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ABSTRACT

The connection between Regge poles and elementary-particle poles was explored in a previous paper, where we made use of the approximation of elastic unitarity and of the assumption of no Castillejo-Dalitz-Dyson (CDD) ambiguity. In this paper we include the CDD ambiguity, and show without recourse to any approximation that the vanishing of $\Gamma(s)$ (the renormalized pion-nucleon proper vertex function with one nucleon off the mass shell) is the necessary and sufficient condition for the elementary nucleon to lie on the Regge trajectory. From the condition $\Gamma(s) \equiv 0$ it also follows that $Z_1 = Z_2 = Z_2\delta M = 0$, where Z_1 is the vertex-renormalization constant, Z_2 is the nucleon wave-function renormalization constant, and δM is the nucleon self-mass. But the vanishing of $\Gamma(s)$ does not always follow from $Z_1 = Z_2 = Z_2\delta M = 0$. Therefore, the so far widely recognized compositeness condition, $Z_1 = Z_2 = 0$, or $Z_2 = 0$ with the finite self-mass, is not sufficient to

Reggeize the elementary nucleon. Finally, Kaus and Zachariasen's criterion for Reggeization is discussed. Their criterion is not necessary to Reggeize the elementary nucleon; it should be so modified as to be $\Gamma(s)\, K(s) \equiv 0$ and $g_B^{\,2} = 0$, where $g_B^{\,2}$ is the residue of the extra pole in the pion-nucleon scattering amplitude and $K(s)$ is the pion-nucleon form factor with one nucleon off the mass shell.

I. INTRODUCTION AND SUMMARY

Let us consider the pion-nucleon scattering. At first we neglect the complication due to spins and isospins. But they are taken into account in the last section. If the nucleon is the elementary particle, the ℓth partial-wave amplitude $T_\ell(s)$ is always written as

$$T_\ell(s) = \delta_{\ell 0}\, \Gamma(s)\, S_F'(s)\, \Gamma(s) + f_\ell(s) , \tag{1.1}$$

where $\Gamma(s)$ is the renormalized π-N proper vertex function with one nucleon off the mass shell, $S_F'(s)$ is the renormalized nucleon propagator, and s is the total energy squared. The vertex function $\Gamma(s)$ is related to the form factor $K(s)$ through

$$\Gamma(s)\, S_F'(s) = K(s)\, S_F(s) , \tag{1.2}$$

where $S_F(s) = 1/(M^2-s)$ is the free nucleon propagator with the nucleon mass M^2, so $T_\ell(s)$ is rewritten as

$$T_\ell(s) = \delta_{\ell 0} \left[\Gamma K \big/ (M^2-s)\right] + f_\ell(s) . \tag{1.3}$$

Jin and MacDowell[1,2] showed quite generally that as one increases the strength of the force, Γ and f_0 develop a pole at the same position where S_F' has a zero, before a pole of T_0 on the second Riemann sheet reaches the first sheet; thus the poles of Γ and f_0 are not poles of T_0 and K (cancellation theorem). By making use of the analytic continuation of f_ℓ into the complex angular-momentum plane, they further suggested that the pole of Γ is associated with a Regge pole, i.e., if f_ℓ has a Regge pole at $\ell = \alpha(s)$ with the proper signature, Γ has a pole at the point where the Regge trajectory crosses the value $\ell = 0$. By the cancellation theorem, however, this pole exactly cancels the Regge pole, so that it does not appear in T_0.

In a previous paper[3] we took this interpretation for the vertex function pole for granted, and explored the connection between the elementary-particle pole and the Regge pole. The basic idea was as follows: The scattering amplitude T_ℓ contains the Kronecker delta singularity in the ℓ plane, due to the existence of the elementary particle. If the elementary particle lay on the Regge trajectory, such a singularity should disappear; and the inverse of this would be true. Our analysis following this idea, however, was restricted by assumptions of elastic unitarity and no Castillejo-Dalitz-Dyson (CDD) ambiguity. In the present paper we include the CDD ambiguity, and show without recourse to any approximation that the vanishing of Γ is the necessary and sufficient condition for the elementary nucleon to lie on the Regge trajectory. From the condition $\Gamma \equiv 0$ it also follows that $Z_1 = Z_2 = Z_2\delta M = 0$, where Z_1 is the vertex-renormalization constant, Z_2 is the nucleon wave-function renormalization constant, and δM is the nucleon self-mass. But the vanishing of Γ does not always follow from $Z_1 = Z_2 = Z_2\delta M = 0$.

