2.38}$$

we find a constraint on $N = 2m+1$:

$$\rho_3 \, \frac{m_s M_{\rm P} \langle\widetilde{x}_{(4)}\rangle\langle\widetilde{X}_{(4)}\rangle}{(10^{12}\,{\rm GeV})^4} \left(\frac{\widetilde{\Lambda}_1}{M_{\rm P}}\right)^{m} \left(\frac{\Lambda_1}{M_{\rm P}}\right)^{m} < 10^{-62}. \tag{4.2.39}$$

Finally, the ρ_2 term sets an additional constraint on ℓ and N:

$$\delta V[a] \sim m_s \rho_2 \frac{\Lambda_L^\ell \Lambda_1^m}{M_{\rm P}^{\ell+m-1}} \left|\langle J_L^{(4)}\rangle\langle\widetilde{X}_{(4)}\rangle\right| \cos\left(\frac{a}{f_a} + \theta_0\right), \tag{4.2.40}$$

$$\rho_2 \, \frac{m_s M_{\rm P} \langle J_L\rangle\langle\widetilde{X}_{(4)}\rangle}{(10^{12}\,{\rm GeV})^4} \left(\frac{\Lambda_L}{M_{\rm P}}\right)^{\ell} \left(\frac{\Lambda_1}{M_{\rm P}}\right)^{m} < 10^{-62}. \tag{4.2.41}$$

As long as β is neither very large nor very small, Eqs. (4.2.35), (4.2.37), (4.2.39) and (4.2.41) provide the most restrictive constraints on m, ℓ and r. A wide range of values is allowed for each of the parameters, as we discuss in more detail below.

4.2.4 Benchmark Models:

In this section we consider the quality of the axion potential in three particular models, with $f_a = 10^{17}$ GeV, $f_a = 10^{12}$ GeV and $f_a = 10^{9}$ GeV. For simplicity, we take $\Lambda_1 \sim M_B \sim f_a$ and $\Lambda_{i\neq 1} \sim \widetilde{\Lambda}_i$ for each model, and we allow all QCD singlet scalar fields to acquire $\mathcal{O}(m_s)$ expectation values. Choices for each of these scales are shown in Table 4.3.

Model **B1** is particularly susceptible to gravitational disruptions, as the scales Λ_i and $\widetilde{\Lambda}_i$ are taken to be relatively close to the Planck scale $M_{\rm P} \sim 10^{19}$ GeV. In this model even exponential

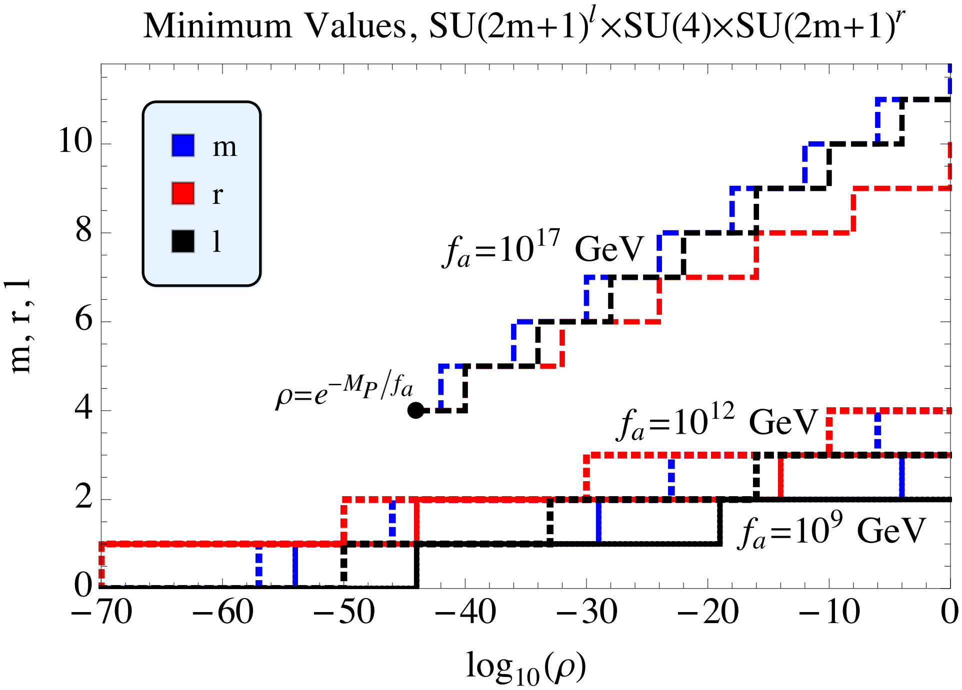

Figure 4.2: Minimum values for m, ℓ and r consistent with $\left|\bar{\theta}\right| < 10^{-10}$ are shown as a function of $\rho_{1\ldots4}$. For the first benchmark model with $f_a = 10^{17}\,\text{GeV}$, we show only values of $\rho \gtrsim \exp(-M_{\text{P}}/f_a) \approx 10^{-43.4}$. The $f_a = 10^{12}\,\text{GeV}$ and $f_a = 10^9\,\text{GeV}$ models are depicted using dotted and solid lines, respectively.

suppression of the constants $\rho_i \sim \exp(-M_{\text{P}}/f_a) \sim 10^{-44}$ cannot account for the high quality of the axion potential, and large values of N, ℓ and r are required. Models **B2** and **B3** have values of $f_a \lesssim 10^{12}\,\text{GeV}$ consistent with the axion dark matter hypothesis; with its smaller values of Λ_i and $\widetilde{\Lambda}_i$, model **B3** is more adept at suppressing gravitational corrections.

In Figure 4.2 we show minimum values for $m \equiv \frac{N-1}{2}$, ℓ, and r consistent with $\left|\bar{\theta}\right| < 10^{-10}$ for the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite axion, as a function of the parameters ρ_i. A wide range is shown for ρ, to accommodate both exponentially suppressed and $\mathcal{O}(1)$ values. In the $\rho_i = \mathcal{O}(1)$ limit, the minimal gauge groups for the three benchmark models are:

$$\begin{aligned} &\textbf{B1:} \quad SU(23)^{11} \times SU(4) \times SU(23)^9 \\ &\textbf{B2:} \quad SU(9)^3 \times SU(4) \times SU(9)^4 \\ &\textbf{B3:} \quad SU(7)^2 \times SU(4) \times SU(7)^3. \end{aligned} \tag{4.2.42}$$

Naturally, if after SUSY breaking the scalar fields $\widetilde{J}_{L,R}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{K}_R$ do not acquire expectation values, then the $U(1)_{\text{PQ}}$ violation induced by W_g affects the axion potential only at loop level, and smaller values for N, ℓ and r are permitted. In the limit where ρ is exponentially suppressed, $\left|\bar{\theta}\right| < 10^{-10}$ no longer constrains m, ℓ or r. Although Eqs. (4.2.35), (4.2.37), (4.2.39) and (4.2.41) are valid only for $m \geq 2$, $r \geq 1$ and $\ell \geq 0$, smaller values for m and r are shown in Figure 4.2 to indicate where ρ is small enough that compositeness is no longer necessary.

4.3 Dynamically Generated W_{tree}

As described in Section 4.2, the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite accidental axion has a high-quality scalar potential and most of the important scales are derived from the confining dynamics, with the exception of M_B in the tree-level superpotential. This is a relatively minor shortcoming: f_a is determined by the relationship between M_B, Λ_1, and $\beta^2 = \langle \widetilde{X} \rangle / \langle \widetilde{Y} \rangle$,

$$f_a^2 = 2 \left| \frac{\Lambda_1^{N-1}}{M_B^{N-3}} \left(\beta^2 + \frac{1}{\beta^2} \right) \right|, \tag{4.3.1}$$

and the scale $M_B \ll M_{\text{P}}$ is added "by hand" in the tree-level superpotential. In this section we show how the M_B term in W_{tree} can be dynamically generated by the s-confinement of an $Sp(2N-4)$ gauge group, so that all of the important mass scales are determined by strong dynamics.

A gauge theory with $2N$ quarks ψ charged under $Sp(2N-4)$ in the fundamental representation s-confines [88] to form mesons $M_{ij} = \epsilon_{ab} \psi_i^a \psi_j^b$, with the superpotential

$$W_d = \frac{\text{Pf}\, M}{\Lambda_0^{2N-1}}. \tag{4.3.2}$$

We break the $SU(2N)$ flavor symmetry by gauging its $SU(N)_1 \times SU(N)_2 = G_1 \times G_2$ subgroup:

$$\Box \longrightarrow (\Box, \mathbf{1}) \oplus (\mathbf{1}, \Box) \qquad\qquad \psi_i^a \longrightarrow (\psi_1)_\alpha^a \oplus (\psi_2)_\beta^a, \tag{4.3.3}$$

where α and β correspond respectively to the $SU(N)_1$ and $SU(N)_2$ gauge indices. The meson $M \sim \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ decomposes into irreducible representations of $G_1 \times G_2$:

$$\mathcal{M}_1^{\alpha_1 \alpha_2} = \frac{(\psi_1)_a^{\alpha_1} (\psi_1)_b^{\alpha_2} \epsilon_{ab}}{\Lambda_0}, \qquad \overline{Q}_1^{\alpha\beta} = \frac{(\psi_1)_a^{\alpha} (\psi_2)_b^{\beta} \epsilon_{ab}}{\Lambda_0}, \qquad \mathcal{M}_2^{\beta_1 \beta_2} = \frac{(\psi_2)_a^{\beta_1} (\psi_2)_b^{\beta_2} \epsilon_{ab}}{\Lambda_0}, \tag{4.3.4}$$

where Λ_0 is the confinement scale of $Sp(2N-4)$. In terms of these operators the dynamically generated superpotential is

$$W_d = \frac{\text{Pf}\,(\psi^2)}{\Lambda_0^{2N-3}} = \frac{(\Lambda_0)^N}{\Lambda_0^{2N-3}} \left[\mathcal{M}_1^m \overline{Q}_1 \mathcal{M}_2^m + \mathcal{M}_1^{m-1} \overline{Q}_1^3 \mathcal{M}_2^{m-1} + \ldots + \mathcal{M}_1 \overline{Q}_1^{2m-1} \mathcal{M}_2 + \overline{Q}_1^{2m+1} \right], \tag{4.3.5}$$

in the case where $N = 2m+1$ is odd. Combinatoric factors for each term in the expansion of $\text{Pf}\, M$ such as $\overline{Q}_1^N \equiv \det \overline{Q}_1$ have been suppressed.

To match this theory with the $A + 4Q + N\overline{Q}$ model, the M_1 and M_2 degrees of freedom must be removed. This is achieved by adding the following matter fields charged under $SU(N)_1 \times SU(N)_2$:

$$2A' + 4Q + \chi + N\overline{Q}_2 = 2(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \mathbf{1}) \oplus 4(\Box, \mathbf{1}) \oplus (\mathbf{1}, \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}) \oplus N(\mathbf{1}, \overline{\Box}). \tag{4.3.6}$$

In the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite model, the $SU(4)$ and $SU(N)$ family symmetries of the Q and $\overline{Q}_2$ are gauged. The full matter content of the theory is shown in Figure 4.3.

