

UC Berkeley

UC Berkeley Previously Published Works

Title

Hecke operators and analytic Langlands correspondence for curves over local fields

Permalink

<https://escholarship.org/uc/item/46j8z6m3>

Journal

Duke Mathematical Journal, 172(11)

ISSN

0012-7094

Authors

Etingof, Pavel
Frenkel, Edward
Kazhdan, David

Publication Date

2023-08-15

DOI

10.1215/00127094-2022-0068

Peer reviewed

HECKE OPERATORS AND ANALYTIC LANGLANDS CORRESPONDENCE FOR CURVES OVER LOCAL FIELDS

PAVEL ETINGOF, EDWARD FRENKEL, and DAVID KAZHDAN

In memory of Isadore Singer

Abstract

We construct analogues of the Hecke operators for the moduli space of G -bundles on a curve X over a local field F with parabolic structures at finitely many points. We conjecture that they define commuting compact normal operators on the Hilbert space of half-densities on this moduli space. In the case $F = \mathbb{C}$, we also conjecture that their joint spectrum is in a natural bijection with the set of ${}^L G$ -opers on X with real monodromy. This may be viewed as an analytic version of the Langlands correspondence for complex curves. Furthermore, we conjecture an explicit formula relating the eigenvalues of the Hecke operators and the global differential operators. Assuming the compactness conjecture, this formula follows from a certain system of differential equations satisfied by the Hecke operators, which we prove here for $G = \mathrm{PGL}_n$.

Contents

1. Introduction	2015
2. The abelian case	2032
3. Generalities on D -modules and integral transforms	2042
4. The case of PGL_n	2053
5. General case	2059
References	2068

1. Introduction

The Langlands program for a curve X over a finite field \mathbb{F}_q studies, in the unramified case, the joint spectrum of the commuting Hecke operators acting on the space of

DUKE MATHEMATICAL JOURNAL

Vol. 172, No. 11, © 2023 DOI [10.1215/00127094-2022-0068](https://doi.org/10.1215/00127094-2022-0068)

Received 26 March 2021. Revision received 15 April 2022.

First published online 27 July 2023.

2020 *Mathematics Subject Classification*. 11R39.

L^2 -functions on the groupoid of \mathbb{F}_q -points of the stack Bun_G of G -bundles on X with its natural measure. It aims to express this spectrum in terms of the Galois data associated to X and the Langlands dual group ${}^L G$.

In this article, which is a continuation of [8], we study an analogue of this problem when \mathbb{F}_q is replaced by a local field F . We define analogues of Hecke operators on a dense subspace of the Hilbert space \mathcal{H}_G of half-densities on Bun_G and conjecture that they extend to commuting compact normal operators on \mathcal{H}_G . Investigation of these operators is a hybrid of functional analysis and the study of the algebraic structure of Bun_G .

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on Bun_G introduced in [3], as well as their complex conjugates. This enables us to describe, subject to the compactness conjecture formulated below, the joint spectrum of these operators in terms of the ${}^L G$ -opers on X with *real monodromy*, which play the role of the Galois data appearing in the case of curves over a finite field. We consider this description as an analogue of the Langlands correspondence in the case of complex curves. In the case of $G = \text{SL}_2$, a similar conjecture about eigenfunctions of the differential operators on Bun_G was proposed earlier by Teschner [29].

1.1. Main objects

Let F be a local field, let G be a split connected reductive group over F , let B be its Borel subgroup, let X be a smooth projective curve over F , and let $S \subset X$ be a reduced divisor defined over F . We denote by $\text{Bun}_G(X, S)$ the moduli stack of pairs (\mathcal{F}, r_S) , where \mathcal{F} is a G -bundle on X and r_S is a reduction to B of the restriction $\mathcal{F}|_S$ of \mathcal{F} to S .¹

Let $\text{Bun}_G^s(X, S) \subset \text{Bun}_G(X, S)$ be the substack of *regularly stable* pairs (\mathcal{F}, r_S) , that is, stable pairs whose group of automorphisms is equal to the center $Z(G)$ of G .

We will assume throughout this paper that $\text{Bun}_G^s(X, S)$ is *open and dense* in $\text{Bun}_G(X, S)$, which means that we are considering one of the following cases:

- (1) the genus of X is greater than 1, and S is arbitrary;
- (2) X is an elliptic curve and $|S| \geq 1$;
- (3) $X = \mathbb{P}^1$ and $|S| \geq 3$.

The stack $\text{Bun}_G^s(X, S)$ is a $Z(G)$ -gerbe over a smooth analytic F -variety $\text{Bun}_G^\circ(X, S)$, which is the corresponding coarse moduli space. For our purposes, $\text{Bun}_G^\circ(X, S)$ is a good substitute for $\text{Bun}_G^s(X, S)$ because all objects we need (such as line bundles or differential operators) naturally descend from $\text{Bun}_G^s(X, S)$ to $\text{Bun}_G^\circ(X, S)$.

¹Just as in the case of curves over \mathbb{F}_q , the theory naturally extends to incorporate reductions to other parabolic subgroups of G . We also expect that it extends to wild ramification.

Langlands asked in [24] whether it is possible to develop an analytic theory of automorphic functions for complex curves. In [8] and the present work, we are developing such a theory (for curves over \mathbb{C} and other local fields), understood as a spectral theory of commuting Hecke operators acting on the Hilbert space $\mathcal{H}_G(X, S)$ of half-densities on $\text{Bun}_G^\circ(X, S)$ for pairs (X, S) satisfying the above condition, as well as global differential operators on $\text{Bun}_G(X, S)$ when $F = \mathbb{C}$ or \mathbb{R} .

To define the Hilbert space $\mathcal{H}_G(X, S)$, we introduce the $\mathbb{R}_{>0}$ -line bundle $\Omega_{\text{Bun}}^{1/2}$ of half-densities on $\text{Bun}_G^\circ(X, S)$.

Namely, our local field F is equipped with the norm map $x \mapsto \|x\|$ (the Haar measure on F multiplies by $\|\lambda\|$ under rescaling by $\lambda \in F$). For instance, for $F = \mathbb{R}$ we have $\|x\| = |x|$, for $F = \mathbb{C}$ we have $\|x\| = |x|^2$, and for $F = \mathbb{Q}_p$ we have $\|x\| = p^{-v(x)}$, where $v(x)$ is the p -adic valuation of x (see [30]). Using the norm map, we associate to any line bundle \mathcal{L} on any smooth algebraic F -variety \mathbb{Y} a complex line bundle $\|\mathcal{L}\|$ on the analytic F -variety $Y := \mathbb{Y}(F)$ with the structure group $\mathbb{R}_{>0}$. Namely, if the transition functions of \mathcal{L} are $\{g_{\alpha\beta}\}$, then the transition functions of $\|\mathcal{L}\|$ are $\{\|g_{\alpha\beta}\|\}$ (in particular, if $F = \mathbb{C}$, then these are $\{|g_{\alpha\beta}|^2\}$). (Note that for Archimedean fields the line bundle $\|\mathcal{L}\|$ has a C^∞ -structure.)

It is shown in [3, Section 4] (see also [25]) that the canonical line bundle K_{Bun} over $\text{Bun}_G(X, S)$ has a square root $K_{\text{Bun}}^{1/2}$. The restriction of $K_{\text{Bun}}^{1/2}$ to $\text{Bun}_G^s(X, S)$ descends to a line bundle on $\text{Bun}_G^\circ(X, S)$ for which we will use the same notation $K_{\text{Bun}}^{1/2}$. We then set

$$\Omega_{\text{Bun}}^{1/2} := \|K_{\text{Bun}}^{1/2}\|. \tag{1.1}$$

Alternatively, $\Omega_X^{1/2}$ can be defined as the square root of the line bundle $\|K_X\|$ (since the structure group of the latter is $\mathbb{R}_{>0}$, this is well defined). This shows that the line bundle $\Omega_{\text{Bun}}^{1/2}$ does not depend on the choice of $K_{\text{Bun}}^{1/2}$.

Let $V_G(X, S)$ be the space of smooth compactly supported sections of $\Omega_{\text{Bun}}^{1/2}$ over $\text{Bun}_G^\circ(X, S)(F)$. We denote by $\langle \cdot, \cdot \rangle$ the positive definite Hermitian form on $V_G(X, S)$ given by

$$\langle v, w \rangle := \int_{\text{Bun}_G^\circ(X, S)(F)} v \cdot \bar{w}, \quad v, w \in V_G(X, S),$$

and define $\mathcal{H}_G(X, S)$ as the *Hilbert space* completion of $V_G(X, S)$.

1.2. Hecke operators

From now on, for brevity, we will drop (X, S) in our notation when no confusion could arise (i.e., write Bun_G for $\text{Bun}_G(X, S)$, \mathcal{H}_G for $\mathcal{H}_G(X, S)$, and so on). We are going to define analogues of the Hecke operators for curves over a local field. For non-Archimedean local fields, these operators were suggested by Braverman and the

third author in [5]. For $G = \mathrm{PGL}_2$, $X = \mathbb{P}^1$, Hecke operators were studied by Kontsevich in [23], and he also knew that such operators could be defined in general. In his letters to us (2019), he conjectured compactness and the Hilbert–Schmidt property of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over \mathbb{C} could be used to construct an analogue of the Langlands correspondence was suggested by Langlands [24], who sought to construct them in the case when $G = \mathrm{GL}_2$, X is an elliptic curve, and $S = \emptyset$ (however, for an elliptic curve X we can only define Hecke operators if $S \neq \emptyset$; see [14, Section 3]).

In the case when G is a torus, the Hecke operators and their spectra on \mathcal{H}_G were completely described in [14, Section 2]. We recall these results for $G = \mathrm{GL}_1$ in Section 2.1 below.

From now on (except in Section 2 below), we will assume that G is a connected simple algebraic group over F . All of our results and conjectures generalize in a straightforward way to connected semisimple algebraic groups over F .

Let P_+^\vee be the set of dominant integral coweights of G . For $\lambda \in P_+^\vee$, we denote by $\overline{Z}(\lambda)$ the Hecke correspondence

$$\overline{q} : \overline{Z}(\lambda) \rightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G \times (X \setminus S)$$

describing the λ -modifications of stable G -bundles at points of $X \setminus S$. More precisely, $\overline{Z}(\lambda)$ classifies triples $(\mathcal{F}, \mathcal{P}, x, t)$, where \mathcal{F} and \mathcal{P} are principal G -bundles on X with a reduction to B at S , x is a closed point in $X \setminus S$, and

$$t : \mathcal{F}|_{X \setminus x} \xrightarrow{\sim} \mathcal{P}|_{X \setminus x}$$

is an isomorphism satisfying the following condition. Choose a formal coordinate z at x and a trivialization of \mathcal{F} on the formal disk $D_x := \mathrm{Spec} \mathbb{C}[[z]]$; then the restriction of t to $D_x^\times := \mathrm{Spec} \mathbb{C}((z))$ naturally gives rise to a point in the affine Grassmannian $\mathrm{Gr} = G((z))/G[[z]]$. The condition is that this point belongs to the closure $\overline{\mathrm{Gr}}_\lambda$ of the $G[[z]]$ -orbit $\mathrm{Gr}_\lambda = G[[z]] \cdot \lambda(z)G[[z]]$ (this condition does not depend on the above choices). Further, denote by $Z(\lambda)$ the open part of $\overline{Z}(\lambda)$ satisfying the condition that this point belongs to Gr_λ itself, and let q be the restriction of \overline{q} to $Z(\lambda)$.

Let

$$\begin{aligned} p_{1,2} &: \mathrm{Bun}_G \times \mathrm{Bun}_G \times (X \setminus S) \rightarrow \mathrm{Bun}_G, \\ p_3 &: \mathrm{Bun}_G \times \mathrm{Bun}_G \times (X \setminus S) \rightarrow X \setminus S \end{aligned}$$

be the natural projections, let $\overline{q}_i := p_i \circ \overline{q}$, and let $q_i := p_i \circ q$. Thus, the fibers of the morphism $\overline{q}_2 \times \overline{q}_3$ are isomorphic to $\overline{\mathrm{Gr}}_\lambda$ and the fibers of the morphism $q_2 \times q_3$ are isomorphic to Gr_λ . Let K_2 be the relative canonical line bundle of the morphism $q_2 \times q_3$.

Let ρ be the integral weight of G such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for all simple coroots α_i^\vee of G . If the set $\{(\lambda, \rho), \lambda \in P_+^\vee\}$ only contains integers, then Beilinson and Drinfeld [3, Section 4] construct a square root $K_{\text{Bun}}^{1/2}$ of K_{Bun} equipped with a trivialization of its fiber at the trivial G -bundle. If this set contains half-integers, they construct such an object for each choice of a square root $K_X^{1/2}$ of K_X . In the latter case, we will assume throughout that a choice of $K_X^{1/2}$ has been made. We will denote by γ the isomorphism class of this $K_X^{1/2}$ and by $K_{\text{Bun}}^{1/2}$ the corresponding line bundle on Bun_G (equipped with a trivialization of its fiber at the trivial G -bundle). The following result is essentially due to Beilinson and Drinfeld [3]. In Section 5.4 below, we explain how to derive it from [3, (241)] (see also [5, Theorem 2.4]).

THEOREM 1.1

There exists a canonical isomorphism

$$a : q_1^*(K_{\text{Bun}}^{1/2}) \xrightarrow{\sim} q_2^*(K_{\text{Bun}}^{1/2}) \otimes K_2 \otimes q_3^*(K_X^{-(\lambda, \rho)}), \tag{1.2}$$

where ρ is the half-sum of positive roots.

The isomorphism a gives rise to an isomorphism

$$a^2 : q_1^*(K_{\text{Bun}}) \xrightarrow{\sim} q_2^*(K_{\text{Bun}}) \otimes (K_2)^2 \otimes q_3^*(K_X^{-2(\lambda, \rho)}) \tag{1.3}$$

which does not depend on the choice of $K_X^{1/2}$ (and $K_{\text{Bun}}^{1/2}$). Using the formula $\|a\| = \|a^2\|^{1/2}$, we then obtain a canonical isomorphism

$$\|a\| : q_1^*(\Omega_{\text{Bun}}^{1/2}) \xrightarrow{\sim} q_2^*(\Omega_{\text{Bun}}^{1/2}) \otimes \Omega_2 \otimes q_3^*(\Omega_X^{-(\lambda, \rho)}). \tag{1.4}$$

Here $\Omega_X^{1/2}$ (resp., $\Omega_{\text{Bun}}^{1/2}$) is defined as the square root of the line bundle $\|K_X\|$ (resp., $\|K_{\text{Bun}}\|$); since the structure group of the latter is $\mathbb{R}_{>0}$, these square roots are well defined. Also, $\Omega_2 := \|K_2\|$, the C^∞ line bundle of densities along the fibers of $q_2 \times q_3$.

Thus, $\|a\|$ does not depend on the choice of $K_X^{1/2}$ (and $K_{\text{Bun}}^{1/2}$). This implies that our definition of the Hecke operators given below also does not depend on this choice. However, we need to make these choices to describe the algebra of global differential operators on Bun_G (see Section 1.3) and to state the differential equations (1.13) satisfied by the Hecke operators. Hence, for the sake of uniformity of exposition, we have made these choices from the beginning.

Now let

$$U_G(\lambda) := \{ \mathcal{F} \in \text{Bun}_G^\circ \mid q_2(q_1^{-1}(\mathcal{F})) \subset \text{Bun}_G^\circ \}. \tag{1.5}$$

This is an open subset of Bun_G° , which is dense if the dimension of Bun_G is sufficiently large (e.g., for $G = \text{PGL}_2$ it is sufficient that $\dim \text{Bun}_{\text{PGL}_2} > 1$).

Suppose that $U_G(\lambda) \subset \text{Bun}_G^\circ$ is dense. Let $V_G(\lambda) \subset V_G$ be the subspace of half-densities f with compact support such that $\text{supp}(f) \subset U_G(\lambda)$. For $\mathcal{P} \in \text{Bun}_G^\circ(F)$, let

$$Z_{\mathcal{P},x} := (q_2 \times q_3)^{-1}(\mathcal{P} \times x), \quad x \in (X \setminus S)(F).$$

According to the above definition of $Z(\lambda)$, the variety $Z_{\mathcal{P},x}$ is isomorphic to the $G[[z]]$ -orbit Gr_λ in the affine Grassmannian of G . Hence it is proper and has rational singularities. The isomorphism (1.4) and the results of Theorem 2.5 and Section 2.6 of [5] imply that for any $f \in V_G(\lambda)$ and $x \in (X \setminus S)(F)$, the restriction of the pullback $q_1^*(f)$ to $Z_{\mathcal{P},x}$ is a compactly supported measure with values in the line $(\Omega_{\text{Bun}}^{1/2})_{\mathcal{P}} \otimes (\Omega_X^{-(\lambda,\rho)})_x$. Therefore, the integral

$$(\widehat{H}_\lambda(x) \cdot f)(\mathcal{P}) := \int_{Z_{\mathcal{P},x}} q_1^*(f) \tag{1.6}$$

is absolutely convergent for all $f \in V_G(\lambda)$ and $\mathcal{P} \in \text{Bun}_G^\circ(F)$. In fact, it belongs to $V_G \otimes (\Omega_X^{-(\lambda,\rho)})_x$.

Since $V_G \subset \mathcal{H}_G$, this integral defines a *Hecke operator*

$$\widehat{H}_\lambda(x) : V_G(\lambda) \rightarrow \mathcal{H}_G \otimes (\Omega_X^{-(\lambda,\rho)})_x.$$

As x varies along $X \setminus S$, the operators $\widehat{H}_\lambda(x)$ combine into a section of the line bundle $\Omega_X^{-(\lambda,\rho)}$ over $X \setminus S$ with values in operators $V_G(\lambda) \rightarrow \mathcal{H}_G$. We denote it by \widehat{H}_λ .

Remark 1.1

We conjecture (and can prove in a number of cases) that the integrals defining $\widehat{H}_\lambda(f)$ are absolutely convergent for all $f \in V_G$.

Remark 1.2

There are two basic differences between the cases of finite and local fields. In the case of a curve X over a finite field F , the set $\text{Bun}_G(F)$ of isomorphism classes $[\mathcal{P}]$ of G -bundles is a countable set with a natural (Tamagawa) measure such that the measure of $[\mathcal{P}]$ is equal to $1/|\text{Aut}(\mathcal{P})(F)|$. On the other hand, for a curve X over a local field F , neither $\text{Bun}_G(F)$ nor the fibers of the Hecke correspondence carry any natural measures. To overcome this problem, we replace the space of functions on $\text{Bun}_G(F)$ by the space of half-densities and use the isomorphism (1.4) to write down formula (1.6) for the action of Hecke operators.

The second difference is that for a curve over a finite field F , the integrals (1.6) are finite sums, so the corresponding Hecke operators are well defined on the space of all functions on $\text{Bun}_G(F)$. On the other hand, for a curve X over a local field F there is no obvious nontrivial space of half-densities stable under these Hecke operators.

One could try to consider the space of continuous half-densities with respect to the natural topology on $\text{Bun}_G(F)$, but this space is equal to $\{0\}$. The reason is that the natural topology on $\text{Bun}_G(F)$ is not Hausdorff.²

The absence of such a subspace creates serious analytic difficulties in defining the notion of the spectrum of the Hecke operators because they are initially defined only on a dense subspace $V_G(\lambda)$ of \mathcal{H}_G (under our assumption that $\text{Bun}_G^s(X, S)$ is dense in $\text{Bun}_G(X, S)$). However, we expect (and can prove for $G = \text{PGL}_2$, $X = \mathbb{P}^1$; see [9]) that these operators extend to bounded operators on \mathcal{H}_G , which are moreover compact and normal, as we state in the following conjecture.

CONJECTURE 1.2

Suppose that $\lambda \neq 0$.

- (1) For any identification $(\Omega_X^{1/2})_x \cong \mathbb{C}$, the operators $\widehat{H}_\lambda(x) : V_G(\lambda) \rightarrow \mathcal{H}_G$ extend to a family of commuting compact normal operators on \mathcal{H}_G , which we denote by $H_\lambda(x)$.
- (2) $H_\lambda(x)^\dagger = H_{-w_0(\lambda)}(x)$.
- (3) $\bigcap_{\lambda, x} \text{Ker} H_\lambda(x) = \{0\}$.

Remark 1.3

- (i) It is easy to see that

$$\langle H_\lambda f, g \rangle = \langle f, H_{-w_0(\lambda)} g \rangle$$

for all $f \in V_G(\lambda)$, $g \in V_G(-w_0(\lambda))$. Therefore, part (2) of the conjecture immediately follows from part (1).

- (ii) If $S \neq \emptyset$, then we expect that the statement of part (3) with a fixed λ can be obtained from (1) by considering the limit of $H_\lambda(x)$ when x tends to a point of S . This will be discussed in more detail in a follow-up paper.

We will refer to Conjecture 1.2 as the *compactness conjecture*. In [9], we prove this conjecture in the case $G = \text{PGL}_2$, $X = \mathbb{P}^1$, $|S| > 3$. From now on, we will assume the validity of the compactness conjecture.

Denote by \mathbb{H}_G the commutative algebra generated by operators $H_\lambda(x)$, $\lambda \in P_+^\vee$, $x \in (X \setminus S)(F)$, and by $\text{Spec}(\mathbb{H}_G)$ their joint spectrum.

COROLLARY 1.3

We have an orthogonal decomposition

²For a simply connected group G , the closures of any two points in $\text{Bun}_G(F)$ have a nonempty intersection, and therefore the only continuous functions on $\text{Bun}_G(F)$ are constants.

$$\mathcal{H}_G = \widehat{\bigoplus_{s \in \text{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s)},$$

where $\mathcal{H}_G(s), s \in \text{Spec}(\mathbb{H}_G)$ are finite-dimensional joint eigenspaces of \mathbb{H}_G in \mathcal{H}_G .