Therefore, the widely recognized compositeness condition, $Z_1 = Z_2 = 0$, or $Z_2 = 0$ with the finite self-mass, is not sufficient to Reggeize the elementary nucleon.

In Sec. II, we collect several well-known formulas for Γ, S_F', and f_ℓ, and summarize the proof of the Jin-MacDowell cancellation theorem, which plays a crucial role in the process of Reggeization. Section III is devoted to the $\Gamma = 0$ limit. Kaus and Zachariasen's criterion for Reggeization is discussed in Sec. IV. Their criterion is not necessary to Reggeize the elementary nucleon; it should be so modified as to be $\Gamma K = 0$ and $g_B^2 = 0$, where g_B^2 is the residue of the extra pole in T_0, introduced by them. In Sec. V, we take into account spins and isospins of the pion and the nucleon. But the conclusion obtained above remains true. The vanishing of Γ does not follow from $Z_1 = Z_2 = Z_2 \delta M = 0$.

II. THE CANCELLATION THEOREM

We assume that S_F' has the Lehmann spectral representation with no subtraction,

$$S_F'(s) = \frac{1}{M^2 - s} + \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \, \frac{\sigma(s')}{s' - s} \quad , \tag{2.1}$$

or with one subtraction,

$$S_F'(s) = \frac{1}{M^2 - s} + d + \frac{s - M^2}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \, \frac{\sigma(s')}{(s' - M^2)(s' - s)} \, , \tag{2.2}$$

where d is the subtraction constant, and μ is the pion mass. Since $\sigma(s) \geq 0$, S_F' is the Herglotz function. Then we have

$$\frac{S_F(s)}{S_F'(s)} = 1 + \frac{s - M^2}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{\overline{\sigma}(s')}{s' - s} - \sum_n C_n \frac{s - M^2}{s - s_n} \, , \tag{2.3}$$

$$\overline{\sigma}(s) = |S_F / S_F'|^2 \, \sigma(s) \quad ,$$

where s_n are real zeros of S_F', and $C_n \geq 0$. In the elastic region, $\overline{\sigma}(s)$ is

$$\begin{aligned} \overline{\sigma}(s) &= \rho(s) \, |K(S_F / S_F')|^2 / (s-M^2)^2 \\ &= \rho(s) \, |\Gamma(s)|^2 / (s-M^2)^2 \quad , \end{aligned} \tag{2.4}$$

$\rho(s)$ being the pion-nucleon, $\ell = 0$, phase-space factor. The wave-function renormalization constant of the nucleon is given by

$$Z_2 = \lim_{s \to \infty} S_F / S_F' \tag{2.5}$$

$$= 1 - \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds \ \overline{\sigma}(s) - \sum_n C_n .$$

Since $0 \leq Z_2 \leq 1$, we have the inequality

$$1 \geq \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds \ \frac{\rho |\Gamma|^2}{(s-M^2)^2} , \tag{2.6}$$

from which it follows[4] that

$$\lim_{s \to \infty} s |\Gamma/s|^2 \to 0 . \tag{2.7}$$

The nucleon self-mass δM^2 is given by the formula[5]

$$Z_2 \delta M^2 = \lim_{s \to \infty} (s-M^2) \left[Z_2 - (S_F / S_F') \right] \tag{2.8}$$

$$= \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds \ (s-M^2) \ \overline{\sigma}(s) + \sum_n C_n \ (s_n - M^2) .$$

If $S_F' \to \infty$ as $s \to \infty$, then we get $Z_2 = Z_2 \delta M^2 = 0$, from which at least one of the s_n's must be smaller than M^2. In fact, from (2.2) one can see that S_F' has one zero for $s \leq M^2$ when $S_F' \to \infty$ as $s \to \infty$.