Gauge-invariant operators of the form $(A' \psi_1^2)$ and $(\chi \psi_2^2)$ can be added as marginal operators in a tree-level superpotential:

$$W_{\text{tree}} = \lambda_i (A_i')^{\alpha_1 \alpha_2} (\psi_1)_{\alpha_1}^{a_1} (\psi_1)_{\alpha_2}^{a_2} \epsilon_{a_1 a_2} + \lambda_0 \chi^{\beta_1 \beta_2} (\psi_2)_{\beta_1}^{a_1} (\psi_2)_{\beta_2}^{a_2} \epsilon_{a_1 a_2}, \tag{4.3.7}$$

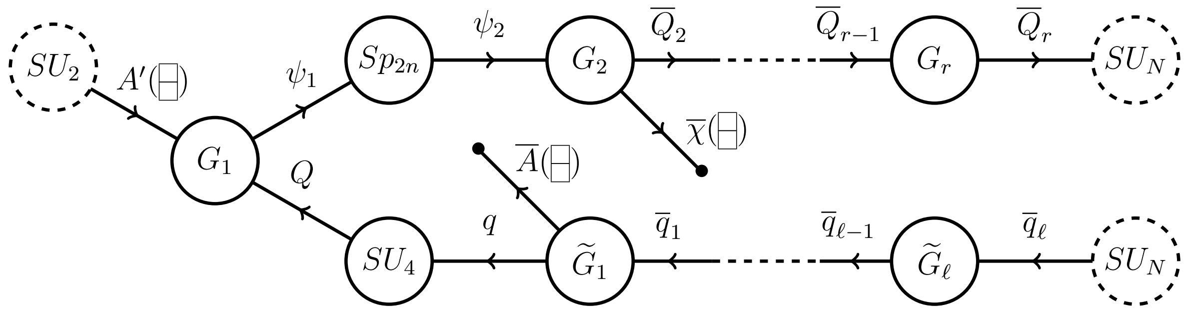

Figure 4.3: The matter content of the $SU(N)^\ell \times SU(4) \times Sp(2n) \times SU(N)^r$ composite axion model is depicted in the moose diagram above, with $Sp_{2n} \equiv Sp(2N-4)$. The $SU(2)$ family symmetry of the A' fields is broken explicitly by the tree-level superpotential Eq. (4.3.7).

where the indices i, a, α and β correspond to $SU(2)$, $Sp(2N-4)$, $SU(N)_1$ and $SU(N)_2$, respectively, and λ_i and λ_0 are dimensionless coupling constants. After $Sp(2N-4)$ confines, W_{tree} becomes

$$W_{\text{tree}} = \lambda_i \Lambda_0 (A'_i)^{\alpha_1\alpha_2} \mathcal{M}_1^{\alpha_1\alpha_2} + \lambda_0 \Lambda_0 \chi^{\beta_1\beta_2} \mathcal{M}_2^{\beta_1\beta_2}. \tag{4.3.8}$$

This is extremely convenient: in the limit where $\Lambda_0 \gg \Lambda_1$, the fields M_1, M_2, χ, and the linear combination "$(A'_1 + A'_2)$" all acquire large masses and decouple. One linear combination of A'_1 and A'_2 remains massless, which we define as A:

$$A \equiv \frac{\lambda_2 A_1 - \lambda_1 A_2}{\mathcal{N}}, \tag{4.3.9}$$

with some normalization factor $\mathcal{N}$.

The dynamically generated superpotential simplifies greatly when we consider the fact that $\mathcal{M}_1$ and $\mathcal{M}_2$ have $\mathcal{O}(\Lambda_0)$ masses from W_{tree}:

$$\frac{\partial W}{\partial A'_i} = \lambda_i \Lambda_0 \mathcal{M}_1, \qquad \frac{\partial W}{\partial \chi} = \lambda_0 \Lambda_0 \mathcal{M}_2. \tag{4.3.10}$$

After integrating out the heavy fields, the superpotential becomes

$$W = \frac{\overline{Q}_1^N}{\Lambda_0^{N-3}}. \tag{4.3.11}$$

Not only is this the desired tree-level superpotential for the composite axion model, but all of the extra matter fields A', χ, $\mathcal{M}_1$ and $\mathcal{M}_2$ have decoupled, leaving only A and $\overline{Q}_1$ as infrared degrees of freedom. In Eq. (4.3.1) M_B is replaced by Λ_0, so that

$$f_a^2 = 2\left|\frac{\Lambda_1^{N-1}}{\Lambda_0^{N-3}}\left(\beta^2 + \frac{1}{\beta^2}\right)\right|. \tag{4.3.12}$$

Every important scale other than M_{P} is now determined solely by confining dynamics.

The nonzero $Sp(2N-4)^2$-$U(1)_B$ anomaly breaks $U(1)_B$ explicitly, as can be seen from the W_d of Eq. (4.3.5). Although in principle the new fields χ and A' provide two additional anomaly-free $U(1)$ symmetries, these are broken by the tree-level superpotential Eq. (4.3.7), and only the $SU(N)_L \times SU(N)_R \times U(1)_A \times U(1)_C \times U(1)_R$ global symmetry remains. Introducing

$$\delta W_{\text{tree}} = \frac{(\overline{A}\overline{q}_1^2\overline{q}_2^2\ldots\overline{q}_\ell^2)}{M_A^{2\ell-2}} + \frac{(\overline{q}_1^N)}{M_C^{N-3}} + \frac{(A^m Q)(A^{m-1}Q^3)}{M_R^{N-1}} + \frac{(\overline{A}^m q)(\overline{A}^{m-1} q^3)}{M_r^{N-1}} \tag{4.3.13}$$

	$Sp(2N-4)$	$SU(N)_1$	$SU(N)_2$	$SU(N)_3$	$SU(4)$	$SU(2)$	$U(1)_{\text{PQ}}$
ψ_1	$\square$	$\overline{\square}$					$-2/N$
ψ_2	$\square$		$\square$				$+2/N$
A'		$\begin{smallmatrix}\square\\\square\end{smallmatrix}$				$\mathbf{2}$	$4/N$
χ			$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$			$\mathbf{1}$	$-4/N$
Q		$\square$			$\square$		$\frac{2-N}{N}$
$\overline{Q}_2$			$\overline{\square}$	$\square$			0

Table 4.4: A subset of the matter fields in the $Sp(2N-4)$ model are shown with their Peccei-Quinn charges. All of the non-Abelian groups except for $SU(2)$ are gauged.

with $M_A \sim M_C \sim M_R \sim M_r \sim M_{\text{P}}$ is sufficient to give masses to the additional pNGBs. In Table 4.4, the Peccei-Quinn charges of each field is shown.

Axion Quality: Of the new superpotential terms which break $U(1)_{\text{PQ}}$, the leading terms are

$$W_g \sim \frac{\chi^m \overline{Q}_2 \overline{Q}_3 \ldots \overline{Q}_r}{M_{\text{P}}^{m+r-4}} + \sum_p \frac{(A_1^{m-p} A_2^p Q)(q\overline{q}_1 \overline{q}_2 \ldots \overline{q}_\ell)}{M_{\text{P}}^{m+\ell-1}} \tag{4.3.14}$$

As χ has a mass of $\mathcal{O}(\Lambda_0)$ and no expectation value, the χ^m interaction has no tree-level effect on the axion potential. The only effects are loop-induced and receive additional suppression.

One linear combination in the $(A_1^{m-p} A_2^p Q)$ sum corresponds to the infrared operator $(A^m Q)$, which has the expectation value $\langle X_1 \rangle$. This term is already included in the W_g of Eq. (4.2.25). Every other term in the sum includes a power of the massive combination $(\lambda_1 A_1 + \lambda_2 A_2)$, which has no expectation value, and is therefore less disruptive to the axion potential than the effects already considered in Eq. (4.2.25).

Aside from the replacement of M_B by Λ_0, the quality factors calculated in Section 4.2.3 are largely unchanged. Operators involving $\overline{Q}_1$ are the exception: now that $\overline{Q}_1 = \psi_1 \psi_2 / \Lambda_0$, a suppression of Λ_0 / M_{P} is added to the operators involving J_R and K_R, marginally improving Eqs. (4.2.35) and (4.2.37):

$$\rho_1 \frac{m_s M_{\text{P}} \left| \langle \widetilde{J}_L \rangle \langle \widetilde{J}_R \rangle \right|}{(10^{12}\,\text{GeV})^4} \left(\frac{\Lambda_0}{M_{\text{P}}} \right) \left(\frac{\Lambda_L^\ell \Lambda_R^r}{M_{\text{P}}^{\ell+r}} \right) \; < \; 10^{-62} \tag{4.3.15}$$

$$\rho_4 \frac{m_s M_{\text{P}}^2 \left| \langle \widetilde{K}_R \rangle \right|}{(10^{12}\,\text{GeV})^4} \left(\frac{\Lambda_0}{M_{\text{P}}} \right)^2 \left(\frac{\Lambda_R}{M_{\text{P}}} \right)^{2r} \; < \; 10^{-62}. \tag{4.3.16}$$

For many values of ρ_i this decreases the minimum value for r by one, as can be seen from the three benchmark models at $\rho_i = \mathcal{O}(1)$:

$$\begin{array}{ll} \textbf{B1:} & SU(23)^{11} \times SU(4) \times Sp(42) \times SU(23)^9 \\ \textbf{B2:} & SU(9)^3 \times SU(4) \times Sp(14) \times SU(9)^3 \\ \textbf{B3:} & SU(7)^2 \times SU(4) \times Sp(10) \times SU(7)^2. \end{array} \tag{4.3.17}$$

Alternate Confinement Order: Thus far, we have required that $\Lambda_0 > \Lambda_1$, simply because the dual of $SU(N) : 2A + 4Q + (2N-4)\overline{Q}$ with the tree-level superpotential $W_{\text{tree}} \sim A\overline{Q}^2$ does not appear in the literature. In principle the infrared behavior of the $2A+4Q+(2N-4)\overline{Q}$ theory with $W_{\rm tree} \neq 0$ can be determined using "deconfinement" techniques [76] and a sequence of dualities: a similar calculation [92] has been completed for $A + FQ + (N + F - 4)\overline{Q}$ with a superpotential of the form $W \sim A\overline{Q}^2$.

Without calculating the degrees of freedom and the superpotential in the infrared dual of $SU(N) : 2A + 4Q + (2N - 4)\overline{Q}$, it is not known how the scale f_a is set in the dual theory. If in the $\Lambda_0 \ll \Lambda_1$ limit $U(1)_{\rm PQ}$ is still broken at the scale $f_a^2 \sim \Lambda_1^{N-1}/\Lambda_0^{N-3}$, then $f_a \sim 10^{12}$ GeV can be achieved with much smaller values of Λ_0 and Λ_1, significantly improving the axion quality.

4.4 Conclusions

In the composite axion model based on the gauge group $SU(N)^\ell \times SU(4) \times SU(N)^r$, a $U(1)_{\rm PQ}$ is spontaneously broken by the vacuum expectation values of the $SU(4)$-charged hadrons $X_1 = (A^m Q)$ and $Y_1 = (A^{m-1}Q^3)$, simultaneously producing the QCD axion and breaking $SU(4)$ to $SU(3)_c$. All important scales in the axion model are generated dynamically from confinement, and are naturally small compared to the Planck scale.

By calculating the disruption to the axion potential $V[a]$ induced by Planck-scale effects, we have demonstrated that the composite model is successful at preserving the quality of the axion potential even when large expectation values are permitted for all of the $U(1)_{\rm PQ}$-charged QCD-singlet scalar fields. In realistic models incorporating SUSY breaking with positive quadratic terms for these scalars such that no large expectation values result, the quality of the axion potential will improve significantly for any given N, ℓ and r, as the terms in W_g disrupt the axion potential to a lesser degree. It would be worthwhile to further investigate such constructions.

It is likely that the success of the $SU(N)^\ell \times SU(4) \times SU(N)^r$ composite axion can be replicated by embedding $SU(3)_c$ within the $SU(N)_R$ flavor symmetry of the $A + 4Q + N\overline{Q}$ model. In this case $U(1)_{\rm PQ}$ will be more closely associated with the $U(1)_B$ flavor symmetry of Table 5.1 rather than $U(1)_A$, and the axion will be generated from a linear combination of $(\overline{Q}_i^N)$ baryons.

Chapter 5

A High Quality Composite Axion

The following is based on a previously published paper by the author and Tim M.P. Tait [3].

5.1 A New Hope

Axion models with sufficiently protected Peccei-Quinn symmetries have proven so far to be an illusive target. Quite apart from the aspiration that an axion arises as an"accidental" consequence of some simple extension to the Standard Model, most existing solutions to the axion quality problem are the results of clever and deliberate model-building. "Cleverness" as a pejorative term [8] is certainly a valid criticism of the axion model developed in Chapter 4: after invoking the strongly coupled $Sp(2n)$ to generate the terms in the superpotential needed to instigate spontaneous $U(1)_{\text{PQ}}$ breaking, the moose diagram Figure 4.3 represents a theory that is substantially more complex than the Standard Model. While we would by no means propose this as a rubric for aesthetics, Figure 4.3 shows roughly one node for every order of magnitude in the strong CP problem ($|\bar{\theta}| < 10^{-10}$).

A much simpler and more appealing model is presented in this chapter, based on an $SU(5) \times SU(5)$ confining supersymmetric gauge theory with local $B - L$ symmetry. The Standard Model matter fields and interactions are easily embedded, and we show that the axion quality is preserved even with the addition of new fields. Certain mesons in the theory are identified as composite Higgs fields, ameliorating the μ problem of the MSSM by coincidentally generating a TeV scale value for $\mu \sim f_a^2/M_{\text{P}}$. Gauge coupling unification is preserved, and the remaining non-Higgs mesons provide composite messengers for gauge-mediated supersymmetry breaking.

These positive developments are natural consequences of the structure of the axion model, with little to no additional model-building effort. Only the Higgs Yukawa couplings must be added "by hand" to the superpotential to complete the MSSM: all of the undesirable baryon and lepton violating superpotential operators are forbidden by the gauged $B - L$ symmetry. Taking all these beneficial qualities into account, we argue that the model described in this chapter represents one of the most compelling known solutions to the axion quality problem.