Let $\mathbb{H}_G(x)$ be the subalgebra of \mathbb{H}_G generated by $H_\lambda(x), \lambda \in P_+^\vee$ for a fixed $x \in (X \setminus S)(F)$.

PROPOSITION 1.4 ([5])

There is an algebra isomorphism $\mathbb{H}_G(x) \simeq \mathbb{C}[P_+^\vee]$. In particular, $H_\lambda(x) \cdot H_\mu(x) = H_{\lambda+\mu}(x)$.

Proof

If F is a non-Archimedean field, then this is equation (3.4) of [5], which is proved in Lemmas 3.5 and 3.9 of [5]. The same proof works for a general local field. \square

Remark 1.4

Note the difference with the case of a curve over \mathbb{F}_q , where the Satake isomorphism naturally identifies the Hecke algebra at a point x with $\text{Rep}^L G$. Thus, in this case the product of the Hecke operators corresponding to $\lambda, \mu \in P_+^\vee$ is in general *not* equal to the Hecke operator corresponding to $\lambda + \mu$; there are correction terms corresponding to lower weights.

Remark 1.5

Using an analogous correspondence

$$Z_r(\lambda) \rightarrow \text{Bun}_G \times \text{Bun}_G \times \text{Sym}^r(X \setminus S)$$

for a positive integer $r > 1$, the above construction can be generalized to yield operators $H_\lambda(D), D \in \text{Sym}^r(X \setminus S)(F)$. We expect that an analogue of the compactness Conjecture 1.2 holds for them. Moreover, if $D = \sum_i m_i x_i, x_i \in X(F)$, then $H_\lambda(D) = \prod_i H_\lambda(x_i)^{m_i}$.

1.3. The case $F = \mathbb{C}$

At the moment, we have a conjectural description of the spectra $\text{Spec}(\mathbb{H}_G)$ only if $F = \mathbb{C}$ and, in some cases, for $F = \mathbb{R}$. We will now describe our conjecture for $F = \mathbb{C}$. Note that in this case we have $\|z\| = |z|^2$. For simplicity, we only consider here the case when $S = \emptyset$ (and hence $g > 1$). But all of our results and conjectures have natural generalizations to the case of an arbitrary S . The case of $G = PGL_2, X = \mathbb{P}^1$, and $|S| > 3$ will be discussed in detail in our next paper [9].

In a previous paper [8], we studied the action of global holomorphic and antiholomorphic differential operators on Bun_G on a dense subspace of the Hilbert space \mathcal{H}_G in the case of a simply connected simple group G . Here we generalize the setup to the case of an arbitrary connected simple G . Then Bun_G has finitely many connected components labeled by $\pi_1(G)$, which we will denote by Bun_G^β , $\beta \in \pi_1(G)$.

Definition 1.5

Let D_G^β be the algebra of global algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\text{Bun}}^{1/2}$ over a connected component Bun_G^β of Bun_G . Beilinson and Drinfeld [3] have proved that these algebras are isomorphic to each other. Therefore, we will also use the notation D_G for D_G^β .

On the other hand, let $\text{Op}_{L_G}(X)$ be the space of (holomorphic) L_G -opers on X defined in [3] and [4]. As shown in [3, Section 3.4], $\text{Op}_{L_G}(X)$ is a union of connected components, each isomorphic to the affine space $\text{Op}_{L_{G_{\text{ad}}}}(X)$, where $L_{G_{\text{ad}}} = L_G/Z(L_G)$ is the group of adjoint type associated to L_G (here $Z(L_G)$ denotes the finite center of L_G). The group $H^1(X, Z(L_G))$ naturally acts on the set of these components by changing the underlying L_G -bundle, and this action is simply transitive. As shown in [3, Section 3.4.2], if the set $\{(\lambda, \rho), \lambda \in P_+^\vee\}$ only contains integers, then there is a canonical choice of a component, and if the set $\{(\lambda, \rho), \lambda \in P_+^\vee\}$ contains half-integers, then there is a canonical choice of a component $\text{Op}_{L_G}^\gamma(X)$ for each isomorphism class γ of $K_X^{1/2}$.

For the sake of uniformity of notation, in both cases we will denote the canonical component by $\text{Op}_{L_G}^\gamma(X)$, with the understanding that in the latter case γ is the isomorphism class of $K_X^{1/2}$ we have chosen before Theorem 1.1. The following theorem is proved in [3, Theorem 3.3.2, Sections 2.2.5 and 2.7.4].

THEOREM 1.6

The algebra D_G is commutative and $\text{Spec}(D_G)$ is isomorphic to the affine space $\text{Op}_{L_G}^\gamma(X)$.

For a given $\chi \in \text{Op}_{L_G}^\gamma(X)$, the system of differential equations for the eigenfunctions of this algebra,

$$P \cdot f = \chi(P)f, \quad P \in D_G, \tag{1.7}$$

where f denotes a local holomorphic section of $K_{\text{Bun}}^{1/2}$, is known as the *quantum Hitchin integrable system*. Its local solutions are the same as the homomorphisms from the twisted D -module Δ_χ on Bun_G to $K_{\text{Bun}}^{1/2}$, where

$$\Delta_\chi := \mathcal{D}'_{\text{Bun}_G} \otimes_{D_G} \mathbb{C}_\chi \tag{1.8}$$

and $\mathcal{D}'_{\text{Bun}_G}$ is the sheaf of (holomorphic) differential operators acting on $K_{\text{Bun}}^{1/2}$ (in (1.8) we consider the diagonal embedding $D_G \rightarrow \mathcal{D}'_{\text{Bun}_G}$ taking the sum over all components Bun_G^β of Bun_G).

Remark 1.6

If $S \neq \emptyset$, then the algebra of global holomorphic differential operators acting on $K_{\text{Bun}}^{1/2}$ is noncommutative, but it contains a commutative subalgebra whose spectrum is isomorphic to the space of L_G -opers on X with regular singularity and regular unipotent monodromy at the points of S (see [8, Section 6]). There is a similar algebra in the case of higher level structures at S (this is analogous to the wild ramification in the case of curves over \mathbb{F}_q), with the corresponding L_G -opers having irregular singularities at the points of S . We expect that the theory has a generalization to this case as well.

It is shown in [3] that Δ_χ is a holonomic D -module. It also has regular singularities (see Theorem 1.12(1) below). Moreover, there is an open substack Bun_G^{vs} of *very stable bundles* (i.e., G -bundles \mathcal{P} such that the vector bundle $\mathfrak{g}_{\mathcal{P}} \otimes K_X$ does not admit nonzero sections taking nilpotent values everywhere on X) such that the restriction of Δ_χ to Bun_G^{vs} is a vector bundle with a projectively flat connection. But its rank grows exponentially with the genus of the curve X and it has highly nontrivial monodromy around the divisor $\text{Bun}_G \setminus \text{Bun}_G^{\text{vs}}$. Therefore, it does not make sense to look for individual holomorphic solutions of the system (1.8).

However, as explained in [29] and in [8, Section 1.5], it makes sense to couple the system (1.8) to its antiholomorphic analogue and look for single-valued solutions of the resulting system of differential equations on the locus Bun_G^{vs} of very stable bundles in Bun_G° . These are the automorphic functions of the analytic theory. It is natural to try to interpret them as eigenfunctions of the algebra

$$\mathcal{A}_G := D_G \otimes_{\mathbb{C}} \overline{D}_G.$$

This is a nontrivial task because elements of \mathcal{A}_G correspond to unbounded operators on the Hilbert space \mathcal{H}_G . Initially, they are defined on the dense subspace $V_G \subset \mathcal{H}_G$. In [8, Definition 1.7], we introduced a Schwartz space $S(\mathcal{A}_G) \subset \mathcal{H}_G$ and conjectured that the elements of \mathcal{A}_G can be extended to $S(\mathcal{A}_G)$ so that their closures form a family of commuting normal operators. Moreover, we conjectured that their joint spectrum is the set $\text{Op}_{L_G}(X)_{\mathbb{R}}$ of L_G -opers on X with real monodromy (see below). In the case of $G = \text{SL}_2$, a similar conjecture was proposed earlier in [29].

In [8], we were able to prove these conjectures in the simplest nontrivial case. However, in the general case it is a daunting task to prove them directly. This is where the integral Hecke operators come in handy. Like differential operators, they are also initially defined on a dense subspace of the Hilbert space \mathcal{H}_G . But unlike the differential operators, we do expect Hecke operators to extend to mutually commuting continuous operators on the entire \mathcal{H}_G , which are moreover *normal*, *compact*, and have trivial common kernel. This is the statement of our compactness Conjecture 1.2. It implies that \mathcal{H}_G decomposes into a (completed) direct sum of mutually orthogonal *finite-dimensional eigenspaces* of the Hecke operators.

Next, we would like to say that the algebra \mathcal{A}_G preserves the subspaces $\mathcal{H}_G(s)$. To do this, observe that \mathcal{A}_G acts on the space V_G^\vee of distributions on Bun_G° . It follows from the definition of \mathcal{H}_G that \mathcal{H}_G is naturally realized as a subspace of V_G^\vee . Hence we can apply elements of \mathcal{A}_G to vectors in the eigenspaces $\mathbb{H}_G(s)$ of the Hecke operators, viewed as *distributions*.

CONJECTURE 1.7

Every eigenspace $\mathbb{H}_G(s)$ of the Hecke operators, viewed as a subspace of V_G^\vee , is an eigenspace of \mathcal{A}_G .

Remark 1.7

- (i) There is a weaker version of Conjecture 1.7, in which V_G is replaced by the space of smooth functions with compact support on some open dense set $U \subset \text{Bun}_G^\circ$. For practical purposes, such a weak version is almost as good but might be easier to prove. For example, we may take U to be the above locus Bun_G^{vs} of very stable bundles in Bun_G° , on which the eigenfunctions of the Hecke operators are smooth (in fact, this is true for any local field F ; we will explain this in a follow-up paper). Then the conjecture is that these eigenfunctions satisfy the differential equations of the quantum Hitchin system combined with its complex conjugate system (which are smooth on this locus) in the classical sense.
- (ii) The algebra \mathcal{A}_G acts on both $V_G(\lambda)$ and V_G . Using an identification (see [3]) of D_G with a quotient of the center at the critical level (see [12], [13]), one can show that the action of \mathcal{A}_G commutes with the Hecke operators $\widehat{H}_\lambda(x) : V_G(\lambda) \rightarrow V_G$. This property, however, is not sufficient for proving Conjecture 1.7. But in the case $G = \text{PGL}_n$ we can derive it from the system of differential equations in Theorem 1.18. We expect that a similar argument also proves Conjecture 1.7 for a general G .

In the rest of this subsection, we will assume the validity of Conjecture 1.7. It implies the following statement. Let

$$\mathfrak{S}(\mathcal{A}_G) := \bigoplus_{s \in \text{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s) \subset \mathcal{H}_G.$$

According to Conjecture 1.7, the algebra \mathcal{A}_G acts on it.

COROLLARY 1.8

The closures of the actions of the elements of \mathcal{A}_G on $\mathfrak{S}(\mathcal{A}_G)$ form a family of commuting normal operators, and each $\mathcal{H}_G(s)$ is an eigenspace of these normal operators.

Remark 1.8

It is clear that the subspace $\mathfrak{S}(\mathcal{A}_G) \subset \mathcal{H}_G$ is contained in the Schwartz space $S(\mathcal{A}_G)$ defined in [8]. We expect that $S(\mathcal{A}_G)$ is a completion of $\mathfrak{S}(\mathcal{A}_G)$ corresponding to a specific growth condition on the coefficients.

Corollary 1.8 implies that we can define the joint spectrum of the commutative algebra \mathcal{A}_G . It follows from the definition that $\text{Spec}(\mathcal{A}_G)$ is naturally realized as a subset of $\text{Op}_{L_G}^\gamma(X) \times \overline{\text{Op}}_{L_G}^\gamma(X)$.

Definition 1.9

An L_G -oper χ is called an *oper with real monodromy* (or *real oper* for short) if the monodromy representation $\pi_1(X, p_0) \rightarrow L_G(\mathbb{C})$ corresponding to χ is isomorphic to its complex conjugate.

Denote by $\text{Op}_{L_G}^\gamma(X)_{\mathbb{R}}$ the subset of real L_G -opers in $\text{Op}_{L_G}^\gamma(X)$.

Remark 1.9

It follows from the above definition that the image of the monodromy representation corresponding to a real L_G -oper is contained in a real form $L_{G_{\mathbb{R}}}$ of L_G . Moreover, this form is inner to the split real form since for every algebraic representation V of L_G , the corresponding monodromy ρ_V is isomorphic to $\overline{\rho}_V$. We conjecture that in fact the form $L_{G_{\mathbb{R}}}$ is the split real form $L_G(\mathbb{R})$, in other words, that real L_G -opers are precisely the L_G -opers χ for which the image of the monodromy representation of χ in $L_G(\mathbb{C})$ is contained, up to conjugation, in $L_G(\mathbb{R})$. This is known to be the case for $L_G = \text{PGL}_2$ and SL_2 (see [11], [18], [20]).

We can prove that $L_{G_{\mathbb{R}}}$ is split in the case when $S \neq \emptyset$. Indeed, the residue of the oper at each parabolic point is regular nilpotent, so the monodromy is regular unipotent. Thus our real form $L_{G_{\mathbb{R}}}$ contains a real regular unipotent element, hence a real

Borel subgroup (the unique Borel subgroup containing it), and hence it is quasisplit. But an inner quasisplit real form must be split.

For $\chi \in \text{Op}_{L_G}^\vee(X)$, we denote, following Section 3.4 of [8], by χ^* the L_G -oper obtained by applying to χ the Chevalley involution on L_G (it belongs to the same component $\text{Op}_{L_G}^\vee(X)$). Theorem 4.2 of [8] implies the following.

THEOREM 1.10

As a subset of $\text{Op}_{L_G}^\vee(X) \times \overline{\text{Op}_{L_G}^\vee(X)}$, the joint spectrum $\text{Spec}(\mathcal{A}_G)$ consists of pairs $(\chi, \bar{\chi}^)$ where $\chi \in \text{Op}_{L_G}^\vee(X)_\mathbb{R}$. Thus, $\text{Spec}(\mathcal{A}_G)$ is naturally realized as a subset of $\text{Op}_{L_G}^\vee(X)_\mathbb{R}$.*

CONJECTURE 1.11

- (1) *The set $\text{Op}_{L_G}^\vee(X)_\mathbb{R}$ is discrete.*
- (2) *$\text{Spec}(\mathcal{A}_G) = \text{Op}_{L_G}^\vee(X)_\mathbb{R}$.*

Remark 1.10

- (i) In [8], we proved Conjecture 1.11 in the case $G = \text{SL}_2$, $X = \mathbb{P}^1$, $|S| = 4$, directly, without relying on the compactness conjecture.
- (ii) Conjecture 1.11(1) is known in the case when $G = \text{PGL}_2$ (see [11, Section 7]).

In [3, Section 5.1.1], Beilinson and Drinfeld attached to every $\chi \in \text{Op}_{L_G}^\vee(X)$ a D -module on Bun_G , which is a Hecke eigensheaf with respect to the flat L_G -bundle corresponding to χ ; namely,

$$\Delta_\chi^0 := K_{\text{Bun}}^{-1/2} \otimes \Delta_\chi, \tag{1.9}$$

where Δ_χ is given by (1.8) (see also [8, Section 3.2] and Remark 4.2 below). The first part of the following statement has been proved in [1, Corollary 11.6.7]. The second part has been proved in [10, Theorem 11.2.1.2 and Remark 11.2.1.3] (in the case $G = \text{PGL}_n$ this also follows from the results of [17]).

THEOREM 1.12

- (1) Δ_χ^0 *has regular singularities.*
- (2) Δ_χ^0 *is irreducible on each connected component of Bun_G .*

Recall that Bun_G has connected components Bun_G^β labeled by $\beta \in \pi_1(G)$. Thus, we have a natural direct sum decomposition

$$\mathcal{H}_G = \bigoplus_{\beta \in \pi_1(G)} \mathcal{H}_G^\beta. \tag{1.10}$$

We also have

$$\mathcal{S}(\mathcal{A}_G) = \bigoplus_{\beta \in \pi_1(G)} \mathcal{S}(\mathcal{A}_G)_\beta, \quad \mathcal{S}(\mathcal{A}_G)_\beta := \mathcal{S}(\mathcal{A}_G) \cap \mathcal{H}_G^\beta.$$

The action of \mathcal{A}_G on $\mathcal{S}(\mathcal{A}_G)$ preserves the direct summands $\mathcal{S}(\mathcal{A}_G)_\beta$. Hence we can talk about the joint eigenvalues of \mathcal{A}_G on each of them and the corresponding multiplicities.

The following statement is proved in Section 1.5 of [8] in the case when G is simply connected, and the proof generalizes in a straightforward way to the case of an arbitrary connected simple Lie group G .

PROPOSITION 1.13

Suppose that $\chi \in \text{Op}_{LG}^\gamma(X)$ corresponds to an element of $\text{Spec}(\mathcal{A}_G)$ and Δ_χ^0 has regular singularities and is irreducible. Then the multiplicity of χ in $\mathcal{S}(\mathcal{A}_G)_\beta$ is equal to 1 for all $\beta \in \pi_1(G)$.

1.4. The case $G = \text{PGL}_n$

According to Theorem 1.12, the spectrum of \mathcal{A}_G is simple on each $\mathcal{S}(\mathcal{A}_G)_\beta \subset \mathcal{H}_G^\beta$, $\beta \in \pi_1(\text{PGL}_n) = \mathbb{Z}/n\mathbb{Z}$.

For $\beta \in \mathbb{Z}/n\mathbb{Z}$, let $\psi_{\chi,\beta}$ be a nonzero generator of the one-dimensional eigenspace of \mathcal{A}_G in \mathcal{H}_G^β corresponding to the eigenvalue χ . Let

$$\mathcal{E}_\chi := \text{span}\{\psi_{\chi,\beta}, \beta \in \mathbb{Z}/n\mathbb{Z}\}.$$

Now consider the Hecke operators $H_{\omega_1}(x), x \in X$, and the corresponding operator-valued section H_{ω_1} of the line bundle $\Omega_X^{-(n-1)/2}$. These operators act from \mathcal{H}_G^β to $\mathcal{H}_G^{\beta+1}$. Corollary 1.8 then implies that $H_{\omega_1}(x)$ and H_{ω_1} preserve the n -dimensional subspace $\mathcal{E}_\chi \subset \mathcal{H}_G$. Moreover, we can normalize the vectors $\psi_{\chi,\beta}$ in such a way that $\Psi_\chi^1 := \sum_{\beta \in \mathbb{Z}/n\mathbb{Z}} \psi_{\chi,\beta}$ is an eigenvector of H_{ω_1} . Then the vectors

$$\Psi_\chi^\epsilon := \sum_{\beta \in \mathbb{Z}/n\mathbb{Z}} \epsilon^\beta \psi_{\chi,\beta}, \quad \epsilon \in \mu_n$$

(where μ_n is the group of n th roots of unity) are also eigenvectors of H_{ω_1} and hence form an eigenbasis of \mathcal{E}_χ . If Φ_χ is a section of $\Omega_X^{-(n-1)/2}$ representing the eigenvalue of H_{ω_1} on Ψ_χ^1 , then the section of $\Omega_X^{-(n-1)/2}$ representing the eigenvalue of H_{ω_1} on Ψ_χ^ϵ is $\epsilon \Phi_\chi$.

Thus, to each $\chi \in \text{Spec } \mathcal{A}_G$ corresponds a collection $\{\Psi_\chi^\epsilon\}$ of eigenvalues of H_{ω_1} , which is naturally a μ_n -torsor. We will now write a conjectural formula for these eigenvalues.

Recall that χ is an SL_n -oper in $\text{Op}_{SL_n}^\gamma(X)_{\mathbb{R}}$. Let $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ be the corresponding holomorphic flat vector bundle on X with $\det(\mathcal{V}_{\omega_1}, \nabla_\chi)$ identified with the trivial flat line bundle. The definition of an oper provides us with an embedding of a line bundle

$$\kappa_{\omega_1} : K_X^{(n-1)/2} \hookrightarrow \mathcal{V}_{\omega_1},$$

and hence an embedding

$$\tilde{\kappa}_{\omega_1} : \mathcal{O}_X \hookrightarrow \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2}.$$

Therefore, we obtain a section

$$s_{\omega_1} := \tilde{\kappa}_{\omega_1}(1) \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2}).$$

In the same way, we obtain a section

$$s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-(n-1)/2}) = \Gamma(X, \mathcal{V}_{\omega_1}^* \otimes K_X^{-(n-1)/2}).$$

By our assumption that $\chi \in \text{Op}_{SL_n}^\gamma(X)_{\mathbb{R}}$, the monodromy representations associated to χ and $\bar{\chi}$ are isomorphic. This means that $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ and $(\bar{\mathcal{V}}_{\omega_1}, \bar{\nabla}_\chi)$ are isomorphic as C^∞ flat vector bundles on X . Hence we have a nondegenerate pairing

$$h_\chi(\cdot, \cdot) : (\mathcal{V}_{\omega_1}, \nabla_\chi) \otimes (\bar{\mathcal{V}}_{\omega_{n-1}}, \bar{\nabla}_\chi) \rightarrow (\mathcal{C}_X^\infty, d)$$

of C^∞ flat vector bundles X . Since $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ and $(\bar{\mathcal{V}}_{\omega_{n-1}}, \bar{\nabla}_\chi)$ are associated to flat SL_n -bundles, their determinants are identified with the trivial flat line bundle. We will require that $h_\chi(\cdot, \cdot)$ induce the canonical pairing on the corresponding determinant line bundles. The set of $h_\chi(\cdot, \cdot)$ normalized this way is a μ_n -torsor, which we denote by $\{\epsilon h_\chi(\cdot, \cdot)\}$.

