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Perturbations of tides and traveling waves for the Korteweg–de Vries equation

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ABSTRACT OF THE DISSERTATION

Perturbations of tides and traveling waves
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by

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This work explores the existence and behavior of solutions to the Korteweg–de Vries equation on the line for large perturbations of certain classical solutions. First, we show that given a suitable solution $V(t,x)$, KdV is globally well-posed for initial data $u(0,x) \in V(0,x) + H^{-1}(\mathbb{R})$. Our conditions on V do include regularity but do not impose any assumptions on spatial asymptotics. In particular, we show that smooth periodic and step-like profiles $V(0,x)$ satisfy our hypotheses.

Our second main objective is to prove a variational characterization of KdV multisolitons. Maddocks and Sachs [110] used that n-solitons are local constrained minimizers of the polynomial conserved quantities in order to prove that n-solitons are orbitally stable in $H^n(\mathbb{R})$. We show that multisolitons are the *unique global* constrained minimizers for this problem. We then use this characterization to provide a new proof of the orbital stability result from [110] via concentration compactness.

The dissertation of Thierry Michel Laurens is approved.

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PUBLICATIONS

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CHAPTER 1

Introduction

1.1 Multisolitons

The Korteweg–de Vries (KdV) equation

$$\frac{d}{dt}u = -u''' + 6uu' \tag{1.1.1}$$

(where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ and $u' = \partial_x u$) was derived over a century ago as a model for surface waves in a shallow channel of water. Although the equation was first proposed by Boussinesq [27], it did not gain traction until Korteweg and de Vries [100] used the explicit solutions

$$Q_{\beta,c}(t,x) = -2\beta^2 \operatorname{sech}^2[\beta(x - 4\beta^2 t - c)] \tag{1.1.2}$$

to explain the empirical observation of solitary traveling waves. Here, $\beta > 0$ controls the amplitude and speed of the wave and $c \in \mathbb{R}$ dictates the initial position of the peak.

Prior to [100], Boussinesq [28] had already discovered the solitary traveling waves (1.1.2) and began to study their variational properties. After first noting that the momentum and energy functionals

$$E_1(u) = \int \tfrac{1}{2}u^2\,dx \quad \text{and} \quad E_2(u) = \int \left[\tfrac{1}{2}(u')^2 + u^3\right] dx \tag{1.1.3}$$

are conserved for all solutions to (1.1.1), he then observed that the profiles (1.1.2) are critical points of E_2 with E_1 constrained. As we will discuss further below, this opens an avenue towards demonstrating that solitary traveling waves are stable, thus providing an explanation as to why they are readily observed in nature.

The solutions (1.1.2) are now commonly referred to as *solitons*, due to their particle-like behavior during interactions. This name was coined by Kruskal and Zabusky [140] when they numerically observed that two colliding solitons emerge with unchanged profiles and speeds. The interaction is nevertheless nonlinear, and this is manifested in a spatial shift of both waves in comparison to their initial trajectories.

We now know that this interaction can be modeled by a single explicit solution $Q_{\beta_1,\beta_2,c_1,c_2}$ which resembles two solitons as $t \to \pm\infty$ with parameters $\beta_1 \neq \beta_2$. In fact, together with the single soliton solutions (1.1.2), these are the beginning of an infinite family of solutions called *n-solitons* which describe the interaction of an arbitrary number n of distinct soliton profiles (see (2.1.1) for details).

In [107], Lax studied the two-soliton solutions $Q_{\beta_1,\beta_2,c_1,c_2}$ in an effort to explain the particle-like behavior of solitons. Just as the solitons (1.1.2) are constrained minimizers of E_2, he found that the existence of two-solitons is the symptom of another conserved quantity. Indeed, although he does not explicitly state it (until his later work [108]), his ODE for the two-soliton is the Euler–Lagrange equation for a critical point of

$$E_3(u) = \int \left[\tfrac{1}{2}(u'')^2 + 5u(u')^2 + \tfrac{5}{2}u^4\right] dx \tag{1.1.4}$$

with E_1 and E_2 constrained. This marked the first step towards proving the stability of two-solitons, which would explain Kruskal and Zabusky's observation in [140].

While the stability of single traveling waves is rather common for physical models, their stability under collisions is quite remarkable. This is closely related to the conservation of (1.1.4), which does not follow immediately from elementary physical considerations, unlike the momentum and energy (1.1.3). In fact, the authors of [115] discovered that the functionals (1.1.3) and (1.1.4) are the beginning of an infinite sequence of conserved quantities $E_n(u)$ defined on the L^2-based Sobolev spaces $H^{n-1}(\mathbb{R})$ (see (2.1.2) and (2.2.1)–(2.2.3) for details). Nowadays, we recognize this as a consequence of the complete integrability of KdV.

The family of multisoliton solutions go hand in hand with these conserved quantities. In general, the n-soliton is known to be a critical point for the following variational problem:

Problem 1.1.1. Minimize E_{n+1} over $H^n(\mathbb{R})$ with $E_1, \ldots, E_n$ constrained.

This variational problem has been heavily studied within the context of multisoliton stability. The first result was obtained by Benjamin [14], who proved that solitons are orbitally stable in $H^1(\mathbb{R})$: solutions that start close to a soliton profile in $H^1(\mathbb{R})$ remain close to a soliton profile for all time. This was the introduction of a widely applicable variational argument (cf. [138]) based on the fact that solitons are local constrained minimizers of E_2. Benjamin's seminal work contained some mathematical gaps, but these were later resolved by Bona [22].

Solitons are not merely local minimizers for this problem, but are *global* minimizers [1]. In order to employ this to deduce orbital stability though, we need to know that profiles that almost minimize E_2 are close to a minimizing soliton. In general, we cannot expect minimizing sequences to admit convergent subsequences, because the set of minimizing solitons is translation-invariant and hence non-compact. This issue was solved by Cazenave and Lions [35] for a variety of NLS-like equations by a concentration compactness principle: minimizing sequences are precompact *modulo translations*. This powerful method is now the dominant way of proving orbital stability, but it has not yet been successfully applied to this variational problem because it requires a global understanding.

Nevertheless, Maddocks and Sachs [110] discovered that n-solitons are orbitally stable in $H^n(\mathbb{R})$. First, they showed that n-solitons are indeed local minimizers of E_{n+1} with $E_1, \ldots, E_n$ constrained. Then their argument relied on a careful study of the Hessian of E_{n+1} on the manifold of minimizing n-solitons in directions tangent and perpendicular to the constraints. This local analysis then implied the global result by employing the commuting flows of $E_1, \ldots, E_n$.

However, the variational problem remains unsolved: are multisolitons *global* constrained minimizers of E_{n+1}? If so, are they unique? In particular, affirmative answers to this problem would be a significant step towards applying concentration compactness to prove the orbital stability of multisolitons. More generally, we would like to understand all solutions to this natural problem because they are fundamental objects for KdV.

In Chapter 2, we will answer these questions in the affirmative (cf. Theorem 2.1.3):

Theorem 1.1.2 ([105])**.** *If the constraints for $E_1, \ldots, E_n$ are attainable by a multisoliton of degree at most n, then the set of such multisolitons are the only global minimizers of E_{n+1} over $H^n(\mathbb{R})$ with $E_1, \ldots, E_n$ constrained.*

Together with an appropriate concentration compactness principle to analyze minimizing sequences (cf. Theorem 2.5.2), we will also recover the result of [110] (cf. Theorem 2.1.4):

Theorem 1.1.3 ([105])**.** *Each n-soliton is orbitally stable in $H^n(\mathbb{R})$.*

In addition to providing a complete answer when the constraints are attainable by a multisoliton of degree at most n, our methods also enable us to study the variational problem and minimizing sequences for other possible constraints. In this case, there are no global constrained minimizers and minimizing sequences exhibit different (and previously undiscovered) behavior; see Theorems 2.1.5 and 2.1.6 for details.

Evidently, the statement of Theorem 1.1.3 is predicated on the fact that for arbitrary initial data $u_0 \in H^n(\mathbb{R})$ with $n \geq 1$ there is a corresponding global solution $u(t)$ that remains in $H^n(\mathbb{R})$ for all $t \in \mathbb{R}$. Well-posedness for initial data in $H^s(\mathbb{R})$ (and $H^s(\mathbb{R}/\mathbb{Z})$) has been a fundamental line of investigation for KdV. The derivative in the nonlinearity of KdV prevents straightforward contraction mapping arguments from closing, so preliminary results produced continuous dependence in a weaker norm than the space of initial data. One of the first results to overcome this loss of derivatives phenomenon was obtained by Bona and Smith [24] who proved global well-posedness for $s \geq 3$. In the following decades, an extensive list of methods has been developed in the effort to lower the regularity s; see, for example, [23, 26, 38, 42, 70, 87, 89, 90, 99, 129, 134, 136]. Recently, a new low-regularity method was introduced in [97] that yields global well-posedness for $s \geq -1$ on both the line and the circle, a result that is sharp in both topologies. In the $\mathbb{R}/\mathbb{Z}$ case this result was already known [86].

Nevertheless, the $n = 1$ case [14] of Theorem 1.1.3 came nearly two decades before the corresponding well-posedness result [89]. This is because well-posedness plays only a small role in the proof of Theorem 1.1.3. Indeed, the crux of the problem is to prove Theorem 1.1.3 for Schwartz solutions, and the result for $H^n(\mathbb{R})$ solutions then follows immediately once well-posedness in $H^n(\mathbb{R})$ is known.

1.2 Periodic spatial asymptotics

Alongside the solitons (1.1.2), Korteweg and de Vries [100] also exhibited periodic traveling wave solutions of KdV. These are the spatially periodic analogues of solitons, and they can be expressed in terms of the Jacobian elliptic cosine function $\operatorname{cn}(z; k)$ as

$$V(t, x) = \eta - h \operatorname{cn}^2 \left[\sqrt{\tfrac{h}{2k^2}}(x - ct); k \right]. \tag{1.2.1}$$

Here, $k \in [0, 1)$ is the elliptic modulus, $h > 0$ is the wave height, and the trough level η and wave speed c are determined by

$$\eta = \tfrac{h}{k^2}\left[\tfrac{E(k)}{K(k)} - 1 + k^2\right] \quad \text{and} \quad c = \tfrac{2h}{k^2}\left[2 - k^2 - \tfrac{3E(k)}{K(k)}\right],$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind. In view of the representation (1.2.1), Korteweg and de Vries [100] dubbed these solutions *cnoidal waves.*

The result analogous to Theorem 1.1.3 for cnoidal waves would be orbital stability with respect to co-periodic perturbations. This topic has been heavily studied. In the pioneering paper [113], McKean proved orbital stability of cnoidal waves for $C^k(\mathbb{R}/L\mathbb{Z})$ perturbations using energy arguments, where L denotes the period of the underlying cnoidal wave. Orbital stability in $H^1(\mathbb{R}/L\mathbb{Z})$ was later proven using variational methods [8], and (in)stability results in $H^1(\mathbb{R}/L\mathbb{Z})$ have since been obtained for generalized KdV (gKdV), fractional KdV, and various other families of nonlinear dispersive equations; see, for example, [6, 9, 12, 17, 37, 44, 77, 84, 117, 118]. Orbital stability has also been demonstrated at the lower regularity $L^2(\mathbb{R}/L\mathbb{Z})$ [32, 56, 81] and even $L^2(\mathbb{R}/nL\mathbb{Z})$ with $n > 1$ [119]. All of these orbital stability results pertain to spatially periodic perturbations.

What about stability with respect to localized perturbations? This physically important problem has also received much attention, but only within the context of spectral (linear) stability, where the linearized equation is considered as an operator on $L^2(\mathbb{R})$ or $L^2(\mathbb{R}/\mathbb{Z})$ perturbations. Spectral stability for $L^2(\mathbb{R})$ perturbations of cnoidal waves was established in [25], and spectral and modulational (in)stability for KdV-like equations have been explored in [10, 17, 31, 78, 80, 82, 83, 119].

As in the case of solitons, a nonlinear (in)stability theory necessarily relies upon our ability to solve the equation for such initial data. However, as solutions in $H^s(\mathbb{R}/\mathbb{Z})$ spaces are spatially periodic and solutions in $H^s(\mathbb{R})$ spaces decay at infinity, localized perturbations of a periodic background have been excluded by traditional well-posedness considerations.

More broadly, we would like to understand other classes of waveforms that are of physical interest. For example, this includes other asymptotically periodic functions, such as wave dislocation where the periods as $x \to \pm\infty$ may not align, and waves with altogether different periodic asymptotics as $x \to \pm\infty$. Quasi- and almost periodic spatial asymptotics have also received much attention in the literature (see, for example, [21, 36, 43, 45, 46, 54, 55, 135]). Evidently, all of these important classes of initial data are excluded by classical analysis.

Our objective in Chapter 4 will be to extend low-regularity methods for well-posedness to the regime of exotic spatial asymptotics. Specifically, we will show that that given a sufficiently regular solution $V(t, x)$, KdV is well-posed for $H^{-1}(\mathbb{R})$ perturbations of V (cf. Theorems 4.1.2 and 4.1.3):

Theorem 1.2.1 ([106]). *Given a solution $V : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ that is sufficiently regular, the KdV equation* (1.1.1) *with initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ is globally well-posed in the following sense: $u(t) = V(t) + q(t)$ where $V(t)$ solves KdV, and the equation*

$$\frac{d}{dt} q = -q''' + 6qq' + 6(Vq)' \tag{1.2.2}$$

for $q(t)$ with initial data in $H^{-1}(\mathbb{R})$ is globally well-posed.

The specific hypotheses satisfied by the background wave V are listed in Definition 4.1.1. These conditions do include regularity, but do not impose any assumptions on spatial asymptotics.

There are rich classes of initial data that satisfy these hypotheses. In Chapter 4 we will show that this includes the important case of smooth periodic initial data $V(0, x)$ (including the cnoidal waves (1.2.1)):

Corollary 1.2.2 ([106]). *Given $V(0) \in H^5(\mathbb{R}/\mathbb{Z})$, the KdV equation* (1.1.1) *is globally well-posed for initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ in the sense of Theorem 1.2.1.*

Just as $H^{-1}(\mathbb{R})$ is the lowest regularity for which we can hope to have well-posedness in the case $V \equiv 0$ [116], we expect that Theorem 1.2.1 is sharp in the class of $H^s(\mathbb{R})$ spaces. There is a known technique [97, Cor. 5.3] for extending $H^{-1}(\mathbb{R})$ well-posedness to $H^s(\mathbb{R})$ with $s > -1$ using equicontinuity, and so $H^{-1}(\mathbb{R})$ is the key space for establishing well-posedness.

We believe that Theorem 1.2.1 can also be applied to classes of quasi-periodic initial data, or any other class amenable to complete integrability methods. The known results on exotic backgrounds use integrable methods like the inverse scattering transform, which are well suited to verify the hypotheses in Definition 4.1.1.

In all cases, the presence of the background wave V breaks the macroscopic conservation laws of KdV. If q is a regular solution to (1.2.2) then the momentum functional E_1 evolves according to

$$\frac{d}{dt} \int q(t,x)^2 \, dx = 6 \int V'(t,x) q(t,x)^2 \, dx. \tag{1.2.3}$$

In fact, in the next section we will discuss a class of solutions for which the term V' has a sign and there is no cancellation in the integral; the resulting growth in the momentum is manifested in a dispersive shock that develops in the long-time asymptotics [51, Fig. 1]. Despite this lack of conservation, we are able to adapt low-regularity methods for well-posedness to the equation (1.2.2) for q because these conserved quantities do not blow up in finite time.

1.3 Step-like initial data

Another physically important class of initial data consists of waveforms that are step-like, in the sense that $u(0, x)$ approaches different constant values as $x \to \pm\infty$. These arise in the study of bore propagation (cf. [16, 34, 60, 71, 124, 139]) and rarefaction waves (cf. [7, 60, 109, 120, 142]).

In Chapter 3, we will analyze step-like initial data in order to apply our general well-posedness result Theorem 1.2.1. Consider the smooth step function

$$W(x) = c_1 \tanh(x) + c_2 \quad \text{with } c_1, c_2 \in \mathbb{R} \text{ fixed,} \tag{1.3.1}$$

which exponentially decays to its asymptotic values. As $-u$ is proportional to the water wave height, W models an incoming tide if $c_1 > 0$ and an outgoing tide if $c_1 < 0$.

A classical result in the study of step-like asymptotics is (cf. Theorem 3.1.1):

Theorem 1.3.1 ([104]). *Fix an integer $s \geq 3$. The KdV equation* (1.1.1) *with initial data $u(0) \in W + H^s(\mathbb{R})$ is globally well-posed in the following sense: $u(t) = W + q(t)$ and the equation*

$$\frac{d}{dt} q = -(q + W)''' + 6(q + W)(q + W)'$$

for $q(t)$ with initial data in $H^s(\mathbb{R})$ is globally well-posed.

Theorem 1.3.1 is not new (as we will discuss in Section 3.1), but we will use its statement to formulate our main result. Applying Theorem 1.3.1 to the initial data $q(0) \equiv 0$, we conclude that given W there is a unique global solution $V(t) = W + q(t)$ to KdV (1.1.1) with initial data W, and $t \mapsto V(t) - W$ is a continuous function into $H^s(\mathbb{R})$ for all $s \geq 3$. The main thrust of this work is to show that such $V(t)$ satisfy the hypotheses of Theorem 1.2.1 (cf. Corollary 4.1.4):

Corollary 1.3.2 ([104]). *Given initial data $V(0) = W$ of the form* (1.3.1)*, the KdV equation* (1.1.1) *is globally well-posed for $u(0) \in V(0) + H^{-1}(\mathbb{R})$ in the sense of Theorem 1.2.1.*

The formulation of Theorem 1.3.1 is inspired by the result [24] of Bona and Smith in the case $W \equiv 0$. They proved well-posedness in $H^s(\mathbb{R})$ for $s \geq 3$ by approximating KdV by a family of BBM equations. Well-posedness for the BBM equation was shown in [15] and follows directly from standard ODE arguments. In fact, in [15, §3] the authors also discuss how to extend their well-posedness result for the BBM equation to step-like initial data, which together with the original argument from [24] can also be used to prove Theorem 1.3.1. However, it is our proof of Theorem 1.3.1, not the statement, that provides the necessary ingredients for Corollary 1.3.2.

In addition to providing input for Corollary 1.3.2, our proof of Theorem 1.3.1 has other benefits to offer. Already in the case $W \equiv 0$, we obtain a new proof of the Bona–Smith result [24] using low-regularity methods. Our proof of Theorem 1.3.1 is significantly shortened in this case, and features a number of advantages. One main advantage is that the *a priori* estimates for our approximate equations are the same as those for KdV. In [24], the authors approximate KdV by the flow

$$\frac{d}{dt}u = (1 - \varepsilon\partial_x^2)^{-1}\{-u''' + 6uu'\} \tag{1.3.2}$$

in the limit $\varepsilon \to 0^+$. The linear term $-u'''$ is relatively harmless; the upshot here is that the nonlinear term $6uu'$ receives two degrees of smoothing, thus allowing straightforward contraction mapping arguments to close. On the other hand, this perturbation breaks the conservation of the quantities E_n, and the recovery of *a priori* estimates for the H^3-norm of $u(t)$ is rather subtle. By comparison, our approximate flows also conserve the quantities E_n

(as we will see in the next section), and so our *a priori* estimates are the same as those for KdV.

The second main advantage of our methods for proving Theorem 1.3.1 is that the convergence of the approximate solutions is demonstrated in a transparent way. In [24], the authors project the initial data $u_\varepsilon(0)$ for the approximate equation (1.3.2) onto low frequencies in a carefully chosen ε-dependent way. Convergence then follows from a delicate trade-off between the initial smoothing and a loss of derivatives in the *a priori* estimates for their approximate equation. Conversely, our argument will feature a more modern approach: first we prove convergence at a lower regularity (in order to absorb any loss of derivatives), and then we recover convergence at higher regularity via an *a priori* equicontinuity result. The statement of equicontinuity pertains to the evolution of the high frequencies of the solution $u(t)$ for fixed initial data $u(0)$, rather than varying the initial data $u_\varepsilon(0)$ and controlling the growth of all frequencies.

1.4 The method of commuting flows

To prove the results presented in Sections 1.2 and 1.3, we will employ the method of commuting flows introduced in [97] to achieve well-posedness in $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{R}/\mathbb{Z})$. This method has been developed in several subsequent papers, both in applications to other phenomena of KdV at low regularity [94, 121] and in achieving sharp well-posedness results for other completely integrable systems [30, 72, 73, 95]. Most recently, the author proved sharp well-posedness for the Benjamin–Ono equation in collaboration with Killip and Vişan [93], a problem on which no progress had been made for 15 years despite much effort.

In the method of commuting flows, the dynamics of KdV is approximated by the Hamiltonian flow associated to a certain family of functionals. Recall that KdV is a Hamiltonian system, in the sense that the equation (1.1.1) is induced by the functional E_2 (defined in (1.1.3)) via the Poisson structure

$$\{F, G\} = \int \frac{\delta F}{\delta q}(x) \Big(\frac{\delta G}{\delta q}\Big)'(x)\, dx.$$

Here, we are using the notation

$$dF|_q(f) = \frac{d}{ds}\Big|_{s=0} F(q + sf) = \int \frac{\delta F}{\delta q}(x) f(x)\, dx$$

for the derivative of the functional $F(q)$. Concomitant with this are the convenient notations

$$q(t) = e^{tJ\nabla H} q(0) \quad \text{for the solution to} \quad \frac{dq}{dt} = \partial_x \frac{\delta H}{\delta q},$$

and

$$\frac{d}{dt} F(q(t)) = \{F(q), H(q)\} \quad \text{for the quantity } F(q) \text{ with } q(t) = e^{tJ\nabla H} q(0).$$

This Poisson structure is the bracket associated to the almost complex structure $J := \partial_x$ and the L^2-pairing. Additionally, the momentum functional E_1 generates translations (in accordance with its name), and the conservation of E_1 under KdV can be expressed as $\{E_1, E_2\} = 0$.

In fact, the whole sequence of conserved quantities E_n Poisson commute with each other, in the sense that

$$\{E_m, E_n\} = 0 \quad \text{for all } m, n. \tag{1.4.1}$$

Indeed, the existence of such a family is a necessary condition for a system to be completely integrable. The identity (1.4.1) tells us that the flows corresponding to E_m and E_n commute (see, for example, [11, §39]), a remarkable property that is not possessed by generic nonlinear equations. However, these are not the commuting flows that we will be using to approximate KdV, because they are functionally independent and effectively flow in orthogonal directions. As we will see below, our choice of commuting flows is constructed from the entire sequence of conserved quantities E_n.

In the theory of finite-dimensional Hamiltonian systems, an ODE is called completely integrable if it possesses sufficiently many (depending only on the dimension) conserved quantities satisfying (1.4.1) that are functionally independent. For such systems, it turns out that we can always formally solve the equations of motion and write down a formula for the solution. Specifically, we can find a change of variables, called *action-angle coordinates*, where the action coordinates are conserved in time and the angles evolve linearly; undoing this change of variables then provides our formal solution. As a consequence, every conserved quantity for the system can be written in terms of the action coordinates alone. This process is known as Liouville integration, and the exact procedure heavily depends on the example at hand.

For PDEs however, the phase space is an infinite-dimensional function space and thus the conditions for a system to be completely integrable are unclear; for example, when is an infinite family of conserved quantities sufficiently many? Nevertheless, KdV is universally accepted as an example of an integrable system. This is because it possesses not only infinitely many conserved quantities E_n, but also enjoys every other consequence that an integrable system should have.

In particular, the authors of [63] discovered a change of variables that linearizes the KdV flow. (Strictly speaking these variables are not action-angle coordinates, but the latter are easily expressed in terms of the former; see [141] for details.) This nonlinear transformation takes the solution $u = u(t)$ of KdV for fixed t, constructs the corresponding one-dimensional Schrödinger operator $-\partial_x^2 + u$, and outputs its scattering data. Specifically, given a potential $u(x)$ in the weighted Lebesgue space

$$L^1_2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} |f(x)|(1 + |x|^2)\, dx < \infty \right\}, \tag{1.4.2}$$

the operator $-\partial_x^2 + u$ on $L^2(\mathbb{R})$ has purely absolutely continuous spectrum $[0, \infty)$ and finitely many simple negative eigenvalues $-\beta_1^2, \dots, -\beta_N^2$. For such u we can construct the transmission and reflection coefficients $T(k; u)$ and $R(k; u)$ at frequencies $k \in \mathbb{R}$. In general $|T|$ and $|R|$ are bounded by 1 for $k \in \mathbb{R}$, while multisolitons are distinguished by having $R \equiv 0$ and $|T| \equiv 1$ on $\mathbb{R}$. Additionally, the transmission coefficient $T(k; u)$ extends to a meromorphic function of k in the upper half-plane $\mathbb{C}^+$ that is continuous down to $\mathbb{R}$ and whose only singularities are simple poles at the square roots $i\beta_1, \dots, i\beta_N$ of the eigenvalues.

In order to have a full set of coordinates we must also consider the norming constants $c_1, \dots, c_N > 0$, defined by $e^{\beta_n x}|\psi_n(x)| \to c_n$ as $x \to +\infty$ where ψ_n is an L^2-normalized eigenfunction with eigenvalue $-\beta_n^2$. It turns out that all of these objects together uniquely characterize the potential u (see Section 2.2 for details). Moreover, they evolve very simply in time: if $u(t)$ solves KdV, then $T(k;u)$, $|R(k;u)|$, and the eigenvalues $-\beta_n^2$ remain constant while $\arg R(k;u)$ and $\log c_n$ evolve linearly.

As in the finite-dimensional case, all conserved quantities for KdV should admit an expression solely in terms of the action coordinates. In particular, this means that the functionals E_n can be written in terms of $T(k;u)$, $|R(k;u)|$, and the eigenvalues $-\beta_n^2$. The exact formulas were discovered by Zakharov and Faddeev in [141], and the sequence begins with

$$\begin{aligned} E_1(u) &= \tfrac{4}{\pi} \int_{-\infty}^{\infty} k^2 \log|a(k;u)|\, dk + \tfrac{8}{3} \sum_{m=1}^{N} \beta_m^3, \\ E_2(u) &= \tfrac{16}{\pi} \int_{-\infty}^{\infty} k^4 \log|a(k;u)|\, dk - \tfrac{32}{5} \sum_{m=1}^{N} \beta_m^5, \\ E_3(u) &= \tfrac{64}{\pi} \int_{-\infty}^{\infty} k^6 \log|a(k;u)|\, dk + \tfrac{128}{7} \sum_{m=1}^{N} \beta_m^7. \end{aligned} \tag{1.4.3}$$

Here, $a(k;u) = 1/T(k;u)$ is simply the reciprocal of the transmission coefficient; this function is holomorphic in $\mathbb{C}^+$ with simple zeros at $i\beta_1, \dots, i\beta_N$. These formulas are valid for u Schwartz, where we have both the energies $E_n(u)$ and the scattering data $a(k;u)$ and $\beta_1, \dots, \beta_N$ at our disposal. In fact, these formulas will be a key tool in proving Theorem 1.1.2.

Conversely, $a(k;u)$ is conserved because it can be represented solely in terms of the conserved quantities E_n. Indeed, Zakharov and Faddeev [141] also showed that for u Schwartz we have

$$\log a(i\kappa;u) = \tfrac{1}{2\kappa}\big(\textstyle\int u\, dx\big) - \tfrac{1}{4\kappa^3} E_1(u) + \tfrac{1}{16\kappa^5} E_2(u) + O(\kappa^{-7}) \tag{1.4.4}$$

as $\kappa \to +\infty$. The coefficients appearing on the right-hand side of this expansion are exactly the sequence of polynomial conserved quantities E_n. The first coefficient is also conserved, but it is a Casimir and hence is automatically conserved under any Hamiltonian flow.

To make sense of $\log a(i\kappa; u)$, the presence of $\int u\,dx$ in the expansion (1.4.4) would curtail our attention to functions u that are at least conditionally integrable. In order to work at lower regularity, we will consider the renormalization

$$\alpha(\kappa, u) := -\log a(i\kappa; u) + \tfrac{1}{2\kappa}\int u\,dx. \tag{1.4.5}$$

As both $a(k; u)$ and $\int u\,dx$ are conserved, we also expect α to be conserved whenever it is defined. In fact, in [97] the authors show that α is a real-analytic function of q on bounded subsets of H^{-1} and is conserved for all $\kappa > 0$ sufficiently large.

From the expansion (1.4.4) we obtain

$$\alpha(\kappa, u) = \tfrac{1}{4\kappa^3}E_1(u) - \tfrac{1}{16\kappa^5}E_2(u) + O(\kappa^{-7}) \tag{1.4.6}$$

as $\kappa \to \infty$, at least for u Schwartz. Rearranging this expression, we might expect that the dynamics of the Hamiltonians

$$H_\kappa(u) := -16\kappa^5\alpha(\kappa, u) + 4\kappa^2 E_1(u) \tag{1.4.7}$$

approximate that of KdV as $\kappa \to \infty$. Indeed, in order to prove well-posedness in H^{-1}, the authors of [97] demonstrated that the flows induced by the Hamiltonians (1.4.7) are well-posed in H^{-1}, commute with each other, and converge to that of KdV in H^{-1} as $\kappa \to \infty$. It then follows that the data-to-solution map extends continuously from smooth initial data to a jointly continuous map $\mathbb{R}_t \times H^{-1} \to H^{-1}$; this is the meaning of well-posedness in [97].

We will employ the commuting flows of the Hamiltonians H_κ in (1.4.7) to prove our general well-posedness result Theorem 1.2.1. Our approximate equation for the perturbation q is obtained by evolving V and $u = V + q$ under the H_κ flow:

$$q_\kappa(t) = e^{tJ\nabla H_\kappa}(q(0) + V(0)) - e^{tJ\nabla H_\kappa}V(0).$$

In Chapter 4 we will show that for V sufficiently regular, this flow is well-posed in $H^{-1}(\mathbb{R})$ and converges to the equation (1.2.2) for q in $H^{-1}(\mathbb{R})$ as $\kappa \to \infty$ uniformly on bounded time intervals.

The major structural difference between our argument and that in [97] is that we cannot assume the existence of regular solutions to (1.2.2). Although some results in this direction do exist (e.g. [52, 53]), we would need to significantly increase our assumptions on the background wave V in order to employ them. Instead of showing that $q_\kappa(t)$ converges to the solution $q(t)$ of (1.2.2) as $\kappa \to \infty$, we show that $q_\kappa(t)$ is a Cauchy sequence as $\kappa \to \infty$ and we define the limit to be an H^{-1} solution of (1.2.2). This is the right notion of solution, because we show that the solution map is jointly continuous $\mathbb{R}_t \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ (cf. Theorem 4.1.2) and on a dense subset of initial data it coincides with the classical notion of solution (cf. Theorem 4.1.3). This approach is consonant with the notion of well-posedness in [97].

In the $V \equiv 0$ case [97], a key step in proving convergence of the H_κ flows is the conservation of the functional $\alpha(\varkappa, q)$ under both the H_κ and KdV flows. However, the appearance of the background wave V breaks this conservation, just as we already saw happen for the momentum in (1.2.3). Instead, we will show (cf. Proposition 4.3.1) that $\alpha(\varkappa, q)$ grows at most exponentially in time under the H_κ flow, provided that V is sufficiently regular and we choose κ to be, say, twice as large as $\varkappa$. We will then be able to match this new dependence between the energy parameters κ and $\varkappa$ in the proof of convergence (cf. Proposition 4.5.2).

CHAPTER 2

Multisolitons are the unique constrained minimizers of the polynomial conserved quantities

2.1 Introduction

Multisolitons are the most well-understood family of explicit solutions to the KdV equation (1.1.1). These solutions describe the interaction of an arbitrary number of the soliton profiles (1.1.2) with distinct amplitudes.

Definition 2.1.1 (Multisoliton solutions)**.** Fix $N \geq 1$. Given $\beta_1, \ldots, \beta_N > 0$ distinct and $c_1, \ldots, c_N \in \mathbb{R}$, the *multisoliton of degree* N (or N*-soliton*) with these parameters is

$$Q_{\boldsymbol{\beta},\boldsymbol{c}}(x) = -2 \tfrac{d^2}{dx^2} \log \det[A(x)], \tag{2.1.1}$$

where $A(x)$ is the $N \times N$ matrix with entries

$$A_{jk}(x) = \delta_{jk} + \tfrac{1}{\beta_j + \beta_k} e^{-\beta_j(x-c_j) - \beta_k(x-c_k)}.$$

The unique solution to KdV (1.1.1) with initial data $u(0,x) = Q_{\boldsymbol{\beta},\boldsymbol{c}}(x)$ is

$$u(t,x) = Q_{\boldsymbol{\beta},\boldsymbol{c}(t)}(x), \quad \text{where} \quad c_j(t) = c_j + 4\beta_j^2 t.$$

We define the *multisoliton of degree zero* to be the zero function.

Formula (2.1.1) was first discovered in [88] as part of a study of potentials for one-dimensional Schrödinger operators with vanishing reflection coefficient. Multisolitons have since been thoroughly examined by means of inverse scattering theory; see, for example, [63, 64, 74, 133, 137, 143].

Multisolitons are closely related to the polynomial conserved quantities E_n of KdV. These are an infinite sequence of functionals that are conserved under the KdV flow [115]. Their densities are defined recursively by [141]:

$$\sigma_1(x) = u(x), \qquad \sigma_{m+1}(x) = -\sigma_m'(x) - \sum_{j=1}^{m-1} \sigma_j(x)\sigma_{m-j}(x).$$

For even m the density σ_m is a complete derivative, but for odd m we obtain a nontrivial conserved quantity

$$E_n(u) = (-1)^n \tfrac{1}{2} \int_{-\infty}^{\infty} \sigma_{2n+1}(x)\, dx \tag{2.1.2}$$

whose density is a polynomial in $u, u', \dots, u^{(n-1)}$. The first three functionals in this sequence are given in (1.1.3) and (1.1.4). When evaluated at an N-soliton, the value of E_n is given by [64, 141]:

$$E_n(Q_{\boldsymbol{\beta},\boldsymbol{c}}) = (-1)^{n+1} \tfrac{2^{2n+1}}{2n+1} \sum_{m=1}^{N} \beta_m^{2n+1}, \tag{2.1.3}$$

which is independent of $\boldsymbol{c}$.

Multisolitons enjoy special variational properties with respect to these conserved quantities. Indeed, Maddocks and Sachs [110] used that the n-soliton is a local minimizer for the following variational problem in order to prove that n-solitons are orbitally stable in $H^n(\mathbb{R})$.

Problem 2.1.2. Given an integer $n \geq 0$ and constraints $e_1, \dots, e_n$, minimize $E_{n+1}(u)$ over the set

$$\mathscr{C}_{\boldsymbol{e}} = \{u \in H^n(\mathbb{R}) : E_1(u) = e_1,\ \dots,\ E_n(u) = e_n\}.$$

In the case $n = 0$ there are no constraints, and the minimizer of the L^2-norm over the space $L^2(\mathbb{R})$ is simply the zero-soliton $q(x) \equiv 0$. We include this trivial observation because it will provide a convenient base case for an induction argument.

Despite the success [110] of Maddocks and Sachs, the variational problem remains unsolved: are multisolitons *global* constrained minimizers of E_{n+1}? If so, are they unique? In particular, affirmative answers to this problem would be a significant step towards applying concentration compactness to prove the orbital stability of multisolitons.

More generally, we would like to understand all solutions to this natural problem because they are basic building blocks. Indeed, special solutions to completely integrable models elucidate an avenue to a low-complexity understanding of the dynamics. A critical point for this variational problem must satisfy the Euler–Lagrange equation

$$\nabla E_{n+1}(u) = \lambda_1 \nabla E_1(u) + \lambda_2 \nabla E_2(u) + \cdots + \lambda_n \nabla E_n(u). \tag{2.1.4}$$

Solutions to these equations are called *algebro-geometric solutions*, and they are fundamental objects for integrable systems [66,67]; in fact, solitons were discovered via (2.1.4). Naturally, we would like to understand all critical points for this variational problem.

In order to state our main results, we introduce the following notation. Given $n \geq 1$, we define the set of feasible constraints

$$\mathcal{F}^n = \{(e_1, \dots, e_n) \in \mathbb{R}^n : \mathscr{C}_{\boldsymbol{e}} \neq \emptyset\}$$

which are attainable by some function in $H^n(\mathbb{R})$. We also define the set of constraints attainable by multisolitons of degree at most $N \geq 0$:

$$\mathcal{M}_N^n = \big\{(e_1, \dots, e_n) \in \mathbb{R}^n : \exists\ M \leq N \text{ and } \boldsymbol{\beta}, \boldsymbol{c} \in \mathbb{R}^M \text{ with } Q_{\boldsymbol{\beta},\boldsymbol{c}} \in \mathscr{C}_{\boldsymbol{e}}\big\}.$$

First we prove that when the constraints are attainable by a multisoliton of degree at most n, those multisolitons are the unique global minimizers:

Theorem 2.1.3 (Variational characterization)**.** *Fix an integer $n \geq 1$. Given constraints $(e_1, \dots, e_n) \in \mathcal{M}_n^n$, there exists a unique integer $N \leq n$ and parameters $\beta_1 > \cdots > \beta_N > 0$ so that the multisoliton $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ lies in $\mathscr{C}_{\boldsymbol{e}}$ for some (and hence all) $\boldsymbol{c} \in \mathbb{R}^N$. Moreover, we have*

$$E_{n+1}(u) \geq E_{n+1}(Q_{\boldsymbol{\beta},\boldsymbol{c}}) \quad \textit{for all } u \in \mathscr{C}_{\boldsymbol{e}},$$

with equality if and only if $u = Q_{\boldsymbol{\beta},\boldsymbol{c}}$ for some $\boldsymbol{c} \in \mathbb{R}^N$.

Together with an appropriate concentration compactness principle (Theorem 2.5.2) to analyze minimizing sequences, we also prove the orbital stability result of [110] via concentration compactness:

Theorem 2.1.4 (Orbital stability)**.** *Fix an integer $n \geq 1$ and distinct positive parameters $\beta_1, \dots, \beta_n$. Given $\varepsilon > 0$ there exists $\delta > 0$ so that for every initial data $u(0) \in H^n(\mathbb{R})$ satisfying*

$$\inf_{\boldsymbol{c} \in \mathbb{R}^n} \|u(0) - Q_{\boldsymbol{\beta},\boldsymbol{c}}\|_{H^n} < \delta,$$

the corresponding solution $u(t)$ of KdV (1.1.1) satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\boldsymbol{c} \in \mathbb{R}^n} \|u(t) - Q_{\boldsymbol{\beta},\boldsymbol{c}}\|_{H^n} < \varepsilon.$$

Many refinements have been made since the discovery of Theorem 2.1.4. While we choose to work in the space $H^n(\mathbb{R})$ because it is amenable to the functionals $E_1, \dots, E_{n+1}$, the regularity of Theorem 2.1.4 in the scope of $H^s(\mathbb{R})$ spaces has since been significantly lowered [5, 98, 112, 114]. Moreover, the time-evolution of the parameter $\boldsymbol{c}$ has been shown to remain close to the usual evolution $c_j + 4\beta_j^2 t$ uniformly for $t > 0$ [3]. In addition to Lyapunov stability statements like Theorem 2.1.4, the asymptotic stability of multisolitons has also been studied [111, 112, 114].

Recently, the $n = 2$ cases of Theorems 2.1.3 and 2.1.4 were resolved by Albert and Nguyen [4]. They also showed that for $n = 1$ we have

$$\mathcal{M}_1^1 = \{e_1 : e_1 \geq 0\} = \mathcal{F}^1,$$

but for $n = 2$ we have

$$\begin{aligned} \mathcal{M}_2^2 &= \left\{(e_1, e_2) : e_1 > 0,\ e_2 \in \left[-\tfrac{32}{5}(\tfrac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}},\ -2^{-\frac{2}{3}} \tfrac{32}{5}(\tfrac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}}\right)\right\} \cup \{(0,0)\}, \\ \mathcal{F}^2 &= \left\{(e_1, e_2) : e_1 > 0,\ e_2 \geq -\tfrac{32}{5}(\tfrac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}}\right\} \cup \{(0,0)\}. \end{aligned} \tag{2.1.5}$$

These sets are depicted in Fig. 2.1. Theorem 2.1.3 says that for each $(e_1, e_2) \in \mathcal{M}_2^2$ the constrained minimizers of E_3 are multisolitons of degree at most 2. Moreover, by the $n = 1$

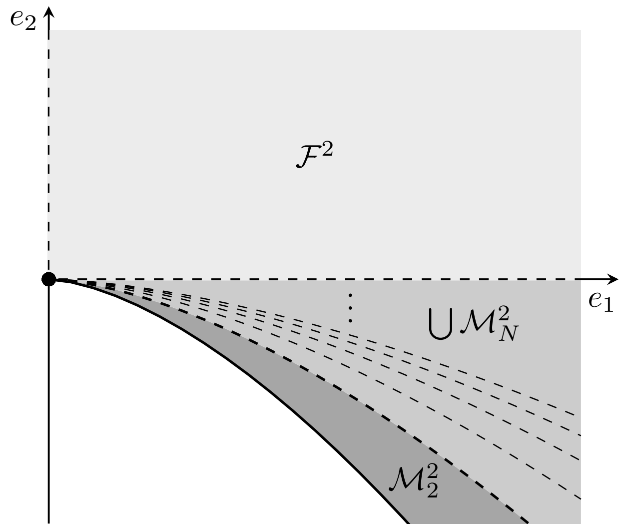

Figure 2.1: The sets (2.1.5) and (2.1.6) of constraints. The three shaded regions correspond to qualitatively different behavior for minimizing sequences.

case of Theorem 2.1.3, we know that for the constraints $e_1 > 0$ and $e_2 = -\frac{32}{5}(\frac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}}$ on the boundary of $\mathcal{M}_2^2$ the minimizer is a single soliton. Likewise, in the case of $n = 3$ constraints, the boundary $\mathcal{M}_3^3 \setminus (\operatorname{int} \mathcal{M}_3^3)$ in $\mathbb{R}^3$ looks like the graph of a continuous function on $\mathcal{M}_2^2$, and so on. In general, we will show that $\mathcal{M}_n^n$ is homeomorphic to the half-open simplex of parameters $\boldsymbol{\beta} \in \mathbb{R}^n$ corresponding to multisolitons of degree at most n (cf. Lemma 2.3.4).