5.2 The Composite Model

Before specializing to the $SU(5)$, $B - L$ model promised in the introduction, we present the generic framework for the composite axion model as a solution to the axion quality problem, with

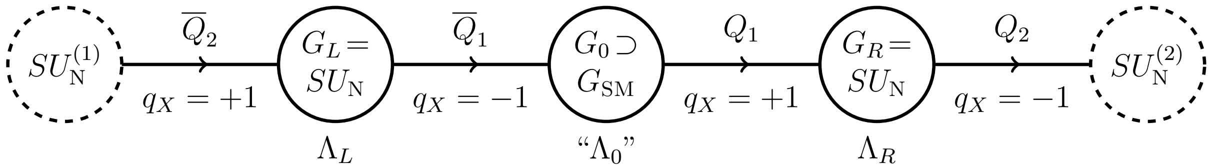

Figure 5.1: Moose diagram indicating the charges of bifundamental matter fields $\overline{Q}_{1,2}$ and $Q_{1,2}$ under the gauge group $SU(N)_L \times SU(N)_{SM} \times SU(N)_R \times U(1)_X$ and global $SU(N)_1 \times SU(N)_2$ global symmetries. The Standard Model $SU(3)_c \times SU(2)_L \times U(1)_Y$ is a subgroup of G_0.

the generic results shown in Figure 5.2. The minimal $SU(5)$ $B-L$ version which fits so well with the MSSM is developed in Section 5.2.2

Conjectured dualities [68, 72] allow one to analyze the low energy behavior of supersymmetric gauge theories. In particular, an $SU(N_c)$ gauge theory with $N_f = N_c$ flavors of quarks $(Q + \overline{Q})$ in the (anti-)fundamental representation is expected to confine at a characteristic scale Λ, such that the low energy degrees of freedom are described by the gauge-singlet operators

$$M = (Q\overline{Q}), \qquad B = (Q^N), \qquad \overline{B} = (\overline{Q}^N), \tag{5.2.1}$$

subject to the quantum-modified constraint

$$\det M - B\overline{B} = \Lambda^{2N}. \tag{5.2.2}$$

The constraint Eq. (5.2.2) guarantees that the global $SU(N_f) \times SU(N_f) \times U(1)$ symmetry is spontaneously broken, either by $\langle M \rangle \neq 0$ or $\langle B\overline{B} \rangle \neq 0$. Similar behavior has been demonstrated in theories with product gauge groups of the form $SU(N) \times SU(N) \times \ldots \times SU(N)$ with bifundamental matter [86]. We show that a composite axion emerges in a subset of these theories, with sufficiently high axion quality.

We invoke the gauge group $SU(N)_L \times SU(N)_{SM} \times SU(N)_R \times U(1)_X$, where $SU(N)_{SM}$ contains the Standard Model $SU(3)_c \times SU(2)_L \times U(1)_Y$ either as a gauged subgroup or as an $SU(5)$ grand unified theory. The strongly coupled $SU(N)_{L,R}$ confine at the characteristic scales $\Lambda_{L,R} \gg$ TeV, but the Abelian $U(1)_X$ is weakly coupled[1]. The bifundamental fields $\overline{Q}_{1,2}$ and $Q_{1,2}$ have $U(1)_X$ charges ± 1, as depicted in the moose diagram of Figure 5.1, with $U(1)_{\text{PQ}}$ charges shown in Table 5.1.

Below the scales Λ_L and Λ_R, the low energy degrees of freedom are described by the composite operators satisfying equations of motion:

$$\begin{aligned} &\overline{M} = (\overline{Q}_2\overline{Q}_1) \qquad \overline{B}_1 = (\overline{Q}_1^N) \qquad \overline{B}_2 = (\overline{Q}_2^N) \qquad \Lambda_L^{2N} = \det \overline{M} - \overline{B}_1\overline{B}_2 \\ &M = (Q_1Q_2) \qquad B_1 = (Q_1^N) \qquad B_2 = (Q_2^N) \qquad \Lambda_R^{2N} = \det M - B_1B_2. \end{aligned} \tag{5.2.3}$$

In the absence of a superpotential, this model respects the global $SU(N)_1 \times SU(N)_2$ symmetries shown in Figure 5.1, as well the gauged $U(1)_X$. There is also a conserved $U(1)_R$, under which the gauginos have charge $+1$ and all of the $\overline{Q}_{1,2}$ and $Q_{1,2}$ are neutral, which remains unbroken everywhere on the moduli space.

In the regime where G_0 is weakly coupled, there is another nearly exact global symmetry, $U(1)_{\text{PQ}}$, which is broken only by the G_0^2-$U(1)_{\text{PQ}}$ anomaly. Due to the locally conserved $U(1)_X$, there

[1] The axion construction leaves the charges of the MSSM matter under $U(1)_X$ largely undetermined. We explore several alternatives below.

	$SU(N)_1$	G_L	G_0	G_R	$SU(N)_2$	$U(1)_X$	$U(1)_{\rm PQ}$
$\overline{Q}_2$	$\square$	$\overline{\square}$				1	$-(1-\alpha)/N$
$\overline{Q}_1$		$\square$	$\overline{\square}$			-1	$(1-\alpha)/N$
Q_1			$\square$	$\overline{\square}$		1	$(1+\alpha)/N$
Q_2				$\square$	$\overline{\square}$	-1	$-(1+\alpha)/N$
$\overline{M}$	$\square$		$\overline{\square}$			0	0
M			$\square$		$\overline{\square}$	0	0
$\overline{B}_2$						N	$-1+\alpha$
$\overline{B}_1$						$-N$	$1-\alpha$
B_1						N	$1+\alpha$
B_2						$-N$	$-1-\alpha$

Table 5.1: $U(1)_{\rm PQ}$ charges and representations under the gauged $G_L \times G_0 \times G_R$ and the global $SU(N)_1 \times SU(N)_2$ symmetries are indicated for the bifundamental quarks (upper half) and composite operators resulting from $G_L \times G_R$ confinement (lower half).

is no unique assignment of Peccei–Quinn charges: rotations under $U(1)_{\rm PQ}$ can always be combined with a global $U(1)_X$ transformation to define a new, equally valid Peccei–Quinn symmetry. This degeneracy is parameterized by the parameter α in Table 5.1.

On the quantum-deformed moduli space described by Eq. (5.2.3), the global $SU(N)_1 \times SU(N)_2 \times U(1)_X \times U(1)_{\rm PQ}$ symmetry must be broken to a subgroup. Furthermore, if the low energy limit of this theory is to approach the Standard Model, then it must be true that $\det M = \det \overline{M} = 0$; otherwise, $SU(3)_c$ would be broken in the vacuum. The vacuum therefore must be engineered to lie on the $\langle B_1 B_2 \rangle \neq 0$, $\langle \overline{B}_1 \overline{B}_2 \rangle \neq 0$ branch of the moduli space, where $U(1)_X$ and $U(1)_{\rm PQ}$ are both spontaneously broken, and the $U(1)_X$ vector supermultiplet acquires a mass by "eating" a combination of the chiral superfields. This is accomplished by including a term in the superpotential of the form:

$$\frac{\left(\overline{Q}_2 \overline{Q}_1\right) \left(Q_1 Q_2\right)}{M_*} \tag{5.2.4}$$

which after confinement generates a mass term for the mesons, $W \sim \mu \overline{M} M$, lifting the mesonic flat directions. If not otherwise present, this term is expected to be induced by quantum gravitational effects.

A unique definition of the Peccei–Quinn charges emerges once $U(1)_X$ is broken: by canonically normalizing the kinetic terms of the (would-be) Nambu–Goldstone bosons of $U(1)_{\rm PQ}$ and $U(1)_X$, the parameter α of Table 5.1 is related to the vacuum expectation values (VEVs) of the baryons as

$$\alpha = \frac{\bar{v}_1^2 + \bar{v}_2^2 - v_1^2 - v_2^2}{f_X^2}, \tag{5.2.5}$$

where

$$\bar{v}_i^2 = 2 \left| \frac{\langle \overline{B}_i \rangle}{\Lambda_L^{N-1}} \right|^2, \qquad v_i^2 = 2 \left| \frac{\langle B_i \rangle}{\Lambda_R^{N-1}} \right|^2, \qquad f_X^2 = \bar{v}_1^2 + \bar{v}_2^2 + v_1^2 + v_2^2, \tag{5.2.6}$$

and where the axion decay constant f_a is

$$f_a^2 = f_X^2 \left(1 - \alpha^2\right). \tag{5.2.7}$$

With this normalization, a $U(1)_{\rm PQ}$ rotation by a phase θ is achieved by the linear shift

$$a \to a + \theta f_a. \tag{5.2.8}$$

Although the products $v_1 v_2$ and $\bar{v}_1 \bar{v}_2$ are set by the quantum modified constraints,

$$\bar{v}_1 \bar{v}_2 = 2\left|\Lambda_L^2\right|, \qquad\qquad v_1 v_2 = 2\left|\Lambda_R^2\right|, \tag{5.2.9}$$

the values of the decay constants f_a and f_X vary along the flat directions within the allowed ranges

$$f_X^2 \geq 4\left|\Lambda_L^2\right| + 4\left|\Lambda_R^2\right|, \qquad\qquad f_a^2 \leq f_X^2. \tag{5.2.10}$$

The case $f_a \ll f_X$ is achieved in the limits $\Lambda_L \gg \Lambda_R$ or $\Lambda_L \ll \Lambda_R$, as $\alpha \to \pm 1$. Conversely, the special case $v_1^2 + v_2^2 = \bar{v}_1^2 + \bar{v}_2^2$ corresponds to $f_a = f_X$.

5.2.1 Axion Quality

To examine the axion quality, we introduce operators characterized by $M_{\rm P}$ which represent an effective field theory description of the low energy residual effects of quantum gravity. It is convenient to introduce a set of rescaled composite operators with mass dimension +1:

$$\overline{\mathcal{M}} = \frac{(\overline{Q}_2\overline{Q}_1)}{\Lambda_L} \qquad \mathcal{M} = \frac{(Q_1 Q_2)}{\Lambda_R} \qquad \overline{\mathcal{B}}_i = \frac{(\overline{Q}_i^N)}{\Lambda_L^{N-1}} \qquad \mathcal{B}_i = \frac{(Q_i^N)}{\Lambda_R^{N-1}}. \tag{5.2.11}$$

The effective gravitational superpotential violating all of the global symmetries takes the form:

$$W_g = \lambda_1 \frac{(\overline{Q}_1^N)(Q_1^N)}{M_{\rm P}^{2N-3}} + \lambda_2 \frac{(\overline{Q}_2^N)(Q_2^N)}{M_{\rm P}^{2N-3}} + \lambda_3 \frac{(\overline{Q}_2^N)(\overline{Q}_1^N)}{M_{\rm P}^{2N-3}} + \lambda_4 \frac{(Q_1^N)(Q_2^N)}{M_{\rm P}^{2N-3}} + \rho_1 \frac{(\overline{Q}_2\overline{Q}_1)(Q_1Q_2)}{M_{\rm P}} + \dots \tag{5.2.12}$$

$$= \left(\frac{\Lambda_L^{N-1}\Lambda_R^{N-1}}{M_{\rm P}^{2N-3}}\right)\left\{\lambda_1 \overline{\mathcal{B}}_1 \mathcal{B}_1 + \lambda_2 \overline{\mathcal{B}}_2 \mathcal{B}_2 + \lambda_3 \overline{\mathcal{B}}_1 \overline{\mathcal{B}}_2 + \lambda_4 \mathcal{B}_1 \mathcal{B}_2\right\} + \rho_1 \left(\frac{\Lambda_L \Lambda_R}{M_{\rm P}}\right) \overline{\mathcal{M}}\mathcal{M} + \dots, \tag{5.2.13}$$

with parameters λ_i and ρ_i encoding the UV physics. Of the operators listed above, only the two associated with λ_1 and λ_2 violate $U(1)_{\rm PQ}$. All of the lower-dimensional operators such as $(\overline{Q}_2\overline{Q}_1)(Q_1Q_2)$ are neutral under $U(1)_{\rm PQ}$, and thus not harmful to the axion quality.

In a supersymmetric vacuum, the leading $U(1)_{\rm PQ}$ violation appears with $M_{\rm P}^{4N-6}$ suppression in the Lagrangian: for example, within terms such as

$$\left|\frac{\partial W_g}{\partial \mathcal{B}_1}\right|^2 = \left|\frac{\Lambda_L^{N-1}\Lambda_R^{N-1}}{M_{\rm P}^{2N-3}}\right|^2 \left|\lambda_1 \overline{\mathcal{B}}_1 + \lambda_4 \mathcal{B}_2\right|^2, \tag{5.2.14}$$

implying a perturbation to the axion potential on the order of

$$Q f_a^4 \sim |\lambda_1 \lambda_4| \left(\frac{\sqrt{\Lambda_L \Lambda_R}}{M_{\rm P}}\right)^{4N-4} M_{\rm P}^2 \langle \overline{\mathcal{B}}_1 \rangle \langle \mathcal{B}_2 \rangle. \tag{5.2.15}$$

Taking $\Lambda_L \approx \Lambda_R \approx f_a \approx 10^{11}$ GeV as a benchmark and ignoring $\mathcal{O}(1)$ factors, the quality factor

$$Q \sim |\lambda_1 \lambda_4|\, 10^{48-32N} \tag{5.2.16}$$

satisfies the bound given in Eq. (??) for $N > 3$, even when the λ_i are $\mathcal{O}(1)$.

