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Stepwise Square Integrability for Nilradicals of Parabolic Subgroups and Maximal Amenable Subgroups

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Abstract

In a series of recent papers ([15], [16], [17], [18]) we extended the notion of square integrability, for representations of nilpotent Lie groups, to that of stepwise square integrability. There we discussed a number of applications based on the fact that nilradicals of minimal parabolic subgroups of real reductive Lie groups are stepwise square integrable. In Part I we prove stepwise square integrability for nilradicals of arbitrary parabolic subgroups of real reductive Lie groups. This is technically more delicate than the case of minimal parabolics. We further discuss applications to Plancherel formulae and Fourier inversion formulae for maximal exponential solvable subgroups of parabolics and maximal aamenable subgroups of real reductive Lie groups. Finally, in Part II, we extend a number of those results to (infinite dimensional) direct limit parabolics.

Part I: Finite Dimensional Theory

1 Stepwise Square Integrable Representations

There is a very precise theory of square integrable representations of nilpotent Lie groups due to Moore and the author [7]. It is based on the Kirillov's general representation theory [2] for nilpotent Lie groups, in which he introduced coadjoint orbit theory to the subject. When a nilpotent Lie group has square integrable representations its representation theory, Plancherel and Fourier inversion formulae, and other aspects of real analysis, become explicit and transparent.

Somewhat later it turned out that many familiar nilpotent Lie groups have foliations, in fact semidirect product towers composed of subgroups that have square integrable representations. These include nilradicals of minimal parabolic subgroups, e.g. the group of strictly upper triangular real or complex matrices. All the analytic benefits of square integrability carry over to stepwise square integrable nilpotent Lie groups.

In order to indicate our results here we must recall the notions of square integrability and stepwise square integrability in sufficient detail to carry them over to nilradicals of arbitrary parabolic subgroups of real reductive Lie groups.

A connected simply connected Lie group N with center Z is called *square integrable*, or is said to *have square integrable reprsentations*, if it has unitary representations π whose coefficients $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$. C.C. Moore and the author worked out the structure and representation theory of these groups [7]. If N has one such square integrable representation then there is a certain polynomial function $\mathrm{Pf}(\lambda)$ on the linear dual space $\mathfrak{z}^*$ of the Lie algebra of Z that is key to harmonic analysis on N. Here $\mathrm{Pf}(\lambda)$ is the Pfaffian of the antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ given by $b_\lambda(x, y) = \lambda([x, y])$. The

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square integrable representations of N are the π_λ (corresponding to coadjoint orbits $\mathrm{Ad}^*(N)\lambda$) where $\lambda \in \mathfrak{z}^*$ with $\mathrm{Pf}(\lambda) \neq 0$, Plancherel almost irreducible unitary representations of N are square integrable, and, up to an explicit constant, $|\mathrm{Pf}(\lambda)|$ is the Plancherel density on the unitary dual $\widehat{N}$ at π_λ. Concretely,

Theorem 1.1. [7] *Let N be a connected simply connected nilpotent Lie group that has square integrable representations. Let Z be its center and $\mathfrak{v}$ a vector space complement to $\mathfrak{z}$ in $\mathfrak{n}$, so $\mathfrak{v}^* = \{\gamma \in \mathfrak{n}^* \mid \gamma|_\mathfrak{z} = 0\}$. If f is a Schwartz class function $N \to \mathbb{C}$ and $x \in N$ then*

$$f(x) = c \int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_x f) |\mathrm{Pf}(\lambda)| d\lambda \tag{1.2}$$

where $c = d!2^d$ with $2d = \dim \mathfrak{n}/\mathfrak{z}$, $r_x f$ is the right translate $(r_x f)(y) = f(yx)$, and Θ is the distribution character

$$\Theta_{\pi_\lambda}(f) = c^{-1} |\mathrm{Pf}(\lambda)|^{-1} \int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi) d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N). \tag{1.3}$$

Here f_1 is the lift $f_1(\xi) = f(\exp(\xi))$ of f from N to $\mathfrak{n}$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$.

More generally, we will consider the situation where

(1.4)
$N = L_1 L_2 \dots L_{m-1} L_m$ where
(a) each factor L_r has unitary representations with coefficients in $L^2(L_r/Z_r)$,
(b) each $N_r := L_1 L_2 \dots L_r$ is a normal subgroup of N with $N_r = N_{r-1} \rtimes L_r$ semidirect,
(c) if $r \geqq s$ then $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$

The conditions of (1.4) are sufficient to construct the representations of interest to us here, but not sufficient to compute the Pfaffian that is the Plancherel density. For that, in the past we used the *strong computability condition*

(1.5) Decompose $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$ (vector space direct) where $\mathfrak{s} = \oplus \mathfrak{z}_r$ and $\mathfrak{v} = \oplus \mathfrak{v}_r$; then $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}_s$ for $r > s$.

The problem is that the strong computability condition (1.5) can fail for some non–minimal real parabolics, but we will see that, for the Plancherel density, we only need the *weak computability condition*

(1.6) Decompose $\mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r$, direct sum of ideals, where $\mathfrak{l}''_r \subset \mathfrak{z}_r$ and $\mathfrak{v}_r \subset \mathfrak{l}'_r$; then $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v}_s$ for $r > s$.

where we retain $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$.

In the setting of (1.4), (1.5) and (1.6) it is useful to denote

(1.7)
(a) $d_r = \frac{1}{2}\dim(\mathfrak{l}_r/\mathfrak{z}_r)$ so $\frac{1}{2}\dim(\mathfrak{n}/\mathfrak{s}) = d_1 + \dots + d_m$, and $c = 2^{d_1+\dots+d_m} d_1! d_2! \dots d_m!$
(b) $b_{\lambda_r} : (x, y) \mapsto \lambda([x, y])$ viewed as a bilinear form on $\mathfrak{l}_r/\mathfrak{z}_r$
(c) $S = Z_1 Z_2 \dots Z_m = Z_1 \times \dots \times Z_m$ where Z_r is the center of L_r
(d) P: polynomial $P(\lambda) = \mathrm{Pf}(b_{\lambda_1})\mathrm{Pf}(b_{\lambda_2}) \dots \mathrm{Pf}(b_{\lambda_m})$ on $\mathfrak{s}^*$
(e) $\mathfrak{t}^* = \{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\}$
(f) $\pi_\lambda \in \widehat{N}$ where $\lambda \in \mathfrak{t}^*$: irreducible unitary representation of $N = L_1 L_2 \dots L_m$ as follows.

Construction 1.8. [16] Given $\lambda \in \mathfrak{t}^*$, in other words $\lambda = \lambda_1 + \dots + \lambda_m$ where $\lambda_r \in \mathfrak{z}_r$ with each $\mathrm{Pf}(b_{\lambda_r}) \neq 0$, we construct $\pi_\lambda \in \widehat{N}$ by recursion on m. If $m = 1$ then π_λ is a square integrable representation of $N = L_1$. Now assume $m > 1$. Then we have the irreducible unitary representation $\pi_{\lambda_1+\dots+\lambda_{m-1}}$ of $L_1 L_2 \dots L_{m-1}$. and (1.4(c)) shows that L_m stabilizes the untary equivalence class of

$\pi_{\lambda_1+\dots+\lambda_{m-1}}$. Since L_m is topologically contractible the Mackey obstruction vanishes and $\pi_{\lambda_1+\dots+\lambda_{m-1}}$ extends to an irreducible unitary representation $\pi'_{\lambda_1+\dots+\lambda_{m-1}}$ on N on the same Hilbert space. View the square integrable representation π_{λ_m} of L_m as a representation of N whose kernel contains $L_1L_2\dots L_{m-1}$. Then we define $\pi_\lambda = \pi'_{\lambda_1+\dots+\lambda_{m-1}}\hat{\otimes}\pi_{\lambda_m}$. ◊

Definition 1.9. The representations π_λ of (1.7(f)), constructed just above, are the *stepwise square integrable* representations of N relative to the decomposition (1.4). If N has stepwise square integrable representations relative to (1.4) we will say that N is *stepwise square integrable.* ◊

Remark 1.10. Construction 1.8 of the stepwise square integrable representations π_λ uses (1.4(c)), $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ for $r > s$, so that L_r stabilizes the unitary equivalence class of $\pi_{\lambda_1+\dots+\lambda_{r-1}}$. The condition (1.5), $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$, enters the picture in proving that the polynomial P of (1.7(d)) is the Pfaffian $\mathrm{Pf} = \mathrm{Pf}_\mathfrak{n}$ of b_λ on $\mathfrak{n}/\mathfrak{s}$. However we don't need that, and the weaker (1.6) is sufficient to show that P is the Plancherel density. See Theorem 1.12 below. ◊

Lemma 1.11. [16] *Assume that N has stepwise square integrable representations. Then Plancherel measure is concentrated on the set $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$ of all stepwise square integrable representations.*

Theorem 1.1 extends to the stepwise square integrable setting, as follows.

Theorem 1.12. [16] *Let N be a connected simply connected nilpotent Lie group that satisfies* (1.4) *and* (1.6)*. Then Plancherel measure for N is concentrated on $\{\pi_\lambda \mid \lambda \in \mathfrak{t}^*\}$. If $\lambda \in \mathfrak{t}^*$, and if u and v belong to the representation space $\mathcal{H}_{\pi_\lambda}$ of π_λ, then the coefficient $f_{u,v}(x) = \langle u, \pi_\nu(x)v\rangle$ satisfies*

$$||f_{u,v}||^2_{L^2(N/S)} = \frac{||u||^2||v||^2}{|P(\lambda)|}. \tag{1.13}$$

The distribution character Θ_{π_λ} of π_λ satisfies

$$\Theta_{\pi_\lambda}(f) = c^{-1}|P(\lambda)|^{-1}\int_{\mathcal{O}(\lambda)} \widehat{f_1}(\xi)d\nu_\lambda(\xi) \text{ for } f \in \mathcal{C}(N) \tag{1.14}$$

where $\mathcal{C}(N)$ is the Schwartz space, f_1 is the lift $f_1(\xi) = f(\exp(\xi))$, $\widehat{f_1}$ is its classical Fourier transform, $\mathcal{O}(\lambda)$ is the coadjoint orbit $\mathrm{Ad}^*(N)\lambda = \mathfrak{v}^* + \lambda$, and $d\nu_\lambda$ is the translate of normalized Lebesgue measure from $\mathfrak{v}^*$ to $\mathrm{Ad}^*(N)\lambda$. The Plancherel formula on N is

$$f(x) = c\int_{\mathfrak{t}^*} \Theta_{\pi_\lambda}(r_x f)|P(\lambda)|d\lambda \text{ for } f \in \mathcal{C}(N). \tag{1.15}$$