CONJECTURE 1.14

We have

$$\{\Phi_\chi^\epsilon\} = \{\epsilon h_\chi(s_{\omega_1}, \overline{s_{\omega_{n-1}}})\}$$

as μ_n -torsors of global sections of the line bundle $\Omega_X^{-(n-1)/2}$ on X .

We will prove a slightly weaker form of this conjecture (Corollary 1.19)—with roots of unity ϵ replaced by nonzero complex numbers—by showing that both sides satisfy the same system of differential equations which has a unique solution up to a scalar.

To explain this, we need an alternative description of the component $\text{Op}_{SL_n}^\gamma(X)$.

Consider n th order differential operators $P : K_X^{-(n-1)/2} \rightarrow K_X^{(n+1)/2}$ (here, if n is even, we use our chosen square root $K_X^{1/2}$) such that

- (1) $\text{symb}(P) \in H^0(X, \mathcal{O}_X)$ is equal to 1;
- (2) the operator $P - (-1)^n P^*$ has order $n - 2$ (here $P^* : K_X^{-(n-1)/2} \rightarrow K_X^{(n+1)/2}$ is the algebraic adjoint operator; see [2, Section 2.4]).

These operators form an affine space, which we denote by $D_n^\gamma(X)$. Locally,

$$P = \partial_z^n + v_{n-2}(z)\partial_z^{n-2} + \dots + v_0(z).$$

The following statement is proved in [4, Section 2.8].

LEMMA 1.15

There is a bijection $\text{Op}_{\text{SL}_n}^\gamma(X) \simeq D_n^\gamma(X)$,

$$\chi \in \text{Op}_{\text{SL}_n}^\gamma(X) \quad \mapsto \quad P_\chi \in D_n^\gamma(X),$$

such that the sections $s_{\omega_1} \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2})$ and $s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-(n-1)/2})$ satisfy

$$P_\chi \cdot s_{\omega_1} = 0, \quad P_\chi^* \cdot s_{\omega_{n-1}} = 0,$$

where P_χ^* is the algebraic adjoint of P_χ (here we use the \mathcal{D}_X -module structures on \mathcal{V}_{ω_1} and $\mathcal{V}_{\omega_{n-1}}$ corresponding to the oper connection of χ).

The flat vector bundle $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ is known to be irreducible (see [4]). Therefore, we obtain the following.

COROLLARY 1.16

We have that $h_{\chi, \omega_1}(s_{\omega_1}, \overline{s_{\omega_{n-1}}})$ is a unique, up to a scalar, section Φ of $\Omega_X^{-(n-1)/2}$ which is a solution of the system of differential equations

$$P_\chi \cdot \Phi = 0, \quad \overline{P_\chi^*} \cdot \Phi = 0. \tag{1.11}$$

On the other hand, let $\mathcal{V}_{\omega_1}^{\text{univ}}$ be the universal vector bundle over $\text{Op}_{\text{SL}_n}^\gamma(X) \times X$ with a partial connection ∇^{univ} along X , such that

$$(\mathcal{V}_{\omega_1}^{\text{univ}}, \nabla^{\text{univ}})|_{X \times X} = (\mathcal{V}_{\omega_1}, \nabla_\chi).$$

Let $\mathcal{V}_{\omega_1, X}^{\text{univ}} := \pi_*(\mathcal{V}_{\omega_1}^{\text{univ}})$, where $\pi : \text{Op}_{\text{SL}_n}^\gamma(X) \times X \rightarrow X$ is the projection and π_* is the \mathcal{O} -module pushforward functor. The connection ∇^{univ} makes $\mathcal{V}_{\omega_1, X}^{\text{univ}}$ into a left \mathcal{D}_X -module.

Recall (see Definition 1.5) that we have denoted by D_{PGL_n} the algebra of global holomorphic differential operators acting on each component $\text{Bun}_{\text{PGL}_n}^\beta$ of $\text{Bun}_{\text{PGL}_n}$. By Theorem 1.6, we have an isomorphism

$$D_{\mathrm{PGL}_n} \simeq \mathrm{Fun} \mathrm{Op}_{\mathrm{SL}_n}^\vee(X).$$

Hence D_{PGL_n} naturally acts on $\mathcal{V}_{\lambda, X}^{\mathrm{univ}}$ and this action commutes with the action of \mathcal{D}_X . Using the above realization of PGL_n -opers as n th order differential operators, we construct the “universal PGL_n -oper” as follows.

LEMMA 1.17

There is a unique n th order differential operator

$$\sigma : K_X^{-(n-1)/2} \rightarrow D_{\mathrm{PGL}_n} \otimes K_X^{(n+1)/2} \tag{1.12}$$

satisfying the following property: for any $\chi \in \mathrm{Op}_{\mathrm{SL}_n}^\vee(X) = \mathrm{Spec} D_{\mathrm{PGL}_n}$, applying the corresponding homomorphism $D_{\mathrm{PGL}_n} \rightarrow \mathbb{C}$, we obtain P_χ .

The following is one of the main results of this paper, which will be proved in Section 4.

THEOREM 1.18

The Hecke operator \widehat{H}_{ω_1} , viewed as an operator-valued section of $\Omega_X^{-(n-1)/2} = K_X^{-(n-1)/2} \otimes \overline{K}_X^{(n-1)/2}$, satisfies the system of differential equations

$$\sigma \cdot \widehat{H}_{\omega_1} = 0, \quad \overline{\sigma} \cdot \widehat{H}_{\omega_1} = 0. \tag{1.13}$$

COROLLARY 1.19

Each of the eigenvalues Φ_χ^ϵ of the Hecke operator H_{ω_1} is equal to a scalar multiple of $h_\chi(s_{\omega_1}, \overline{s_{\omega_{n-1}}})$.

Proof of Corollary 1.19 from Theorem 1.18

Equations (1.13) imply that the eigenvalues of H_{ω_1} satisfy equations (1.11). That is because if v is an eigenvector of $\mathcal{A}_{\mathrm{PGL}_n}$ with the eigenvalue of D_{PGL_n} corresponding to a holomorphic SL_n -oper χ , then according to Theorem 1.10, the eigenvalue of $\overline{D}_{\mathrm{PGL}_n}$ on v corresponds to the antiholomorphic SL_n -oper $\overline{\chi}^*$. Furthermore, it is clear that the n th order operator $\overline{K}_X^{-(n-1)/2} \rightarrow \overline{K}_X^{(n+1)/2}$ associated to $\overline{\chi}^*$ is \overline{P}_χ^* . Corollary 1.16 then implies Corollary 1.19.

Corollary 1.19 describes the eigenvalues of the Hecke operator H_{ω_1} for $G = \mathrm{PGL}_n$ up to scalar multiples (it is slightly weaker than Conjecture 1.14, which describes these eigenvalues up to multiplication by n th roots of unity). The eigenvalues of the other Hecke operators H_λ for $G = \mathrm{PGL}_n$ can be found from the eigenvalues of H_{ω_1} using Proposition 1.4.

Moreover, Theorem 1.18 also implies Conjecture 1.7. Indeed, it can be shown that the differential equations (1.13) imply a suitable version of commutativity of the algebra \mathcal{A}_G with the Hecke operators. This will be discussed in more detail in [9].

In Section 4, we will derive Theorem 1.18 from the theorem of Beilinson and Drinfeld [3] describing the action of the *Hecke functor* \mathbb{H}_{ω_1} on the left D -module $\mathcal{D}_{\text{BunPGL}_n} \otimes K_{\text{Bun}}^{-1/2}$ on $\text{Bun}_{\text{PGL}_n}$. We will also discuss a generalization of these results to an arbitrary simple Lie group G .

This shows a deep connection between the geometric/categorical Langlands correspondence and the analytic/function-theoretic one. \square

1.5. Structure of the paper

The paper is organized as follows. In Section 2, we consider the abelian case, $G = \text{GL}_1$, in which one can already see the main ingredients of our construction in the non-abelian case. In Section 3, we formulate some basic results on the compatibility between the natural operations on functions (pullback, pushforward, and integral transforms) and the corresponding functors between categories of twisted D -modules. We then use these results in the following sections to relate the Hecke operators and Hecke functors and derive differential equations on the Hecke operators. In Section 4, we prove that the Hecke operators satisfy the system (1.13) of differential equations corresponding to the “universal SL_n -oper” (Theorem 1.18), using a Hecke eigensheaf property established in [3]. In Section 5, we formulate the analogues of Theorem 1.18 and Corollary 1.19 describing the eigenvalues of the Hecke operators in the case of an arbitrary simple Lie group G . We then outline the proof of these results generalizing the argument we used in Section 4 in the case of PGL_n .

2. The abelian case

In this section, we consider the case $G = \text{GL}_1$. Though the spectral problem here is rather simple (standard Fourier analysis on a torus), it provides a useful illustration of our general method. Indeed, one can already observe in it all the essential ingredients of the picture for a general group G . Hence it is instructive to consider it as a blueprint for the general construction.

We first recall the results of [8] and [14] on the eigenvalues of the corresponding Hecke operators and the global differential operators. We will then prove that the Hecke operators satisfy a system of differential equations, which are analogous to equations (1.13). This system will enable us to prove a relation between the eigenvalues of the Hecke operators and the global differential operators and lead to an alternative, and simpler, proof of the results of [8] and [14] describing these eigenvalues. The differential equations follow from a theorem describing the image of the sheaf \mathcal{D} of differential operators on the Jacobian of the curve X under the action

of the corresponding *Hecke functor*. Thus, these differential equations link the geometric/categorical Langlands correspondence and the analytic Langlands correspondence. A generalization of this equation, and this link, to $G = \mathrm{PGL}_n$ will be presented in Section 4.

2.1. The global picture

Let $G = \mathrm{GL}_1$. In this case, the role of Bun_G° is played by the Picard scheme $\mathrm{Pic}(X)$, which is a fine moduli space of line bundles on X . The canonical line bundle on $\mathrm{Pic}(X)$ is isomorphic to the trivial line bundle, and we will fix such an isomorphism. Hence we have a positive Hermitian inner product on the space of smooth functions on $\mathrm{Pic}(X)$. Let $L^2(\mathrm{Pic}(X))$ be its completion. This is our Hilbert space $\mathcal{H}_{\mathrm{GL}_1}$. The Hecke operators on $\mathcal{H}_{\mathrm{GL}_1}$ are easy to define as they do not involve integration. Their spectrum was described in [14, Section 2] (not only for GL_1 but also for an arbitrary complex torus), whereas the spectrum of the algebra of global differential operators was described in [8, Section 5]. We recall these descriptions below.

Denote by H_p the Hecke operator associated to a point $p \in X$ and the defining one-dimensional representation of GL_1 . It acts on $L^2(\mathrm{Pic}(X))$ as the pullback with respect to the map sending a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$,

$$\mathcal{L} \mapsto \mathcal{L}(p). \quad (2.1)$$

These operators obviously commute with each other for different $p \in X$.

Recall that $\mathrm{Pic}(X)$ is a disjoint union of connected components $\mathrm{Pic}^n(X)$, $n \in \mathbb{Z}$, labeled by the degrees of line bundles. The Hecke operators shift the degree by 1. Let us fix a point $p_0 \in X$ once and for all. The map (2.1) with $p = p_0$ identifies $\mathrm{Pic}^n(X)$ and $\mathrm{Pic}^{n+1}(X)$ for all $n \in \mathbb{Z}$. This implies (see [14, Section 2.1] for more details) that the spectral theory of the operators H_p , $p \in X$, on $L^2(\mathrm{Pic}(X))$ is equivalent to the spectral theory of the operators

$${}_{p_0}H_p := H_p \circ H_{p_0}^{-1}, \quad p \in X,$$

acting on

$$\mathcal{H}_{\mathrm{GL}_1}^0 := L^2(\mathrm{Pic}^0(X)).$$

Note that ${}_{p_0}H_{p_0}$ is the identity operator.

Next, the Hecke operators ${}_{p_0}H_p$ commute with the algebra $\mathcal{A}_{\mathrm{GL}_1}$ of global differential operators on the Jacobian $\mathrm{Pic}^0(X)$, which is generated by the translation vector fields (both holomorphic and antiholomorphic). Hence they share the same eigenfunctions, namely, the Fourier harmonics on the Jacobian viewed as a real torus. We will now describe a relation between their eigenvalues following [14, Section 2.4] and [8, Section 5].

To describe the eigenfunctions and eigenvalues of the operators ${}_{p_0}H_p$ (following [14, Section 2.4]), recall that as a real torus,

$$\text{Pic}^0(X) \simeq H^1(X, \mathbb{R})^*/H_1(X, \mathbb{Z}), \tag{2.2}$$

where the embedding $H_1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{R})^*$ is defined by sending $\beta \in H_1(X, \mathbb{Z})$ to the linear functional on $H^1(X, \mathbb{R})$,

$$H^1(X, \mathbb{R}) \ni c \mapsto \int_{\beta} c = \int_{\beta} (\omega_c + \bar{\omega}_c), \quad \omega_c \in H^0(X, \Omega^{1,0}). \tag{2.3}$$

Given $\gamma \in H^1(X, \mathbb{Z})$, denote by φ_{γ} the harmonic representative of the image of γ in $H^1(X, \mathbb{R})$. Then

$$\varphi_{\gamma} = \omega_{\gamma} + \bar{\omega}_{\gamma}, \quad \omega_{\gamma} \in H^0(X, \Omega^{1,0}). \tag{2.4}$$

Note that these are precisely the smooth one-forms on X whose integrals over all one-cycles in X are integers.

Now view φ_{γ} as a linear functional on $H^1(X, \mathbb{R})^*$ and set

$$f_{\gamma} = e^{2\pi i \varphi_{\gamma}}, \quad \gamma \in H^1(X, \mathbb{Z}). \tag{2.5}$$

LEMMA 2.1

For any $\gamma \in H^1(X, \mathbb{Z})$, f_{γ} is a well-defined function on $\text{Pic}^0(X)$ given by (2.2).

The functions $f_{\gamma}, \gamma \in H^1(X, \mathbb{Z})$ are the *Fourier harmonics* of the torus $\text{Pic}^0(X)$. They form an orthogonal basis of the Hilbert space $\mathcal{H}_{\text{GL}_1}^0$.

The following statement is proved in [14, Theorem 2.4].

PROPOSITION 2.2

The function $f_{\gamma}, \gamma \in H^1(X, \mathbb{Z})$, is an eigenfunction of the Hecke operators ${}_{p_0}H_p$. The eigenvalue $F_{\gamma}(p)$ of ${}_{p_0}H_p$ on f_{γ} is given by the formula

$$F_{\gamma}(p) = \exp\left(\int_{p_0}^p 2\pi i (\omega_{\gamma} + \bar{\omega}_{\gamma})\right), \tag{2.6}$$

where the integral is taken over any path connecting p_0 to p (the integral does not depend on the choice of this path).

Now consider the algebra of global differential operators on $\text{Pic}^0(X)$,

$$\mathcal{A}_{\text{GL}_1} = D_{\text{GL}_1} \otimes \bar{D}_{\text{GL}_1},$$

where D_{GL_1} (resp., \overline{D}_{GL_1}) is the algebra of global holomorphic (resp., antiholomorphic) differential operators on $\text{Pic}^0(X)$. Since $\text{Pic}^0(X)$ is compact and has an abelian group structure, we have

$$D_{GL_1} = \text{Sym}(\Theta \text{Pic}^0(X)), \quad \overline{D}_{GL_1} = \text{Sym}(\overline{\Theta} \text{Pic}^0(X)),$$

where $\Theta \text{Pic}^0(X)$ (resp., $\overline{\Theta} \text{Pic}^0(X)$) is the commutative Lie algebra of holomorphic (resp., antiholomorphic) translation vector fields on $\text{Pic}^0(X)$. We have

$$\Theta \text{Pic}^0(X) \simeq H^0(X, \Omega^{1,0})^*, \quad \overline{\Theta} \text{Pic}^0(X) \simeq H^0(X, \Omega^{0,1})^*.$$

Therefore, the eigenvalues of the algebra \mathcal{A}_{GL_1} on any joint eigenvector are encoded by a pair (\mathbf{a}, \mathbf{b}) , where $\mathbf{a} \in H^0(X, \Omega^{1,0})$ and $\mathbf{b} \in H^0(X, \Omega^{0,1})$.

We associate to \mathbf{a} a holomorphic GL_1 -oper, that is, the trivial line bundle on X with the holomorphic connection $d + \mathbf{a}$. And we associate to \mathbf{b} an antiholomorphic GL_1 -oper, that is, the trivial line bundle on X with the antiholomorphic connection $d + \mathbf{b}$.

PROPOSITION 2.3 ([8, Theorem 5.4])

(1) *The eigenvalues of \mathcal{A}_{GL_1} on f_γ are given by the pairs*

$$(\mathbf{a}, \mathbf{b}) = (2\pi i \omega_\gamma, 2\pi i \overline{\omega}_\gamma).$$

(2) *The GL_1 -opers $d + 2\pi i \omega_\gamma, \gamma \in H^1(X, \mathbb{Z})$ are all holomorphic GL_1 -opers on X with real monodromy (i.e., the monodromy representation takes values in $GL_1(\mathbb{R}) \subset GL_1(\mathbb{C})$).*

(3) *The spectrum of \mathcal{A}_{GL_1} on $\mathcal{H}_{GL_1}^0$ is in bijection with the set of holomorphic GL_1 -opers on X with real monodromy.*

Combining Propositions 2.2 and 2.3, we obtain the following relation between the eigenvalues of the Hecke operators and \mathcal{A}_{GL_1} .

THEOREM 2.4

Let $F(p)$ and (\mathbf{a}, \mathbf{b}) be the eigenvalues of the Hecke operators ${}_{p_0}H_p, p \in X$, and \mathcal{A}_{GL_1} on a joint eigenfunction in $L^2(\text{Pic}^0(X))$, respectively. Then

$$F(p) = \exp\left(\int_{p_0}^p (\mathbf{a} + \mathbf{b})\right) \tag{2.7}$$

and $\mathbf{b} = -\overline{\mathbf{a}}$.

This is equivalent to the following statement. Denote by ∂ and $\bar{\partial}$ the holomorphic and antiholomorphic de Rham differentials, respectively.

PROPOSITION 2.5

The function $F(p)$ is the single-valued solution of the differential equation

$$dF = (\mathbf{a} + \mathbf{b})F \tag{2.8}$$

or, equivalently, the system

$$\partial F = \mathbf{a}F, \quad \bar{\partial} F = \mathbf{b}F \tag{2.9}$$

normalized so that $F(p_0) = 1$.

Remark 2.1

Theorem 2.4 does not specify the possible values of the one-forms \mathbf{a} and \mathbf{b} that appear in this relation (which we already know from Propositions 2.2 and 2.3). But these values can be readily obtained from the relation (2.7).

Indeed, single-valuedness of the function $F(p)$ given by (2.7) implies that the integrals of the one-form $\mathbf{a} + \mathbf{b}$ over all one-cycles in X are integer multiples of $2\pi i$. This is equivalent to

$$\mathbf{a} + \mathbf{b} = 2\pi i\varphi_\gamma, \tag{2.10}$$

where $\varphi_\gamma = \omega_\gamma + \bar{\omega}_\gamma$ is the harmonic one-form introduced above, for some $\gamma \in H^1(X, \mathbb{Z})$. On the other hand, the self-adjointness on $L^2(\text{Pic}^0(X))$ of the operators of the form $\xi - \bar{\xi}$ and $(\xi + \bar{\xi})/i$, where ξ is any holomorphic translation vector field on $\text{Pic}^0(X)$, implies that

$$\mathbf{b} = -\bar{\mathbf{a}}. \tag{2.11}$$

Combining formulas (2.10) and (2.11), we obtain that

$$\mathbf{a} = 2\pi i\omega_\gamma, \quad \mathbf{b} = 2\pi i\bar{\omega}_\gamma$$

for some $\gamma \in H^1(X, \mathbb{Z})$.

Therefore, we can derive Propositions 2.2 and 2.3 from Theorem 2.4. Thus, Theorem 2.4 (or equivalently, Proposition 2.5) is the key statement that yields explicit formulas for the eigenvalues of both Hecke operators and the global differential operators.

In the rest of this section, we will give an alternative proof of Proposition 2.5 using the action of the Hecke functors ${}_{p_0}\mathbf{H}_p$ (categorical versions of the Hecke operators ${}_{p_0}H_p$) on the sheaf $\mathcal{D}_{\text{GL}_1}$ of differential operators on $\text{Pic}^0(X)$. This will be our blueprint for proving analogous statements for a general group G .

2.2. *Hecke operators and Hecke functors*

Consider the GL_1 version of the Hecke correspondence

$$\mathcal{H}ecke = \{(\mathcal{L}, \mathcal{L}(p_0 - p), p)\} \subset \text{Pic}^0(X) \times \text{Pic}^0(X) \times X, \tag{2.12}$$

and let $q_1, q_2 : \mathcal{H}ecke \rightarrow \text{Pic}^0(X)$ be the two projections and $q_3 : \mathcal{H}ecke \rightarrow X$. We have

$$q_2 \times q_3 : \mathcal{H}ecke \xrightarrow{\sim} \text{Pic}^0(X) \times X$$

and with this isomorphism, q_1 becomes the map

$$q_1 : \text{Pic}^0(X) \times X \rightarrow \text{Pic}^0(X), \tag{2.13}$$

$$\mathcal{L} \mapsto \mathcal{L}(p - p_0). \tag{2.14}$$

Denote by ${}_{p_0}\widehat{H}_p$ the restriction of ${}_{p_0}H_p$ to the dense subspace of C^∞ -functions on $\text{Pic}^0(X)$. This is the operator of pulling back a C^∞ -function under the map q_1 and restricting the result to $\text{Pic}^0(X) \times p$. As p varies along X , these operators combine into a single operator

$${}_{p_0}\widehat{H} : C^\infty(\text{Pic}^0(X)) \rightarrow C^\infty(\text{Pic}^0(X) \times X).$$

We are going to relate it to the *Hecke functor*

$${}_{p_0}\mathbf{H} := q_1^* : \text{Mod}(\mathcal{D}_{\text{Pic}^0(X)}) \rightarrow \text{Mod}(\mathcal{D}_{\text{Pic}^0(X) \times X}),$$

the D -module pullback functor.