Now let us turn our attention to (1.3). Several authors[6,7] showed that in the elastic region f_ℓ and Γ satisfy unitarity by themselves:

$$\text{Im } f_\ell = \rho f_\ell f_\ell^* \quad , \tag{2.9}$$

$$\text{Im } \Gamma = \rho \Gamma f_0^* = \rho f_0 \Gamma^* \quad . \tag{2.10}$$

These relations are the same as those of T_ℓ and K. In fact, a combined use of (1.2), (1.3), (2.3), and (2.4), together with unitarity relations of T_0 and K, yields (2.9) and (2.10). From these relations (2.9) and (2.10) we see that f_0 and Γ could have a pole at the same position on the second sheet, given by $1 + 2i\rho f_0 = 0$. If this pole moves onto the first sheet, f_0 and Γ develop a pole at the same position, say $s = s_0$. Then the integral path in (2.3) is deformed, yielding a pole term

$$\frac{s - M^2}{\pi} \oint ds' \, \frac{\Gamma(s')\, \rho\, (s')\, \Gamma^{II}(s')}{(s'-M^2)^2 (s' - s)} = -C_0 \, \frac{s - M^2}{s - s_0} \, , \tag{2.11}$$

$$C_0 = \left(\frac{\gamma/g_0}{M^2 - s_0} \right)^2 \quad , \tag{2.12}$$

$$\Gamma^{II}(s) = \Gamma(s) \Big/ \left[1 + 2i\rho(s) f_0(s) \right] \quad , \tag{2.13}$$

where γ_0 and $-g_0^{\,2}$ are residues of the pole s_0 of Γ and f_0, respectively. From the Herglotz property of S_F' and from unitarity of f_0, this pole s_0 can lie only in the interval $M^2 \lesssim s \lesssim (M+\mu)^2$. Thus, S_F' develops a zero at the same point s_0. Therefore, from (1.2) we see that K has no pole at $s = s_0$, and hence

$$
\begin{aligned}
(M^2 - s_0)\, C_0 &= \lim_{s \to \infty} (s - s_0)\Big/\Gamma K \\
&= \gamma/K(s_0) \quad .
\end{aligned}
\tag{2.14}
$$

The equations (2.12) and (2.14) give

$$
K(s_0)\gamma\Big/(M^2 - s_0) = g_0^{\,2} \quad , \tag{2.15}
$$

which shows, as is seen from (1.3), that the pole s_0 of Γ and f_0 is not the pole of T_0. This is the cancellation theorem due to Jin and MacDowell,[2] and plays an important role in the next section.

Other zeros s_n $(n \neq 0)$ in S_F' which can lie only in the region $s \lesssim M^2$ and $(M+\mu)^2 \lesssim s$ are not poles of Γ but zeros of K, corresponding to CDD zeros. Therefore, C_n $(n \neq 0)$ in (2.3) are given by

$$
C_n = \frac{\Gamma(s_n)\Big/K'(s_n)}{M^2 - s_n} \quad (n \neq 0). \tag{2.16}
$$

III. THE $\Gamma(s) = 0$ LIMIT

In the Reggeized world the Kronecker delta term in T_ℓ should disappear, i.e., $\Gamma K \equiv 0$. We assume that K has no pole. Then it cannot be identically zero if $g \neq 0$, g being the pion-nucleon coupling constant.[8] Since we are not interested in the $g = 0$ case, the condition $\Gamma K \equiv 0$ is nothing but the condition $\Gamma \equiv 0$. In the following we assume the existence of K, even if $\Gamma = 0$.[9] When K has a pole, we can take the limit $K \equiv 0$ but $g \neq 0$. This case was considered by Kaus and Zachariasen,[10] and is discussed in the next section.

Now consider the dispersion relation of Γ:

$$\Gamma(s) = g - \frac{s - M^2}{s - s_0} \frac{\gamma}{M^2 - s_0} + \frac{s - M^2}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \frac{\operatorname{Im} \Gamma(s')}{(s' - M^2)(s' - s)} . \tag{3.1}$$

Here Γ needs at most one subtraction, because the asymptotic behavior of Γ is given by (2.7). From (3.1) one can see that the limit $\Gamma \equiv 0$ yields[11]

$$s_0 = M^2 , \tag{3.2}$$

and

$$\lim_{s_0 \to M^2} \gamma \big/ (M^2 - s_0) = g . \tag{3.3}$$

On the other hand the cancellation theorem (2.15) gives

$$g_0^2 = g^2 \quad . \tag{3.4}$$

Since the pole s_0 with the residue g_0^2 lies on the Regge trajectory, Eqs. (3.2) and (3.4) mean that the elementary nucleon lies on this Regge trajectory.