Theorem 1.12 is proved in [16] for groups N that satisfy (1.4) together with (1.5). We will need it for (1.4) together with the somewhat less restrictive (1.6). The only point where the argument needs a slight modification is in the proof of (1.13). The action of L_m on $\mathfrak{l}_1 + \dots + \mathfrak{l}_{m-1}$ is unipotent, so there is an L_m–invariant measure preserving decomposition $N_m/S_m = (L_1/Z_1) \times \dots \times (N_m/Z_m)$. The case $m = 1$ is the property $|f_{u,v}|^2_{L^2(L_1/Z_1)} = \frac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda)|} < \infty$ of coefficients of square integrable representations. By induction on m, $|f_{u,v}|^2_{L^2(N_{m-1}/S_{m-1}))} = \frac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|}$ for N_{m-1}. Let π' be the extension of $\pi \in \widehat{N_{m-1}}$ to N. Let $u, v \in \mathcal{H}_{\pi_{\lambda_1+\dots\lambda_{m-1}}}$ and write v_y for $\pi'_{\lambda_1+\dots+\lambda_{m-1}}(y)v$. Let $u', v' \in \mathcal{H}_{\pi_{\lambda_m}}$.

$$\begin{aligned}
||f_{u\otimes u',v\otimes v'}||^2_{L^2(N/S)} &= \int_{N/S} |\langle u, \pi'_{\lambda_1+\dots\lambda_{m-1}}(xy)v\rangle|^2 |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(xyS_m)\\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi'_{\lambda_1+\dots\lambda_{m-1}}(xy)v\rangle|^2 d(xS_{m-1})\right) d(yZ_m)\\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi'_{\lambda_1+\dots\lambda_{m-1}}(x)v_y\rangle|^2 d(xS_{m-1})\right) d(yZ_m)\\
&= \int_{L_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 \left(\int_{N_{m-1}/S_{m-1}} |\langle u, \pi_{\lambda_1+\dots\lambda_{m-1}}(x)v_y\rangle|^2 d(xS_{m-1})\right) d(yZ_m)\\
&= \frac{||u||^2||v_y||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|}\int_{N_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(yZ_m)\\
&= \frac{||u||^2||v||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_{m-1})|}\int_{N_m/Z_m} |\langle u', \pi_{\lambda_m}(y)v'\rangle|^2 d(yZ_m) = \frac{||u\otimes u'||^2||v\otimes v'||^2}{|\mathrm{Pf}(\lambda_1)\dots\mathrm{Pf}(\lambda_m)|} < \infty.
\end{aligned}$$

Thus Theorem 1.12 is valid as stated.

The first goal of this note is to show that if N is the nilradical of a parabolic subgroup Q of a real reductive Lie group, then N is stepwise square integrable, specifically that it satisfies (1.4) and (1.6), so that Theorem 1.12 applies to it. That is Theorem 4.10. The second goal is to examine applications to Fourier analysis on the parabolic Q and to some infinite dimensional parabolics.

In Section 2 we recall the restricted root machinery used in [16] to show that nilradicals of minimal parabolics are stepwise square integrable. In Section 3 we make a first approximation to refine that machinery to apply to general parabolics. That is enough to see that they satisfy (1.4) and construct the stepwise square integrable representations, but not enough to compute the Plancherel density. Then in Section 4 we complete the argument, proving (1.6) to compute the Plancherel density and verify the estimates and inversion formula of Theorem 1.12 for arbitrary parabolic subgroups of real reductive Lie groups. The main result is Theorem 4.10.

In Section 5 we apply Theorem 4.10 to obtain explicit Plancherel and Fourier inversion formulae for the maximal exponential solvable subgroups AN in real real parabolic subgroups $Q = MAN$, following the lines of the minimal parabolic case studied in [17]. The key point here is computation of the Dixmier–Pukánszky operator D for the group AN. Recall that D is a pseudo–differential operator that compensates lack of unimodularity in AN.

There are technical obstacles to extending our results to non-minimal parabolics $Q = MAN$, many involving the orbit types for noncompact reductive groups M, but in Section 6 we do carry out the extension to the maximal amenable subgroups $(M \cap K)AN$. This covers all the maximal amenable subgroups of G that satisfy a certain technical condition [6].

Finally, in Section 6 we look at direct limit parabolics in direct limit locally reductive Lie groups.

2 Specialization to Minimal Parabolics

In order to prove our result for nilradicals of arbitrary parabolics we need to study the construction that gives the decomposition $N = L_1 L_2 \dots L_m$ of 1.4 and the form of the Pfaffian polynomials for the individual the square integrable layers L_r .

Let G be a connected real reductive Lie group, $G = KAN$ an Iwasawa decompsition, and $Q = MAN$ the corresponding minimal parabolic subgroup. Complete $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Now we have root systems

- $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$: roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$ (ordinary roots),
- $\Delta(\mathfrak{g}, \mathfrak{a})$: roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ (restricted roots),
- $\Delta_0(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}) \mid 2\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\}$ (nonmultipliable restricted roots).

The choice of $\mathfrak{n}$ is the same as the choice of a positive restricted root systen $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Define

$$\begin{aligned} &\beta_1 \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximal positive restricted root and} \\ &\beta_{r+1} \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ is a maximum among the roots of } \Delta^+(\mathfrak{g}, \mathfrak{a}) \text{ orthogonal to all } \beta_i \text{ with } i \leqq r \end{aligned} \tag{2.1}$$

The resulting roots (we usually say *root* for *restricted root*) β_r , $1 \leqq r \leqq m$, are mutually strongly orthogonal, in particular mutually orthogonal, and each $\beta_r \in \Delta_0(\mathfrak{g}, \mathfrak{a})$. For $1 \leqq r \leqq m$ define

$$\begin{aligned} &\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\} \text{ and} \\ &\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \dots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}. \end{aligned} \tag{2.2}$$

We know [16, Lemma 6.1] that if $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then either $\alpha \in \{\beta_1, \dots, \beta_m\}$ or α belongs to exactly one of the sets Δ_r^+ . Further [16, Lemma 6.2] if $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ then either $\alpha \in \{\beta_1, \dots, \beta_m\}$ or α belongs to exactly one of the sets Δ_r^+ .

The layers are are the

$$\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{\Delta_r^+} \mathfrak{g}_\alpha \text{ for } 1 \leqq r \leqq m \tag{2.3}$$

Denote

(2.4) $\qquad s_{\beta_r}$ is the Weyl group reflection in β_r and $\sigma_r : \Delta(\mathfrak{g},\mathfrak{a}) \to \Delta(\mathfrak{g},\mathfrak{a})$ by $\sigma_r(\alpha) = -s_{\beta_r}(\alpha)$.

Then σ_r leaves β_r fixed and preserves Δ_r^+. Further, if $\alpha, \alpha' \in \Delta_r^+$ then $\alpha + \alpha'$ is a (restricted) root if and only if $\alpha' = \sigma_r(\alpha)$, and in that case $\alpha + \alpha' = \beta_r$.

From this it follows [16, Theorem 6.11] that $N = L_1 L_2 \dots L_m$ satisfies (1.4) and (1.5), so it has stepwise square integrable representations. Further [16, Lemma 6.4] the L_r are Heisenberg groups in a sense that if $\lambda_r \in \mathfrak{z}_r^*$ with $\mathrm{Pf}_{\mathfrak{l}_r}(\lambda_r) \neq 0$ then $\mathfrak{l}_r / \ker \lambda_r$ is an ordinary Heisenberg group of dimension $\dim \mathfrak{v}_r + 1$.

3 Intersection with an Arbitrary Real Parabolic

Every parabolic subgroup of G is conjugate to a parabolic that contains the minimal parabolic $Q = MAN$. Let Ψ denote the set of simple roots for the positive system $\Delta^+(\mathfrak{g},\mathfrak{a})$. Then the parabolic subgroups of G that contain Q are in one to one correspondence with the subsets $\Phi \subset \Psi$, say $Q_\Phi \leftrightarrow \Phi$, as follows. Denote $\Psi = \{\psi_i\}$ and set

$$\text{(3.1)} \qquad \begin{aligned} \Phi^{red} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta(\mathfrak{g},\mathfrak{a}) \mid n_i = 0 \text{ whenever } \psi_i \notin \Phi\right\} \\ \Phi^{nil} &= \left\{\alpha = \sum\nolimits_{\psi_i \in \Psi} n_i \psi_i \in \Delta^+(\mathfrak{g},\mathfrak{a}) \mid n_i > 0 \text{ for some } \psi_i \notin \Phi\right\}. \end{aligned}$$

Then, on the Lie algebra level, $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi + \mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ where

$$\text{(3.2)} \qquad \begin{aligned} &\mathfrak{a}_\Phi = \{\xi \in \mathfrak{a} \mid \psi(\xi) = 0 \text{ for all } \psi \in \Phi\} = \Phi^\perp, \\ &\mathfrak{m}_\Phi + \mathfrak{a}_\Phi \text{ is the centralizer of } \mathfrak{a}_\Phi \text{ in } \mathfrak{g}, \text{ so } \mathfrak{m}_\Phi \text{ has root system } \Phi^{red}, \text{ and} \\ &\mathfrak{n}_\Phi = \sum\nolimits_{\alpha \in \Phi^{nil}} \mathfrak{g}_\alpha, \text{ nilradical of } \mathfrak{q}_\Phi, \text{ sum of the positive } \mathfrak{a}_\Phi\text{–root spaces.} \end{aligned}$$

Since $\mathfrak{n} = \sum_r \mathfrak{l}_r$, as given in (2.3) we have

$$\text{(3.3)} \qquad \mathfrak{n}_\Phi = \sum\nolimits_r (\mathfrak{n}_\Phi \cap \mathfrak{l}_r) = \sum\nolimits_r \left((\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi) + \sum\nolimits_{\Delta_r^+} (\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi)\right).$$

As $\mathrm{ad}\,(\mathfrak{m})$ is irreducible on each restricted root space, if $\alpha \in \{\beta_r\} \cup \Delta_r^+$ then $\mathfrak{g}_\alpha \cap \mathfrak{n}_\Phi$ is 0 or all of $\mathfrak{g}_\alpha$.