2.3. *Another derivation of the differential equations*

We will now derive the differential equations (2.9) appearing in Proposition 2.5 using Corollary 3.3 from Section 3.1, in which we will take $Z = \text{Pic}^0(X) \times X, Y = \text{Pic}^0(X)$ and let q_1 be the map $Z \rightarrow Y$ given by (2.14). Denote the corresponding section $1_{Z \rightarrow Y}$ (see Section 3.1) by 1_{q_1} . The equations (2.9) will follow from Corollary 3.3 and the fact that 1_{q_1} satisfies the differential equation of Proposition 2.9 below.

To prove Proposition 2.9, denote by \mathcal{D}_{GL_1} the sheaf of holomorphic differential operators on $\text{Pic}^0(X)$. The algebra

$$D_{GL_1} = \Gamma(\text{Pic}^0(X), \mathcal{D}_{GL_1}) = \text{Sym}(\Theta_{\text{Pic}^0(X)})$$

acts by endomorphisms of \mathcal{D}_{GL_1} from the right, and this action commutes with the left action of \mathcal{D}_{GL_1} . We know that

$$\text{Spec } D_{GL_1} \simeq H^0(X, \Omega^{1,0}) = \text{Op}_{GL_1}(X).$$

Recall that a (holomorphic) GL_1 -oper on X is a holomorphic connection ∇ on the trivial line bundle on X , so we can write

$$\nabla = \nabla_{\mathbf{a}} := d + \mathbf{a}, \tag{2.15}$$

where $\mathbf{a} \in H^0(X, \Omega^{1,0})$.

The trivial line bundle \mathcal{V} on $\text{Op}_{GL_1}(X) \times X$ is equipped with a partial connection ∇^{univ} along X , whose restriction to $\nabla_{\mathbf{a}} \times X \subset \text{Op}_{GL_1}(X) \times X$ is the connection $\nabla_{\mathbf{a}}$ on X . Thus, $(\mathcal{V}, \nabla^{\text{univ}})$ is the universal GL_1 -oper on $\text{Op}_{GL_1}(X) \times X$. We can write an explicit formula for ∇^{univ} .

Let W be a finite-dimensional vector space, and let $A = \text{Fun } W = \text{Sym } W^*$. The canonical element of $W^* \otimes W$ gives rise to an element of $A \otimes W$. Taking $W = H^0(X, \Omega^{1,0})$, we obtain a holomorphic one-form σ with values in $\text{Fun } \text{Op}_{GL_1}(X)$. Then

$$\nabla^{\text{univ}} = d + \sigma. \tag{2.16}$$

Explicitly, if $\{\omega_i, i = 1, \dots, g\}$ is a basis of $H^0(X, \Omega^{1,0})$ and $\{b_i, i = 1, \dots, g\}$ is the dual basis in $H^0(X, \Omega^{1,0})^* \simeq \Theta \text{Pic}^0(X)$, then

$$\sigma = \sum_{i=1}^g b_i \omega_i. \tag{2.17}$$

Let $\mathcal{V}_X := \pi_*(\mathcal{V})$, where π is the projection $\text{Op}_{GL_1}(X) \times X \rightarrow X$. The connection ∇^{univ} makes \mathcal{V}_X into a left \mathcal{D}_X -module.

Moreover, the unit $1 \in \text{Fun } \text{Op}_{GL_1}(X)$ gives rise to a global section of \mathcal{V}_X , which we denote by $1_{\mathcal{V}_X}$. In addition, \mathcal{V}_X is equipped with an action of $\text{Fun } \text{Op}_{GL_1}(X) \simeq D_{GL_1}$ which commutes with the action of \mathcal{D}_X .

This allows us to define the following D -module on $\text{Pic}^0(X) \times X$:

$$\mathcal{D}_{GL_1} \boxtimes_{D_{GL_1}} \mathcal{V}_X.$$

The algebra D_{GL_1} acts on it by endomorphisms which commute with the action of the sheaf $\mathcal{D}_{\text{Pic}^0(X) \times X}$.

Define its global section s by the formula

$$s := 1 \boxtimes 1_{\mathcal{V}_X}.$$

The element (2.17) gives rise to a linear operator

$$\sigma : \mathcal{D}_{GL_1} \boxtimes_{D_{GL_1}} \mathcal{V}_X \rightarrow \mathcal{D}_{GL_1} \boxtimes_{D_{GL_1}} (\mathcal{V}_X \otimes K_X)$$

(namely, we interpret the b_i as elements of $\text{Sym}(\Theta \text{Pic}^0(X)) = D_{GL_1}$).

The \mathcal{D}_X -module structure on $\mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} \mathcal{V}_X$ allows us to define the action of the holomorphic de Rham differential ∂_X along X on sections of $\mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} \mathcal{V}_X$,

$$\partial_X : \mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} \mathcal{V}_X \rightarrow \mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} (\mathcal{V}_X \otimes K_X).$$

Formula (2.16) readily implies the following.

LEMMA 2.6

The section s satisfies the equation

$$\partial_X s = \sigma \cdot s. \tag{2.18}$$

COROLLARY 2.7

The D -module $\mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} \mathcal{V}_X$ is isomorphic to $\mathcal{D}_{\text{GL}_1} \boxtimes \mathcal{O}_X$ with the action of \mathcal{D}_X on the second factor modified so that the holomorphic de Rham differential ∂_X acts as follows:

$$\partial_X \mapsto 1 \boxtimes \partial_X + \sigma, \tag{2.19}$$

where $\sigma \mapsto \sum_{i=1}^g b_i \boxtimes \omega_i$.

Now recall the D -module $q_1^*(\mathcal{D}_X)$ on $\text{Pic}^0(X) \times X$ and its section 1_{q_1} . The algebra $\mathcal{D}_{\text{GL}_1}$ naturally acts on $q_1^*(\mathcal{D}_X)$ by endomorphisms commuting with the D -module structure.

THEOREM 2.8

There is an isomorphism of D -modules on $\text{Pic}^0(X) \times X$ equipped with a commuting action of $\mathcal{D}_{\text{GL}_1}$,

$$q_1^*(\mathcal{D}_{\text{GL}_1}) \simeq \mathcal{D}_{\text{GL}_1} \boxtimes_{\mathcal{D}_{\text{GL}_1}} \mathcal{V}_X, \tag{2.20}$$

under which the section 1_{q_1} is mapped to s .

Proof

By Corollary 2.7, we need to prove that $q_1^*(\mathcal{D}_{\text{GL}_1}) \simeq \mathcal{D}_{\text{GL}_1} \boxtimes \mathcal{O}_X$, with the modified action of \mathcal{D}_X on the second factor.

Let A be an abelian variety. It comes equipped with the group homomorphism $m : A \times A \rightarrow A$. We claim that $m^*(\mathcal{D}_A) \simeq \mathcal{D}_A \boxtimes \mathcal{O}_A$, with \mathcal{D}_A corresponding to the first factor acting on itself from the left, and \mathcal{D}_A corresponding to the second factor

acting on \mathcal{O}_A in a modified way so that the corresponding action of the holomorphic de Rham differential is given by the formula

$$\partial_A \mapsto 1 \boxtimes \partial_A + \sigma_A, \tag{2.21}$$

where

$$\sigma_A = \sum_i b_i \boxtimes \omega_i.$$

Here $\{\omega_i\}$ is a basis in the space $T_1^*A = H^0(A, K_A)$ and $\{b_i\}$ is the dual basis in the dual vector space T_1A , which we identify with the space $\Theta(A)$ of holomorphic translation vector fields on A (so the b_i can be viewed as global differential operators on A).

To see this, we invoke formula (3.2) from the next section which shows that

$$m^*(\mathcal{D}_A) = \mathcal{D}_{A \times A} / (\mathcal{D}_{A \times A} \cdot \Theta_{A \times A/A}), \tag{2.22}$$

where $\Theta_{A \times A/A}$ is the sheaf of vertical vector fields for the morphism $m : A \times A \rightarrow A$. It is generated by the global translation vector fields on $A \times A$ of the form $\xi \boxtimes 1 - 1 \boxtimes \xi$, where $\xi \in \Theta(A)$. The quotient (2.22) can be identified with $\mathcal{D}_A \boxtimes \mathcal{O}_A$, but then the action of the vector field $1 \boxtimes \xi$ corresponding to the element ξ in the second factor \mathcal{D}_A is given by the action of $\xi \boxtimes 1$ corresponding to the element ξ in the first factor \mathcal{D}_A , which coincides with the above description.

Now we apply this to the case $A = \text{Pic}^0(X)$ (so $\mathcal{D}_A = \mathcal{D}_{\text{GL}_1}$) and observe that

$$q_1^*(\mathcal{D}_{\text{GL}_1}) \simeq (m^*(\mathcal{D}_{\text{GL}_1}))|_{\text{Pic}^0(X) \times X},$$

where m is the multiplication map on $\text{Pic}^0(X)$ and X is embedded into the second factor $\text{Pic}^0(X)$ via the Abel–Jacobi map $X \rightarrow \text{Pic}^0(X)$ sending $p \mapsto \mathcal{O}_X(p - p_0)$.

The theorem now follows from the fact that the linear map from

$$H^0(\text{Pic}^0(X), K_{\text{Pic}^0(X)}) \simeq H^0(X, K_X)$$

to $H^0(X, K_X)$ corresponding to the pullback of a one-form under the Abel–Jacobi map is the identity, so that formula (2.21) becomes (2.19). □

Using Theorem 2.8 and Lemma 2.6, we obtain the sought-after differential equation on 1_{q_1} .

PROPOSITION 2.9

We have $\partial_X 1_{q_1} = \sigma \cdot 1_{q_1}$.

Corollary 3.3, which is proved in Section 3.1 below, gives us a way to produce differential equations on the operator of pullback of functions from differential equations satisfied by the unit section of the \mathcal{D} -module pullback of the sheaf of differential operators. More precisely, we will use Corollary 3.3 in the setting where the map $p : Z \rightarrow Y$ is $q_1 : \text{Pic}^0(X) \times X \rightarrow \text{Pic}^0(X)$, so that $\mathbf{p}_{C^\infty}^{-1} = p_0 \widehat{H}$ and $1_{Z \rightarrow Y} = 1_{q_1}$. Combined with Proposition 2.9, Corollary 3.3 gives us the main result of this section.

THEOREM 2.10

The Hecke operator $p_0 \widehat{H}$, viewed as a smooth function on X with values in operators on $C^\infty(\text{Pic}^0(X))$, satisfies the system of differential equations

$$\partial_X \cdot p_0 \widehat{H} = \sigma \cdot p_0 \widehat{H}, \quad \bar{\partial}_X \cdot p_0 \widehat{H} = \bar{\sigma} \cdot p_0 \widehat{H}. \tag{2.23}$$

This theorem implies Proposition 2.5. Thus, we have obtained an alternative proof of Proposition 2.5 which relies on the Hecke eigensheaf property (Theorem 2.8) of the sheaf $\mathcal{D}_{\text{GL}_1}$ (in the sense of Remark 2.3 below).

In other words, the differential equation on the eigenvalues of the Hecke operators follows from the description of the action of the Hecke functor on the sheaf $\mathcal{D}_{\text{GL}_1}$.

Remark 2.2

The equations (2.23) yield the following explicit formula for the Hecke operators in terms of the translation vector fields:

$$p_0 H_p = \exp\left(\int_{p_0}^p (\sigma + \bar{\sigma})\right) = \exp\left(\int_{p_0}^p \left(\sum_{i=1}^g b_i \omega_i + \sum_{i=1}^g \bar{b}_i \bar{\omega}_i\right)\right), \tag{2.24}$$

where the b_i and \bar{b}_i are viewed as translation vector fields on $\text{Pic}^0(X)$ (see (2.17)), whereas the ω_i and $\bar{\omega}_i$ are one-forms on X which are integrated over a path connecting the points p_0 and p . Integrality properties of the eigenvalues of the b_i and \bar{b}_i then imply that the exponential of this integral does not depend on the choice of such path. This way we obtain another interpretation of the relation between the eigenvalues from Theorem 2.4. However, the path-independence of (2.24) is not obvious, and this is why we prefer to express this relation in terms of the system (2.23).

Remark 2.3

For $\mathfrak{a} \in \text{Op}_{\text{GL}_1}(X) = \text{Spec } D_{\text{GL}_1}$, let $I_{\mathfrak{a}}$ be the corresponding ideal in D_{GL_1} . Define the following D -module on $\text{Pic}^0(X)$:

$$\Delta_{\mathfrak{a}} := \mathcal{D}_{\text{GL}_1} / (\mathcal{D}_{\text{GL}_1} \cdot I_{\mathfrak{a}}).$$

Theorem 2.8 implies that

$$q_1^*(\Delta_{\mathbf{a}}) \simeq \Delta_{\mathbf{a}} \boxtimes \mathcal{V}_{\mathbf{a}},$$

where $\mathcal{V}_{\mathbf{a}} = (\mathcal{O}_X, \nabla_{\mathbf{a}})$ is the flat line bundle corresponding to \mathbf{a} (see (2.15)). The last isomorphism expresses the fact that $\Delta_{\mathbf{a}}$ is the restriction to $\text{Pic}^0(X)$ of a *Hecke eigensheaf* on $\text{Pic}(X)$ with the eigenvalue $\mathcal{V}_{\mathbf{a}}$. In fact, Theorem 2.8 is equivalent to this statement for all $\mathbf{a} \in \text{Op}_{\text{GL}_1}(X)$. In this sense, $\mathcal{D}_{\text{GL}_1}$ is the universal Hecke eigensheaf parameterized by all GL_1 -opers on X .

Here is a categorical interpretation. Recall that a D -module version of the Fourier–Mukai transform (see [26], [28]) establishes an equivalence between the category of coherent D -modules on $\text{Pic}^0(X)$ and the category of coherent sheaves on Loc_{GL_1} , the moduli space of flat line bundles on X (it is isomorphic to an affine bundle over $\text{Pic}^0(X)$). The space $\text{Op}_{\text{GL}_1}(X)$ can be realized as a subvariety of Loc_{GL_1} (it is the fiber over the point corresponding to the trivial line bundle in $\text{Pic}^0(X)$). Let $\text{Coh}(\text{Op}_{\text{GL}_1}(X))$ be the category of coherent sheaves supported (scheme-theoretically) on $\text{Op}_{\text{GL}_1}(X) \subset \text{Loc}_{\text{GL}_1}$. The restriction of the above Fourier–Mukai transform to $\text{Coh}(\text{Op}_{\text{GL}_1}(X))$ gives rise to an equivalence E between this (abelian) category and the category of D -modules \mathcal{K} on $\text{Pic}^0(X)$ with finite global presentation, that is, such that there is an exact sequence

$$\mathcal{D}_{\text{GL}_1}^{\oplus m} \rightarrow \mathcal{D}_{\text{GL}_1}^{\oplus n} \rightarrow \mathcal{K} \rightarrow 0.$$

This equivalence E takes an object \mathcal{F} of $\text{Coh}(\text{Op}_{\text{GL}_1}(X))$ to

$$E(\mathcal{F}) := \mathcal{D}_{\text{GL}_1} \otimes_{\mathcal{D}_{\text{GL}_1}} F, \quad F := \Gamma(\text{Op}_{\text{GL}_1}(X), \mathcal{F}),$$

where we use the fact that $\mathcal{D}_{\text{GL}_1} \simeq \text{Fun Op}_{\text{GL}_1}(X)$ (see [16, Section 2]).

It follows from this definition that $E(\mathcal{O}_{\mathbf{a}}) = \Delta_{\mathbf{a}}$ and $E(\mathcal{O}_{\text{Op}_{\text{GL}_1}(X)}) = \mathcal{D}_{\text{GL}_1}$.

3. Generalities on D -modules and integral transforms

In this section, we discuss the compatibility between natural functors on the categories of (twisted) \mathcal{D} -modules and the corresponding operations on sections of line bundles. Though most of the results of this section are fairly straightforward, we were unable to find them in the literature. We expect that these results are likely to have other applications, so it is worthwhile to record them here.

We note that closely related topics have been discussed in the works by D’Agnolo and Schapira [6] and Goncharov [21], and in fact, we will use a result of [6] below. Also, in writing this section we have benefited from the advice of P. Schapira.

First, we discuss the pullback functor (Section 3.1), then the pullback functor in the setting of twisted differential operators (Section 3.2), then the pushforward

functor (Section 3.3), and finally the integral transform functors associated to correspondences (Section 3.4). We will use the results of the last subsection (specifically, Corollary 3.12) to prove Theorem 1.18 in Section 4. Roughly speaking, the differential equations (1.13) on the Hecke operator will follow from certain properties of the corresponding Hecke functor established in [3].

Remark 3.1

In the case of a curve over a finite field \mathbb{F}_q , one can pass from Hecke eigensheaves to Hecke eigenfunctions using Grothendieck’s *faisceaux-fonctions* correspondence. Namely, taking the trace of the Frobenius (a topological generator of the Galois group of \mathbb{F}_q) on the stalks of a Hecke eigensheaf on Bun_G , we obtain a function on $\text{Bun}_G(\mathbb{F}_q)$. Crucially, this function is a Hecke eigenfunction because Grothendieck’s correspondence is compatible with natural operations on sheaves and functions.

In the case of a curve defined over \mathbb{C} , there is no Frobenius as the field of complex numbers is algebraically closed. Instead, we construct Hecke eigenfunctions as single-valued bilinear combinations of sections of the corresponding Hecke eigensheaf Δ_χ (see formula (1.8)) and sections of a complex conjugate sheaf, as explained in Section 1.5 of [8] and Section 1.3 above. One could argue that this procedure is what replaces taking the traces of the Frobenius in the case of a curve over \mathbb{C} , but the question remains why the resulting function is an eigenfunction of the Hecke operators. The answer is that the results of this section enable us to derive the Hecke eigenfunction property (and to compute the corresponding eigenvalues) from the Hecke eigensheaf property of Δ_χ , using the cyclicity of the Hecke eigensheaf Δ_χ (viewed as a twisted D -module on Bun_G) and the cyclicity of the corresponding “Hecke eigenvalues” (viewed as twisted D -modules on the curve X). Thus, the results established in this section may be viewed as an analogue in the complex case of the compatibility of Grothendieck’s correspondence with natural operations on sheaves and functions.

3.1. Pullback

For a smooth complex manifold Y , denote by \mathcal{O}_Y and \mathcal{D}_Y the sheaves of holomorphic functions and differential operators on Y , respectively (in the analytic topology). Let $\text{Mod}(\mathcal{D}_Y)$ be the category of left \mathcal{D}_Y -modules.

Remark 3.2

In what follows, we can take as \mathcal{D}_Y the sheaf of algebraic differential operators on Y and the category of modules over it. Then we have analogous statements as well.

Given sheaves of algebras \mathcal{A} and \mathcal{B} on Y , an $(\mathcal{A}, \mathcal{B})$ -bimodule is, by definition, a sheaf of modules over $\mathcal{A} \otimes \mathcal{B}^{\text{opp}}$ on Y .

We need to recall some facts about the pullback functor for D -modules. Let $p : Z \rightarrow Y$ be a submersion of smooth complex manifolds. Denote by p^{-1} the sheaf-theoretic pullback functor.

The \mathcal{O} -module pullback functor p^* is defined as follows. If \mathcal{F} is an \mathcal{O}_Y -module, then the \mathcal{O}_Z -module $p^*(\mathcal{F})$ is

$$p^*(\mathcal{F}) := \mathcal{O}_Z \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{F}).$$

If \mathcal{F} is a left \mathcal{D}_Y -module, then $p^*(\mathcal{F})$ has a natural structure of a left \mathcal{D}_Z -module. To explain this, let $\Theta_{Z/Y}$ be the relative tangent sheaf of the morphism $p : Z \rightarrow Y$ (its sections are vertical vector fields on Z with respect to p). This is a sheaf of Lie algebras. It acts by commutator on \mathcal{D}_Z , and this action preserves the left ideal $(\mathcal{D}_Z \cdot \Theta_{Z/Y}) \subset \mathcal{D}_Z$. Moreover, we have

$$(\mathcal{D}_Z / (\mathcal{D}_Z \cdot \Theta_{Z/Y}))^{\Theta_{Z/Y}} \simeq p^{-1}(\mathcal{D}_Y) \tag{3.1}$$

and hence

$$\mathcal{D}_{Z \rightarrow Y} := \mathcal{D}_Z / (\mathcal{D}_Z \cdot \Theta_{Z/Y}) \tag{3.2}$$

is naturally a $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule.

Remark 3.3

Since p is a submersion,

$$\mathcal{D}_{Z \rightarrow Y} \simeq p^*(\mathcal{D}_Y) = \mathcal{O}_Z \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{D}_Y) \tag{3.3}$$

as an $(\mathcal{O}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule, and this is how $\mathcal{D}_{Z \rightarrow Y}$ is usually defined. Defining it by formula (3.2) for such p has the advantage that it makes the \mathcal{D}_Z -module structure on it manifest.