Inversely, from (3.2) and (3.4) it follows that $\Gamma \equiv 0$. To show this, we make use of (2.5). We rewrite Z_2 as

$$Z_2 = (1 - C_0) - \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds \quad \overline{\sigma}(s) - \sum_{n \neq 0} C_n \quad , \tag{3.5}$$

where C_0 is given from (2.12) and (2.14), by,

$$C_0 = \left[g_0 \Big/ K(s_0) \right]^2 \quad . \tag{3.6}$$

Now, in the limits $s_0 = M^2$ and $g_0^2 = g^2$, we have $C_0 = 1$. Then, from (3.5) it follows that $\overline{\sigma}(s) = 0$, $Z_2 = 0$, and $C_n = 0$ $(n \neq 0)$, because these are all nonnegative quantities. In the elastic region $\overline{\sigma}(s)$ is given by (2.4) and hence $\Gamma = 0$. Therefore, by analytic continuation Γ vanishes everywhere. We have required the existence of K, even if $\Gamma = 0$. Thus the Kronecker delta term disappears, and the amplitude T_ℓ now reduces to f_ℓ.

In conclusion we have shown quite generally that the vanishing of Γ is the necessary and sufficient condition for the nucleon to lie on the Regge trajectory. In this proof, the existence of K has been required, even in the limit $\Gamma = 0$.

Let us make a few remarks concerning the limit $\Gamma = 0$:

(a) From $\Gamma = 0$ it follows that $Z_1 = Z_2 = Z_2 \delta M^2 = 0$, where Z_1 is defined by $\lim_{s \to \infty} \Gamma(s) = Z_1 g$. But the inverse of this is not always true. The last condition $Z_2 \delta M^2 = 0$ was used by Gerstein and Deshpande[12] as one of compositeness conditions. They showed that if $Z_2 = 0$, and if the self-mass δM^2 is finite, then $\overline{\sigma}(s) = 0$ and $s_0 = M^2$ follow. However, from (2.8) it is clear that when one of s_n in (2.8) appears in the region $s < M^2$, i.e., when S_F' becomes infinite as $s \to \infty$, their conclusion never follows. If S_F' satisfies the unsubtracted dispersion relation, the condition $Z_2 \delta M^2 = 0$ is equivalent to the condition $\Gamma = 0$ in the scalar theory. However, if spins of the pion and the nucleon are taken into account, the latter does not again follow from the former, as was shown by Deshpande.[12]

(b) Since the pole s_0 of Γ coincides with the zero of S_F', the limit $s_0 = M^2$ is possible only if the integral in the Lehmann representation becomes divergent or the subtraction constant tends to infinity, or both. This means $S_F' \equiv \infty$. Then from $S_F' \Gamma = S_F K =$ finite we get $\Gamma = 0$, or from (2.3) $\overline{\sigma}(s)$ must vanish and hence $\Gamma = 0$. This is an alternative proof of obtaining $\Gamma = 0$. In this case the limit $g_0^2 = g^2$ is derived from $s_0 = M^2$ through $\Gamma = 0$.

(c) Other zeros s_n $(n \neq 0)$ in S_F' are zeros of K which cannot lie in the interval $M^2 < s < (M+\mu)^2$. We conjecture that as s_0 approaches M^2 those zeros s_n go to infinity. This may be seen in the following argument: When S_F' has the unsubtracted Lehmann representation (2.1), we have the equation

$$\frac{1}{(s_0 - M^2)(s_n - M^2)} = -\frac{P}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\sigma(s')}{(s' - s_0)(s' - s_n)}, \quad (3.7)$$

where $M^2 \leqslant s_0 \leqslant (M+\mu)^2$ and $(M+\mu)^2 \leqslant s_n$. Now, in the limit $s_0 = M^2$ the integral in (2.1) becomes divergent. However, we suppose that the integral in (3.7) be finite.[13] Then s_n must go to plus infinity. Since $K(s_n) = 0$, this means $K(s) \to 0$ as $s \to \infty$. When S_F' has the once-subtracted Lehmann representation (2.2), one of the s_n's, say s_1, may lie in the region $s \leqslant M^2$, and the other zeros s_n $(n \neq 0, 1)$ can lie only in the region $(M+\mu)^2 \leqslant s$. Then we have

$$\frac{1}{(s_0 - M^2)(s_n - M^2)} = -\frac{2P\int_{(M+\mu)^2}^{\infty} ds' \dfrac{\sigma(s')}{(s' - M^2)(s' - s_0)(s' - s_n)}}{\left[d^2 + 4I(s_0)\right]^{\frac{1}{2}} + \left[d^2 + 4I(s_n)\right]^{\frac{1}{2}}}, \quad (3.8)$$

where

$$I(s) = \frac{P}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \frac{\sigma(s')}{(s' - M^2)(s' - s)} . \qquad (3.9)$$