Lemma 3.4. *Suppose $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$. Then $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$.*

Proof. Since $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$, the root β_r has form $\sum_{\psi \in \Phi} n_\psi \psi$ with each $n_\psi \geqq 0$ and $n_\psi = 0$ for $\psi \notin \Phi$. If $\alpha \in \Delta_r^+$ it has form $\sum_{\psi \in \Psi} n'_\psi \psi$ with $0 \leqq n'_\psi \leqq n_\psi$ for each $\psi \in \Psi$. In particular $n'_\psi = 0$ for $\psi \notin \Phi$. Now every root space of $\mathfrak{l}_r$ is contained in $\mathfrak{m}_\Psi$. In particular $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = 0$. □

Remark 3.5. We can define a partial order on $\{\beta_i\}$ by: $\beta_{i+1} \succ \beta_i$ when the set of positive roots of which β_{i+1} is a maximum is contained in the corresponding set for β_i. This is only a consideration when one further disconnects the Dynkin diagram by deleting a node at which $-\beta_i$ attaches, which doesn't happen for type A. If $\beta_s \succ \beta_r$ in this partial order, and $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi = 0$, then $\mathfrak{g}_{\beta_s} \cap \mathfrak{n}_\Phi = 0$ as well, so $\mathfrak{l}_s \cap \mathfrak{n}_\Phi = 0$. ♢

Lemma 3.6. *Suppose $\mathfrak{g}_{\beta_r} \cap \mathfrak{n}_\Phi \neq 0$. Define $J_r \subset \Delta_r^+$ by $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{g}_{\beta_r} + \sum_{J_r} \mathfrak{g}_\alpha$. Decompose $J_r = J'_r \cup J''_r$ (disjoint) where $J'_r = \{\alpha \in J_r \mid \sigma_r \alpha \in J_r\}$ and $J''_r = \{\alpha \in J_r \mid \sigma_r \alpha \notin J_r\}$. Then $\mathfrak{g}_{\beta_r} + \sum_{J''_r} \mathfrak{g}_\alpha$ belongs to a single $\mathfrak{a}_\Phi$–root space in $\mathfrak{n}_\Phi$, i.e. $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$, for every $\alpha \in J''_r$.*

Proof. Two restricted roots $\alpha = \sum_\Psi n_i \psi_i$ and $\alpha' = \sum_\Psi n'_i \psi_i$ have the same restriction to $\mathfrak{a}_\Phi$ if and only if $n_i = n'_i$ for all $\psi_i \notin \Phi$. Now suppose $\alpha \in J''_r$ and $\alpha' = \sigma_r \alpha$. Then $n_i > 0$ for some $\psi_i \notin \Phi$ but $n'_i = 0$ for all $\psi_i \notin \Phi$. Thus α and $\beta_r = \alpha + \sigma_r \alpha$ have the same ψ_i–coefficient $n_i = n_i + n'_i$ for every $\psi_i \notin \Phi$. In other words the corresponding restricted root spaces are contained in the same $\mathfrak{a}_\Phi$–root space. □

Lemma 3.7. *Suppose $\mathfrak{l}_r \cap \mathfrak{n}_\Phi \neq 0$. Then the algebra $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ has center $\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha$, and $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = (\mathfrak{g}_{\beta_r} + \sum_{J_r''} \mathfrak{g}_\alpha) + (\sum_{J_r'} \mathfrak{g}_\alpha))$. Further, $\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \left(\sum_{J_r''} \mathfrak{g}_\alpha\right) \oplus \left(\mathfrak{g}_{\beta_r} + \left(\sum_{J_r'} \mathfrak{g}_\alpha\right)\right)$ direct sum of ideals.*

Proof. This is immediate from the statements and proofs of Lemmas 3.4 and 3.6. □

Following the cascade construction (2.1) it will be convenient to define sets of simple restricted roots

(3.8) $$\Psi_1 = \Psi \text{ and } \Psi_{s+1} = \{\psi \in \Psi \mid \langle \psi, \beta_i \rangle = 0 \text{ for } 1 \leqq i \leqq s\}.$$

Note that Ψ_r is the simple root system for $\{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \alpha \perp \beta_i \text{ for } i < r\}$.

Lemma 3.9. *If $r > s$ then $[\mathfrak{l}_r \cap \mathfrak{n}_\Phi \,,\, \mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha] = 0$.*

Proof. Suppose that $\alpha \in J_s''$. Express α and $\sigma_s\alpha$ as sums of simple roots, say $\alpha = \sum n_i \psi_i$ and $\sigma_s \alpha = \sum n_i' \psi_i$. Then, $n_i' = 0$ for all $\psi_i \in \Psi_s \cap \Phi^{nil}$ and $\beta_s = \sum (n_i + n_i')\psi_i$. In other words the coefficient of ψ_i is the same for α and β_s whenever $\psi_i \in \Psi_s \cap \Phi^{nil}$. Now let $\gamma \in (\{\beta_r\} \cup \Delta_r^+) \cap \Phi^{nil}$ where $r > s$, and express $\gamma = \sum c_i \psi_i$. Then $c_{i_0} > 0$ for some $\beta_{i_0} \in (\Psi_r \cap \Phi^{nil})$. Note $\Psi_r \subset \Psi_s$, so $c_{i_0} > 0$ for some $\beta_{i_0} \in (\Psi_s \cap \Phi^{nil})$. Also, $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s$ because $r > s$. If $\gamma + \alpha$ is a root then its ψ_{i_0}–coefficient is greater than that of β_s, which is impossible. Thus $\gamma + \alpha$ is not a root. The lemma follows. □

We look at a particular sort of linear functional on $\sum_r \bigl(\mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha\bigr)$. Choose $\lambda_r \in \mathfrak{g}_{\beta_r}^*$ such that b_{λ_r} is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$. Set $\lambda = \sum \lambda_r$. We know that (1.4(c)) holds for the nilradical of the minimal parabolic $\mathfrak{q}$ that contains $\mathfrak{q}_\Phi$. In view of Lemma 3.9 it follows that $b_\lambda(\mathfrak{l}_r, \mathfrak{l}_s) = \lambda([\mathfrak{l}_r, \mathfrak{l}_s] = 0$ for $r > s$. For this particular type of λ, the bilinear form b_λ has kernel $\sum_r \bigl(\mathfrak{g}_{\beta_s} + \sum_{J_s''} \mathfrak{g}_\alpha\bigr)$ and is nondegenerate on $\sum_r \sum_{J_r'} \mathfrak{g}_\alpha$.

At this point, the decomposition $N_\Phi = (L_1 \cap N_\Phi)(L_2 \cap N_\Phi) \ldots (L_m \cap N_\Phi)$ satisfies the first two conditions of (1.4):

(a) each factor $L_r \cap N_\Phi$ has unitary representations with coefficients in $L^2((L_r \cap N_\Phi)/(center))$, and

(b) each $N_r \cap N_\Phi := (L_1 \cap N_\Phi) \ldots (L_r \cap N_\Phi)$ is a normal subgroup of N_Φ with $N_r \cap N_\Phi = (N_{r-1} \cap N_\Phi) \rtimes (L_r \cap N_\Phi)$ semidirect.

With Lemma 3.9 this is enough to carry out Construction 1.8 of our representations π_λ of N_Φ. However it is not enough for (1.4(c)) and (1.6). For that we will group the $L_r \cap N_\Phi$ in a way that gives us (1.6) in such a way that (1.4(c)) follows from Lemma 3.9. This will be done in the next section.

4 Extension to Arbitrary Parabolic Nilradicals

In this section we address (1.4(c)) and (1.6), completing the proof that N_Φ has a decomposition that leads to stepwise square integrable representations.

We start with some combinatorics. Denote sets of indices as follows. q_1 is the first index of (1.4) (usually 1) such that $\beta_{q_1}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_1 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_1}|_{\mathfrak{a}_\Phi}\}.$$

Then q_2 is the first index of (1.4) such that $q_2 \notin I_1$ and $\beta_{q_2}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_2 = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_2}|_{\mathfrak{a}_\Phi}\}.$$

Continuing, q_k is the first index of (1.4) such that $q_k \notin (I_1 \cup \cdots \cup I_{k-1})$ and $\beta_{q_k}|_{\mathfrak{a}_\Phi} \neq 0$; define

$$I_k = \{i \mid \beta_i|_{\mathfrak{a}_\Phi} = \beta_{q_k}|_{\mathfrak{a}_\Phi}\}$$

as long as possible. Write ℓ for the last index k that leads to a nonempty set I_k. Then, in terms of the index set of (1.4), $I_1 \cup \cdots \cup I_\ell$ consists of all the indices i for which $\beta_i|_{\mathfrak{a}_\Phi} \neq 0$.

For $1 \leqq j \leqq \ell$ define

(4.1) $$\mathfrak{l}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{l}_i \cap \mathfrak{n}_\Phi) = \Bigl(\sum\nolimits_{i \in I_j} \mathfrak{l}_i\Bigr) \cap \mathfrak{n}_\Phi \text{ and } \mathfrak{l}_{\Phi,j}^{compl} = \sum\nolimits_{k \geqq j} \mathfrak{l}_{\Phi,k}.$$

Lemma 4.2. *If $k \geqq j$ then $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$. For each index j, $\mathfrak{l}_{\Phi,j}$ and $\mathfrak{l}_{\Phi,j}^{compl}$ are subalgebras of $\mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi,j}$ is an ideal in $\mathfrak{l}_{\Phi,j}^{compl}$.*

Proof. As we run along the sequence $\{\beta_1, \beta_2, \dots\}$ the coefficients of the simple roots are weakly decreasing, so in particular the coefficients of the roots in $\Psi \setminus \Phi$ are weakly decreasing. If $r \in I_k$, $s \in I_j$ and $k > j$ now $r > s$. Using $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s$ (and thus $[(\mathfrak{l}_r \cap \mathfrak{n}_\Phi), (\mathfrak{l}_s \cap \mathfrak{n}_\Phi)] \subset \mathfrak{l}_s \cap \mathfrak{n}_\Phi)$ for $r > s$ it follows that $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$ for $k > j$.

Now suppose $k = j$. If $r = s$ then $[\mathfrak{l}_r, \mathfrak{l}_r] = \mathfrak{g}_{\beta_r}$, so we may assume $r > s$, and thus $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}_s \subset \mathfrak{l}_{\Phi,j}$. It follows that $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{l}_{\Phi,j}$ for $k = j$.

Now it is immediate that $\mathfrak{l}_{\Phi,j}$ and $\mathfrak{l}_{\Phi,j}^{compl}$ are subalgebras of $\mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi,j}$ is an ideal in $\mathfrak{l}_{\Phi,j}^{compl}$. □

Lemma 4.3. *If $k > j$ then $[\mathfrak{l}_{\Phi,k}\, , \mathfrak{l}_{\Phi,j}] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} = 0$.*

Proof. This is implicit in Theorem 1.12, which gives (1.6), but we give a direct proof for the convenience of the reader. Let $\mathfrak{g}_\gamma \subset \mathfrak{l}_{\Phi,k}$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}_j$ with $[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha] \cap \sum_{i \in I_j} \mathfrak{g}_{\beta_i} \neq 0$. Then $[\mathfrak{g}_\gamma, \mathfrak{g}_\alpha] = \mathfrak{g}_{\beta_i}$ where $\mathfrak{g}_\gamma \subset \mathfrak{l}_r$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}_i$, so $\mathfrak{g}_\gamma = \mathfrak{g}_{\beta_i - \alpha} \subset \mathfrak{l}_r \cap \mathfrak{l}_i = 0$. That contradiction proves the lemma. □

Given $r \in I_j$ we use the notation of Lemma 3.6 to decompose

$$\mathfrak{l}_r \cap \mathfrak{n}_\Phi = \mathfrak{l}'_r + \mathfrak{l}''_r \text{ where } \mathfrak{l}'_r = \mathfrak{g}_{\beta_r} + \sum\nolimits_{J'_r} \mathfrak{g}_\alpha \text{ and } \mathfrak{l}''_r = \sum\nolimits_{J''_r} \mathfrak{g}_\alpha \,. \tag{4.4}$$