The isomorphism (3.3) implies that the \mathcal{O} -module pullback $p^*(\mathcal{F})$ of a left \mathcal{D}_Y -module \mathcal{F} can be written as

$$p^*(\mathcal{F}) \simeq \mathcal{D}_{Z \rightarrow Y} \otimes_{p^{-1}(\mathcal{D}_Y)} p^{-1}(\mathcal{F}),$$

and hence the \mathcal{O}_Z -module structure on $p^*(\mathcal{F})$ naturally extends to a \mathcal{D}_Z -module structure. Thus, we obtain the pullback functor for D -modules, which we denote in the same way as the \mathcal{O} -module pullback:

$$p^* : \text{Mod}(\mathcal{D}_Y) \rightarrow \text{Mod}(\mathcal{D}_Z).$$

On the other hand, let \mathcal{C}_Y^∞ and \mathcal{C}_Z^∞ be the sheaves of \mathbb{C} -valued C^∞ -functions on the smooth real manifolds underlying Y and Z , respectively (in the analytic topology). Consider the internal Hom sheaf

$$\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)$$

on Z in the category of sheaves of vector spaces. By definition, for an open subset $U \subset Z$,

$$\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)(U) := \text{Hom}(p^{-1}(\mathcal{C}_Y^\infty)|_U, \mathcal{C}_Z^\infty|_U) \tag{3.4}$$

(as is well known, the presheaf defined by this formula is a sheaf). In particular, we have a special global section $\mathbf{p}_{C^\infty}^{-1}$ of $\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)$ over Z , which corresponds to the pullback of smooth functions from Y to Z . More precisely, for any open $U \subset Z$, the restriction of $\mathbf{p}_{C^\infty}^{-1}$ to U maps $f \in (p^{-1}(\mathcal{C}_Y^\infty))(U) = C^\infty(p(U))$ to its pullback to U via p (note that $p(U)$ is open because p is a submersion).

LEMMA 3.1

The sheaf $\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)$ has a natural $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule structure, which is defined as follows: for every open $U \subset Z$, given $P \in \mathcal{D}_Z(U)$, $Q \in (p^{-1}(\mathcal{D}_Y))(U) = \mathcal{D}_Y(p(U))$, and $\phi \in \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)(U)$, which is a compatible system $\{\phi_{U'} \in \text{Hom}(C^\infty(p(U')), C^\infty(U'))\}$ for open subsets $U' \subset U$,

$$\phi \mapsto P \circ \phi \circ Q,$$

where $P \circ \phi \circ Q$ stands for the compatible system $\{P|_{U'} \circ \phi_{U'} \circ Q|_{p(U')} \mid U' \subset U\}$.

On the other hand, we have the $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule $\mathcal{D}_{Z \rightarrow Y}$ given by (3.2). The unit $1_Y \in \mathcal{D}_Y$ gives rise to a global section $1_{Z \rightarrow Y}$ of $\mathcal{D}_{Z \rightarrow Y}$.

PROPOSITION 3.2

There is a unique injective homomorphism of $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodules

$$\mathcal{D}_{Z \rightarrow Y} \rightarrow \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)$$

sending $1_{Z \rightarrow Y}$ to $\mathbf{p}_{C^\infty}^{-1}$.

Proof

Let

$$\mathcal{A}_{Z,Y} := \mathcal{D}_Z \cdot \mathbf{p}_{C^\infty}^{-1}.$$

For every open $U \subset Z$, we have

$$\xi \cdot \mathbf{p}_{C^\infty}^{-1}(f) = 0, \quad \forall \xi \in \Theta_{Z/Y}(U), \forall f \in C^\infty(p(U)).$$

It follows that $\mathcal{A}_{Z,Y}$ is naturally isomorphic to the right-hand side of (3.2) and hence to $\mathcal{D}_{Z \rightarrow Y}$ as a left \mathcal{D}_Z -module. The isomorphism (3.1) defines the structure of a $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule on both $\mathcal{A}_{Z,Y}$ and $\mathcal{D}_{Z \rightarrow Y}$. It is clear that the former is compatible with the $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$ -bimodule structure on $\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)$. \square

Remark 3.4

Concretely, we can choose a sufficiently fine open covering of Z so that each neighborhood is isomorphic to the product of two balls $B_m \in \mathbb{C}^m$ and $B_n \in \mathbb{C}^n$ and the restriction of the map p to it is isomorphic to the projection $B_m \times B_n \rightarrow B_m$. Let $\{y_1, \dots, y_m\}$ and $\{x_1, \dots, x_n\}$ be coordinates on B_m and B_n . It is clear that the spaces of sections of both $\mathcal{A}_{Z,Y}$ and $\mathcal{D}_{Z \rightarrow Y}$ are both isomorphic to $\mathcal{H}ol(y_1, \dots, y_m, x_1, \dots, x_n) \otimes \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}]$, where $\mathcal{H}ol(y_1, \dots, y_m, x_1, \dots, x_n)$ denotes the space of holomorphic functions on $B_m \times B_n$.

COROLLARY 3.3

Suppose that $P \cdot 1_{Z \rightarrow Y} = 0$ for some $P \in \Gamma(Z, \mathcal{D}_Z)$. Then

$$P \cdot \mathbf{p}_{C^\infty}^{-1} = 0, \quad \overline{P} \cdot \mathbf{p}_{C^\infty}^{-1} = 0. \tag{3.5}$$

3.2. Pullback in the twisted setting

The results of Section 3.1 can be generalized to twisted differential operators. Namely, let \mathcal{L} be a holomorphic line bundle on Y . Recall that in Section 1.1, using the norm map $a \mapsto \|a\| = |a|^2$ from \mathbb{C}^\times to $\mathbb{R}_{>0}$, we associated to \mathcal{L} a C^∞ complex line bundle

$$\|\mathcal{L}\| = |\mathcal{L}|^2$$

with the structure group $\mathbb{R}_{>0}$ on Y , viewed as a complex analytic variety. Clearly,

$$|\mathcal{L}|^2 \simeq (\mathcal{L} \otimes \overline{\mathcal{L}}) \otimes_{\mathcal{O}_Y \otimes \overline{\mathcal{O}_Y}} \mathcal{C}_Y^\infty. \tag{3.6}$$

In other words, if the transition functions of \mathcal{L} are $\{g_{\alpha\beta}\}$, then the transition functions of $|\mathcal{L}|^2$ are $\{|g_{\alpha\beta}|^2\}$.

Consider the sheaf of (twisted) differential operators acting on \mathcal{L} (see [2]),

$$\mathcal{D}_{Y,\mathcal{L}} := \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1}.$$

Likewise, we have the sheaf $\mathcal{D}_{Z,p^*\mathcal{L}}$ of differential operators acting on the line bundle $p^*(\mathcal{L})$ on Z .

The role of $\mathcal{D}_{Z \rightarrow Y}$ is now played by

$$\begin{aligned} \mathcal{D}_{Z \rightarrow Y, \mathcal{L}} &:= p^*(\mathcal{L}) \otimes_{\mathcal{O}_Z} p^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1}) \\ &\simeq p^*(\mathcal{L}) \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \rightarrow Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{L}), \end{aligned} \tag{3.7}$$

which is naturally a $(\mathcal{D}_{Z, p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y, \mathcal{L}}))$ -bimodule. The unit $1_Y \in \mathcal{D}_Y$ gives rise to a global section $1_{Z \rightarrow Y, \mathcal{L}}$ of $\mathcal{D}_{Z \rightarrow Y, \mathcal{L}}$.

On the other hand, let $\mathcal{C}_{Y, |\mathcal{L}|^2}^\infty$ and $\mathcal{C}_{Z, |p^*(\mathcal{L})|^2}^\infty$ be the sheaves of C^∞ -sections of the line bundles $|\mathcal{L}|^2$ on Y and $|p^*(\mathcal{L})|^2$ on Z , respectively. Then $\mathcal{D}_{Y, \mathcal{L}}$ naturally acts on $\mathcal{C}_{Y, |\mathcal{L}|^2}^\infty$ and $\mathcal{D}_{Z, p^*(\mathcal{L})}$ naturally acts on $\mathcal{C}_{Z, |p^*(\mathcal{L})|^2}^\infty$.

Consider the internal Hom sheaf $\mathcal{H}om(p^{-1}(\mathcal{C}_{Y, |\mathcal{L}|^2}^\infty), \mathcal{C}_{Z, |p^*(\mathcal{L})|^2}^\infty)$ on Z and its global section $\mathbf{p}_{\mathcal{L}}^{-1}$ corresponding to the natural pullback map

$$\mathbf{p}_{\mathcal{L}}^{-1} : C^\infty(Y, |\mathcal{L}|^2) \rightarrow C^\infty(Z, |p^*(\mathcal{L})|^2). \tag{3.8}$$

The sheaf $\mathcal{H}om(p^{-1}(\mathcal{C}_{Y, |\mathcal{L}|^2}^\infty), \mathcal{C}_{Z, |p^*(\mathcal{L})|^2}^\infty)$ has the structure of a $(\mathcal{D}_{Z, p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y, \mathcal{L}}))$ -bimodule defined in the same way as in Lemma 3.1.

PROPOSITION 3.4

There is a unique injective homomorphism of $(\mathcal{D}_{Z, p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y, \mathcal{L}}))$ -bimodules

$$\mathcal{D}_{Z \rightarrow Y, \mathcal{L}} \rightarrow \mathcal{H}om(p^{-1}(\mathcal{C}_{Y, |\mathcal{L}|^2}^\infty), \mathcal{C}_{Z, |p^*(\mathcal{L})|^2}^\infty)$$

sending $1_{Z \rightarrow Y, \mathcal{L}}$ to $\mathbf{p}_{\mathcal{L}}^{-1}$.

Proof

According to the definition (3.7), we have

$$\mathcal{D}_{Z \rightarrow Y, \mathcal{L}} \simeq p^*(\mathcal{L}) \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \rightarrow Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{L}). \tag{3.9}$$

Moreover, isomorphisms (3.2) and (3.1) imply similar isomorphisms in the twisted case (note that $\Theta_{Z/Y}$ naturally embeds into $\mathcal{D}_{Z, p^*(\mathcal{L})}$):

$$\mathcal{D}_{Z \rightarrow Y, \mathcal{L}} \simeq \mathcal{D}_{Z, p^*(\mathcal{L})} / (\mathcal{D}_{Z, p^*(\mathcal{L})} \cdot \Theta_{Z/Y}), \tag{3.10}$$

$$p^{-1}(\mathcal{D}_{Y, \mathcal{L}}) \simeq (\mathcal{D}_{Z, p^*(\mathcal{L})} / (\mathcal{D}_{Z, p^*(\mathcal{L})} \cdot \Theta_{Z/Y}))^{\Theta_{Z/Y}}. \tag{3.11}$$

Now we argue in the same way as in the proof of Proposition 3.2. Let

$$\mathcal{A}_{Z, Y, \mathcal{L}} := \mathcal{D}_{Z, p^*(\mathcal{L})} \cdot \mathbf{p}_{\mathcal{L}}^{-1}.$$

For every open $U \subset Z$, we have

$$\xi \cdot \mathbf{p}_{\mathcal{L}}^{-1}(f) = 0, \quad \forall \xi \in \Theta_{Z/Y}(U), \forall f \in C^\infty(p(U), \mathcal{L}).$$

By restricting to sufficiently small open subsets, we obtain from Proposition 3.2 that $\mathcal{A}_{Z,Y,\mathcal{L}}$ is naturally isomorphic to the right-hand side of (3.10) and hence to $\mathcal{D}_{Z \rightarrow Y, \mathcal{L}}$ as a left $\mathcal{D}_{Z,p^*(\mathcal{L})}$ -module. Using the isomorphism (3.11), we obtain a $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$ -bimodule structure on both $\mathcal{A}_{Z,Y,\mathcal{L}}$ and $\mathcal{D}_{Z \rightarrow Y, \mathcal{L}}$ and therefore an isomorphism between $\mathcal{A}_{Z,Y,\mathcal{L}}$ and $\mathcal{D}_{Z \rightarrow Y, \mathcal{L}}$ as $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$ -bimodules, sending $1_{Z \rightarrow Y, \mathcal{L}}$ to $\mathbf{p}_{\mathcal{L}}^{-1}$. By construction, the former is precisely the $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$ -submodule of $\mathcal{H}om(p^{-1}(\mathcal{C}_{Y,|\mathcal{L}|^2}^\infty), \mathcal{C}_{Z,|p^*(\mathcal{L})|^2}^\infty)$ generated by $\mathbf{p}_{\mathcal{L}}^{-1}$. \square

3.3. Pushforward

Suppose that $p : Z \rightarrow Y$ is a submersion with compact fibers, and denote by $K_{Z/Y}$ the corresponding relative canonical bundle. Then

$$\Omega_{Z/Y} := |K_{Z/Y}|^2$$

is the C^∞ line bundle of relative densities. Let K_Y and K_Z be the canonical line bundles on Y and Z , respectively, and let $\Omega_Y := |K_Y|^2$ and $\Omega_Z := |K_Z|^2$ be the C^∞ line bundles of densities on Y and Z , respectively. We have

$$\Omega_{Z/Y} \simeq \Omega_Z \otimes_{\mathcal{O}_Z} p^*(\Omega_Y^{-1}).$$

Denote by p_* the sheaf-theoretic pushforward functor. Let again \mathcal{L} be a line bundle on Y . Consider the internal Hom sheaf (where \otimes stands for $\otimes_{\mathcal{C}_Y^\infty}$ or $\otimes_{\mathcal{C}_Z^\infty}$)

$$\mathcal{H}om(p_*(\mathcal{C}_{Z,|p^*(\mathcal{L})|^2 \otimes \Omega_Z}^\infty), \mathcal{C}_{Y,|\mathcal{L}|^2 \otimes \Omega_Y}^\infty) \simeq \mathcal{H}om(p_*(\mathcal{C}_{Z,|p^*(\mathcal{L})|^2 \otimes \Omega_{Z/Y}}^\infty), \mathcal{C}_{Y,|\mathcal{L}|^2}^\infty)$$

on Y and its global section $\mathbf{p}_!^{\mathcal{L}}$ corresponding to the integration map

$$\mathbf{p}_!^{\mathcal{L}} : C^\infty(Z, |p^*(\mathcal{L})|^2 \otimes \Omega_{Z/Y}) \rightarrow C^\infty(Y, |\mathcal{L}|^2). \tag{3.12}$$

Set

$$\mathcal{M} := p^*(\mathcal{L}) \otimes_{\mathcal{O}_Z} K_{Z/Y}$$

so that

$$|\mathcal{M}|^2 = |p^*(\mathcal{L})|^2 \otimes_{\mathcal{C}_Z^\infty} \Omega_{Z/Y}.$$

LEMMA 3.5

The sheaf $\mathcal{H}om(p_*(\mathcal{C}_{Z,|\mathcal{M}|^2}^\infty), \mathcal{C}_{Y,|\mathcal{L}|^2}^\infty)$ has a natural $(\mathcal{D}_{Y,\mathcal{L}}, p_*(\mathcal{D}_{Z,\mathcal{M}}))$ -bimodule structure, which is defined as follows: for every open $U \subset Y$, given $R \in (p_*(\mathcal{D}_{Z,\mathcal{M}}))(U) = \mathcal{D}_{Z,\mathcal{M}}(p^{-1}(U))$, $S \in \mathcal{D}_{Y,\mathcal{L}}(U)$, and $\phi \in \mathcal{H}om(p_*(\mathcal{C}_{Z,|\mathcal{M}|^2}^\infty), \mathcal{C}_{Y,|\mathcal{L}|^2}^\infty)(U)$, which is a compatible system $\{\phi_{U'} \in \text{Hom}(C^\infty(p^{-1}(U')), |\mathcal{M}|^2), C^\infty(U', |\mathcal{L}|^2)\}$ for open subsets $U' \subset U$,

$$\phi \mapsto S \circ \phi \circ R,$$

where $S \circ \phi \circ R$ stands for the compatible system $\{S|_{U'} \circ \phi_{U'} \circ R|_{p^{-1}(U')} \mid U' \subset U\}$.

On the other hand, denote by p_*^D the (derived) D -module pushforward functor

$$p_*^D : D^b(\mathcal{D}_Z) \rightarrow D^b(\mathcal{D}_Y).$$

If \mathcal{F} is a \mathcal{D}_Z -module, then it follows from the definition (see [22, Section 4.6]) that

$$p_*^D(\mathcal{F}) = R^\bullet p_*(\mathcal{D}_{Y \leftarrow Z} \overset{L}{\otimes}_{\mathcal{D}_Z} \mathcal{F}), \tag{3.13}$$

where

$$\mathcal{D}_{Y \leftarrow Z} := K_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \rightarrow Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(K_Y^{-1})$$

is a sheaf on Z with a natural structure of a $(p^{-1}(\mathcal{D}_Y), \mathcal{D}_Z)$ -bimodule.

The right action of \mathcal{D}_Y on K_Y gives rise to a canonical isomorphism

$$\mathcal{D}_Y^{\text{opp}} \simeq K_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} K_Y^{-1},$$

which yields an identification (see [22, Remark 4.18])

$$\mathcal{D}_{Y \leftarrow Z} \simeq p^{-1}(\mathcal{D}_Y) \otimes_{p^{-1}(\mathcal{O}_Y)} K_{Z/Y}. \tag{3.14}$$

Let us apply $p_*^{D,0} := H^0 p_*^D$ to the left \mathcal{D}_Z -module $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1}$. Since it is free, $\overset{L}{\otimes}$ in formula (3.13) is the ordinary \otimes . Note also that it carries a commuting right $\mathcal{D}_{Z,\mathcal{M}}$ -module structure and hence is a $(\mathcal{D}_Z, \mathcal{D}_{Z,\mathcal{M}})$ -bimodule. This implies that the sheaf $p_*^{D,0}(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1})$ is a $(\mathcal{D}_Y, p_*(\mathcal{D}_{Z,\mathcal{M}}))$ -bimodule.

Setting

$$\mathcal{D}_{Y \leftarrow Z, \mathcal{M}} := \mathcal{D}_{Y \leftarrow Z} \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1} \simeq p^{-1}(\mathcal{D}_Y) \otimes_{p^{-1}(\mathcal{O}_Y)} p^*(\mathcal{L}^{-1}),$$

we obtain an isomorphism

$$p_*^{D,0}(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1}) \simeq p_*(\mathcal{D}_{Y \leftarrow Z, \mathcal{M}}).$$

Define the following $(\mathcal{D}_{Y, \mathcal{L}}, p_*(\mathcal{D}_{Z, \mathcal{M}}))$ -bimodule on Y :

$$\widetilde{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}} := \mathcal{L} \otimes_{\mathcal{O}_Y} p_*^{D,0}(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1}) \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} p_*(\mathcal{D}_{Y \leftarrow Z, \mathcal{M}}).$$

It follows from the definition of the functor $p_*^{D,0}$ that the unit $1_Z \in \mathcal{D}_Z$ gives rise to a global section $1_{Y \leftarrow Z, \mathcal{L}}$ of $\widetilde{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}}$.

PROPOSITION 3.6

There is a unique injective homomorphism of $(\mathcal{D}_{Y, \mathcal{L}}, p_*(\mathcal{D}_{Z, \mathcal{M}}))$ -bimodules

$$\widetilde{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}} \rightarrow \mathcal{H}om(p_*(\mathcal{C}_{Z, |\mathcal{M}|^2}^\infty), \mathcal{C}_{Y, |\mathcal{L}|^2}^\infty)$$

sending $1_{Y \leftarrow Z, \mathcal{L}}$ to $\mathbf{p}_1^\mathcal{L}$.

Proof

Let us prove the statement in the case $\mathcal{L} = \mathcal{O}_Y$, so that $\mathcal{M} = K_{Z/Y}$.

Recall the relative tangent sheaf $\Theta_{Z/Y}$. The sheaf $p_*(\Theta_{Z/Y})$ naturally acts on $p_*(K_{Z/Y})$ and on $p_*(\mathcal{C}_{Z, \Omega_{Z/Y}}^\infty)$ by Lie derivatives. Hence $\Theta_{Z/Y}$ embeds into $p_*(\mathcal{D}_{Z, K_{Z/Y}})$ as a subsheaf of Lie algebras, and so it acts on $p_*(\mathcal{D}_{Z, K_{Z/Y}})$ by commutators. For every open $U \subset Y$, we have

$$\mathbf{p}_1^\mathcal{L}(\xi \cdot f) = 0, \quad \forall \xi \in p_*(\Theta_{Z/Y})(U), \forall f \in C^\infty(p^{-1}(U), \Omega_{Z/Y}).$$

It follows that

$$p_*(\mathcal{D}_{Z, K_{Z/Y}})^{\text{opp}} \cdot \mathbf{p}_1^\mathcal{L} \simeq p_*(\mathcal{D}_{Z, K_{Z/Y}}) / (p_*(\Theta_{Z/Y}) \cdot p_*(\mathcal{D}_{Z, K_{Z/Y}})).$$

Formula (3.14) implies that

$$p_*(\mathcal{D}_{Z, K_{Z/Y}}) / (p_*(\Theta_{Z/Y}) \cdot p_*(\mathcal{D}_{Z, K_{Z/Y}})) \simeq \widetilde{\mathcal{D}}_{Y \leftarrow Z, K_{Z/Y}}.$$

This completes the proof of the proposition for $\mathcal{L} = \mathcal{O}_Y$, so that $\mathcal{M} = K_{Y/Z}$. For a general line bundle \mathcal{L} , the statement of the proposition is derived in a similar way (cf. Propositions 3.2 and 3.4). □

3.4. Integral transforms

Consider a correspondence

$$\begin{array}{ccc}
 & Z & \\
 p_1 \swarrow & & \searrow p_2 \\
 Y_1 & & Y_2
 \end{array} \tag{3.15}$$

Assume that p_1 and p_2 are submersions with compact fibers. Let \mathcal{L}_1 and \mathcal{L}_2 be line bundles on Y_1 and Y_2 , respectively, such that

$$p_1^*(\mathcal{L}_1) \simeq p_2^*(\mathcal{L}_2) \otimes_{\mathcal{O}_Z} K_{Z/Y_2}. \tag{3.16}$$

In what follows, we will fix such an isomorphism.