Albert and Nguyen's analysis does not easily extend to the general case however, because it makes crucial use of the fact that for $n = 2$ all solutions of the Euler–Lagrange equation (2.1.4) are one- or two-solitons [2]. Much is known about the ODEs (2.1.4); they are completely integrable Hamiltonian systems and thus can formally be solved. Nevertheless, for $n \geq 3$ it is open whether all solutions to the Euler–Lagrange equation (2.1.4) are multisolitons of degree at most n. (Specifically, it is difficult to show that if the Lagrange multipliers $\lambda_1, \ldots, \lambda_n$ do not correspond to an n-soliton then solutions of (2.1.4) must be singular; see [2, §6] for details.)

Another advantage of our method is that it enables us to study the variational problem and minimizing sequences even when the constraints $(e_1, \ldots, e_n) \notin \mathcal{M}_n^n$ are not attainable by a multisoliton of degree at most n. After obtaining the $n = 2$ case of Theorems 2.1.3 and 2.1.4, Albert and Nguyen [4] made the reasonable conjecture for the remaining case $(e_1, e_2) \in \mathcal{F}^2 \setminus \mathcal{M}_2^2$ that minimizing sequences resemble a collection of solitons with at least two β parameters equal. Using our methods, we prove that this is partially true provided that the constraints are attainable by some multisoliton:

Theorem 2.1.5. *Given constraints $(e_1, \ldots, e_n) \in \mathcal{M}_N^n \setminus \mathcal{M}_{N-1}^n$ for some $N \geq n+1$, the infimum of $E_{n+1}(u)$ over $u \in \mathscr{C}_{\boldsymbol{e}}$ is finite but not attained.*

Moreover, if $\{q_k\}_{k \geq 1} \subset \mathscr{C}_{\boldsymbol{e}}$ is a minimizing sequence:

$$E_1(q_k) \to e_1, \quad \ldots, \quad E_n(q_k) \to e_n, \quad E_{n+1}(q_k) \to \inf_{u \in \mathscr{C}_{\boldsymbol{e}}} E_{n+1}(u) \quad \text{as } k \to \infty,$$

then there exist parameters $\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^J$ of total degree $\sum_{j=1}^J \# \boldsymbol{\beta}_n^j = N$ taking at most n distinct values so that along a subsequence we have

$$\inf_{\boldsymbol{c}^1, \ldots, \boldsymbol{c}^J} \left\| q_k - \sum_{j=1}^J Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j} \right\|_{H^n} \to 0 \quad \text{as } k \to \infty.$$

We must have multiple multisolitons in the conclusion of Theorem 2.1.5, because two multisolitons with a common β parameter necessarily become infinitely separated as $k \to \infty$. On the other hand, we cannot guarantee N single-soliton profiles as originally conjectured in [4], because two distinct values of the minimizing parameters $\boldsymbol{\beta}$ can correspond to a multisoliton that does not resemble well-separated solitons.

Theorem 2.1.5 still does not account for all of the remaining feasible constraints $\mathcal{F}^n$! In general, we can compute the boundary of $\mathcal{M}_N^n$ by finding the extrema of E_{n+1} for $\beta_1, \ldots, \beta_N > 0$ distinct. For $n = 2$, it is not difficult to show that

$$\mathcal{M}_N^2 = \left\{ (e_1, e_2) : e_1 > 0,\ e_2 \in \left[-\tfrac{32}{5} (\tfrac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}},\ -N^{-\frac{2}{3}} \tfrac{32}{5} (\tfrac{3}{8})^{\frac{5}{3}} e_1^{\frac{5}{3}} \right) \right\} \cup \{(0,0)\}$$

for all $N \geq 2$. Indeed, this is closely related to the elementary inequality

$$N^{-\frac{2}{3}} \left(\sum_{m=1}^N \beta_m^3 \right)^{\frac{5}{3}} \leq \sum_{m=1}^N \beta_m^5 \leq \left(\sum_{m=1}^N \beta_m^3 \right)^{\frac{5}{3}}$$

which expresses the equivalence of the ℓ^3- and ℓ^5-norms on $\mathbb{R}^N$. The sets $\mathcal{M}_N^2$ are depicted in the phase diagram of Fig. 2.1, and can also be understood in terms of the single dimensionless variable $e_2 e_1^{-5/3}$. Note that this implies that the set $\mathcal{M}_N^n \setminus \mathcal{M}_{N-1}^n$ in Theorem 2.1.5 is nonempty for all $n \geq 2$ and $N \geq 1$, since increasing N always introduces new values of e_2. Moreover, we see that the set

$$\bigcup_{N\geq 0} \mathcal{M}_N^2 = \big\{(e_1, e_2) : e_1 > 0,\ e_2 \in \big[-\tfrac{32}{5}\big(\tfrac{3}{8}\big)^{\frac{5}{3}} e_1^{\frac{5}{3}}, 0\big)\big\} \cup \{(0,0)\} \tag{2.1.6}$$

misses a large portion of the feasible constraints $\mathcal{F}^2$ in (2.1.5).

We discover that in the remaining case when the constraints $(e_1, \dots, e_n) \in \mathcal{F}^n \setminus \bigcup_{N\geq 0} \mathcal{M}_N^n$ are not attainable by any multisoliton, Albert–Nguyen's conjecture cannot be true and entirely different behavior is exhibited. To illustrate this point, we provide the following characterization of Schwartz minimizing sequences in the case $n = 2$. (Recall from Section 1.4 the definition of the scattering data $a(k; u)$ and $\beta_1, \dots, \beta_N$ for Schwartz potentials $u(x)$.)

Theorem 2.1.6. *Given constraints $(e_1, e_2) \in \mathcal{F}^2$ with $(e_1, e_2) \notin \mathcal{M}_N^2$ for all N, the infimum of $E_3(u)$ over $u \in \mathscr{C}_{\boldsymbol{e}} \cap \mathcal{S}(\mathbb{R})$ is finite but not attained.*

Moreover, if $\{q_j\}_{j\geq 1} \subset \mathscr{C}_{\boldsymbol{e}} \cap \mathcal{S}(\mathbb{R})$ is a minimizing sequence:

$$E_1(q_j) \to e_1, \quad E_2(q_j) \to e_2, \quad E_3(q_j) \to \inf_{u \in \mathscr{C}_{\boldsymbol{e}}} E_3(u) \quad \text{as } j \to \infty,$$

then $\beta_{j,m} \to 0$ as $j \to \infty$ for all m, and $k \mapsto \log|a(k; q_j)|$ converges in the sense of distributions to the even extension of a unique Dirac delta distribution (i.e. $c_0(d\delta_{k_0}(k) + d\delta_{-k_0}(k))$ for unique constants $c_0, k_0 \geq 0$).

What might such a minimizing sequence look like? We can exhibit a family of these sequences via an ansatz inspired by the Wigner–von Neumann example of a Schrödinger potential with a positive eigenvalue. Given parameters $c > 0$ and $k \geq 0$, it is straightforward to check that the sequence

$$q_n(x) = \sqrt{\tfrac{c}{n}} e^{-\frac{x^2}{2n^2}} \cos(2kx)$$

obeys

$$E_1(q_n) \to \tfrac{\sqrt{\pi}}{4}c, \quad E_2(q_n) \to \sqrt{\pi}ck^2, \quad E_3(q_n) \to 4\sqrt{\pi}ck^4$$

as $n \to \infty$. In the proof of Theorem 2.1.6 we will explicitly compute the constrained infimum of E_3, and it is given by $e_2^2 e_1^{-1}$ (cf. Lemma 2.7.2). We see that the limit of $E_3(q_n)$ above is exactly equal to this quantity, and so $\{q_n\}_{n\geq 1}$ is a Schwartz minimizing sequence. By Theorem 2.1.6, we deduce that this sequence has vanishing β parameters and $\log|a(k;q_n)|$ converging to the even extension of a delta distribution.

This chapter is organized as follows. In Section 2.2 we recall some scattering theory and facts about the energy functionals E_n. At the center of this section lies the Zakharov–Faddeev trace formulas (2.2.7) for the polynomial conserved quantities $E_n(u)$, which generalize the formula (2.1.3) for the case where $u = Q_{\boldsymbol{\beta},\boldsymbol{c}}$ is a multisoliton. Explicitly, the first few formulas in this sequence are given by (1.4.3). For multisolitons q we have $|a(k;q)| \equiv 1$ on $\mathbb{R}$, and so the $\log|a(k;q)|$ moments vanish and we recover (2.1.3).

In Section 2.3, we further analyze the functionals $E_1, \dots, E_{n+1}$ on the manifold of multisolitons and use this to describe the set $\mathcal{M}_n^n$ of constraints.

The proof of Theorem 2.1.3 is then presented in Section 2.4. A key step is realizing that in order to minimize E_{n+1}, a minimizer q must satisfy $\log|a(k;q)| \equiv 0$ on $\mathbb{R}$ (cf. (2.2.4) and (2.2.7)). We know that $\log|a(k;q)|$ can be brought all the way down to zero since the constraints can be met solely by the moments of $\boldsymbol{\beta} \in \mathbb{R}^n$; this is why we must assume that $(e_1, \dots, e_n) \in \mathcal{M}_n^n$ in Theorem 2.1.3. Next, we prove that $|a(k;q)| \equiv 1$ on $\mathbb{R}$ if and only if q is a multisoliton. The “if” statement is already known from scattering theory (cf. (2.2.9)). For the reverse implication, we use some classical complex analysis to characterize $k \mapsto a(k;q)$ on $\mathbb{C}^+$ and conclude that q is a multisoliton (cf. Lemma 2.4.6). It then remains to show that on the manifold of multisolitons, there is a unique minimizing set of β parameters. First, we can rule out the case of $N \geq n+1$ parameters by a variational argument (cf. Lemma 2.3.1). For the remaining $N \leq n$ unknown parameters $\beta_1, \dots, \beta_N$, the formulas (2.1.3) for the n constraints provide a nonlinear system of equations, which we show has a unique solution in

Lemma 2.3.2.

In Section 2.5 we apply a concentration compactness principle (Theorem 2.5.2) to minimizing sequences in order to prove Theorem 2.1.4.

In Section 2.6, we prove Theorem 2.1.5 by adapting the methods of Sections 2.3 to 2.5 to allow for repeated values in the β parameters.

Finally, in Section 2.7 we prove Theorem 2.1.6. The proof is again based on the trace formulas (1.4.3). The condition $(e_1, e_2) \notin \mathcal{M}_N^2$ for all N requires that the $\log|a(k;q_j)|$ moments are nonvanishing as $j \to \infty$. Consequently, $\{q_j\}_{j\geq 1}$ is a minimizing sequence for a constrained moment problem for measures, and such minimizers are finite linear combinations of point masses. This particular moment problem for $n = 2$ is easily solved using the Cauchy–Schwarz inequality, but for general n it is a Stieltjes moment problem. (We recommend [130] for an introduction to this classical analysis result.)

2.2 Preliminaries

In this section, we recall some facts about the energy functionals E_n and results from scattering theory for future reference. In particular, this will enable us to formulate the Zakharov–Faddeev trace formulas (2.2.7) that lie at the heart of our analysis.

The functionals (1.1.3) and (1.1.4) are the beginning of an infinite sequence of polynomial conserved quantities (2.1.2). We will only need certain properties of these functionals however, rather than their exact formula.

Proposition 2.2.1 ([115]). *Given an integer $n \geq 0$, there exists a functional of the form*

$$E_{n+1}(u) = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\left(u^{(n)}\right)^2 + P_{n+1}(u)\right] dx \tag{2.2.1}$$

that is conserved for Schwartz solutions $u(t)$ to the KdV equation (1.1.1)*, where P_{n+1} is a polynomial in $u, u', \ldots, u^{(n-1)}$. Each term of P_{n+1} is of the form $c_{\alpha_1\ldots\alpha_d} u^{(\alpha_1)} \cdots u^{(\alpha_d)}$ with*

$d \geq 3$ *and obeys*

$$\sum_{j=1}^{d} \alpha_j = 2n + 4 - 2d \quad \textit{and} \quad 0 \leq \alpha_j \leq n - 1. \tag{2.2.2}$$

Each term of P_{n+1} is of cubic or higher order, and the condition (2.2.2) simply says that they share the same scaling symmetry as the quadratic term $\frac{1}{2}(u^{(n)})^2$. In particular, this requires that the degree of P_{n+1} is at most $n + 2$.

When combined with Sobolev embedding, a classical argument (cf. [108, Th. 3.1] in the periodic case) yields estimates for the functionals (2.2.1):

Lemma 2.2.2. *Given $n \geq 0$, we have*

$$E_{n+1}(u) \lesssim_{\|u\|_{H^n}} 1 \quad \textit{and} \quad \|u\|_{H^n} \lesssim_{E_1(u),\ldots,E_{n+1}(u)} 1 \tag{2.2.3}$$

uniformly for $u \in \mathcal{S}(\mathbb{R})$. Moreover, $E_{n+1} : H^n(\mathbb{R}) \to \mathbb{R}$ is continuous.

Here, we are using the familiar L^2-based Sobolev spaces $H^s(\mathbb{R})$ (and the L^p-based spaces $W^{j,p}(\mathbb{R})$) of real-valued functions on $\mathbb{R}$. In addition to these classes, the scattering theory that we need to state our trace formulas will require that we work in the weighted L^1-spaces

$$L^1_j(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{-\infty}^{\infty} |f(x)|(1 + |x|^j)\, dx < \infty \right\}$$

with $j \geq 1$. When we need a common ground, we will use the class $\mathcal{S}(\mathbb{R})$ of Schwartz functions.

Given a potential $q \in L^1_1(\mathbb{R})$ and $k \in \mathbb{R} \setminus \{0\}$, the Jost functions $f_j(x;k)$ are the unique solutions to the corresponding Schrödinger equation

$$-f_j'' + q f_j = k^2 f_j, \quad j = 1, 2$$

with the asymptotics

$$f_1(x;k) \sim e^{ikx} \quad \text{as } x \to +\infty, \qquad f_2(x;k) \sim e^{-ikx} \quad \text{as } x \to -\infty.$$

The transmission and reflection coefficients $T_j(k)$ and $R_j(k)$ are then uniquely determined by

$$T_1(k)f_2(x;k) = R_1(k)f_1(x;k) + f_1(x;-k),$$
$$T_2(k)f_1(x;k) = R_2(k)f_2(x;k) + f_2(x;-k).$$

Forward scattering theory tells us that the transmission and reflection coefficients satisfy the following properties. Proofs of these facts can be found in many introductory texts on the subject; however, we recommend the paper [47, §2 Th. 1] of Deift and Trubowitz for a complete and self-contained proof.

Proposition 2.2.3 (Forward scattering theory). *If $q \in L^1_2(\mathbb{R})$, then the scattering matrix*

$$S(k) := \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}$$

extends to a continuous function of $k \in \mathbb{R}$ satisfying the following properties:

(i) (Symmetry) For all $k \in \mathbb{R}$,

$$T_1(k) \equiv T_2(k) =: T(k).$$

(ii) (Unitarity) The matrix $S(k)$ is unitary for all $k \in \mathbb{R}$:

$$T(k)\overline{R_2(k)} + R_1(k)\overline{T(k)} \equiv 0, \qquad |T(k)|^2 + |R_j(k)|^2 \equiv 1 \quad \textit{for } j = 1, 2.$$

(iii) (Analyticity) $T(k)$ is meromorphic in the open upper half-plane $\mathbb{C}^+$ and is continuous down to $\mathbb{R}$. Moreover, $T(k)$ has a finite number of poles $i\beta_1, \ldots, i\beta_N$, all of which are simple and on the imaginary axis, and $-\beta_1^2, \ldots, -\beta_N^2$ are the bound states of the operator $-\partial_x^2 + q$.

(iv) (Asymptotics) We have

$$T(k) = 1 + O(\tfrac{1}{k}) \quad \textit{as } |k| \to \infty \textit{ uniformly for } \operatorname{Im} k \geq 0,$$
$$R_j(k) = O(\tfrac{1}{k}) \quad \textit{as } |k| \to \infty,\ k \in \mathbb{R}.$$

(v) (Rate at $k=0$) $T(k)$ is nonvanishing for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, and either

(a) $|T(k)| \geq c > 0$ for all $k \in \overline{\mathbb{C}^+}$, or

(b) $T(k) = T'(0)k + o(k)$ for $k \in \overline{\mathbb{C}^+}$ with $T'(0) \neq 0$ and $R_j(k) = -1 + R_j'(0)k + o(k)$ for $k \in \mathbb{R}$.

(vi) (Reality) For all $k \in \mathbb{R}$,

$$\overline{T(k)} = T(-k), \quad \overline{R_j(k)} = R_j(-k) \quad \textit{for } j = 1, 2.$$

Our trace formulas are most conveniently stated in terms of the reciprocal of the transmission coefficient:

$$a(k;q) := \tfrac{1}{T(k)}.$$

For $q \in L^1_2(\mathbb{R})$, Proposition 2.2.3 tells us that $k \mapsto a(k;q)$ is a holomorphic function on the open upper-half plane $\mathbb{C}^+$ and is continuous down to $\mathbb{R}$. It has finitely many zeros $i\beta_1, \ldots, i\beta_N$ in $\mathbb{C}^+$, all of which are simple and on the imaginary axis. Moreover, we have the boundary conditions

$$|a(k;q)| \geq 1 \quad \text{for all } k \in \mathbb{R}, \tag{2.2.4}$$

$$|a(k;q) - 1| = O(\tfrac{1}{|k|}) \quad \text{as } |k| \to \infty \text{ uniformly for } \operatorname{Im} k \geq 0, \tag{2.2.5}$$

along with the reality condition

$$\overline{a(k;q)} = a(-\overline{k};q) \quad \text{for all } k \in \mathbb{C}^+. \tag{2.2.6}$$

For $u \in \mathcal{S}(\mathbb{R})$, the Zakharov–Faddeev trace formulas [141] provide an alternative representation of the polynomial conserved quantities:

$$E_n(u) = \tfrac{2^{2n}}{\pi} \int_{-\infty}^{\infty} k^{2n} \log|a(k;u)|\, dk + (-1)^{n+1} \tfrac{2^{2n+1}}{2n+1} \sum_{m=1}^{N} \beta_m^{2n+1}. \tag{2.2.7}$$

The measure $\log|a(k;u)|\, dk$ on $\mathbb{R}$ is nonnegative and even, and the first terms on the RHS are its even moments (starting with the second), which are finite for $u \in \mathcal{S}(\mathbb{R})$ [141]. The second terms are the odd moments of the distinct positive numbers $\beta_1, \dots, \beta_N$ (starting with the third) and are alternating in sign.

Later, we will deduce that a constrained minimizer q of E_{n+1} must have certain scattering data due to the trace formulas (2.2.7). Consequently, it will be useful to know when we can reconstruct the potential q from the scattering data [47, §5 Th. 3]:

Proposition 2.2.4 (Inverse scattering theory)**.** *A matrix*

$$S(k) := \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}, \quad k \in \mathbb{R}$$

is the scattering matrix for some $q \in L^1_2(\mathbb{R})$ without bound states if and only if

(i) (Symmetry) For all $k \in \mathbb{R}$,

$$T_1(k) \equiv T_2(k) =: T(k).$$

(ii) (Unitarity) The matrix $S(k)$ is unitary for all $k \in \mathbb{R}$:

$$T(k)\overline{R_2(k)} + R_1(k)\overline{T(k)} \equiv 0, \qquad |T(k)|^2 + |R_j(k)|^2 \equiv 1 \quad \text{for } j = 1, 2.$$

(iii) (Analyticity) $T(k)$ is analytic in the open upper half-plane $\mathbb{C}^+$ and is continuous down to $\mathbb{R}$.

(iv) (Asymptotics) We have

$$T(k) = 1 + O(\tfrac{1}{k}) \quad \text{as } |k| \to \infty \text{ uniformly for } \operatorname{Im} k \geq 0,$$
$$R_j(k) = O(\tfrac{1}{k}) \quad \text{as } |k| \to \infty,\ k \in \mathbb{R}.$$

(v) (Rate at $k = 0$) $T(k)$ is nonvanishing for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, and either

(a) $|T(k)| \geq c > 0$ for all $k \in \overline{\mathbb{C}^+}$, or

(b) $T(k) = T'(0)k + o(k)$ for $k \in \overline{\mathbb{C}^+}$ with $T'(0) \neq 0$ and $R_j(k) = -1 + R_j'(0)k + o(k)$ for $k \in \mathbb{R}$.

(vi) (Reality) For all $k \in \mathbb{R}$,

$$\overline{T(k)} = T(-k), \quad \overline{R_j(k)} = R_j(-k) \quad \text{for } j = 1, 2.$$

(vii) (Fourier decay) The Fourier transforms $F_j := \widehat{R}_j$, $j = 1, 2$ are absolutely continuous and

$$\int_{-\infty}^{a} |F_1'(\kappa)|(1+\kappa^2)\, d\kappa < \infty \quad \text{and} \quad \int_{a}^{\infty} |F_2'(\kappa)|(1+\kappa^2)\, d\kappa < \infty$$

for all $a \in \mathbb{R}$.

The characterization in Proposition 2.2.4 is most easily stated in terms of potentials q without bound states. This does not pose a problem though, because there is an explicit formula for modifying q in order to prescribe any number of bound states [47, §3 Th. 6]. Rather than the explicit formula for q, we will only need to keep track of the changes in the transmission coefficient:

Proposition 2.2.5 (Adding bound states). *Let $q(x) \in L^1_1(\mathbb{R})$ be a potential without bound states and $\beta_1, \ldots, \beta_N > 0$ distinct. Then there exists a potential $q(x; +N) \in L^1_1(\mathbb{R})$ with the N bound states $-\beta_1^2, \ldots, -\beta_N^2$. Moreover, the transmission coefficient is related to that of $q(x)$ via*

$$T(k; +N) = T(k) \prod_{m=1}^{N} \frac{k + i\beta_m}{k - i\beta_m}. \tag{2.2.8}$$

Within this narrative, multisolitons are characterized by having vanishing reflection coefficients. In view of the preceding, this identifies the formula for $a(k; q)$:

Corollary 2.2.6 (Characterization of multisolitons). *Given distinct $\beta_1, \ldots, \beta_N > 0$ and $q \in \mathcal{S}(\mathbb{R})$, we have $q = Q_{\boldsymbol{\beta}, \boldsymbol{c}}$ for some $\boldsymbol{c} \in \mathbb{R}^N$ if and only if*

$$a(k; q) = \prod_{m=1}^{N} \frac{k - i\beta_m}{k + i\beta_m}. \tag{2.2.9}$$

Notice that the formula (2.2.9) is a finite Blaschke product from $\mathbb{C}^+$ to the unit disk, with zeros that are distinct and lie only on the imaginary axis. In particular, we see that multisolitons $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ satisfy $|a(k;Q_{\boldsymbol{\beta},\boldsymbol{c}})| \equiv 1$ on $\mathbb{R}$, and so the $\log|a|$ moments in the trace formulas (2.2.7) vanish. Consequently, the functionals E_n for a multisoliton $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ are independent of $\boldsymbol{c}$.

2.3 The polynomial conserved quantities

The purpose of this section is to analyze the functionals $E_1,\dots,E_{n+1}$ restricted to the manifold of multisolitons, and to then use this to describe the set $\mathcal{M}_n^n$ of constraints.

First, we will prove that as long as we have at least $n+1$ distinct positive β parameters at our disposal, we can decrease E_{n+1} while preserving $E_1,\dots,E_n$. This surprisingly powerful fact will turn out to be an integral part of our analysis.

Lemma 2.3.1. *Fix $n \geq 1$, and suppose $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ is a multisoliton of degree $N \geq n+1$. Then there exist $\widetilde{\beta}_1,\dots,\widetilde{\beta}_N > 0$ distinct so that*

$$E_m(Q_{\widetilde{\boldsymbol{\beta}},\boldsymbol{c}}) = E_m(Q_{\boldsymbol{\beta},\boldsymbol{c}}) \quad \textit{for } m = 1,\dots,n, \quad \textit{but} \quad E_{n+1}(Q_{\widetilde{\boldsymbol{\beta}},\boldsymbol{c}}) < E_{n+1}(Q_{\boldsymbol{\beta},\boldsymbol{c}}).$$

Proof. We will apply the implicit function theorem to the first $n+1$ trace formulas (2.2.7) as functions of $\boldsymbol{\beta}$. Reorder $\boldsymbol{\beta}$ so that $\beta_1 > \dots > \beta_N > 0$. Define the function $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ by

$$f(x_1,\dots,x_{n+1}) = \begin{pmatrix} x_1^3 + x_2^3 + \dots + x_{n+1}^3 \\ x_1^5 + x_2^5 + \dots + x_{n+1}^5 \\ \vdots \\ x_1^{2n+1} + x_2^{2n+1} + \dots + x_{n+1}^{2n+1} \end{pmatrix}. \tag{2.3.1}$$

This function has derivative matrix

$$Df(\beta_1,\dots,\beta_{n+1}) = \left(\begin{array}{ccc|c} 3\beta_1^2 & \dots & 3\beta_n^2 & 3\beta_{n+1}^2 \\ 5\beta_1^4 & \dots & 5\beta_n^4 & 5\beta_{n+1}^4 \\ \vdots & \ddots & \vdots & \vdots \\ (2n+1)\beta_1^{2n} & \dots & (2n+1)\beta_n^{2n} & (2n+1)\beta_{n+1}^{2n} \end{array}\right). \tag{2.3.2}$$

The left $n \times n$ block matrix is a Vandermonde matrix after pulling out common factors from each row and column, and thus has determinant

$$3 \cdot 5 \cdots (2n+1)\, \beta_1^2 \cdots \beta_n^2 \prod_{j<k} (\beta_k^2 - \beta_j^2). \tag{2.3.3}$$

This is nonvanishing as $\beta_1, \dots, \beta_n$ are positive and distinct. The implicit function theorem then implies that there exists $\varepsilon > 0$ and C^1 functions $x_1(x_{n+1}), \dots, x_n(x_{n+1})$ defined on $(\beta_{n+1} - \varepsilon, \beta_{n+1} + \varepsilon)$ so that

$$f(x_1(x_{n+1}), \dots, x_n(x_{n+1}), x_{n+1}) \equiv \begin{pmatrix} \beta_1^3 + \cdots + \beta_{n+1}^3 \\ \beta_1^5 + \cdots + \beta_{n+1}^5 \\ \vdots \\ \beta_1^{2n+1} + \cdots + \beta_{n+1}^{2n+1} \end{pmatrix} \tag{2.3.4}$$

for $x_{n+1} \in (\beta_{n+1} - \varepsilon, \beta_{n+1} + \varepsilon)$.

It remains to show that we can pick x_{n+1} in a way that decreases the next odd moment. To this end, we will compute its derivative at $x_{n+1} = \beta_{n+1}$:

$$\begin{aligned} &\frac{d}{dx_{n+1}}\bigg|_{x_{n+1}=\beta_{n+1}} \left[x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3} \right] \\ &= (2n+3) \left[\begin{pmatrix} \beta_1^{2n+2} & \dots & \beta_n^{2n+2} \end{pmatrix} \begin{pmatrix} x_1'(\beta_{n+1}) \\ \vdots \\ x_n'(\beta_{n+1}) \end{pmatrix} + \beta_{n+1}^{2n+2} \right]. \end{aligned} \tag{2.3.5}$$

The derivative of $x_1(x_{n+1}), \dots, x_n(x_{n+1})$ is determined by differentiating (2.3.4) at $x_{n+1} = \beta_{n+1}$. This yields

$$\begin{pmatrix} x_1'(\beta_{n+1}) \\ \vdots \\ x_n'(\beta_{n+1}) \end{pmatrix} = - \begin{pmatrix} 3\beta_1^2 & \dots & 3\beta_n^2 \\ \vdots & \ddots & \vdots \\ (2n+1)\beta_1^{2n} & \dots & (2n+1)\beta_n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} 3\beta_{n+1}^2 \\ \vdots \\ (2n+1)\beta_{n+1}^{2n} \end{pmatrix}. \tag{2.3.6}$$

Inserting this into (2.3.5), we obtain an expression solely in terms of $\beta_1, \dots, \beta_{n+1}$. In order to compute this, we will leverage that it is a Schur complement for the derivative matrix (2.3.2) but with an appended row. Specifically, if we define the $(n+1)\times(n+1)$ block matrix

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} = \left(\begin{array}{ccc:c} 3\beta_1^2 & \dots & 3\beta_n^2 & 3\beta_{n+1}^2 \\ 5\beta_1^4 & \dots & 5\beta_n^4 & 5\beta_{n+1}^4 \\ \vdots & \ddots & \vdots & \vdots \\ \hdashline (2n+3)\beta_1^{2n+2} & \dots & (2n+3)\beta_n^{2n+2} & (2n+3)\beta_{n+1}^{2n+2} \end{array}\right),$$

then LHS(2.3.5) is now given by

$$d - cA^{-1}b.$$

On the other hand, applying one step of Gaussian elimination to our block matrix yields

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & 0 \\ cA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & b \\ 0 & d - cA^{-1}b \end{pmatrix}.$$

Taking the determinant of both sides, we deduce that LHS(2.3.5) is equal to

$$\det \begin{pmatrix} A & b \\ c & d \end{pmatrix} (\det A)^{-1}.$$

Both terms above can be computed by the Vandermonde determinant formula: the determinant of A is given by (2.3.3) for n and the determinant of the block matrix is given by (2.3.3) for $n+1$.

Altogether, we conclude

$$\begin{aligned} &\frac{d}{dx_{n+1}}\bigg|_{x_{n+1}=\beta_{n+1}} \left[x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3}\right] \\ &= (2n+3)\beta_{n+1}^2 \prod_{j=1}^{n} (\beta_{n+1}^2 - \beta_j^2). \end{aligned}$$

The RHS is nonvanishing since $\beta_1, \ldots, \beta_{n+1}$ are positive and distinct. As $\beta_1, \ldots, \beta_N$ were distinct to begin with, we conclude that there exists $x_{n+1} \in (\beta_{n+1} - \varepsilon, \beta_{n+1} + \varepsilon)$ sufficiently close to β_{n+1} so that the values $x_1(x_{n+1})$, ..., $x_n(x_{n+1})$, x_{n+1}, and β_{n+2} remain distinct, and

$$(-1)^n \left[x_1(x_{n+1})^{2n+3} + \cdots + x_n(x_{n+1})^{2n+3} + x_{n+1}^{2n+3}\right]$$

is strictly less than its value at $x_{n+1} = \beta_{n+1}$. This quantity is the value of E_{n+1} for the multisoliton with β parameters $x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}$. Replacing $\beta_1, \ldots, \beta_{n+1}$ by $x_1(x_{n+1}), \ldots, x_n(x_{n+1}), x_{n+1}$ in $\boldsymbol{\beta}$, we obtain new distinct parameters $\widetilde{\beta}_1 > \cdots > \widetilde{\beta}_N > 0$ (with $\widetilde{\beta}_j = \beta_j$ for $j \geq n+2$) so that the multisoliton $Q_{\widetilde{\boldsymbol{\beta}},\boldsymbol{c}}$ decreases E_{n+1} while preserving $E_1, \ldots, E_n$. □

Our next step is to find the unique set of distinct β parameters with at most n values that attain the constraints. As $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, it only remains to show that there is at most one solution:

Lemma 2.3.2. *Fix $n \geq 1$. Given constraints $e_1, \ldots, e_n$, there is at most one choice of $N \leq n$ and $\beta_1 > \cdots > \beta_N > 0$ so that*

$$E_m(Q_{\boldsymbol{\beta},\boldsymbol{c}}) = e_m \quad \text{for } m = 1, \ldots, n \text{ and any } \boldsymbol{c} \in \mathbb{R}^N.$$

We will follow the clever argument from [131], which we learned about from [49, §3]. In fact, the result in [131] is even more general: it is shown that any n power sums of n distinct positive real numbers has at most one solution (up to permutation), in addition to some generalizations. However, we will provide a complete and self-contained proof here for future reference (in Corollary 2.3.3).

Proof. Suppose towards a contradiction that there exist $\beta_1 > \cdots > \beta_N > 0$ and $\widetilde{\beta}_1 > \cdots > \widetilde{\beta}_{\widetilde{N}} > 0$ with $\widetilde{N} \leq N \leq n$ such that $E_k(Q_{\boldsymbol{\beta},\boldsymbol{c}}) = E_k(Q_{\widetilde{\boldsymbol{\beta}},\boldsymbol{c}})$ for $k = 1, \ldots, n$. By the trace formulas (2.2.1), this requires that

$$\sum_{m=1}^{N} \beta_m^{2k+1} = \sum_{m=1}^{\widetilde{N}} \widetilde{\beta}_m^{2k+1} \quad \text{for } k = 1, \ldots, n. \tag{2.3.7}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}^n$ given by $f(x) = (x^3, x^5, \ldots, x^{2n+1})$. After canceling common terms and moving everything to the LHS, we obtain

$$\sum_{j=1}^{M} \varepsilon_j f(\beta_j) = 0$$

for some $\beta_1 > \cdots > \beta_M > 0$, $M \leq 2N$, and signs $\varepsilon_j \in \{\pm 1\}$.

Next, we append $2N - M$ copies of $\beta_j := 0$ and $\varepsilon_j := -1$ for $j = M+1, \dots, 2N$ so that $\sum_{j=1}^{2N} \varepsilon_j = 0$. Using summation by parts, this allows us to write

$$0 = \sum_{j=1}^{M} \varepsilon_j f(\beta_j) + \sum_{j=M+1}^{2N} (-1) f(0) = \sum_{j=1}^{2N} \varepsilon_j f(\beta_j) = \sum_{j=1}^{2N-1} \alpha_j [f(\beta_j) - f(\beta_{j+1})],$$

where $\alpha_j = \sum_{k=1}^{j} \varepsilon_k$. By the fundamental theorem of calculus, we obtain

$$0 = \int_0^{\beta_1} \phi(s) f'(s) \, ds \tag{2.3.8}$$

for the step function ϕ which takes value α_j on the interval (β_{j+1}, β_j). Let $I_1, \dots, I_m$ denote the disjoint intervals in $[0, \beta_1]$ (in consecutive order) on which ϕ is nonvanishing and has constant sign. Note that we can have at most n such intervals; indeed, j must increment by two in order for α_j to change sign, which together with the first and last indices j account for all of the $2N \leq 2n$ parameters. The equality (2.3.8) then tells us that the rows of the $m \times n$ matrix A with entries

$$a_{jk} = \int_{I_j} |\phi(s)| (f'(s))_k \, ds, \quad j \in \{1, \dots, m\}, \; k \in \{1, \dots, n\}$$

are linearly dependent.

We claim that A is a strictly totally positive matrix—i.e. all of the minors of A are strictly positive—which will contradict the linear dependence among the rows. Given two subsets of indices $J \subset \{1, \dots, m\}$ and $K \subset \{1, \dots, n\}$, we can write the corresponding minor of A as

$$\begin{aligned} &\operatorname{minor}_{J,K} \begin{pmatrix} \int_{I_1} |\phi(s)| (f'(s))_1 \, ds & \dots & \int_{I_1} |\phi(s)| (f'(s))_n \, ds \\ \vdots & \ddots & \vdots \\ \int_{I_m} |\phi(s)| (f'(s))_1 \, ds & \dots & \int_{I_m} |\phi(s)| (f'(s))_n \, ds \end{pmatrix} \\ &= \int_{I_{j_1}} \cdots \int_{I_{j_{|J|}}} \operatorname{minor}_{J,K} \begin{pmatrix} |\phi(s_1)| (f'(s_1))_1 & \dots & |\phi(s_1)| (f'(s_1))_n \\ \vdots & \ddots & \vdots \\ |\phi(s_m)| (f'(s_m))_1 & \dots & |\phi(s_m)| (f'(s_m))_n \end{pmatrix} ds_{j_{|J|}} \dots ds_{j_1}. \end{aligned}$$

(Indeed, expanding the determinant on the LHS into a sum over permutations of the matrix entries, each term is a product of $|J|$ integrals which we combine into one $|J|$-fold integral.) The integrand on the RHS above can be factored as

$$\operatorname{minor}_{J,K}\left\{\begin{pmatrix}|\phi(s_1)| & & \\ & \ddots & \\ & & |\phi(s_m)|\end{pmatrix}\begin{pmatrix}(f'(s_1))_1 & \dots & (f'(s_1))_n \\ \vdots & \ddots & \vdots \\ (f'(s_m))_1 & \dots & (f'(s_m))_n\end{pmatrix}\right\}.$$

The first matrix has positive determinant because ϕ is nonvanishing on each I_j by construction. Therefore, it suffices to show that $f'(s)$ is a strictly totally positive kernel, i.e. for all $0 < s_1 < \cdots < s_\ell$ and $k_1 < \cdots < k_\ell$ the matrix

$$\big((f'(s_i))_{k_j}\big)_{1\le i\le \ell,\, 1\le j\le \ell} = \begin{pmatrix}(2k_1+1)s_1^{2k_1} & \dots & (2k_\ell+1)s_1^{2k_\ell} \\ \vdots & \ddots & \vdots \\ (2k_1+1)s_\ell^{2k_1} & \dots & (2k_\ell+1)s_\ell^{2k_\ell}\end{pmatrix} \tag{2.3.9}$$

has positive determinant. Note that when $k_1, \dots, k_\ell$ is an arithmetic progression, this matrix is essentially a Vandermonde matrix with rows and columns multiplied by constants. We will follow the classical argument for Vandermonde matrices.

First, we claim that the determinant is nonzero. Suppose towards a contradiction that the determinant vanishes. Then the columns would be linearly dependent, and so there would exist $\lambda_1, \dots, \lambda_\ell \in \mathbb{R}$ so that

$$\sum_{j=1}^{\ell} \lambda_j s_i^{2k_j} = 0 \quad \text{for } i = 1, \dots, \ell.$$

(We absorbed the coefficients $2k_j + 1$ into λ_j.) This means that the polynomial

$$P(x) := \sum_{j=1}^{\ell} \lambda_j x^{2k_j} \tag{2.3.10}$$

has ℓ positive roots $s_1, \dots, s_\ell$. To obtain a contradiction, we will prove that nontrivial polynomials of the form (2.3.10) can have at most $\ell - 1$ positive roots by induction on ℓ. The base case $\ell = 1$ is immediate. For the inductive step, note that if $P(x)$ has ℓ positive roots, then $x^{-2k_1}P(x)$ is a polynomial with the same ℓ positive roots. By Rolle's theorem, the polynomial $(x^{-2k_1}P(x))'$ therefore has $\ell-1$ positive roots. On the other hand, $(x^{-2k_1}P(x))'$ is a polynomial of the form (2.3.10) for $\ell-1$, and so this contradicts the inductive hypothesis.

Lastly, we show that the determinant is positive. Now that we know the determinant is nonzero, its sign must be is independent of the choice of $0 < s_1 < \cdots < s_\ell$ and $k_1 < \cdots < k_\ell$ (but may depend on ℓ). Therefore, we may pick $k_j = j$ for $j = 1, \dots, \ell$ so that we essentially have a Vandermonde matrix with determinant

$$\det \begin{pmatrix} 3s_1^2 & 5s_1^4 & \dots & (2\ell+1)s_1^{2\ell} \\ 3s_2^2 & 5s_2^4 & \dots & (2\ell+1)s_2^{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 3s_\ell^2 & 5s_\ell^4 & \dots & (2\ell+1)s_\ell^{2\ell} \end{pmatrix} = 3 \cdot 5 \cdots (2\ell+1)\, s_1^2 \cdots s_\ell^2 \prod_{j<k}(s_k^2 - s_j^2).$$

This is positive for any ℓ since $0 < s_1 < \cdots < s_\ell$. □

For future reference (cf. Lemma 2.5.7), we note that the proof of Lemma 2.3.2 can allow for repeated β parameters, as long as the total number of values is still at most n.

Corollary 2.3.3. *Fix $n \geq 1$. Given constraints $e_1, \dots, e_n$, there is at most one choice of $N \geq 1$ and $\beta_1 \geq \cdots \geq \beta_N > 0$ attaining at most n distinct values that satisfies*

$$(-1)^{m+1}\tfrac{2^{2m+1}}{2m+1} \sum_{j=1}^{N} \beta_j^{2m+1} = e_m \quad \textit{for } m = 1, \dots, n.$$

Proof. We repeat the proof of Lemma 2.3.2. Construct the step function ϕ so that (2.3.8) holds. It only remains to show that there are still at most n intervals I_j on which ϕ is nonvanishing and has constant sign. If $\beta_m = \widetilde{\beta}_{\widetilde{m}}$ for some m and $\widetilde{m}$, then these terms can be canceled from (2.3.7) while retaining equality. Consequently, the only new possibility for ϕ is that there may be a run of β_m parameters with the same value and the same sign ε_j. This increases the size of the jumps of ϕ but does not affect the number of sign changes, and so the claim follows. □

By Lemma 2.3.2, the map $(\beta_1, \ldots, \beta_n) \mapsto (E_1(Q_{\boldsymbol{\beta},\boldsymbol{c}}), \ldots, E_n(Q_{\boldsymbol{\beta},\boldsymbol{c}}))$ from the half-open simplex

$$\{(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \exists N \text{ with } \beta_1 > \cdots > \beta_N > 0,\ \beta_{N+1} = \cdots = \beta_n = 0\} \tag{2.3.11}$$

into $\mathcal{M}_n^n$ has a well-defined inverse. In fact, the inverse is also continuous:

Lemma 2.3.4. *The function $(\beta_1, \ldots, \beta_n) \mapsto (E_1(Q_{\boldsymbol{\beta},\boldsymbol{c}}), \ldots, E_n(Q_{\boldsymbol{\beta},\boldsymbol{c}}))$ is a homeomorphism from the simplex* (2.3.11) *onto $\mathcal{M}_n^n$.*

Proof. Let

$$\Phi(\beta_1, \ldots, \beta_n) = \left(\tfrac{8}{3} \sum_{m=1}^{n} \beta_m^3,\ -\tfrac{32}{5} \sum_{m=1}^{n} \beta_m^5,\ \ldots,\ (-1)^{n-1} \tfrac{2^{2n+1}}{2n+1} \sum_{m=1}^{n} \beta_m^{2n+1} \right)$$

denote this function, which maps into $\mathcal{M}_n^n$ by definition of $\mathcal{M}_n^n$. Each component of Φ is a polynomial, and so Φ is smooth. By Lemma 2.3.2, we also know that Φ is a bijection from the simplex (2.3.11) onto the set of constraints $\mathcal{M}_n^n$.