Here J'_r consists of roots $\alpha \in \Delta_r^+$ such that $\mathfrak{g}_\alpha + \mathfrak{g}_{\beta_r - \alpha'} \subset \mathfrak{n}_\Phi$, and J''_r consists of roots $\alpha \in \Delta_r^+$ such that $\mathfrak{g}_\alpha \subset \mathfrak{n}_\Phi$ but $\mathfrak{g}_{\beta_r - \alpha'} \not\subset \mathfrak{n}_\Phi$. For $1 \leqq j \leqq \ell$ define

$$\mathfrak{z}_{\Phi,j} = \sum\nolimits_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}''_i) \tag{4.5}$$

and decompose

$$\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} + \mathfrak{l}''_{\Phi,j} \text{ where } \mathfrak{l}'_{\Phi,j} = \sum\nolimits_{i \in I_j} \mathfrak{l}'_i \text{ and } \mathfrak{l}''_{\Phi,j} = \sum\nolimits_{i \in I_j} \mathfrak{l}''_i \,. \tag{4.6}$$

Lemma 4.7. *Recall $\mathfrak{l}_{\Phi,j}^{compl} = \sum_{k \geqq j} \mathfrak{l}_{\Phi,k}$ from (4.1). For each j, both $\mathfrak{z}_{\Phi,j}$ and $\mathfrak{l}''_{\Phi,j}$ are central ideals in $\mathfrak{l}_{\Phi,j}^{compl}$, and $\mathfrak{z}_{\Phi,j}$ is the center of $\mathfrak{l}_{\Phi,j}$.*

Proof. Lemma 3.6 shows that $\alpha|_{\mathfrak{a}_\Phi} = \beta_i|_{\mathfrak{a}_\Phi}$ whenever $i \in I_j$ and $\mathfrak{g}_\alpha \subset \mathfrak{l}''_{\Phi,j}$. If $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}''_i] \neq 0$ it contains some $\mathfrak{g}_\delta$ such that $\mathfrak{g}_\delta \subset \mathfrak{l}_{\Phi,j}$ and at least one of the coefficients of δ along roots of $\Psi \setminus \Phi$ is greater than that of β_i. As $\mathfrak{g}_\delta \subset \mathfrak{l}_i$ that is impossible. Thus $\mathfrak{l}''_{\Phi,j}$ is a central ideal in $\mathfrak{l}_{\Phi,j}^{compl}$. The same is immediate for $\mathfrak{z}_{\Phi,j} = \sum_{i \in I_j} (\mathfrak{g}_{\beta_i} + \mathfrak{l}''_i)$. In particular $\mathfrak{z}_{\Phi,j}$ is central in $\mathfrak{l}_{\Phi,j}$. But the center of $\mathfrak{l}_{\Phi,j}$ can't be any larger, by definition of $\mathfrak{l}'_{\Phi,j}$. □

Decompose

$$\mathfrak{n}_\Phi = \mathfrak{z}_\Phi + \mathfrak{v}_\Phi \text{ where } \mathfrak{z}_\Phi = \sum_j \mathfrak{z}_{\Phi,j}\ ,\ \mathfrak{v}_\Phi = \sum_j \mathfrak{v}_{\Phi,j} \text{ and } \mathfrak{v}_{\Phi,j} = \sum_{i \in I_j} \sum_{\alpha \in J'_i} \mathfrak{g}_\alpha \,. \tag{4.8}$$

Then Lemma 4.7 gives us (1.6) for the $\mathfrak{l}_{\Phi,j}$: $\mathfrak{l}_{\Phi,j} = \mathfrak{l}'_{\Phi,j} \oplus \mathfrak{l}''_{\Phi,j}$ with $\mathfrak{l}''_{\Phi,j} \subset \mathfrak{z}_{\Phi,j}$ and $\mathfrak{v}_{\Phi,j} \subset \mathfrak{l}'_{\Phi,j}$.

Lemma 4.9. *For generic $\lambda_j \in \mathfrak{z}_{\Phi,j}^*$ the kernel of b_{λ_j} on $\mathfrak{l}_{\Phi,j}$ is just $\mathfrak{z}_{\Phi,j}$, in other words b_{λ_j} is is nondegenerate on $\mathfrak{v}_{\Phi,j} \simeq \mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j}$. In particular $L_{\Phi,j}$ has square integrable representations.*

Proof. From the definition of $\mathfrak{l}'_{\Phi,j}$, the bilinear form b_{λ_j} on $\mathfrak{l}_{\Phi,j}$ annihilates the center $\mathfrak{z}_{\Phi,j}$ and is nondegenerate on $\mathfrak{v}_{\Phi,j}$. Thus the corresponding representation π_{λ_j} of $L_{\Phi,j}$ has coefficients that are square integrable modulo its center. □

Now we come to our first main result:

Theorem 4.10. *Let G be a real reductive Lie group and Q a real parabolic subgroup. Express $Q = Q_\Phi$ in the notation of* (3.1) *and* (3.2)*. Then its nilradical N_Φ has decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ that satisfies the conditions of* (1.4) *and* (1.6) *as follows. The center $Z_{\Phi,j}$ of $L_{\Phi,j}$ is the analytic subgroup for $\mathfrak{z}_{\Phi,j}$ and*

(a) *each factor $L_{\Phi,j}$ has unitary representations with coefficients in $L^2(L_{\Phi,j}/Z_{\Phi,j})$, and*

(b) *each $N_{\Phi,j} := L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,j}$ is a normal subgroup of N_Φ*
with $N_{\Phi,j} = N_{\Phi,j-1} \rtimes L_{\Phi,j}$ semidirect,

(c) $[\mathfrak{l}_{\Phi,k}, \mathfrak{z}_{\Phi,j}] = 0$ *and* $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_\Phi + \mathfrak{l}''_{\Phi,j}$ *for $k > j$.*

In particular N_Φ has stepwise square integrable representations relative to the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$.

Proof. Statement (a) is the content of Lemma 4.9, and statement (b) follows from Lemma 4.2. The first part of (c), $[\mathfrak{l}_{\Phi,k}, \mathfrak{z}_{\Phi,j}] = 0$ for $k > j$, is contained in Lemma 4.7. The second part, $[\mathfrak{l}_{\Phi,k}, \mathfrak{l}_{\Phi,j}] \subset \mathfrak{v}_\Phi + \mathfrak{l}''_{\Phi,j}$ for $k > j$, follows from Lemma 4.3. □

5 The Maximal Exponential–Solvable Subgroup $A_\Phi N_\Phi$

In this section we extend the considerations of [17, §4] from minimal parabolics to the exponential–solvable subgroups $A_\Phi N_\Phi$ of real parabolics $Q_\Phi = M_\Phi A_\Phi N_\Phi$. It turns out that the of Plancherel and Fourier inversion formulae of N_Φ go through, with only small changes, to the non–unimodular solvable group $A_\Phi N_\Phi$. We follow the development in [17, §4].

Let H be a separable locally compact group of type I. Then [3, §1] the Fourier inversion formula for H has form

$$f(x) = \int_{\widehat{H}} \operatorname{trace} \pi(D(r(x)f))d\mu_H(\pi) \tag{5.1}$$

where D is an invertible positive self adjoint operator on $L^2(H)$, conjugation semi–invariant of weight equal to that of the modular function δ_H, and μ is a positive Borel measure on the unitary dual $\widehat{H}$. When H is unimodular, D is the identity and (5.1) reduces to the usual Fourier inversion formula for H. In general the semi–invariance of D compensates any lack of unimodularity. See [3, §1] for a detailed discusison including a discussion of the domains of D and $D^{1/2}$. Here $D \otimes \mu$ is unique up to normalization of Haar measure, but (D, μ) is not unique, except of course when we fix one of them, such as in the unimodular case when we take $D = 1$. Given such a pair (D, μ) we refer to D as a *Dixmier–Pukánszky operator* and to μ as the associated Plancherel measure.

The goal of this section is to describe a "best" choice of the Dixmier–Pukánszky operator for $A_\Phi N_\Phi$ in terms of the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ that gives stepwise square integrable representations of N_Φ.

Let δ denote the modular function of Q_Φ. Its kernel contains $M_\Phi N_\Phi$ because $\mathrm{Ad}(M_\Phi)$ is reductive with compact center and $\mathrm{Ad}(N_\Phi)$ is unipotent. Thus $\delta(man) = \delta(a)$, and if $\xi \in \mathfrak{a}_\Phi$ then $\delta(\exp(\xi)) = \exp(\operatorname{trace}(\operatorname{ad}(\xi)))$. Note that δ also is the modular function for $A_\Phi N_\Phi$.

Lemma 5.2. *Let $\xi \in \mathfrak{a}_\Phi$. Then each $\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j}$ is even, and*
(i) *the trace of* $\operatorname{ad}(\xi)$ *on* $\mathfrak{l}_{\Phi,j}$ *is* $\frac{1}{2}(\dim(\mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)$ *for any $j_0 \in I_j$,*
(ii) *the trace of* $\operatorname{ad}(\xi)$ *on $\mathfrak{n}_\Phi$, on $\mathfrak{a}_\Phi + \mathfrak{n}_\Phi$ and on $\mathfrak{q}_\Phi$ is* $\frac{1}{2}\sum_j(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)$, *and*
(iii) *the determinant of* $\mathrm{Ad}(\exp(\xi))$ *on $\mathfrak{n}_\Phi$, on $\mathfrak{a}_\Phi + \mathfrak{n}_\Phi$, and on $\mathfrak{q}_\Phi$, is* $\prod_j \exp(\beta_{j_0}(\xi))^{\frac{1}{2}(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j}}$.

Proof. We use the notation of (4.4), (4.5) and (4.6). It is immediate that $\dim \mathfrak{l}_r + \dim(\mathfrak{g}_{\beta_r} + \mathfrak{l}''_r)$ is even. Sum over $r \in I_j$ to see that $\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j}$ is even.

The trace of $\operatorname{ad}(\xi)$ on $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ is $(\dim \mathfrak{g}_{\beta_r})\beta_r(\xi)$ on $\mathfrak{g}_{\beta_r}$, plus $\frac{1}{2}\sum_{\alpha \in J'_r}(\dim \mathfrak{g}_\alpha)\beta_r(\xi)$ (for the pairs $\mathfrak{g}_\alpha, \mathfrak{g}'_\alpha \in \Delta^+_r \cap \Phi^{nil}$ that pair into $\mathfrak{g}_{\beta_r}$), plus $\sum_{\alpha \in J''_r}(\dim \mathfrak{g}_\alpha)\beta_r(\xi)$ (since $\alpha \in J''_r$ implies $\alpha|_{\mathfrak{a}_\Phi} = \beta_r|_{\mathfrak{a}_\Phi}$).

Now the trace of $\mathrm{ad}\,(\xi)$ on $\mathfrak{l}_r \cap \mathfrak{n}_\Phi$ is

$$(\tfrac{1}{2}\dim \mathfrak{g}_{\beta_r} + \tfrac{1}{2}\dim \mathfrak{l}'_r + \dim \mathfrak{l}''_r)\beta_r(\xi) = \tfrac{1}{2}(\dim(\mathfrak{l}_r \cap \mathfrak{n}_\Phi) + \dim(\mathfrak{g}_{\beta_r} + \mathfrak{l}''_r)\beta_r(\xi)$$

summing over $r \in I_j$ we arrive at assertion (i). Then sum over j for (ii) and exponentiate for (iii). □

We reformulate Lemma 5.2 as

Lemma 5.3. *The modular function $\delta = \delta_{Q_\Phi}$ of $Q_\Phi = M_\Phi A_\Phi N_\Phi$ is*

$$\delta(man) = \prod_j \exp(\beta_{j_0}(\log a))^{\frac{1}{2}(\dim \mathfrak{l}_{\Phi,j} + \dim \mathfrak{z}_{\Phi,j})}.$$

The modular function $\delta_{A_\Phi N_\Phi}$ is $\delta|_{A_\Phi N_\Phi}$.