Consider the Hom sheaf on Y_2 ,

$$\mathcal{H}om(p_{2*}p_1^{-1}(\mathcal{C}_{Y_1, \mathcal{L}_1}^\infty), \mathcal{C}_{Y_2, \mathcal{L}_2}^\infty). \tag{3.17}$$

Combining Lemmas 3.1 and 3.5, we obtain that it is a $(\mathcal{D}_{Y_2, \mathcal{L}_2}, p_{2*}p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodule.

Recall that we have a section

$$\begin{aligned}
 \mathbf{p}_{2!}^{\mathcal{L}_2} &\in \mathcal{H}om(p_{2*}(\mathcal{C}_{Z, |p_2^*(\mathcal{L}_2)|^2 \otimes \Omega_{Z/Y_2}}^\infty), \mathcal{C}_{Y_2, |\mathcal{L}_2|^2}^\infty) \\
 &= \mathcal{H}om(p_{2*}(\mathcal{C}_{Z, |p_1^*(\mathcal{L}_1)|^2}^\infty), \mathcal{C}_{Y_2, |\mathcal{L}_2|^2}^\infty)
 \end{aligned}$$

(here we apply the isomorphism (3.16)). We also have a section

$$\mathbf{p}_{1, \mathcal{L}_1}^{-1} \in \mathcal{H}om(p_1^{-1}(\mathcal{C}_{Y_1, |\mathcal{L}_1|^2}^\infty), \mathcal{C}_{Z, |p_1^*(\mathcal{L}_1)|^2}^\infty)$$

which gives rise to a section

$$p_{2*}(\mathbf{p}_{1, \mathcal{L}_1}^{-1}) \in \mathcal{H}om(p_{2*}p_1^{-1}(\mathcal{C}_{Y_1, |\mathcal{L}_1|^2}^\infty), p_{2*}(\mathcal{C}_{Z, |p_1^*(\mathcal{L}_1)|^2}^\infty)).$$

Denote by $H_{Z, Y_1, Y_2}^{\mathcal{L}_1, \mathcal{L}_2}$, or H_Z for short, the composition

$$H_Z := \mathbf{p}_{2!}^{\mathcal{L}_2} \circ p_{2*}(\mathbf{p}_{1, \mathcal{L}_1}^{-1}),$$

which is a section of the sheaf (3.17). This is the *integral transform* associated to the correspondence Z and the line bundles \mathcal{L}_1 and \mathcal{L}_2 .

On the other hand, recall the $(\mathcal{D}_{Z, p^*(\mathcal{L}_1)}, p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodule $\mathcal{D}_{Z \rightarrow Y_1, \mathcal{L}_1}$ and the $(\mathcal{D}_{Y, \mathcal{L}_2}, p_{2*}(\mathcal{D}_{Z, \mathcal{M}_2}))$ -bimodule $\widehat{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}_2}$, where

$$\mathcal{M}_2 = p_2^*(\mathcal{L}_2) \otimes_{\mathcal{O}_Z} K_{Z/Y_2}.$$

Formula (3.16) implies that

$$\mathcal{D}_{Z,p^*(\mathcal{L}_1)} \simeq \mathcal{D}_{Z,\mathcal{M}_2}.$$

Therefore, we can form the tensor product

$$\mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\mathcal{L}_1, \mathcal{L}_2} := \widetilde{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}_2} \otimes_{p_{2*}(\mathcal{D}_{Z, \mathcal{M}_2})} p_{2*}(\mathcal{D}_{Z \rightarrow Y_1, \mathcal{L}_1}), \tag{3.18}$$

which is naturally a $(\mathcal{D}_{Y_2, \mathcal{L}_2}, p_{2*} p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodule. The canonical sections of $\widetilde{\mathcal{D}}_{Y \leftarrow Z, \mathcal{M}_2}$ and $\mathcal{D}_{Z \rightarrow Y_1, \mathcal{L}_1}$ introduced above give us a global section of $\mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\mathcal{L}_1, \mathcal{L}_2}$, which we denote by $1_{Y_2 \leftarrow Z \rightarrow Y_1}$.

Propositions 3.4 and 3.6 imply the following.

PROPOSITION 3.7

There is a unique homomorphism of $(\mathcal{D}_{Y_2, \mathcal{L}_2}, p_{2*} p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodules

$$\mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\mathcal{L}_1, \mathcal{L}_2} \rightarrow \mathcal{H}om(p_{2*} p_1^{-1}(\mathcal{C}_{Y_1, |\mathcal{L}_1|^2}^\infty), \mathcal{C}_{Y_2, |\mathcal{L}_2|^2}^\infty)$$

sending $1_{Y_2 \leftarrow Z \rightarrow Y_1}$ to H_Z .

This proposition has the following obvious corollary.

COROLLARY 3.8

Suppose that $P \cdot 1_{Y_2 \leftarrow Z \rightarrow Y_1} = 0$ for some $P \in \Gamma(Y_2, \mathcal{D}_{Y_2, \mathcal{L}_2})$. Then

$$P \cdot H_Z = 0, \quad \overline{P} \cdot H_Z = 0.$$

In other words, holomorphic differential equations satisfied by the section $1_{Y_2 \leftarrow Z \rightarrow Y_1}$ give rise to differential equations on the integral transform H_Z obtained from the correspondence Z .

Now we want to connect this to the D -module integral transform functor associated to the correspondence Z . Namely, we have the functor

$$\mathbf{H}_Z^\bullet : D^b(\mathcal{D}_{Y_1}) \rightarrow D^b(\mathcal{D}_{Y_2}), \tag{3.19}$$

$$\mathcal{E} \mapsto p_{2*}^D(p_1^*(\mathcal{E})). \tag{3.20}$$

Let

$$\mathcal{H}_Z := \mathcal{L}_2 \otimes_{\mathcal{O}_{Y_2}} \mathbf{H}_Z^0(\mathcal{D}_{Y_1} \otimes_{\mathcal{O}_{Y_1}} \mathcal{L}^{-1}). \tag{3.21}$$

This is a $(\mathcal{D}_{Y_2, \mathcal{L}_2}, p_{2*} p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodule. The unit 1_{Y_1} gives rise to a global section of \mathcal{H}_Z , which we denote by ψ_Z .

Recall formula (3.13) for the functor p_{2*}^D . The next result follows from [6, Proposition 2.12].

PROPOSITION 3.9 ([6])

Suppose that the map $Z \rightarrow Y_1 \times Y_2$ is a closed embedding. Then $\mathcal{D}_{Y_2 \leftarrow Z} \overset{L}{\otimes}_{\mathcal{D}_Z} \mathcal{D}_{Z \rightarrow Y_1}$ is concentrated in cohomological degree 0.

This proposition has the following corollary.

COROLLARY 3.10

If $Z \rightarrow Y_1 \times Y_2$ is a closed embedding, then there is an isomorphism of $(\mathcal{D}_{Y_2, \mathcal{L}_2}, p_{2*} p_1^{-1}(\mathcal{D}_{Y_1, \mathcal{L}_1}))$ -bimodules

$$\mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\mathcal{L}_1, \mathcal{L}_2} \simeq \mathcal{H}_Z \tag{3.22}$$

under which $1_{Y_2 \leftarrow Z \rightarrow Y_1}$ is mapped to ψ_Z .

Combining Corollaries 3.8 and 3.10, we obtain the following.

COROLLARY 3.11

Suppose that $Z \rightarrow Y_1 \times Y_2$ is a closed embedding. If $P \cdot \psi_Z = 0$ for some $P \in \Gamma(Y_2, \mathcal{D}_{Y_2, \mathcal{L}_2})$, then

$$P \cdot H_Z = 0, \quad \overline{P} \cdot H_Z = 0.$$

In our proof of Theorem 1.18 in the next section, we will need the following variant of Corollary 3.11. Let \mathcal{L}'_2 be another line bundle on Y_2 , and let

$$\mathcal{D}_{Y_2, \mathcal{L}_2}^{\mathcal{L}'_2} := \mathcal{L}'_2 \otimes_{\mathcal{O}_{Y_2}} \mathcal{L}_2^{-1} \otimes_{\mathcal{O}_{Y_2}} \mathcal{D}_{Y_2, \mathcal{L}_2},$$

the sheaf of differential operators acting from \mathcal{L}_2 to \mathcal{L}'_2 on Y_2 .

COROLLARY 3.12

Suppose that $Z \rightarrow Y_1 \times Y_2$ is a closed embedding. If $P \cdot \psi_Z = 0$ for some $P \in \Gamma(Y_2, \mathcal{D}_{Y_2, \mathcal{L}_2}^{\mathcal{L}'_2})$, then

$$P \cdot H_Z = 0, \quad \overline{P} \cdot H_Z = 0.$$

4. The case of PGL_n

In this section, we consider the case of $G = \text{PGL}_n$ (so that ${}^L G = \text{SL}_n$) and the Hecke correspondence $\overline{Z}(\omega_1)$ associated to $\lambda = \omega_1$, the first fundamental coweight of PGL_n .

There are two advantages in this case that we will exploit: (1) $\overline{Z}(\omega_1) = Z(\omega_1)$, that is, the fibers of $q_2 \times q_3$ are isomorphic to a closed $G[[z]]$ -orbit Gr_{ω_1} , which is smooth and compact (in fact, $\text{Gr}_{\omega_1} \simeq \mathbb{P}^{n-1}$); and (2) the special section (corresponding to the oper Borel reduction) of the universal oper bundle \mathcal{V}_{ω_1} satisfies an n th order differential equation (see Lemma 1.15). Hence we will be able to apply Corollary 3.12 in the case of the Hecke correspondence $Z(\omega_1)$ to derive the differential equations (1.13) on the Hecke operator \widehat{H}_{ω_1} and thus prove Theorem 1.18.

Note that we also have the multiplicity-one property (see Proposition 1.13 and Theorem 1.12). We will use the notation of Section 1.2.

4.1. Hecke functor

We follow the definition of the Hecke functor $\mathbf{H}_{\omega_1}^\bullet$ given in [3] (where it is denoted by $T_{\omega_1}^\bullet$), taking into account the fact that in this case the fibers of the morphism $q_2 \times q_3$ are isomorphic to $\overline{\text{Gr}}_{\omega_1} \simeq \mathbb{P}^{n-1}$ and hence are smooth. The following definition is taken from [3, Section 5.2.4].

Definition 4.1 ([3])

The Hecke functor

$$\mathbf{H}_{\omega_1}^\bullet : D^b(\mathcal{D}_{\text{BunPGL}_n}) \rightarrow D^b(\mathcal{D}_{\text{BunPGL}_n \times X})$$

is defined as follows. For a left D -module M on BunPGL_n ,

$$\mathbf{H}_{\omega_1}^\bullet(M) := (q_2 \times q_3)_*^D(q_1^*(M)), \tag{4.1}$$

where $(q_2 \times q_3)_*^D$ denotes the derived direct image functor for D -modules.

Denote by $\mathbf{H}_{\omega_1}^i$ the corresponding i th cohomology functor. We now recall a theorem of Beilinson and Drinfeld [3] describing the action of $\mathbf{H}_{\omega_1}^i$ on a specific D -module on BunPGL_n .

If n is even, then to define this D -module, we need to pick a square root $K_X^{1/2}$ of the canonical line bundle on X (as in Section 1.4). To it, one associates a specific square root

$$\mathcal{L} = K_{\text{Bun}}^{1/2} \tag{4.2}$$

of the canonical line bundle on BunPGL_n following [3, Section 4] (see also [25]). If n is odd, then the construction of \mathcal{L} does not require any choices.

Let $\mathcal{D}_{\text{BunPGL}_n}$ be the sheaf of differential operators on BunPGL_n . Then

$$\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}$$

is a left D -module on $\text{Bun}_{\text{PGL}_n}$ equipped with the commuting right action of the algebra

$$D_{\text{PGL}_n} = D_{\text{PGL}_n}^\beta = \Gamma(\text{Bun}_{\text{PGL}_n}^\beta, \mathcal{L} \otimes \mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}).$$

Recall from Definition 1.5 that the right-hand side does not depend on β . Moreover, we have

$$D_{\text{PGL}_n} \simeq \text{Fun Op}_{\text{SL}_n}^\gamma(X)$$

according to the result of [3] which is quoted in Theorem 1.6 above. Here γ denotes the isomorphism class of $K_X^{1/2}$, as in Theorem 1.6.

4.2. Hecke eigensheaf property

Recall the left D -module $\mathcal{V}_{\omega_1, X}^{\text{univ}}$ on X obtained from the universal oper bundle. It is equipped with a commuting action of the above algebra D_{PGL_n} . Furthermore, by definition of SL_n -opers, we have an embedding

$$\kappa_{\omega_1}^{\text{univ}} : K_X^{(n-1)/2} \hookrightarrow \mathcal{V}_{\omega_1, X}^{\text{univ}}$$

and hence a section

$$s_{\omega_1}^{\text{univ}} \in \Gamma(X, K_X^{-(n-1)/2} \otimes \mathcal{V}_{\omega_1, X}^{\text{univ}}).$$

Recall the n th order differential operator (1.12). Lemmas 1.15 and 1.17 imply the following differential equation on $s_{\omega_1}^{\text{univ}}$ (this is an analogue of equation (2.18) in the case of GL_1):

$$\sigma \cdot s_{\omega_1}^{\text{univ}} = 0. \tag{4.3}$$

We will now use this equation to derive the system (1.13).

Consider the isomorphism (1.2):

$$a : q_1^*(\mathcal{L}^{1/2}) \xrightarrow{\sim} q_2^*(\mathcal{L}) \otimes K_2 \otimes q_3^*(K_X^{-(n-1)/2}). \tag{4.4}$$

It gives rise to a section $\psi_{Z(\omega_1)}$ of $(\mathcal{L} \boxtimes K_X^{-(n-1)/2}) \otimes \mathbf{H}_{\omega_1}^0(\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1})$.

The first part of the following theorem is Theorem 5.2.9 of [3]. The second part follows from Theorems 5.4.11 and 5.4.12 and Proposition 8.1.5 of [3].

THEOREM 4.2 ([3])

We have $\mathbf{H}_{\omega_1}^i(\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}) = 0$ for $i \neq 0$, and

$$\mathbf{H}_{\omega_1}^0(\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}) \boxtimes_{D_{\text{PGL}_n}} \mathcal{V}_{\omega_1, X}^{\text{univ}}$$

as left $\mathcal{D}_{\text{BunPGL}_n} \boxtimes \mathcal{D}_X$ -modules equipped with a commuting action of D_{PGL_n} .

Moreover, the section $\psi_{Z(\omega_1)}$ of

$$\begin{aligned} & (\mathcal{L} \boxtimes K_X^{-(n-1)/2}) \otimes \mathbf{H}_{\omega_1}^0(\mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}) \\ & \simeq (\mathcal{L} \otimes \mathcal{D}_{\text{BunPGL}_n} \otimes \mathcal{L}^{-1}) \boxtimes_{D_{\text{PGL}_n}} (K_X^{-(n-1)/2} \otimes \mathcal{V}_{\omega_1, X}^{\text{univ}}) \end{aligned}$$

coincides with $1 \boxtimes s_{\omega_1}^{\text{univ}}$.

Theorem 4.2 and (4.3) immediately imply the following.

COROLLARY 4.3

The section $\psi_{Z(\omega_1)}$ satisfies

$$\sigma \cdot \psi_{Z(\omega_1)} = 0. \tag{4.5}$$

Now we derive Theorem 1.18 from Corollary 4.3 using Corollary 3.12 in the case of the Hecke correspondence $Z(\omega_1)$.

4.3. Proof of Theorem 1.18

We are going to derive Theorem 1.18 from Corollary 3.12 with Z , Y_1 , Y_2 , \mathcal{L}_1 , \mathcal{L}_2 , and \mathcal{L}'_2 defined below. Initially, we would like to take $Y_1 = \text{Bun}_{\text{PGL}_n}^\circ$, $Y_2 = \text{Bun}_{\text{PGL}_n}^\circ \times X$, and $Z = Z(\omega_1)$ (see Section 1.2). However, in order for the morphism $Z \rightarrow Y_1 \times Y_2$ to be a closed embedding and the morphism $Z \rightarrow Y_2$ to be proper, we need to restrict $Z(\omega_1)$ to open dense subsets on both sides, as we now explain.

Recall from formula (1.5) that we have an open dense subvariety of $\text{Bun}_{\text{PGL}_n}^\circ$,

$$U_{\text{PGL}_n}(\omega_1) = \{ \mathcal{F} \in \text{Bun}_{\text{PGL}_n}^\circ \mid q_2(q_1^{-1}(\mathcal{F})) \subset \text{Bun}_{\text{PGL}_n}^\circ \}, \tag{4.6}$$

which is dense by our assumption. It follows from [27, Lemma 5.9]³ that there exist open dense subsets

$$Y_1 \subset U_{\text{PGL}_n}(\omega_1), \quad Y_2 \subset \text{Bun}_{\text{PGL}_n}^\circ \times X$$

such that the restriction Z of $Z(\omega_1)$ to $Y_1 \times Y_2$ is a closed embedding and the corresponding map $q_2 \times q_3 : Z \rightarrow Y_2$ is proper. Denote by p_1 and p_2 the maps q_1 and $q_2 \times q_3$ restricted to $Y_1 \times Y_2$, respectively. Finally, set $\mathcal{L}_1 = K_{\text{Bun}}^{1/2}$ and $\mathcal{L}_2 = K_{\text{Bun}}^{1/2} \boxtimes K_X^{-(n-1)/2}$. Thus, we have $|\mathcal{L}_1|^2 = \Omega_{\text{Bun}}^{1/2}$ and $|\mathcal{L}_2|^2 = \Omega_{\text{Bun}}^{1/2} \boxtimes \Omega_X^{-(n-1)/2}$.

³We thank Tony Pantev for this reference and a helpful discussion. Note that the correspondence considered in [27] differs from the Hecke correspondence $Z(\omega_1)$ in that one of the two bundles is dualized; but since the dual of a stable bundle is stable, we can use Lemma 5.9 of [27] in our setting.

Then we have the isomorphism (3.16) which follows from the isomorphism (4.4) (see also formula (4.2)).

Now we are in the setting of Corollary 3.12, with $\mathcal{L}'_2 := K_{\text{Bun}}^{1/2} \boxtimes K_X^{(n+1)/2}$. Let H_Z be the corresponding integral transform operator. We then obtain from Corollary 3.12 and equation (4.5) that H_Z satisfies the system of differential equations

$$\sigma \cdot H_Z = 0, \quad \bar{\sigma} \cdot H_Z = 0. \tag{4.7}$$

Next, we relate H_Z to our Hecke operator \widehat{H}_{ω_1} . Recall from Section 1.2 that the operator \widehat{H}_{ω_1} is defined as the integral transform via the correspondence $Z(\omega_1)$ from the space $V_{\text{PGL}_n}(\omega_1)$ of smooth compactly supported sections of $|\mathcal{L}_1|^2 = \Omega_{\text{Bun}}^{1/2}$ on $U_{\text{PGL}_n}(\omega_1)$ to the space $V_{\text{PGL}_n} \otimes \Gamma(X, \Omega_X^{-(n-1)/2})$ of smooth sections of $|\mathcal{L}'_2|^2 = \Omega_{\text{Bun}}^{1/2} \boxtimes \Omega_X^{(n+1)/2}$ on $\text{Bun}_{\text{PGL}_n}^\circ \times X$ (in fact, the image consists of compactly supported sections). Let \widetilde{H}_{ω_1} be the restriction of \widehat{H}_{ω_1} to the space of smooth sections of $|\mathcal{L}_1|^2$ that are compactly supported on $Y_1 \subset U_{\text{PGL}_n}(\omega_1)$, followed by the restriction to Y_2 of the resulting section of $|\mathcal{L}'_2|^2$ on $\text{Bun}_G^\circ \times X$. Thus, \widetilde{H}_{ω_1} acts from the space of smooth compactly supported sections of $|\mathcal{L}_1|^2$ on Y_1 to the space of smooth sections of $|\mathcal{L}'_2|^2$ on Y_2 .

It follows from the above definition that \widetilde{H}_{ω_1} is the restriction of H_Z to smooth compactly supported sections of $|\mathcal{L}_1|^2$ on Y_1 . Therefore, \widetilde{H}_{ω_1} also satisfies the system (4.7). Since Y_1 is dense in $U_{\text{PGL}_n}(\omega_1)$ and Y_2 is dense in $\text{Bun}_{\text{PGL}_n}^\circ \times X$, this implies that \widehat{H}_{ω_1} also satisfies this system of equations. Thus, we obtain the system (1.13). This completes the proof of Theorem 1.18. □

Remark 4.1

It is possible to write an explicit formula for the differential operator σ similar to formula (2.17) in the case of GL_1 . For example, let $G = \text{PGL}_2$, so ${}^L G = \text{SL}_2$. Then the space $\text{Op}_{\text{SL}_2}^\vee(X)$ is an affine space over the vector space $H^0(X, K_X^2)$. Let us pick a point $\chi_0 \in \text{Op}_{\text{SL}_2}^\vee(X)$ and use it to identify $\text{Op}_{\text{SL}_2}^\vee(X)$ with $H^0(X, K_X^2)$. Let $\{\varphi_i, i = 1, \dots, 3g - 3\}$ be a basis of $H^0(X, K_X^2)$, and let $\{F_i, i = 1, \dots, 3g - 3\}$ be the set of generators of the polynomial algebra $\text{Fun Op}_{\text{SL}_2}^\vee(X)$ dual to this basis, that is,

$$F_i(\chi_0 + \varphi_j) = \delta_{ij}.$$

Let $\{D_i, i = 1, \dots, 3g - 3\}$ be the global holomorphic differential operators on $\text{Bun}_{\text{PGL}_2}$ corresponding to the F_i under the isomorphism $\text{Fun Op}_{\text{SL}_2}^\vee(X) \simeq D_{\text{PGL}_2}$.