It remains to show that Φ^{-1} is continuous. Fix an open subset $V \subset \mathcal{M}_n^n$ and let $\bigcup_{m=1}^{\infty} K_m$ be a compact exhaustion of $\mathcal{M}_n^n$. Recall the elementary topology fact that if $f : X \to Y$ is a continuous bijection between topological spaces with X compact and Y Hausdorff, then f is a homeomorphism. As the map Φ is also proper, then Φ is a homeomorphism from $\Phi^{-1}(K_m)$ to K_m for all m. Therefore $\Phi^{-1}(V \cap K_m)$ is relatively open in $\Phi^{-1}(K_m)$ for all m, and hence $\Phi^{-1}(V)$ is open. □

Given constraints $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, let $\beta_1 > \cdots > \beta_N > 0$ be the unique set of parameters with $N \leq n$ and $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ satisfying the constraints. We define

$$C(e_1, \ldots, e_n) := E_{n+1}(Q_{\boldsymbol{\beta},\boldsymbol{c}}) \tag{2.3.12}$$

to be the value of the next functional for these parameters. In proving Theorem 2.1.3, we will ultimately show that $C(e_1, \ldots, e_n)$ is the minimum of E_{n+1} subject to the constraints $e_1, \ldots, e_n$.

In order to do this, we will first need some properties of C:

Lemma 2.3.5. *The function* $C : \mathcal{M}_n^n \to \mathbb{R}$ *defined in* (2.3.12) *is continuous and is decreasing in each variable. Moreover, on the interior of* $\mathcal{M}_n^n$, $C(e_1, \dots, e_n)$ *is continuously differentiable and satisfies* $\frac{\partial C}{\partial e_j} < 0$ *for* $j = 1, \dots, n$.

Proof. We write

$$C(e_1, \dots, e_n) = (-1)^n \tfrac{2^{2n+3}}{2n+3} \big(\beta_1^{2n+3} + \cdots + \beta_n^{2n+3}\big), \tag{2.3.13}$$

where $(\beta_1, \dots, \beta_n)$ is the unique solution to

$$e_1 = \tfrac{8}{3} \sum_{m=1}^{n} \beta_m^3, \quad e_2 = -\tfrac{32}{5} \sum_{m=1}^{n} \beta_m^5, \quad \dots, \quad e_n = (-1)^{n-1} \tfrac{2^{2n+1}}{2n+1} \sum_{m=1}^{n} \beta_m^{2n+1} \tag{2.3.14}$$

in the simplex (2.3.11), guaranteed by Lemma 2.3.2. Note that C is continuous as the composition of the inverse of the homeomorphism in Lemma 2.3.4 with a polynomial.

Consequently, it suffices to show that $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \dots, n$ on the interior of $\mathcal{M}_n^n$. By Lemma 2.3.4, the interior of $\mathcal{M}_n^n$ corresponds to the set of n positive parameters $\beta_1 > \cdots > \beta_n > 0$. In other words, the interior of $\mathcal{M}_n^n$ is the set of constraints $(e_1, \dots, e_n)$ which correspond to n-solitons.

We will now compute $\frac{\partial C}{\partial e_j}$ assuming $\beta_1 > \cdots > \beta_n > 0$. Differentiating the constraints (2.3.14) with respect to e_j, we see that

$$\begin{pmatrix} 8\beta_1^2 & 8\beta_2^2 & \dots & 8\beta_n^2 \\ -32\beta_1^4 & -32\beta_2^4 & \dots & -32\beta_n^4 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-1}2^{2n+1}\beta_1^{2n} & (-1)^{n-1}2^{2n+1}\beta_2^{2n} & \dots & (-1)^{n-1}2^{2n+1}\beta_n^{2n} \end{pmatrix} \begin{pmatrix} \frac{\partial \beta_1}{\partial e_j} \\ \frac{\partial \beta_2}{\partial e_j} \\ \vdots \\ \frac{\partial \beta_n}{\partial e_j} \end{pmatrix} \tag{2.3.15}$$

is equal to the jth coordinate vector $(0, \dots, 0, 1, 0, \dots, 0)$. This is a Vandermonde matrix after pulling out common factors from each row and column, and thus it has determinant

$$8 \cdot (-32) \cdots (-1)^{n-1} 2^{2n+1}\, \beta_1^2 \cdots \beta_n^2 \prod_{j<k} (\beta_k^2 - \beta_j^2).$$

This expression is nonvanishing since $\beta_1 > \cdots > \beta_n > 0$, and so we conclude that the partial derivatives $\frac{\partial \beta_k}{\partial e_j}$ exist and are uniquely determined by the matrix product (2.3.15). On the other hand, differentiating (2.3.13) with respect to e_j yields

$$\frac{\partial C}{\partial e_j} = (-1)^n 2^{2n+3} \begin{pmatrix} \beta_1^{2n+2} & \dots & \beta_n^{2n+2} \end{pmatrix} \begin{pmatrix} \frac{\partial \beta_1}{\partial e_j} \\ \vdots \\ \frac{\partial \beta_n}{\partial e_j} \end{pmatrix}.$$

We can determine the column vector on the RHS via (2.3.15). Collecting these equations for $j = 1, \dots, n$, we obtain the matrix equation

$$\begin{aligned} &\begin{pmatrix} \frac{\partial C}{\partial e_1} & \dots & \frac{\partial C}{\partial e_n} \end{pmatrix} \begin{pmatrix} 8\beta_1^2 & \dots & 8\beta_n^2 \\ \vdots & \ddots & \vdots \\ (-1)^{n-1}2^{2n+1}\beta_1^{2n} & \dots & (-1)^{n-1}2^{2n+1}\beta_n^{2n} \end{pmatrix} \\ &= (-1)^n 2^{2n+3} \begin{pmatrix} \beta_1^{2n+2} & \dots & \beta_n^{2n+2} \end{pmatrix}, \end{aligned} \tag{2.3.16}$$

where we moved the matrix to the LHS to avoid inverting it. We have already seen that this matrix is invertible, and so we conclude that the partial derivatives $\frac{\partial C}{\partial e_j}$ exist and are uniquely determined by the above equality.

In order to compute the derivatives $\frac{\partial C}{\partial e_j}$, we will harness the classical role of Vandermonde matrices in polynomial interpolation. Specifically, reading off the n components of the equality (2.3.16), we see that the derivatives $\frac{\partial C}{\partial e_j}$ are the coefficients C_j of the polynomial

$$P(x) := 8C_1 x - 32C_2 x^2 + \cdots + (-1)^{n-1}2^{2n+1}C_n x^n - (-1)^n 2^{2n+3} x^{n+1}$$

which satisfies

$$P(\beta_m^2) = 0 \quad \text{for } m = 1, \dots, n.$$

As $\beta_1 > \cdots > \beta_n > 0$, there is only one such polynomial, namely,

$$P(x) = (-1)^{n+1} 2^{2n+3} x \prod_{m=1}^{n} (x - \beta_m^2).$$

Therefore, the coefficients C_j are given by Vieta's formulas:

$$\begin{aligned} 8C_1 &= (-1)^{n+1}2^{2n+3}(-1)^n \prod_{j=1}^{n} \beta_j^2, \\ -32C_2 &= (-1)^{n+1}2^{2n+3}(-1)^{n-1} \sum_{k=1}^{n} \prod_{j\neq k} \beta_j^2, \\ &\vdots \\ (-1)^{n-1}2^{2n+1}C_n &= (-1)^{n+1}2^{2n+3}(-1) \sum_{j=1}^{n} \beta_j^2, \end{aligned} \tag{2.3.17}$$

where the RHS for C_j involves the $(n-j+1)$st elementary symmetric polynomial in $\beta_1^2, \dots, \beta_n^2$. In particular, we see that each $\frac{\partial C}{\partial e_j} = C_j$ is given by $(-1)^{2n+1} = -1$ times a strictly positive quantity, and hence is strictly negative as desired. □

In order to employ that C is decreasing, we will also need to know that the set $\mathcal{M}_n^n$ is downward closed within $\bigcup_{N\geq 0} \mathcal{M}_N^n$ in the following sense:

Lemma 2.3.6. *If the constrains $\widetilde{e}_1, \dots, \widetilde{e}_n$ are in $\mathcal{M}_N^n$ for some N and*

$$\widetilde{e}_1 \leq e_1, \quad \dots, \quad \widetilde{e}_n \leq e_n$$

for some $(e_1, \dots, e_n) \in \mathcal{M}_n^n$, then $(\widetilde{e}_1, \dots, \widetilde{e}_n) \in \mathcal{M}_n^n$.

Proof. Let $\widetilde{\beta}_1, \dots, \widetilde{\beta}_N > 0$ be the β parameters of the multisoliton which witnesses the constraints $\widetilde{e}_1, \dots, \widetilde{e}_n$. The case $N \leq n$ is immediate, so assume that $N \geq n+1$. Let

$$\alpha_j = \sum_{m=1}^{N} \widetilde{\beta}_m^j, \quad j = 3, 5, \dots, 2n+1$$

denote the odd moments of $\widetilde{\beta}_1, \dots, \widetilde{\beta}_N$. Define

$$\Gamma := \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : x_1, \dots, x_N \geq 0, \ \sum_{m=1}^{N} x_m^j = \alpha_j \text{ for } j = 3, 5, \dots, 2n+1 \right\}$$

to be the set of parameters in $\mathbb{R}^N$ satisfying the constraints, which is nonempty because it contains $(\widetilde{\beta}_1, \dots, \widetilde{\beta}_N)$.

It suffices to show that the intersection of Γ with the n-dimensional boundary face $\{(x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0\}$ is nonempty, since a point $(x_1, \ldots, x_n, 0, \ldots, 0)$ provides the n-soliton parameters that we seek. The case $n = 1$ is immediate as Γ is just an ℓ^3-sphere, and so we may assume that Γ is the intersection of $n \geq 2$ constraints.

To accomplish this, consider the set "between" the constraints

$$\Omega := \Big\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1, \ldots, x_N \geq 0, \\ (-1)^k \sum_{m=1}^{N} x_m^{2k+1} \leq (-1)^k \alpha_{2k+1} \text{ for } k = 1, \ldots, n\Big\}.$$

Unlike Γ, we already know that the intersection of Ω with the boundary face $\{(x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0\}$ is nonempty by premise. Indeed, as $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, then there exist $x_1, \ldots, x_n$ so that

$$(-1)^{k+1} \tfrac{2^{2k+1}}{2k+1} \sum_{m=1}^{n} \beta_m^{2k+1} = e_k \geq \widetilde{e}_k = (-1)^{k+1} \tfrac{2^{2k+1}}{2k+1} \alpha_{2k+1} \quad \text{for } k = 1, \ldots, n.$$

(This premise is in fact necessary, as the sets Ω and Γ may not intersect the boundary face $\{(x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0\}$ in general; cf. Lemma 2.6.1.) Note that Ω is also bounded, because each coordinate x_j is bounded by $\alpha_5^{1/5}$ since $n \geq 2$. As $\Omega \cap \{(x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0\}$ is nonempty, closed, and bounded, there exists some point $(\beta_1, \ldots, \beta_n, 0, \ldots, 0)$ in this intersection that minimizes the $(n + 1)$st odd moment $(-1)^n \sum x_m^{2n+3}$. This point must actually lie in Γ, because Lemma 2.3.5 tells us that the value $(-1)^n \sum x_m^{2n+3}$ is an individually decreasing function of the moments on $\Omega \cap \{(x_1, \ldots, x_N) : x_{n+1}, \ldots, x_N = 0\}$. $\square$

2.4 Global minimizers

We will prove Theorem 2.1.3 over the course of this section by induction on n. We begin with the base case $n = 0$. The conclusion is immediate since

$$E_1(u) = \int_{-\infty}^{\infty} \tfrac{1}{2} u^2 \, dx \geq 0 \quad \text{for all } u \in L^2(\mathbb{R}),$$

with equality if and only if u is equal to the zero-soliton $q(x) \equiv 0$.

Next, we turn to the inductive step. Suppose that $n \geq 1$ and that Theorem 2.1.3 holds for $1, 2, \ldots, n-1$. This inductive hypothesis yields the following fact for $\mathcal{M}_n^n$:

Lemma 2.4.1. *The set $\mathcal{M}_n^n$ is a relatively open subset of the feasible constraints $\mathcal{F}^n$ (with respect to the topology on $\mathbb{R}^n$).*

Proof. Given $(e_1, \ldots, e_n)$ in the interior of $\mathcal{M}_n^n$, we know that the numer of β parameters is n by Lemma 2.3.4. Therefore, for each $k \leq n-1$ we can increase and decrease E_{k+1} while preserving $E_1, \ldots, E_k$ by Lemma 2.3.1 since n is strictly larger than k. Moreover, Lemma 2.3.4 implies that the set $\mathcal{M}_n^n \smallsetminus (\operatorname{int} \mathcal{M}_n^n)$ corresponds to multisolitons of degree at most $n-1$, and thus lie on the boundary of $\mathcal{F}^n$ by the inductive hypothesis that Theorem 2.1.3 holds for each $k = 1, \ldots, n-1$. □

Next, we will prove the first half of the inductive step: that multisolitons of degree at most n are global constrained minimizers.

Theorem 2.4.2. *Given constraints $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, there exists a unique integer $N \leq n$ and parameters $\beta_1 > \cdots > \beta_N > 0$ so that the multisoliton $Q_{\boldsymbol{\beta},\boldsymbol{c}}$ lies in $\mathscr{C}_{\boldsymbol{e}}$ for some (and hence all) $\boldsymbol{c} \in \mathbb{R}^N$. Moreover, we have*

$$E_{n+1}(u) \geq E_{n+1}(Q_{\boldsymbol{\beta},\boldsymbol{c}}) \quad \text{for all } u \in \mathscr{C}_{\boldsymbol{e}}.$$

Proof. We first prove the inequality for u Schwartz. In this case, the trace formula (2.2.7) allows us to write

$$E_{n+1}(u) = \tfrac{2^{2n+2}}{\pi} \int_{-\infty}^{\infty} k^{2n+2} \log |a(k;u)| \, dk + (-1)^n \tfrac{2^{2n+3}}{2n+3} \sum_{m=1}^{N} \beta_m^{2n+3}.$$

The integrand is nonnegative by (2.2.4), so we can omit the integral to obtain the inequality

$$\geq (-1)^n \tfrac{2^{2n+3}}{2n+3} \sum_{m=1}^{N} \beta_m^{2n+3}.$$

Note that the β parameters do not satisfy the constraints e_j, but rather the smaller constraints $e_j - \frac{2^{2j}}{\pi}\int k^{2j}\log|a|\,dk$ because the moments of $\log|a(k;u)|$ may not vanish. Nevertheless, as $\mathcal{M}_n^n$ is downward closed by Lemma 2.3.6, we know that these constraints are still attainable by a multisoliton of degree at most n and so Lemma 2.3.1 implies that

$$\geq C\big(e_1 - \tfrac{4}{\pi}\textstyle\int k^2\log|a|\,dk,\ \ldots,\ e_n - \tfrac{2^{2n}}{\pi}\int k^{2n}\log|a|\,dk\big).$$

Finally, C is individually decreasing in each variable by Lemma 2.3.5, and so we conclude

$$\geq C(e_1,\ldots,e_n)$$

as desired.

For general $u \in H^n(\mathbb{R})$, we approximate by a sequence of Schwartz functions. The constraints $e_1,\ldots,e_n$ and minimum value C for the approximate functions will converge by the continuity of $E_1,\ldots,E_n : H^n(\mathbb{R}) \to \mathbb{R}$ and $C : \mathcal{M}_n^n \to \mathbb{R}$, the latter of which we proved in Lemma 2.3.5. Moreover, the constraints $e_1,\ldots,e_n$ for the approximate functions eventually lie in $\mathcal{M}_n^n$ by Lemma 2.4.1. $\square$

To conclude the inductive step of Theorem 2.1.3, it remains to show that any other constrained minimizer must also be a multisoliton with the right β parameters:

Theorem 2.4.3. *Suppose we have constraints $(e_1,\ldots,e_n) \in \mathcal{M}_n^n$ and that $q \in \mathscr{C}_{\boldsymbol{e}}$ minimizes $E_{n+1}(u)$ over $\mathscr{C}_{\boldsymbol{e}}$. Then $q = Q_{\boldsymbol{\beta},\boldsymbol{c}}$ for some $\boldsymbol{c} \in \mathbb{R}^N$, where $\boldsymbol{\beta} \in \mathbb{R}^N$ are the unique parameters satisfying the constraints guaranteed by Theorem 2.4.2.*

We break the proof of Theorem 2.4.3 into steps, with the overarching assumption that $q \in H^n(\mathbb{R})$ is a constrained minimizer of E_{n+1}.

In order to analyze q using the trace formulas, we first need to know that q is sufficiently regular so that we may construct $a(k;q)$:

Lemma 2.4.4. *If q is a constrained minimizer in the sense of Theorem 2.4.3, then q is Schwartz.*

We will use that q solves the Euler–Lagrange equation (cf. (2.4.1)) to show that q is both infinitely smooth and exponentially decaying. As we will see shortly, smoothness follows from classical ODE theory because $q \in H^n(\mathbb{R})$ *a priori*. On the other hand, exponential decay is more delicate: even though we know $q(x) \to 0$ as $x \to \pm\infty$ (since $q \in H^1(\mathbb{R})$), there *do* exist multipliers $\lambda_1, \dots, \lambda_n$ so that (2.4.1) admits algebraically decaying solutions as $x \to \pm\infty$. For example, $u(x) = 2x^{-2}$ is a (meromorphic) solution to (2.4.1) for all $n \geq 1$ with multipliers $\lambda_1 = \cdots = \lambda_n = 0$. This is the beginning of an infinite family of solutions called the *algebro-geometric solutions* to the stationary KdV hierarchy (see [66, §1.3] for details). This does not pose an obstruction here because the multipliers must be negative for a minimizer (cf. (2.4.3)), as is the case for any multisoliton.

Proof. As a critical point of E_{n+1}, q satisfies the Euler–Lagrange equation

$$\nabla E_{n+1}(q) = \lambda_1 \nabla E_1(q) + \lambda_2 \nabla E_2(q) + \cdots + \lambda_n \nabla E_n(q) \tag{2.4.1}$$

for some Lagrange multipliers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. This assumes that the gradients $\nabla E_1(q)$, $\dots$, $\nabla E_n(q)$ are linearly independent; however, the other case is analogous, since a linear dependence can be written as an equation of the form (2.4.1) for some smaller n.

First we show that q is infinitely smooth. As $q \in H^n(\mathbb{R})$, q only solves (2.4.1) in the sense of distributions *a priori*. The highest order term in (2.4.1) is $q^{(2n)}$, and it only appears in $\nabla E_{n+1}(q)$. Isolating this term, we obtain

$$q^{(2n)} = P\big(q, q', \dots, q^{(2n-2)}\big) \tag{2.4.2}$$

for a polynomial P. Note that product terms $q^{(\gamma_1)} \cdots q^{(\gamma_d)}$ satisfy $\sum \gamma_j \leq 2n-2$ by the scaling requirement (2.2.2). In particular, if $q \in H^s$ with $s \geq n$, then RHS(2.4.2) is in H^{-s+2}. (For example, $qq^{(2n-2)} \in H^{-s+2}$ because $q^{(2n-2)} \in H^{s-(2n-2)} \subset H^{-s+2}$ and $q \in H^s \subset H^{s-2}$.) Beginning with $q \in H^n$, the equation (2.4.2) tells us that $q^{(2n)}$ is in H^{-n+2}, and so we conclude that $q \in H^{n+2}$. Now taking $q \in H^{n+2}$ as input, the equation (2.4.2) then tells us that $q^{(2n)}$ is in H^{-n+4}, and so we conclude that $q \in H^{n+4}$. Iterating, we conclude that q is in H^s for all $s > 0$ and hence is smooth.

Next, we claim that q decays exponentially as $x \to \pm\infty$. Our salvation here is that because q is a minimizer (and not merely a critical point), we have restrictions on the Lagrange multipliers. If the constraints are in $\mathcal{M}_n^n \smallsetminus (\operatorname{int} \mathcal{M}_n^n)$, then q is a minimizer of E_m for some $m \leq n$ and thus q is a multisoliton of degree at most m by the inductive hypothesis that Theorem 2.1.3 holds for $m-1$. So assume that the constraints $(e_1, \ldots, e_n)$ are in the interior of $\mathcal{M}_n^n$. Consequently, we know from Lemma 2.3.5 that the minimum value $C(e_1, \ldots, e_n)$ is a C^1 function in a neighborhood of $(e_1, \ldots, e_n)$ and

$$\lambda_j = \frac{\partial C}{\partial e_j}(e_1, \ldots, e_n) < 0 \quad \text{for } j = 1, \ldots, n. \tag{2.4.3}$$

The equality above is a general fact about Lagrange multipliers called the *envelope theorem*, and has applications to economics. (Cf. [132, Th. 1.F.4] and the corollary in Ex. 2 for a proof. As we know that all of the derivatives exist, this purely algebraic proof for the finite dimensional case still applies.)

From the quadratic terms of the energies (2.2.1), we see that the linear part of the Euler–Lagrange equation (2.4.1) is

$$Lu := (-1)^n u^{(2n)} + (-1)^n \lambda_n u^{(2n-2)} + \cdots + \lambda_2 u'' - \lambda_1 u.$$

The constant coefficients of this operator are alternating in sign by (2.4.3), and consequently it has no purely imaginary eigenvalues. Indeed, if ξ is purely imaginary then all the terms in the polynomial

$$(-1)^n \xi^{2n} + (-1)^n \lambda_n \xi^{2n-2} + \cdots + \lambda_2 \xi^2 - \lambda_1$$

are nonnegative, and so the polynomial is bounded below by $-\lambda_1 > 0$. As the Euler–Lagrange equation (2.4.1) is an ODE of order $2n$, we may view it as a first-order system of ODEs in the variables $(q, q', \ldots, q^{(2n-1)}) \in \mathbb{R}^{2n}$. We just showed that the origin in $\mathbb{R}^{2n}$ is a hyperbolic fixed point for this system, and so the stable manifold theorem [39, Ch. 13 Th. 4.1] tells us that there exists a corresponding stable manifold in a neighborhood of the origin. We already know that $q^{(j)}(x) \to 0$ as $x \to \pm\infty$ for all $j \geq 0$ (since $q \in H^{j+1}$), and so eventually $(q, q', \ldots, q^{(2n-1)})$ remains in a small neighborhood of the origin in $\mathbb{R}^{2n}$ for all x sufficiently large. By [39, Ch. 13 Th. 4.1], this can only happen if $q(x)$ is on the stable manifold and hence decays exponentially as $x \to \pm\infty$. $\square$

Now that we know $q \in \mathcal{S}(\mathbb{R})$, we have the trace formulas (2.2.7) at our disposal. Next, we show that the constrained minimizer q must satisfy $|a(k; q)| \equiv 1$ on $\mathbb{R}$:

Lemma 2.4.5. *Let $q \in \mathcal{S}(\mathbb{R})$ such that $|a(k; q)| \neq 1$ for some $k \in \mathbb{R}$. Then there exists some $\widetilde{q} \in \mathcal{S}(\mathbb{R})$ with*

$$E_m(\widetilde{q}) = E_m(q) \quad \text{for } m = 1, \ldots, n, \quad \text{but} \quad E_{n+1}(\widetilde{q}) < E_{n+1}(q).$$

In particular, a constrained minimizer q in the sense of Theorem 2.4.3 must satisfy $|a(k; q)| = 1$ for all $k \in \mathbb{R}$.

Proof. We will only modify the transmission coefficient of q and leave the bound states $-\beta_1^2, \ldots, \beta_N^2$ unchanged. Specifically, we will wiggle $\log |a(k; q)|$ via the implicit function theorem in a way that decreases its $(n + 1)$st moment in the trace formulas (2.2.7) while keeping the first n moments constant. Then we will reconstruct the new potential $\widetilde{q}$ using inverse scattering theory.

Let $\psi_1, \ldots, \psi_{n+1} \in C_c^\infty(\mathbb{R})$ be even functions to be chosen later. Define the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by

$$f(x_1, \ldots, x_{n+1}) = \begin{pmatrix} \int k^2 \big[\log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \big] \, dk \\ \int k^4 \big[\log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \big] \, dk \\ \vdots \\ \int k^{2n} \big[\log |a(k; q)| + x_1 \psi_1(k) + \cdots + x_{n+1} \psi_{n+1}(k) \big] \, dk \end{pmatrix}.$$

This function has derivative matrix

$$Df(0,\dots,0) = \left(\begin{array}{ccc|c} \int k^2\psi_1(k)\,dk & \dots & \int k^2\psi_n(k)\,dk & \int k^2\psi_{n+1}(k)\,dk \\ \int k^4\psi_1(k)\,dk & \dots & \int k^4\psi_n(k)\,dk & \int k^4\psi_{n+1}(k)\,dk \\ \vdots & \ddots & \vdots & \vdots \\ \int k^{2n}\psi_1(k)\,dk & \dots & \int k^{2n}\psi_n(k)\,dk & \int k^{2n}\psi_{n+1}(k)\,dk \end{array}\right)$$

at the origin. If we replace each ψ_j by the even extension $\frac{1}{2}(d\delta_{-k_j} + d\delta_{k_j})$ of a Dirac delta mass at $k_j > 0$, then the left $n \times n$ block matrix becomes the Vandermonde matrix

$$\begin{pmatrix} k_1^2 & k_2^2 & \dots & k_n^2 \\ k_1^4 & k_2^4 & \dots & k_n^4 \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{2n} & k_2^{2n} & \dots & k_n^{2n} \end{pmatrix} \quad \text{with determinant} \quad k_1^2 \cdots k_n^2 \prod_{i<j}(k_j^2 - k_i^2).$$

This determinant is nonvanishing provided that we pick the k_i positive and distinct.

As $a(k;q)$ is a continuous and even function of $k \in \mathbb{R}$ by the reality condition (2.2.6), we may pick $n+1$ distinct points $k_1, \dots, k_{n+1}$ in $\{k > 0 : |a(k;q)| \neq 1\}$. We will take $\psi_1, \dots, \psi_n$ to be mollifications of $\frac{1}{2}(d\delta_{-k_j} + d\delta_{k_j})$ for $j = 1, \dots, n$ by a smooth and even function. If we take the mollifier to have sufficiently small support, then $\psi_1, \dots, \psi_n$ will have disjoint supports within $\{k \neq 0 : |a(k;q)| \neq 1\}$. Taking the support of the mollifier to be even smaller if necessary, the above computation shows that the left $n \times n$ block of $Df(0, \dots, 0)$ is invertible. Now the implicit function theorem implies that there exists $\varepsilon > 0$ and C^1 functions $x_1(x_{n+1})$, $\dots$, $x_n(x_{n+1})$ defined on $(-\varepsilon, \varepsilon)$ so that

$$f(x_1(x_{n+1}), \dots, x_n(x_{n+1}), x_{n+1}) \equiv f(0, \dots, 0) = \begin{pmatrix} \int k^2 \log|a(k;q)|\,dk \\ \int k^4 \log|a(k;q)|\,dk \\ \vdots \\ \int k^{2n} \log|a(k;q)|\,dk \end{pmatrix} \tag{2.4.4}$$

for $x_{n+1} \in (-\varepsilon, \varepsilon)$.

It remains to show that we can pick x_{n+1} in a way that decreases the next $\log|a|$ moment. To this end, we compute the derivative

$$\frac{d}{dx_{n+1}}\bigg|_{x_{n+1}=0} \int k^{2n+2}\big[\log|a| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1}\big]\,dk$$

$$= \int k^{2n+2}\left[\begin{pmatrix}\psi_1 & \dots & \psi_n\end{pmatrix}\begin{pmatrix}x_1'(0)\\ \vdots \\ x_n'(0)\end{pmatrix} + \psi_{n+1}\right]dk.$$

The derivative of $x_1(x_{n+1}), \dots, x_n(x_{n+1})$ is determined by differentiating (2.4.4) at $x_{n+1} = 0$. This yields

$$\begin{pmatrix}x_1'(0)\\ \vdots \\ x_n'(0)\end{pmatrix} = -\begin{pmatrix}\int k^2\psi_1 & \dots & \int k^2\psi_n\\ \vdots & \ddots & \vdots \\ \int k^{2n}\psi_1 & \dots & \int k^{2n}\psi_n\end{pmatrix}^{-1}\begin{pmatrix}\int k^2\psi_{n+1}\\ \vdots \\ \int k^{2n}\psi_{n+1}\end{pmatrix}.$$

Recall that the matrix above is invertible by our choice of $\psi_1, \dots, \psi_n$. Inserting this into the derivative of the $(2n+2)$nd moment, the resulting matrix product is supported on the union of the supports of $\psi_1, \dots, \psi_n$ which is disjoint from the support of the other term ψ_{n+1} in the integrand. Therefore, we may pick another smooth and even function ψ_{n+1} supported in a sufficiently small neighborhood of $\pm k_{n+1}$ so that the whole integral is nonzero. It then follows that there exists $x_{n+1} \in (-\varepsilon, \varepsilon)$ sufficiently small so that

$$\log|a(k;q)| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1} \tag{2.4.5}$$

is nonnegative for $k \in \mathbb{R}$, decreases E_{n+1}, and preserves $E_1, \dots, E_n$.

It only remains to show that the density (2.4.5) corresponds to $\log|a(k;\widetilde{q})|$ for some $\widetilde{q} \in \mathcal{S}(\mathbb{R})$. To accomplish this, we will reconstruct $\widetilde{q}$ from its scattering data by verifying properties (i)-(vii) of Proposition 2.2.4. First, we require that the transmission coefficient satisfies

$$\tfrac{1}{|T(k;\widetilde{q})|} = |a(k;\widetilde{q})| = \exp\big\{\log|a(k;q)| + x_1(x_{n+1})\psi_1 + \cdots + x_n(x_{n+1})\psi_n + x_{n+1}\psi_{n+1}\big\}$$

for $k \in \mathbb{R}$. We have $|T(k;\widetilde{q})| \leq 1$ because (2.4.5) is nonnegative. As $a(k,q)$ extended to a bounded holomorphic to all of $\mathbb{C}^+$, it is in the Hardy space $\mathscr{H}^\infty(\mathbb{C}^+)$. By [65, Ch. II Th. 4.4], given $g \in L^\infty(\mathbb{R})$ nonnegative, there exists a holomorphic function $h \in \mathscr{H}^\infty(\mathbb{C}^+)$ with $|h(k)| = g(k)$ for almost every $k \in \mathbb{R}$ if and only if

$$\int_{-\infty}^{\infty} \frac{\log|g(k)|}{1+k^2}\,dk > -\infty.$$

In particular, this property was satisfied by $g(k) = |a(k;q)|$. As we have smoothly modified $\log|a(k;q)|$ on a compact subset, this condition is also satisfied for $g(k) = |a(k;\widetilde{q})|$ and so there must also exist a holomorphic extension $a(k;\widetilde{q}) := h(k)$ to $\mathbb{C}^+$. This ensures that $T(k;\widetilde{q}) = \frac{1}{a(k;\widetilde{q})}$ satisfies the analyticity condition (iii).

Next, we set $T_1(k;\widetilde{q}) = T_2(k;\widetilde{q}) = T(k;\widetilde{q})$ in accordance with the symmetry condition (i). We then require the modulus of the reflection coefficients satisfy $|R_1(k;\widetilde{q})| = |R_2(k;\widetilde{q})| = \sqrt{1-|T(k;\widetilde{q})|^2}$ and the phases satisfy

$$\frac{\arg R_1(k;\widetilde{q}) + \arg R_2(k;\widetilde{q})}{2} = \tfrac{\pi}{2} - \arg T(k;\widetilde{q})$$

for $k > 0$ to ensure that the unitary condition (ii) holds. We also need to construct R_1, R_2 so that condition (v) on the rate at $k = 0$ still holds. We are still free to specify the difference $\arg R_1 - \arg R_2$, which if $T(0;q) = 0$ we take to satisfy $\arg R_1 \to \pi$ and $\arg R_2 \to \pi$ as $k \downarrow 0$. As we have only modified $T(k;q)$ on a compact subset of $\mathbb{R} \setminus \{0\}$, altogether we conclude that the condition (v) is satisfied. We then extend R_1 and R_2 to $k < 0$ according to the reality condition (vi).

Note that the Fourier decay condition (vii) is automatically satisfied because we have perturbed $R_1(k;q)$ and $R_2(k;q)$ smoothly. Likewise, the asymptotics condition (iv) is also satisfied because the coefficients have only been modified on a compact subset of $\mathbb{R} \setminus \{0\}$. The resulting potential $\widetilde{q}$ is also automatically Schwartz. Indeed, inverse scattering theory reconstructs $\widetilde{q}$ via an explicit integral [47, §4 Eq. $(1)_R$] in terms of R_1 and the Jost function $f_1(x;k)$, which have only been smoothly modified on a compact subset of $\mathbb{R} \setminus \{0\}$.

Lastly, we use Proposition 2.2.5 to add the bound states $\beta_1, \ldots, \beta_N$ of q back to $\widetilde{q}$. The formula (2.2.8) for the new transmission coefficient shows that $|a(k;\widetilde{q})|$ is unchanged for $k \in \mathbb{R}$. This, together with the construction (2.4.5) of $\log|a(k;\widetilde{q})|$, shows that $\widetilde{q}$ decreases E_{n+1} while preserving $E_1, \ldots, E_n$ as desired. $\square$

Next, we will show that the requirement $|a(k;q)| \equiv 1$ on $\mathbb{R}$ forces $a(k;q)$ to be the finite Blaschke product (2.2.9) on $\mathbb{C}^+$:

Lemma 2.4.6. *If q is a constrained minimizer in the sense of Theorem 2.4.3, then q is a multisoliton.*

Proof. Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half-plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk. By Lemma 2.4.5, we know that $|a(k;q)| \equiv 1$ on $\mathbb{R}$. We also know by the asymptotics (2.2.5) that $a(k;q)$ tends to 1 as $k \to \infty$ within $\mathbb{C}^+$. Applying the maximum modulus principle to the half-disks $\overline{\mathbb{C}^+} \cap \{z : |z| \leq R\}$ and taking $R \to \infty$, we conclude that $k \mapsto a(k;q)$ maps $\mathbb{C}^+$ into $\overline{\mathbb{D}}$.

In particular, $a(\,\cdot\,;q)$ is in the Hardy space $\mathscr{H}^\infty(\mathbb{C}^+)$. We may therefore apply inner-outer factorization [65, Ch. II Cor. 5.7] to obtain

$$a(k;q) = e^{i\theta} B(k) S(k) F(k),$$

where $\theta \in \mathbb{R}$ is a constant, $B(k)$ is a Blaschke product, $S(k)$ is a singular function, and $F(k)$ is an outer factor.

First, we claim that $F(k)$ is constant. Note that on $\mathbb{R}$ we have $|B(k)| \equiv 1$ everywhere and $|S(k)| \equiv 1$ almost everywhere. Therefore $|F(k)| \equiv 1$ almost everywhere on $\mathbb{R}$. As F is an outer factor, $\log|F|$ in $\mathbb{C}^+$ is given by the Poisson integral over its boundary values. As $\log|F| \equiv 0$ almost everywhere on $\mathbb{R}$, we conclude that $\log|F| \equiv 0$ on $\mathbb{C}^+$ and hence F is constant.

Next, we claim that $S(k)$ is also constant. Recall that if $S(k)$ is a singular function and $|S(k)|$ is continuous from $\mathbb{C}^+$ to any $k \in \mathbb{R} \cup \{\infty\}$, then k is not in the support of the singular measure that defines S. In our case we know that $a(k;q)$ extends continuously to $\mathbb{R}$ and ∞, and so we conclude that the measure for $S(k)$ vanishes identically.

Altogether we now have $a(k;q) \equiv e^{i\phi}B(k)$ for some constant $\phi \in \mathbb{R}$. Taking $k \to +i\infty$ we have $a(k;q) \to 1$ by (2.2.5) and $B(k) \to 1$, and so we conclude that $a(k;q) \equiv B(k)$. By Proposition 2.2.3 we know that the zeros of $a(k;q)$ are purely imaginary and simple. Therefore $a(k;q)$ takes the form (2.2.9), and so we conclude that q is a multisoliton. $\square$

Finally, to conclude the proof of Theorem 2.4.3, we note that the degree of the multisoliton q is at most n. Otherwise, Lemma 2.3.1 would imply that we could replace q by another multisoliton that decreases E_{n+1} while preserving $E_1, \ldots, E_n$, which would contradict that q is a minimizer.

2.5 Orbital stability

The goal of this section is to prove Theorem 2.1.4. It will follow easily from the following property of minimizing sequences:

Theorem 2.5.1. *Fix an integer $n \geq 1$. If $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$ and $\{q_k\}_{k\geq 1} \subset H^n(\mathbb{R})$ is a minimizing sequence:*

$$E_1(q_k) \to e_1, \quad \ldots, \quad E_n(q_k) \to e_n, \quad E_{n+1}(q_k) \to C(e_1, \ldots, e_n) \tag{2.5.1}$$

as $k \to \infty$, then there exists a subsequence which converges in $H^n(\mathbb{R})$ to the manifold of minimizing solitons $\{Q_{\boldsymbol{\beta},\boldsymbol{c}} : \boldsymbol{c} \in \mathbb{R}^N\}$.

We begin the proof of Theorem 2.5.1 by fixing a minimizing sequence $\{q_k\}_{k\geq 1} \subset H^n(\mathbb{R})$ satisfying (2.5.1). The estimates (2.2.3) that prove that $E_1, \ldots, E_{n+1}$ are continuous functionals on $H^n(\mathbb{R})$ show that the sequence $\{q_k\}_{k\geq 1}$ is bounded in $H^n(\mathbb{R})$.

Now that we know $\{q_k\}_{k\geq 1}$ is bounded in $H^n(\mathbb{R})$, we are able to apply a concentration compactness principle adapted to our variational problem. Specifically, we will use the following statement associated to the embedding $H^n(\mathbb{R}) \hookrightarrow W^{n-1,3}(\mathbb{R})$, whose formulation is inspired by [96]. The choice of concentration compactness principle is not unique (cf. [4, §3]), but we will see below that this choice turns out to be efficient (see, for example, the proof of Lemma 2.5.4).

Theorem 2.5.2. *Fix an integer $n \geq 1$. If $\{q_k\}_{k\geq 1}$ is a bounded sequence in $H^n(\mathbb{R})$, then there exist $J^* \in \{0, 1, \dots, \infty\}$, J^*-many profiles $\{\phi^j\}_{j=1}^{J^*} \subset H^n(\mathbb{R})$, and J^*-many sequences $\{x_k^j\}_{j=1}^{J^*} \subset \mathbb{R}$ so that along a subsequence we have the decomposition*

$$q_k(x) = \sum_{j=1}^{J} \phi^j(x - x_k^j) + r_k^J(x) \quad \textit{for all } J \in \{0, \dots, J^*\} \textit{ finite} \tag{2.5.2}$$

that satisfies:

$$\lim_{J\to J^*} \limsup_{k\to\infty} \left\| r_k^J \right\|_{W^{n-1,3}} = 0, \tag{2.5.3}$$

$$\lim_{k\to\infty} \left| \left\| q_k \right\|_{H^n}^2 - \left(\sum_{j=1}^{J} \left\| \phi^j \right\|_{H^n}^2 + \left\| r_k^J \right\|_{H^n}^2 \right) \right| = 0 \quad \textit{for all } J \textit{ finite}, \tag{2.5.4}$$

$$\lim_{J\to J^*} \left| \limsup_{k\to\infty} \left\| q_k \right\|_{W^{n-1,3}}^3 - \sum_{j=1}^{J} \left\| \phi^j \right\|_{W^{n-1,3}}^3 \right| = 0, \tag{2.5.5}$$

$$|x_k^j - x_k^\ell| \to \infty \quad \textit{as } k \to \infty \textit{ whenever } j \neq \ell. \tag{2.5.6}$$

The $n = 1$ case of Theorem 2.5.2 is well-known [75, Prop. 3.1]; for a textbook presentation of such concentration compactness principles, we recommend [96]. While it does not appear that Theorem 2.5.2 for $n \geq 2$ has been recorded in the literature, it can be proved by exactly the same method (e.g. [96, Th. 4.7]) and we omit the details.

We apply this concentration compactness principle to the minimizing sequence $\{q_k\}_{k\geq 1}$ in Theorem 2.5.1. After passing to a subsequence, Theorem 2.5.2 provides a number $J^* \in \{0, 1, \dots, \infty\}$, J^*-many profiles $\{\phi^j\}_{j=1}^{J^*} \subset H^n(\mathbb{R})$, and J^*-many sequences $\{x_k^j\}_{j=1}^{J^*} \subset \mathbb{R}$ so that along a subsequence we have the decomposition (2.5.2) satisfying the properties (2.5.3)–(2.5.6). We will ultimately show that each profile ϕ^j is a constrained minimizer of E_{n+1}, and hence is a multisoliton.