Now, in the limit $s_0 = M^2$ the integral in (2.2) or the subtraction constant d becomes infinite, or both. However, we suppose that the integral in the numerator in (3.8) is finite. Then s_n must go to plus infinity. However, the motion of s_1 is somewhat different from these of s_n's. Since $s_1 \leq M^2$, we have

$$2/(s_1 - M^2) = d - [d^2 + 4I(s_1)]^{\frac{1}{2}} . \qquad (3.10)$$

As s_0 approaches M^2 the right-hand side in (3.10) tends to various values, depending on those of the ratio $I(s_1)/d$. But the case $I(s_1)/d \rightarrow \infty$ should be rejected, because s_1 tends to M^2 and hence $K(s_1) = K(M^2) = g = 0$, contradicting our assumption $g \neq 0$. If we require $I(s_1)/d \rightarrow 0$ as $s_0 \rightarrow M^2$, then s_1 tends to minus infinity, which means $K(s) \rightarrow 0$ as $s \rightarrow \infty$. Thus the CDD zeros in K disappear.

IV. THE KAUS-ZACHARIASEN CASE

Kaus and Zachariasen[10] considered the Reggeization when K has an extra pole near M^2, say s_B. Let us suppose that this pole comes from the second Riemann sheet of K. Then $S_F{}'$ turns out to be

$$S_F{}'(s) = \frac{1}{M^2 - s} + \frac{1}{s_B - s}\left(\frac{\lambda}{s_B - M^2}\right)^2 + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\sigma(s')}{s' - s} , \tag{4.1}$$

or

$$S_F{}'(s) = \frac{1}{M^2 - s} + d + \frac{s - M^2}{s_B - s}\frac{\lambda^2}{(s_B - M^2)^3} + \frac{s - M^2}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\sigma(s')}{(s' - M^2)(s' - s)} , \tag{4.2}$$

where λ is the coupling constant between the nucleon and the new particle with the mass $s_B^{\frac{1}{2}}$, and we set $M^2 < s_B$. The propagator must have one zero s_0 between M^2 and s_B, and may have at most one zero $s_0{}'$ between s_B and $(M+\mu)^2$. These zeros come from the second sheet of Γ, and hence coincide with the poles of Γ. Therefore, by the cancellation theorem these poles are not poles of T_0. The propagator may have also CDD zeros in the region $(M+\mu)^2 \leq s$, and one CDD zero in the region $s \leq M^2$ if $S_F{}' \to \infty$ as $s \to \infty$. But these zeros are not poles of Γ but zeros of K, just like those in the preceding sections. For simplicity, hereafter we consider only the case when there is no zero between s_B and $(M+\mu)^2$.[14] Then Eq. (2.3) for $S_F/S_F{}'$ and the dispersion relation (3.1) for Γ remain unchanged.

Now, in the Reggeized world ΓK should vanish, so that $K = 0$ or $\Gamma = 0$. Since K as well as Γ has a pole, we can take the limit $K = 0$ or $\Gamma = 0$ even if $g \neq 0$. Kaus and Zachariasen considered only the former limit $K = 0$, and neglected erroneously the latter limit $\Gamma = 0$. We assume that ΓK satisfies the once-subtracted dispersion relation

$$\Gamma K = g^2 - \frac{s - M^2}{s - s_0}\,\Gamma_0 - \frac{s - M^2}{s - s_B}\,K_0 + \frac{s - M^2}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{\mathrm{Im}\,(\Gamma K)}{(s' - M^2)(s' - s)}\,, \tag{4.3}$$

where

$$\Gamma_0 = \gamma K(s_0)\Big/(M^2 - s_0) = g_0^{\,2}\,, \tag{4.4}$$

$$K_0 = \lambda g_B \Gamma(s_B)\Big/(s_B - M^2) = -g_B^{\,2}\,. \tag{4.5}$$

Here $-g_B^{\,2}$ and $-\lambda g_B$ are residues of poles of T_0 and K at $s = s_B$, respectively. We rewrite (4.3) as

$$\Gamma K = (g^2 + g_B^{\,2} - g_0^{\,2}) + \frac{g_B^{\,2}(s_B - M^2)}{s - s_B} - \frac{g_0^{\,2}(s_0 - M^2)}{s - s_0}$$

$$+ \frac{s - M^2}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{\mathrm{Im}\,(\Gamma K)}{(s' - M^2)(s' - s)}\,. \tag{4.6}$$