By Lemma 1.15, we have an isomorphism $\text{Op}_{\text{SL}_2}^\vee(X) \simeq D_2^\vee(X)$, where $D_2^\vee(X)$ is the space of projective connections on X (corresponding to our choice of $K_X^{1/2}$). It sends

$$\chi \in \text{Op}_{\text{SL}_2}^\vee(X) \quad \mapsto \quad P_\chi \in D_2^\vee(X).$$

Then we can write

$$\sigma = P_{\chi_0} + \sum_{i=1}^{3g-3} D_i \otimes \varphi_i : K_X^{-1/2} \rightarrow D_{\text{PGL}_2} \otimes K_X^{3/2} \tag{4.8}$$

(it is clear that this operator does not depend on the choice of χ_0).

Note that locally on X , after choosing a local coordinate z , we can write the second-order differential operator $P_{\chi_0} \in D_2^\vee(X)$ in the form

$$P_{\chi_0} = \partial_z^2 + v_0(z).$$

This gives a more concrete realization of the equations (1.13).

Remark 4.2

For $\chi \in \text{Op}_{\text{PGL}_n}^\vee(X) = \text{Spec } D_{\text{PGL}_n}$, let \mathbb{C}_χ be the corresponding one-dimensional D_{PGL_n} -module. In [3, Section 5.1.1], Beilinson and Drinfeld defined the following left D -module on $\text{Bun}_{\text{PGL}_n}$:

$$\Delta_\chi^0 := (\mathcal{D}_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}) \otimes_{D_{\text{PGL}_n}} \mathbb{C}_\chi$$

(this is the D -module from Theorem 1.12). They derived from Theorem 4.2 that $\mathbf{H}_{\omega_1}^i(\Delta_\chi^0) = 0$ for $i \neq 0$ and (see [3, Theorem 5.2.6])

$$\mathbf{H}_{\omega_1}^0(\Delta_\chi^0) \simeq \Delta_\chi^0 \boxtimes (\mathcal{V}_{\omega_1}, \nabla_\chi).$$

This means that Δ_χ^0 is a *Hecke eigensheaf* with respect to the flat SL_n -bundle corresponding to χ (see [3]). In this sense, $\mathcal{D}_{\text{PGL}_n} \otimes \mathcal{L}^{-1}$ is a universal Hecke eigensheaf parameterized by the component $\text{Op}_{\text{SL}_n}^\vee(X)$ of the space of SL_n -opers on X .

Recall the equivalence of categories in the abelian case obtained by restriction of the D -module version of the Fourier–Mukai transform discussed in Remark 2.3. It has a non-abelian analogue (see, e.g., [16]). In the case of $G = \text{PGL}_n$, on one side we have the category of coherent sheaves on $\text{Op}_{\text{PGL}_n}^\vee(X)$. On the other side, we have the category of D -modules \mathcal{K} on $\text{Bun}_{\text{PGL}_n}$ with finite global presentation of the form

$$(\mathcal{D}_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1})^{\oplus m} \rightarrow (\mathcal{D}_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1})^{\oplus r} \rightarrow \mathcal{K} \rightarrow 0.$$

The equivalence E takes an object \mathcal{F} of the former category to

$$E(\mathcal{F}) := (\mathcal{D}_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}) \otimes_{D_{\text{PGL}_n}} F, \quad F := \Gamma(\text{Op}_{\text{PGL}_n}^\vee(X), \mathcal{F}).$$

In particular, $E(\mathcal{O}_{\text{Op}_{\text{PGL}_n}^\vee(X)}) = \mathcal{D}_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}$ and $E(\mathcal{O}_\chi) = \Delta_\chi^0$.

5. General case

In this section, we formulate the analogues of Theorem 1.18 and Corollary 1.19 describing the eigenvalues of the Hecke operators in the case of an arbitrary simple Lie group G and outline their proof following the argument of the previous section.

The case of $G = \mathrm{PGL}_n$ and $\lambda = \omega_1$, which we considered in the previous section, differs from the case of a general group G and dominant integral coweight $\lambda \in P_+^\vee$ in two ways. First, in the case of $G = \mathrm{PGL}_n$ and $\lambda = \omega_1$, we have $\overline{Z}(\lambda) = Z(\lambda)$, that is, the fibers $(\overline{q}_2 \times \overline{q}_3)^{-1}(\mathcal{P}, x)$ of the Hecke correspondence $\overline{Z}(\omega_1)$ are isomorphic to the $\mathrm{PGL}_n[[z]]$ -orbit Gr_{ω_1} in the affine Grassmannian of PGL_n which is smooth and compact. But for general G and $\lambda \in P_+^\vee$, these fibers are isomorphic to the closure of the $G[[z]]$ -orbit Gr_λ and are singular. Second, in the case of $G = \mathrm{PGL}_n$ and $\lambda = \omega_1$, the canonical section of the universal oper bundle satisfies a scalar differential equation of Lemma 1.15, but for general G and λ this is not the case. Hence for general G and λ our construction needs some modifications, which we discuss in this section.

Namely, we formulate an analogue (Conjecture 5.1) of Corollary 1.19 describing the eigenvalues of Hecke operators in terms of real ${}^L G$ -opers; an analogue (Lemma 5.4) of the system (4.3) of differential equations satisfied by the section $s_\lambda^{\mathrm{univ}}$ of the universal oper bundle (twisted by a power of K_X); and an analogue (Conjecture 5.5) of Theorem 1.18 describing differential equations satisfied by the Hecke operators. Conjecture 5.1 follows from Conjecture 5.5 in the same way as in the case of PGL_n .

Thus, the only statement that remains to be proved is Conjecture 5.5. It can be derived from the Hecke eigensheaf property (established in [3] and recalled in Section 5.3) by an argument analogous to the one we used in the proof of Theorem 1.18 in Section 4.3 in the case of PGL_n . However, because the morphism $\overline{q}_2 \times \overline{q}_3$ is not smooth in the general case (in the sense of algebraic geometry), deriving Conjecture 5.5 from this result requires additional care. We leave the details to a follow-up paper.

5.1. Eigenvalues of the Hecke operators

Recall from the discussion before Theorem 1.6 that the space $\mathrm{Op}_{L_G}(X)$ of ${}^L G$ -opers on X has a canonical component $\mathrm{Op}_{L_G}^\gamma(X)$ isomorphic to the affine space $\mathrm{Op}_{L_G^{\mathrm{ad}}}^\gamma(X)$. If the set $\{(\lambda, \rho), \lambda \in P_+^\vee\}$ contains half-integers, then to specify $\mathrm{Op}_{L_G}^\gamma(X)$ we need to choose a square root $K_X^{1/2}$ of the canonical line bundle on X ; then γ denotes the isomorphism class of this $K_X^{1/2}$ which we also use to construct a square root $\mathcal{L} = K_{\mathrm{Bun}}^{1/2}$ of the canonical line bundle on Bun_G (see the discussion before Theorem 1.1). Otherwise, the component $\mathrm{Op}_{L_G}^\gamma(X)$ is well defined without any choices.

For $\lambda \in P_+^\vee$, let V_λ be the corresponding irreducible finite-dimensional representation of ${}^L G$. Given $\chi \in \text{Op}_{{}^L G}^\vee(X)$, we obtain a flat holomorphic vector bundle $(\mathcal{V}_\lambda, \nabla_{\chi, \lambda})$ on X . Moreover, the oper Borel reduction gives rise to an embedding

$$\kappa_\lambda : K_X^{\langle \lambda, \rho \rangle} \hookrightarrow \mathcal{V}_\lambda$$

and hence

$$\tilde{\kappa}_\lambda : \mathcal{O}_X \hookrightarrow K_X^{-\langle \lambda, \rho \rangle} \otimes \mathcal{V}_\lambda.$$

Let

$$s_\lambda := \tilde{\kappa}_\lambda(1) \in \Gamma(X, K_X^{-\langle \lambda, \rho \rangle} \otimes \mathcal{V}_\lambda).$$

Now suppose that $\chi \in \text{Op}_{{}^L G}^\vee(X)_\mathbb{R}$. Then we have an isomorphism of C^∞ flat bundles

$$(\mathcal{V}_\lambda, \nabla_{\chi, \lambda}) \simeq (\overline{\mathcal{V}}_\lambda, \overline{\nabla}_{\chi, \lambda})$$

and hence a pairing

$$h_{\chi, \lambda}(\cdot, \cdot) : (\mathcal{V}_\lambda, \nabla_{\chi, \lambda}) \otimes (\overline{\mathcal{V}}_{-w_0(\lambda)}, \overline{\nabla}_{\chi, -w_0(\lambda)}) \rightarrow (\mathcal{E}_X^\infty, d)$$

as $V_\lambda^* \simeq V_{-w_0(\lambda)}$. Since $\langle -w_0(\lambda), \rho \rangle = \langle \lambda, \rho \rangle$, we have

$$\overline{s_{-w_0(\lambda)}} \in \Gamma(X, \overline{K}_X^{-\langle \lambda, \rho \rangle} \otimes \overline{\mathcal{V}}_\lambda^*).$$

Recall that Bun_G has connected components $\text{Bun}_G^\beta, \beta \in \pi_1(G)$, and we have a direct sum decomposition

$$\mathcal{H}_G = \bigoplus_{\beta \in \pi_1(G)} \mathcal{H}_G^\beta. \tag{5.1}$$

According to Conjecture 1.11, for each $\chi \in \text{Op}_{{}^L G}^\vee(X)_\mathbb{R}$ we have a nonzero eigenspace of \mathcal{A}_G in \mathcal{H}_G^β for all $\beta \in \pi_1(G)$ (which is one-dimensional by Proposition 1.13 and Theorem 1.12). We expect that the Hecke operator H_λ preserves the direct sum of these subspaces (see Section 1.4 in the case of PGL_n). Moreover, one can find out precisely how H_λ permutes different \mathcal{H}_G^β by analyzing the action of the center $Z({}^L G)$ on V_λ (which is naturally identified with the group of characters of $\pi_1(G)$). As in the case of PGL_n , this implies that the eigenvalues $\{\Phi_\lambda(\chi)\}$ of H_λ corresponding to χ form a torsor over the group $\mu_{\alpha(G, \lambda)}$ of roots of unity of some order $\alpha(G, \lambda)$; for example, $\alpha(\text{PGL}_n, \omega_1) = n$ (this is so even for $G = \text{SO}_{4n}/\{\pm I\}$, when $\pi_1(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$). This is why we describe these eigenvalues up to a scalar.

The following conjecture is an analogue of Corollary 1.19 for a general group G .

CONJECTURE 5.1

For $\chi \in \text{Op}_{LG}^\gamma(X)_{\mathbb{R}}$, the section $\Phi_\lambda(\chi) \in \Gamma(X, \Omega_X^{-(\lambda, \rho)})$ is equal to

$$\Phi_\lambda(\chi) = h_{\chi, \lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$$

up to a scalar.

Remark 5.1

As in the case of $G = \text{PGL}_n$ (see Section 1.4), we expect that both the pairing $h_{\chi, \lambda}(\cdot, \cdot)$ and the eigenvalues $\Phi_\lambda(\chi)$ of the Hecke operator H_λ can be normalized up to a root of unity of order $\alpha(G, \lambda)$. Conjecture 5.1 can then be refined to a statement that two torsors over the corresponding group $\mu_{\alpha(G, \lambda)}$ of roots of unity are equal to each other, similarly to Conjecture 1.14 for $G = \text{PGL}_n$.

5.2. Analogue of the system of differential equations

To prove Conjecture 5.1, we need an analogue of the system of differential equations which appear in Corollary 1.16 and Theorem 1.18 in the case of PGL_n and $\lambda = \omega_1$. In that case, we used the possibility to interpret opers in terms of scalar differential operators of order n (see Lemma 1.15). An analogous interpretation is also possible if G is of type B_ℓ or C_ℓ and $\lambda = \omega_1$, but is not known (and most likely does not exist) in general (see [4], [7]). Instead, we replace the equations from Corollary 1.16 and Theorem 1.18 with a statement (Lemma 5.2) about the (twisted) D -module on X obtained by applying *all* possible (twisted) differential operators to the canonical section.

Namely, let

$$\mathcal{D}_{X, -(\lambda, \rho)} := K_X^{-(\lambda, \rho)} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} K_X^{(\lambda, \rho)}$$

be the sheaf of differential operators acting on the line bundle $K_X^{-(\lambda, \rho)}$ on X . Set

$$\mathcal{V}_\lambda^K := K_X^{-(\lambda, \rho)} \otimes_{\mathcal{O}_X} \mathcal{V}_\lambda.$$

The oper connection $\nabla_{\chi, \lambda}$ defines a \mathcal{D}_X -module structure on \mathcal{V}_λ , and therefore a $\mathcal{D}_{X, -(\lambda, \rho)}$ -module structure on \mathcal{V}_λ^K . We will denote this $\mathcal{D}_{X, -(\lambda, \rho)}$ -module by $\mathcal{V}_{\chi, \lambda}^K$. Recall that we have a canonical section $s_\lambda \in \Gamma(X, \mathcal{V}_\lambda^K)$.

LEMMA 5.2

If the monodromy representation of the flat vector bundle $(V_\lambda, \nabla_{\chi, \lambda})$ is irreducible, then

$$\mathcal{D}_{X, -(\lambda, \rho)} \cdot s_\lambda = \mathcal{V}_{\chi, \lambda}^K. \tag{5.2}$$

Proof

Note that $\mathcal{D}_{X,-(\lambda,\rho)} \cdot s_\lambda$ is a nonzero $\mathcal{D}_{X,-(\lambda,\rho)}$ -submodule of $\mathcal{V}_{\chi,\lambda}^K$. If the monodromy representation of the flat vector bundle $(V_\lambda, \nabla_{\chi,\lambda})$ is irreducible, then $(V_\lambda, \nabla_{\chi,\lambda})$ is an irreducible \mathcal{D}_X -module, so $\mathcal{V}_{\chi,\lambda}^K$ is an irreducible $\mathcal{D}_{X,-(\lambda,\rho)}$ -module. Hence (5.2) follows. \square

Let $I_{\lambda,\chi}$ be the left annihilating ideal of s_λ in the sheaf $\mathcal{D}_{X,-(\lambda,\rho)}$. Thus, we have an exact sequence of left $\mathcal{D}_{X,-(\lambda,\rho)}$ -modules

$$0 \rightarrow I_{\lambda,\chi} \rightarrow \mathcal{D}_{X,-(\lambda,\rho)} \rightarrow \mathcal{V}_{\chi,\lambda}^K \rightarrow 0.$$

Lemma 5.2 has the following immediate corollary.

COROLLARY 5.3

Suppose that the monodromy representation of the flat vector bundle $(V_\lambda, \nabla_{\chi,\lambda})$, where $\chi \in \text{Op}_{LG}(X)_{\mathbb{R}}$, is irreducible. Then $h_{\chi,\lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$ is a unique, up to a scalar, nonzero section $\Psi_\lambda(\chi)$ of $\Omega_X^{-(\lambda,\rho)}$ annihilated by the ideals $I_{\lambda,\chi}$ and $\overline{I_{-w_0(\lambda),\chi}}$.

Remark 5.2

For $G = \text{PGL}_n$, the ideal $I_{\lambda,\chi}$ is globally generated by a differential operator of order n . Therefore, in this case, a section annihilated by this ideal is the same as a section satisfying the n th order differential equation (1.11). For a general simple Lie group G , the ideal $I_{\lambda,\chi}$ does not have such a generator. But as the above corollary shows, this is not necessary. What matters is the cyclicity of the D -module $\mathcal{V}_{\chi,\lambda}^K$ (see (5.2)).

We note that it is the cyclicity of two types of twisted D -modules: $\mathcal{V}_{\chi,\lambda}^K$ on X and Δ_χ on Bun_G (see (1.8)), that enables us to link the geometric Langlands correspondence and the analytic one (see Remark 3.1).

Remark 5.3

Note that a nonzero section $\Psi_\lambda(\chi)$ from Corollary 5.3 satisfies

$$(\mathcal{D}_{X,-(\lambda,\rho)} \otimes \overline{\mathcal{D}}_{X,-(\lambda,\rho)}) \cdot \Psi_\lambda(\chi) \simeq \mathcal{V}_{\chi,\lambda}^K \otimes \overline{\mathcal{V}}_{\chi,-w_0(\lambda)}^K.$$

Corollary 5.3 is an analogue of Corollary 1.16. We are going to formulate a conjectural analogue of Theorem 1.18 (Conjecture 5.5) in a similar way.

Let $\mathcal{V}_\lambda^{\text{univ}}$ be the universal vector bundle over $\text{Op}_{LG}^\gamma(X) \times X$ with a partial connection ∇^{univ} along X , such that

$$(\mathcal{V}_\lambda^{\text{univ}}, \nabla^{\text{univ}})|_{X \times X} = (\mathcal{V}_\lambda, \nabla_{\chi,\lambda}), \quad \chi \in \text{Op}_{LG}^\gamma(X).$$

Let $\pi : \text{Op}_{L_G}^{\vee}(X) \times X \rightarrow X$ be the projection and set

$$\mathcal{V}_{X,\lambda}^{\text{univ}} := \pi_*(\mathcal{V}_{\lambda}^{\text{univ}}), \quad \mathcal{V}_{X,\lambda}^{K,\text{univ}} := K_X^{-(\lambda,\rho)} \otimes \mathcal{V}_{X,\lambda}^{\text{univ}}.$$

Then $\mathcal{V}_{X,\lambda}^{K,\text{univ}}$ is naturally a $\mathcal{D}_{X,-(\lambda,\rho)}$ -module on X , equipped with a commuting action of $\text{Fun Op}_{L_G}^{\vee}(X) \simeq D_G$.

Moreover, the oper Borel reduction gives rise to an embedding

$$k_{\lambda}^{\text{univ}} : K_X^{-(\lambda,\rho)} \hookrightarrow \mathcal{V}_{\lambda,X}^{\text{univ}}$$

and hence a canonical section

$$s_{\lambda}^{\text{univ}} \in \Gamma(X, \mathcal{V}_{X,\lambda}^{K,\text{univ}}).$$

Consider the cyclic $(D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)})$ -module $(D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}) \cdot s_{\lambda}^{\text{univ}}$ generated by $s_{\lambda}^{\text{univ}}$. The next lemma follows from the irreducibility of the universal L_G -oper bundle. It is an analogue of Lemma 1.17 for a general group G .

LEMMA 5.4

There is an isomorphism

$$(D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}) \cdot s_{\lambda}^{\text{univ}} \simeq \mathcal{V}_{X,\lambda}^{K,\text{univ}} \tag{5.3}$$

of $(D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)})$ -modules.

Now recall that the Hecke operator \widehat{H}_{λ} is a section of $\Omega_X^{-(\lambda,\rho)}$ with values in operators $V_G(\lambda) \rightarrow V_G$. Hence we can apply to it the sheaf $\mathcal{D}_{X,-(\lambda,\rho)}$ as well as the algebra D_G , through its action on V_G . The two actions commute, and they generate a $\mathcal{D}_{X,-(\lambda,\rho)}$ -module inside the sheaf of C^{∞} -sections of $\Omega_X^{-(\lambda,\rho)}$ on X with values in operators $V_G(\lambda) \rightarrow V_G$. Let us denote this $\mathcal{D}_{X,-(\lambda,\rho)}$ -module by $\langle H_{\lambda} \rangle$.

Similarly, we can apply to H_{λ} the sheaf $\overline{\mathcal{D}}_{X,-(\lambda,\rho)}$ and the algebra \overline{D}_G . Denote the resulting $\overline{\mathcal{D}}_{X,-(\lambda,\rho)}$ -module by $\overline{\langle H_{\lambda} \rangle}$. The following is an analogue of Theorem 1.18 for a general group G .

CONJECTURE 5.5

There are isomorphisms

$$\langle H_{\lambda} \rangle \simeq \mathcal{V}_{X,\lambda}^{K,\text{univ}}, \quad \overline{\langle H_{\lambda} \rangle} \simeq \overline{\mathcal{V}}_{X,\lambda}^{K,\text{univ}} \tag{5.4}$$

of $(D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)})$ -modules (resp., $(\overline{D}_G \otimes \overline{\mathcal{D}}_{X,-(\lambda,\rho)})$ -modules).

In the case $G = \text{PGL}_n$, $\lambda = \omega_1$, this conjecture is equivalent to the statement of Theorem 1.18.

Suppose that the monodromy representation of the flat vector bundle $(V_\lambda, \nabla_{\chi, \lambda})$, where $\chi \in \text{Op}_{LG}(X)_{\mathbb{R}}$, is irreducible. Then Conjecture 5.1 follows from Conjecture 5.5 and Corollary 5.3 in the same way as Corollary 1.19 follows from Theorem 1.18 and Corollary 1.16 in the case $G = \text{PGL}_n, \lambda = \omega_1$.

5.3. Hecke eigensheaf property

It remains to prove Conjecture 5.5. As in the case of PGL_n , we wish to derive Conjecture 5.5 using the formalism of Section 3.4 from the Hecke eigensheaf property established by Beilinson and Drinfeld [3].

We start by recalling the definition of the Hecke functor in the general case from [3].