First, we will treat the case $J^* = 0$:

Lemma 2.5.3. *If $J^* = 0$, then $e_1 = \cdots = e_n = 0$ and $q_k \to 0$ in $H^n(\mathbb{R})$ as $k \to \infty$.*

Proof. The decomposition (2.5.2) reads $q_k = r_k^0$, and so

$$E_2(q_k) = E_2(r_k^0) = \int \left\{ \tfrac{1}{2} \left[(r_k^0)' \right]^2 + (r_k^0)^3 \right\} dx.$$

The second term in the integrand contributes $\|r_k^0\|_{L^3}^3$, which we know vanishes in the limit $k \to \infty$ by (2.5.3). The remaining term is nonnegative, and so we obtain

$$\lim_{k\to\infty} E_2(q_k) \geq 0.$$

On the other hand, we know that this limit is attainable by a multisoliton since $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$, and so there exists $N \leq n$ and $\beta_1 > \cdots > \beta_N > 0$ so that

$$\lim_{k\to\infty} E_2(q_k) = -\tfrac{32}{5} \sum_{m=1}^{N} \beta_m^5 \leq 0.$$

Together, we see that $E_2(q_k) \to 0$ and $N = 0$. The only multisoliton that can attain this value is the zero-soliton $q(x) \equiv 0$, and so we conclude that $e_1 = \cdots = e_n = 0$.

Now we have

$$0 = \lim_{k\to\infty} E_1(q_k) = \lim_{k\to\infty} \tfrac{1}{2} \left\| q_k \right\|_{L^2}^2,$$

and so $q_k \to 0$ in $L^2(\mathbb{R})$. Using the estimates (2.2.3) that prove that $E_1, \ldots, E_{n+1}$ are continuous functionals on $H^n(\mathbb{R})$, we obtain $q_k \to 0$ in $H^n(\mathbb{R})$ as desired. □

Lemma 2.5.3 proves Theorem 2.5.1 in the case $J^* = 0$, and so for the remainder of the section we assume $J^* \geq 1$.

Next, we show that our decomposition accounts for the entirety of the limiting value of each $E_m(q_k)$:

Lemma 2.5.4. *For each $m = 1, \ldots, n+1$ we have*

$$\lim_{J\to J^*} \limsup_{k\to\infty} \left| E_m(q_k) - \left[\sum_{j=1}^{J} E_m(\phi^j) + E_m(r_k^J) \right] \right| = 0.$$

Proof. Fix m, and insert the decomposition (2.5.2) for q_k into the expression (2.2.1) for E_m. By the H^n-norm property (2.5.4), we see the quadratic terms of each E_m cancel, leaving only cubic and higher order terms:

$$\begin{aligned}
&\limsup_{k\to\infty} \left| E_m(q_k) - \left[\sum_{j=1}^{J} E_m(\phi^j) + E_m(r_k^J) \right] \right| \\
&= \limsup_{k\to\infty} \left| \int \left\{ P_m\left(\sum_{j=1}^{J} \phi^j(x - x_k^j) + r_k^J \right) - \left[\sum_{j=1}^{J} P_m(\phi^j) + P_m(r_k^J) \right] \right\} dx \right|.
\end{aligned}$$

Consider an arbitrary term $c_{\alpha_1,\dots,\alpha_d} u^{(\alpha_1)} \cdots u^{(\alpha_d)}$ of $P_m(u)$ for $u = q_k$, ϕ^j, or r_k^J. Expanding all products in the case $u = q_k = \sum \phi^j(x - x_k^j) + r_k^J$, we are left with a term of the form

$$u_1^{(\alpha_1)} \cdots u_d^{(\alpha_d)}$$

where each u_ℓ for $\ell = 1, \dots, d$ is one of the profiles ϕ^j, its translation $\phi^j(x - x_k^j)$, or the remainder r_k^J.

We claim that all of the terms with $u_1, \dots, u_d \neq r_k^J$ cancel; in other words,

$$\lim_{k\to\infty} \left| \int \left\{ P_m\left(\sum_{j=1}^{J} \phi^j(x - x_k^j) \right) - \sum_{j=1}^{J} P_m(\phi^j) \right\} dx \right| = 0 \tag{2.5.7}$$

for all $J \leq J^*$ finite. When all of the u_ℓ are given by the same translated profile $\phi^j(x - x_k^j)$, we can change variables $y = x - x_k^j$ in the integral and recover the corresponding term where $u_1, \dots, u_d = \phi^j(x)$. When there are at least two different translated profiles, the integral vanishes in the limit $k \to \infty$ by the well-separation condition (2.5.6) and approximating each ϕ^j by compactly-supported functions.

It remains to show that the remaining terms (which contain at least one factor of r_k^J) vanish in L^1. Note that by the scaling requirement (2.2.2), each order α_ℓ is at most $m - 2 \leq n - 1$. We estimate the highest order factor of r_k^J in L^3, which is vanishing in the limit $k \to \infty$ and $J \to J^*$ by the small-remainder condition (2.5.3). We then estimate the two other highest order factors ϕ^j or r_k^J in L^3, and the remaining terms are bounded in L^∞ since ϕ^j and r_k^J are uniformly bounded in $H^n \hookrightarrow W^{n-1,\infty}$. $\square$

Next, we show that the quadratic term of $E_{n+1}(r_k^J)$ dominates as $k \to \infty$:

Lemma 2.5.5. *For each $m = 1, \dots, n+1$ we have*

$$\lim_{J \to J^*} \limsup_{k \to \infty} \left| E_m(r_k^J) - \tfrac{1}{2}\big\|(r_k^J)^{(m-1)}\big\|_{L^2}^2 \right| = 0.$$

Proof. Fix m. Using the expression (2.2.1) for E_m, we write

$$E_m(r_k^J) - \tfrac{1}{2}\big\|(r_k^J)^{(m-1)}\big\|_{L^2}^2 = \int P_m\big(r_k^J\big)\, dx.$$

Consider the contribution of an arbitrary term $c_{\alpha_1,\dots,\alpha_d} u^{(\alpha_1)} \cdots u^{(\alpha_d)}$ of $P_m(u)$, where each order α_ℓ is at most $m-2 \le n-1$ by the scaling requirement (2.2.2). We estimate the three highest order factors of r_k^J in L^3, which vanish in the limit $k \to \infty$ and $J \to J^*$ by the small-remainder condition (2.5.3). We then estimate the remaining terms in L^∞, which are all bounded since the sequence r_k^J is uniformly bounded in $H^n \hookrightarrow W^{n-1,\infty}$. Altogether, we conclude that every term vanishes in the limit $k \to \infty$ and $J \to J^*$. □

Next, we show that each profile ϕ^j is a constrained minimizer:

Lemma 2.5.6. *For each $1 \le j \le J^*$ finite, the profile ϕ^j minimizes $E_{n+1}(u)$ over all $u \in \mathscr{C}_{\boldsymbol{e}}$ with the constraints $E_1(\phi^j)$, ..., $E_n(\phi^j)$, and hence is a multisoliton $Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}$ of degree at most n.*

Proof. Suppose towards a contradiction that there exists j for which ϕ^j does not minimize E_{n+1}. Then we can replace ϕ^j by another profile $\widetilde{\phi}^j \in H^n(\mathbb{R})$ with

$$E_m(\widetilde{\phi}^j) = E_m(\phi^j) \quad \text{for } m = 1, \dots, n, \quad \text{but} \quad E_{n+1}(\widetilde{\phi}^j) < E_{n+1}(\phi^j).$$

Construct a new sequence $\{\widetilde{q}_k\}_{k\ge1}$ given by the decomposition (2.5.2), but with $\widetilde{\phi}^j$ in place of ϕ^j. This new sequence still satisfies the properties (2.5.3)–(2.5.6), and so by Lemma 2.5.4 we have

$$\lim_{k\to\infty} E_m(\widetilde{q}_k) = e_m \quad \text{for } m = 1, \dots, n, \quad \text{and} \quad \lim_{k\to\infty} E_{n+1}(\widetilde{q}_k) < \lim_{k\to\infty} E_{n+1}(q_k). \tag{2.5.8}$$

However, this contradicts that $\{q_k\}_{k\geq 1}$ was a minimizing sequence. Therefore, we conclude that each ϕ^j is a constrained minimizer of E_{n+1}.

Applying our variational characterization (Theorem 2.1.3), we conclude that ϕ^j is a multisoliton of degree at most n. $\square$

We now know that our profiles $\{\phi^j\}_{j=1}^{J^*}$ are a (possibly infinite) collection of multisolitons $\{Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}\}_{j=1}^{J^*}$. Concatenate all of the vectors $\boldsymbol{\beta}^j$ to form one (possibly infinite) string $\coprod_{j=1}^{J^*}\boldsymbol{\beta}^j$ of positive numbers which may contain repeated values. Next, we show that all of the parameters together minimize E_{n+1} subject to the constraints $e_1,\dots,e_n$:

Lemma 2.5.7. *The concatenation* $\coprod_{j=1}^{J^*}\boldsymbol{\beta}^j$ *is equal to the unique set of parameters* $\beta_1 > \cdots > \beta_N > 0$ *satisfying the constraints* $e_1,\dots,e_n$. *In particular,* J^* *is finite.*

Proof. Consider the relaxed variational problem where we minimize $E_{n+1}(u)$ over the larger set

$$\{u \in H^n(\mathbb{R}) : E_1(u) \leq e_1,\ \dots,\ E_n(u) \leq e_n\}. \tag{2.5.9}$$

As the minimum value $C(e_1,\dots,e_n)$ is strictly decreasing in each constraint by Lemma 2.3.5 and the set $\mathcal{M}_n^n$ of constraints is downward closed by Lemma 2.3.6, then this relaxed minimization problem enjoys the same conclusions of Theorem 2.1.3. We will ultimately show that the profiles $\{Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}\}_{1\leq j<J^*}$ together form a minimizer for this relaxed problem.

In the proof of Theorem 2.1.3 we only needed to treat the case of finitely many β parameters, but this is easily resolved as follows. Suppose towards a contradiction that $J^* = \infty$. We know that the third moment $\sum \beta_m^3$ of $\boldsymbol{\beta} = \coprod_{j=1}^{J^*}\boldsymbol{\beta}^j$ is finite by the trace formula (2.2.1) for E_1, and so we have $\beta_m \to 0$ as $m \to \infty$. In particular, even though there may be repeated values in $\boldsymbol{\beta}$, there are at least $n+1$ distinct values of $\beta_m > 0$. By Lemma 2.3.1, we may replace the first $n+1$ distinct values of β_m to obtain new parameters $\widetilde{\boldsymbol{\beta}}^j$ so that $E_1,\dots,E_n$ are preserved, E_{n+1} is decreased, and each $\boldsymbol{\beta}^j$ is still a set of multisoliton parameters. Constructing a new sequence $\{\widetilde{q}_k\}_{k\geq 1}$ given by the decomposition (2.5.2) for the profiles $\widetilde{\phi}^j = Q_{\widetilde{\boldsymbol{\beta}}^j \boldsymbol{c}^j}$, we obtain a strictly better choice of minimizing sequence in the sense of (2.5.8). This contradicts that $\{q_k\}_{k\geq 1}$ was a minimizing sequence.

Now that we know $J^* < \infty$, Lemma 2.5.4 implies that

$$\sum_{j=1}^{J^*} E_m(Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}) \leq \limsup_{k\to\infty} \left[E_m(q_k) - E_m(r_k^{J^*}) \right] \tag{2.5.10}$$

for each $m = 1, \dots, n+1$. For $m \leq n$, we have $E_m(q_k) \to e_m$ by construction and

$$\liminf_{k\to\infty} E_m(r_k^{J^*}) \geq 0$$

by Lemma 2.5.5, and so RHS(2.5.10) is at most e_m. In other words, the parameters $\boldsymbol{\beta}$ satisfy the relaxed constraints

$$\sum_{j=1}^{J^*} E_m(Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}) \leq e_m \quad \text{for } m = 1, \dots, n. \tag{2.5.11}$$

For $m = n+1$, we know that $E_m(q_k)$ converges to the minimum value $C(e_1, \dots, e_n)$, and so RHS(2.5.10) is at most $C(e_1, \dots, e_n)$. This yields

$$\sum_{j=1}^{J^*} E_{n+1}(Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}) \leq C(e_1, \dots, e_n). \tag{2.5.12}$$

Strict inequality here should not be possible since C is the minimum value of E_{n+1} over the set (2.5.9). Indeed, by (2.5.4) and (2.5.7) we have

$$\sum_{j=1}^{J^*} E_{n+1}(Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}) = \liminf_{k\to\infty} E_{n+1}\left(\sum_{j=1}^{J^*} Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}(x - x_k^j) \right) \geq C(e_1, \dots, e_n).$$

Therefore, we conclude that equality holds in (2.5.12).

Altogether, we see that the finite collection $\boldsymbol{\beta}$ of parameters is a minimizer for the relaxed variational problem (2.5.9). As the minimum value $C(e_1, \dots, e_n)$ is strictly decreasing in each constraint by Lemma 2.3.5, we must have equality in (2.5.11). There cannot be $n+1$ distinct values in $\boldsymbol{\beta}$, since otherwise we could use Lemma 2.3.1 to replace the first $n+1$ distinct values in a way that preserves $E_1, \dots, E_n$ and decreases E_{n+1} in order to obtain a strictly better minimizing sequence. Now that we know there are at most n distinct values of β_m, Corollary 2.3.3 implies that $\boldsymbol{\beta}$ is equal to the unique set of parameters $\beta_1 > \cdots > \beta_N > 0$ with $N \leq n$ that satisfies the constraints $(e_1, \ldots, e_n) \in \mathcal{M}_n^n$. $\square$

It remains to show that the whole sequence q_k converges strongly to the manifold of minimizing multisolitons. To this end, we will need:

Lemma 2.5.8. *The remainders $r_k^{J^*} \to 0$ in $H^n(\mathbb{R})$ as $k \to \infty$.*

Proof. By Lemma 2.5.7, the profiles $\{Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j}\}_{1 \leq j < J^*}$ satisfy the constraints:

$$\sum_{j=1}^{J^*} E_m(Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j}) = e_m \quad \text{for } m = 1, \ldots, n.$$

Combining this with Lemmas 2.5.4 and 2.5.5, we deduce

$$0 = \lim_{k \to \infty} E_m\big(r_k^{J^*}\big) = \lim_{k \to \infty} \tfrac{1}{2}\big\|(r_k^{J^*})^{(m-1)}\big\|_{L^2}^2$$

for $m = 1, \ldots, n+1$. $\square$

The last ingredient that we will need is the following "molecular decomposition" of multisolitons, which says that our superposition $\sum Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j}(x - x_k^j)$ of well-separated multisolitons is close to the manifold of multisolitons:

Proposition 2.5.9. *Fix integers $n \geq 0$ and $J \geq 1$. Suppose $\boldsymbol{\beta}^j$ and $\boldsymbol{c}^j$ are multisoliton parameters for each $1 \leq j \leq J$, and that all of the components β_m^j of each $\boldsymbol{\beta}^j$ are distinct for all j and m. Then for any collection of J-many sequences $\{x_k^j\}_{j=1}^J \subset \mathbb{R}$ satisfying the well-separation condition (2.5.6), there exists a sequence $\boldsymbol{c}_k$ so that*

$$Q_{\boldsymbol{\beta}, \boldsymbol{c}_k}(x) - \sum_{j=1}^{J} Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j}(x - x_k^j) \to 0 \quad \text{in } H^n(\mathbb{R}) \text{ as } k \to \infty,$$

where $\boldsymbol{\beta}$ is the concatenation $\coprod_{j=1}^{J} \boldsymbol{\beta}^j$.

Proof. The $n = 0$ case is proved in [98, Prop. 3.1]. Given $n \geq 1$, we pick $\boldsymbol{c}_k$ from the $n = 0$ case so that the desired convergence occurs in $L^2(\mathbb{R})$. Note that

$$\left\| Q_{\boldsymbol{\beta},\boldsymbol{c}_k}(x) - \sum_{j=1}^{J} Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}(x - x_k^j) \right\|_{H^{n+1}} \lesssim 1$$

uniformly in k, by the estimates (2.2.3) that prove that $E_1, \ldots, E_{n+2}$ are continuous, the trace formulas (2.2.7), and the well-separation condition (2.5.6). Using the inequality

$$\|f\|_{H^n} \leq \|f\|_{L^2}^{\frac{1}{n+1}} \|f\|_{H^{n+1}}^{\frac{n}{n+1}}$$

(which follows from Hölder's inequality in Fourier variables), we conclude that the sequence converges in $H^n(\mathbb{R})$. □

We are now prepared to finish the proof of Theorem 2.5.1:

Proof of Theorem 2.5.1. It remains to show that the sequence $\{q_k\}_{k\geq 1}$ converges to the manifold $\{Q_{\boldsymbol{\beta},\boldsymbol{c}} : \boldsymbol{c} \in \mathbb{R}^N\}$. So far, we have the decomposition

$$q_k(x) = \sum_{j=1}^{J^*} Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}(x - x_k^j) + r_k^{J^*}(x)$$

with J^* finite. Let $Q_{\boldsymbol{\beta},\boldsymbol{c}_k}$ be the sequence of multisolitons guaranteed by the $H^n(\mathbb{R})$ molecular decomposition (Proposition 2.5.9). We estimate

$$\|q_k - Q_{\boldsymbol{\beta},\boldsymbol{c}_k}\|_{H^n} \leq \left\| Q_{\boldsymbol{\beta},\boldsymbol{c}_k}(x) - \sum_{j=1}^{J^*} Q_{\boldsymbol{\beta}^j,\boldsymbol{c}^j}(x - x_k^j) \right\|_{H^n} + \left\| r_k^{J^*} \right\|_{H^n}.$$

The first term on the RHS converges to zero as $k \to \infty$ by Proposition 2.5.9. The second term on the RHS converges to zero by Lemma 2.5.8. Together, we conclude that

$$\inf_{\boldsymbol{c}\in\mathbb{R}^n} \|q_k - Q_{\boldsymbol{\beta},\boldsymbol{c}}\|_{H^n} \to 0 \quad \text{as} \quad k \to \infty$$

as desired. □

As a corollary, we obtain orbital stability:

Proof of Theorem 2.1.4. Suppose towards a contradiction that orbital stability fails. Then there exists a constant $\varepsilon_0 > 0$, a sequence of initial data $\{q_k(0)\}_{k\geq 1} \subset H^n(\mathbb{R})$, and a sequence of times $\{t_k\}_{k\geq 1} \subset \mathbb{R}$ such that

$$\inf_{\boldsymbol{c}\in\mathbb{R}^n} \|q_k(0) - Q_{\boldsymbol{\beta},\boldsymbol{c}}\|_{H^n} \to 0 \quad \text{as } k \to \infty,$$

but the corresponding solutions $q_k(t)$ to KdV obey

$$\inf_{\boldsymbol{c}\in\mathbb{R}^n} \|q_k(t_k) - Q_{\boldsymbol{\beta},\boldsymbol{c}}\|_{H^n} \geq \varepsilon_0 \quad \text{for all } k. \tag{2.5.13}$$

As $E_1, \dots, E_{n+1}$ are continuous on $H^n(\mathbb{R})$ and are conserved by the KdV flow, we have

$$\lim_{k\to\infty} E_m(q_k(t_k)) = \lim_{k\to\infty} E_m(q_k(0)) = E_m(Q_{\boldsymbol{\beta},\boldsymbol{c}})$$

for each $m = 1, \dots, n+1$. There are n-many β parameters, and so these are exactly the conditions (2.5.1) that the sequence $\{q_k(t_k)\}_{k\geq 1}$ is a minimizing sequence for E_{n+1} with constraints $E_1(Q_{\boldsymbol{\beta},\boldsymbol{c}}), \dots, E_n(Q_{\boldsymbol{\beta},\boldsymbol{c}})$ that are in $\mathcal{M}_n^n$. By Theorem 2.5.1, there exists a subsequence of $\{q_k(t_k)\}_{k\geq 1}$ which converges to the manifold $\{Q_{\boldsymbol{\beta},\boldsymbol{c}} : \boldsymbol{c} \in \mathbb{R}^N\}$ in $H^n(\mathbb{R})$, which contradicts our assumption (2.5.13). $\square$

2.6 Proof of Theorem 2.1.5

In this section, we will adapt the methods of Sections 2.3 to 2.5 in order to prove Theorem 2.1.5. Fix $N \geq n+1$, and consider constraints $(e_1, \dots, e_n) \in \mathcal{M}_N^n$ that are attainable by a multisoliton of degree at most N. We aim to show that minimizing sequences resemble a superposition of multisolitons with at most n distinct amplitudes.

As the constraints are attainable by finitely many parameters, compactness still guarantees that there exists a minimizing set of N-soliton parameters $\beta_1 \geq \cdots \geq \beta_N \geq 0$, provided that we allow for repeated values:

Lemma 2.6.1. *Given constraints* $(e_1, \dots, e_n) \in \mathcal{M}_N^n$*, there exist* $\beta_1 \geq \cdots \geq \beta_N \geq 0$ *which minimize*

$$E_{n+1}(Q_{\boldsymbol{\beta},\boldsymbol{c}}) = (-1)^n \tfrac{2^{2n+3}}{2n+3} \sum_{m=1}^{N} \beta_m^{2n+3}$$

over the set of multisolitons in the constraint set $\mathscr{C}_{\mathbf{e}}$*.*

Proof. Consider the set Γ of parameters

$$\left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : x_1, \dots, x_N \geq 0,\ \sum_{m=1}^{N} x_m^{2j+1} = \alpha_j \text{ for } j = 1, \dots, n \right\} \tag{2.6.1}$$

that satisfy the constraints, where

$$\alpha_j = (-1)^{j+1} \tfrac{2j+1}{2^{2j+1}} e_j$$

are the prescribed odd moments. Note that the set Γ is compact, and it is nonempty since $(e_1, \dots, e_n) \in \mathcal{M}_N^n$. Therefore there exists a minimizer $(\beta_1, \dots, \beta_N)$ of the next odd moment

$$(-1)^n \tfrac{2^{2n+3}}{2n+3} \sum_{m=1}^{N} x_m^{2n+3}$$

in Γ. As the odd moments are symmetric in $\beta_1, \dots, \beta_N$, we may reorder them so that $\beta_1 \geq \cdots \geq \beta_N \geq 0$. $\square$

Unlike in the proof of Lemma 2.3.6, the set Γ cannot reach the boundary $\{(x_1, \dots, x_N) : x_{n+1}, \dots, x_N = 0\}$ since $(e_1, \dots, e_n) \notin \mathcal{M}_n^n$, and so we are no longer able to reduce the number of parameters. Instead, we can employ the implicit function theorem argument from Lemma 2.3.1 to reduce the number of distinct components in the minimizer $(\beta_1, \dots, \beta_N)$:

Lemma 2.6.2. *If* $\beta_1 \geq \cdots \beta_N \geq 0$ *is a minimizer (in the sense of Lemma 2.6.1), then there are at most* n *distinct values of* β_m*.*

Proof. It suffices to show that if there are at least $n+1$ distinct values in $\beta_1, \dots, \beta_N$, then there exist new values $\widetilde{\beta}_1, \dots, \widetilde{\beta}_N$ which preserve $E_1, \dots, E_n$ but decrease E_{n+1}. To prove this, we repeat the proof of Lemma 2.3.1. Rather than recapitulating the whole proof, let us focus on the few minor alterations that need to be made.

For example, consider the case where we have $N \geq n+2$, $\beta_j = \beta_{j+1}$ for some j, and all other β_m are distinct. Replace the function (2.3.1) by

$$f(x_1, \ldots, x_{n+1}) = \begin{pmatrix} x_1^3 + \cdots + x_{j-1}^3 + 2x_j^3 + x_{j+1}^3 + \cdots + x_{n+1}^3 \\ x_1^5 + \cdots + x_{j-1}^5 + 2x_j^5 + x_{j+1}^5 + \cdots + x_{n+1}^5 \\ \vdots \\ x_1^{2n+1} + \cdots + x_{j-1}^{2n+1} + 2x_j^{2n+1} + x_{j+1}^{2n+1} + \cdots + x_{n+1}^{2n+1} \end{pmatrix}.$$

This simply multiplies the jth column of the derivative matrix (2.3.2) by 2. Consequently, the left $n \times n$ submatrix still has nonzero determinant and thus we may apply the implicit function theorem. We can then proceed with the remainder of the proof of Lemma 2.3.1.

In the general case, each column of the derivative matrix (2.3.2) is simply multiplied by a constant. Therefore the left $n \times n$ submatrix is still invertible, and the proof of Lemma 2.3.1 proceeds as before. □

Now that we know that every minimizer must possess at most n distinct β values, Corollary 2.3.3 immediately implies that the minimizer is unique.

We are now prepared to define our candidate value for the infimum of E_{n+1} subject to the constraints $e_1, \ldots, e_n$. We extend the definition of C to $(e_1, \ldots, e_n) \in \mathcal{M}_N^n$ via

$$C(e_1, \ldots, e_n) = (-1)^n \tfrac{2^{2n+3}}{2n+3} \sum_{j=1}^{N} \beta_j^{2n+3}.$$

This quantity still satisfies the properties from Lemma 2.3.5:

Lemma 2.6.3. *The function $C : \mathcal{M}_N^n \to \mathbb{R}$ is continuous and is decreasing in each variable. Moreover, C is defined piecewise on finitely many connected subsets of $\mathcal{M}_N^n$, and on the interior of each such subset $C(e_1, \ldots, e_n)$ is continuously differentiable and satisfies $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, n$.*

Proof. Given a minimizer $\beta_1 \geq \cdots \geq \beta_N \geq 0$, Lemma 2.6.2 implies that there exist multiplicities $m_1, \ldots, m_{\overline{N}}$ and distinct values $\overline{\beta}_1 > \cdots > \overline{\beta}_{\overline{N}} \geq 0$ so that $\overline{N} \leq n$, $\sum m_j = N$, and the string $\beta_1, \ldots, \beta_N$ consists of m_1 copies of $\overline{\beta}_1$, m_2 copies of $\overline{\beta}_2$, and so on. This allows us to write

$$C(e_1, \ldots, e_n) = (-1)^n \tfrac{2^{2n+3}}{2n+3}\big(m_1 \overline{\beta}_1^{2n+3} + \cdots + m_{\overline{N}} \overline{\beta}_{\overline{N}}^{2n+3}\big).$$

We will see that for $\overline{N} = n$ and each fixed choice of multiplicities $m_1, \ldots, m_n$, we have $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, n$ as long as $\overline{\beta}_1 > \cdots > \overline{\beta}_{\overline{N}} > 0$. In this way $C(e_1, \ldots, e_n)$ is a piecewise-defined function, and there are finitely many pieces because the number of possible multiplicities $m_1, \ldots, m_{\overline{N}}$ is finite.

Fix multiplicities $m_1, \ldots, m_{\overline{N}}$, and repeat the computation from Lemma 2.3.5. In fact, in the case $\overline{N} = n$ the same computation applies! Indeed, in Lemma 2.3.5 we computed $\frac{\partial C}{\partial e_j}$ from the equality (2.3.16). We have now multiplied the columns of the matrix on the LHS and each entry on the RHS by the multiplicities $m_1, \ldots, m_n$, but this does not alter the system of equations. In the case $\overline{N} < n$, the system of equations (2.3.14) is overdetermined. However, if we only consider the first $\overline{N}$ constraints, then the computation proceeds with $\overline{N}$ in place of n, and we conclude that C as a function of $e_1, \ldots, e_{\overline{N}}$ (where $m_1, \ldots, m_{\overline{N}}$ are fixed) is C^1 and satisfies $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, \overline{N}$.

We are also able to compute $\frac{\partial \overline{\beta}_k}{\partial e_j}$ as long as $\overline{\beta}_1 > \cdots > \overline{\beta}_n > 0$, which implies that if we wiggle $e_1, \ldots, e_n$ then we can also wiggle $\overline{\beta}_1, \ldots, \overline{\beta}_n$ in a way that still satisfies the constraints. By uniqueness (Corollary 2.3.3), the perturbed values of $\overline{\beta}_1, \ldots, \overline{\beta}_n$ still minimize E_{n+1}. This defines an injective map Φ from the simplex

$$\{(\overline{\beta}_1, \ldots, \overline{\beta}_n) \in \mathbb{R}^n : \overline{\beta}_1 > \cdots > \overline{\beta}_n > 0\} \tag{2.6.2}$$

into $\mathcal{M}_n^n$, and it is smooth up to its boundary. The image of Φ is exactly the interior of one of the components on which $C(e_1, \ldots, e_n)$ is defined by a single formula. The boundary of this component corresponds to some subset of the boundary of the simplex (2.6.2), which means that two values of $\overline{\beta}_j$ are colliding or that $\overline{\beta}_n$ is vanishing. Repeating the proof of

Lemma 2.3.4, we conclude that Φ is a homeomorphism onto this component (including any boundary points it may contain). It then follows that C is continuous by the same argument as in Lemma 2.3.5. $\square$

Next, we show that the set of constraints $\mathcal{M}_N^n$ is still downward closed:

Lemma 2.6.4. *If the constraints $\widetilde{e}_1, \dots, \widetilde{e}_n$ are in $\mathcal{M}_{\widetilde{N}}^n$ for some $\widetilde{N}$ and*

$$\widetilde{e}_1 \leq e_1, \quad \dots, \quad \widetilde{e}_n \leq e_n$$

for some $(e_1, \dots, e_n) \in \mathcal{M}_N^n$, then $(\widetilde{e}_1, \dots, \widetilde{e}_n) \in \mathcal{M}_N^n$.

Proof. Let $\widetilde{\beta}_1 \dots, \widetilde{\beta}_{\widetilde{N}} > 0$ denote the β parameters of the multisoliton which witnesses the constraints $\widetilde{e}_1, \dots, \widetilde{e}_n$, and assume that we are in the nontrivial case $\widetilde{N} \geq N+1$. Repeating the proof of Lemma 2.3.6, we conclude that the set Γ of parameters in $\mathbb{R}^{\widetilde{N}}$ that satisfy the constraints must intersect the boundary $\{(x_1, \dots, x_{\widetilde{N}}) : x_{N+1}, \dots, x_{\widetilde{N}} = 0\}$. Any point in the intersection provides the desired N-soliton parameters. $\square$

We are now prepared to prove that $C(e_1, \dots, e_n)$ is the infimum of E_{n+1}:

Proposition 2.6.5. *Given $(e_1, \dots, e_n) \in \mathcal{M}_N^n$ for some $N \geq n+1$, we have*

$$\inf\{E_{n+1}(u) : u \in \mathscr{C}_{\boldsymbol{e}}\} = C(e_1, \dots, e_n). \tag{2.6.3}$$

Moreover, if $(e_1, \dots, e_n) \notin \mathcal{M}_n^n$, then this infimum is not attained by any $u \in \mathscr{C}_{\boldsymbol{e}}$.

Proof. First, we claim that

$$E_{n+1}(u) \geq C(e_1, \dots, e_n) \quad \text{for all } u \in \mathscr{C}_{\boldsymbol{e}}.$$

We repeat the proof of Theorem 2.4.2. This proof only required Lemmas 2.3.1, 2.3.5 and 2.3.6 as input, and we have established their analogues Lemmas 2.6.2 to 2.6.4 in this new setting.

To prove (2.6.3), it remains to show that $E_{n+1}(u)$ can be arbitrarily close to $C(e_1, \dots, e_n)$ for some choice of $u \in \mathscr{C}_{\boldsymbol{e}}$. Recall that $C(e_1, \dots, e_n)$ is defined in terms of a minimizer

$\beta_1 \geq \cdots \geq \beta_N \geq 0$ in the sense of Lemma 2.6.1. The claim follows by taking u to be an N-soliton with parameters $\widetilde{\beta}_1 > \cdots > \widetilde{\beta}_N > 0$ that converge to the minimizer $\beta_1 \geq \cdots \geq \beta_N \geq 0$ within the set (2.6.1).

Lastly, suppose towards a contradiction that $E_{n+1}(q) = C(e_1, \ldots, e_n)$ for some $q \in \mathscr{C}_{\boldsymbol{e}}$ and $(e_1, \ldots, e_n) \notin \mathcal{M}_n^n$. First, we show that q is Schwartz by repeating the proof of Lemma 2.4.4. In the case where the number $\overline{N}$ of multiplicities is equal to n, we have $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, n$ and the proof of Lemma 2.4.4 carries out unaltered. In the case where $\overline{N} < n$, we recall from the proof of Lemma 2.6.3 that we may regard C as a function of $e_1, \ldots, e_{\overline{N}}$ and we have $\frac{\partial C}{\partial e_j} < 0$ for $j = 1, \ldots, \overline{N}$. In either case, the minimizer q satisfies an Euler–Lagrange equation of the form (2.4.1) with $\lambda_1 < 0$ and all other $\lambda_j \leq 0$, and this is sufficient to conclude that q is Schwartz. Then, by directly applying Lemma 2.4.5, 2.4.6, and 2.3.1 (without alteration!), we see that q is a multisoliton of degree at most n, which contradicts that $(e_1, \ldots, e_n) \notin \mathcal{M}_n^n$. □

When combined with concentration compactness, we can prove that minimizing sequences resemble a superposition of multisolitons with at most n distinct values of β_m:

Proof of Theorem 2.1.5. It only remains to prove the minimizing sequence statement. Fix $(e_1, \ldots, e_n) \in \mathcal{M}_N^n \setminus \mathcal{M}_{N-1}^n$ for some $N \geq n+1$, and suppose that $\{q_k\}_{k\geq 1} \subset H^n(\mathbb{R})$ satisfies

$$E_1(q_k) \to e_1, \quad \ldots, \quad E_n(q_k) \to e_n, \quad E_{n+1}(q_k) \to C(e_1, \ldots, e_n)$$

as $k \to \infty$.

First, we apply our concentration compactness principle. After passing to a subsequence, Theorem 2.5.2 provides us with a number $J^* \in \{0, 1, \ldots, \infty\}$, J^*-many profiles $\{\phi^j\}_{j=1}^{J^*} \subset H^n(\mathbb{R})$, and J^*-many sequences $\{x_k^j\}_{j=1}^{J^*} \subset \mathbb{R}$ so that along a subsequence we have the decomposition (2.5.2) which satisfies the properties (2.5.3)–(2.5.6).

The proof of Theorem 2.5.1 up through Lemma 2.5.6 still applies (without alteration), and so we conclude that each profile ϕ^j is a multisoliton $Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j}$. Repeating the proof of Lemma 2.5.7, we see that the concatenation $\boldsymbol{\beta} = \coprod_{j=1}^{J^*} \boldsymbol{\beta}^j$ minimizes the inequality for E_{n+1}

in Proposition 2.6.5. Therefore $\boldsymbol{\beta}$ is a minimizer in the sense of Lemma 2.6.1, and so by Lemmas 2.6.2 and 2.6.4 we see that J^* is finite, the total degree $\sum \#\boldsymbol{\beta}^j$ is equal to N, and the components of $\boldsymbol{\beta}$ attain at most n distinct values.

We now have

$$\left\| q_k - \sum_{j=1}^{J^*} Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j} \right\|_{H^n} = \left\| r_k^{J^*} \right\|_{H^n}.$$

Repeating the proof of Lemma 2.5.8, we see that the RHS converges to zero as $k \to \infty$. This yields

$$\inf_{\boldsymbol{c}^1, \ldots, \boldsymbol{c}^{J^*}} \left\| q_k - \sum_{j=1}^{J^*} Q_{\boldsymbol{\beta}^j, \boldsymbol{c}^j} \right\|_{H^n} \to 0 \quad \text{as } k \to \infty$$

as desired. □

2.7 Proof of Theorem 2.1.6

The goal of this section is to prove Theorem 2.1.6. Suppose that $(e_1, e_2) \in \mathcal{F}^2$ and $(e_1, e_2) \notin \mathcal{M}_N^2$ for all N. We aim to show that Schwartz minimizing sequences for these constraints have vanishing β parameters and $\log|a|$ converging to the even extension of a Dirac delta distribution.

By the explicit description (2.1.5) and (2.1.6) of $\mathcal{F}^2$ and $\bigcup_{N \geq 0} \mathcal{M}_N^2$, we note that our conditions on (e_1, e_2) are equivalent to $e_1 > 0$ and $e_2 \geq 0$.

First, we find a lower bound for the $\log|a|$ contribution to E_3:

Lemma 2.7.1. *We have*

$$E_3(u) \geq \tfrac{64}{\pi} \tfrac{\gamma_1^2}{\gamma_0} + C\big(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1\big) \quad \text{for all } u \in \mathscr{C}_{\mathbf{e}} \cap \mathcal{S}(\mathbb{R}), \tag{2.7.1}$$

where

$$\gamma_0 = \int_{-\infty}^{\infty} k^2 \log|a(k;u)|\, dk \quad \text{and} \quad \gamma_1 = \int_{-\infty}^{\infty} k^4 \log|a(k;u)|\, dk. \tag{2.7.2}$$

Proof. Fix $u \in \mathcal{S}(\mathbb{R})$. Substituting $x = k^2$ into the trace formulas (2.2.7) and recalling the reality condition (2.2.6), we obtain

$$\begin{aligned}
e_1 &= \tfrac{4}{\pi} \int_0^\infty x^{\frac{1}{2}} \log|a(x^{\frac{1}{2}}; u)| \, dx + \tfrac{8}{3} \sum_{m=1}^N \beta_m^3, \\
e_2 &= \tfrac{16}{\pi} \int_0^\infty x^{\frac{3}{2}} \log|a(x^{\frac{1}{2}}; u)| \, dx - \tfrac{32}{5} \sum_{m=1}^N \beta_m^5, \\
E_3(u) &= \tfrac{64}{\pi} \int_0^\infty x^{\frac{5}{2}} \log|a(x^{\frac{1}{2}}; u)| \, dx + \tfrac{128}{7} \sum_{m=1}^N \beta_m^7.
\end{aligned}$$

The first constraint says that the positive measure

$$d\mu := x^{\frac{1}{2}} \log|a(x^{\frac{1}{2}}; u)| \, dx$$

on $[0, \infty)$ has total mass

$$\gamma_0 := \int_0^\infty 1 \, d\mu(x) = \tfrac{\pi}{4}\Big(e_1 - \tfrac{8}{3} \sum_{m=1}^N \beta_m^3 \Big) \in \big[0, \tfrac{\pi}{4} e_1\big]. \tag{2.7.3}$$

The first constraint also restricts how large the first moment of $d\mu$ can be. As $p \mapsto \|\boldsymbol{\beta}\|_{\ell^p}$ is decreasing, we have

$$\Big(\sum_{m=1}^N \beta_m^5 \Big)^{\frac{1}{5}} \leq \Big(\sum_{m=1}^N \beta_m^3 \Big)^{\frac{1}{3}} \leq \big(\tfrac{3}{8} e_1\big)^{\frac{1}{3}}.$$

This requires that the first moment obeys

$$\gamma_1 := \int_0^\infty x \, d\mu(x) = \tfrac{\pi}{16}\Big(e_2 + \tfrac{32}{5} \sum_{m=1}^N \beta_m^5 \Big) \in \big[\tfrac{\pi}{16} e_2,\ \tfrac{\pi}{16}\big(e_2 + \tfrac{32}{5}(\tfrac{3}{8}e_1)^{\frac{5}{3}}\big)\big]. \tag{2.7.4}$$

In order to bound $E_3(u)$ below, we seek a lower bound for the second moment

$$\gamma_2 := \int_0^\infty x^2 \, d\mu(x).$$

By Cauchy–Schwarz we have

$$\gamma_1 = \int_0^\infty x \, d\mu(x) \leq \Big(\int_0^\infty 1 \, d\mu(x) \Big)^{\frac{1}{2}} \Big(\int_0^\infty x^2 \, d\mu(x) \Big)^{\frac{1}{2}} = \gamma_0^{\frac{1}{2}} \gamma_2^{\frac{1}{2}},$$

and so

$$\gamma_2 \geq \tfrac{\gamma_1^2}{\gamma_0}. \tag{2.7.5}$$

For future reference (cf. (2.7.8)), we note that equality occurs above if and only if the functions $x^{\frac{1}{2}} \log |a(x^{\frac{1}{2}}; u)|$ and $x^{\frac{5}{2}} \log |a(x^{\frac{1}{2}}; u)|$ are proportional. As $k \mapsto \log |a(k; u)|$ is a continuous nonnegative function on $\mathbb{R}$ for u Schwartz, this can only happen when $\log |a(k; u)| \equiv 0$ for $k \in \mathbb{R}$.

The estimate (2.7.5) provides a lower bound for the $\log |a|$ moment of $E_3(u)$. The β moment is then bounded below by the infimum C of $E_3(u)$ subject to the smaller constraints where the $\log |a|$ moments are removed:

$$\tfrac{128}{7} \sum_{m=1}^{N} \beta_m^7 = E_3(Q_{\boldsymbol{\beta},\boldsymbol{c}}) \geq C\big(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1\big).$$

Together, this yields the inequality (2.7.1). □

We will now minimize the lower bound in (2.7.1) over all possible γ_0 and γ_1. At first glance, the first term γ_1^2/γ_0 is smallest when γ_0 is large and γ_1 is small, while the second term $C(e_1 - \frac{4}{\pi}\gamma_0, e_2 - \frac{16}{\pi}\gamma_1)$ is smallest when both γ_0 and γ_1 are small. We will see below that the first term is dominant, which yields the following inequality:

Lemma 2.7.2. *Given constraints $e_1 > 0$ and $e_2 \geq 0$, we have*

$$\inf\{E_3(u) : u \in \mathscr{C}_{\boldsymbol{e}} \cap \mathcal{S}(\mathbb{R})\} = \tfrac{e_2^2}{e_1}. \tag{2.7.6}$$

Moreover, this infimum is not attained by any $u \in \mathscr{C}_{\boldsymbol{e}} \cap \mathcal{S}(\mathbb{R})$.