Then the limit $\Gamma K = 0$ yields

$$g^2 + g_B^2 = g_0^2 , \tag{4.7}$$

$$(s_0 - M^2)(s_B - s_0) = 0, \tag{4.8}$$

and

$$g_B^2 (s_B - M^2) = g_0^2 (s_0 - M^2) . \tag{4.9}$$

From these equations we obtain three solutions:

(i) $s_B = s_0 \neq M^2$, $g_B^2 = g_0^2$, and hence $g^2 = 0$,

(ii) $s_0 = M^2 \neq s_B$, $g_B^2 = 0$, and hence $g^2 = g_0^2$,

(iii) $s_0 = M^2 = s_B$, and in general $g_B^2 \neq 0$, i.e., $g^2 \neq g_0^2$.

The solution (i) contradicts the assumption $g \neq 0$, so this must be rejected. The solution (ii) means that the elementary nucleon lies on the Regge trajectory, whereas the solution (iii) does not mean it, because of $g^2 \neq g_0^2$. Thus we reach the conclusion that in order for the same residue to appear as in the elementary case, we must in addition require $g^2 = g_0^2$ or $g_B^2 = 0$ in the $\Gamma K = 0$ limit. This means that

$$\lim_{\Gamma K \to 0} (s_0 - M^2)\big/(s_B - M^2) = g_B^2\big/g_0^2 = 0 , \tag{4.10}$$

as is seen from (4.9).

Inversely, if we require $g^2 = g_0^2$ and $\lim_{s_0 \to M^2}$ $(s_0 - M^2)\Big/(s_B - M^2) = 0$, then we have $\Gamma K = 0$ and $g_B^2 = 0$. This is seen as follows: From (4.1) or (4.2) we have

$$\frac{s_B - M^2}{s_0 - M^2} = 1 + \left(\frac{\lambda}{s_B - M^2}\right)^2 + \frac{s_B - s_0}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \; \frac{\sigma(s')}{s' - s_0} \; , \qquad (4.11)$$

or

$$2(s_B - M^2)\Big/(s_0 - M^2) = (\tilde{d} + \tilde{\sigma} + 1) + \left[(\tilde{d} + \tilde{\sigma} + 1)^2 + 4\,\lambda^2\Big/(s_B - M^2)^2\right]^{\frac{1}{2}} , \qquad (4.12)$$

where $\tilde{d} = d\,(s_B - s_0)$, (4.13)

$$\tilde{\sigma} = \frac{(s_B - s_0)\,(s_0 - M^2)}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \; \frac{\sigma(s')}{(s' - M^2)\,(s' - s_0)} \; . \qquad (4.14)$$

If $[(s_B - M^2)/(s_0 - M^2)] \to \infty$, then $\lambda^2/(s_B - M^2)^2$ or the integral in (4.11) becomes infinity, or both, or the right-hand side in (4.12) becomes infinity. This means $S_F' \to \infty$. Hence from $S_F'\Gamma = S_F K =$ finite (including zero) we get $\Gamma K = 0$. On the other hand, $g^2 = g_0^2$ and $\Gamma K = 0$ yield $g_B^2 = 0$.

In conclusion we have shown that when K has an extra pole, $\Gamma K = 0$ and $g_B^2 = 0$ are the necessary and sufficient conditions for the nucleon to lie on the Regge trajectory. We have again conjectured that in the limits $\Gamma K = 0$ and $g_B^2 = 0$ the CDD zeros in K go to infinity.