We will use the notation of Section 1.2. Consider the Hecke correspondence $\overline{Z}(\lambda)$. The fibers of the morphism $\overline{q}_2 \times \overline{q}_3$ are isomorphic to the closure $\overline{\text{Gr}}_\lambda$ of the $G[[z]]$ -orbit Gr_λ in the affine Grassmannian of G . We have denoted by $Z(\lambda)$ the open dense part of $\overline{Z}(\lambda)$ such that the fibers of $q_2 \times q_3$ restricted to $Z(\lambda)$ are isomorphic to Gr_λ , and we have denoted by q_i the restriction of the morphism \overline{q}_i to $Z(\lambda)$.

The following definition is taken from [3, Section 5.2.4].

Definition 5.6 ([3])

Let M be a left D -module on Bun_G . Denote by $q_1^*(M)$ the intermediate extension to $Z(\lambda)$ of $q_1^*(M)$. (Locally, we can choose an isomorphism $\overline{Z}(\lambda) \simeq \text{Bun}_G \times \overline{\text{Gr}}_\lambda \times X$ so that \overline{q}_1 is the projection on the first factor; then $q_1^*(M)$ can be identified with the exterior tensor product of $q_1^*(M)$, the irreducible D -module on $\overline{\text{Gr}}_\lambda$, and \mathcal{O}_X .) The Hecke functor is defined by the following formula:

$$\mathbf{H}_\lambda^\bullet(M) := (\overline{q}_2 \times \overline{q}_3)_*^D(q_1^*(M)). \tag{5.5}$$

Now consider the left D -module $\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}$, where $\mathcal{L} = K_{\text{Bun}}^{1/2}$. It is equipped with the commuting right action (see [3, Section 5.1.1]) of

$$D_G = \Gamma(\text{Bun}_G^\beta, \mathcal{L} \otimes \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq \text{Fun Op}_{LG}^\gamma(X).$$

The isomorphism (1.2) gives rise to a section $\psi_{Z(\lambda)}$ of

$$(\mathcal{L} \boxtimes K_X^{-(\lambda, \rho)}) \otimes \mathbf{H}_\lambda^0(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}).$$

The following theorem is due to [3] (see the references before Theorem 4.2 above).

THEOREM 5.7 ([3])

We have $\mathbf{H}_\lambda^i(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) = 0$ for $i \neq 0$, and

$$\mathbf{H}_\lambda^0(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \boxtimes_{D_G} \mathcal{V}_{\lambda, X}^{\text{univ}}$$

as left $(\mathcal{D}_{\text{Bun}_G} \boxtimes \mathcal{D}_X)$ -modules equipped with a commuting action of D_G .

Moreover, the section $\psi_{Z(\lambda)}$ of

$$(\mathcal{L} \boxtimes K_X^{-(\lambda, \rho)}) \otimes \mathbf{H}_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{L} \otimes \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \boxtimes_{D_G} (K_X^{-(\lambda, \rho)} \otimes \mathcal{V}_{\lambda, X}^{\text{univ}})$$

coincides with $1 \boxtimes s_\lambda^{\text{univ}}$.

Theorem 5.7 and Lemma 5.4 imply the following.

COROLLARY 5.8

There is an isomorphism

$$(D_G \otimes \mathcal{D}_{X, -(\lambda, \rho)}) \cdot \psi_{Z(\lambda)} \simeq \mathcal{V}_{X, \lambda}^{K, \text{univ}} \tag{5.6}$$

of $(D_G \otimes \mathcal{D}_{X, -(\lambda, \rho)})$ -modules.

Conjecture 5.5 (and hence Conjecture 5.1) can be derived from Corollary 5.8 by adapting the results of Section 3.4 to the present situation (similarly to what we did in the proof of Theorem 1.18 in Section 4.3 in the case of PGL_n). This requires additional care since the morphism $\bar{q}_2 \times \bar{q}_3$ is not smooth in general (in the sense of algebraic geometry). We leave the details to a follow-up paper.

5.4. Proof of Theorem 1.1

In this subsection, we derive Theorem 1.1 from a local statement about line bundles on the affine Grassmannian given in formula (241) of [3] (it is reproduced in formula (5.9) below). All ingredients are contained in [3]. We include the argument here for completeness.

For a point x of our curve X , let F_x be the formal completion of the field of rational functions on X at x , and let \mathcal{O}_x be its ring of integers. We will also use the notation $\mathcal{O} = \mathbb{C}[[z]]$, $F = \mathbb{C}((z))$. Let $\text{Aut } \mathcal{O}$ be the group of automorphisms of \mathcal{O} . It naturally acts on the formal loop group $G(F)$ preserving the subgroup $G(\mathcal{O})$. Hence we obtain an action of $\text{Aut } \mathcal{O}$ on the affine Grassmannian $\text{Gr} = G(F)/G(\mathcal{O})$, which preserves the $G(\mathcal{O})$ -orbits Gr_λ , $\lambda \in P_+^\vee$.

Consider first the case when the set $\{(\lambda, \rho), \lambda \in P_+^\vee\}$ only contains integers. Let \widehat{M} be the ind-scheme defined in [3, Section 2.8.3], which parameterizes quadruples $(x, t_x, \mathcal{F}, \gamma_x)$, where x is a point of our curve X , t_x is a formal coordinate at x (so that we can identify \mathcal{O}_x with $\mathbb{C}[[t_x]]$), \mathcal{F} is a G -bundle on X , and γ_x is a trivialization of \mathcal{F} on the disk $D_x = \text{Spec } \mathcal{O}_x$. The projection

$$\widehat{M} \rightarrow \text{Bun}_G \times X, \tag{5.7}$$

$$(x, t_x, \mathcal{F}, \gamma_x) \mapsto (\mathcal{F}, x)$$

is a torsor for the group $\text{Aut } O \rtimes G(O)$, which naturally acts on t_x and γ_x . We use it to construct a functor \mathbf{F} sending an (ind-)scheme Y with an action of $\text{Aut } O \rtimes G(O)$ to an (ind-)scheme over $\text{Bun}_G \times X$,

$$\mathbf{F}(Y) = \mathcal{Y} := \widehat{M} \times_{\text{Aut } O \rtimes G(O)} Y.$$

This is a generalization of the Gelfand–Kazhdan functor in [19] from (ind-)schemes equipped with an action of the group $\text{Aut } O$ to (ind-)schemes over X . Applying it to Gr_λ , we obtain the scheme

$$\mathbf{F}(\text{Gr}_\lambda) = \mathcal{G}r_\lambda := \widehat{M} \times_{\text{Aut } O \rtimes G(O)} \text{Gr}_\lambda$$

over $\text{Bun}_G \times X$. Denote by r the projection $\mathcal{G}r_\lambda \rightarrow \text{Bun}_G \times X$.

PROPOSITION 5.9 ([3, Section 5.2.2(ii)])

There is a natural isomorphism $Z(\lambda) \simeq \mathcal{G}r_\lambda$ under which the projection $q_2 \times q_3 : Z(\lambda) \rightarrow \text{Bun}_G \times X$ is identified with r .

Next, in [3, Section 4.6], a local Pfaffian line bundle was defined on Gr . We will denote it by \mathcal{L}_{Gr} . According to the construction, \mathcal{L}_{Gr} is $(\text{Aut } O \rtimes G(O))$ -equivariant, and hence so is its restriction $\mathcal{L}_{\text{Gr}_\lambda}$ to the $G(O)$ -orbit Gr_λ . Applying the above functor \mathbf{F} to it, we obtain a line bundle on $\mathcal{G}r_\lambda$, which we will denote by $\mathcal{L}_{\mathcal{G}r_\lambda}$.

In [3], the line bundle $\mathcal{L}_{\text{Gr}_\lambda}$ was described explicitly. To explain this result, we need to define a certain one-dimensional representation of $\text{Aut } O \rtimes G(O)$. Namely, let us assign to every element ϕ of $\text{Aut } O$ the image of $z \in \mathbb{C}[[z]] = O$ under ϕ . This assignment sets up a bijection between $\text{Aut } O$ and the space of formal power series

$$\phi(z) = \sum_{n \geq 0} \phi_n z^{n+1} \in \mathbb{C}[[z]], \tag{5.8}$$

where ϕ_0 is invertible. In particular, we obtain a canonical homomorphism $\gamma : \text{Aut } O \rightarrow \mathbb{G}_m$, $\phi(z) \mapsto \phi_0$.

Definition 5.10

For $n \in \mathbb{Z}$, let $\sigma(n)$ be the one-dimensional representation of $\text{Aut } O$ on which it acts via the composition of γ and the character of \mathbb{G}_m raising ϕ_0 to the power n . We

extend it to a one-dimensional representation (denoted in the same way) of $\text{Aut } O \times G(O)$.

THEOREM 5.11 ([3, (241)])

There is a canonical isomorphism of $(\text{Aut } O \times G(O))$ -equivariant line bundles on Gr_λ ,

$$\mathcal{L}_{\text{Gr}_\lambda} \simeq K_{\text{Gr}_\lambda} \otimes \mathfrak{o}_\lambda, \tag{5.9}$$

where K_{Gr_λ} is the canonical line bundle on Gr_λ and $\mathfrak{o}_\lambda := \mathfrak{o}(-\langle \lambda, \rho \rangle)$.

We now derive Theorem 1.1 from this result. First, we need the following statement which is proved, for example, in [15, Section 6.4].

LEMMA 5.12

Under the functor \mathbf{F} introduced above, the representation $\mathfrak{o}(n)$ of $\text{Aut } O \times G(O)$ goes to the line bundle $r_X^(K_X^n)$, where r_X is the projection $\mathcal{G}r_\lambda \xrightarrow{r} \text{Bun}_G \times X \rightarrow X$.*

The functor \mathbf{F} also sends K_{Gr_λ} to the relative canonical line bundle of the morphism $r : \mathcal{G}r_\lambda \rightarrow \text{Bun}_G \times X$, which by Proposition 5.9 is the line bundle K_2 introduced in Section 1.2. We also have $r_X = q_3$ under the isomorphism of Proposition 5.9. Therefore, Theorem 5.11 and Lemma 5.12 imply that there is a canonical isomorphism

$$\mathcal{L}_{\mathcal{G}r_\lambda} \simeq K_2 \otimes q_3^*(K_X^{-\langle \lambda, \rho \rangle}). \tag{5.10}$$

On the other hand, under our current assumption that the set $\{\langle \lambda, \rho \rangle, \lambda \in P_+^\vee\}$ only contains integers, Beilinson–Drinfeld construction in [3, Section 4.4.1], produces a square root of the canonical line bundle on Bun_G (normalized by a trivialization of its fiber at the trivial G -bundle). We will denote it by $K_{\text{Bun}}^{1/2}$. The results of [3, Sections 4.4.14, 4.6] imply that under the isomorphism of Proposition 5.9, we have a canonical identification

$$\mathcal{L}_{\mathcal{G}r_\lambda} \simeq q_1^*(K_{\text{Bun}}^{1/2}) \otimes q_2^*(K_{\text{Bun}}^{1/2})^{-1}. \tag{5.11}$$

Combining the isomorphisms (5.10) and (5.11), we obtain the isomorphism (1.2). This completes the proof of Theorem 1.1 under this assumption.

Now suppose that the set $\{\langle \lambda, \rho \rangle, \lambda \in P_+^\vee\}$ contains half-integers. Then we modify the above argument as follows. Define a double cover $\text{Aut}_2 O$ of the group $\text{Aut } O$ as the subgroup of the group $\text{Aut } O \times \mathbb{G}_m$ consisting of pairs (ϕ, w) , where ϕ is given by (5.8) and $w^2 = \phi_0$. Define the homomorphism $\gamma_2 : \text{Aut } O \rightarrow \mathbb{G}_m$ by the formula

$$\gamma_2(\phi, w) = w. \quad (5.12)$$

Next, we fix a square root $K_X^{1/2}$ of the canonical line bundle K_X on X . As explained in [3, Section 4.3.16], we can then extend the above functor \mathbf{F} to a functor \mathbf{F}_2 from (ind-)schemes Y with an action of $\mathrm{Aut}_2 O \times G(O)$ to (ind-)schemes over $\mathrm{Bun}_G \times X$, which has the following defining property. Let $\mathfrak{o}(m)$, $m \in \frac{1}{2}\mathbb{Z}$, be the one-dimensional representation of $\mathrm{Aut}_2 O \times G(O)$ on which $G(O)$ acts trivially and $\mathrm{Aut}_2 O$ acts as the composition of γ_2 given by (5.12) and the one-dimensional representation of \mathbb{G}_m given by $w \mapsto w^{2m}$. Then \mathbf{F}_2 sends $\mathfrak{o}(m)$ to the line bundle $r_X^*(K_X^m)$ for all $m \in \frac{1}{2}\mathbb{Z}$ (here K_X^m stands for $(K_X^{1/2})^{2m}$, where $K_X^{1/2}$ is the chosen square root of K_X).

As shown in [3], a local Pfaffian line bundle $\mathcal{L}_{\mathrm{Gr}}$ on Gr can still be defined in this case, but it is now $(\mathrm{Aut}_2 O \times G(O))$ -equivariant. Moreover, the isomorphism (5.9) then holds as an isomorphism of $(\mathrm{Aut}_2 O \times G(O))$ -equivariant line bundles on Gr_λ and we also have the isomorphism (5.11), where $K_{\mathrm{Bun}}^{1/2}$ denotes the square root of the canonical line bundle on Bun_G associated to the above choice of $K_X^{1/2}$ (see the discussion before Theorem 1.1). Therefore, the same argument proves the isomorphism (1.2) in general. This completes the proof of Theorem 1.1. \square

Acknowledgments. We are grateful to M. Kontsevich for sharing his ideas on Hecke operators, and to A. Braverman, V. Drinfeld, D. Gaiotto, D. Gaitsgory, T. Pantev, and E. Witten for useful discussions. The second author thanks P. Schapira for helpful discussions and references concerning the material of Section 3. We also thank the referees for helpful comments.

Etingof's work was partially supported by National Science Foundation (NSF) grant DMS-1916120. Kazhdan's work was partially supported by European Research Council (ERC) grant 669655.

References

- [1] D. ARINKIN, D. GAITSGORY, D. KAZHDAN, S. RASKIN, N. ROZENBLYUM, and Y. VARSHAVSKY, *The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support*, preprint, [arXiv:2010.01906v2](https://arxiv.org/abs/2010.01906v2) [math.AG]. (2027)
- [2] A. BEILINSON and J. BERNSTEIN, "A proof of Jantzen conjectures" in *I. M. Gelfand Seminar*, Adv. Soviet Math. **16**, Part 1, Amer. Math. Soc., Providence, 1993, 1–50. MR 1237825. (2030, 2046)
- [3] A. BEILINSON and V. DRINFELD, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, preprint, 1991, <http://www.math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf>.

- (2016, 2017, 2019, 2023, 2024, 2025, 2027, 2032, 2043, 2054, 2055, 2058, 2059, 2064, 2065, 2066, 2067, 2068)
- [4] ———, *Opers*, preprint, [arXiv:math/0501398v1](https://arxiv.org/abs/math/0501398v1) [math.AG]. (2023, 2030, 2061)
- [5] A. BRAVERMAN and D. KAZHDAN, *Some examples of Hecke algebras for two-dimensional local fields*, Nagoya Math. J. **184** (2006), 57–84. MR 2285231. DOI 10.1017/S0027763000009314. (2018, 2019, 2020, 2022)
- [6] A. D’AGNOLO and P. SCHAPIRA, *Radon-Penrose transform for \mathcal{D} -modules*, J. Funct. Anal. **139** (1996), no. 2, 349–382. MR 1402769. DOI 10.1006/jfan.1996.0089. (2042, 2053)
- [7] V. G. DRINFELD and V. V. SOKOLOV, *Equations of Korteweg-deVries type and simple Lie algebras* (in Russian), Dokl. Akad. Nauk **258** (1981), no. 1, 11–16; English translation in Math. Dokl. **23** (1981), 457–462. MR 0615463. (2061)
- [8] P. ETINGOF, E. FRENKEL, and D. KAZHDAN, “An analytic version of the Langlands correspondence for complex curves” in *Integrability, Quantization, and Geometry, II: Quantum Theories and Algebraic Geometry*, Proc. Sympos. Pure Math. **103**, Amer. Math. Soc., Providence, 2021, 137–202. MR 4285697. (2016, 2017, 2023, 2024, 2025, 2026, 2027, 2028, 2032, 2033, 2035, 2043)
- [9] ———, *Analytic Langlands correspondence for PGL_2 on \mathbb{P}^1 with parabolic structures over local fields*, Geom. Funct. Anal. **32** (2022), no. 4, 725–831. MR 4472583. DOI 10.1007/s00039-022-00603-w. (2021, 2022, 2032)
- [10] J. FÆRGEAN and S. RASKIN, *Non-vanishing of geometric Whittaker coefficients for reductive groups*, preprint, [arXiv:2207.02955](https://arxiv.org/abs/2207.02955) [math.RT]. (2027)
- [11] G. FALTINGS, *Real projective structures on Riemann surfaces*, Compos. Math. **48** (1983), no. 2, 223–269. MR 0700005. DOI 10.1007/s00229-022-01377-z. (2026, 2027)
- [12] B. FEIGIN and E. FRENKEL, “Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras” in *Infinite Analysis, Part A, B (Kyoto, 1991)*, Adv. Ser. Math. Phys. **16**, World Sci. Publ., River Edge, 1992, 197–215. MR 1187549. DOI 10.1142/s0217751x92003781. (2025)
- [13] E. FRENKEL, *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005), no. 2, 297–404. MR 2146349. DOI 10.1016/j.aim.2004.08.002. (2025)
- [14] ———, *Is there an analytic theory of automorphic functions for complex algebraic curves?*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), no. 042. MR 4098670. DOI 10.3842/SIGMA.2020.042. (2018, 2032, 2033, 2034)
- [15] E. FRENKEL and D. BEN-ZVI, *Vertex Algebras and Algebraic Curves*, 2nd ed., Math. Surveys Monogr. **88**, Amer. Math. Soc., Providence, 2004. MR 2082709. DOI 10.1090/surv/088. (2067)
- [16] E. FRENKEL and C. TELEMAN, *Geometric Langlands correspondence near opers*, J. Ramanujan Math. Soc. **28A** (2013), 123–147. MR 3115191. (2042, 2058)
- [17] D. GAITSGORY, *Outline of the proof of the geometric Langlands conjecture for GL_2* , Astérisque **370** (2015), 1–112. MR 3364744. (2027)

- [18] D. GALLO, M. KAPOVICH, and A. MARDEN, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Ann. of Math. (2) **151** (2000), no. 2, 625–704. MR 1765706. DOI 10.2307/121044. (2026)
- [19] I. M. GELFAND and D. KAZHDAN, *Certain questions of differential geometry and the computation of the cohomologies of the Lie algebras of vector fields* (in Russian), Dokl. Akad. Nauk **200** (1971), 269–272; English translation in Math. Dokl. **12** (1971), 1367–1370. MR 0287566. (2066)
- [20] W. M. GOLDMAN, *Projective structures with Fuchsian holonomy*, J. Differential Geom. **25** (1987), no. 3, 297–326. MR 0882826. DOI 10.4310/jdg/1214440978. (2026)
- [21] A. B. GONCHAROV, *Differential equations and integral geometry*, Adv. Math. **131** (1997), no. 2, 279–343. MR 1483971. DOI 10.1006/aima.1997.1669. (2042)
- [22] M. KASHIWARA, *D-Modules and Microlocal Calculus*, Trans. Math. Monogr. **217**, Amer. Math. Soc., Providence, 2003. MR 1943036. DOI 10.1090/mmono/217. (2049)
- [23] M. KONTSEVICH, “Notes on motives in finite characteristic” in *Algebra, Arithmetic, and Geometry, in Honor of Y. I. Manin, Vol. II*, Progr. Math. **270**, Birkhäuser Boston, Boston, 2010, 213–247. MR 2641191. DOI 10.1007/978-0-8176-4747-6_7. (2018)
- [24] R. P. LANGLANDS, *On the analytic form of the geometric theory of automorphic forms*, preprint, <http://publications.ias.edu/rpl/section/2659> (accessed 7 April 2023). (2017, 2018)
- [25] Y. LASZLO and C. SORGER, *The line bundles on the moduli of parabolic G-bundles over curves and their sections*, Ann. Sci. Éc. Norm. Supér. (4) **30** (1997), no. 4, 499–525. MR 1456243. DOI 10.1016/S0012-9593(97)89929-6. (2017, 2054)
- [26] G. LAUMON, *Transformation de Fourier généralisée*, preprint, arXiv:alg-geom/9603004v1. (2042)
- [27] M. S. NARASIMHAN and S. RAMANAN, “Geometry of Hecke cycles, I” in *C. P. Ramanujam: A Tribute*, Tata Inst. Fund. Res. Stud. Math. **8**, Springer, Berlin, 1978, 291–345. MR 0541029. (2056)
- [28] M. ROTHSTEIN, *Connections on the total Picard sheaf and the KP hierarchy*, Acta Appl. Math. **42** (1996), no. 3, 297–308. MR 1376873. DOI 10.1007/BF01064170. (2042)
- [29] J. TESCHNER, *Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence*, preprint, arXiv:1707.07873v2 [math-ph]. (2016, 2024)
- [30] A. WEIL, “Adèles et groupes algébriques” in *Séminaire Bourbaki, Vol. 5, no. 186*, Soc. Math. France, Paris, 1959, 249–257. MR 1603471. (2017)

Etingof

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; etingof@math.mit.edu; <https://orcid.org/0000-0002-0710-1416>

Frenkel

Department of Mathematics, University of California, Berkeley, California, USA;

frenkel@math.berkeley.edu; <https://orcid.org/0000-0001-6519-8132>

Kazhdan

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel;

kazhdan@math.huji.ac.il