Proof. The domain of (γ_0, γ_1) in $\mathbb{R}^2$ is contained in the rectangle given by the product of the intervals in (2.7.3) and (2.7.4). Let (γ_0, γ_1) be the minimizer of

$$\tfrac{64}{\pi} \tfrac{\gamma_1^2}{\gamma_0} + C(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1)$$

over this compact rectangle. Differentiating with respect to γ_1, we have

$$\tfrac{\partial}{\partial \gamma_1}\big\{ \tfrac{64}{\pi} \tfrac{\gamma_1^2}{\gamma_0} + C(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1)\big\} = \tfrac{128}{\pi} \tfrac{\gamma_1}{\gamma_0} - \tfrac{16}{\pi} \tfrac{\partial C}{\partial e_2}(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1).$$

The derivative $\frac{\partial C}{\partial e_2}$ is nonpositive by Lemma 2.3.5, and so this quantity is positive for all γ_1 in the open interval $(\frac{\pi}{16}e_2,\ \frac{\pi}{16}(e_2+\frac{32}{5}(\frac{3}{8}e_1)^{\frac{5}{3}}))$. Therefore the minimizer must have $\gamma_1 = \frac{\pi}{16}e_2$. Similarly, we have

$$\tfrac{\partial}{\partial\gamma_0}\big\{\tfrac{64}{\pi}\tfrac{\gamma_1^2}{\gamma_0} + C(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1)\big\} = -\tfrac{64}{\pi}\tfrac{\gamma_1^2}{\gamma_0^2} - \tfrac{4}{\pi}\tfrac{\partial C}{\partial e_1}(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1).$$

The derivative $\frac{\partial C}{\partial e_1}$ is $O(\beta_1^2\beta_2^2)$ by the computation (2.3.17), and hence vanishes as $\beta_1, \beta_2 \to 0$. Therefore, taking $\gamma_1 \to \frac{\pi}{16}e_2$ we obtain

$$\tfrac{\partial}{\partial\gamma_0}\big\{\tfrac{64}{\pi}\tfrac{\gamma_1^2}{\gamma_0} + C(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1)\big\} \to -\tfrac{\pi}{4}\tfrac{e_2^2}{\gamma_0^2}$$

for all γ_0 in the open interval $(0, \frac{\pi}{4}e_1)$. Therefore the minimizer has $\gamma_0 = \frac{\pi}{4}e_1$.

Altogether, we conclude that the minimum occurs at $\gamma_0 = \frac{\pi}{4}e_1$, $\gamma_1 = \frac{\pi}{16}e_2$ with value

$$\big\{\tfrac{64}{\pi}\tfrac{\gamma_1^2}{\gamma_0} + C(e_1 - \tfrac{4}{\pi}\gamma_0, e_2 - \tfrac{16}{\pi}\gamma_1)\big\}\Big|_{\gamma_0=\frac{\pi}{4}e_1,\ \gamma_1=\frac{\pi}{16}e_2} = \tfrac{e_2^2}{e_1}.$$

To prove (2.7.6), it remains to show that we can make $E_3(u)$ arbitrarily close to this value. Fix $(\widetilde{\gamma}_0, \widetilde{\gamma}_1)$ in the interior of the rectangle given by the product of the intervals in (2.7.3) and (2.7.4) that is arbitrarily close to the minimizer (γ_0, γ_1). Pick a smooth and even function $k \mapsto \log|a(k;\widetilde{u})|$ with compact support in $\mathbb{R}\setminus\{0\}$ which attains the moments $(\widetilde{\gamma}_0, \widetilde{\gamma}_1)$ (in the sense of (2.7.2)). Arguing as in Lemma 2.4.5, we can then use Proposition 2.2.4 to construct a function $\widetilde{u} \in \mathcal{S}(\mathbb{R})$ (with no bound states) so that $k \mapsto \log|a(k;\widetilde{u})|$ attains the prescribed moments $(\widetilde{\gamma}_0, \widetilde{\gamma}_1)$.

Lastly, suppose that

$$E_3(q) = \tfrac{e_2^2}{e_1} \tag{2.7.7}$$

for some $q \in \mathscr{C}_{\boldsymbol{e}} \cap \mathcal{S}(\mathbb{R})$. Then we would have $\gamma_0 = \frac{\pi}{4}e_1$, $\gamma_1 = \frac{\pi}{16}e_2$ and hence the β moments $\sum\beta_m^3$ and $\sum\beta_m^5$ must vanish. As $e_1 > 0$ then this implies $\log|a(k;u)| \not\equiv 0$, and so we must have strict inequality in (2.7.5):

$$E_3(q) = \tfrac{64}{\pi}\gamma_2 > \tfrac{64}{\pi}\tfrac{\gamma_1^2}{\gamma_0} = \tfrac{e_2^2}{e_1}. \tag{2.7.8}$$

This contradicts the assumption (2.7.7), and so such a minimizer q cannot exist. $\square$

Now that we have found the infimum of E_3, we are prepared to analyze Schwartz minimizing sequences:

Proof of Theorem 2.1.6. Fix a minimizing sequence $\{q_j\}_{j\geq 1} \subset \mathcal{S}(\mathbb{R})$, so that

$$E_1(q_j) \to e_1, \quad E_2(q_j) \to e_2, \quad E_3(q_j) \to \tfrac{e_2^2}{e_1} \quad \text{as } j \to \infty. \tag{2.7.9}$$

In the inequality of Lemma 2.7.2, we see that we have equality in the limit $j \to \infty$. Therefore the moments $\sum_{m\geq 1} \beta_{j,m}^7$ vanish as $j \to \infty$; otherwise, we could construct a strictly better minimizing sequence with no β parameters, because the constraints can be met solely in terms of the $\log|a|$ moments. This implies

$$\beta_{j,m} \leq \Big(\sum_{\ell \geq 1} \beta_{j,\ell}^7 \Big)^{\frac{1}{7}} \to 0 \quad \text{as } j \to \infty$$

for all m.

We pass to an arbitrary subsequence of $\{q_j\}_{j\geq 1}$. We claim that there is a further subsequence with $\log|a|\, dk$ converging to the even extension of a unique point mass, from which it will follow that the whole sequence $\log|a(k;q_j)|\, dk$ converges to the same limit. Consider the measures

$$d\mu_j := x^{\frac{1}{2}} \log|a(x^{\frac{1}{2}}; q_j)|\, dx$$

on $[0,\infty)$. Changing variables $x = k^2$, the convergence (2.7.9) together with the trace formulas (2.2.7) and the reality condition (2.2.6) tell us that moments of $d\mu_j$ obey

$$\begin{aligned} \gamma_0^j &:= \int_0^\infty 1\, d\mu_j(x) \to \tfrac{\pi}{4} e_1 =: \gamma_0, \\ \gamma_1^j &:= \int_0^\infty x\, d\mu_j(x) \to \tfrac{\pi}{16} e_2 =: \gamma_1, \\ \gamma_2^j &:= \int_0^\infty x^2\, d\mu_j(x) \to \tfrac{\gamma_1^2}{\gamma_0}. \end{aligned}$$

We claim that the renormalized measures $d\mu_j / \gamma_0^j$ on $[0,\infty)$ are tight. For $R > 0$ we estimate

$$\tfrac{1}{\gamma_0^j} \mu_j((R,\infty)) = \tfrac{1}{\gamma_0^j} \int_R^\infty d\mu_j(x) \leq \tfrac{1}{R\gamma_0^j} \int_R^\infty x\, d\mu_j(x).$$

Note that $1/\gamma_0^j$ is bounded uniformly for j large since $\gamma_0^j \to \gamma_0 > 0$. Also, the integral on the RHS is bounded uniformly in j since $\gamma_1^j \to \gamma_1$. Together, we conclude that the RHS tends to zero as $R \to \infty$ uniformly in j.

Therefore, by Prokhorov's theorem we may pass to a subsequence along which the probability measures $d\mu_j/\gamma_0^j$ converge weakly to some probability measure $d\mu/\gamma_0$. As $\gamma_0^j \to \gamma_0 > 0$, then the measures $d\mu_j$ converge weakly to $d\mu_j$.

The sequence of second moments γ_2^j converges, and hence is bounded. It then follows that the zeroth and first moments converge to those of μ:

$$\int_0^\infty d\mu_j(x) = \lim_{j\to\infty} \int_0^\infty d\mu_j(x) = \gamma_0, \quad \int_0^\infty x\, d\mu_j(x) = \lim_{j\to\infty} \int_0^\infty x\, d\mu_j(x) = \gamma_1.$$

For the second moments, we use Fatou's lemma (which holds for weakly converging measures) to obtain

$$\tfrac{\gamma_1^2}{\gamma_0} \le \int_0^\infty x^2\, d\mu(x) \le \liminf_{j\to\infty} \int_0^\infty x^2\, d\mu_j(x) = \tfrac{\gamma_1^2}{\gamma_0}.$$

Altogether, we conclude that μ minimizes the second moment lower bound (2.7.5) from the Cauchy–Schwarz inequality. Therefore the distributions $d\mu$ and $x^2\, d\mu(x)$ on $[0,\infty)$ are proportional, and hence μ is a Dirac delta mass. The support and total mass of this distribution are uniquely determined by γ_0 and γ_1. In turn, the limiting distribution

$$\tfrac{1}{2k}(d\mu(k^2) + d\mu(-k^2))$$

of $\log|a|\, dk$ on $\mathbb{R}$ is then uniquely determined by the reality condition (2.2.6). Lastly, we note that weak convergence of measures implies convergence when integrated against bounded continuous test functions by the Portmanteau theorem, and hence implies convergence in distribution. □

CHAPTER 3

Well-posedness for step-like initial data at high regularity

3.1 Introduction

Historically, investigations of the well-posedness problem for KdV have focused on initial data in the L^2-based Sobolev spaces $H^s(\mathbb{R}/\mathbb{Z})$ and $H^s(\mathbb{R})$. This framework necessarily produces solutions that are spatially periodic or decay at infinity. However, as KdV is a model for surface waves in a shallow channel of water, there are other classes of initial data that are of physical interest. In particular, waveforms that are step-like—in the sense that $u(0,x)$ asymptotically approaches distinct constant values as $x \to \pm\infty$—arise in the study of bore propagation (cf. [16, 34, 60, 71, 124, 139]) and rarefaction waves (cf. [7, 60, 109, 120, 142]). Such asymptotic behavior has real physical consequences; we will see below that the polynomial conservation laws are broken, and in the case of an incoming tide there is an infinite influx of energy into the system.

Our objective in this chapter is to extend low-regularity methods for well-posedness to the regime of nonzero spatial asymptotics. We define the smooth step function

$$W(x) = c_1 \tanh(x) + c_2 \quad \text{with } c_1, c_2 \in \mathbb{R} \text{ fixed}, \tag{3.1.1}$$

which exponentially decays to its asymptotic values. As $-u$ is proportional to the water wave height, W models an incoming tide if $c_1 > 0$ and an outgoing tide if $c_1 < 0$. In fact, we can always perform a boost to prescribe c_2 courtesy of the Galilean symmetries of KdV (1.1.1), but we will not make use of this.

A classical result in the study of step-like asymptotics is:

Theorem 3.1.1. *Fix an integer* $s \geq 3$*. The KdV equation* (1.1.1) *with initial data* $u(0) \in W + H^s(\mathbb{R})$ *is globally well-posed in the following sense:* $u(t) = W + q(t)$ *where* $q(t)$ *is the global solution to*

$$\frac{d}{dt}q = -(q+W)''' + 6(q+W)(q+W)' \tag{3.1.2}$$

with initial data $q(0) = u(0) - W$ *in* $H^s(\mathbb{R})$*. Moreover,* $q(t)$ *is in* $C_t H^s([-T,T] \times \mathbb{R})$ *for all* $T > 0$*,* $q(t)$ *is unique in this class, and* $q(t)$ *depends continuously upon the initial data* $q(0)$ *in* $H^s(\mathbb{R})$*.*

Theorem 3.1.1 is not new (as we will discuss below), but we will use its statement to formulate our main result. Applying Theorem 3.1.1 to the initial data $q(0) \equiv 0$, we conclude that given W there is a unique global solution $V(t) = W + q(t)$ to KdV (1.1.1) with initial data W, and $t \mapsto V(t) - W$ is a continuous function into $H^s(\mathbb{R})$ for all $s \geq 3$. The main thrust of this work is to show that KdV is globally well-posed for $H^{-1}(\mathbb{R})$ perturbations of $V(t)$; see Corollary 4.1.4 for details.

Lower regularity than $H^3(\mathbb{R})$ has been obtained in the study of well-posedness for perturbations of a fixed step-like background wave. The first result was recorded in [79], who proved local well-posedness for perturbations in $H^s(\mathbb{R})$, $s > \frac{3}{2}$, and global well-posedness for $s \geq 2$. Local well-posedness was then extended to $s > 1$ in [62] for the same family of background waves. Independently, local well-posedness for $H^2(\mathbb{R})$ perturbations was proved for gKdV in [144], along with global-in-time existence when the background wave is a kink solution and the initial data is small in $H^1(\mathbb{R})$.

Subsequent to our work, a new result [122] for gKdV demonstrates local well-posedness for perturbations in $H^s(\mathbb{R})$, $s > \frac{1}{2}$ and global well-posedness for $s \geq 1$. In addition to a larger class of equations, this work also applies to a wide variety of background waves, including both step-like and periodic asymptotics. In particular, the background wave is not assumed to be time-independent nor an exact solution, but rather is allowed to solve the equation modulo a localized error term.

The primary tool used in the literature to study step-like solutions of KdV has been the inverse scattering transform. In the case of a highly regular step-like background, existence for the Cauchy problem has been examined in [33, 40, 41, 52, 53, 85]. In order to employ the inverse scattering transform these results assume that $u(0) - W$ is integrable against $1+|x|^N$ for some $N \geq 1$, and consequently $H^s(\mathbb{R})$ spaces are not amenable to such methods. Nevertheless, these methods do yield existence for Schwartz class perturbations [52]. Classes of one-sided step-like initial data were treated in [69,127,128] and one-sided step-like elements of $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ were treated in [68]. Despite the lack of assumptions as $x \to -\infty$ (the direction in which radiation propagates), these low-regularity arguments require rapid decay as $x \to +\infty$ and global boundedness from below. By comparison, our argument is symmetric in $\pm x$ and in $\pm u$.

The inverse scattering transform is also used to study the long-time behavior of such solutions; see, for example, [7, 13, 18–20, 51, 76, 91, 92, 101, 102, 120]. The asymptotics are spatially asymmetric and differ in the cases of tidal bores and rarefaction waves.

To prove Theorem 3.1.1, we will employ the method of commuting flows introduced in [97]. This method was used to prove both symplectic non-squeezing [121] and invariance of white noise [94] for KdV on the line. The method of commuting flows has also been adapted to other completely integrable systems [30, 72, 73, 93, 95]. However, aside from the white noise result [94], this marks the first application of the method of commuting flows to nontrivial spatial asymptotics.

Unlike previous applications of the method of commuting flows, the presence of the background wave W breaks all of the conservation laws. A solution u of KdV (1.1.1) necessarily obeys the microscopic conservation law

$$\frac{d}{dt}\left(\tfrac{1}{2}u^2\right) = \left[-uu'' + \tfrac{1}{2}(u')^2 + 2u^3\right]'.$$

For Schwartz solutions u, integrating in space yields macroscopic conservation of the momentum $E_1(u)$ (defined in (1.1.3)). However, if merely $u - W$ is Schwartz, then we obtain

$$\frac{d}{dt} \int \tfrac{1}{2} \left[u(t,x)^2 - u(0,x)^2 \right] \, dx = 2W(x)^3 \Big|_{x=-\infty}^{x=+\infty}. \tag{3.1.3}$$

In the case $c_1 > 0$, $c_2 = 0$ of an incoming tide, the RHS is equal to $4c_1^3 > 0$. The momentum's growth is manifested in a dispersive shock that develops in the long-time asymptotics [51, Fig. 1].

Interpreting W as an incoming or outgoing tide, we will refer to (3.1.2) as *tidal KdV*. To prove Theorem 3.1.1 we will show that tidal KdV is well-posed in $H^s(\mathbb{R})$ for $s \geq 3$. Computations similar to (3.1.3) show that the presence of W in tidal KdV breaks all of the polynomial conservation laws of KdV. Despite this, we are able to adapt the method of commuting flows to tidal KdV because these conserved quantities do not blow up in finite time.

In the case $W \equiv 0$, the authors of [97] introduced the Hamiltonians H_κ defined by (1.4.7), and showed that their flows converge to that of KdV in $H^{-1}(\mathbb{R})$ as $\kappa \to \infty$. These H_κ flows are easier to work with; in particular, well-posedness follows from straightforward ODE arguments. Moreover, two H_κ flows with different energy parameters κ commute with one another, which greatly benefits the proof of convergence as $\kappa \to \infty$.

As the H_κ flows approximate KdV, we will need to construct analogous approximate equations for tidal KdV (3.1.2). Just as how we obtained tidal KdV from KdV, we subtract the background wave W from u to obtain the tidal H_κ flow for $q = u - W$ with Hamiltonian H_κ^W:

$$e^{tJ\nabla H_\kappa^W} q = e^{tJ\nabla H_\kappa}(q + W) - W.$$

This tidal H_κ flow is indeed Hamiltonian, but we will not need the formula for the Hamiltonian; we only formally introduce H_κ^W so that we have a succinct notation for its flow. In proving Corollary 4.1.4 and Theorem 3.1.1, we will show that the tidal H_κ flow is well-posed in $H^s(\mathbb{R})$ for $s \geq 3$, commutes with any other tidal $H_\varkappa$ flow, and converges to tidal KdV in

$H^s(\mathbb{R})$ as $\kappa \to \infty$ uniformly on bounded time intervals.

This chapter is organized as follows. In Section 3.2 we define the diagonal Green's function for perturbations $q \in H^{-1}(\mathbb{R})$ of the background W, which we will use to formulate the tidal H_κ flow. In Section 3.3 we prove *a priori* estimates and global well-posedness for the tidal H_κ flow. As a stepping stone to convergence in H^s-norm, we prove in Section 3.4 that the tidal H_κ flow converges in the weaker H^{-2}-norm. The entirety of Section 3.5 is dedicated to controlling the Fourier tail growth in time. We then combine the low-regularity convergence and Fourier tail control in Section 3.6 to obtain convergence in H^s-norm and conclude our main result.

3.2 Diagonal Green's function

We begin by reviewing our notation and the necessary tools from [97], which can be consulted for further details.

For a Sobolev space $W^{k,p}(\mathbb{R})$ we use the spacetime norm

$$\|q\|_{C_t W^{k,p}(I\times\mathbb{R})} := \sup_{t\in I} \|q(t)\|_{W^{k,p}(\mathbb{R})}$$

for $I \subset \mathbb{R}$ an interval. In addition to the usual Sobolev spaces $W^{k,p}$ and H^s, we define the norm

$$\|f\|^2_{H^s_\kappa(\mathbb{R})} := \int_{\mathbb{R}} (\xi^2 + 4\kappa^2)^s |\hat{f}(\xi)|^2 \, d\xi. \tag{3.2.1}$$

The presence of the factor of four is to make the calculation (3.2.3) an exact identity. Our convention for the Fourier transform is

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx, \qquad \|\hat{f}\|_{L^2} = \|f\|_{L^2}\,.$$

In analogy with the usual H^s spaces, we have the elementary facts

$$\|wf\|_{H^{\pm 1}_\kappa} \lesssim \|w\|_{W^{1,\infty}} \|f\|_{H^{\pm 1}_\kappa}\,, \qquad \|wf\|_{H^{\pm 1}_\kappa} \lesssim \|w\|_{H^1} \|f\|_{H^{\pm 1}_\kappa} \tag{3.2.2}$$

uniformly for $\kappa \geq 1$. We will exclusively use the L^2 pairing $\langle \cdot, \cdot \rangle$; the space H^{-1}_κ is dual to H^1_κ with respect to this pairing, and so the inequalities (3.2.2) for H^{-1}_κ are implied by those for H^1_κ.

We write $\mathfrak{I}_p$ for the Schatten classes (also called trace ideals) of compact operators on the Hilbert space $L^2(\mathbb{R})$ whose singular values are ℓ^p-summable. Of particular importance will be the Hilbert–Schmidt class $\mathfrak{I}_2$: recall that an operator A on $L^2(\mathbb{R})$ is Hilbert–Schmidt if and only if it admits an integral kernel $a(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$, and we have

$$\|A\|_{\mathrm{op}} \leq \|A\|_{\mathfrak{I}_2} = \left(\iint |a(x,y)|^2 \, dx \, dy \right)^{1/2}.$$

The product of two Hilbert–Schmidt operators A and B is of trace class $\mathfrak{I}_1$, the trace is cyclic:

$$\operatorname{tr}(AB) := \iint a(x,y) b(y,x) \, dy \, dx = \operatorname{tr}(BA),$$

and we have the estimate

$$|\operatorname{tr}(AB)| \leq \|A\|_{\mathfrak{I}_2} \|B\|_{\mathfrak{I}_2}.$$

Additionally, Hilbert–Schmidt operators form a two-sided ideal in the algebra of bounded operators, due to the inequality

$$\|BAC\|_{\mathfrak{I}_2} \leq \|B\|_{\mathrm{op}} \|A\|_{\mathfrak{I}_2} \|C\|_{\mathrm{op}}.$$

We notate the resolvent of the Schrödinger operator with zero potential by

$$R_0(\kappa) := \left(-\partial_x^2 + \kappa^2 \right)^{-1} \quad \text{with integral kernel} \quad \langle \delta_x, R_0(\kappa) \delta_y \rangle = \tfrac{1}{2\kappa} e^{-\kappa|x-y|}.$$

The energy parameter κ will always be real and positive. Consequently, $R_0(\kappa)$ will always be positive definite and so we may consider its positive definite square-root $\sqrt{R_0(\kappa)}$.

The following calculation is the basis for all of the analysis that follows.

Lemma 3.2.1 (Key estimate [97, Prop. 2.1])**.** *For $q \in H^{-1}(\mathbb{R})$ we have*

$$\left\| \sqrt{R_0(\kappa)} \, q \sqrt{R_0(\kappa)} \right\|_{\mathfrak{I}_2}^2 = \frac{1}{\kappa} \int \frac{|\hat{q}(\xi)|^2}{\xi^2 + 4\kappa^2} \, d\xi = \frac{1}{\kappa} \|q\|_{H^{-1}_\kappa}^2. \tag{3.2.3}$$

The identity (3.2.3) guarantees that the Neumann series for the resolvent of $-\partial^2 + q$ converges for all κ sufficiently large when q belongs to a bounded subset of $H^{-1}_\kappa(\mathbb{R})$. Consequently, we will always be working within the closed balls

$$B_A(\kappa) := \{q \in H^{-1}(\mathbb{R}) : \|q\|_{H^{-1}_\kappa} \leq A\}, \qquad B_A := \{q \in H^{-1}(\mathbb{R}) : \|q\|_{H^{-1}} \leq A\} \tag{3.2.4}$$

of radius $A > 0$. Note that $B_A(\kappa) \supset B_A$ for $\kappa \geq 1$, and so any result obtained for $B_A(\kappa)$ with $\kappa \geq 1$ also holds for the fixed set B_A. The resolvent construction also works for $q \in B_A(\kappa)$ perturbations of a background wave $V \in L^\infty$:

Lemma 3.2.2 (Resolvents)**.** *Fix $V \in L^\infty(\mathbb{R})$. Given $q \in H^{-1}(\mathbb{R})$, there exists a unique self-adjoint operator corresponding to $-\partial_x^2 + V + q$ with domain $H^1(\mathbb{R})$. Moreover, given $A > 0$ there exists $\kappa_0 > 0$ so that the series*

$$R(\kappa, V) = (-\partial^2 + V + \kappa^2)^{-1} = \sum_{\ell=0}^\infty (-1)^\ell \sqrt{R_0}\big(\sqrt{R_0} V \sqrt{R_0}\big)^\ell \sqrt{R_0} \tag{3.2.5}$$

converges absolutely to a positive definite operator for $\kappa \geq \kappa_0$, and the series

$$R(\kappa, V+q) = \sum_{\ell=0}^\infty (-1)^\ell \sqrt{R(\kappa,V)}\big[\sqrt{R(\kappa,V)}\, q \sqrt{R(\kappa,V)}\big]^\ell \sqrt{R(\kappa,V)} \tag{3.2.6}$$

converges absolutely for $q \in B_A(\kappa)$ and $\kappa \geq \kappa_0$.

Proof. Initially we require that $\kappa \geq 1$. As $V \in L^\infty$, we may define the operator $-\partial^2 + V$ via the quadratic form

$$\phi \mapsto \int \big(|\phi'(x)|^2 + V(x)|\phi(x)|^2\big)\, dx$$

equipped with the domain $H^1(\mathbb{R})$. Using the elementary estimates $\|R_0\|_{\mathrm{op}} \leq \kappa^{-2}$ and $\|V\|_{\mathrm{op}} \leq \|V\|_{L^\infty}$, it is clear that the Neumann series (3.2.5) for $R(\kappa, V)$ is absolutely convergent for all $\kappa^2 \geq 2\,\|V\|_{L^\infty}$. Once we know the series absolutely converges, it is straightforward to verify that multiplying by $-\partial^2 + V + \kappa^2$ produces the identity operator.

Expanding the series (3.2.5) and using the identity (3.2.3) we estimate

$$\Big\|\sqrt{R(\kappa,V)}\, q \sqrt{R(\kappa,V)}\Big\|_{\mathfrak{I}_2}^2 = \operatorname{tr}\{R(\kappa,V) q R(\kappa,V) \overline{q}\}$$

$$\leq \sum_{\ell,m=0}^{\infty} \left\| \sqrt{R_0} V \sqrt{R_0} \right\|_{\mathrm{op}}^{\ell+m} \left\| \sqrt{R_0} q \sqrt{R_0} \right\|_{\mathfrak{I}_2}^2 \leq 4\kappa^{-1} \|q\|_{H_\kappa^{-1}}^2$$

for all $\kappa^2 \geq 2\|V\|_{L^\infty}$, and hence

$$\left\| \sqrt{R(\kappa,V)}\, q \sqrt{R(\kappa,V)} \right\|_{\mathfrak{I}_2} \leq 2\kappa^{-1/2} \|q\|_{H_\kappa^{-1}}. \tag{3.2.7}$$

Consequently, given $q \in B_A(\kappa)$ we have

$$\begin{aligned} \int q(x)|\phi(x)|^2\,dx &\leq \left\| \sqrt{R(\kappa,V)}\, q \sqrt{R(\kappa,V)} \right\|_{\mathrm{op}} \int \left(|\phi'(x)|^2 + |V(x)||\phi(x)|^2 \right) dx \\ &\leq \tfrac{1}{2} \int \left(|\phi'(x)|^2 + |V(x)||\phi(x)|^2 \right) dx \end{aligned}$$

for all $\phi \in H^1(\mathbb{R})$ provided that $\kappa \geq 16A^2$. We conclude that $-\partial^2 + V + q$ is a form-bounded perturbation of $-\partial^2 + V$ with relative norm strictly less than 1; this guarantees that $-\partial^2 + V + q$ exists, is unique, and has the same form domain $H^1(\mathbb{R})$ (cf. [125, Th. X.17]). The estimate (3.2.7) then demonstrates that the series (3.2.6) for $R(\kappa, V+q)$ is absolutely convergent for all $\kappa \geq 16A^2$. □

In [97] the diagonal Green's function—the restriction of the kernel $G(x,y;\kappa,q)$ of the operator $R(\kappa,q)$ to the diagonal—was instrumental in controlling q in H^{-1}. This construction also works for $q \in B_A(\kappa)$ perturbations of V:

Proposition 3.2.3 (Diagonal Green's function)**.** *Fix $V \in L^\infty(\mathbb{R})$. Given $A > 0$ there exists $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$ the diagonal Green's function $g(x;\kappa,V+q) := G(x,x;\kappa,V+q)$ exists for $q \in B_A(\kappa)$, the two functionals*

$$q \mapsto g(x;\kappa,V+q) - g(x;\kappa,V) \quad \textit{and} \quad q \mapsto \frac{1}{g(x;\kappa,V+q)} - \frac{1}{g(x;\kappa,q)} \tag{3.2.8}$$

are real analytic from $B_A(\kappa)$ into $H_\kappa^1(\mathbb{R})$, and we have the estimate

$$\|g(x;\kappa,V+q) - g(x;\kappa,V)\|_{H_\kappa^1} \lesssim \kappa^{-1} \|q\|_{H_\kappa^{-1}} \tag{3.2.9}$$

uniformly for $q \in B_A(\kappa)$ and $\kappa \geq \kappa_0$.

Moreover, in the case where V is the smooth step function W (defined in (3.1.1)), for any integer $s \geq 0$ and $A > 0$ there exists $\kappa_0 > 0$ so that

$$\|g(x;\kappa, W+q) - g(x;\kappa, W)\|_{H^{s+2}_\kappa} \lesssim \kappa^{-1} \|q\|_{H^s_\kappa} \tag{3.2.10}$$

uniformly for $\|q\|_{H^s} \leq A$ and $\kappa \geq \kappa_0$.

Proof. In Fourier variables, we have

$$\|\sqrt{R_0}\delta_x\|^2_{L^2} \lesssim \kappa^{-1}, \qquad \|\sqrt{R_0}\delta_{x+h} - \sqrt{R_0}\delta_x\|^2_{L^2} \leq \int \frac{|e^{i\xi h} - 1|^2}{\xi^2 + 1}\, d\xi \lesssim |h|$$

for $\kappa \geq 1$. This demonstrates that $x \mapsto \sqrt{R_0}\delta_x$ is $\frac{1}{2}$-Hölder continuous as a map from $\mathbb{R}$ to L^2. We initialize κ_0 to be the constant from Lemma 3.2.2. Then from the series (3.2.5) we see that

$$\begin{aligned} &|\langle \delta_x, [R(\kappa, V) - R_0(\kappa)]\, \delta_y\rangle - \langle \delta_{x'}, [R(\kappa, V) - R_0(\kappa)]\, \delta_{y'}\rangle| \\ &\lesssim \kappa^{-1/2}\big(|x - x'|^{1/2} + |y - y'|^{1/2}\big) \sum_{\ell=1}^{\infty} \big(\kappa^{-2}\, \|V\|_{L^\infty}\big)^\ell . \end{aligned}$$

The series converges provided that $\kappa \gg \|V\|^{1/2}_{L^\infty}$. Consequently, the Green's function $G(x,y) = \langle \delta_x, R(\kappa, V)\delta_y\rangle$ is continuous in both x and y, and so we may unambiguously define

$$g(x;\kappa, V) = \tfrac{1}{2\kappa} + \sum_{\ell=1}^{\infty} (-1)^\ell \big\langle \sqrt{R_0}\delta_x, (\sqrt{R_0} V \sqrt{R_0})^\ell \sqrt{R_0}\delta_x \big\rangle. \tag{3.2.11}$$

The zeroth-order term $\frac{1}{2\kappa}$ can be seen directly from the integral kernel for the free resolvent $R_0(\kappa)$.

Similarly, from the series (3.2.6) and the estimate (3.2.7) we have

$$\begin{aligned} &|\langle \delta_x, [R(\kappa, V+q) - R(\kappa, V)]\, \delta_y\rangle - \langle \delta_{x'}, [R(\kappa, V+q) - R(\kappa, V)]\, \delta_{y'}\rangle| \\ &\lesssim \kappa^{-1/2}\big(|x - x'|^{1/2} + |y - y'|^{1/2}\big) \sum_{\ell=1}^{\infty} \big(2\kappa^{-1/2} A\big)^\ell \end{aligned}$$

for all $q \in B_A(\kappa)$. The series converges provided that we also have $\kappa \gg A^2$. Therefore $G(x, y; \kappa, V+q)$ is also a continuous function of x and y and so we may define

$$g(x;\kappa, V+q) = g(x;\kappa, V) + \sum_{\ell=1}^{\infty} (-1)^\ell \big\langle \sqrt{R}\delta_x, (\sqrt{R}\, q \sqrt{R})^\ell \sqrt{R}\delta_x \big\rangle$$

where $R = R(\kappa, V)$. This shows that the first functional of (3.2.8) is real analytic for $q \in B_A(\kappa)$.

Next we check that $g(x; \kappa, V + q) - g(x; \kappa, V)$ is in H^1_κ by duality and the operator estimate (3.2.7):

$$\left| \int f(x)[g(\kappa, V + q) - g(\kappa, V)](x)\, dx \right|$$
$$\leq \sum_{\ell=1}^{\infty} \left\| \sqrt{R(\kappa, V)} f \sqrt{R(\kappa, V)} \right\|_{\mathfrak{I}_2} \left\| \sqrt{R(\kappa, V)}\, q \sqrt{R(\kappa, V)} \right\|_{\mathfrak{I}_2}^{\ell} \lesssim \kappa^{-1} \|f\|_{H^{-1}_\kappa} \|q\|_{H^{-1}_\kappa}.$$

Taking a supremum over all $\|f\|_{H^{-1}_\kappa} \leq 1$ we obtain the estimate (3.2.9).

Now we will show that $g(x; \kappa, V + q)$ is nonvanishing so that the second functional of (3.2.8) is also real analytic. Using the series (3.2.5) we estimate

$$|g(x; \kappa, V) - g(x; \kappa, 0)| \leq \|\sqrt{R_0}\delta_x\|^2_{L^2} \sum_{\ell=1}^{\infty} \left(\kappa^{-2} \|V\|_{L^\infty}\right)^{\ell} \lesssim \kappa^{-3}$$

for $\kappa \gg \|V\|^{1/2}_{L^\infty}$. As $g(x; \kappa, 0) \equiv \frac{1}{2\kappa}$, we can take κ_0 larger if necessary to ensure

$$\tfrac{1}{4\kappa} \leq g(x; \kappa, V) \leq \tfrac{3}{4\kappa}$$

for all $\kappa \geq \kappa_0$. The estimate (3.2.9) combined with the observation

$$\|f\|_{L^\infty} \leq \|f\|^{1/2}_{L^2} \|f'\|^{1/2}_{L^2} \lesssim \kappa^{-1/2} \|f\|_{H^1_\kappa} \tag{3.2.12}$$

then guarantees that there exists $\kappa_0 \gg A^2$ so that

$$\|g(x; \kappa, V + q) - g(x; \kappa, V)\|_{L^\infty} \leq \tfrac{1}{8\kappa}$$

for all $q \in B_A(\kappa)$ and $\kappa \geq \kappa_0$. Consequently, the second functional of (3.2.8) is also real-analytic.

Finally, given $s \geq 0$, we check that $g(x; \kappa, W + q) - g(x; \kappa, W)$ is in H^{s+2}_κ by estimating the first $s + 1$ derivatives in H^1_κ by duality. The Green's function for a translated potential is the translation of the original Green's function:

$$g(x; \kappa, q(\cdot + h)) = g(x + h; \kappa, q) \quad \text{for all } h \in \mathbb{R}. \tag{3.2.13}$$

Differentiating (3.2.13) at $h = 0$ and using the resolvent identity, we have

$$g^{(j)}(x;\kappa, W+q) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\big\langle \delta_x, [\partial^j, R(\kappa,W)(qR(\kappa,W))^{\ell}\delta_x]\big\rangle. \tag{3.2.14}$$

Here, $[A,B] = AB - BA$ denotes the commutator and ∂^j denotes j spatial partial derivatives. Within the summand there are $\ell+1$ factors of $R(\kappa,W)$, and we expand each into the series (3.2.5) in powers of W indexed by m_i. For $j = 0,\dots,s+1$ and $f \in H^{-1}_{\kappa}$, this yields

$$\begin{aligned}&\left|\int f(x)[g(\kappa,W+q) - g(\kappa,W)]^{(j)}(x)\,dx\right| \\ &\leq \sum_{\ell=1}^{\infty}\sum_{m_0,\dots,m_\ell=0}^{\infty}\big|\operatorname{tr}\big\{f[\partial^j, R_0(WR_0)^{m_0}qR_0\cdots qR_0(WR_0)^{m_\ell}]\big\}\big|.\end{aligned}$$

We distribute the derivatives $[\partial^j,\cdot]$ using the product rule. We use the operator estimate (3.2.3) for each factor of $\sqrt{R_0}\,q\sqrt{R_0}$ and estimate the remaining factors in operator norm. Given a multiindex $\sigma \in \mathbb{N}^{\ell}$ with $|\sigma| \leq j$, Hölder's inequality in Fourier variables yields

$$\prod_{i=1}^{\ell}\|q^{(\sigma_j)}\|_{H^{-1}_{\kappa}} \leq \|q^{(|\sigma|)}\|_{H^{-1}_{\kappa}}\|q\|_{H^{-1}_{\kappa}}^{\ell-1} \leq \|q\|_{H^{j-1}_{\kappa}}\|q\|_{H^{-1}_{\kappa}}^{\ell-1}.$$

As $j \leq s+1$, we have

$$\begin{aligned}&\left|\int f(x)[g(\kappa,W+q) - g(\kappa,W)]^{(j)}(x)\,dx\right| \\ &\leq \sum_{\ell=1}^{\infty}\sum_{m_0,\dots,m_\ell=0}^{\infty}\frac{\|f\|_{H^{-1}_{\kappa}}\|q\|_{H^s_{\kappa}}}{\kappa}\left(\frac{\|q\|_{H^{-1}_{\kappa}}}{\kappa^{1/2}}\right)^{\ell-1}\left(\frac{\|W\|_{W^{s+1,\infty}}}{\kappa^2}\right)^{m_0+\cdots+m_\ell}.\end{aligned}$$

First we perform the inner sum over $m_0,\dots,m_\ell$; re-indexing $m = m_0+\cdots+m_\ell$, we have

$$\begin{aligned}\sum_{m_0,\dots,m_\ell\geq 0}\left(\frac{\|W\|_{W^{s+1,\infty}}}{\kappa^2}\right)^{m_0+\cdots+m_\ell} &= \sum_{m=0}^{\infty}\frac{(\ell+m)!}{\ell!\,m!}\left(\frac{\|W\|_{W^{s+1,\infty}}}{\kappa^2}\right)^{m} \\ &\lesssim \left(1 - \frac{\|W\|_{W^{s+1,\infty}}}{\kappa^2}\right)^{\ell+1} \leq 1\end{aligned} \tag{3.2.15}$$

uniformly in ℓ, provided that $\kappa \gg \|W\|_{W^{s+1,\infty}}^{1/2}$. The sum over $\ell \geq 1$ then converges uniformly for $\kappa \gg A^2$, yielding

$$\left|\int f[g(\kappa,W+q) - g(\kappa,W)]^{(j)}\,dx\right| \lesssim \kappa^{-1}\,\|f\|_{H^{-1}_{\kappa}}\,\|q\|_{H^s_{\kappa}} \quad \text{for } j = 0,\dots,s+1.$$

Taking a supremum over $\|f\|_{H^{-1}_{\kappa}} \leq 1$, we obtain the estimate (3.2.10). □

As an offspring of the resolvent $R(\kappa, q)$, the diagonal Green's function comes with some algebraic identities. In particular, in [97, Lem. 2.5–2.6] it is shown that for Schwartz q we have the identities

$$\int \frac{G(x,y;\kappa,q)G(y,x;\kappa,q)}{2g(y;\kappa,q)^2}\,dy = g(x;\kappa,q) \tag{3.2.16}$$

and

$$\begin{aligned}&\int G(x,y;\kappa,q)\big[-f''' + 2qf' + 2(qf)' + 4\kappa^2 f'\big](y)G(y,x;\kappa,q)\,dy \\ &= 2f'(x)g(x;\kappa,q) - 2f(x)g'(x;\kappa,q)\end{aligned} \tag{3.2.17}$$

for all Schwartz f. To show that these hold for general $q \in B_A(\kappa)$, we argue as follows. Given $A > 0$, we pick κ_0 from Proposition 3.2.3. Then both sides are analytic in q, and so equality follows from the proofs [97, Lem. 2.5–2.6] for the Schwartz case.

As is suggested by taking $f = g(\kappa, q)$ in (3.2.17), multiplying by $1/2g(x;\kappa,q)^2$, and integrating in x, the diagonal Green's function satisfies the ODE

$$g'''(\kappa,q) = 2qg'(\kappa,q) + 2\,[qg(\kappa,q)]' + 4\kappa^2 g'(\kappa,q); \tag{3.2.18}$$

see [97, Prop. 2.3] for a proof.

Ultimately, the convergence of the approximate flows will be dominated by the linear and quadratic terms of the series (3.2.5) for the diagonal Green's function. Consequently, we will now record some useful operator identities for these two terms:

Lemma 3.2.4. *For $\kappa \geq 1$ we have the operator identities*

$$16\kappa^5\langle \delta_x, R_0 f R_0 \delta_x\rangle = 16\kappa^4 R_0(2\kappa) f = \big[4\kappa^2 + \partial^2 + R_0(2\kappa)\partial^4\big]\,f, \tag{3.2.19}$$

$$\begin{aligned}16\kappa^5\langle \delta_x, R_0 f R_0 h R_0 \delta_x\rangle = 3fh &- 3[R_0(2\kappa)f''][R_0(2\kappa)h''] \\ &+ 4\kappa^2[R_0(2\kappa)f'][R_0(2\kappa)h'](-5 + R_0(2\kappa)\partial^2) \\ &+ 4\kappa^2[R_0(2\kappa)f][R_0(2\kappa)h](5\partial^2 + 2R_0(2\kappa)\partial^4),\end{aligned} \tag{3.2.20}$$

where $R_0 = R_0(\kappa)$.

Proof. From the integral kernel formula for $R_0(\kappa)$ we see that $\langle \delta_x, R_0 f R_0 \delta_x \rangle = \kappa^{-1} R_0(2\kappa) f$, which demonstrates the first equality of (3.2.19). The second equality follows from the symbol identity

$$\frac{16\kappa^4}{\xi^2 + 4\kappa^2} = 4\kappa^2 - \xi^2 + \frac{\xi^4}{\xi^2 + 4\kappa^2}$$

in Fourier variables.