More serious perturbations to the axion potential emerge when supersymmetry breaking is taken into account. Supersymmetry breaking induces an "A-term" potential,

$$-\mathcal{L}_A = \left(\frac{\Lambda_L^{N-1}\Lambda_R^{N-1}}{M_{\mathrm{P}}^{2N-3}}\right)\left\{A_1\lambda_1\overline{\mathcal{B}}_1\mathcal{B}_1 + A_2\lambda_2\overline{\mathcal{B}}_2\mathcal{B}_2 + A_3\lambda_3\overline{\mathcal{B}}_1\overline{\mathcal{B}}_2 + A_4\lambda_4\mathcal{B}_1\mathcal{B}_2\right\} + h.c, \tag{5.2.17}$$

where the mass scales A_i are in principle calculable once a particular mechanism of supersymmetry breaking is specified. To remain agnostic concerning the details of supersymmetry-breaking, we assume that the A_i should be of roughly the same magnitude as the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gaugino masses.

Both the A_1 and A_2 terms in Eq. (5.2.17) perturb the axion potential:

$$\delta V(a) = 2\frac{\Lambda_L^{N-1}\Lambda_R^{N-1}}{M_{\mathrm{P}}^{2N-3}}\left\{\left|A_1\lambda_1\langle\overline{\mathcal{B}}_1\rangle\langle\mathcal{B}_1\rangle\right|\cos\left(2\frac{a}{f_a}+\varphi_1\right) + \left|A_2\lambda_2\langle\overline{\mathcal{B}}_2\rangle\langle\mathcal{B}_2\rangle\right|\cos\left(2\frac{a}{f_a}+\varphi_2\right)\right\}. \tag{5.2.18}$$

Again taking $\Lambda_{L,R} \approx f_a \approx 10^{11}$ GeV, the constraint on the quality factor Eq. (??) can be written as

$$\frac{\lambda_i A_i}{10^4\,\mathrm{GeV}}\left(\frac{10^{19}\,\mathrm{GeV}}{M_{\mathrm{P}}}\right)^{2N-3}\left(\frac{\Lambda_L\Lambda_R}{10^{22}\,\mathrm{GeV}^2}\right)^{N-1}\frac{\langle\overline{\mathcal{B}}_i\rangle\langle\mathcal{B}_i\rangle}{10^{22}\,\mathrm{GeV}^2}\cdot 10^{-16N} \lesssim 10^{-76} \tag{5.2.19}$$

for $i = 1, 2$, indicating that models with $N \geq 5$ are free from fine-tuning as long as the characteristic scales $\Lambda_{L,R}$ and f_a are not much larger than 10^{11} GeV.

In Figure 5.2 we plot the maximum values of λ_i consistent with Eq. (5.2.19), for given values of f_a, N, and the other parameters, with the simplifying assumptions $A_1 \approx A_2$ and $\lambda_1 \approx \lambda_2$. It is convenient to label the vacua with the following parameterization:

$$\tan\beta_L = \frac{\bar{v}_2}{\bar{v}_1} \qquad \tan\beta_R = \frac{v_2}{v_1} \qquad \sin^2 2\gamma = \frac{f_a^2}{f_X^2} = 1-\alpha^2. \tag{5.2.20}$$

All of the dimensionful parameters except for A_i and M_{P} are now expressed in terms of f_a:

$$\bar{v}_1 = \frac{\cos\beta_L}{2\cos\gamma}f_a \qquad \bar{v}_2 = \frac{\sin\beta_L}{2\cos\gamma}f_a \qquad v_1 = \frac{\cos\beta_R}{2\sin\gamma}f_a \qquad v_2 = \frac{\sin\beta_R}{2\sin\gamma}f_a, \tag{5.2.21}$$

so that the axion quality condition is expressed:

$$\frac{Qf_a^4}{M_{\mathrm{P}}^4} = 8\left(\frac{f_a^2}{8M_{\mathrm{P}}^2\sin 2\gamma}\right)^N(\sin 2\beta_L\sin 2\beta_R)^{\frac{N-1}{2}}\left(\frac{\lambda_1 A_1\cos\beta_L\cos\beta_R + \lambda_2 A_2\sin\beta_L\sin\beta_R}{M_{\mathrm{P}}}\right) \lesssim 10^{-88}. \tag{5.2.22}$$

Because $\beta_{L,R}$ label degenerate vacua on the moduli space defined by Eq. (5.2.3), particularly large or small values of $\tan\beta_{L,R}$ are typically unnatural. On the other hand, γ is primarily determined by the ratio Λ_L/Λ_R:

$$\tan\gamma = \frac{\Lambda_L}{\Lambda_R}\sqrt{\frac{\sin 2\beta_L}{\sin 2\beta_R}}, \tag{5.2.23}$$

so large or small values of $\tan\gamma$ are more easily tolerated from a naturalness perspective. As we see from Eq. (5.2.22), the best axion quality is achieved for $\tan\gamma \approx 1$, when $f_a \approx f_X$ and $\Lambda_L \approx \Lambda_R$.

We show the maximum tolerable $\lambda_1 \approx \lambda_2$ as a function of f_a for a few choices of N, $\tan\beta_L = \tan\beta_R$, and $\sin 2\gamma$ in Figure 5.2. While effective field theory would suggest that generic theories

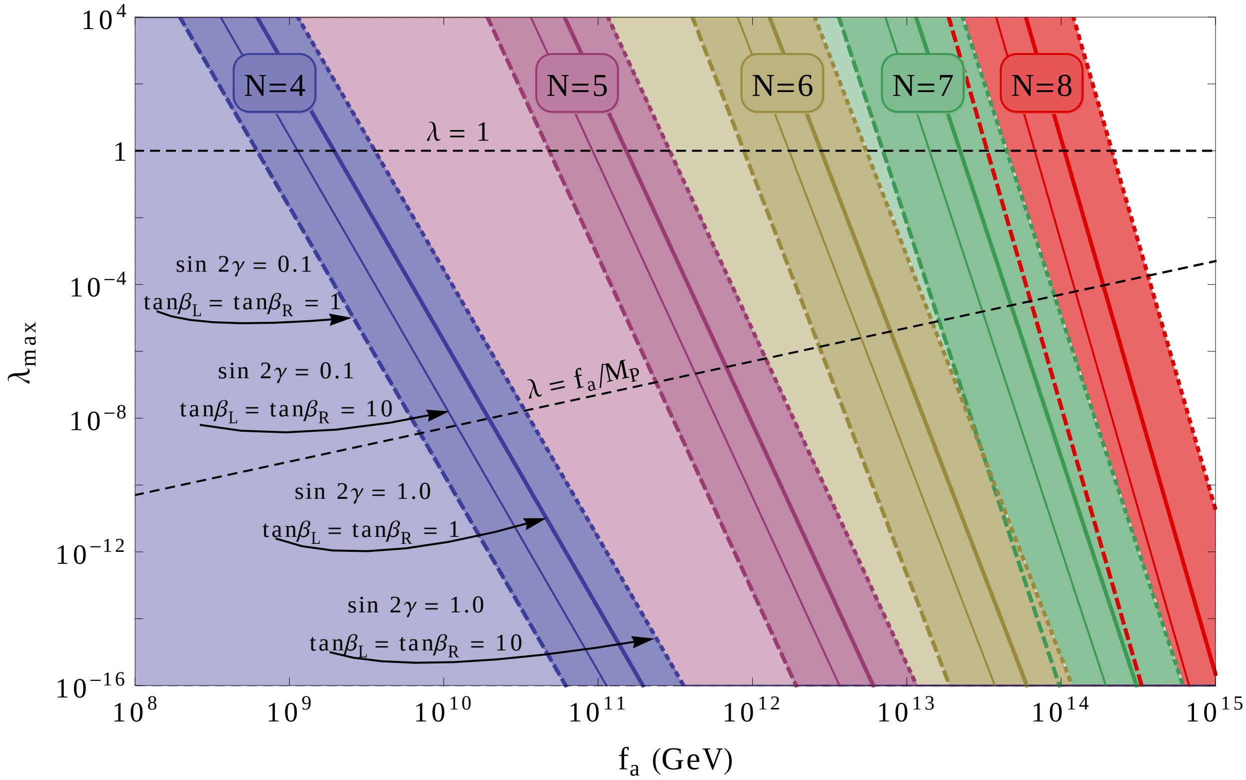

Figure 5.2: Maximum values of $\lambda_1 \approx \lambda_2$ consistent with Eq. (5.2.22) for given values of f_a and $N = 4, 5, 6, 7, 8$. The region to the left of each line indicates the axion models which return $|\bar{\theta}| < 10^{-11}$ without any fine tuning. From left to right within each band of a given N, models are indicated with: $\sin 2\gamma = 0.1$, $\tan\beta_L = \tan\beta_R = 1$ (thin, dashed); $\sin 2\gamma = 0.1$, $\tan\beta_L = \tan\beta_R = 10$ (thin, solid); $\sin 2\gamma = \tan\beta_L = \tan\beta_L = 1$ (thick, solid); and $\sin 2\gamma = 1$, $\tan\beta_L = \tan\beta_L = 10$ (thin, dotted). In each case $A_1 \approx A_2 = 10^5$ GeV.

of quantum gravity should produce $\lambda_{1,2} \sim \mathcal{O}(1)$, in [39, 40, 43] it is argued that wormhole-induced $U(1)_{\rm PQ}$ violation yields suppressed values of $\lambda_i \sim \exp(-S_w)$, where the wormhole action S_w depends logarithmically on the axion decay constant, $S_w \sim a - b \ln \frac{f_a}{M_{\rm P}}$. For typical cases the resulting suppression in λ_i is modest: values as small as $\lambda \sim 10^{-7}$ are achieved in [43] for $f_a \sim 10^{12}$ GeV. For $N = 5$ such that G_0 is large enough to contain the SM, $\mathcal{O}(1)$ λ's are consistent with $f_a \lesssim 10^{11}$ GeV.

Generally, the high axion quality observed in Eq. (5.2.19) is preserved even when new fields are coupled to the model provided that they are neutral under $U(1)_X$. Problems arise if there are fields S with $U(1)_X$ charges:

$$q_S = \pm N, \pm\frac{N}{2}, \pm\frac{N}{3}, \ldots, \pm\frac{N}{N-1}, \tag{5.2.24}$$

for which case W_g includes gauge-invariant terms $S^p B_{1,2}$ or $S^p \overline{B}_{1,2}$ for some power $p < N$.

5.2.2 $U(1)_{B-L}$ as $U(1)_X$

From Eq. (5.2.19) we see the remarkable fact that for $f_a \lesssim 10^{11}$ GeV and $\mathcal{O}(1)$ values in the couplings λ_i, sufficient protection of the axion quality requires $N \geq 5$: precisely the right size to fit the entire Standard Model within G_0. In this section we take $G_0 = SU(5)$ to be a global symmetry

	$SU(5)_1$	SU_3	SU_2	$U(1)_Y$	$SU(5)_2$	$U(1)_{B-L}$	$U(1)_{\rm PQ}$
$\overline{\mathcal{M}}^{(3)}$	$\mathbf{5}$	$\overline{\mathbf{3}}$		$1/3$		0	0
$\overline{\mathcal{M}}^{(2)}$	$\mathbf{5}$		$\mathbf{2}$	$-1/2$		0	0
$\mathcal{M}^{(3)}$		$\mathbf{3}$		$-1/3$	$\overline{\mathbf{5}}$	0	0
$\mathcal{M}^{(2)}$			$\mathbf{2}$	$1/2$	$\overline{\mathbf{5}}$	0	0
Q_L		$\mathbf{3}$	$\mathbf{2}$	$1/6$		$+1/3$	0
$\bar{u}_R$		$\overline{\mathbf{3}}$		$-2/3$		$-1/3$	0
$\bar{d}_R$		$\overline{\mathbf{3}}$		$1/3$		$-1/3$	0
L			$\mathbf{2}$	$-1/2$		$+1$	0
$\bar{e}_R$				$+1$		-1	0
$\bar{\nu}_R$				0		-1	0
$\mathcal{B}_1$, $\overline{\mathcal{B}}_2$				0		$5q$	$\pm 1 + \alpha$
$\overline{\mathcal{B}}_1$, $\mathcal{B}_2$				0		$-5q$	$\pm 1 - \alpha$

Table 5.2: Transformation representations of the superfields for the $U(1)_X = U(1)_{B-L}$ model.

with a gauged $SU(3)_c \times SU(2)_L \times U(1)_Y$ subgroup, and we identify $U(1)_X$ as the $B-L$ symmetry of the Standard Model. The mesons $\mathcal{M}(\mathbf{5})$ and $\overline{\mathcal{M}}(\overline{\mathbf{5}})$ decompose into irreducible representations of $SU(3) \times SU(2) \times U(1)$:

$$\mathcal{M}(\mathbf{5}) \longrightarrow \mathcal{M}^{(3)}(\mathbf{3},\mathbf{1})_{-\frac{1}{3}} \oplus \mathcal{M}^{(2)}(\mathbf{1},\mathbf{2})_{\frac{1}{2}} \tag{5.2.25}$$

$$\overline{\mathcal{M}}(\overline{\mathbf{5}}) \longrightarrow \overline{\mathcal{M}}^{(3)}(\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \oplus \overline{\mathcal{M}}^{(2)}(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}. \tag{5.2.26}$$

Table 5.2 indicates the representations of the composites under the SM, plus three generations of MSSM matter and three right-handed neutrinos necessary to cancel the $U(1)_{B-L}$ gauge anomaly.