V. SPIN-ONE-HALF PARTICLE

We take into account spins and isospins in the pion-nucleon scattering. We make use of the Gell-Mann-Low[15] representation for the nucleon propagator:

$$S_F'(-i\gamma p) = \frac{1}{M + i\gamma p} + \frac{1}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\sigma_+(W')}{W' + i\gamma p} - \frac{1}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\sigma_-(W')}{W' - i\gamma p} . \tag{5.1}$$

Since $\sigma_\pm(W) \geqslant 0$ for $W \geqslant M + \mu$, the function

$$S_F'(W) = \frac{1}{M - W} + \frac{1}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\sigma_+(W')}{W' - W} - \frac{1}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\sigma_-(W')}{W' + W} \tag{5.2}$$

obtained by replacing $-i\gamma p$ in (5.1) with W is the Herglotz function. Then, the function S_F/S_F' can be written as

$$S_F/S_F' = 1 + \frac{W - M}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\bar{\sigma}_+(W')}{W' - W} - \frac{W - M}{\pi}\int_{M+\mu}^{\infty} dW' \frac{\bar{\sigma}_-(W')}{W' + W}$$

$$- \sum_n \frac{W - M}{W - W_n} C_n , \tag{5.3}$$

$$\bar{\sigma}_\pm(W) \equiv |S_F/S_F'|^2 \sigma_\pm(W) , \tag{5.4}$$

where W_n are real zeros of S_F', and $C_n \geq 0$. Let us write the form factor with one nucleon off the mass shell in the form

$$\bar{u}(p')\, \tau_\alpha i\gamma_5 K(-i\gamma p) = \bar{u}(p')\, \tau_\alpha i\gamma_5\, [\Lambda_+(p)\, K(W) + \Lambda_-(p)\, K(-W)]\ , \tag{5.5}$$

where

$$\Lambda_\pm(p) = (W \mp i\gamma p)/2W\ . \tag{5.6}$$

Here $K(W)$ is normalized to g at $W = M$. In the elastic region, $\sigma_\pm(W)$ is then given by

$$\sigma_\pm(W) = \frac{3}{16\pi}\ \frac{\rho_\pm(W)}{(W \mp M)^2}\ |K(\pm W)|^2\ , \tag{5.7}$$

where

$$\rho_\pm(W) = q(W)\ [(W \mp M)^2 - \mu^2]\ /\ W^2\ , \tag{5.8}$$

$$2Wq(W) = \left\{(W - M)^2 - \mu^2\right\}^{\frac{1}{2}} \left\{(W + M)^2 - \mu^2\right\}^{\frac{1}{2}}\ . \tag{5.9}$$

If we define the proper vertex function by

$$\Gamma(W) = K(W)\, S_F(W)/S_F'(W) , \tag{5.10}$$

then $\bar{\sigma}_{\pm}(W)$ in the elastic region is

$$\bar{\sigma}_{\pm}(W) = \frac{3}{16\pi} \frac{\rho_{\pm}(W)}{(W \mp M)^2} |\Gamma(\pm W)|^2 . \tag{5.11}$$

Now, the partial-wave amplitude, $T_{j\pm}(W)$, for a state with total angular momentum $j = \ell \pm \frac{1}{2}$ and isospin $I = \frac{1}{2}$ can be written as[16]

$$T_{j\pm}(W) = \frac{\pm 3}{16\pi} \frac{(M \mp W)^2 - \mu^2}{W^2} \frac{\Gamma(\pm W)\, K(\pm W)}{M \mp W} \delta_{j\frac{1}{2}} + f_{j\pm}(W) . \tag{5.12}$$

Here we have normalized $T_{j\pm}(W)$ to

$$T_{j\pm}(W) = e^{i\delta_{j\pm}} \sin \delta_{j\pm}/q \tag{5.13}$$

above the threshold, $\delta_{j\pm}$ being the phase shift. We assume that $f_{j\pm}(W)$ is meromorphic in the j plane. The form factor $K(W)$ satisfies unitarity relations[17]

$$\text{Im } K(W_+) = q(W_+)\, T^*_{\frac{1}{2}+}(W_+)\, K(W_+) \,, \tag{5.14}$$

$$\text{Im } K(-W_+) = q(-W_+)\, T^*_{\frac{1}{2}+}(-W_+)\, K(-W_+) \,, \tag{5.15}$$

for W, $M + \mu \lesssim W \lesssim M + 2\mu$, where $W_+ = W + i\epsilon$, and ϵ is an infinitesimally small positive number. By virtue of the MacDowell relation[18]

$$T_{\frac{1}{2}+}(-W) = -\, T_{\frac{1}{2}-}(W) \,, \tag{5.16}$$

Eq. (5.15) can be rewritten as

$$\text{Im } K(-W_+) = q(W_+)\, T^*_{\frac{1}{2}-}(W_+)\, K(-W_+) \,. \tag{5.17}$$