Now we turn to the second identity (3.2.20). In [97, Appendix] the Fourier transform of LHS(3.2.20) is found to be

$$\mathcal{F}\big(\text{LHS(3.2.20)}\big)(\xi) = \frac{8\kappa^4}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{[\xi^2 + (\xi-\eta)^2 + \eta^2 + 24\kappa^2] \hat{f}(\xi-\eta) \hat{h}(\eta)}{(\xi^2 + 4\kappa^2)((\xi-\eta)^2 + 4\kappa^2)(\eta^2 + 4\kappa^2)} \, d\eta.$$

The operator identity (3.2.20) then follows from the equality

$$\begin{aligned} \frac{8\kappa^4 \, [\xi^2 + (\xi-\eta)^2 + \eta^2 + 24\kappa^2]}{(\xi^2 + 4\kappa^2)((\xi-\eta)^2 + 4\kappa^2)(\eta^2 + 4\kappa^2)} &= 3 - \frac{3\eta^2(\xi-\eta)^2}{((\xi-\eta)^2 + 4\kappa^2)(\eta^2 + 4\kappa^2)} \\ - \frac{20\kappa^2 \, [-\eta(\xi-\eta) + \xi^2]}{((\xi-\eta)^2 + 4\kappa^2)(\eta^2 + 4\kappa^2)} &+ \frac{4\kappa^2 \xi^2 \, [\eta(\xi-\eta) + 2\xi^2]}{(\xi^2 + 4\kappa^2)((\xi-\eta)^2 + 4\kappa^2)(\eta^2 + 4\kappa^2)}. \end{aligned}$$

□

We will also need to know that after extracting the linear and quadratic terms from $\kappa^5 g(\kappa, q+W)$, the remainder tends to zero as $\kappa \to \infty$:

Lemma 3.2.5. *Given an integer $s \geq 1$ and $A > 0$, we have*

$$\begin{aligned} \kappa^5 \big\| \big\{ g(\kappa, q+W) + \langle \delta_x, R_0(q+W) R_0 \delta_x \rangle \\ - \langle \delta_x, R_0(q+W) R_0(q+W) R_0 \delta_x \rangle \big\}^{(s+1)} \big\|_{L^2} \to 0 \quad \text{as } \kappa \to \infty \end{aligned} \tag{3.2.21}$$

uniformly for $\|q\|_{H^s} \leq A$.

Proof. We estimate the sth derivative in H^1 by duality. Differentiating the translation identity (3.2.14) at $h = 0$, we have

$$g^{(s)}(x; \kappa, W+q) = \sum_{\ell=0}^{\infty} (-1)^\ell \big\langle \delta_x, [\partial^s, R(\kappa, W)(qR(\kappa, W))^\ell \delta_x] \big\rangle.$$

Within the summand there are $\ell+1$ factors of $R(\kappa, W)$, and we expand each into the series (3.2.5) in powers of W indexed by m_i. For $f \in H^{-1}$ this yields

$$\kappa^5 \left| \int f(x) \big\{ g(\kappa, q+W) + \langle \delta_x, R_0(q+W)R_0\delta_x \rangle - \langle \delta_x, R_0(q+W)R_0(q+W)R_0\delta_x \rangle \big\}^{(s)} \, dx \right|$$
$$\leq \kappa^5 \sum_{\substack{\ell \geq 0,\ m_0,\dots,m_\ell \geq 0 \\ \ell+m_0+\cdots+m_\ell \geq 3}} \Big| \operatorname{tr}\big\{ f[\partial^s, R_0(WR_0)^{m_0} qR_0 \cdots qR_0(WR_0)^{m_\ell}] \big\} \Big|. \tag{3.2.22}$$

We distribute the derivatives $[\partial^s, \cdot]$ using the product rule. We then use the operator estimate (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1} \|f\|_{L^2}$ to put the highest order q in L^2. In the instance that there are no factors of q, we put the highest order W term in L^2 and use that W' is in H^{s-1}. We then estimate all other terms in operator norm; the remaining factors of q have at most $s-1$ derivatives, and thus may be estimated in L^∞ via the embedding $H^1 \hookrightarrow L^\infty$. This yields

$$\text{RHS(3.2.22)} \lesssim \kappa^5 \sum_{\substack{\ell \geq 0,\ m_0,\dots,m_\ell \geq 0 \\ \ell+m_0+\cdots+m_\ell \geq 3}} \frac{\|f\|_{H^{-1}}}{\kappa^{1/2}} \frac{\max\{\|q\|_{H^s}, \|W'\|_{H^{s-1}}\}}{\kappa^{3/2}} \times \left(\frac{\max\{\|q\|_{H^s}, \|W\|_{W^{s,\infty}}\}}{\kappa^2} \right)^{\ell+m_0+\cdots+m_\ell-1}.$$

We re-index $m = m_0 + \cdots + m_\ell$ and sum over $\ell + m \geq 3$ as in (3.2.15). The sum converges provided $\kappa \gg \|q\|_{H^s}^{1/2}$ and $\kappa \gg \|W\|_{W^{s,\infty}}^{1/2}$. The condition $\ell + m \geq 3$ guarantees that when we sum over the parenthetical term we gain a factor $\lesssim (\kappa^{-2})^2$, and so we obtain

$$\text{RHS(3.2.22)} \lesssim \kappa^{-1} \|f\|_{H^{-1}}$$

uniformly for $\|q\|_{L^2} \leq A$ and $\kappa \geq \kappa_0(A)$. The claim (3.2.21) follow by taking a supremum over $\|f\|_{H^{-1}} \leq 1$. $\square$

3.3 Tidal H_κ flow

The argument of [97] relies upon the Hamiltonians H_κ defined in (1.4.7), whose flows approximate that of KdV as $\kappa \to \infty$. Specifically, in [97, Prop. 3.2] it is shown that the H_κ

flow can be expressed in terms of the diagonal Green's function as

$$\frac{d}{dt}u = 16\kappa^5 g'(\kappa, u) + 4\kappa^2 u'. \tag{3.3.1}$$

Moreover, the flows at any two energy parameters κ and $\varkappa$ commute:

$$\{H_\kappa, H_\varkappa\} = 0. \tag{3.3.2}$$

We need an analogous approximate flow for step-like initial data. Mimicking how we obtained tidal KdV from KdV, we subtract the background W from the function u to obtain the tidal H_κ flow

$$\frac{d}{dt}q = 16\kappa^5 g'(\kappa, q + W) + 4\kappa^2 (q + W)' \tag{3.3.3}$$

for $q := u - W$. The tidal H_κ flow is also Hamiltonian; however, we will not need the exact formula for its Hamiltonian.

In this section we will show that the tidal H_κ flow is globally well-posed in H^s for all integers $s \geq 0$. We restrict our attention to integer s since the result for non-integer $s \geq 0$ follows from interpolation. Once we obtain well-posedness, the commutativity (3.3.2) of the H_κ flows implies that any two tidal H_κ flows commute with each other.

We begin with local well-posedness. The H_κ flows are easier to work with because local well-posedness follows from a contraction mapping argument.

Lemma 3.3.1. *Given an integer $s \geq -1$ and $A > 0$, there exists a constant κ_0 so that for $\kappa \geq \kappa_0$ the tidal H_κ flows* (3.3.3) *with initial data in the closed ball $B_A^s \subset H^s(\mathbb{R})$ of radius A are locally well-posed.*

Proof. Fix an integer $s \geq -1$. The solution $q(t)$ to the tidal H_κ flow satisfies the integral equation

$$q(t) = e^{t4\kappa^2\partial_x} q(0) + \int_0^t e^{(t-\tau)4\kappa^2\partial_x} \left[16\kappa^5 g'(\kappa, q(\tau) + W) + 4\kappa^2 W'\right] d\tau.$$

A contraction mapping argument proves local well-posedness, provided we have the Lipschitz estimate

$$\begin{aligned}&\|g'(\kappa, q+W) - g'(\kappa, \tilde{q}+W)\|_{H^s}\\ &\lesssim \|[g(\kappa, q+W) - g(\kappa, W)] - [g(\kappa, \tilde{q}+W) - g(\kappa, W)]\|_{H^{s+2}} \lesssim \|q-\tilde{q}\|_{H^s}\end{aligned}$$

uniformly on bounded subsets of H^s.

Fix $A > 0$. It suffices to show that $f \mapsto d[g(\kappa, \cdot + W)]|_q(f)$ is bounded $H^s \to H^{s+2}$ uniformly for $\|q\|_{H^s} \leq A$. Using the resolvent identity we calculate

$$d[g(\kappa, \cdot + W)]|_q(f) = -\langle \delta_x, R(\kappa, q+W) f R(\kappa, q+W)\delta_x\rangle.$$

Just as we did for the single resolvent $\langle \delta_x, R(\kappa, q+W)\delta_x\rangle$ in (3.2.10), we estimate the first $s+1$ derivatives in H^1 by duality and expand each resolvent into a series. We conclude that there exists a constant κ_0 such that

$$\|d[g(\kappa, \cdot + W)]|_q(f)\|_{H^{s+2}} \lesssim \|f\|_{H^s}$$

uniformly for $q \in B_A^s$ and $\kappa \geq \kappa_0$. □

In order to obtain global well-posedness, we will prove *a priori* estimates in H^s for all integers $s \geq 0$. Our energy arguments are inspired by those of Bona and Smith [24]. The family of BBM equations which Bona–Smith uses to approximate the KdV flow does not conserve the polynomial conserved quantities of KdV. One benefit of our method is that in the case $W \equiv 0$, the H_κ flows do conserve these quantities (as is suggested by the asymptotic expansion (1.4.6) and Poisson commutativity (1.4.1)), and consequently the *a priori* estimates are identical to that of KdV. In particular, in the case $W \equiv 0$ we obtain a new proof of the Bona–Smith theorem using the low-regularity methods from [97]. (This is not subsumed by [97, Cor. 5.3], which only addresses $H^s(\mathbb{R})$ for $s \in [-1, 0)$.)

Our energy arguments are much simplified in the case $\kappa = \infty$, where the tidal H_κ flow becomes tidal KdV. Our manipulations are motivated by the corresponding tidal KdV terms at $\kappa = \infty$, where operations involving commutators and cycling the trace correspond to more elementary operations involving integration by parts. In particular, the reason for the restriction $s \geq 3$ is the same as in [24]: when estimating $\frac{d}{dt}\|q^{(s)}(t)\|_{L^2}^2$ under the KdV flow, $s = 3$ is the smallest integer for which the nonlinear contribution can be estimated in terms of $\|q^{(s)}(t)\|_{L^2}^2$ provided that we already control $q(t)$ in H^{s-1}.

We begin with $s = 0$:

Proposition 3.3.2. *Given $A, T > 0$ there exist constants C and κ_0 such that solutions $q_\kappa(t)$ to the tidal H_κ flow* (3.3.3) *obey*

$$\|q(0)\|_{L^2} \leq A \quad \implies \quad \|q_\kappa(t)\|_{L^2} \leq C \quad \textit{for all } |t| \leq T \textit{ and } \kappa \geq \kappa_0.$$

Proof. By approximation and local well-posedness we may assume that $q(0) \in H^\infty$. Let

$$E_1(t) := \tfrac{1}{2} \int q_\kappa(t,x)^2 \, dx.$$

This is the first polynomial conserved quantity of the KdV hierarchy, and in the case $W \equiv 0$ one can directly show that $\frac{d}{dt}E_1 = 0$ under the H_κ flow using the ODE (3.2.18) satisfied by the diagonal Green's function.

To counteract the factor of κ^5 in the tidal H_κ flow and obtain a bound for all κ large, we will extract the linear and quadratic terms. Using the translation identity (3.2.14), we write

$$\begin{aligned}
&\frac{d}{dt}E_1 \\
&= \int q_\kappa \big\{ -16\kappa^5 \langle \delta_x, R_0 q_\kappa' R_0 \delta_x \rangle + 4\kappa^2 q_\kappa' \big\} \, dx && (3.3.4) \\
&\quad + \int q_\kappa \big\{ -16\kappa^5 \langle \delta_x, R_0 W' R_0 \delta_x \rangle + 4\kappa^2 W' \big\} \, dx && (3.3.5) \\
&\quad + 16\kappa^5 \int q_\kappa \langle \delta_x, [\partial, R_0 q_\kappa R_0 q_\kappa R_0] \delta_x \rangle \, dx && (3.3.6) \\
&\quad + 16\kappa^5 \int q_\kappa \big\{ \langle \delta_x, [\partial, R_0 W R_0 q_\kappa R_0] \delta_x \rangle + \langle \delta_x, [\partial, R_0 q_\kappa R_0 W R_0] \delta_x \rangle \big\} \, dx && (3.3.7) \\
&\quad + 16\kappa^5 \int q_\kappa \langle \delta_x, [\partial, R_0 W R_0 W R_0] \delta_x \rangle \, dx && (3.3.8)
\end{aligned}$$

$$+ 16\kappa^5 \int q_\kappa \big\{ g(\kappa, q_\kappa + W) + \langle \delta_x, R_0(q_\kappa + W) R_0 \delta_x \rangle \\ - \langle \delta_x, R_0(q_\kappa + W) R_0 (q_\kappa + W) R_0 \delta_x \rangle \big\}' \, dx. \tag{3.3.9}$$

We will estimate the terms (3.3.4)–(3.3.9) separately.

The first linear contribution (3.3.4) vanishes. Indeed, using the first operator identity of (3.2.19) we write

$$(3.3.4) = \int q_\kappa \big\{ -16\kappa^4 R_0(2\kappa) q_\kappa' + 4\kappa^2 q_\kappa' \big\} \, dx.$$

This vanishes because the integrand is odd in Fourier variables, or equivalently the integrand is a total derivative.

Now we estimate the linear contribution (3.3.5) from W. Using the operator identity (3.2.19) we write

$$\begin{aligned} |(3.3.5)| &= \left| \int q_\kappa \big\{ -W''' - \big[R_0(2\kappa) W^{(5)} \big] \big\} \, dx \right| \\ &\lesssim \|q_\kappa\|_{L^2} \left(\|W'''\|_{L^2} + \kappa^{-2} \|W^{(5)}\|_{L^2} \right) \lesssim E_1^{1/2} \lesssim E_1 + 1. \end{aligned}$$

Note that W' is Schwartz, and we allow our implicit constants to depend on the fixed function W.

The first quadratic contribution (3.3.6) also vanishes. Distributing the derivative $[\partial, \cdot]$ and noting that $[\partial, R_0] = 0$, we write

$$(3.3.6) = 16\kappa^5 \big(\operatorname{tr}\{ q_\kappa R_0 [\partial, q_\kappa] R_0 q_\kappa R_0 \} + \operatorname{tr}\{ q_\kappa R_0 q_\kappa R_0 [\partial, q_\kappa] R_0 \} \big).$$

Both of these terms vanish by cycling the trace.

Next we turn to the second quadratic contribution (3.3.7). By linearity and cycling the trace, we can "integrate by parts" to write

$$\begin{aligned} (3.3.7) &= 16\kappa^5 \big(-\operatorname{tr}\{ [\partial, q_\kappa] R_0 W R_0 q_\kappa R_0 \} + \operatorname{tr}\{ q_\kappa [\partial, R_0 q_\kappa R_0 W R_0] \} \big) \\ &= 16\kappa^5 \operatorname{tr}\{ q_\kappa R_0 q_\kappa R_0 [\partial, W] R_0 \}. \end{aligned}$$

Using the estimate (3.2.3) and the observations $\|\sqrt{R_0}\|_{\mathrm{op}} \lesssim \kappa^{-1}$ and $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1}\|f\|_{L^2}$, we estimate

$$|(3.3.7)| \lesssim \kappa^5 \left\|\sqrt{R_0}q_\kappa\sqrt{R_0}\right\|_{\mathfrak{I}_2}^2 \left\|\sqrt{R_0}W'\sqrt{R_0}\right\|_{\mathrm{op}} \lesssim \|W'\|_{L^\infty} E_1.$$

The quadratic W contribution (3.3.8) is easily estimated. We distribute the derivative and estimate

$$|(3.3.8)| \lesssim \kappa^5 \left\|\sqrt{R_0}q_\kappa\sqrt{R_0}\right\|_{\mathfrak{I}_2} \left\|\sqrt{R_0}W'\sqrt{R_0}\right\|_{\mathfrak{I}_2} \left\|\sqrt{R_0}W\sqrt{R_0}\right\|_{\mathrm{op}}.$$

Using the identity (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1}\|f\|_{L^2}$, we obtain

$$|(3.3.8)| \lesssim E_1^{1/2} \lesssim E_1 + 1.$$

For the series tail (3.3.9), we integrate by parts once to put the derivative on q_κ and we write

$$|(3.3.9)| \leq 16\kappa^5 \sum_{\substack{\ell\geq 0,\ m_0,\dots,m_\ell\geq 0\\ \ell+m_0+\dots+m_\ell\geq 3}} \left|\operatorname{tr}\left\{q_\kappa' R_0(WR_0)^{m_0}qR_0\cdots qR_0(WR_0)^{m_\ell}]\right\}\right|.$$

Observe that the summand vanishes for $m_0+\dots+m_\ell = 0$ by writing $q_\kappa' = [\partial, q_\kappa]$ and cycling the trace, and so we may insert the condition $m_0+\dots+m_\ell \geq 1$ in the summation. We use the operator estimate (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1}\|f\|_{L^2}$ to put each factor of q in L^2, and we put all other factors in operator norm:

$$\lesssim \kappa^5 \sum_{\substack{\ell\geq 0,\ m_0+\dots+m_\ell\geq 1\\ \ell+m_0+\dots+m_\ell\geq 3}} \frac{\|q_\kappa'\|_{H^{-1}}}{\kappa^{1/2}} \left(\frac{\|q\|_{L^2}}{\kappa^{3/2}}\right)^\ell \left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m_0+\dots+m_\ell}.$$

We split the sum into $\ell = 0$, $\ell = 1$, $\ell = 2$, and $\ell \geq 3$ terms. We then re-index $m = m_0+\dots+m_\ell$, sum over $m \geq 1$ as in (3.2.15), and then sum in ℓ. The sum converges provided $\kappa \gg E_1^{1/3}(t)$ and $\kappa \gg \|W\|_{L^\infty}^{1/2}$. The conditions $m \geq 1$ and $\ell + m \geq 3$ guarantee that when we sum over the two parenthetical terms we gain a factor $\lesssim (\kappa^{-3/2})^2(\kappa^{-2})$, and so we obtain

$$\lesssim \kappa^{-1/2}\|q_\kappa\|_{L^2}$$

for all κ large. Taking a supremum over $\|f\|_{H^{-1}} \leq 1$ and restricting to κ sufficiently large, we conclude there exists $\kappa_0(E_1(t))$ such that

$$|(3.3.9)| \leq E_1^{1/2} \lesssim E_1 + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(E_1(t)).$$

Altogether, we have shown that there exist constants C and $\kappa_0(E_1(t))$ such that

$$\left|\frac{d}{dt}E_1\right| \leq C(E_1 + 1) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(E_1(t)).$$

Grönwall's inequality then yields the bound

$$E_1(t) \leq (E_1(0)+1)e^{CT} - 1 \quad \text{uniformly for } |t| \leq T,\ \kappa \geq \kappa_0\big((E_1(0)+1)e^{CT} - 1\big),$$

which concludes the proof. $\square$

Next, we control the growth of the H^1-norm:

Proposition 3.3.3. *Given $A, T > 0$ there exist constants C and κ_0 such that solutions $q_\kappa(t)$ to the tidal H_κ flow* (3.3.3) *obey*

$$\|q(0)\|_{H^1} \leq A \quad \implies \quad \|q_\kappa(t)\|_{H^1} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.$$

Proof. By approximation and local well-posedness we may assume that $q(0) \in H^\infty$. Let

$$E_2(t) := \int \left\{ \tfrac{1}{2}(q_\kappa'(t,x))^2 + q_\kappa(t,x)^3 \right\} dx$$

denote the next polynomial conserved quantity of KdV.

We multiply the tidal H_κ flow (3.3.3) by $-q_\kappa'' + 3q_\kappa^2$ and integrate in space to obtain an expression for $\frac{d}{dt}E_2$. We then integrate by parts to remove the derivative from $g(\kappa, q_\kappa + W) - g(\kappa, W)$, expand both diagonal Green's functions using the relation (3.2.16), and apply the identity (3.2.17) to obtain

$$\begin{aligned} &\frac{d}{dt}E_2 \\ &= -\int q_\kappa'' \big[16\kappa^5 g'(\kappa, W) + 4\kappa^2 W'\big]\, dx \end{aligned} \tag{3.3.10}$$

$$+ 4\kappa^2 \int \left\{3W' q_\kappa^2 + 16\kappa^5 q_\kappa' \big[g(\kappa, q_\kappa + W) - g(\kappa, W)\big]\right\} dx \tag{3.3.11}$$

$$+ 16\kappa^5 \int \big[2W q_\kappa' + 2(W q_\kappa)'\big]\big[g(\kappa, q_\kappa + W) - g(\kappa, W)\big]\, dx. \tag{3.3.12}$$

Note that in the case $W \equiv 0$, all three integrals vanish and E_2 is conserved as expected. We will estimate the terms (3.3.10)–(3.3.12) separately.

We begin with the term (3.3.10). We integrate by parts once, expand $g(\kappa, W)$ in a series, and extract the linear term:

$$\begin{aligned}(3.3.10) = \int q_\kappa' \big[-16\kappa^5 \langle \delta_x, R_0 W'' R_0 \delta_x\rangle + 4\kappa^2 W''\big]\, dx \\ + 16\kappa^5 \sum_{m\geq 2} (-1)^m \operatorname{tr}\left\{q_\kappa' [\partial^2, R_0 (W R_0)^m]\right\}.\end{aligned}$$

For the first term we use the operator identity (3.2.19) to estimate

$$\begin{aligned}&\left|\int q_\kappa' \big[-16\kappa^5 \langle \delta_x, R_0 W'' R_0 \delta_x\rangle + 4\kappa^2 W''\big]\, dx\right| \\ &= \left|\int q_\kappa' \big[- W^{(4)} - R_0(2\kappa) W^{(6)}\big]\, dx\right| \lesssim \|q_\kappa'\|_{L^2} \left(\|W^{(4)}\|_{L^2} + \kappa^{-2} \|W^{(6)}\|\right).\end{aligned}$$

For the second term we distribute the two derivatives $[\partial^2, \cdot]$, use the estimate (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$ to put q_κ' and the highest order W term in L^2, and put the remaining terms in operator norm:

$$\begin{aligned}&16\kappa^5 \sum_{m\geq 2} \left|\operatorname{tr}\left\{q_\kappa' [\partial^2, R_0 (W R_0)^m]\right\}\right| \\ &\lesssim \kappa^5 \sum_{m\geq 2} m^2 \frac{\|q_\kappa'\|_{L^2}}{\kappa^{3/2}} \frac{\|W'\|_{H^1}}{\kappa^{3/2}} \left(\frac{\|W\|_{W^{1,\infty}}}{\kappa^2}\right)^{m-1} \lesssim \|W'\|_{H^1} \|W\|_{W^{1,\infty}} \|q_\kappa'\|_{L^2}\end{aligned}$$

uniformly for $\kappa \gg \|W\|_{W^{1,\infty}}^{1/2}$. Altogether we conclude

$$|(3.3.10)| \lesssim \|q_\kappa'\|_{L^2}^2 + 1$$

uniformly for κ large.

Next we turn to the term (3.3.11). Expanding $g(\kappa, q_\kappa + W)$ and extracting the terms that are linear and quadratic in q_κ and W, we write

$$
\begin{aligned}
&(3.3.11)\\
&= -64\kappa^7 \int q_\kappa' \langle \delta_x, R_0 q_\kappa R_0 \delta_x \rangle \, dx && (3.3.13)\\
&\quad + 64\kappa^7 \int q_\kappa' \langle \delta_x, R_0 q_\kappa R_0 q_\kappa R_0 \delta_x \rangle \, dx && (3.3.14)\\
&\quad + 4\kappa^2 \int \left\{ 16\kappa^5 q_\kappa' \big(\langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x \rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x \rangle \big) + 3W' q_\kappa^2 \right\} dx && (3.3.15)\\
&\quad + 64\kappa^7 \sum_{\substack{\ell \geq 1,\ m_0, \dots, m_\ell \geq 0 \\ \ell + m_0 + \dots + m_\ell \geq 3}} (-1)^{\ell + m_0 + \dots + m_\ell} \operatorname{tr}\left\{ q_\kappa' R_0 (W R_0)^{m_0} q_\kappa R_0 q_\kappa R_0 (W R_0)^{m_\ell} \right\}. && (3.3.16)
\end{aligned}
$$

The terms (3.3.13) and (3.3.14) vanish by cycling the trace:

$$
\begin{aligned}
(3.3.13) &= -64\kappa^7 \operatorname{tr}\{ [\partial, q_\kappa] R_0 q_\kappa R_0 \} = 0,\\
(3.3.14) &= 64\kappa^7 \operatorname{tr}\{ [\partial, q_\kappa] R_0 q_\kappa R_0 q_\kappa R_0 \} = 0.
\end{aligned}
$$

For the term (3.3.15), we integrate by parts to replace $3W' q_\kappa^2$ by $-6W q_\kappa q_\kappa'$. We then use the operator identity (3.2.20) and the estimates $\| R_0(2\kappa) \partial^j \|_{\mathrm{op}} \lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on L^∞ by the explicit kernel formula for R_0 and Young's inequality) to conclude

$$
|(3.3.15)| \lesssim \| q_\kappa' \|_{L^2}^2 + 1.
$$

For the tail (3.3.16) we estimate

$$
|(3.3.16)| \leq 64\kappa^7 \sum_{\substack{\ell \geq 1,\ m_0, \dots, m_\ell \geq 0 \\ \ell + m_0 + \dots + m_\ell \geq 3}} \left| \operatorname{tr}\left\{ q_\kappa' R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (W R_0)^{m_\ell} \right\} \right|.
$$

We put q_κ' and one other q_κ in L^2 via the estimate (3.2.3) and put the remaining terms in operator norm. We have $\|q\|_{L^2} \lesssim 1$ uniformly for $|t| \leq T$ and κ large by Proposition 3.3.2, and so we obtain

$$
\lesssim \kappa^7 \sum_{\substack{\ell \geq 1,\ m_0, \dots, m_\ell \geq 0 \\ \ell + m_0 + \dots + m_\ell \geq 3}} \frac{\|q_\kappa'\|_{L^2}}{\kappa^3} \left(\frac{\|q_\kappa\|_{L^\infty}}{\kappa^2} \right)^{\ell - 1} \left(\frac{\|W\|_{L^\infty}}{\kappa^2} \right)^{m_0 + \dots + m_\ell}.
$$

The condition $\ell + m_0 + \cdots + m_\ell \geq 3$ yields a gain $\lesssim (\kappa^{-2})^2$ when we sum over the two parenthetical terms, and so we obtain

$$\lesssim \|q_\kappa'\|_{L^2} \lesssim \|q_\kappa'\|_{L^2}^2 + 1$$

provided that $\kappa \gg \|q_\kappa\|_{L^\infty}^{1/2}$ and $\kappa \gg \|W\|_{L^\infty}^{1/2}$. From Proposition 3.3.2 we know that

$$\|q_\kappa\|_{L^2} \lesssim 1, \qquad \|q_\kappa\|_{L^\infty} \leq \|q_\kappa\|_{L^2}^{1/2} \|q_\kappa'\|_{L^2}^{1/2} \lesssim \|q_\kappa'\|_{L^2}^{1/2} \tag{3.3.17}$$

for $\kappa \geq \kappa_0(T, \|q(0)\|_{L^2})$ sufficiently large, and so altogether we conclude

$$|(3.3.11)| \lesssim \|q_\kappa'\|_{L^2}^2 + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(\|q_\kappa\|_{H^1}).$$

It remains to estimate the term (3.3.12). Expanding $g(\kappa, q_\kappa + W) - g(\kappa, q_\kappa + W)$ and extracting the linear term, we write

$$\begin{aligned} &|(3.3.12)| \\ &\leq \left| 32\kappa^5 \int \left[W q_\kappa' + (W q_\kappa)' \right] \langle \delta_x, R_0 q_\kappa R_0 \delta_x \rangle \, dx \right| && (3.3.18) \\ &\quad + 32\kappa^5 \sum_{\substack{\ell \geq 1,\ m_0, \ldots, m_\ell \geq 0 \\ \ell + m_0 + \cdots + m_\ell \geq 2}} \left| \operatorname{tr} \left\{ \left[W q_\kappa' + (W q_\kappa)' \right] R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (W R_0)^{m_\ell} \right\} \right|. && (3.3.19) \end{aligned}$$

For the first term (3.3.18) we use the operator identity (3.2.19) to write

$$(3.3.18) = 8\kappa^2 \int [W q_\kappa' + (W q_\kappa)'] q_\kappa \, dx + \int [W q_\kappa' + (W q_\kappa)'][q_\kappa'' + R_0(2\kappa)\partial^2 q_\kappa''] \, dx.$$

The first integral vanishes because the integrand is a total derivative. For the second integral, we integrate by parts to obtain

$$\begin{aligned} \int [W q_\kappa' + (W q_\kappa)'][q_\kappa'' + R_0(2\kappa)\partial^2 q_\kappa''] \, dx = &- \int [2W' q_\kappa' + W'' q_\kappa][q_\kappa' + R_0(2\kappa)\partial^2 q_\kappa'] \, dx \\ &+ \int \left\{ W q_\kappa' [R_0(2\kappa)\partial^2 q_\kappa''] - W q_\kappa'' [R_0(2\kappa)\partial^2 q_\kappa''] \right\} dx. \end{aligned}$$

Those terms without q_κ'' can be estimated using Cauchy–Schwarz and the observation that $\|R_0(2\kappa)\partial^2\|_{\text{op}} \lesssim 1$. For the remaining terms, we "integrate by parts" in Fourier variables:

$$\left| \int \left\{ W q_\kappa' [R_0(2\kappa)\partial^2 q_\kappa''] - W q_\kappa'' [R_0(2\kappa)\partial^2 q_\kappa''] \right\} dx \right|$$

$$= (2\pi)^{-\frac{1}{2}} \left| \iint \widehat{W}(\xi-\eta)\widehat{q_\kappa'}(\eta)\widehat{q_\kappa'}(\xi) \frac{i(\xi-\eta)\xi^2}{\xi^2+4\kappa^2}\, d\xi\, d\eta \right|$$

$$\lesssim \iint \left| \widehat{W'}(\xi-\eta)\widehat{q_\kappa'}(\eta)\widehat{q_\kappa'}(\xi) \right| d\xi\, d\eta \lesssim \|\widehat{W'}\|_{L^1} \|q_\kappa'\|_{L^2}^2 \lesssim \|W'\|_{H^1} \|q_\kappa'\|_{L^2}^2.$$

In the last inequality, we used Cauchy–Schwarz to estimate

$$\int \left|\widehat{W'}(\xi)\right| d\xi \leq \left(\int \frac{d\xi}{\xi^2+1} \right)^{\frac{1}{2}} \left(\int (\xi^2+1) \left|\widehat{W'}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}.$$

Together, we conclude

$$|(3.3.18)| \lesssim \|q_\kappa'\|_{L^2}^2 + 1.$$

For the tail (3.3.19) we put $Wq_\kappa' + (Wq_\kappa)'$ and one q_κ in L^2 using the estimate (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$, and we put all other terms in operator norm to obtain

$$|(3.3.19)| \lesssim \kappa^5 \sum_{\substack{\ell \geq 1,\ m_0,\dots,m_\ell \geq 0 \\ \ell + m_0 + \cdots + m_\ell \geq 2}} \frac{\|q_\kappa\|_{H^1}}{\kappa^3} \left(\frac{\|q_\kappa\|_{L^\infty}}{\kappa^2} \right)^{\ell-1} \left(\frac{\|W\|_{L^\infty}}{\kappa^2} \right)^{m_0+\cdots+m_\ell} \lesssim \|q_\kappa\|_{H^1}$$

provided that $\kappa \gg \|q_\kappa\|_{L^\infty}^{1/2}$ and $\kappa \gg \|W\|_{L^\infty}^{1/2}$. Note the condition $\ell + m_0 + \cdots + m_\ell \geq 2$ yielded a gain $\lesssim \kappa^{-2}$ when we summed over the parenthetical terms. Recalling our control (3.3.17) over the L^∞-norm of q_κ, we conclude

$$|(3.3.19)| \lesssim \|q_\kappa'\|_{L^2}^2 + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(\|q_\kappa'\|_{L^2}).$$

Altogether we have obtained

$$\left| \frac{d}{dt} E_2 \right| \lesssim \|q_\kappa'\|_{L^2}^2 + 1 \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(\|q_\kappa'\|_{L^2}).$$

We use E_2 and the estimates (3.3.17) to bound q_κ' in L^2:

$$\|q_\kappa'\|_{L^2}^2 \lesssim E_2 + \left| \int q_\kappa^3\, dx \right| \lesssim E_2 + \|q_\kappa'\|_{L^2}^{1/2}.$$

Together, we conclude that there exists a constant $C = C(T, A)$ such that

$$\|q_\kappa'(t)\|_{L^2}^2 \leq C + C\, \|q_\kappa'\|_{L^2}^{1/2} + C \int_0^t \|q_\kappa'(s)\|_{L_x^2}^2\, ds.$$

For $\|q_\kappa'(t)\|_{L^2}^2 \gtrsim C^{4/3}$ we can apply Grönwall's inequality to obtain $\|q_\kappa'(t)\|_{L^2}^2 \lesssim 1$ for $|t| \leq T$, and so we conclude

$$\|q_\kappa'(t)\|_{L^2}^2 \leq C(T, \|q(0)\|_{H^1}) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(T, \|q(0)\|_{H^1}). \qquad \square$$

The last space for which we need to rely upon the corresponding polynomial conserved quantity to obtain an *a priori* estimate is H^2. Starting with H^3, the energy arguments are much simplified and the *a priori* estimates are proven inductively.

Proposition 3.3.4. *Given $A, T > 0$ there exist constants C and κ_0 such that solutions $q_\kappa(t)$ to the tidal H_κ flow* (3.3.3) *obey*

$$\|q(0)\|_{H^2} \leq A \quad \implies \quad \|q_\kappa(t)\|_{H^2} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.$$

Proof. By approximation and local well-posedness we may assume that $q(0) \in H^\infty$. Let

$$E_3(t) := \int \left\{ \tfrac{1}{2}(q_\kappa''(t,x))^2 + 5q_\kappa(t,x)(q_\kappa'(t,x))^2 + \tfrac{5}{2}q_\kappa(t,x)^4 \right\} dx$$

denote the third energy in the KdV hierarchy of conserved quantities.

We multiply the tidal H_κ flow (3.3.3) by $q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa'' + 10q_\kappa^3$ and integrate in space to obtain an expression for $\frac{d}{dt}E_3$. We then integrate by parts to remove the derivative from $g(\kappa, q_\kappa + W) - g(\kappa, W)$, expand both diagonal Green's functions using the relation (3.2.16), and apply the identity (3.2.17) to obtain

$$\begin{aligned}\frac{d}{dt}E_3 = &\int \left[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa'' + 10q_\kappa^3\right]\left[16\kappa^5 g'(\kappa, W) + 4\kappa^2 W'\right] dx \\ &+ 32\kappa^5 \int \left[- q_\kappa' q_\kappa'' - 2q_\kappa q_\kappa''' + 15q_\kappa^2 q_\kappa'\right] g(\kappa, W)\, dx \\ &+ 2\kappa^5 \int \left[2\kappa^2(-q_\kappa''' + 6q_\kappa q_\kappa') - 2Wq_\kappa''' - W'q_\kappa'' + 12Wq_\kappa q_\kappa' + 3W'q_\kappa^2\right] \\ &\qquad\qquad\qquad \times \left[g(\kappa, q_\kappa + W) - g(\kappa, W)\right] dx.\end{aligned}$$

Note that in the case $W \equiv 0$, all three integrals vanish and E_3 is conserved as expected.

In order to exhibit cancellation in the limit $\kappa \to \infty$, we expand $g(\kappa, q_\kappa + W) - g(\kappa, W)$ in powers of q_κ and W and regroup terms:

$$
\begin{aligned}
&\frac{d}{dt}E_3 \\
&= \int \big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa''\big]\big[16\kappa^5 g'(\kappa, W) + 4\kappa^2 W'\big] && (3.3.20)\\
&\quad - 32\kappa^5 \int \big[q_\kappa' q_\kappa'' + 2q_\kappa q_\kappa'''\big]\big[g(\kappa, W) + \langle \delta_x, R_0 W R_0 \delta_x\rangle\big] && (3.3.21)\\
&\quad + 64\kappa^7 \int (-q_\kappa''' + 6q_\kappa q_\kappa')\big[g(\kappa, q_\kappa) - \tfrac{1}{2\kappa}\big] && (3.3.22)\\
&\quad + 32\kappa^5 \int \big\{-2\kappa^2 q_\kappa'''\big[\langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x\rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x\rangle\big] \\
&\qquad\qquad + [2W q_\kappa''' + W' q_\kappa'']\langle \delta_x, R_0 q_\kappa R_0 \delta_x\rangle + [q_\kappa' q_\kappa'' + 2q_\kappa q_\kappa''']\langle \delta_x, R_0 W R_0 \delta_x\rangle\big\} && (3.3.23)\\
&\quad + 8\kappa^2 \int \big\{5W' q_\kappa^3 - 12\kappa^3\big[4W q_\kappa q_\kappa' + W' q_\kappa^2\big]\langle \delta_x, R_0 q_\kappa R_0 \delta_x\rangle \\
&\qquad\qquad + 48\kappa^5 q_\kappa q_\kappa'\big[\langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x\rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x\rangle\big]\big\} && (3.3.24)\\
&\quad + 64\kappa^7 \sum_{\substack{\ell \geq 1,\, m_0 + \cdots + m_\ell \geq 1 \\ \ell + m_0 + \cdots + m_\ell \geq 3}} (-1)^{\ell + m_0 + \cdots + m_\ell} \operatorname{tr}\{(-q_\kappa''' + 6q_\kappa q_\kappa')R_0 (W R_0)^{m_0} \\
&\qquad\qquad \times q_\kappa R_0 \cdots q_\kappa R_0 (W R_0)^{m_\ell}\} && (3.3.25)\\
&\quad + 16\kappa^5 \sum_{\substack{\ell \geq 1,\, m_0 + \cdots + m_\ell \geq 0 \\ \ell + m_0 + \cdots + m_\ell \geq 2}} (-1)^{\ell + m_0 + \cdots + m_\ell} \operatorname{tr}\{[-4W q_\kappa''' - 2W' q_\kappa'' + 24 W q_\kappa q_\kappa' \\
&\qquad\qquad + 6W' q_\kappa^2] R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (W R_0)^{m_\ell}\}. && (3.3.26)
\end{aligned}
$$

Note that in (3.3.22) we extracted the terms from (3.3.25) with no factors of W, which is reflected in the condition $m_0 + \cdots + m_\ell \geq 1$. We will estimate each of the terms (3.3.20)–(3.3.26) separately.

For the term (3.3.20) we expand $g(\kappa, W)$ in powers of W:

$$
\begin{aligned}
|(3.3.20)| \leq &\left| \int \big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa''\big]\big[-\langle \delta_x, R_0 W' R_0 \delta_x\rangle + 4\kappa^2 W'\big] \right| \\
&+ 16\kappa^5 \sum_{m \geq 2} \big| \operatorname{tr}\big\{\big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa''\big][\partial, R_0 (W R_0)^m]\big\}\big|.
\end{aligned}
$$

For the integral, we use the operator identity (3.2.19) to write

$$
\int \big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10q_\kappa q_\kappa''\big]\big[-\langle \delta_x, R_0 W' R_0 \delta_x\rangle + 4\kappa^2 W'\big]
$$

$$= \int \big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10 q_\kappa q_\kappa''\big]\big[-W''' - R_0(2\kappa)W^{(5)}\big].$$

For the term $q_\kappa^{(4)}$ we integrate by parts twice. As W' is Schwartz and $\|q_\kappa\|_{H^1} \lesssim 1$ by Proposition 3.3.3, then Cauchy–Schwarz yields

$$\left| \int \big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10 q_\kappa q_\kappa''\big]\big[-W''' - R_0(2\kappa)W^{(5)}\big] \right| \lesssim \|q_\kappa''\|_{L^2} + 1 \lesssim \|q_\kappa''\|_{L^2}^2 + 1.$$

For the tail, we again integrate by parts twice for $q_\kappa^{(4)}$. We then estimate the q_κ terms and the highest order W term in L^2 using the estimate (3.2.3) and $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1}\|f\|_{L^2}$ and the remaining terms in L^∞. This yields

$$\begin{aligned} &16\kappa^5 \sum_{m\geq 2} \big| \operatorname{tr}\big\{\big[q_\kappa^{(4)} - 5(q_\kappa')^2 - 10 q_\kappa q_\kappa''\big][\partial, R_0(WR_0)^m]\big\}\big| \\ &\lesssim \kappa^5 \sum_{m\geq 2} \frac{\|q_\kappa''\|_{L^2}\|W'\|_{H^2}}{\kappa^3}\left(\frac{\|W\|_{W^{1,\infty}}}{\kappa^2}\right)^{m-1} \lesssim \|q_\kappa''\|_{L^2} \lesssim \|q_\kappa''\|_{L^2}^2 + 1 \end{aligned}$$

provided that $\kappa \gg \|W\|_{W^{1,\infty}}^{1/2}$.

For the term (3.3.21) we integrate by parts once to write

$$|(3.3.21)| \leq 32\kappa^5 \sum_{m\geq 2} \big| \operatorname{tr}\big\{\big[\tfrac{1}{2}(q_\kappa')^2 - 2 q_\kappa q_\kappa''\big][\partial, R_0(WR_0)^m]\big\}\big|.$$

We estimate the q_κ terms and the one factor of W' in L^2 using the estimate (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1}\|f\|_{L^2}$, we estimate the remaining terms in operator norm. By Proposition 3.3.3 we have

$$\|q_\kappa'\|_{L^\infty} \leq \|q_\kappa'\|_{L^2}^{1/2}\|q_\kappa''\|_{L^2}^{1/2} \lesssim \|q_\kappa''\|_{L^2}^{1/2} \lesssim \|q_\kappa''\|_{L^2} + 1.$$

Together, we obtain

$$|(3.3.21)| \lesssim \kappa^5 \sum_{m\geq 2} \frac{(\|q_\kappa''\|_{L^2} + 1)\|W'\|_{L^2}}{\kappa^3}\left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m-1} \lesssim \|q_\kappa''\|_{L^2} \lesssim \|q_\kappa''\|_{L^2}^2 + 1$$

provided that $\kappa \gg \|W\|_{L^\infty}^{1/2}$.