Consider semi-invariance of the polynomial P of (1.7(d)), which by definition is the product of factors $\mathrm{Pf}_{\mathfrak{l}_{\Phi,j}}$. Using (4.8) and Lemma 4.9, calculate with bases of the $\mathfrak{v}_{\Phi,j}$ as in [17, Lemma 4.4] to arrive at

Lemma 5.4. *Let $\xi \in \mathfrak{a}_\Phi$ and $a = \exp(\xi) \in A_\Phi$. Then $\mathrm{ad}\,(\xi)P = \left(\frac{1}{2}\sum_j \dim(\mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j})\beta_{j_0}(\xi)\right) P$ and $\mathrm{Ad}(a)P = \left(\prod_j (\exp(\beta_{j_0}(\xi)))^{\frac{1}{2}\sum_j \dim(\mathfrak{l}_{\Phi,j}/\mathfrak{z}_{\Phi,j})}\right) P$.*

Definition 5.5. The *quasi-center* of $\mathfrak{n}_\Phi$ is $\mathfrak{s}_\Phi = \sum_j \mathfrak{z}_{\Phi,j}$. Fix a basis $\{e_t\}$ of $\mathfrak{s}_\Phi$ consisting of ordinary root vectors, $e_t \in \mathfrak{g}_{\alpha_t}$. The *quasi-center determinant* relative to the choice of $\{e_t\}$ is the polynomial function $\mathrm{Det}_{\mathfrak{s}_\Phi}(\lambda) = \prod_t \lambda(e_t)$ on $\mathfrak{s}^*_\Phi$. ♢

Let $a \in A_\Phi$ and compute $(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}_\Phi})(\lambda) = \mathrm{Det}_{\mathfrak{s}_\Phi}(\mathrm{Ad}^*(a)^{-1}\lambda) = \prod_t \lambda(\mathrm{Ad}(a)e_t)$. Each $e_t \in \mathfrak{z}_{\Phi,j}$ is multiplied by $\exp(\beta_{j_0}(\log a))$. So $(\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}_\Phi})(\lambda) = \left(\prod_j \exp(\beta_{j_0}(\log a))^{\dim \mathfrak{z}_{\Phi,j}}\right)\mathrm{Det}_{\mathfrak{s}_\Phi}(\lambda)$. Now

Lemma 5.6. *If $\xi \in \mathfrak{a}_\Phi$ then $\mathrm{Ad}(\exp(\xi))\mathrm{Det}_{\mathfrak{s}_\Phi} = \left(\prod_j \exp(\beta_{j_0}(\xi))^{\dim \mathfrak{z}_{\Phi,j}}\right)\mathrm{Det}_{\mathfrak{s}_\Phi}$ $(j_0 \in I_j)$.*

Combining Lemmas 5.2, 5.3 and 5.6 we have

Proposition 5.7. *The product $P \cdot \mathrm{Det}_{\mathfrak{s}_\Phi}$ is an $\mathrm{Ad}(Q_\Phi)$-semi-invariant polynomial on $\mathfrak{s}^*_\Phi$ of degree $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{s}_\Phi)$ and of weight equal to the weight of the modular function δ_{Q_Φ}.*

Denote $V_\Phi = \exp(\mathfrak{v}_\Phi)$ and $S_\Phi = \exp(\mathfrak{s}_\Phi)$. Then $V_\Phi \times S_\Phi \to N_\Phi$, by $(v, s) \mapsto vs$, is an analytic diffeomorphism. Define

(5.8) $\quad D:$ Fourier transform of $P \cdot \mathrm{Det}_{\mathfrak{s}_\Phi}$ acting on $\mathrm{A_\Phi N_\Phi = A_\Phi V_\Phi S_\Phi}$ by acting on the $\mathrm{S_\Phi}$ variable.

Theorem 5.9. *The operator D of (5.8) is an invertible self–adjoint differential operator of degree $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{d}_\Phi)$ on $L^2(A_\Phi N_\Phi)$ with dense domain the Schwartz space $\mathcal{C}(A_\Phi N_\Phi)$, and it is $\mathrm{Ad}(M_\Phi A_\Phi N_\Phi)$ semi–invariant of weight equal to that of the modular function. In other words, $|D|$ is a Dixmier–Pukánszky operator on $A_\Phi N_\Phi$ with domain equal to the space of rapidly decreasing C^∞ functions.*

Proof. Since it is the Fourier transform of a real polynomial, D is a differential operator which is invertible and self–adjoint on $L^2(A_\Phi N_\Phi)$. Its degree as differential operator is the same as the degree of the polynomial. Further it has dense domain $\mathcal{C}(A_\Phi N_\Phi)$. By Proposition 5.7 its degree is $\frac{1}{2}(\dim \mathfrak{n}_\Phi + \dim \mathfrak{s}_\Phi)$ and D is $\mathrm{Ad}(M_\Phi A_\Phi N_\Phi)$ semi–invariant as claimed. □

The action of $\mathfrak{a}_\Phi$ on $\mathfrak{z}_{\Phi,j}$ is scalar, $\mathrm{ad}\,(\alpha)\zeta = \beta_{j_0}(\alpha)\zeta$ where (as before) $j_0 \in I_j$. So the isotropy algebra $(\mathfrak{a}_\Phi)_\lambda$ is the same at every $\lambda \in \mathfrak{t}^*_\Phi$, given by $(\mathfrak{a}_\Phi)_\lambda = \{\alpha \in \mathfrak{a}_\Phi \mid \text{ every } \beta_{j_0}(\alpha) = 0\}$. Thus the (A_Φ)–stabilizer on $\mathfrak{t}^*_\Phi$ is

(5.10) $$A'_\Phi := \{\exp(\alpha) \mid \text{ every } \beta_{j_0}(\alpha) = 0\}, \text{ independent of choice of } \lambda \in \mathfrak{t}^*_\Phi.$$

Given $\lambda \in \mathfrak{t}^*_\Phi$, in other words give a stepwise square integrable representation π_λ where $\lambda \in \mathfrak{s}^*_\Phi$, we write $\pi^\dagger_\lambda$ for the extension of π_λ to a representation of $A'_\Phi N_\Phi$ on the same Hilbert space. That extension exists because A'_Φ is a vector group, thus contractible to a point, so $H^2(A'_\Phi; \mathbb{C}') = H^2(point; \mathbb{C}') = \{1\}$, and the Mackey obstruction vanishes. Now the representations of $A'_\Phi N_\Phi$ corresponding to π_λ are the

(5.11) $$\pi_{\lambda,\phi} := \mathrm{Ind}\,_{A'_\Phi N_\Phi}^{A_\Phi N_\Phi}(\exp(i\phi) \otimes \pi^\dagger_\lambda) \text{ where } \phi \in \mathfrak{a}'_\Phi.$$

Note also that

(5.12) $$\pi_{\lambda,\phi} \cdot \mathrm{Ad}(an) = \pi_{\mathrm{Ad}^*(a)\lambda,\phi} \text{ for } a \in A_\Phi \text{ and } n \in N_\Phi.$$

The resulting Plancherel formula (5.1), $f(x) = \int_{\widehat{H}} \mathrm{trace}\,\pi(D(r(x)f))d\mu_H(\pi)$, $H = A_\Phi N_\Phi$, is

Theorem 5.13. *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Given $\pi_{\lambda,\phi} \in \widehat{A_\Phi N_\Phi}$ as described in* (3.1) *and* (3.2) *let $\Theta_{\pi_{\lambda,\phi}} : h \mapsto \operatorname{trace} \pi_{\lambda,\phi}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\phi}}$ is a tempered distribution. If $f \in \mathcal{C}(A_\Phi N_\Phi)$ then*

$$f(x) = c \int_{(\mathfrak{a}'_\Phi)^*} \left(\int_{\mathfrak{s}^*_\Phi/\mathrm{Ad}^*(A_\Phi)} \Theta_{\pi_{\lambda,\phi}}(D(r(x)f)) |\mathrm{Pf}(\lambda)| d\lambda \right) d\phi$$

where $c > 0$ depends on normalizations of Haar measures

Proof. We compute along the lines of the computation of [4, Theorem 2.7] and [5, Theorem 3.2].

$$\begin{aligned}
&\operatorname{trace} \pi_{\lambda,\phi}(Dh) \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \delta(x)^{-1} \operatorname{trace} \int_{N_\Phi A'_\Phi} (Dh)(x^{-1}nax) \cdot (\pi^\dagger_\lambda \otimes \exp(i\phi))(na)\, dn\, da\, dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \operatorname{trace} \int_{N_\Phi A'_\Phi} (Dh)(nx^{-1}ax) \cdot (\pi^\dagger_\lambda \otimes \exp(i\phi))(xnx^{-1}a)\, dn\, da\, dx.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{(\mathfrak{a}'_\Phi)^*} \operatorname{trace} \pi_{\lambda,\phi}(Dh)\, d\phi \\
&\quad = \int_{\widehat{A'_\Phi}} \int_{x \in A_\Phi/A'_\Phi} \operatorname{trace} \int_{N_\Phi A'_\Phi} (Dh)(nx^{-1}ax)(\pi^\dagger_\lambda \otimes \exp(i\phi))(xnx^{-1}a)\, dn\, da\, dx\ d\phi \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \int_{\widehat{A'_\Phi}} \operatorname{trace} \int_{N_\Phi A'_\Phi} (Dh)(nx^{-1}ax)(\pi^\dagger_\lambda \otimes \exp(i\phi))(xnx^{-1}a)\, dn\, da\ d\phi\ dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \operatorname{trace} \int_{N_\Phi} (Dh)(n)\pi^\dagger_\lambda(xnx^{-1}) dn\, dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \operatorname{trace} \int_{N_\Phi} (Dh)(n)(\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)(n) dn\, dx \qquad (5.14) \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \operatorname{trace} (\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)(Dh))\, dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} (\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)_*(D)\ \operatorname{trace} (\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)(h) dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} (\pi^\dagger_\lambda)_*(\mathrm{Ad}(x) \cdot D) \operatorname{trace} (\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)(h)\, dx \\
&\quad = \int_{x \in A_\Phi/A'_\Phi} \delta_{A_\Phi N_\Phi}(x) \operatorname{trace} (\mathrm{Ad}(x^{-1}) \cdot \pi^\dagger_\lambda)(h)\, dx = \int_{\lambda' \in \mathrm{Ad}^*(A_\Phi)\lambda} \operatorname{trace} \pi^\dagger_{\lambda'}(h) |\mathrm{Pf}(\lambda')| d\lambda'.
\end{aligned}$$