The $B-L$ charges of the baryons $\mathcal{B}_i$ and $\overline{\mathcal{B}}_i$ are left in terms of a constant $q \neq 0$ which parameterizes their size relative to the canonical charges of the MSSM matter. While generic values of q are phenomenologically viable, certain choices would permit low-dimensional $U(1)_{\rm PQ}$-violating operators and spoil the axion quality. The problematic q can be identified by considering all of the low-dimensional $SU(5)_{\rm SM}$ singlet operators with nonzero $B-L$ charge:

$$(\bar{\nu}_R)_{-1}, \qquad (\bar{\nu}_R^n)_{-n}, \qquad (L\overline{\mathcal{M}}^{(2)})_{+1}, \qquad (\bar{d}_R\mathcal{M}^{(3)})_{-1/3}, \qquad (\overline{\mathcal{M}}^{(3)}Q_L L)_{+1/3}, \tag{5.2.27}$$

where the subscripts indicate the $B-L$ charge of each operator. Since none of these carry PQ charge, the superpotential operator constructed by multiplying any of them by a baryon superfield would violate $U(1)_{\rm PQ}$ unacceptably. To avoid this issue, we restrict ourselves to the cases where $q \neq \pm\frac{n}{5}$, for $n = 0, 1, 2, 3, 4$, and also $q \neq \pm\frac{1}{3}$.

Composite Higgs Doublets

The identification of $X = B-L$ has positive implications for the superpotential, notably by forbidding many of the operators that would mediate highly constrained B and/or L violation such as proton decay [93]. The allowed low energy effective superpotential has the form:

$$W = \mu\overline{\mathcal{M}}^{(2)}\mathcal{M}^{(2)} + \mu'\overline{\mathcal{M}}^{(3)}\mathcal{M}^{(3)} + y_u Q_L\mathcal{M}^{(2)}\bar{u}_R + y_d Q_L\overline{\mathcal{M}}^{(2)}\bar{d}_R + y_e L\overline{\mathcal{M}}^{(2)}\bar{e}_R + y_\nu L\mathcal{M}^{(2)}\bar{\nu}_R, \tag{5.2.28}$$

containing mass terms for the doublet and triplet mesons, and Yukawa interactions for the doublets with the MSSM matter.

The mesons $\overline{\mathcal{M}}^{(2)}$ and $\mathcal{M}^{(2)}$ have the same gauge representations as the MSSM Higgs superfields H_d and H_u. We take the economical route of interpreting the lightest $\overline{\mathcal{M}}^{(2)} + \mathcal{M}^{(2)}$ pair of the five flavors of $SU(2)_L$ doublet mesons as composite MSSM Higgs superfields, which potentially offers insight into the μ problem of the MSSM. The terms in Eq. (5.2.28) descend from non-renormalizable composite operators in the UV theory. In the case of the μ terms, these operators are dimension-4 and violate the $U(1)_R$ symmetry. If generated by quantum gravitational residuals, the natural mass scale for μ and μ' would thus be:

$$W_g \sim \frac{(\overline{Q}_2\overline{Q}_1)(Q_1Q_2)}{M_{\mathrm{P}}} \longrightarrow \frac{\Lambda_L\Lambda_R}{M_{\mathrm{P}}}\left(\overline{\mathcal{M}}^{(2)}\mathcal{M}^{(2)} + \overline{\mathcal{M}}^{(3)}\mathcal{M}^{(3)}\right) \longrightarrow \mu, \mu' \sim \frac{\Lambda_L\Lambda_R}{M_{\mathrm{P}}}. \tag{5.2.29}$$

This is $\mu \sim \mathcal{O}(\mathrm{TeV})$ for our benchmark choice of $\Lambda_L \approx \Lambda_R \approx 10^{11}$ GeV.

The Yukawa interactions of Eq. (5.2.28) similarly correspond to dimension five operators in the UV. Realizing the large couplings necessary for the heavy quarks requires that they be generated at a lower scale $M_F \ll M_{\mathrm{P}}$:

$$W = y'_u \frac{Q_L(Q_1Q_2)\bar{u}_R}{M_F} + y'_d \frac{Q_L(\overline{Q}_2\overline{Q}_1)\bar{d}_R}{M_F} + y'_e \frac{L(\overline{Q}_2\overline{Q}_1)\bar{e}_R}{M_F}, \tag{5.2.30}$$

where $y_t \sim 1$ requires $M_F \sim \Lambda_R$ (and y_b requires Λ_L is not much larger). Unlike the dynamics generating the μ terms, the Yukawa interactions are compatible with the $U(1)_R$ symmetry, which allows for the disparate scales to remain technically natural.

The presence of the four additional $\mathcal{M}^{(2)}$ and $\overline{\mathcal{M}}^{(2)}$ in Eq. (5.2.30) poses a potential phenomenological problem. In the absence of any additional structure, the $y'_{u,d,e}$ couplings of the matter fields with the heavier $SU(2)_L$ doublets will generally introduce flavor-changing neutral currents (FCNC). A number of potential solutions exist in the literature. For example, by imposing minimal flavor violation [94] on Eq. (5.2.30), the $\mathcal{M}^{(2)}$ and $\overline{\mathcal{M}}^{(2)}$ can have masses as small as a few TeV. Or, as we discuss in Section 5.3, a discrete symmetry can be imposed (even if broken at M_{P}) to forbid the $y'_{u,d,e}$ couplings for all of the mesons except for H_u and H_d.

Color-Triplet Mesons

As illustrated in Eq. (5.2.29), we expect that gravitational effects induce electroweak scale $\mathcal{O}(\frac{\Lambda_L\Lambda_R}{M_{\mathrm{P}}})$ supersymmetric masses for each of the five pairs of $\overline{\mathcal{M}}^{(3)}\mathcal{M}^{(3)}$ color triplets. Generically, color triplets with weak scale masses are very tightly constrained, especially because the interactions

$$W_{\mathrm{bad}} \sim Q_L\overline{\mathcal{M}}^{(3)}L + \bar{u}_R\mathcal{M}^{(3)}\bar{e}_R + \bar{d}_R\mathcal{M}^{(3)} + \overline{\mathcal{M}}^{(3)}\bar{u}_R\overline{\mathcal{M}}^{(3)} + \ldots, \tag{5.2.31}$$

if present, would mediate fast proton decay. Fortunately, every term in Eq. (5.2.31) is forbidden upon gauging $U(1)_X = U(1)_{B-L}$. Thus, $\mathcal{M}^{(3)}$ and $\overline{\mathcal{M}}^{(3)}$ are distinct from the Higgs color triplets which typically appear in $SU(5)$ grand unified theories. In Section 5.3 we explore the possibility that they could (along with the extra $SU(2)_L$ doublets) serve as messengers for gauge-mediated supersymmetry breaking.

	$SU(5)_1$	SU_3	SU_2	$U(1)_Y$	$SU(5)_2$	$U(1)_X$	$U(1)_{\rm PQ}$
$\overline{\mathcal{M}}^{(3)}$	$\mathbf{5}$	$\overline{\mathbf{3}}$		$1/3$		0	0
$\overline{\mathcal{M}}^{(2)}$	$\mathbf{5}$		$\mathbf{2}$	$-1/2$		0	0
$\mathcal{M}^{(3)}$		$\mathbf{3}$		$-1/3$	$\overline{\mathbf{5}}$	0	0
$\mathcal{M}^{(2)}$			$\mathbf{2}$	$1/2$	$\overline{\mathbf{5}}$	0	0
$\mathcal{B}_1$, $\overline{\mathcal{B}}_2$				0		5	$\pm 1+\alpha$
$\overline{\mathcal{B}}_1$, $\mathcal{B}_2$				0		-5	$\pm 1-\alpha$
Q_L		$\mathbf{3}$	$\mathbf{2}$	$1/6$		$+q$	0
$\bar{u}_R$		$\overline{\mathbf{3}}$		$-2/3$		$+q$	0
$\bar{d}_R$		$\overline{\mathbf{3}}$		$1/3$		$-3q$	0
L			$\mathbf{2}$	$-1/2$		$-3q$	0
$\bar{e}_R$				$+1$		$+q$	0
$\bar{\nu}_R$				0		$5q$	0
H_u			$\mathbf{2}$	$1/2$		$-2q$	0
H_d			$\mathbf{2}$	$-1/2$		$2q$	0

Table 5.3: Charges of the matter fundamental superfields and Higgs doublets and composite baryons and mesons in the "5/-3/1" $U(1)_X$ model.

5.2.3 Alternatives to $B-L$

In addition to $B-L$, there are a number of other acceptable anomaly-free $U(1)_X$ charge assignments for the Standard Model matter. While none are as attractive as $B-L$, in this section we sketch three alternatives: a "5/-3/1" pattern of $U(1)_X$ charges within each generation; every matter superfield neutral under $U(1)_X$; and a $L_i - L_j$ model.

5/-3/1 Model

An alternative charge assignment is shown in Table 5.3: Q_L, $\bar{u}_R$ and $\bar{e}_R$ fields have $U(1)_X$ charge q; L and $\bar{d}_R$ have charge $-3q$; and the $\bar{\nu}_R$ has charge $5q$ to cancel the $U(1)_X^3$. anomaly. Forbidding all $U(1)_{\rm PQ}$-violating operators of dimension less than 10 requires:

$$q \neq \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{5}{2}, \pm\frac{5}{3}, \tag{5.2.32}$$

but otherwise q is a free parameter describing a family of models. With this charge assignment the undesirable baryon and lepton number violating operators LH_u, $LL\bar{e}_R$, $QL\bar{d}_R$ and $\bar{u}_R\bar{d}_R\bar{d}_R$ are all forbidden, and proton decay occurs via the dimension 5 operator $W \sim \bar{u}_R\bar{u}_R\bar{d}_R\bar{e}_R/M_{\rm P}$.

Unlike in the $B-L$ model, $U(1)_X$ forbids the mesons $\mathcal{M}^{(2)}$ and $\overline{\mathcal{M}}^{(2)}$ from having Yukawa interactions with MSSM matter unless $q=0$. Thus, additional fundamental Higgs doublets $H_u + H_d$ with $U(1)_X$ charges $\pm 2q$ must be added to generate quark and lepton masses,

$$W_H = \mu H_u H_d + y_u Q_L H_u \bar{u}_R + y_d Q_L H_d \bar{d}_R + y_e L H_d \bar{e}_R + y_\nu L H_u \bar{\nu}_R. \tag{5.2.33}$$

As in the MSSM with fundamental Higgs doublets, there is no *a priori* reason for μ to be at the weak scale.

Renormalizable couplings between the mesons $\mathcal{M}$ and $\overline{\mathcal{M}}$ and the MSSM fields are mediated exclusively by gauge interactions. Direct couplings in the superpotential are suppressed, beginning with the dimension-7 operators $(\mathcal{M}\overline{\mathcal{M}})H_uH_d$. Direct couplings which would allow the mesons to decay entirely into the Standard Model depend sensitively on q, with the operators permitting prompt decay also typically violating $U(1)_{\rm PQ}$ and forbidden by Eq. (5.2.32). As consequence, the lightest mesons tend to have long lifetimes, and for some values of q can be absolutely stable and bounded by the strong constraints on colored or charged cosmological relic particles.

q $= 0$: Neutral MSSM

In the limit $q \to 0$, the MSSM decouples from $U(1)_X$. This assignment allows for Yukawa interactions between the mesons and MSSM matter, permitting $\mathcal{M}^{(2)}$ and $\overline{\mathcal{M}}^{(2)}$ to play the role of the MSSM Higgs doublets, with $\mathcal{O}(\Lambda_L\Lambda_R/M_{\rm P})$ supersymmetric masses as in Eq. (5.2.29). However, $U(1)_X$ no longer forbids the problematic operators of Eq. (5.2.31) or

$$W'_{\rm bad} \sim L\mathcal{M}^{(2)} + LL\bar{e}_R + QL\bar{d}_R + \bar{u}_R\bar{d}_R\bar{d}_R. \tag{5.2.34}$$

Among the potentially disastrous consequences of $W'_{\rm bad}$ is a short proton lifetime. This problem is averted in the MSSM by imposing a $\mathbb{Z}_2$ R parity, which ensures that the superpotential respects the $B-L$ global symmetry. Upon imposing R parity or some other discrete symmetry on the $q = 0$ model, the superpotential comes to resemble that of the $B-L$ axion model in all respects except one: if $q = 0$ the right-handed neutrino is a singlet under the gauge symmetries, at which point it can be safely omitted.