The term (3.3.22) vanishes. Indeed, after integrating by parts and adding a total derivative we have

$$(3.3.22) = -64\kappa^7 \int (-q_\kappa'' + 3q_\kappa^2) g'(\kappa, q_\kappa)\, dx = -4\kappa^2 \int (-q_\kappa'' + 3q_\kappa^2)\big[16\kappa^5 g'(\kappa, q_\kappa) + 4\kappa^2 q_\kappa'\big]\, dx.$$

The integral on the RHS is $\frac{d}{dt} E_2$ in the case $W \equiv 0$ and hence vanishes, as we observed in Proposition 3.3.3.

For the term (3.3.23), we integrate by parts to write

$$\begin{aligned}(3.3.23) = 32\kappa^5 \int \Big\{ & 2\kappa^2 q_\kappa'' \big[\langle \delta_x, R_0 W R_0 q_\kappa' R_0 \delta_x\rangle + \langle \delta_x, R_0 q_\kappa' R_0 W R_0 \delta_x\rangle\big] \\ & - 2W q_\kappa'' \langle \delta_x, R_0 q_\kappa' R_0 \delta_x\rangle - q_\kappa' q_\kappa'' \langle \delta_x, R_0 W R_0 \delta_x\rangle \\ & + 2\kappa^2 q_\kappa'' \big[\langle \delta_x, R_0 W' R_0 q_\kappa R_0 \delta_x\rangle + \langle \delta_x, R_0 q_\kappa R_0 W' R_0 \delta_x\rangle\big] \\ & - W' q_\kappa'' \langle \delta_x, R_0 q_\kappa R_0 \delta_x\rangle - 2 q_\kappa q_\kappa'' \langle \delta_x, R_0 W' R_0 \delta_x\rangle \Big\}\, dx.\end{aligned}$$

We use the operator identities (3.2.19) and (3.2.20). Observe that the leading order contributions as $\kappa \to \infty$ (i.e. $4\kappa^2 f$ in (3.2.19) and $3fh$ in (3.2.20)) cancel out. The remainder is easily estimated, yielding

$$|(3.3.23)| \lesssim \|q_\kappa''\|_{L^2}^2 + 1.$$

For the term (3.3.24) we write

$$\begin{aligned}(3.3.24) = 8\kappa^2 \int \Big\{ & 48\kappa^5 q_\kappa q_\kappa' \big[\langle \delta_x, R_0 W R_0 q_\kappa R_0 \delta_x\rangle + \langle \delta_x, R_0 q_\kappa R_0 W R_0 \delta_x\rangle\big] \\ & - 15 W q_\kappa^2 q_\kappa' - 24\kappa^3 W q_\kappa q_\kappa' \langle \delta_x, R_0 q_\kappa R_0 \delta_x\rangle + 12\kappa^3 W q_\kappa^2 \langle \delta_x, R_0 q_\kappa' R_0 \delta_x\rangle \Big\}.\end{aligned}$$

We use the operator identities (3.2.19) and (3.2.20). Observe that the leading order contributions as $\kappa \to \infty$ (i.e. $4\kappa^2 f$ in (3.2.19) and $3fh$ in (3.2.20))) cancel out. The remainder is easily estimated, yielding

$$|(3.3.24)| \lesssim \|q_\kappa''\|_{L^2}^2 + 1.$$

For the tail (3.3.25), we integrate by parts once to obtain

$$|(3.3.25)| \lesssim \kappa^7 \sum_{\substack{\ell \geq 1,\ m_0 + \cdots + m_\ell \geq 1 \\ \ell + m_0 + \cdots + m_\ell \geq 3}} \Big| \operatorname{tr}\big\{(-q_\kappa'' + 3q_\kappa^2)[\partial, R_0 (W R_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (W R_0)^{m_\ell}]\big\}\Big|.$$

We put $-q_\kappa''+3q_\kappa^2$ and the highest order q_κ in L^2 using the identity (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1}\|f\|_{L^2}$, and we estimate the remaining terms in operator norm:

$$\lesssim \kappa^7 \sum_{\substack{\ell\geq 1,\ m_0+\dots+m_\ell\geq 1\\ \ell+m_0+\dots+m_\ell\geq 3}} \frac{\|q_\kappa''\|_{L^2}+1}{\kappa^3}\left(\frac{\|q_\kappa\|_{H^1}}{\kappa^2}\right)^{\ell-1}\left(\frac{\|W\|_{W^{1,\infty}}}{\kappa^2}\right)^{m_0+\dots+m_\ell}.$$

We re-index $m=m_0+\dots+m_\ell$ and sum over $\ell+m$ as in (3.2.15). The condition $\ell+m_0+\dots+m_\ell\geq 3$ guarantees a gain $\lesssim(\kappa^{-2})^2$ when we sum over the two parenthetical terms, and so we obtain an acceptable bound.

For the tail (3.3.26), we estimate

$$\begin{aligned}|(3.3.26)| \lesssim \kappa^5 \sum_{\substack{\ell\geq 1,\ m_0,\dots,m_\ell\geq 0\\ \ell+m_0+\dots+m_\ell\geq 2}} \Big|\operatorname{tr}\Big\{&\big[-4Wq_\kappa'''-2W'q_\kappa''+24Wq_\kappa q_\kappa'+6W'q_\kappa^2\big]\\ &\times R_0(WR_0)^{m_0}q_\kappa R_0\cdots q_\kappa R_0(WR_0)^{m_\ell}\Big\}\Big|.\end{aligned}$$

For the term q_κ''' we integrate by parts once. We then put the square-bracketed term and the highest order factor of q_κ in L^2 using the identity (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1}\|f\|_{L^2}$, and we estimate the remaining terms in operator norm:

$$|(3.3.26)| \lesssim \kappa^5 \sum_{\substack{\ell\geq 1,\ m_0,\dots,m_\ell\geq 0\\ \ell+m_0+\dots+m_\ell\geq 2}} \frac{\|q_\kappa''\|_{L^2}+1}{\kappa^{3/2}}\left(\frac{\|q_\kappa\|_{H^1}}{\kappa^{3/2}}\right)^{\ell}\left(\frac{\|W\|_{W^{1,\infty}}}{\kappa^2}\right)^{m_0+\dots+m_\ell}.$$

We re-index $m=m_0+\dots+m_\ell$ and sum over $\ell+m$ as in (3.2.15). The condition $\ell+m_0+\dots+m_\ell\geq 2$ guarantees a gain $\lesssim \kappa^{-3/2}\cdot\kappa^{-2}$ when we sum over the two parenthetical terms, and so we conclude

$$|(3.3.25)| \lesssim \|q_\kappa''\|_{L^2}+1 \lesssim \|q_\kappa''\|_{L^2}^2+1$$

provided that κ is sufficiently large (independently of $\|q_\kappa''\|_{L^2}$).

Altogether, we have obtained

$$\left|\frac{d}{dt}E_3\right| \lesssim \|q_\kappa''\|_{L^2}^2+1 \quad \text{uniformly for } |t|\leq T \text{ and } \kappa\geq\kappa_0,$$

where κ_0 depends only on T and $\|q(0)\|_{H^1}$. Using Proposition 3.3.3, we can then bound

$$\|q_\kappa''\|_{L^2}^2 \lesssim E_3 + \left|\int q_\kappa(q_\kappa')^2\,dx\right| + \left|\int q_\kappa^4\,dx\right| \lesssim E_3+1.$$

Together, we conclude that there exists a constant $C = C(T, A)$ such that

$$\|q_\kappa''(t)\|_{L^2}^2 \leq C + C\int_0^t \|q_\kappa''(s)\|_{L_x^2}^2\, ds$$

uniformly for $|t| \leq T$ and $\kappa \geq \kappa_0$. Grönwall's inequality then yields

$$\|q_\kappa''(t)\|_{L^2}^2 \leq C(T, \|q(0)\|_{H^2}) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(T, \|q(0)\|_{H^1}), \qquad \square$$

as desired.

For H^s, $s \geq 3$ we proceed by induction:

Proposition 3.3.5. *Given an integer $s \geq 3$ and $A, T > 0$ there exist constants C and κ_0 such that solutions $q_\kappa(t)$ to the tidal H_κ flow* (3.3.3) *obey*

$$\|q(0)\|_{H^s} \leq A \quad \implies \quad \|q_\kappa(t)\|_{H^s} \leq C \quad \text{for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.$$

Proof. We induct on s, with the base case given by Proposition 3.3.4. Assume the result holds for $s - 1$.

By approximation and local well-posedness we may assume that $q(0) \in H^\infty$. We define

$$F_s(t) := \tfrac{1}{2}\int (q_\kappa^{(s)}(t,x))^2\, dx.$$

Expanding $g(\kappa, q_\kappa + W)$ in powers of q_κ and W, we write

$$\begin{aligned}
&\frac{d}{dt}F_s \\
&= \int q_\kappa^{(s)}\big\{-16\kappa^5\langle\delta_x, R_0 q_\kappa^{(s+1)} R_0\delta_x\rangle + 4\kappa^2 q_\kappa^{(s+1)}\big\}\, dx && (3.3.27)\\
&\quad + \int q_\kappa^{(s)}\big\{-16\kappa^5\langle\delta_x, R_0 W^{(s+1)} R_0\delta_x\rangle + 4\kappa^2 W^{(s+1)}\big\}\, dx && (3.3.28)\\
&\quad + 16\kappa^5 \int q_\kappa^{(s)}\langle\delta_x, [\partial^{s+1}, R_0 q_\kappa R_0 q_\kappa R_0]\delta_x\rangle\, dx && (3.3.29)\\
&\quad + 16\kappa^5 \int q_\kappa^{(s)}\big\{\langle\delta_x, [\partial^{s+1}, R_0 W R_0 q_\kappa R_0]\delta_x\rangle + \langle\delta_x, [\partial^{s+1}, R_0 q_\kappa R_0 W R_0]\delta_x\rangle\big\}\, dx && (3.3.30)\\
&\quad + 16\kappa^5 \int q_\kappa^{(s)}\langle\delta_x, [\partial^{s+1}, R_0 W R_0 W R_0]\delta_x\rangle\, dx && (3.3.31)
\end{aligned}$$

$$
+ 16\kappa^5 \int q_\kappa^{(s)} \big\{ g(\kappa, q_\kappa + W) + \langle \delta_x, R_0(q_\kappa + W) R_0 \delta_x \rangle \\
- \langle \delta_x, R_0(q_\kappa + W) R_0 (q_\kappa + W) R_0 \delta_x \rangle \big\}^{(s+1)} \, dx. \tag{3.3.32}
$$

We will estimate the terms (3.3.27)–(3.3.32) separately.

The first linear contribution (3.3.27) vanishes. To see this, we use the first operator identity of (3.2.19) to write

$$
(3.3.27) = \int q_\kappa^{(s)} \big\{ -16\kappa^4 R_0(2\kappa) q_\kappa^{(s+1)} + 4\kappa^2 q_\kappa^{(s+1)} \big\} \, dx = 0.
$$

In the last equality we noted that the integrand is odd in Fourier variables, or equivalently that the integrand of (3.3.27) is a total derivative.

Now we estimate the linear contribution (3.3.28) from W. Using the operator identity (3.2.19) and recalling that W' is Schwartz, we estimate

$$
\begin{aligned}
|(3.3.28)| &= \left| \int q_\kappa^{(s)} \{ -W^{(s+3)} - [R_0(2\kappa) W^{(s+5)}] \} \, dx \right| \\
&\lesssim \| q_\kappa^{(s)} \|_{L^2} \big(\| W^{(s+3)} \|_{L^2} + \kappa^{-2} \| W^{(s+5)} \|_{L^2} \big) \lesssim F_s^{1/2} \lesssim F_s + 1.
\end{aligned}
$$

In the first quadratic term (3.3.29) we distribute the derivatives $[\partial^{s+1}, \cdot]$. For the terms with $q_\kappa^{(s+1)}$, we "integrate by parts" to write

$$
\begin{aligned}
&16\kappa^5 \big(\operatorname{tr} \big\{ q_\kappa^{(s)} R_0 q_\kappa^{(s+1)} R_0 q_\kappa R_0 \big\} + \operatorname{tr} \big\{ q_\kappa^{(s)} R_0 q_\kappa R_0 q_\kappa^{(s+1)} R_0 \big\} \big) \\
&= 16\kappa^5 \operatorname{tr} \big\{ \big[\partial, q_\kappa^{(s)} R_0 q_\kappa^{(s)} R_0 \big] q_\kappa R_0 \big\} = -16\kappa^5 \operatorname{tr} \big\{ q_\kappa^{(s)} R_0 q_\kappa^{(s)} R_0 [\partial, q_\kappa] R_0 \big\}.
\end{aligned}
$$

This leaves

$$
|(3.3.29)| \lesssim \kappa^5 \sum_{j=1}^{s} \big| \operatorname{tr} \big\{ q_\kappa^{(s)} R_0 q_\kappa^{(j)} R_0 q_\kappa^{(s+1-j)} R_0 \big\} \big| + \big| \operatorname{tr} \big\{ q_\kappa^{(s)} R_0 q_\kappa^{(s-1)} R_0 q_\kappa^{(s-1)} R_0 \big\} \big|.
$$

The last term only appears in the case $s = 3$, but we can see that it vanishes by writing $q_\kappa^{(s)} = [\partial, q_\kappa^{(s-1)}]$ and cycling the trace. Note that all copies of q_κ now have at most s derivatives. We put the two highest order factors of q_κ in L^2 using the identity (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$. As $s \geq 3$, the third factor $q_\kappa^{(j)}$ has order $j \leq s - 2$

and may be estimated in operator norm because $\|q_\kappa^{(j)}\|_{L^\infty} \leq \|q_\kappa\|_{H^{s-1}} \lesssim 1$ by inductive hypothesis. This yields

$$|(3.3.29)| \lesssim \|q_\kappa^{(s)}\|_{L^2}^2 + \|q_\kappa^{(s)}\|_{L^2} \lesssim F_s + 1.$$

The second quadratic contribution (3.3.30) is similar. For the terms with $q_\kappa^{(s+1)}$ we "integrate by parts" to write

$$\begin{aligned} &16\kappa^5 \big(\operatorname{tr}\big\{ q_\kappa^{(s)} R_0 q_\kappa^{(s+1)} R_0 W R_0 \big\} + \operatorname{tr}\big\{ q_\kappa^{(s)} R_0 W R_0 q_\kappa^{(s+1)} R_0 \big\} \big) \\ &= 16\kappa^5 \operatorname{tr}\big\{ \big[\partial, q_\kappa^{(s)} R_0 q_\kappa^{(s)} R_0\big] W R_0 \big\} = -16\kappa^5 \operatorname{tr}\big\{ q_\kappa^{(s)} R_0 q_\kappa^{(s)} R_0 [\partial, W] R_0 \big\}. \end{aligned}$$

In all cases we put the two factors of q_κ in L^2 using the identity (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$, and the remaining factors in operator norm. This yields

$$|(3.3.30)| \lesssim \|q_\kappa^{(s)}\|_{L^2}^2 + \|q_\kappa^{(s)}\|_{L^2} \lesssim F_s + 1.$$

The quadratic W contribution (3.3.31) is easily estimated. We put $q_\kappa^{(s)}$ and the higher order W term in L^2 using the identity (3.2.3) and the observation $\|f\|_{H_\kappa^{-1}} \lesssim \kappa^{-1} \|f\|_{L^2}$, and we put the remaining factor of W in L^∞. This yields

$$|(3.3.31)| \lesssim \|q_\kappa^{(s)}\|_{L^2} \lesssim F_s + 1.$$

Next we turn to the series tail (3.3.32). Applying the tail convergence (3.2.21) to $q = q_\kappa$, we know there exists a constant $\kappa_0(F_s(t))$ so that

$$\begin{aligned} 16\kappa^5 \big\| \big\{ & g(\kappa, q_\kappa + W) + \langle \delta_x, R_0(q_\kappa + W) R_0 \delta_x \rangle \\ & - \langle \delta_x, R_0(q_\kappa + W) R_0 (q_\kappa + W) R_0 \delta_x \rangle \big\}^{(s+1)} \big\|_{L^2} \leq 1 \end{aligned}$$

uniformly for $\kappa \geq \kappa_0(F_s(t))$. Therefore, by Cauchy–Schwarz we have

$$|(3.3.32)| \leq (2F_s)^{1/2} \lesssim F_s + 1 \quad \text{uniformly for } \kappa \geq \kappa_0(F_s(t)).$$

Altogether, we have shown that there exists a constant $C = C(T, A)$ such that

$$\left| \frac{d}{dt} F_s \right| \leq C(F_s + 1) \quad \text{uniformly for } |t| \leq T \text{ and } \kappa \geq \kappa_0(F_s(t)).$$

Grönwall's inequality then yields the bound

$$F_s(t) \leq (F_s(0)+1)e^{CT} - 1 \quad \text{uniformly for } |t| \leq T,\ \kappa \geq \kappa_0\big((F_s(0)+1)e^{CT} - 1\big),$$

which concludes the inductive step. □

As a consequence, we are able to upgrade local well-posedness to global well-posedness:

Corollary 3.3.6. *Given an integer $s \geq 0$ and $A, T > 0$, there exists a constant κ_0 so that for $\kappa \geq \kappa_0$ the tidal H_κ flows* (3.3.3) *with initial data in the closed ball $B_A^s \subset H^s(\mathbb{R})$ of radius A are globally well-posed.*

Proof. Fix $A, T > 0$, let C be the constant guaranteed by Propositions 3.3.2 to 3.3.5, and consider the closed ball $B_C^s \subset H^s$ of radius C. By local well-posedness (cf. Lemma 3.3.1) we know there exists $\delta > 0$ such that the integral equation is a contraction on $C_t B_C^s([-\delta, \delta] \times \mathbb{R})$, and hence there exists a unique fixed point q_κ. However, by Propositions 3.3.2 to 3.3.5 we know that $q_\kappa(t)$ is in B_C^s as long as $|t| \leq T$. Therefore, we may iterate the contraction argument to construct a unique solution in $C_t H^s([-T, T] \times \mathbb{R})$ that depends continuously upon the initial data. □

3.4 Convergence at low regularity

Ultimately, we want to show that for initial data in H^s with $s \geq 3$ the solutions $q_\kappa(t)$ to the tidal H_κ flows converge in H^s. Although the linear and quadratic terms of the tidal H_κ flow formally converge to tidal KdV as $\kappa \to \infty$, the remainder contains $q_\kappa^{(5)}$ (cf. (3.2.19)). Consequently, we will first demonstrate convergence in H^{-2} so that we may absorb these five extra derivatives:

Proposition 3.4.1. *Given $T > 0$ and a bounded set $Q \subset H^3$ of initial data, the corresponding solutions $q_\kappa(t)$ to the tidal H_κ flows* (3.3.3) *are Cauchy in $C_t H^{-2}([-T, T] \times \mathbb{R})$ as $\kappa \to \infty$ uniformly for $q(0) \in Q$.*

Proof. In the following all spacetime norms will be taken over the slab $[-T,T]\times\mathbb{R}$. Let κ_0 denote the constant from Corollary 3.3.6 for $s=3$, so that for $\kappa\geq\kappa_0$ the solutions $q_\kappa(t)$ to the H_κ flows exist in C_tH^3.

Consider the difference $q_\kappa - q_\varkappa$ of two of these solutions with $\varkappa\geq\kappa\geq\kappa_0$. Recall that the tidal H_κ and tidal $H_\varkappa$ flows commute (cf. (3.3.2)). Letting H_κ^W denote the tidal H_κ flow Hamiltonian, this allows us to write

$$q_\varkappa(t) = e^{tJ\nabla H_\varkappa^W}q(0) = e^{tJ\nabla(H_\varkappa^W - H_\kappa^W)}e^{tJ\nabla H_\kappa^W}q(0).$$

Consequently, we estimate

$$\|q_\kappa - q_\varkappa\|_{C_tH^{-1}} \leq \sup_{q\in Q_T^*(\kappa)}\sup_{\varkappa\geq\kappa}\|e^{tJ\nabla(H_\varkappa^W - H_\kappa^W)}q - q\|_{C_tH^{-1}},$$

for the set

$$Q_T^*(\kappa) := \{e^{tJ\nabla H_\kappa^W}q(0) : |t|\leq T,\ q(0)\in Q\}$$

of tidal H_κ flows. By the fundamental theorem of calculus, it suffices to show that under the difference flow $H_\varkappa^W - H_\kappa^W$ we have

$$\sup_{q\in Q_T^*(\kappa)}\sup_{\varkappa\geq\kappa}\left\|\frac{dq}{dt}\right\|_{C_tH^{-2}} \to 0 \quad \text{as } \kappa\to\infty.$$

Note that $Q_T^*(\kappa)$ is a bounded subset of H^3 by the *a priori* estimate of Proposition 3.3.5.

Given initial data $q(0)\in Q_T^*(\kappa)$, let $q(t)$ denote the corresponding solution to the difference flow $H_\varkappa^W - H_\kappa^W$. Then q solves

$$\frac{d}{dt}q = 16\varkappa^5 g'(\kappa, q+W) + 4\varkappa^2(q+W)' - 16\kappa^5 g'(\varkappa, q+W) - 4\kappa^2(q+W)'.$$

To exhibit cancellation in the limit $\varkappa,\kappa\to\infty$, we expand $g'(\kappa,q+W)$ into a series in q and W and extract the linear and quadratic terms:

$$\begin{aligned}&\frac{d}{dt}q\\ &= \big\{-16\varkappa^5\langle\delta_x, R_0(\varkappa)(q+W)R_0(\varkappa)\delta_x\rangle + 4\varkappa^2(q+W)\\ &\qquad + 16\kappa^5\langle\delta_x, R_0(\kappa)(q+W)R_0(\kappa)\delta_x\rangle - 4\kappa^2(q+W)\big\}'\end{aligned} \tag{3.4.1}$$

$$+\big\{16\varkappa^5\langle\delta_x, R_0(\varkappa)(q+W)R_0(\varkappa)(q+W)R_0(\varkappa)\delta_x\rangle$$
$$- 16\kappa^5\langle\delta_x, R_0(\kappa)(q+W)R_0(\kappa)(q+W)R_0(\kappa)\delta_x\rangle\big\}' \tag{3.4.2}$$
$$+\sum(\text{terms with 3 or more } q \text{ or } W). \tag{3.4.3}$$

We will show that each of the terms (3.4.1)–(3.4.3) converge to zero.

For the linear term (3.4.1), we use the operator identity (3.2.19) to estimate

$$\begin{aligned}\|(3.4.1)\|_{H^{-2}} &= \|[-R_0(2\varkappa)+R_0(2\kappa)](q+W)^{(5)}\|_{H^{-2}}\\ &\lesssim (\varkappa^{-2}+\kappa^{-2})\big(\|q^{(5)}\|_{H^{-2}}+\|W^{(5)}\|_{H^{-2}}\big) \lesssim \kappa^{-2}\big(\,\|q\|_{H^3}+\|W'''\|_{L^2}\,\big)\end{aligned}$$

uniformly for $\varkappa \geq \kappa$. As $q \in Q_T^*(\kappa)$ is bounded in H^3, we conclude that

$$\sup_{q\in Q_T^*(\kappa)}\ \sup_{\varkappa\geq\kappa}\ \|(3.4.1)\|_{C_tH^{-2}} \to 0 \quad \text{as } \kappa\to\infty.$$

For the quadratic term (3.4.2), we add and subtract the corresponding tidal KdV term $6(q+W)(q+W)'$ and estimate

$$\begin{aligned}\|(3.4.2)\|_{H^{-2}} \lesssim &\,\big\|16\varkappa^5\langle\delta_x, R_0(\varkappa)(q+W)R_0(\varkappa)(q+W)R_0(\varkappa)\delta_x\rangle - 3(q+W)^2\big\|_{H^{-1}}\\ &+\big\|16\kappa^5\langle\delta_x, R_0(\kappa)(q+W)R_0(\kappa)(q+W)R_0(\kappa)\delta_x\rangle - 3(q+W)^2\big\|_{H^{-1}}.\end{aligned}$$

Using the operator identity (3.2.20) and the estimates $\|R_0(2\kappa)\partial^j\|_{\text{op}} \lesssim \kappa^{j-2}$ for $j=0,1,2$ (the estimate for $j=0$ is also true as an operator on L^∞ by the explicit kernel formula for R_0 and Young's inequality), one can easily prove by duality that

$$\big\|16\kappa^5\langle\delta_x, R_0(\kappa)fR_0(\kappa)gR_0(\kappa)\delta_x\rangle - 3fg\big\|_{L^2} \lesssim \kappa^{-2}\,\|f\|_{W^{2,\infty}}\,\|g\|_{H^2}\,.$$

Moreover, the roles of f and g can be exchanged since the identity (3.2.20) is symmetric in f and g. Therefore, expanding the products $(q+W)(q+W)$ we have

$$\|(3.4.2)\|_{H^{-2}} \lesssim (\varkappa^2+\kappa^2)\big(\,\|q\|_{H^3}^2 + \|W\|_{W^{3,\infty}}\,\|q\|_{H^3} + \|W\|_{W^{2,\infty}}\,\|W'\|_{H^2}\,\big).$$

As $q \in Q_T^*(\kappa)$ is bounded in H^3, we conclude that

$$\sup_{q\in Q_T^*(\kappa)}\ \sup_{\varkappa\geq\kappa}\ \|(3.4.2)\|_{C_tH^{-2}} \to 0 \quad \text{as } \kappa\to\infty.$$

It only remains to show that the tails (3.4.3) converge to zero in C_tH^{-2}. In fact, by (3.2.21) we have convergence in the stronger C_tL^2-norm:

$$\sup_{q\in Q_T^*(\kappa)} \sup_{\varkappa\geq\kappa} \|(3.4.3)\|_{C_tL^2} \to 0 \quad \text{as } \kappa\to\infty. \qquad \square$$

3.5 Equicontinuity

We want to upgrade the H^{-2} convergence of the previous section to H^s, $s\geq 3$. This will be accomplished via the estimate

$$\|q_\varkappa - q_\kappa\|_{H^s}^2 \lesssim (N+1)^{s+2} \|q_\varkappa - q_\kappa\|_{H^{-2}}^2 + \|q_\varkappa - q_\kappa\|_{H^s(|\xi|\geq N)}^2 .$$

In this section, we will show that we can pick N sufficiently large so that the second term on the RHS is arbitrarily small uniformly for $\kappa,\varkappa$ large. It then follows from Proposition 3.4.1 that the first term on the RHS converges to zero as $\kappa,\varkappa\to\infty$.

Uniform control over Fourier tails is closely related to equicontinuity. Given a Banach space $X(\mathbb{R})$ of functions on $\mathbb{R}$, we call a set $Q\subset X$ *equicontinuous* if

$$\sup_{q\in Q} \|q(\cdot+h)-q(\cdot)\|_X \to 0 \quad \text{as } h\to 0.$$

Note that for the supremum norm this definition coincides with the equicontinuity criterion of the Arzelà–Ascoli theorem. It is natural to call this property equicontinuity as previous authors have [50, 61, 123, 126], because it appears in the Kolmogorov–Riesz compactness theorem for the case $X = L^p$ [29, Th. 4.26]. In particular, it follows that a precompact subset of $H^s(\mathbb{R})$ is equicontinuous in $H^s(\mathbb{R})$.

For $X = H^s$, the Fourier transform provides us with the following characterization of equicontinuity:

Lemma 3.5.1 (Equicontinuity [97, §4])**.** *A bounded subset $Q\subset H^s(\mathbb{R})$ is equicontinuous if and only if*

$$\sup_{q\in Q} \int_{|\xi|\geq N} (\xi^2+1)^s|\hat q(\xi)|^2\,d\xi \to 0 \quad \text{as } N\to\infty.$$

In the case $s = -1$, this is also equivalent to

$$\sup_{q \in Q} \|q\|^2_{H^{-1}_\kappa} \to 0 \quad \text{as } \kappa \to \infty.$$

It would suffice to show that the tidal H_κ flows $\{q_\kappa(t) : |t| \le T,\ \kappa \ge \kappa_0\}$ are equicontinuous in $H^s(\mathbb{R})$. With the presence of the background wave W in tidal KdV, we expect that the H^s-norm of $q_\kappa(t)$ may grow in time and thus we must estimate its growth. Expanding the diagonal Green's function in powers of q_κ and W, we are able to control the linear and quadratic terms as we would for tidal KdV; however, it remains to control the higher order contributions which vanish in the limit $\kappa \to \infty$. Consequently, instead of honest equicontinuity for the tidal H_κ flows $q_\kappa(t)$, we will require $\kappa \ge N$ in Proposition 3.5.4 so that $O(\kappa^{-1})$ contributions as $\kappa \to \infty$ are also $O(N^{-1})$ as $N \to \infty$.

In order to control the Fourier tail growth we will use a smooth Littlewood–Paley decomposition. We define Littlewood–Paley pieces via the following L^2-based partition of unity. Fix a C^∞ function $\phi : \mathbb{R} \to [0, 1]$ that satisfies

$$\phi(\xi) = \begin{cases} 1 & |\xi| \le 1, \\ 0 & |\xi| \ge 2. \end{cases}$$

Then the function

$$\psi(\xi) := \sqrt{\phi(\xi) - \phi(2\xi)} \quad \text{satisfies} \quad \sum_{N \in 2^{\mathbb{Z}}} \psi^2(\tfrac{\xi}{N}) = 1 \quad \text{for all } \xi \neq 0.$$

Sums over capitalized indices will always be over the set $2^{\mathbb{Z}} := \{2^n : n \in \mathbb{Z}\}$. For Schwartz functions f we define

$$\widehat{P_N f}(\xi) = \psi(\tfrac{\xi}{N}) \hat{f}(\xi), \qquad \widehat{P^2_{\ge N} f}(\xi) = \sum_{K \ge N} \psi^2(\tfrac{\xi}{K}) \hat{f}(\xi), \qquad P^2_{<N} = 1 - P^2_{\ge N}.$$

Our choice of partition of unity ensures that the square sum $\sum P_N^2 f$ converges to f in L^p for $p \in (1, \infty)$. We choose a square-sum decomposition because we will ultimately measure $\|P_{\ge N} q_\kappa^{(s)}\|^2_{L^2}$, which we may write as the L^2-pairing of $P^2_{\ge N} q_\kappa^{(s)}$ and $q_\kappa^{(s)}$.

We remark that directly estimating the growth of $\|P_{\geq N} q^{(s)}\|_{L^2}^2$ would fail due to the quadratic term of tidal KdV. Indeed, if we compute $\frac{d}{dt}\|P_{\geq N} q^{(s)}\|_{L^2}^2$ under the tidal KdV flow, we obtain a term of the form

$$\int \left(P_{\geq N}^2 q^{(s)}\right) \left(3q^2\right)^{(s+1)} \, dx.$$

Decomposing each factor of $q = P_{\geq N}^2 q + P_{<N}^2 q$, the terms with at least one copy of $P_{\geq N}^2 q$ can be estimated by two factors of $\|P_{\geq N} q^{(s)}\|_{L^2}$. However, the high-low-low term

$$\int \left(P_{\geq N}^2 q^{(s)}\right) \left[3\left(P_{<N}^2 q\right)\left(P_{<N}^2 q\right)\right]^{(s+1)} \, dx$$

only contributes one factor of $\|P_{\geq N} q^{(s)}\|_{L^2}$, which does not guarantee that initially small Fourier tails remain small.

To overcome this, we introduce a more gradual high-frequency cutoff. Given an integer $s \geq 3$ and a Schwartz function f, we define the Fourier multiplier

$$\widehat{\Pi_{\geq N} f}(\xi) = m_{\mathrm{hi}}(\tfrac{\xi}{N})\widehat{f}(\xi), \qquad m_{\mathrm{hi}}(\xi) = \sum_{K<1} K^s \psi^2(\tfrac{\xi}{K}) + \sum_{K \geq 1} \psi^2(\tfrac{\xi}{K}). \tag{3.5.1}$$

The power of s in the definition (3.5.1) will provide us with the replacement (3.5.6) for the Bernstein inequality satisfied by $P_{\geq N}^2$. We also define

$$\widehat{\Pi_{<N} f}(\xi) = \sqrt{1 - m_{\mathrm{hi}}^2(\tfrac{\xi}{N})}\widehat{f}(\xi) \qquad \text{so that} \qquad \Pi_{<N}^2 + \Pi_{\geq N}^2 = 1.$$

For the Littlewood–Paley operators we have the familiar Bernstein inequalities

$$\begin{aligned} &\|P_N f^{(j)}\|_{L^p} \sim N^j \|P_N f\|_{L^p} \quad \text{for } p \in (1,\infty), \ j \in \mathbb{Z}, \\ &\|P_N f^{(j)}\|_{L^\infty} \lesssim N^j \|P_N f\|_{L^\infty} \quad \text{for } j > 0. \end{aligned} \tag{3.5.2}$$

Summing over $N \in 2^{\mathbb{N}}$, we obtain the high and low frequency projection estimates

$$\|P_{<N}^2 f^{(j)}\|_{L^p} \lesssim N^j \|P_{<N}^2 f\|_{L^p} \quad \text{for } p \in [1,\infty], \ j > 0, \tag{3.5.3}$$

$$\|P_{\geq N}^2 f\|_{L^p} \lesssim N^{-j} \|P_{\geq N}^2 f^{(j)}\|_{L^p} \quad \text{for } p \in (1,\infty), \ j > 0. \tag{3.5.4}$$

We will now obtain analogous Bernstein inequalities for our projection operators $\Pi_{\geq N}$ and $\Pi_{<N}$:

Lemma 3.5.2. *Fix an integer $s \geq 3$. Then the operators $\Pi_{\geq N}$ defined in* (3.5.1) *are bounded on L^p for $p \in [1, \infty]$ uniformly in N, and we have the estimates*

$$\|\Pi_{<N}^2 q^{(s+j)}\|_{L^p} \lesssim N^j \|P_{<2N}^2 q^{(s)}\|_{L^p} \quad \textit{for } p \in [1, \infty],\ j > 0, \tag{3.5.5}$$

$$\|\Pi_{\geq N}^2 q^{(s-j)}\|_{L^p} \lesssim N^{-j} \|\Pi_{\geq N} q^{(s)}\|_{L^p} \quad \textit{for } p \in (1, \infty),\ 0 < j \leq s. \tag{3.5.6}$$

Proof. Boundedness on L^p follows from Young's inequality. Indeed, if we let

$$m_{\text{lo}}(\tfrac{\xi}{N}) = \sqrt{1 - m_{\text{hi}}^2(\tfrac{\xi}{N})}$$

denote the Fourier symbol of $\Pi_{<N}$, then we have $m_{\text{lo}} \in C_c^\infty$ and

$$\|\Pi_{<N} f\|_{L^p} = \left\| N^d m_{\text{lo}}^\vee(N \cdot) * f \right\|_{L^p} \lesssim \left\| N^d m_{\text{lo}}^\vee(N \cdot) \right\|_{L^1} \|f\|_{L^p} = \|m_{\text{lo}}^\vee\|_{L^1} \|f\|_{L^p}$$

for any $p \in [1, \infty]$.

For the inequality (3.5.5) we may now assume that q is Schwartz by approximation. We use the Bernstein inequality (3.5.2) to estimate

$$\begin{aligned} \|\Pi_{<N}^2 q^{(s+j)}\|_{L^p} &\leq \sum_{K<N} \|P_K^2 q^{(s+j)}\|_{L^p} \lesssim \sum_{K<N} K^j \|P_K^2 q^{(s)}\|_{L^p} \\ &\lesssim \sum_{K<N} K^j \|P_{<2N}^2 q^{(s)}\|_{L^p} \lesssim N^j \|P_{<2N}^2 q^{(s)}\|_{L^p}. \end{aligned}$$

Note that in the second line we inserted the operator $P_{<2N}^2$ since $P_K^2 P_{<2N}^2 = P_K^2$ for $K < N$, and then used the boundedness of the operators P_K^2.

For the inequality (3.5.6), we use the Bernstein inequalities (3.5.2) and (3.5.4) to estimate

$$\begin{aligned} \|\Pi_{\geq N}^2 q^{(s-j)}\|_{L^p} &\leq \sum_{K<N} \tfrac{K^s}{N^s} \|P_K^2 \Pi_{\geq N} q^{(s-j)}\|_{L^p} + \|P_{\geq N}^2 \Pi_{\geq N} q^{(s-j)}\|_{L^p} \\ \lesssim \sum_{K<N} \tfrac{K^{s-j}}{N^s} \|\Pi_{\geq N} q^{(s)}\|_{L^p} &+ N^{-j} \|P_{\geq N}^2 \Pi_{\geq N} q^{(s)}\|_{L^p} \lesssim N^{-j} \|\Pi_{\geq N} q^{(s)}\|_{L^p} \end{aligned}$$

for Schwartz q. Note that in the second line we spent a factor of K^j to insert j derivatives on q, and then used the boundedness of the operators P_K. □

Next, we will prove an estimate for a commutator involving $\Pi_{\geq N}$ and $\Pi_{<N}$:

Lemma 3.5.3. *Let $\widetilde{P}_M^2 = \sum_{K=M/4}^{4M} P_K^2$ denote a fattened Littlewood–Paley projection. Then for all bounded functions $w \in L^\infty(\mathbb{R}^2)$ and Schwartz functions f, g, h we have*

$$\begin{aligned}\Big| \iint & \big[\big(\widehat{P_M^2 \Pi_{\geq N}^2 f}\big)(\xi) \big(\widehat{\Pi_{<N}^2 h}\big)(\xi - \eta) \\ & - \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} f}\big)(\xi) \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(\xi - \eta) \big] \big(\widehat{P_{<\frac{M}{8}}^2 g}\big)(\eta) w(\xi, \eta)\, d\xi\, d\eta \Big| \\ \lesssim & \|w\|_{L^\infty} \|P_M \Pi_{\geq N} f\|_{L^2} \|P_{<\frac{M}{8}}^2 g'\|_{H^1} \big(\tfrac{M^2}{N^3} \|\widetilde{P}_M^2 \Pi_{<N}^2 h\|_{L^2} + \|P_M \Pi_{\geq N} \Pi_{<N} h\|_{L^2} \big)\end{aligned}$$

uniformly for κ large.

Proof. Within the square brackets, we are interchanging a factor of $P_M \Pi_{\geq N}$ and $\Pi_{<N}$ between f and h. We change to Fourier variables and break this maneuver into two steps, first moving $P_M \Pi_{\geq N}$ and then moving $\Pi_{<N}$:

$$\begin{aligned}\Big| \iint & \big[\big(\widehat{P_M^2 \Pi_{\geq N}^2 f}\big)(\xi) \big(\widehat{\Pi_{<N}^2 h}\big)(\xi - \eta) \\ & - \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} f}\big)(\xi) \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(\xi - \eta) \big] \big(\widehat{P_{<\frac{M}{8}}^2 g}\big)(\eta) w(\xi, \eta)\, d\xi\, d\eta \Big| \\ = \iint & \big(\widehat{P_M \Pi_{\geq N} f}\big)(\xi) \big[\psi(\tfrac{\xi}{M}) m_{\mathrm{hi}}(\tfrac{\xi}{N}) - \psi(\tfrac{\xi-\eta}{M}) m_{\mathrm{hi}}(\tfrac{\xi-\eta}{N}) \big] \\ & \times \big(\widehat{\Pi_{<N}^2 h}\big)(\xi - \eta) \big(\widehat{P_{<\frac{M}{8}}^2 g}\big)(\eta) w(\xi, \eta)\, d\xi\, d\eta \qquad (3.5.7) \\ + \iint & \big(\widehat{P_M \Pi_{\geq N} f}\big)(\xi) \big[m_{\mathrm{lo}}(\tfrac{\xi-\eta}{N}) - m_{\mathrm{lo}}(\tfrac{\xi}{N}) \big] \\ & \times \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(\xi - \eta) \big(\widehat{P_{<\frac{M}{8}}^2 g}\big)(\eta) w(\xi, \eta)\, d\xi\, d\eta, \qquad (3.5.8)\end{aligned}$$

where ψ, m_{hi}, and m_{lo} are the Fourier multipliers for the operators P_M, $\Pi_{\geq N}$, and $\Pi_{<N}$ respectively. Observe that the RHS of the desired inequality vanishes for $M \geq 8N$. Consequently, we will estimate the terms (3.5.7) and (3.5.8) for $M \leq 4N$ and note that they vanish for $M \geq 8N$.

Observe that the integrand of the first term (3.5.7) is supported in the region $\frac{M}{2} \leq |\xi| \leq 2M$, $|\eta| \leq \frac{M}{8}$. On this region we have

$$|\xi - \eta| \geq |\xi| - |\eta| \geq \tfrac{M}{2} - \tfrac{M}{8} \geq \tfrac{M}{4}, \qquad |\xi - \eta| \leq |\xi| + |\eta| \leq 2M + \tfrac{M}{8} \leq 4M.$$

Therefore we can insert $\sum_{K=M/4}^{4M}\psi^2(\frac{\xi-\eta}{K})$ into the integrand, which is the Fourier multiplier for the fattened Littlewood–Paley projection $\widetilde{P}_M^2=\sum_{K=M/4}^{4M}P_K^2$ applied to h. Now $\widetilde{P}_M^2\Pi_{<N}^2 h$ vanishes for $M\geq 8N$, and so we may assume $M\leq 4N$.