Summing over $\overline{\lambda} = \mathrm{Ad}^*(A_\Phi)(\lambda) \in \mathfrak{t}^*/\mathrm{Ad}^*(A_\Phi)$ we now have

$$\begin{aligned}
&\int_{\overline{\lambda} \in \mathfrak{t}^*_\Phi/\mathrm{Ad}^*(A_\Phi)} \left(\int_{(\mathfrak{a}'_\Phi)^*} \operatorname{trace} \pi_{\lambda,\phi}(Dh)\, d\phi \right) d\overline{\lambda} \\
&\qquad = \int_{\overline{\lambda} \in \mathfrak{t}^*_\Phi/\mathrm{Ad}^*(A_\Phi)} \left(\int_{\lambda' \in \mathrm{Ad}^*(A_\Phi)\lambda} \operatorname{trace} \pi^\dagger_{\lambda'}(h) |\mathrm{Pf}(\lambda')| d\lambda' \right) d\overline{\lambda} \qquad (5.15) \\
&\qquad = \int_{\lambda \in \mathfrak{s}^*_\Phi} \operatorname{trace} \pi_\lambda(h) |\mathrm{Pf}(\lambda)| d\lambda = h(1).
\end{aligned}$$

Let h denote any right translate of f. The theorem follows. ∎

6 The Maximal Amenable Subgroup $U_\Phi A_\Phi N_\Phi$

In this section we extend our results on N_Φ and $A_\Phi N_\Phi$ to the maximal amenable subgroups

$$E_\Phi := U_\Phi A_\Phi N_\Phi \text{ where } U_\Phi \text{ is a maximal compact subgroup of } M_\Phi\,.$$

Of course if $\Phi = \emptyset$, i.e. if Q_Φ is a minimal parabolic, then $U_\Phi = M_\Phi$. We start by recalling the classification of maximal amenable subgroups in real reductive Lie groups.

Recall the definition. A *mean* on a locally compact group H is a linear functional μ on $L^\infty(H)$ of norm 1 and such that $\mu(f) \geqq 0$ for all real–valued $f \geqq 0$. H is *amenable* if it has a left–invariant mean. There are more than a dozen useful equivalent conditions. Solvable groups and compact groups are amenable, as are extensions of amenable groups by amenable subgroups. In particular if U_Φ is a maximal compact subgroup of M_Φ then $E_\Phi := U_\Phi A_\Phi N_\Phi$ is amenable.

We'll need a technical condition [6, p. 132]. Let H be the group of real points in a linear algebraic group whose rational points are Zariski dense, let A be a maximal $\mathbb{R}$–split torus in H, let $Z_H(A)$ denote the centralizer of A in H, and let H_0 be the algebraic connected component of the identity in H. Then H is *isotropically connected* if $H = H_0 \cdot Z_H(A)$. More generally we will say that a subgroup $H \subset G$ is *isotropically connected* if the algebraic hull of $\mathrm{Ad}_G(H)$ is isotropically connected. The point is Moore's theorem

Proposition 6.1. [6, Theorem 3.2]. *The groups $E_\Phi := U_\Phi A_\Phi N_\Phi$ are maximal amenable subgroups of G. They are isotropically connected and self–normalizing. As Φ runs over the $2^{|\Psi|}$ subsets of Ψ the E_Φ are mutually non–conjugate. An amenable subgroup $H \subset G$ is contained in some E_Φ if and only if it is isotropically connected.*

Now we need some notation and definitions.

$$\text{if } \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \text{ then } [\alpha] = [\alpha]_\Phi = \{\gamma \in \Delta^+(\mathfrak{g},\mathfrak{a}) \mid \gamma|_{\mathfrak{a}_\Phi} = \alpha|_{\mathfrak{a}_\Phi}\} \text{ and } \mathfrak{g}_{[\alpha]} = \textstyle\sum_{\gamma\in[\alpha]} \mathfrak{g}_\gamma\,.$$

Recall [12, Theorem 8.3.13] that the various $\mathfrak{g}_{[\alpha]}$, $\alpha \notin \Phi^{red}$, are $\mathrm{ad}\,(\mathfrak{m}_\Phi)$–invariant and are absolutely irreducible as $\mathrm{ad}\,(\mathfrak{m}_\Phi)$–modules.

Definition 6.2. The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ of Theorem 4.10 is *strongly invariant* if each $\mathrm{ad}\,(\mathfrak{m}_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, equivalently if each $\mathrm{Ad}(M_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$, in other words whenever $\mathfrak{z}_{\Phi,j} = \mathfrak{g}_{[\beta_{j_0}]}$. The decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is *weakly invariant* if each $\mathrm{Ad}(U_\Phi)\mathfrak{z}_{\Phi,j} = \mathfrak{z}_{\Phi,j}$. ◊

Here are some special cases.

(1) If Φ is empty, i.e. if Q_Φ is a minimal parabolic, then the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is strongly invariant.

(2) If $|\Psi \setminus \Phi| = 1$, i.e. if Q_Φ is a maximal parabolic, then $N_\Phi = L_{\Phi,1}$, strongly invariant.

(3) Let $G = SL(6;\mathbb{R})$ with simple roots $\Psi = \{\psi_1,\dots,\psi_5\}$ in the usual order and $\Phi = \{\psi_1,\psi_4,\psi_5\}$. Then $\beta_1 = \psi_1 + \dots + \psi_5$, $\beta_2 = \psi_2 + \psi_3 + \psi_4$ and $\beta_3 = \psi_3$. Note $\beta_1|_{\mathfrak{a}_\Phi} = \beta_2|_{\mathfrak{a}_\Phi} \neq \beta_3|_{\mathfrak{a}_\Phi} = (\psi_3+\psi_4)|_{\mathfrak{a}_\Phi}$. Thus $\mathfrak{n}_\Phi = \mathfrak{l}_{\Phi,1} + \mathfrak{l}_{\Phi_2}$ with $\mathfrak{l}_{\Phi,1} = (\mathfrak{l}_1 + \mathfrak{l}_2) \cap \mathfrak{n}_\Phi$ and $\mathfrak{l}_{\Phi_2} = \mathfrak{g}_{\beta_3}$. Now $\mathfrak{g}_{[\beta_3]} \neq \mathfrak{z}_{\Phi,2}$ so the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is not strongly invariant.

(4) In the example just above, $[\beta_3] = \{\psi_3, \psi_3+\psi_4, \psi_3+\psi_4+\psi_5\}$. The semisimple part $[\mathfrak{m}_\Phi,\mathfrak{m}_\Phi]$ of $\mathfrak{m}_\Phi$ is direct sum of $\mathfrak{m}_1 = \mathfrak{sl}(2;\mathbb{R})$ with simple root ψ_1 and $\mathfrak{m}_{4,5} = \mathfrak{sl}(3;\mathbb{R})$ with simple roots ψ_4 and ψ_5. The action of $[\mathfrak{m}_\Phi,\mathfrak{m}_\Phi]$ on $\mathfrak{g}_{[\beta_3]}$ is trivial on $\mathfrak{m}_1$ and the usual (vector) representation of $\mathfrak{m}_{4,5}$. That remains irreducible on the maximal compact $\mathfrak{so}(3)$ in $\mathfrak{m}_{4,5}$. It follows that here the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is not weakly invariant.

Lemma 6.3. *Let $F = \exp(i\mathfrak{a}) \cap K$. Then F is an elementary abelian 2–group of cardinality $\leqq 2^{\dim \mathfrak{a}}$. In particular, F is finite, and if $x \in F$ then $x^2 = 1$. Further, F is central in M_Φ (thus also in U_Φ), $U_\Phi = FU_\Phi^0$, $E_\Phi = FE_\Phi^0$ and $M_\Phi = FM_\Phi^0$.*

Proof. Let θ be the Cartan involution of G for which $K = G^\theta$. If $x \in F$ then $x = \theta(x) = x^{-1}$ so $x^2 = 1$. Now F is an elementary abelian 2–group of cardinality $\leqq 2^{\dim \mathfrak{a}}$, in particular F is finite.

Let G_u denote the compact real form of $G_{\mathbb{C}}$ such that $G \cap G_u = K$, and let $(A_\Phi)_u$ denote the torus subgroup $\exp(i\mathfrak{a}_\Phi)$. The centralizer $Z_{G_u}((A_\Phi)_u)$ is connected. Let $x \in U_\Phi$. It belongs to a maximal torus $(H_\Phi)_u(A_\Phi)_u)$ of $Z_{G_u}((A_\Phi)_u)$. As $x \in K$ we max choose $(H_\Phi)_u$ to be invariant under θ. In other words $(H_\Phi)_u$ is a compact real form of a group $(H_\Phi)_{\mathbb{C}}$ where $H_\Phi \subset M_\Phi$. Here $H_\Phi = H'_\Phi H''_\Phi$ where $H'_\Phi = K \cap H_\Phi$ and $H''_\Phi \subset A$. Express $x = x_1 x_2$ where $x_1 \in H'_\Phi$ and $x_2 \in H''_\Phi$. Note that $H''_\Phi \subset A \cap M_\Phi$ is connected so $x_2 \in U^0_\Phi$. Also, $H'_\Phi = \exp(\mathfrak{k} \cap \mathfrak{h}_\Phi)(K \cap (\exp(i\mathfrak{h}''_\Phi)\exp(i\mathfrak{a}_\Phi))) = \exp(\mathfrak{k} \cap \mathfrak{h}_\Phi)(K \cap (\exp(i\mathfrak{a}_\Phi)) = F\exp(\mathfrak{k} \cap \mathfrak{h}_\Phi)$. Now $x \in FU^0_\Phi$. We have proved $U_\Phi \subset FU^0_\Phi$. Since U_Φ is a maximal compact subgroup of M_Φ and the latter has only finitely many topological components it follows that $M_\Phi \subset FM^0_\Phi$. Since $F \subset M \subset U_\Phi \subset M_\Phi$ now $U_\Phi = FU^0_\Phi$ and $M_\Phi = FM^0_\Phi$. As E_Φ is the semidirect product of U_Φ with an exponential solvable (thus topologically contractible) group it also follows that $E_\Phi = FE^0_\Phi$. □

Lemma 6.4. *The action of F on $\mathfrak{s}^*_\Phi$ is trivial.*

Proof. We know that the action of F is trivial on each $\mathfrak{z}^*_j$ [17, Proposition 3.6]. The action of M_Φ is absolutely irreducible on every $\mathfrak{a}_\Phi$–root space [12, Theorem 8.13.3]. Using Lemma 3.6 we see that the action of F is trivial on each $\mathfrak{z}_{\Phi,j}$, thus trivial on their sum $\mathfrak{s}_\Phi$, by duality trivial on $\mathfrak{s}^*_\Phi$. □

When $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant we can proceed more or less as in [17]. Set

$$\mathfrak{r}^*_\Phi = \{\lambda \in \mathfrak{s}^*_\Phi \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{–orbit on } \mathfrak{s}^*_\Phi\}. \tag{6.5}$$