$L_i - L_j$ Models

The Standard Model also admits anomaly-free $U(1)$ symmetries for which charges not are uniform across all three generations. The combinations of $L_\mu - L_\tau$ and $L_e - L_\tau$ are among the phenomenologically interesting alternatives. Models of this type are typically consistent with a composite H_u and H_d, but as in the MSSM, an R parity must be imposed on such models to ensure that all of the B and L violating operators of Eq. (5.2.34) are forbidden.

5.3 Gauge-Mediated Supersymmetry Breaking

Beyond the usual MSSM superfields, there are relatively few additional light degrees of freedom:

- The four baryons $\overline{\mathcal{B}}_{1,2}$ and $\mathcal{B}_{1,2}$ contain at most two light fields in the $\langle\mathcal{B}_i\rangle \neq 0$, $\langle\overline{\mathcal{B}}_i\rangle \neq 0$ vacuum. There is a chiral multiplet containing the composite axion.
- For $U(1)_X$ gauge coupling $g_X \ll 1$, there is a $U(1)_X$ vector supermultiplet with a mass $m_X \sim g_X f_X$, where f_X is typically $\sim f_a$.
- The mesons $\mathcal{M}$ and $\overline{\mathcal{M}}$ have $\mathcal{O}(\Lambda_L\Lambda_R/M_{\rm P})$ vectorlike masses. In the $B-L$ model and its variants, the lightest such $SU(2)_L$ doublets are identified as the MSSM H_u and H_d leaving four heavier $\mathcal{M}^{(2)} + \overline{\mathcal{M}}^{(2)}$ pairs, and five color triplets $\mathcal{M}^{(3)} + \overline{\mathcal{M}}^{(3)}$.

In this section we explore how these mesons may be utilized as messengers of supersymmetry breaking.

We parameterize the supersymmetry-breaking in a secluded sector as a set of one or more chiral superfields X_i acquiring F-term expectation values,

$$\langle X\rangle = \mathcal{X} + \theta^2\mathcal{F}_X, \tag{5.3.1}$$

with $\mathcal{F}_X \neq 0$. Introducing superpotential terms of the form $W \sim X\overline{\mathcal{M}}^{(3,2)}\mathcal{M}^{(3,2)}$ communicates supersymmetry breaking to the MSSM [95, 96]. In the UV theory this superpotential originates from dimension-5 operators $(\overline{Q}_2\overline{Q}_1)X(Q_1Q_2)/M_S^2$, reducing in the IR to

$$W_s = \lambda_3'^{ij}\left(\frac{\Lambda_L\Lambda_R}{M_S^2}\right)X\overline{\mathcal{M}}_i^{(3)}\mathcal{M}_j^{(3)} + \lambda_2'^{ij}\left(\frac{\Lambda_L\Lambda_R}{M_S^2}\right)X\overline{\mathcal{M}}_i^{(2)}\mathcal{M}_j^{(2)}, \tag{5.3.2}$$

where the indices $i, j = 1\dots5$, for some scale $M_S \gtrsim \sqrt{\Lambda_L\Lambda_R}$ which we take to be small compared to $M_{\rm P}$. It is convenient to absorb the factors of $\Lambda_L\Lambda_R/M_S^2$ into the definitions of $\lambda_{2,3}$:

$$\lambda_{2,3}^{ij} = \frac{\Lambda_L\Lambda_R}{M_S^2}\lambda_{2,3}'^{ij}. \tag{5.3.3}$$

As with the Yukawa couplings of Eq. (5.2.30), the superpotential W_s respects a global $U(1)_R$ symmetry under which the mesons $\mathcal{M}$ and $\overline{\mathcal{M}}$ are neutral, and X has charge $+2$.

As discussed in Section 5.2.2, Yukawa-like couplings between the matter fields and the four heavy $\mathcal{M}^{(2)} + \overline{\mathcal{M}}^{(2)}$ may introduce unacceptable flavor-changing neutral currents. A standard solution is to impose a "messenger parity" on the model, under which the Higgs $H_{u,d}$ are even, and the messengers $\mathcal{M}^{(2,3)}$ and $\overline{\mathcal{M}}^{(2,3)}$ are odd. Thus, the direct couplings between messenger $SU(2)_L$ doublets and the matter fields are forbidden, and the problematic flavor-changing neutral currents are avoided.[2] Imposing this $\mathbb{Z}_2$ symmetry reduces Eq. (5.3.2) to:

$$W_s = \lambda_3^{1,1}X\overline{\mathcal{M}}_1^{(3)}\mathcal{M}_1^{(3)} + \lambda_2^{1,1}XH_dH_u + \sum_{i=2\dots5}\sum_{j=2\dots5}\left(\lambda_3^{ij}X\overline{\mathcal{M}}_i^{(3)}\mathcal{M}_j^{(3)} + \lambda_2^{ij}X\overline{\mathcal{M}}_i^{(2)}\mathcal{M}_j^{(2)}\right), \tag{5.3.4}$$

where, if the messenger parity is derived from the global symmetries of the quarks Q_2 and $\overline{Q}_2$, we take the $SU(3)_c$ triplets $\overline{\mathcal{M}}_1^{(3)}$ and $\mathcal{M}_1^{(3)}$ to be even under the $\mathbb{Z}_2$ messenger parity.

Since the mesons come in complete $SU(5)$ multiplets, gauge unification at a scale $M_{\rm GUT}$ is preserved due to the fact that $\mathcal{M}^{(3)} + \mathcal{M}^{(2)}$ and $\overline{\mathcal{M}}^{(3)} + \overline{\mathcal{M}}^{(2)}$ form complete $SU(5)_{\rm SM}$ multiplets. Following [97], the gauge coupling strength $\alpha_{\rm GUT}$ at the unification scale $M_{\rm GUT}$ is modified by

$$\delta\alpha_{\rm GUT}^{-1} = -\frac{N_f}{2\pi}\ln\frac{M_{\rm GUT}}{\mathcal{X}} \tag{5.3.5}$$

where $N_f = N_c = 5$. Requiring that $SU(5)_{\rm SM}$ remains perturbative up to the unification scale imposes a lower bound on $\mathcal{X}$:

$$\mathcal{X} \gtrsim 10^{-13} \times M_{\rm GUT} \approx 2\ \text{TeV}. \tag{5.3.6}$$

In addition to Eq. (5.3.2), the meson messengers also acquire $U(1)_R$ violating mass terms from the Planck scale effects, $\mu_{2,3} \sim \Lambda_L\Lambda_R/M_{\rm P}$, leading to a scalar mass matrix:

$$\begin{pmatrix} \mathcal{M}^\dagger_{(2,3)} & \overline{\mathcal{M}}_{(2,3)} \end{pmatrix}\begin{pmatrix} (\lambda_{2,3}\mathcal{X}+\mu_{2,3})^\dagger(\lambda_{2,3}\mathcal{X}+\mu_{2,3}) & (\lambda_{2,3}\mathcal{F}_X)^\dagger \\ \lambda_{2,3}\mathcal{F}_X & (\lambda_{2,3}\mathcal{X}+\mu_{2,3})(\lambda_{2,3}\mathcal{X}+\mu_{2,3})^\dagger \end{pmatrix}\begin{pmatrix} \mathcal{M}_{(2,3)} \\ \overline{\mathcal{M}}^\dagger_{(2,3)} \end{pmatrix}. \tag{5.3.7}$$

[2]The messenger parity is a discrete subgroup of the $SU(5)_1 \times SU(5)_2$ flavor symmetry, and can be derived from the breaking pattern $SU(5)_{1,2} \to SU(4)_{1,2} \times U(1)$ with $\mathbb{Z}_2 \subset \mathbb{Z}_4 \subset U(1)$, where $H_{u,d}$ and the corresponding $SU(3)_c$ triplets are invariant under the action of $\mathbb{Z}_4$.

Performing $SU(4)_{1,2} \times U(1)_{1,2}$ rotations on the fields $\overline{\mathcal{M}}^{(2)}$ and $\mathcal{M}^{(2)}$, the matrices $(\lambda_2 \mathcal{X} + \mu_2)$ and $(\lambda_2 \mathcal{F}_X)$ can be simultaneously diagonalized and made real:

$$M_i = (\lambda_2 \mathcal{X} + \mu_2)_{ii}, \qquad\qquad F_i = (\lambda_2 \mathcal{F}_X)_{ii}, \tag{5.3.8}$$

with eigenvalues $M_i^2 \pm F_i$. This basis also diagonalizes the scalar mass matrix of $\overline{\mathcal{M}}^{(3)}$ and $\mathcal{M}^{(3)}$ in the special case $\lambda_2 = \lambda_3$ and $\mu_2 = \mu_3$ (but not in general). Positivity of the (squared) messenger masses imposes a constraint on the F-term VEV of the superfield X:

$$\mathcal{F}_X < \frac{\mu_2^2}{\lambda_2} + 2\mu_2 \mathcal{X} + \lambda_2 \mathcal{X}^2 \tag{5.3.9}$$

for each pair of λ_2^{ii} and μ_2^{ii} in the diagonal basis. Note that due to the compositeness of the messengers, the couplings $\lambda_{2,3}$ are suppressed by a factor $\Lambda_L \Lambda_R / M_S^2$ which may be much smaller than unity.

To produce the correct electroweak scale, the M^2 and F terms for H_u and H_d must coincide. Taking $\lambda_2^{1,1} \sim \frac{\Lambda_L \Lambda_R}{M_S^2}$ and $\mu_2^{1,1} \sim \frac{\Lambda_L \Lambda_R}{M_{\rm P}}$, this condition implies a relationship between the scales M_S, $\mathcal{X}$ and $\mathcal{F}_X$:

$$\mathcal{F}_X \sim \Lambda_L \Lambda_R \left(\frac{\mathcal{X}}{M_S} + \frac{M_S}{M_{\rm P}} \right)^2 . \tag{5.3.10}$$

Taking the simplifying case $\sqrt{\Lambda_L \Lambda_R} \sim f_a \sim 10^{11}\,\text{GeV}$ and $M_S \gtrsim f_a$ in the limit $\mathcal{X} < 10^5\,\text{GeV}$, Eq. (5.3.10) reduces to the condition $\sqrt{\mathcal{F}_X} \sim \frac{f_a M_S}{M_{\rm P}}$. An investigation of the extensions to the composite axion model satisfying this constraint would be an interesting opportunity for future work.

Chapter 6

Conclusion

In Chapter 5 we explored a composite axion model in which an accidental Peccei–Quinn symmetry naturally emerges as a solution to the strong CP problem. Gravitational perturbations to the axion scalar potential are shown to be sufficiently suppressed in the $N_c = 5$ model to permit an axion decay constant of $f_a \lesssim 3 \times 10^{11}\,\text{GeV}$, even under the pessimistic assumptions that supersymmetry breaking induces the most dangerous $U(1)_{\text{PQ}}$-violating A-term potential, and that the higher-dimensional operators representing quantum gravitational effects are parameterized by $\mathcal{O}(1)$ coupling constants.

In addition to providing a satisfactory solution to the axion quality problem, this composite framework is easily extended to any model of axion-like particles (ALPs) with masses much smaller than the scale of spontaneous symmetry breaking.

The general $SU(N)_L \times SU(N)_R \times U(1)_X$ axion model allows the Standard Model matter fields to carry nearly any anomaly-free $U(1)_X$ charge assignment without negatively affecting the axion quality. In particular, attractive features emerge when $U(1)_X$ is associated with gauging the Standard Model $B - L$ global symmetry. The leading terms in the superpotential are those of the MSSM, with none of the problematic B or L violating operators that would otherwise need to be forbidden by invoking a discrete "matter parity". Additionally, if the Higgs H_u and H_d are taken to be the lightest of the $SU(2)_L$ charged mesons from $SU(5)_L$ and $SU(5)_R$ confinement, the dimension-4 gravitationally-induced operator naturally generates an electroweak scale μ term for $f_a \sim 10^{11}$ GeV. Other choices of $U(1)_X$ charge assignments share this feature, that the $SU(2)_L$ charged mesons have the same quantum numbers as H_u and H_d, and could therefore produce a composite Higgs with a TeV scale μ term.