Here we have taken the cuts of $q(W)$ in the W plane between $-(M + \mu)$ and $-(M - \mu)$ and between $M - \mu$ and $M + \mu$, so that $q(W)$ is real for W, $|W| > M + \mu$, and $q(-W) = -q(W)$ when $W > M + \mu$. One can show that Γ and $f_{j\pm}$ satisfy unitarity relations by themselves,

$$\text{Im } \Gamma(W_+) = q(W_+)\, f^*_{\frac{1}{2}+}(W_+)\, \Gamma(W_+) \,, \tag{5.18}$$

$$\text{Im } \Gamma(-W_+) = q(-W_+)\, f^*_{\frac{1}{2}+}(-W_+)\, \Gamma(-W_+) \,, \tag{5.19}$$

and

$$\text{Im}\, f_{j\pm}(W_+) = q(W_+)\, |f_{j\pm}(W_+)|^2 , \qquad (5.20)$$

for W, $M + \mu \leq W \leq M + 2\mu$. By virtue of (5.16), $f_{j\pm}(W)$ also has the same reflection property, so that (5.19) can be rewritten as

$$\text{Im}\, \Gamma(-W_+) = q(W_+)\, f^*_{\frac{1}{2}-}(W_+)\, \Gamma(-W_+) . \qquad (5.21)$$

In quite the same way as in the preceding sections, the Kronecker delta singularity in $T_{j\pm}(W)$ must disappear in the Reggeized world; from (5.12) it then follows that $\Gamma(W) \not\equiv 0$. We shall not take the case when $K(W)$ has an extra pole. A generalization of the case is easy, just like Sec. IV. Now, the dispersion relation of $\Gamma(W)$ normalized to g at $W = M$ is

$$\Gamma(W) = g + \frac{W - M}{\pi} \int_{M+\mu}^{\infty} dW' \frac{\text{Im}\, \Gamma(W')}{(W' - M)(W' - W)} + \frac{W - M}{\pi} \int_{-\infty}^{-(M+\mu)} dW'$$
$$\times \frac{\text{Im}\, \Gamma(W')}{(W' - M)(W' - W)} - \frac{W - M}{W - W_0} \frac{\gamma}{M - W_0} , \qquad (5.22)$$

where W_0 is a pole of $\Gamma(W)$ coming from the second Riemann sheet, and γ is its residue. From the same argument as in the preceding section, this pole W_0 is also a pole of $f_{\frac{1}{2}+}(W)$, but not a pole of $T_{\frac{1}{2}+}(W)$

and $K(W)$, so the equality

$$K(W_0)\, \gamma/(M - W_0) = g_0^{\,2} \tag{5.23}$$

is valid, where $-g_0^{\,2}$ is the residue of the pole in $(16\pi/3) f_{\frac{1}{2}+}(W) \times W^2/[(M - W)^2 - \mu^2]$. From the reflection property, $f_{\frac{1}{2}-}(W)$ has a pole at $W = -W_0$, but not a pole of $T_{\frac{1}{2}-}(W)$ and $K(-W)$, by virtue of (5.23). Now Eqs. (5.22) and (5.23), together with the condition $\Gamma(W) \equiv 0$, lead to $M = W_0$ and $g^2 = g_0^{\,2}$. Since the pole of $f_{\frac{1}{2}\pm}(W)$ lies on the Regge trajectory, these equalities mean that the elementary nucleon pole now lies on the Regge trajectory. The inverse of this can be shown to be true, in quite the same way as in the preceding section. Thus, all the results obtained in the scalar nucleon case are also valid in the fermion case. Note that in the fermion case, the limit $Z_2 \delta M = 0$ is not equivalent to $\Gamma = 0$, even when S_F' satisfies the unsubtracted Lehmann representation.

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FOOTNOTES AND REFERENCES

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$$\Gamma(s) \simeq g\,(M^2 - s_0)\Big/(s - s_0) \equiv gf\,(s, M^2) \;.$$

If we take the limits $s = s_0$ and $M^2 = s_0$ along a line $(M^2 - s_0)\Big/(s - s_0) = \alpha$ in the (s, M^2) plane, then we have $\lim_{s \to s_0} f\,(s, M^2) = \alpha$, α being an arbitrary constant. At this point, the coupling constant should be defined in the limit along a line $(M^2 - s_0) / (s - s_0) = 1$. The author is indebted to Professor Tosio Kato for this point.

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