Then $\mathfrak{r}^*_\Phi$ is dense, open and U_Φ–invariant in $\mathfrak{s}^*_\Phi$. By definition of principal orbit the isotropy subgroups of U_Φ at the various points of $\mathfrak{r}^*_\Phi$ are conjugate, and we take a measurable section σ to $\mathfrak{r}^*_\Phi \to \mathfrak{r}^*_\Phi \backslash U_\Phi$ on whose image all the isotropy subgroups are the same,

$$U'_\Phi : \text{ isotropy subgroup of } U_\Phi \text{ at } \sigma(U_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}^*_\Phi. \tag{6.6}$$

In view of Lemma 6.4 the principal isotropy subgroups U'_Φ are specified by the work of W.–C. and W.–Y. Hsiang [1] on the structure and classification of principal orbits of compact connected linear groups. With a glance back at (5.10) we have

$$U'_\Phi A'_\Phi : \text{ isotropy subgroup of } U_\Phi A_\Phi \text{ at } \sigma(U_\Phi A_\Phi(\lambda)), \text{ independent of } \lambda \in \mathfrak{r}^*_\Phi. \tag{6.7}$$

The first consequence, as in [17, XX], is

Theorem 6.8. *Suppose that $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant. Let $f \in \mathcal{C}(U_\Phi N_\Phi)$ Given $\lambda \in \mathfrak{r}^*_\Phi$ let $\pi^\dagger_\lambda$ denote the extension of π_λ to a representation of $U'_\Phi N_\Phi$ on the space of π_λ. Then the Plancherel density at $\mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi}(\pi^\dagger_\lambda \otimes \gamma)$, $\gamma \in \widehat{U'_\Phi}$, is $(\dim\gamma)|P(\lambda)|$ and the Plancherel Formula for $U_\Phi N_\Phi$ is*

$$f(un) = c\int_{\mathfrak{r}^*_\Phi/\mathrm{Ad}^*(U_\Phi)} \sum\nolimits_{\gamma \in \widehat{U'_\Phi}} \mathrm{trace}\,\mathrm{Ind}_{U'_\Phi N_\Phi}^{U_\Phi N_\Phi} r(un)(f)\cdot\dim(\gamma)\cdot|P(\lambda)|d\lambda$$

where $c = 2^{d_1+\dots+d_m}d_1!d_2!\dots d_m!$, from (1.7).

Combining Theorems 5.13 and 6.8 we have

Theorem 6.9. *Let $Q_\Phi = M_\Phi A_\Phi N_\Phi$ be a parabolic subgroup of the real reductive Lie group G. Let U_Φ be a maxixmal compact subgroup of M_Φ, so $E_\Phi = U_\Phi A_\Phi N_\Phi$ is a maximal amenable subgroup of Q_Φ. Suppose that the decomposition $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant. Given $\lambda \in \mathfrak{r}^*_\Phi$, $\gamma \in \mathfrak{a}'_\Phi$ and $\gamma \in \widehat{U'_\Phi}$ denote*

$$\pi_{\lambda,\phi,\gamma} = \mathrm{Ind}_{U'_\Phi A'_\Phi N_\Phi}{}^{U_\Phi A_\Phi N_\Phi} \in \widehat{E_\Phi}.$$

Let $\Theta_{\pi_{\lambda,\phi,\gamma}} : h \mapsto \mathrm{trace}\,\pi_{\lambda,\phi,\gamma}(h)$ denote its distribution character. Then $\Theta_{\pi_{\lambda,\phi,\gamma}}$ is a tempered distribution on the maximal amenable subgroup E_Φ. If $f \in \mathcal{C}(E_\Phi)$ then

$$f(x) = c\int_{(\mathfrak{a}'_\Phi)^*}\left(\int_{\mathfrak{r}^*_\Phi/\mathrm{Ad}^*(U_\Phi A_\Phi)} \Theta_{\pi_{\lambda,\phi,\gamma}}(D(r(x)f))|P(\lambda)|d\lambda\right)d\phi$$

where $c = (\frac{1}{2\pi})^{\dim\mathfrak{a}_\Phi/2}\, 2^{d_1+\dots+d_m}d_1!d_2!\dots d_m!$.

When weak invariance fails we replace the $\mathfrak{z}_{\Phi,j}$ by the larger

(6.10) $$\widetilde{\mathfrak{z}_{\Phi,j}} = \sum\nolimits_{\alpha \in Y_j} \mathfrak{g}_\alpha \text{ where } Y_j = \{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}) \mid \alpha|_{\mathfrak{a}_\Phi} = \beta_{j_0}|_{\mathfrak{a}_\Phi}\}.$$

Note that $\widetilde{z_{\Phi,j}}$ is an irreducible $\mathrm{Ad}(M_\Phi^0)$–module. We need to show that we can replace $\mathfrak{s}_\Phi = \sum \mathfrak{z}_{\Phi,j}$ by

$$\widetilde{\mathfrak{s}_\Phi} := \sum \widetilde{\mathfrak{z}_{\Phi,j}}$$

in our Plancherel formulae. The key is

Lemma 6.11. *Let $\lambda_j \in \widetilde{\mathfrak{z}_{\Phi,j}}^*$. Split $\widetilde{\mathfrak{z}_{\Phi,j}} = \mathfrak{z}_{\Phi,j} + \mathfrak{w}_{\Phi,j}$ where $\mathfrak{w}_{\Phi,j} = \widetilde{\mathfrak{z}_{\Phi,j}} \cap \mathfrak{v}_\Phi$ is the sum of the $\mathfrak{g}_\alpha$ that occur in $\widetilde{\mathfrak{z}_{\Phi,j}}$ but not in $\mathfrak{z}_{\Phi,j}$. Then the Pfaffian $\mathrm{Pf}_j(\lambda_j) = \mathrm{Pf}_j(\lambda_j|_{\mathfrak{z}_{\Phi,j}})$.*

Proof. Write $\lambda_j = \lambda_{\mathfrak{z},j} + \lambda_{\mathfrak{w},j}$ where $\lambda_{\mathfrak{z},j}(\mathfrak{w}_{\Phi,j}) = 0 = \lambda_{\mathfrak{w},j}(\mathfrak{z}_{\Phi,j})$. Let $\mathfrak{g}_\gamma, \mathfrak{g}_\delta \subset \mathfrak{l}_{\Phi,j}$ with $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \neq 0$. Then $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \subset \mathfrak{l}_{\Phi,j}$, so $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] \cap \mathfrak{w}_{\Phi,j} = 0$, in particular $\lambda_{\mathfrak{w},j}([\mathfrak{g}_\gamma, \mathfrak{g}_\delta]) = 0$. In other words $\lambda_j([\mathfrak{g}_\gamma, \mathfrak{g}_\delta]) = \lambda_j|_{\mathfrak{z}_{\Phi,j}}([\mathfrak{g}_\gamma, \mathfrak{g}_\delta])$. Now $b_{\lambda_j|_{\mathfrak{z}_{\Phi,j}}} = b_{\lambda_j}$, so their Pfaffians are the same. ☐

In order to extend Theorems 6.8 and 6.9 we now need only make some trivial changes to (6.5), (6.6, (6.7) and the measurable section:

- $\widetilde{\mathfrak{r}_\Phi}^* = \{\lambda \in \widetilde{\mathfrak{s}_\Phi}^* \mid P(\lambda) \neq 0 \text{ and } \mathrm{Ad}(U_\Phi)\lambda \text{ is a principal } U_\Phi\text{–orbit on } \widetilde{\mathfrak{s}_\Phi}^*\}$.
- $\widetilde{\sigma}$: measurable section to $\widetilde{\mathfrak{r}_\Phi}^* \to \widetilde{\mathfrak{r}_\Phi}^* \backslash U_\Phi$ on whose image all the isotropy subgroups are the same.
- U'_Φ : isotropy subgroup of U_Φ at $\widetilde{\sigma}(U_\Phi(\lambda))$, independent of $\lambda \in \widetilde{\mathfrak{r}_\Phi}^*$.
- $U'_\Phi A'_\Phi$: isotropy subgroup of $U_\Phi A_\Phi$ at $\widetilde{\sigma}(U_\Phi A_\Phi(\lambda))$, independent of $\lambda \in \widetilde{\mathfrak{r}_\Phi}^*$.

Then Theorems 6.8 and 6.9 extend *mutatis mutandis* without the condition that $N_\Phi = L_{\Phi,1}L_{\Phi,2}\dots L_{\Phi,\ell}$ is weakly invariant.

Part II: Infinite Dimensional Theory

7 Direct limit parabolics

In this section we carry our results on N_Φ and $U_\Phi N_\Phi$ over to a class of infinite dimensional Lie groups, the direct limits $N_{\Phi,\infty} = \varinjlim N_{\Phi,n}$, where $\{N_{\Phi,n}\}$ is a strict direct system of nilradicals of a system of appropriately aligned parabolics $Q_{\Phi,n} = M_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$. In order to do this we must adjust ordering in the decompositions (1.4) of the connected simply connected nilpotent Lie groups $N_{\Phi,n}$ so that they fit together as n increases. We do that by reversing the indices and keeping the L_r constant as n goes to infinity. First, we suppose that

(7.1) $\{N_n\}$ is a strict direct system of connected simply connected nilpotent Lie groups,

in other words the connected simply connected nilpotent Lie groups N_n have the property that N_n is a closed analytic subgroup of N_ℓ for all $\ell \geqq n$. As usual, Z_r denotes the center of L_r. For each n, we require that

(7.2)
$N_n = L_1 L_2 \cdots L_{m_n}$ where
(a) L_r is a closed analytic subgroup of N_n for $1 \leqq r \leqq m_n$ and
(b) each L_r has unitary representations with coefficients in $L^2(L_r/Z_r)$.
(x) $L_{p,q} = L_{p+1}L_{p+2}\cdots L_q (p < q)$ and $N_{\ell,n} = L_{m_\ell+1}L_{m_\ell+2}\cdots L_{m_n} = L_{m_\ell,m_n} (\ell < n)$;
(c) $N_{\ell,n}$ is normal in N_n and $N_n = N_r \ltimes N_{r,n}$ semidirect product,
(d) decompose $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n}_n = \mathfrak{s}_n + \mathfrak{l}_n^{compl}$ where $\mathfrak{s}_n = \bigoplus_{r \leqq m_n} \mathfrak{z}_r$ and $\mathfrak{l}_n^{compl} = \bigoplus_{r \leqq m_n} \mathfrak{v}_r$; then $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{l}''_s + \mathfrak{v}$ for $r < s$ where $\mathfrak{l}_r = \mathfrak{l}'_r \oplus \mathfrak{l}''_r$ direct sum of ideals with $\mathfrak{l}''_r \subset \mathfrak{z}_r$ and $\mathfrak{v}_r \subset \mathfrak{l}'_r$

With this setup we can follow the lines of the constructions in [16, Section 5] as indicated in §1 above. Denote

(7.3) $$P_n(\gamma_n) = \mathrm{Pf}_1(\lambda_1)\mathrm{Pf}_2(\lambda_2)\cdots\mathrm{Pf}_{m_n}(\lambda_{m_n}) \text{ where } \lambda_r \in \mathfrak{z}_r^* \text{ and } \gamma_n = \lambda_1 + \cdots + \lambda_{m_n}$$

and the nonsingular set

(7.4) $$\mathfrak{t}_n^* = \{\gamma_n \in \mathfrak{s}_n^* \mid P_n(\gamma_n) \neq 0\}.$$

When $\gamma_n \in \mathfrak{t}_n^*$ the stepwise square integrable representation $\pi_{\gamma_n} \in \widehat{N_n}$ is constructed recursively as in 1.8 with the indices reversed: $\pi_{\gamma_n} = \pi'_{\gamma_{n-1}} \widehat{\otimes} \pi_{\lambda_n}$, and $\mathcal{H}_{\pi_{\gamma_n}} = \mathcal{H}_{\pi_{\gamma_{n-1}}} \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_n}}$ is its representation space.