The low energy phenomenology largely resembles the MSSM plus a chiral superfield containing the standard QCD axion, axino, and a saxion. More unique are the presence of meson fields in vectorlike color triplet and electroweak doublet representations. In theories in which the lightest weak doublet pair are identified as the MSSM Higgs superfields, they will have $\sim$ TeV masses. Their detailed phenomenology depends on the $U(1)_X$ charge assignments and some choices of (perhaps slightly broken) global symmetries, and their presence indicates that the Large Hadron Collider could potentially uncover clues to higher scale physics. Alternatively, some of these fields could play the role of messengers, leading to a picture in which supersymmetry-breaking is mediated by gauge interactions.

Among the many opportunities for future work, some promising directions include developing the supersymmetry breaking sector, explaining the pattern of Yukawa couplings in the MSSM, or exploring the cosmological implications of the composite model in the early universe.

Appendix A

Properties of the QCD Axion

These appendices originally appeared in [2], by the author and Tim M.P. Tait.

A.1 Axion Quality

To leading order in a, the QCD-induced axion potential $V(a)$ has the form

$$V(a) = V_0 - \frac{1}{2} m_a^2 \left(a + f_a \bar{\theta}\right), \tag{A.1.1}$$

which is minimized when $\langle\theta\rangle \equiv (a/f_a + \bar{\theta})$ is equal to zero. It is convenient to define the shifted field $\alpha \equiv a + f_a \bar{\theta}$, so that $\langle\theta\rangle = \langle\alpha\rangle/f_a$. Explicit $U(1)_{\mathrm{PQ}}$ violation elsewhere in the theory adds corrections to $V(a)$,

$$\delta V(a) = Q f_a^4 \cos\left(\kappa\left(\frac{a}{f_a} + \bar{\theta}\right) + \theta_0\right), \tag{A.1.2}$$

which for small values of $\langle\theta\rangle$ is approximately

$$\delta V(a) = Q f_a^4 \left(1 - \frac{1}{2}\left(\frac{\kappa\alpha}{f_a}\right)^2\right) \cos\theta_0 - Q f_a^4 \left(\frac{\kappa\alpha}{f_a}\right) \sin\theta_0. \tag{A.1.3}$$

As θ_0 is determined by the precise manner in which $U(1)_{\mathrm{PQ}}$ is broken, we do not assume that it is smaller than $\mathcal{O}(1)$. Combining Eqs. (A.1.1) and (A.1.2), $V(a)$ becomes

$$V(\alpha) = \left(V_0 + Q f_a^4 \cos\theta_0\right) - \left(Q f_a^3 \kappa \sin\theta_0\right) \alpha - \frac{1}{2}\left(m_a^2 + Q f_a^2 \kappa^2 \cos\theta_0\right) \alpha^2, \tag{A.1.4}$$

so that the expectation value $\langle\alpha\rangle$ shifts away from zero:

$$\langle\alpha\rangle = -\frac{Q f_a^3 \kappa \sin\theta_0}{m_a^2 + Q f_a^2 \kappa^2 \cos\theta_0}. \tag{A.1.5}$$

Experimental measurements of $\langle\theta\rangle$ set an upper bound $\langle\alpha\rangle < f_a\,|\theta_{\mathrm{max}}|$. Assuming $|\theta_{\mathrm{max}}|\,\kappa \ll \sin\theta_0$, the corresponding bound on Q is

$$Q < \frac{m_a^2}{f_a^2} \frac{|\theta_{\mathrm{max}}|}{\kappa\,|\sin\theta_0|}. \tag{A.1.6}$$

Using $m_a^2 = m_\pi^2 f_\pi^2 / f_a^2$ and assuming $\kappa \sin\theta_0 = \mathcal{O}(1)$, Eq. (A.1.6) implies

$$Q \lesssim 10^{-62} \left(\frac{10^{12}\ \mathrm{GeV}}{f_a}\right)^4. \tag{A.1.7}$$

A.2 Axion Assignment in a General Vacuum

Suppose there exist many fields Φ_i, each with a Peccei-Quinn charge q_i. Let us define the charge-normalized expectation value

$$v_i \equiv q_i \sqrt{2} \langle \Phi_i \rangle \tag{A.2.1}$$

for each field Φ_i. If there are n fields with nonzero expectation values, then let us define $n-1$ fields η_i and the axion a, with the following assignment:

$$\begin{aligned}
\Phi_1 &= \left(\frac{\phi_1}{\sqrt{2}} + \langle \Phi_1 \rangle \right) \exp\left(\frac{iq_1}{f_a} (a + \alpha_1 \eta_1) \right) && \text{(A.2.2)} \\
\Phi_2 &= \left(\frac{\phi_2}{\sqrt{2}} + \langle \Phi_2 \rangle \right) \exp\left(\frac{iq_2}{f_a} (a + \beta_1 \eta_1 + \beta_2 \eta_2) \right) && \text{(A.2.3)} \\
\Phi_3 &= \left(\frac{\phi_3}{\sqrt{2}} + \langle \Phi_3 \rangle \right) \exp\left(\frac{iq_3}{f_a} (a + \gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3) \right) && \text{(A.2.4)} \\
&\vdots \\
\Phi_{n-1} &= \left(\frac{\phi_{n-1}}{\sqrt{2}} + \langle \Phi_{n-1} \rangle \right) \exp\left(\frac{iq_{n-1}}{f_a} (a + \alpha_1^{(n-1)} \eta_1 + \ldots + \alpha_{n-1}^{(n-1)} \eta_{n-1}) \right) && \text{(A.2.5)} \\
\Phi_n &= \left(\frac{\phi_n}{\sqrt{2}} + \langle \Phi_n \rangle \right) \exp\left(\frac{iq_n}{f_a} (a + \alpha_1^{(n)} \eta_1 + \ldots + \alpha_{n-1}^{(n)} \eta_{n-1}) \right) && \text{(A.2.6)}
\end{aligned}$$

In the sequence above, the first appearance of each field η_i is in the phase of Φ_i. The field Φ_n does not introduce any new η_i fields.

Let us define the following $(n-1)$ constants:

$$\begin{aligned}
x_1 &= \beta_1 = \gamma_1 = \delta_1 = \ldots = \alpha_1^{(n-1)} = \alpha_1^{(n)} && \text{(A.2.7)} \\
x_2 &= \gamma_2 = \delta_2 = \ldots = \alpha_2^{(n-1)} = \alpha_2^{(n)} && \text{(A.2.8)} \\
x_3 &= \delta_3 = \ldots = \alpha_3^{(n-1)} = \alpha_3^{(n)} && \text{(A.2.9)} \\
&\vdots \\
x_{n-2} &= \alpha_{n-2}^{(n-1)} = \alpha_{n-2}^{(n)} && \text{(A.2.10)} \\
x_{n-1} &= \alpha_{n-1}^{(n)}. && \text{(A.2.11)}
\end{aligned}$$

These equalities follow from the vanishing of the kinetic cross terms, which also give the following relationships between the x_i and $\{\alpha_1, \beta_2, \gamma_3, \ldots, \alpha_{n-1}^{(n-1)}\}$:

$$\begin{aligned}
0 &= 1 + x_1 \alpha_1 && \text{(A.2.12)} \\
0 &= 1 + x_1^2 + x_2 \beta_2 && \text{(A.2.13)} \\
0 &= 1 + x_1^2 + x_2^2 + x_3 \gamma_3 && \text{(A.2.14)} \\
&\vdots \\
0 &= 1 + x_1^2 + \ldots + x_{n-2}^2 + x_{n-1} \alpha_{n-1}^{(n-1)}. && \text{(A.2.15)}
\end{aligned}$$

Finally, we require that the kinetic terms $(\partial_\mu \eta_i)^2$ and $(\partial_\mu a)^2$ are canonically normalized. This leads to the remaining n constraints:

$$\frac{f_a^2}{v_1^2} = 1 + \alpha_1^2 \quad \text{(A.2.16)}$$

$$\frac{f_a^2}{v_2^2} = 1 + x_1^2 + \beta_2^2 \quad \text{(A.2.17)}$$

$$\frac{f_a^2}{v_3^2} = 1 + x_1^2 + x_2^2 + \gamma_3^2 \quad \text{(A.2.18)}$$

$$\vdots$$

$$\frac{f_a^2}{v_{n-1}^2} = 1 + x_1^2 + x_2^2 + \ldots + x_{n-2}^2 + (\alpha_{n-1}^{(n-1)})^2 \quad \text{(A.2.19)}$$

$$\frac{f_a^2}{v_n^2} = 1 + x_1^2 + x_2^2 + \ldots + x_{n-2}^2 + x_{n-1}^2. \quad \text{(A.2.20)}$$

These systems of equations have the solutions:

$$\alpha_1^2 = \frac{f_a^2 - v_1^2}{v_1^2} \qquad x_1^2 = \frac{v_1^2}{f_a^2 - v_1^2} \quad \text{(A.2.21)}$$

$$\beta_2^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2)}{v_2^2(f_a^2 - v_1^2)} \qquad x_2^2 = \frac{v_2^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2)(f_a^2 - v_1^2)} \quad \text{(A.2.22)}$$

$$\gamma_3^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2 - v_3^2)}{v_3^2(f_a^2 - v_1^2 - v_2^2)} \qquad x_3^2 = \frac{v_3^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2 - v_3^2)(f_a^2 - v_1^2 - v_2^2)}, \quad \text{(A.2.23)}$$

and so on. The general solution is

$$(\alpha_i^{(i)})^2 = \frac{f_a^2(f_a^2 - v_1^2 - v_2^2 - \ldots - v_i^2)}{v_i^2(f_a^2 - v_1^2 - v_2^2 - \ldots - v_{i-1}^2)} \quad \text{(A.2.24)}$$

$$x_i^2 = \frac{v_i^2 f_a^2}{(f_a^2 - v_1^2 - v_2^2 - \ldots - v_i^2)(f_a^2 - v_1^2 - v_2^2 - \ldots - v_{i-1}^2)}, \quad \text{(A.2.25)}$$

for $i = 1 \ldots (n-1)$. Each $\alpha_i^{(i)}$ and x_i must also obey

$$\alpha_i^{(i)} x_i < 0, \quad \text{(A.2.26)}$$

but the signs of $\alpha^{(i)}$ and x_i are otherwise arbitrary.

Finally, the axion decay constant is:

$$f_a^2 = v_1^2 + v_2^2 + \ldots + v_{n-1}^2 + v_n^2. \quad \text{(A.2.27)}$$

Appendix B

S-Confining Product Gauge Groups

This appendix originally appeared in [1] by the author.

B.1 Derivation of classical constraints

In this appendix we find the classical constraints between gauge singlet operators in the $A+4Q+N\overline{Q}$ model, along with the coefficients in the dynamically generated superpotential. It is useful to consider a particular non-trivial solution of the D flatness conditions.

B.1.1 D-Flat Directions

The auxiliary gluon scalar fields have interactions from the Kähler potential given by $V = \frac{1}{2}D^a D^a$, where

$$D^a = -g\left(Q_i^{\star\alpha}(T^a_{\Box})^\beta_\alpha Q^i_\beta + \overline{Q}^{\star j}_\alpha (T^a_{\overline{\Box}})^\alpha_\beta \overline{Q}^\alpha_j + A^{\star\beta\alpha}(T^a_{\substack{\Box\\\Box}})^{\delta\epsilon}_{\alpha\beta} A_{\delta\epsilon}\right). \tag{B.1.1}$$

Ground state solutions are given by $D^a D^a = 0$. Equation (B.1.1) can be simplified by replacing $T_{\overline{\Box}}$ and $T_{\substack{\Box\\\Box}}$ with $T^a_{\Box}$:

$$(T^a_{\overline{\Box}})^\alpha_\beta = -(T^a_{\Box})^\beta_\alpha, \qquad\qquad (T^a_{\substack{\Box\\\Box}})^{\delta\epsilon}_{\alpha\beta} = (T^a_{\Box})^\delta_\alpha \delta^\epsilon_\beta + \delta^\delta_\alpha (T^a_{\Box})^\epsilon_\beta. \tag{B.1.2}$$

With this substitution, we may write D^a as

$$D^a \;=\; -g\left(Q_i^{\star\alpha} Q^i_\beta - \overline{Q}^\alpha_j \overline{Q}^{\star j}_\beta + 2A^{\star\alpha\gamma}A_{\gamma\beta}\right)(T^a_{\Box})^\beta_\alpha. \tag{B.1.3}$$

The indices i and j refer to $SU(4)_L$ and $SU(N)_R$, respectively, while α, β and γ correspond to the gauge group. The generators $T^a_{\Box}$ span the set of traceless $N \times N$ matrices, so if the fields satisfy

$$Q_i^{\star\alpha} Q^i_\beta - \overline{Q}^\alpha_j \overline{Q}^{\star j}_\beta + 2A^{\star\alpha\gamma}A_{\gamma\beta} = \rho\delta^\alpha_\beta \tag{B.1.4}$$

for any constant ρ, then $D^a = 0$. It is useful to define the matrices d, $\bar{d}$, and d_A as follows:

$$d^\alpha_\beta = Q_i^{\star\alpha} Q^i_\beta, \qquad\qquad \bar{d}^\alpha_\beta = \overline{Q}^\alpha_j \overline{Q}^{\star j}_\beta, \qquad\qquad (d_A)^\alpha_\beta = A^{\star\alpha\gamma}A_{\gamma\beta}, \tag{B.1.5}$$

so that Eq. (B.1.4) can be written as

$$d^\alpha_\beta - \bar{d}^\alpha_\beta + 2(d_A)^\alpha_\beta = \rho\delta^\alpha_\beta. \tag{B.1.6}$$

Each d term defined above is invariant under the $SU(4)_L \times SU(N)_R$ flavor transformations.