The parameter space for our representations of the direct limit Lie group $N = \varinjlim N_n$ is

(7.5) $$\mathfrak{t}^* = \bigcup_{n>0} \left\{ \gamma = \sum \lambda_r \in \mathfrak{s}^* \,\middle|\, \gamma_\ell \in \mathfrak{t}_\ell^* \text{ for } \ell \leqq n \text{ and } \lambda_r = 0 \in \mathfrak{z}_r^* \text{ for } r > m_n \right\}$$

where $\mathfrak{s}^* := \bigcup_{\ell>0} \mathfrak{s}_\ell^* = \sum_{r>0} \mathfrak{z}_r^*$. The representations π_γ of N are defined as above: given $\gamma = \sum \lambda_r \in \mathfrak{t}^*$ we have the index $n = n(\gamma)$ defined by $\gamma_\ell \in \mathfrak{t}_\ell^*$ for $\ell \leqq n(\gamma)$ and $\lambda_r = 0 \in \mathfrak{z}_r^*$ for $\ell > m_{n(\gamma)}$. Express

(7.6) $$N = N_{n(\gamma)} \ltimes N_{n(\gamma),\infty} \text{ semidirect product, where } N_{n(\gamma),\infty} = \prod\nolimits_{r>m_{n(\gamma)}} L_r\,.$$

In particular the closed normal subgroup $N_{n(\gamma),\infty}$ satisfies $N_{n(\gamma)} \cong N/N_{n(\gamma),\infty}$, and we denote

(7.7) $$\pi_\gamma\text{: lift to } N \text{ of the stepwise square integrable } \pi_{\lambda_1+\cdots+\lambda_{m_{n(\gamma)}}} \in \widehat{N_{n(\gamma)}}\,.$$

The representation space of π_γ is the projective (jointly continuous) tensor product

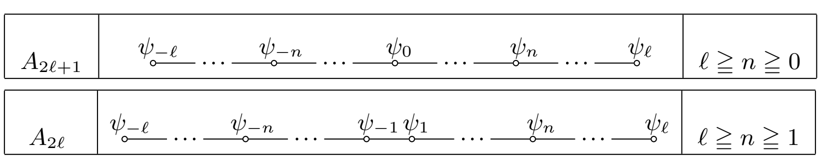
(7.8) $$\mathcal{H}_{\pi_\gamma} = \mathcal{H}_{\pi_{\lambda_1}} \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_2}} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{H}_{\pi_{\lambda_{n(\gamma)}}}$$

These representations π_γ are the *limit stepwise square integrable* representations of N. medskip

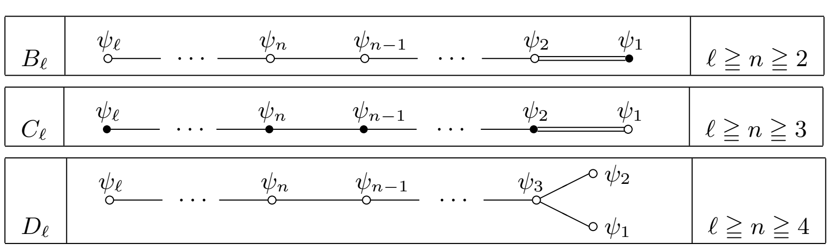
8 Direct Limit Groups

We adapt the constructions (7.7) and (7.8) to limits of nilradicals of parabolic subgroups. That requires some alignment of root systems so that the direct limit respects the restricted root structures, in particular the strongly orthogonal root structures, of the N_n. We enumerate the set $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{a}_n)$ of nonmultipliable simple restricted roots so that, in the Dynkin diagram, for type A we spread from the center of the diagram. For types B, C and D, ψ_1 is the *right* endpoint, In other words for $\ell \geqq n$ Ψ_ℓ is constructed from Ψ_n adding simple roots to the *left* end of their Dynkin diagrams. Thus

(8.1)



(8.2)



We describe this by saying that G_ℓ *propagates* G_n. For types B, C and D this is the same as the notion of propagation in [8] and [9].

The direct limit groups obtained this way are $SL(\infty;\mathbb{C})$, $SO(\infty;\mathbb{C})$, $Sp(\infty;\mathbb{C})$, $SL(\infty;\mathbb{R})$, $SL(\infty;\mathbb{H})$, $SU(\infty,q)$ with $q \leqq \infty$, $SO(\infty,q)$ with $q \leqq \infty$, $Sp(\infty,q)$ with $q \leqq \infty$, $Sp(\infty;\mathbb{R})$ and $SO^*(2\infty)$.

Let $\{G_n\}$ be a direct system of real semisimple Lie groups in which G_ℓ propagates G_n for $\ell \geqq n$. Then the corresponding simple restricted root systems satisfy $\Psi_n \subset \Psi_\ell$ as indicated in (8.1) and (8.2). Consider conditions on a family $\Phi = \{\Phi_n\}$ of subsets $\Phi_n \subset \Psi_n$ such that $G_n \hookrightarrow G_\ell$ maps the corresponding parabolics $Q_{\Phi,n} \hookrightarrow Q_{\Phi,\ell}$. Then we have

$$Q_{\Phi,\infty} := \varinjlim Q_{\Phi,n} \text{ inside } G_\infty := \varinjlim G_n\,. \tag{8.3}$$

Express $Q_{\Phi,n} = M_{\Phi,n}A_{\Phi,n}N_{\Phi,n}$ and $Q_{\Phi,\ell} = M_{\Phi,\ell}A_{\Phi,\ell}N_{\Phi,\ell}$. Then $M_{\Phi,n} \hookrightarrow M_{\Phi,\ell}$ is equivalent to $\Phi_n \subset \Phi_\ell$, $A_{\Phi,n} \hookrightarrow A_{\Phi,\ell}$ is implicit in the condition that G_ℓ propagates G_n, and $N_{\Phi,n} \hookrightarrow N_{\Phi,\ell}$ is equivalent to $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$. As before let $U_{\Phi,n}$ denote a maximal compact subgroup of $M_{\Phi,n}$; we implicitly assume that $U_{\Phi,n} \hookrightarrow U_{\Phi,\ell}$ whenever $M_{\Phi,n} \hookrightarrow M_{\Phi,\ell}$.

We will extend some of our results from the finite dimensional setting to these subgroups of $Q_{\Phi,\infty}$.

$$\begin{aligned}
&N_{\Phi,\infty} := \varinjlim N_{\Phi,n} \text{ maximal locally unipotent subgroup, requiring } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell),\\
&A_{\Phi,\infty} := \varinjlim A_{\Phi,n}\,,\\
&U_{\Phi,\infty} := \varinjlim U_{\Phi,n} \text{ maximal lim–compact subgroup, requiring } \Phi_n \subset \Phi_\ell,\\
&U_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim U_{\Phi,n}N_{\Phi,n} \text{ requiring } \Phi_n = \Phi_\ell\,.
\end{aligned} \tag{8.4}$$

We will also say something, but not much, about

$$\begin{aligned}
&A_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim A_{\Phi,n}N_{\Phi,n} \text{ max. exponential solvable subgroup where } (\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell),\\
&U_{\Phi,\infty}A_{\Phi,\infty}N_{\Phi,\infty} := \varinjlim U_{\Phi,n}A_{\Phi,n}N_{\Phi,n} \text{ maximal amenable subgroup where } \Phi_n = \Phi_\ell\,.
\end{aligned} \tag{8.5}$$

The difficulty with the two limit groups of (8.5) is that we don't have a Dixmier–Pukánszky operator, so we don't have a Fourier inversion formula.

Start with $N_{\Phi,\infty}$. For that we must assume $(\Psi_n \setminus \Phi_n) \subset (\Psi_\ell \setminus \Phi_\ell)$. In view of the propagation assumption on the G_n the maximal set of strongly orthogonal non–multipliable roots in $\Delta^+(\mathfrak{g}_n, \mathfrak{a}_n)$ is increasing in n. It is obtained by cascading up (we reversed the indexing from the finite dimensional setting) has form $\{\beta_1, \dots, \beta_{r_n}\}$. Following ideas of Section 4 we construct the sets $I_{n,k}$ of indices for which the β_i have the same restriction to $\mathfrak{a}_{\Phi,n}$ and all belong to $\Delta(\mathfrak{g}_n, \mathfrak{a}_n)$. Note that $I_{n,k}$ can increase as n increases, for example in some cases the Φ stop growing, i.e. where there is an index n_0 such that $\Phi_n = \Phi_{n_0} \neq \emptyset$ for $n \geqq n_0$. This happens when $\Delta(\mathfrak{g}_n, \mathfrak{a}_n)$ is of type A_n with each $\Psi = \{\psi_1\}$. Thus we also denote $I_{\infty,k} = \bigcup_n I_{n,k}$.

As in (4.1), define

$$\begin{aligned}
\mathfrak{l}_{\Phi,n,j} &= \sum_{i \in I_{n,j}} (\mathfrak{l}_i \cap \mathfrak{n}_{\Phi,n}) = \Big(\sum_{i \in I_{n,j}} \mathfrak{l}_i\Big) \cap \mathfrak{n}_{\Phi,n} \text{ and } \mathfrak{l}^{compl}_{\Phi,n,j} = \sum_{k \geqq j} \mathfrak{l}_{\Phi,n,k},\\
\mathfrak{l}_{\Phi,\infty,j} &= \sum_{i \in I_{\infty,j}} (\mathfrak{l}_i \cap \mathfrak{n}_{\Phi}) = \Big(\sum_{i \in I_{\infty,j}} \mathfrak{l}_i\Big) \cap \mathfrak{n}_{\Phi} \text{ and } \mathfrak{l}^{compl}_{\Phi,\infty,j} = \sum_{k \geqq j} \mathfrak{l}_{\Phi,\infty,k}\,.
\end{aligned} \tag{8.6}$$

$L_{\Phi,n,j}$ denotes the analytic subgroup with Lie algebra $\mathfrak{l}_{\Phi,n,j}$ and $L_{\Phi,\infty,j} = \varinjlim_n L_{\Phi,n,j}$ has Lie algebra $\mathfrak{l}_{\Phi,\infty,j}$. We have this set up so that $N_{\Phi,\infty} = \varinjlim_n N_{\Phi,n} = \varinjlim_j L_{\Phi,\infty,j} = \varinjlim_j \varinjlim_n L_{\Phi,n,j}$.

Edited up to here.

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