By rotating the $SU(N)$ color basis, it is possible to block-diagonalize the matrix A such that the only non-zero entries are $A_{12} = -A_{21} = \sigma_1$, $A_{34} = -A_{43} = \sigma_2$, etc. For even $SU(N = 2m)$, this continues until $\sigma_m = A_{N-1,N}$. In this basis, the d_A matrix is diagonal and equal to

$$(d_A)^\alpha_\beta = \text{Diag}(|\sigma_1|^2, |\sigma_1|^2, |\sigma_2|^2, |\sigma_2|^2, \ldots, |\sigma_m|^2, |\sigma_m|^2), \tag{B.1.7}$$

with $\text{Pf}\, A = \sigma_1\sigma_2 \ldots \sigma_m$. For odd $N = 2m+1$, $\sigma_m = A_{N-2,N-1}$, and $A_{jN} = 0$ for all $j = 1 \ldots N$. The d_A matrix is again diagonal, but with $(d_A)^N_N = 0$.

$$(d_A)^\alpha_\beta = \text{Diag}(|\sigma_1|^2, |\sigma_1|^2, |\sigma_2|^2, |\sigma_2|^2, \ldots, |\sigma_m|^2, |\sigma_m|^2, 0). \tag{B.1.8}$$

The Pfaffian, $\text{Pf}\, A$, is not defined for odd-dimensional matrices.

It is not generally possible to simultaneously diagonalize d_A, d, and $\bar{d}$. This is a departure from SUSY QCD: in this case, if $\bar{d}$ is diagonal, then $d^\alpha_\beta = \bar{d}^\alpha_\beta + \rho\delta^\alpha_\beta$ must also be diagonal. Once d_A is added, this condition is relaxed.

B.1.2 Special Cases

In this section we consider the $\langle\phi\rangle \gg \Lambda$ limit along particular flat directions in which d_A, d, and $\bar{d}$ happen to be diagonal. Let us begin with the $N = 2m$ case:

$$A = \begin{pmatrix} 0 & \sigma_1 & & & & & \\ -\sigma_1 & 0 & & & & & \\ & & 0 & \sigma_2 & & & \\ & & -\sigma_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \sigma_m \\ & & & & & -\sigma_m & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} v_1 & 0 & & \\ 0 & v_2 & 0 & \\ & 0 & v_3 & 0 \\ & & 0 & v_4 \\ 0 & 0 & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} \bar{v}_1 & 0 & & & \\ 0 & \bar{v}_2 & & & \\ & 0 & \ddots & & \\ & & & \ddots & 0 \\ & & & 0 & \bar{v}_N \end{pmatrix}. \tag{B.1.9}$$

In this vacuum, the matrices d_A, d and $\bar{d}$ are:

$$\begin{aligned} d_A &= \text{Diag}\left(|\sigma_1|^2, |\sigma_1|^2, |\sigma_2|^2, |\sigma_2|^2, \ldots, |\sigma_m|^2, |\sigma_m|^2\right) && \text{(B.1.10)} \\ d &= \text{Diag}\left(|v_1|^2, |v_2|^2, |v_3|^2, |v_4|^2, 0, \ldots, 0\right) && \text{(B.1.11)} \\ \bar{d} &= \text{Diag}\left(|\bar{v}_1|^2, |\bar{v}_2|^2, |\bar{v}_3|^2, \ldots, |\bar{v}_{N-1}|^2, |\bar{v}_N|^2\right), && \text{(B.1.12)} \end{aligned}$$

subject to the constraint

$$d^\alpha_\alpha - \bar{d}^\alpha_\alpha + 2(d_A)^\alpha_\alpha = \rho. \tag{B.1.13}$$

In the classical limit, the gauge-invariant operators are

$$J=\begin{pmatrix} \bar v_1 v_1 & 0 & & \\ 0 & \bar v_2 v_2 & 0 & \\ & 0 & \bar v_3 v_3 & 0 \\ & & 0 & \bar v_4 v_4 \\ 0 & 0 & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K=\begin{pmatrix} 0 & \hat\sigma_1 & & & & & \\ -\hat\sigma_1 & 0 & & & & & \\ & & 0 & \hat\sigma_2 & & & \\ & & -\hat\sigma_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \hat\sigma_m \\ & & & & & -\hat\sigma_m & 0 \end{pmatrix}, \tag{B.1.14}$$

$$V = \begin{pmatrix} 0 & V_{12} & 0 & 0 \\ -V_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{34} \\ 0 & 0 & -V_{34} & 0 \end{pmatrix}, \quad \begin{aligned} U &= \sigma_1\sigma_2\ldots\sigma_m \\ \mathcal{W} &= v_1 v_2 v_3 v_4 \sigma_3\ldots\sigma_m \\ Z &= \bar v_1 \bar v_2 \bar v_3 \ldots \bar v_N, \end{aligned} \tag{B.1.15}$$

where we define

$$V_{12} \equiv (v_1 v_2)\sigma_2\sigma_3\ldots\sigma_m\,, \qquad V_{34} \equiv \sigma_1(v_3 v_4)\sigma_3\ldots\sigma_m\,, \qquad \hat\sigma_i \equiv \sigma_i \bar v_{2i-1}\bar v_{2i} \tag{B.1.16}$$

for $i = 1\ldots m$.

In the $N = 2m+1$ case we add a row and column to A, with $A_{\alpha,N} = A_{N,\beta} = 0$ for all α and β. The form of $\overline{Q}$ is left unchanged, but we add a nontrivial N^{th} row to Q^i_N with entries $q^i \neq 0$. With these modifications, the matrices d_A, d and $\bar d$ become

$$\begin{aligned} d_A &= \mathrm{Diag}\left(|\sigma_1|^2, |\sigma_1|^2, |\sigma_2|^2, |\sigma_2|^2, \ldots, |\sigma_m|^2, |\sigma_m|^2, 0\right) & \text{(B.1.17)} \\ d &= \mathrm{Diag}\left(|v_1|^2, |v_2|^2, |v_3|^2, |v_4|^2, 0, \ldots, 0, \sum_i |q_i|^2\right) & \text{(B.1.18)} \\ \bar d &= \mathrm{Diag}\left(|\bar v_1|^2, |\bar v_2|^2, |\bar v_3|^2, \ldots, |\bar v_{N-1}|^2, |\bar v_N|^2\right), & \text{(B.1.19)} \end{aligned}$$

and the gauge-invariant operators are

$$J=\begin{pmatrix} \bar v_1 v_1 & 0 & & \\ 0 & \bar v_2 v_2 & 0 & \\ & 0 & \bar v_3 v_3 & 0 \\ & & 0 & \bar v_4 v_4 \\ 0 & 0 & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \\ \bar v_N q_1 & \bar v_N q_2 & \bar v_N q_3 & \bar v_N q_4 \end{pmatrix}, \quad K=\begin{pmatrix} 0 & \hat\sigma_1 & & & & & & 0 \\ -\hat\sigma_1 & 0 & & & & & & \\ & & 0 & \hat\sigma_2 & & & & \\ & & -\hat\sigma_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \hat\sigma_M & 0 \\ & & & & & -\hat\sigma_M & 0 & 0 \\ 0 & & & & & 0 & 0 & 0 \end{pmatrix}, \tag{B.1.20}$$

$$\begin{aligned} X^i &= \sigma_1\sigma_2\ldots\sigma_M q_i \\ Z &= \bar v_1 \ldots \bar v_N \end{aligned}, \quad Y_i = \begin{cases} i=1: & \sigma_1 v_3 v_4 \sigma_3\ldots\sigma_M q_2 \\ i=2: & -\sigma_1 v_3 v_4 \sigma_3\ldots\sigma_M q_1 \\ i=3: & v_1 v_2 \sigma_2\sigma_3\ldots\sigma_M q_4 \\ i=4: & -v_1 v_2\sigma_2\sigma_3\ldots\sigma_M q_3 \end{cases}. \tag{B.1.21}$$

Classical constraints The dynamically generated superpotential has the form $W \sim A^{N-2}Q^4\bar{Q}^N$. For odd N, there are three ways to contract the gauge indices:

$$W_d = \frac{\alpha}{\Lambda^b}\Big(X^iY_iZ + \beta_1\epsilon_{i_1\ldots i_4}\epsilon^{j_1\ldots j_N}X^{i_1}(K_{j_1j_2}\ldots K_{j_{N-4}j_{N-3}})J^{i_2}_{j_{N-2}}J^{i_3}_{j_{N-1}}J^{i_4}_{j_N} + \beta_2\epsilon^{j_1\ldots j_N}Y_i(K_{j_1j_2}\ldots K_{j_{N-2}j_{N-1}})J^i_{j_N}\Big), \tag{B.1.22}$$

while for even N there are five terms:

$$W_d = \frac{\alpha}{\Lambda^b}\Big(U\mathcal{W}Z + \gamma_1\epsilon_{i_1\ldots i_4}V^{i_1i_2}V^{i_3i_4}Z + \gamma_2\epsilon_{j_1\ldots j_N}\epsilon_{i_1\ldots i_4}U\left(K_{j_1j_2}\ldots K_{j_{N-5},j_{N-4}}\right)\left(J^{i_1}_{j_{N-3}}\ldots J^{i_4}_{j_N}\right) + \gamma_3\epsilon_{j_1\ldots j_N}\epsilon_{i_1\ldots i_4}V^{i_1i_2}\left(K_{j_1j_2}\ldots K_{j_{N-3},j_{N-2}}\right)\left(J^{i_3}_{j_{N-1}}J^{i_4}_{j_N}\right) + \gamma_4\mathcal{W}\mathrm{Pf}\,K\Big). \tag{B.1.23}$$

The relationships between the coefficients are determined by matching the equations of motion from W_d to the classical constraints on the operators.

In the classical limit for even N, it follows from Eq. (B.1.15) that

$$\mathrm{Pf}\,V = V_{12}V_{34} = (\sigma_1\sigma_2v_1v_2v_3v_4)(\sigma_3\ldots\sigma_m)^2 = U\cdot Z, \tag{B.1.24}$$

for example, so that

$$\gamma_1 = -\frac{1}{2^22!}. \tag{B.1.25}$$

Applying this technique to other products of gauge invariant operators, we find

$$\gamma_2 = -\frac{1}{2^{m-2}(m-2)!4!}, \qquad \gamma_3 = +\frac{1}{4\cdot 2^{m-1}(m-1)!}, \qquad \gamma_4 = -1. \tag{B.1.26}$$

For odd N the relevant classical constraints have the form

$$X^iZ = -\beta_2\epsilon^{j_1\ldots j_N}(K_{j_1j_2}\ldots K_{j_{N-2}j_{N-1}})J^i_{j_N} \tag{B.1.27}$$

$$Y_iZ = -\beta_1\epsilon_{ii_2i_3i_4}\epsilon^{j_1\ldots j_N}(K_{j_1j_2}\ldots K_{j_{N-4}j_{N-3}})J^{i_2}_{j_{N-2}}J^{i_3}_{j_{N-1}}J^{i_4}_{j_N}. \tag{B.1.28}$$

Based on Eqs. (B.1.20) and (B.1.21),

$$X^iZ = (\sigma_1\ldots\sigma_mq_i)(\bar{v}_1\ldots\bar{v}_N) \tag{B.1.29}$$

$$Y_iZ = (\sigma_1v_3v_4\sigma_3\ldots\sigma_mq_2)(\bar{v}_1\ldots\bar{v}_N), \tag{B.1.30}$$

which when matched to the corresponding products of J and K imply that

$$\beta_1 = -\frac{1}{2^{m-1}(m-1)!3!}, \qquad \beta_2 = -\frac{1}{2^mm!}. \tag{B.1.31}$$

In both cases the overall constant α has no effect on the equations of motion, and cannot be calculated from the classical constraints.

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