Next, we will estimate the first term (3.5.7). By the fundamental theorem of calculus,

$$\begin{aligned}\big|\psi(\tfrac{\xi}{M})m_{\mathrm{hi}}(\tfrac{\xi}{N})-\psi(\tfrac{\xi-\eta}{M})m_{\mathrm{hi}}(\tfrac{\xi-\eta}{N})\big| &\leq \int_0^1 s|\eta|\big|\big(\psi(\tfrac{\cdot}{M})m_{\mathrm{hi}}(\tfrac{\cdot}{N})\big)'(\xi-s\eta)\big|\,ds\\ &\lesssim |\eta|\tfrac{M^{s-1}}{N^s}\quad \text{for } M\leq N.\end{aligned}$$

In the last inequality, we note that $\psi(\frac{\cdot}{M})m_{\mathrm{hi}}(\frac{\cdot}{N})$ is a function with amplitude M^s/N^s supported in an annulus of width M; indeed, for $M\leq N$ we have

$$\big|\big(\psi(\tfrac{\xi}{M})m_{\mathrm{hi}}(\tfrac{\xi}{N})\big)'\big|\leq \big|\psi(\tfrac{\xi}{M})'m_{\mathrm{hi}}(\tfrac{\xi}{N})\big|+\big|\psi(\tfrac{\xi}{M})m_{\mathrm{hi}}(\tfrac{\xi}{N})'\big|\lesssim M^{-1}\cdot\tfrac{M^s}{N^s}+1\cdot\tfrac{M^{s-1}}{N^s}.$$

This yields

$$\begin{aligned}|(3.5.7)| &\lesssim \|w\|_{L^\infty}\tfrac{M^{s-1}}{N^s}\|\widehat{P_M\Pi_{\geq N}f}\|_{L^2}\|\widehat{P^2_{<\frac{M}{8}}g'}\|_{L^1}\|\widehat{\widetilde{P}_M^2\Pi_{<N}^2h}\|_{L^2}\\ &\lesssim \|w\|_{L^\infty}\tfrac{M^{s-1}}{N^s}\|P_M\Pi_{\geq N}f\|_{L^2}\|P^2_{<\frac{M}{8}}g'\|_{H^1}\|\widetilde{P}_M^2\Pi_{<N}^2h\|_{L^2}.\end{aligned}$$

In the last inequality, we used Cauchy–Schwarz to estimate

$$\int\big|\big(\widehat{P^2_{<\frac{M}{8}}g'}\big)(\xi)\big|\,d\xi\leq\left(\int\frac{d\xi}{\xi^2+1}\right)^{\frac12}\left(\int(\xi^2+1)\big|\big(\widehat{P^2_{<\frac{M}{8}}g'}\big)(\xi)\big|^2\,d\xi\right)^{\frac12}.$$

For the second term (3.5.8), we note that the Fourier support of $\Pi_{\geq N}\Pi_{<N}h$ is bounded by N; in particular, $P_M\Pi_{\geq N}\Pi_{<N}h$ vanishes for $M\geq 8N$. For $M\leq 4N$ we estimate

$$\big|m_{\mathrm{lo}}(\tfrac{\xi-\eta}{N})-m_{\mathrm{lo}}(\tfrac{\xi}{N})\big|\leq\int_0^1 s|\eta|\big|\big(m_{\mathrm{lo}}(\tfrac{\cdot}{N})\big)'(\xi-s\eta)\big|\,ds\lesssim|\eta|N^{-1}.$$

This yields

$$\begin{aligned}|(3.5.8)| &\lesssim \|w\|_{L^\infty}N^{-1}\|\widehat{P_M\Pi_{\geq N}f}\|_{L^2}\|\widehat{P^2_{<\frac{M}{8}}g'}\|_{L^1}\|\widehat{P_M\Pi_{\geq N}\Pi_{<N}h}\|_{L^2}\\ &\lesssim \|w\|_{L^\infty}\tfrac{M^{s-1}}{N^s}\|P_M\Pi_{\geq N}f\|_{L^2}\|P^2_{<\frac{M}{8}}g'\|_{H^1}\|P_M\Pi_{\geq N}\Pi_{<N}h\|_{L^2}.\end{aligned}$$

Combining this with the estimate of (3.5.7), the claim follows. □

We are now equipped to prove our equicontinuity statement. Let $Q(N) \subset H^s$ for $N \in 2^{\mathbb{N}}$ be bounded sets of initial data that satisfy

$$Q(M) \supset Q(N) \text{ for } M \leq N, \quad \text{and} \quad \lim_{N\to\infty} \sup_{q(0)\in Q(N)} \|\Pi_{\geq N} q(0)\|_{H^s} = 0. \tag{3.5.9}$$

Proposition 3.5.4. *Fix an integer $s \geq 3$ and define the corresponding projection operator* (3.5.1)*. Given $T > 0$ and bounded sets $Q(N) \subset H^s$ of initial data satisfying* (3.5.9)*, the corresponding solutions $q_\kappa(t)$ to the tidal H_κ flow* (3.3.3) *obey*

$$\lim_{N\to\infty} \sup_{q(0)\in Q(N)} \sup_{\kappa\geq N} \|\Pi_{\geq N} q_\kappa(t)\|_{C_t H^s([-T,T]\times\mathbb{R})} = 0.$$

Proof. Expanding $g(\kappa, q_\kappa + W)$ in powers of q_κ and W, we write

$$\frac{d}{dt}\left(\|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2\right)$$

$$= \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\left\{-16\kappa^5\langle\delta_x, R_0 q_\kappa R_0 \delta_x\rangle + 4\kappa^2 q_\kappa\right\}^{(s+1)} dx \tag{3.5.10}$$

$$+ \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\left\{-16\kappa^5\langle\delta_x, R_0 W R_0 \delta_x\rangle + 4\kappa^2 W\right\}^{(s+1)} dx \tag{3.5.11}$$

$$+ 16\kappa^5 \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\langle\delta_x, R_0 q_\kappa R_0 q_\kappa R_0 \delta_x\rangle^{(s+1)} dx \tag{3.5.12}$$

$$+ 16\kappa^5 \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\left\{\langle\delta_x, (R_0 W R_0 q_\kappa R_0 + R_0 q_\kappa R_0 W R_0)\delta_x\rangle\right\}^{(s+1)} dx \tag{3.5.13}$$

$$+ 16\kappa^5 \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\langle\delta_x, R_0 W R_0 W R_0 \delta_x\rangle^{(s+1)} dx \tag{3.5.14}$$

$$\begin{aligned} &+ 16\kappa^5 \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\big\{g(\kappa, q_\kappa + W) + \langle\delta_x, R_0(q_\kappa + W)R_0\delta_x\rangle \\ &\qquad\qquad - \langle\delta_x, R_0(q_\kappa + W)R_0(q_\kappa + W)R_0\delta_x\rangle\big\}^{(s+1)} dx. \end{aligned} \tag{3.5.15}$$

We will estimate the terms (3.5.10)–(3.5.15) separately.

The first linear term (3.5.10) vanishes. To see this, we use the first operator identity of (3.2.19) to write

$$(3.5.10) = \int \left(\Pi_{\geq N}^2 q_\kappa^{(s)}\right)\left\{-16\kappa^4 R_0(2\kappa) q_\kappa + 4\kappa^2 q_\kappa\right\}^{(s+1)} dx = 0.$$

In the last equality we note that the integrand is odd in Fourier variables, or equivalently that the integrand is a total derivative because differentiation commutes with the Fourier multipliers $\Pi_{\geq N}$ and R_0.

Now we estimate the linear contribution (3.5.11) from W. Using the operator identity (3.2.20), we write

$$\begin{aligned}|(3.5.11)| &= \left|\int \big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)\big\{ -W^{(s+3)} - R_0(2\kappa)W^{(s+5)}\big\}\,dx\right| \\ &\lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}\big(\|\Pi_{\geq N} W^{(s+3)}\|_{L^2} + \kappa^{-2}\|W^{(s+5)}\|_{L^2}\big).\end{aligned}$$

Recalling that W' is Schwartz and $\kappa \geq N$, we obtain

$$\lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}\cdot N^{-2} \lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|^2_{L^2} + N^{-4}.$$

Next, we turn to the first quadratic contribution (3.5.12), which is nonvanishing due to the presence of the frequency cutoff $\Pi^2_{\geq N}$. We write

$$\begin{aligned}|(3.5.12)| &= 16\kappa^5\left|\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0\big[\partial^{s+1}, q_\kappa R_0 q_\kappa R_0\big]\right\}\right| \\ &\lesssim \sum_{j=0}^{s+1}\kappa^5\left|\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 q_\kappa^{(j)} R_0 q_\kappa^{(s+1-j)} R_0\right\}\right|.\end{aligned}$$

Decomposing the highest order $q_\kappa = \Pi^2_{\geq N}q_\kappa + \Pi^2_{<N}q_\kappa$ we have

$$\begin{aligned}&|(3.5.12)| \\ &\lesssim \sum_{j=0}^{\lfloor\frac{s+1}{2}\rfloor}\kappa^5\Big|\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 q_\kappa^{(j)} R_0 \big(\Pi^2_{\geq N} q_\kappa^{(s+1-j)}\big) R_0\right\} \\ &\qquad\qquad + \operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 \big(\Pi^2_{\geq N} q_\kappa^{(s+1-j)}\big) R_0 q_\kappa^{(j)} R_0\right\}\Big| \qquad (3.5.16)\\ &\quad + \sum_{j=0}^{\lfloor\frac{s+1}{2}\rfloor}\kappa^5\Big|\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 q_\kappa^{(j)} R_0 \big(\Pi^2_{< N} q_\kappa^{(s+1-j)}\big) R_0\right\} \\ &\qquad\qquad + \operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 \big(\Pi^2_{< N} q_\kappa^{(s+1-j)}\big) R_0 q_\kappa^{(j)} R_0\right\}\Big|. \qquad (3.5.17)\end{aligned}$$

First we will estimate the high-frequency contribution (3.5.16). We can "integrate by parts" to eliminate the terms with $q_\kappa^{(s+1)}$. Specifically, by cycling the trace we have

$$\begin{aligned}&\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0\big(\Pi^2_{\geq N} q_\kappa^{(s+1)}\big)R_0 q_\kappa R_0\right\} + \operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 q_\kappa R_0\big(\Pi^2_{\geq N} q_\kappa^{(s+1)}\big)R_0\right\} \\ &= \operatorname{tr}\left\{\big[\partial, \big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0\big] q_\kappa R_0\right\} = -\operatorname{tr}\left\{\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0\big(\Pi^2_{\geq N} q_\kappa^{(s)}\big)R_0 q_\kappa' R_0\right\}.\end{aligned}$$

For the remaining terms we use the Hilbert–Schmidt norm estimate (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1}\|f\|_{L^2}$ to put the two highest order terms in L^2, and we put the remaining terms in operator norm:

$$|(3.5.16)| \lesssim \sum_{j=1}^{\lfloor\frac{s+1}{2}\rfloor} \|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2}\|q_\kappa^{(j)}\|_{L^\infty}\|\Pi_{\geq N}^2 q_\kappa^{(s+1-j)}\|_{L^2}.$$

As $s \geq 3$ then the index j is at most $s-1$, and so the term $\|q_\kappa^{(j)}\|_{L^\infty}$ is uniformly bounded for $|t| \leq T$ and $\kappa \geq \kappa_0$ by the embedding $H^1 \hookrightarrow L^\infty$ and the *a priori* estimate of Proposition 3.3.5. The remaining term $\|\Pi_{\geq N}^2 q_\kappa^{(s+1-j)}\|_{L^2}$ either matches the first factor $\|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2}$ or is $\lesssim N^{-1}$ by the Bernstein inequality (3.5.6). Altogether we conclude

$$|(3.5.16)| \lesssim \|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2}^2 + \|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2}\cdot N^{-1} \lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 + N^{-2}.$$

The low-frequency contribution (3.5.17) requires more manipulation. We will push one factor of $\Pi_{\geq N}$ onto the low-frequency term and the resulting frequency cancellation will yield an acceptable contribution. As $\Pi_{\geq N}$ is not a sharp frequency cutoff, we divide the first factor $\Pi_{\geq N}^2 q_\kappa^{(s)}$ into its frequency scales:

$$\begin{aligned}|(3.5.17)| \lesssim \sum_M \sum_{j=0}^{\lfloor\frac{s+1}{2}\rfloor} \kappa^5\Big| &\operatorname{tr}\left\{\left(P_M^2\Pi_{\geq N}^2 q_\kappa^{(s)}\right)R_0 q_\kappa^{(j)} R_0\left(\Pi_{<N}^2 q_\kappa^{(s+1-j)}\right)R_0\right\} \\ &+ \operatorname{tr}\left\{\left(P_M^2\Pi_{\geq N}^2 q_\kappa^{(s)}\right)R_0\left(\Pi_{<N}^2 q_\kappa^{(s+1-j)}\right)R_0 q_\kappa^{(j)} R_0\right\}\Big|.\end{aligned} \tag{3.5.18}$$

Consider the first summand of RHS(3.5.18). We split $q_\kappa^{(j)} = P_{\geq\frac{M}{8}}^2 q_\kappa^{(j)} + P_{<\frac{M}{8}}^2 q_\kappa^{(j)}$ into high and low frequencies; the high-frequency contribution can be estimated directly, and for the low-frequency term we trade factors of $P_M\Pi_{\geq N}$ and $\Pi_{<N}$ between $q_\kappa^{(s)}$ and $q_\kappa^{(s+1-j)}$ to create a commutator:

$$\kappa^5 \operatorname{tr}\left\{\left(P_M^2\Pi_{\geq N}^2 q_\kappa^{(s)}\right)R_0 q_\kappa^{(j)} R_0\left(\Pi_{<N}^2 q_\kappa^{(s+1-j)}\right)R_0\right\}$$

$$= \kappa^5 \operatorname{tr}\left\{\left(P_M^2\Pi_{\geq N}^2 q_\kappa^{(s)}\right)R_0\left(P_{\geq\frac{M}{8}}^2 q_\kappa^{(j)}\right)R_0\left(\Pi_{<N}^2 q_\kappa^{(s+1-j)}\right)R_0\right\} \tag{3.5.19}$$

$$+ \kappa^5 \operatorname{tr}\left\{\left(P_M\Pi_{\geq N}\Pi_{<N} q_\kappa^{(s)}\right)R_0\left(P_{<\frac{M}{8}}^2 q_\kappa^{(j)}\right)R_0\left(P_M\Pi_{\geq N}\Pi_{<N} q_\kappa^{(s+1-j)}\right)R_0\right\} \tag{3.5.20}$$

$$+ \kappa^5 \operatorname{tr}\big\{\big[\big(\Pi_{<N}^2 q_\kappa^{(s+1-j)}\big)R_0\big(P_M^2\Pi_{\geq N}^2 q_\kappa^{(s)}\big)R_0 - \big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s+1-j)}\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big]\big(P_{<\frac{M}{8}}^2 q_\kappa^{(j)}\big)R_0\big\}. \tag{3.5.21}$$

For the term (3.5.19) we put the two highest order terms in L^2 and the lowest order term in L^∞. This yields

$$|(3.5.19)| \lesssim \begin{cases} \min\{\frac{M^s}{N^s},1\}\|P_M^2\Pi_{\geq N}q_\kappa^{(s)}\|_{L^2}\cdot M^{-2}\cdot N & \text{if } j=0,\\ \min\{\frac{M^s}{N^s},1\}\|P_M^2\Pi_{\geq N}q_\kappa^{(s)}\|_{L^2}\cdot M^{-1}\cdot 1 & \text{if } j\geq 1.\end{cases}$$

For the term (3.5.20), we can now integrate by parts for the $j=0$ case:

$$\begin{aligned}
&\operatorname{tr}\big\{\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big(P_{<\frac{M}{8}}^2 q_\kappa\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s+1)}\big)R_0\big\}\\
&\quad + \operatorname{tr}\big\{\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s+1)}\big)R_0\big(P_{<\frac{M}{8}}^2 q_\kappa\big)R_0\big\}\\
&= \operatorname{tr}\big\{\big[\partial, \big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big]\big(P_{<\frac{M}{8}}^2 q_\kappa\big)R_0\big\}\\
&= -\operatorname{tr}\big\{\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}q_\kappa^{(s)}\big)R_0\big(P_{<\frac{M}{8}}^2 q_\kappa'\big)R_0\big\},
\end{aligned}$$

which is now the summand for $j=1$. For $j\geq 1$ we put the two highest order terms in L^2 and the lowest order term in L^∞ to obtain

$$|(3.5.20)| \lesssim \begin{cases} \|P_M\Pi_{\geq N}q_\kappa^{(s)}\|_{L^2}^2\cdot 1 & \text{if } j=1,\\ \|P_M\Pi_{\geq N}q_\kappa^{(s)}\|_{L^2}\cdot 1\cdot N^{-1}\min\{\frac{M^s}{N^s},1\} & \text{if } j\geq 2.\end{cases}$$

For the commutator term (3.5.21) we will apply the estimate of Lemma 3.5.3 to the functions $f=q_\kappa^{(s)}$, $g=q_\kappa^{(j)}$, and $h=q_\kappa^{(s+1-j)}$. Writing the trace as an iterated integral and changing to Fourier variables, we have

$$\begin{aligned}
(3.5.21) = \kappa^5 \operatorname{tr}\big\{\big[&\big(\Pi_{<N}^2 h\big)R_0\big(P_M^2\Pi_{\geq N}^2 f\big)R_0\\
&- \big(P_M\Pi_{\geq N}\Pi_{<N}h\big)R_0\big(P_M\Pi_{\geq N}\Pi_{<N}f\big)R_0\big]\big(P_{<\frac{M}{8}}^2 g\big)R_0\big\}
\end{aligned}$$

$$= \frac{\kappa^5}{(2\pi)^{\frac{3}{2}}} \iiint \big[\big(\widehat{\Pi^2_{<N} h}\big)(\xi_1 - \xi_3)\big(\widehat{P_M^2 \Pi^2_{\geq N} f}\big)(\xi_3 - \xi_2) - \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(\xi_1 - \xi_3)\big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} f}\big)(\xi_3 - \xi_2)\big] \times \frac{\big(\widehat{P^2_{<\frac{M}{8}} g}\big)(\xi_2 - \xi_1)}{(\xi_3^2 + \kappa^2)(\xi_2^2 + \kappa^2)(\xi_1^2 + \kappa^2)}\, d\xi_1\, d\xi_2\, d\xi_3.$$

Changing variables $\eta_1 = \xi_2 - \xi_1$, $\eta_2 = \xi_3 - \xi_2$, $\eta_3 = \xi_3$, this becomes

$$(3.5.21) = \frac{\kappa^5}{(2\pi)^{\frac{3}{2}}} \iiint \big[\big(\widehat{\Pi^2_{<N} h}\big)(-\eta_1 - \eta_2)\big(\widehat{P_M^2 \Pi^2_{\geq N} f}\big)(\eta_2) - \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(-\eta_1 - \eta_2)\big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} f}\big)(\eta_2)\big] \times \frac{\big(\widehat{P^2_{<\frac{M}{8}} g}\big)(\eta_1)\, d\eta_1\, d\eta_2\, d\eta_3}{(\eta_3^2 + \kappa^2)((\eta_3 - \eta_2)^2 + \kappa^2)((\eta_3 - \eta_1 - \eta_2)^2 + \kappa^2)}.$$

The functions f, g, and h are now independent of η_3, and so we may evaluate the η_3 integral using residue calculus:

$$(3.5.21) = \frac{\kappa^4}{2(2\pi)^{\frac{1}{2}}} \iiint \big[\big(\widehat{\Pi^2_{<N} h}\big)(-\eta_1 - \eta_2)\big(\widehat{P_M^2 \Pi^2_{\geq N} f}\big)(\eta_2) - \big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} h}\big)(-\eta_1 - \eta_2)\big(\widehat{P_M \Pi_{\geq N} \Pi_{<N} f}\big)(\eta_2)\big] \times \frac{\big(\widehat{P^2_{<\frac{M}{8}} g}\big)(\eta_1)(24\kappa^2 + \eta_1^2 + \eta_2^2 + (\eta_1 + \eta_2)^2)}{(\eta_1^2 + 4\kappa^2)(\eta_2^2 + 4\kappa^2)((\eta_1 + \eta_2)^2 + \kappa^2)}\, d\eta_1\, d\eta_2.$$

This is now of the form of Lemma 3.5.3 for the multiplier

$$w(\xi, \eta) = \frac{\kappa^4(24\kappa^2 + \eta_1^2 + \eta_2^2 + (\eta_1 + \eta_2)^2)}{2(2\pi)^{\frac{1}{2}}(\eta_1^2 + 4\kappa^2)(\eta_2^2 + 4\kappa^2)((\eta_1 + \eta_2)^2 + \kappa^2)}.$$

Moreover, this multiplier is bounded uniformly in κ:

$$\|w\|_{L^\infty} = \tfrac{3}{16}(2\pi)^{-\frac{1}{2}} \quad \text{for all } \kappa > 0.$$

Therefore, by Lemma 3.5.3 and the Bernstein inequalities (3.5.6) and (3.5.5) we have

$$|(3.5.21)| \lesssim \begin{cases} \|P_M \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2} \|\widetilde{P}_M^2 \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2} + \|P_M \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 & j = 0, \\ \frac{M^{s-1}}{N^s} \|P_M \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2} + N^{-1} \|P_M \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 & j \geq 1, \end{cases}$$

for $M \leq 4N$.

We repeat the decomposition (3.5.19)–(3.5.21) for the second term in the summand of RHS(3.5.18). At each step we obtain the same estimates; indeed, although we cannot commute the operators within the trace, we still obtain the same integral because w was symmetric in ξ and η.

Altogether, we obtain the following estimate of the low-frequency quadratic contribution (3.5.17):

$$|(3.5.17)| \lesssim \sum_{M} \|\widetilde{P}_M \Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 + \sum_{M \leq 4N} \tfrac{M}{N^2} + \sum_{M \geq N} \tfrac{1}{M} \lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 + N^{-1}.$$

In the last inequality, we noted that the sum of the multipliers in Fourier variables is bounded.

For the quadratic term (3.5.13) involving q_κ and W we can repeat the decomposition (3.5.16)–(3.5.21). Previously we put $q_\kappa^{(0)}$ in L^∞ and not L^2 since it was the lowest order term, and consequently the same estimates apply because $W \in L^\infty$ and W' is Schwartz.

The quadratic term (3.5.14) for W can be estimated directly. Extracting the leading term as $\kappa \to \infty$, we write

$$\begin{aligned} &(3.5.14) \\ &= \int \big(\Pi_{\geq N}^2 q_\kappa^{(s)}\big)(3W^2)^{(s+1)}\,dx && (3.5.22) \\ &\quad + \int \big(\Pi_{\geq N}^2 q_\kappa^{(s)}\big)\big\{16\kappa^5 \langle \delta_x, R_0 W R_0 W R_0 \delta_x\rangle - 3W^2\big\}^{(s+1)}\,dx. && (3.5.23) \end{aligned}$$

For (3.5.22) we distribute the $s+1$ derivatives and move one $\Pi_{\geq N}$ off of q_κ:

$$\begin{aligned} |(3.5.22)| &\lesssim \sum_{j=0}^{s+1} \left| \int \big(\Pi_{\geq N} q_\kappa^{(s)}\big)\, \Pi_{\geq N}\big(W^{(j)} W^{(s+1-j)}\big)\,dx \right| \\ &\lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2} \cdot N^{-1} \lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 + N^{-2}. \end{aligned}$$

In the second line we noted that $W^{(j)}W^{(s+1-j)}$ is Schwartz since W' is Schwartz and $W \in L^\infty$ is smooth. For (3.5.23) we use the operator identity (3.2.20) and the estimates $\|R_0(2\kappa)\partial^j\|_{\mathrm{op}}$

$\lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on L^∞ by the explicit kernel formula for R_0 and Young's inequality) to prove by duality that

$$\left\| 16\kappa^5 \langle \delta_x, R_0(\kappa) f R_0(\kappa) h R_0(\kappa)\delta_x\rangle - 3fg \right\|_{L^2} \lesssim \kappa^{-2} \left\| f \right\|_{W^{2,\infty}} \left\| h \right\|_{H^2}.$$

Moreover, the roles of f and h can be exchanged since the identity (3.2.20) is symmetric in f and h. Distributing the $s+1$ derivatives and recalling $\kappa \geq N$, we estimate

$$|(3.5.23)| \lesssim N^{-2} \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2} \left\| W \right\|_{W^{s+3,\infty}} \left\| W' \right\|_{H^{s+3}} \lesssim \|\Pi_{\geq N} q_\kappa^{(s)}\|_{L^2}^2 + N^{-4}.$$

Finally, we estimate the tail (3.5.15) using Cauchy–Schwarz and (3.2.21):

$$|(3.5.15)| \lesssim \|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2} \cdot o(1) \lesssim \|\Pi_{\geq N}^2 q_\kappa^{(s)}\|_{L^2}^2 + o(1)$$

uniformly for $\kappa \geq N$ as $N \to \infty$. Note that $o(1)$ as $\kappa \to \infty$ implies $o(1)$ as $N \to \infty$ due to the restriction $\kappa \geq N$.

Altogether, we have shown there exists a constant C such that

$$\left| \frac{d}{dt} \|\Pi_{\geq N} q_\kappa^{(s)}(t)\|_{L^2}^2 \right| \leq C \|\Pi_{\geq N} q_\kappa^{(s)}(t)\|_{L^2}^2 + o(1) \quad \text{as } N \to \infty,$$

uniformly for $|t| \leq T$, $\kappa \geq N$, and $q(0) \in Q(N)$. By Grönwall's inequality, we then have

$$\|\Pi_{\geq N} q_\kappa^{(s)}(t)\|_{L^2}^2 \leq e^{CT} \|\Pi_{\geq N} q^{(s)}(0)\|_{L^2}^2 + o(1) \quad \text{as } N \to \infty,$$

uniformly for $|t| \leq T$, $\kappa \geq N$, and $q(0) \in Q(N)$. By (3.5.9), the term $\|\Pi_{\geq N} q^{(s)}(0)\|_{L^2}$ converges to zero as $N \to \infty$ uniformly for $q(0) \in Q(N)$. Therefore we conclude

$$\sup_{q(0)\in Q(N)} \sup_{\kappa \geq N} \|\Pi_{\geq N} q_\kappa(t)\|_{C_t H^s([-T,T]\times\mathbb{R})} \to 0 \quad \text{as } N \to \infty,$$

as desired. □

3.6 Well-posedness

The goal of this section is to prove our main result Theorem 3.1.1. The first step is show that the tidal H_κ flows converge in H^s as $\kappa \to \infty$ by combining the low-regularity convergence of Proposition 3.4.1 and the uniform Fourier tail control from Proposition 3.5.4:

Proposition 3.6.1. *Fix an integer $s \geq 3$ and $T > 0$. Given bounded sets $Q(\kappa) \subset H^s$ of initial data satisfying* (3.5.9)*, the corresponding tidal H_κ solutions $q_\kappa(t)$ are Cauchy in $C_t H^s([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$ uniformly for $q(0) \in Q(\kappa)$.*

Proof. In the following all spacetime norms will be over the slab $[-T,T] \times \mathbb{R}$. Splitting at a large frequency N to be chosen, we estimate

$$\|q_\varkappa - q_\kappa\|_{C_t H^s}^2 \lesssim (N+1)^{s+2} \|q_\varkappa - q_\kappa\|_{C_t H^{-2}}^2 + \|q_\varkappa - q_\kappa\|_{C_t H^s(|\xi| \geq N)}^2 . \tag{3.6.1}$$

For the second term we estimate

$$\|q_\varkappa - q_\kappa\|_{C_t H^s(|\xi|\geq N)}^2 \leq 2\big(\|\Pi_{\geq N} q_\varkappa\|_{C_t H^s}^2 + \|\Pi_{\geq N} q_\kappa\|_{C_t H^s}^2 \big). \tag{3.6.2}$$

Fix $\varepsilon > 0$. First, by Proposition 3.5.4 we take $N = N_0$ sufficiently large to ensure that RHS(3.6.2) is bounded by $\varepsilon/2$ for all $\varkappa, \kappa \geq N_0$. With N_0 fixed, we then use Proposition 3.4.1 to pick $\kappa_0 \geq N_0$ so that the first term of RHS(3.6.1) is bounded by $\varepsilon/2$ for all $\varkappa, \kappa \geq \kappa_0$. Together, we conclude that $\|q_\varkappa - q_\kappa\|_{H^s}^2 \leq \varepsilon$ for all $\varkappa, \kappa \geq \kappa_0$. □

Next, we show that the limits guaranteed by Proposition 3.6.1 solve tidal KdV:

Proposition 3.6.2. *Fix an integer $s \geq 3$ and $T > 0$. Given initial data $q(0) \in H^s(\mathbb{R})$, there exists a corresponding solution $q(t)$ to tidal KdV* (3.1.2) *in $(C_t H^s \cap C_t^1 H^{s-3})([-T,T] \times \mathbb{R})$.*

Proof. In the following all spacetime norms will be taken over the slab $[-T,T] \times \mathbb{R}$. Applying Proposition 3.6.1 to the single function $Q = \{q(0)\}$, we define $q(t)$ to be $\lim_{\kappa\to\infty} q_\kappa(t)$ which we know exists in $C_t H^s$. It remains to show that $\frac{d}{dt} q$ is in $C_t H^{s-3}$ and is equal to the RHS of tidal KdV (3.1.2). We already know that the RHS(3.1.2) is in $C_t H^{s-3}$, so it suffices to show that $\frac{d}{dt} q_\kappa$ converges to RHS(3.1.2) in the lower regularity norm $C_t H^{-1}$.

We will extract the linear and quadratic terms of the tidal H_κ flow to witness its convergence to tidal KdV. Using the translation identity (3.2.14), we write

$$\begin{aligned} &\frac{d}{dt} q_\kappa \\ &= -16\kappa^5 \langle \delta_x, R_0 q_\kappa' R_0 \delta_x \rangle + 4\kappa^2 q_\kappa' \end{aligned} \tag{3.6.3}$$

$$- 16\kappa^5\langle\delta_x, R_0W'R_0\delta_x\rangle + 4\kappa^2 W' \tag{3.6.4}$$

$$+ 16\kappa^5\langle\delta_x, [\partial, R_0q_\kappa R_0q_\kappa R_0]\delta_x\rangle \tag{3.6.5}$$

$$+ 16\kappa^5\big\{\langle\delta_x, [\partial, R_0WR_0q_\kappa R_0]\delta_x\rangle + \langle\delta_x, [\partial, R_0q_\kappa R_0WR_0]\delta_x\rangle\big\} \tag{3.6.6}$$

$$+ 16\kappa^5\langle\delta_x, [\partial, R_0WR_0WR_0]\delta_x\rangle \tag{3.6.7}$$

$$\begin{aligned} &+ 16\kappa^5\big\{g(\kappa, q_\kappa + W) + \langle\delta_x, R_0(q_\kappa + W)R_0\delta_x\rangle \\ &\qquad\qquad - \langle\delta_x, R_0(q_\kappa + W)R_0(q_\kappa + W)R_0\delta_x\rangle\big\}'. \end{aligned} \tag{3.6.8}$$

We will show that the first five terms (3.6.3)–(3.6.7) converge in C_tH^{-1} to the terms of tidal KdV (3.1.2), and the tail (3.6.8) converges to zero as $\kappa \to \infty$.

We begin with the linear contribution (3.6.3) from q_κ. Using the operator identity (3.2.19) we write

$$(3.6.3) = -q_\kappa''' - R_0(2\kappa)\partial^2(q_\kappa - q)''' - R_0(2\kappa)\partial^2 q'''.$$

As $q_\kappa \to q$ in C_tH^s, the first term on the RHS converges to $-q'''$ in C_tH^{s-3} and the second term converges to zero in C_tH^{s-3} since $\|R_0(2\kappa)\partial^2\|_{\mathrm{op}} \lesssim 1$ uniformly in κ. The last term converges to zero since the operator $R_0(2\kappa)\partial^2$ is readily seen via Fourier variables to converge strongly to zero as $\kappa \to \infty$. As the regularity $s - 3 \geq 0$ is greater than -1, we conclude

$$(3.6.3) \to -q''' \quad \text{in } C_tH^{-1} \text{ as } \kappa \to \infty.$$

For the linear contribution (3.6.4) from W, we again use the operator identity (3.2.19) we write

$$(3.6.4) = -W''' - R_0(2\kappa)\partial^2 W'''.$$

As W' is Schwartz and the operator $R_0(2\kappa)\partial^2$ converges strongly to zero as $\kappa \to \infty$, the second term converges to zero in C_tH^s and hence in C_tH^{-1}. Consequently,

$$(3.6.4) \to -W''' \quad \text{in } C_tH^{-1} \text{ as } \kappa \to \infty.$$

Next we turn to the first quadratic term (3.6.5). We write

$$(3.6.5) = 6q_\kappa q_\kappa' + \big\{16\kappa^5\langle\delta_x, [\partial, R_0q_\kappa R_0q_\kappa R_0]\delta_x\rangle - 6q_\kappa q_\kappa'\big\}.$$

As $q_\kappa \to q$ in $C_t H^s$, then the first term of the RHS above converges to $6qq'$ in $C_t H^{s-1}$ and hence in $C_t H^{-1}$ as well. For the second term we estimate in H^{-1} by duality. For $\phi \in H^1$ we distribute the derivative $[\partial, \cdot]$ using the product rule and use the operator identity (3.2.20) to obtain

$$\begin{aligned}&\int \big\{16\kappa^5\langle \delta_x, [\partial, R_0 q_\kappa R_0 q_\kappa R_0]\delta_x\rangle - 6q_\kappa q_\kappa'\big\}\phi\, dx\\ &= \int \big\{-6[R_0(2\kappa)q_\kappa''][R_0(2\kappa)q_\kappa''']\phi + 8\kappa^2[R_0(2\kappa)q_\kappa'][R_0(2\kappa)q_\kappa''](-5\phi + R_0(2\kappa)\partial^2\phi)\\ &\qquad\qquad + 8\kappa^2[R_0(2\kappa)q_\kappa][R_0(2\kappa)q_\kappa'](5\phi'' + 2R_0(2\kappa)\partial^2\phi'')\big\}\, dx.\end{aligned}$$

For each term on the RHS, we put two terms in L^2 and the remaining term in L^∞. For those terms with ϕ'' we integrate by parts once, we put all factors of ϕ' in L^2, and we put ϕ in $L^\infty \supset H^1$. We put the highest order q_κ term in L^2 and the lower order term in L^2 or L^∞ as needed. Using $\|R_0(2\kappa)\partial^j\|_{\mathrm{op}} \lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on L^∞ by the explicit kernel formula for R_0 and Young's inequality), we obtain

$$\left|\int \big\{16\kappa^5\langle \delta_x, [\partial, R_0 q_\kappa R_0 q_\kappa R_0\delta_x\rangle - 6q_\kappa q_\kappa\big\}\phi\, dx\right| \lesssim \kappa^{-2}\, \|\phi\|_{H^1}\, \|q_\kappa\|_{H^s}^2 .$$

Taking a supremum over $\|\phi\|_{H^1} \le 1$, we conclude

$$(3.6.5) \to 6qq' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$

The second quadratic term (3.6.6) is similar, but now we must put W in L^∞. First we write

$$\begin{aligned}(3.6.6) = 6(Wq_\kappa)' + \big\{&16\kappa^5\langle \delta_x, [\partial, R_0 W R_0 q_\kappa R_0]\delta_x\rangle\\ &+ 16\kappa^5\langle \delta_x, [\partial, R_0 q_\kappa R_0 W R_0]\delta_x\rangle - 6(Wq_\kappa)'\big\}.\end{aligned}$$

As $q_\kappa \to q$ in $C_t H^s$, the first term of the RHS above converges to $6(Wq)'$ in $C_t H^{s-1}$ and hence in $C_t H^{-1}$ as well. For the second term we estimate in H^{-1} by duality. For $\phi \in H^1$ we distribute the derivatives $[\partial, \cdot]$ fusing the product rule and use the operator identity (3.2.20). For the term $\langle \delta_x, R_0 W R_0 q_\kappa' R_0 \delta_x\rangle$ this yields

$$\int \big\{16\kappa^5\langle \delta_x, R_0 W R_0 q_\kappa' R_0\delta_x\rangle - 3Wq_\kappa'\big\}\phi\, dx$$

$$= \int \big\{-3[R_0(2\kappa)W''][R_0(2\kappa)q_\kappa''']\phi + 4\kappa^2[R_0(2\kappa)W'][R_0(2\kappa)q_\kappa''](-5\phi + R_0(2\kappa)\partial^2\phi)$$
$$+ 4\kappa^2[R_0(2\kappa)W][R_0(2\kappa)q_\kappa'](5\phi'' + 2R_0(2\kappa)\partial^2\phi'')\big\}\, dx.$$

This equality also holds for the second term $\langle \delta_x, R_0 q_\kappa' R_0 W R_0 \delta_x\rangle$ because the identity (3.2.20) is symmetric in f and h. For those terms with ϕ'' we integrate by parts once to obtain ϕ' which we put in L^2, we put all factors of W in L^∞, and we put the remaining terms in L^2. This yields

$$\left| \int \big\{16\kappa^5 \langle \delta_x, R_0 W R_0 q_\kappa' R_0 \delta_x\rangle - 3Wq_\kappa'\big\}\phi\, dx\right| \lesssim \kappa^{-2}\, \|\phi\|_{H^1}\, \|q_\kappa\|_{H^s}\,,$$

and similarly for the term $\langle \delta_x, R_0 q_\kappa' R_0 W R_0 \delta_x\rangle$. The remaining two contributions from $\langle \delta_x, R_0 W' R_0 q_\kappa R_0 \delta_x\rangle$ and $\langle \delta_x, R_0 q_\kappa R_0 W' R_0 \delta_x\rangle$ are even easier, since W' is Schwartz and q_κ has one less derivative. Taking a supremum over $\|\phi\|_{H^1} \le 1$, we conclude

$$(3.6.6) \to 6(Wq)' \quad \text{in } C_tH^{-1} \text{ as } \kappa \to \infty.$$

The third quadratic term (3.6.7) is similar. We write

$$(3.6.7) = 6WW' + \big\{16\kappa^5 \langle \delta_x, [\partial, R_0 W R_0 W R_0]\delta_x\rangle - 6WW'\big\}.$$

We easily estimate the second term above using the operator identity (3.2.20) and noting that $W \in L^\infty$ and W' is Schwartz. This yields

$$(3.6.7) \to 6WW' \quad \text{in } C_tH^{-1} \text{ as } \kappa \to \infty.$$

Lastly, we show that the tail (3.6.8) converges to zero in C_tH^{-1}. We will estimate in H^{-1} by duality. For $\phi \in H^1$ we write

$$\left|\int \phi \cdot (3.6.8)\, dx\right| \le 16\kappa^5 \sum_{\substack{\ell \ge 0,\ m_0,\dots,m_\ell \ge 0 \\ \ell + m_0 + \dots + m_\ell \ge 3}} \Big| \operatorname{tr}\big\{\phi[\partial, R_0 (WR_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (WR_0)^{m_\ell}]\big\}\Big|.$$

Recall that we first expanded $g(\kappa, q_\kappa + W)$ in powers of q_κ, the ℓth term having ℓ-many factors of $q_\kappa R(\kappa, W)$, and then expanded each $R(\kappa, W)$ into a series in W indexed by m_i. The condition $\ell + m_0 + \cdots + m_\ell \geq 3$ reflects that we have already accounted for all of the summands with one and two q_κ or W. We distribute the derivative $[\partial, \cdot]$, use the estimate (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1} \|f\|_{L^2}$ to put ϕ and all copies of q_κ in L^2, and then estimate W in operator norm to obtain

$$\lesssim \kappa^5 \sum_{\substack{\ell \geq 0,\ m_0, \ldots, m_\ell \geq 0 \\ \ell + m_0 + \cdots + m_\ell \geq 3}} \frac{\|\phi\|_{L^2}}{\kappa^{3/2}} \left(\frac{\|q_\kappa\|_{H^1}}{\kappa^{3/2}} \right)^\ell \left(\frac{\|W\|_{W^{1,\infty}}}{\kappa^2} \right)^{m_0 + \cdots + m_\ell}.$$

We first sum over the indices $m_0, \ldots, m_\ell \geq 0$ as we did in (3.2.15) using that $W \in W^{1,\infty}$, and then we sum over $\ell \geq 1$ since q_κ is bounded in $C_t H^s$ for all κ large. In doing so, the condition $\ell + m_0 + \cdots + m_\ell \geq 3$ guarantees that summing over the two pararenthetical terms yields a gain $\lesssim (\kappa^{-3/2})^3$, from which we conclude

$$\lesssim \kappa^{-1} \|\phi\|_{H^1}.$$

Taking a supremum over $\|\phi\|_{H^1} \leq 1$ we obtain

$$(3.6.8) \to 0 \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty. \qquad \square$$

We now use a classical L^2 energy argument to show that we have unconditional uniqueness for initial data in H^s, $s \geq 3$:

Lemma 3.6.3. *Fix $T > 0$. Given an initial data $q(0) \in H^3$, there is at most one corresponding solution to tidal KdV* (3.1.2) *in $(C_t H^3 \cap C^1_t L^2)([-T, T] \times \mathbb{R})$.*

Proof. Suppose $q(t)$ and $\tilde{q}(t)$ are both in $(C_t H^3 \cap C^1_t L^2)([-T, T] \times \mathbb{R})$, solve tidal KdV, and have the same initial data $q(0) = \tilde{q}(0)$. From the differential equation, we see that the difference obeys

$$\left| \frac{d}{dt} \int \tfrac{1}{2}(q - \tilde{q})^2 \, dx \right| = \left| \int (q - \tilde{q}) \{ -(q - \tilde{q})''' + 3(q^2 - \tilde{q}^2)' + [6W(q - \tilde{q})]' \} \, dx \right|.$$

The first term $(q-\tilde{q})'''$ contributes a total derivative and vanishes, while the remaining terms can be integrated by parts to obtain

$$\begin{aligned} &= \left| \int (q-\tilde{q})^2 \{ \tfrac{3}{2}(q+\tilde{q})' + 3W' \}(t,x)\,dx \right| \\ &\leq \left(\tfrac{3}{2}\|q'\|_{L^\infty} + \tfrac{3}{2}\|\tilde{q}'\|_{L^\infty} + 3\|W'\|_{L^\infty} \right) \|q-\tilde{q}\|_{L^2}^2 . \end{aligned}$$

Estimating $\|q'\|_{L^\infty} \lesssim \|q\|_{H^2}$, $\|\tilde{q}'\|_{L^\infty} \lesssim \|\tilde{q}\|_{H^2}$ and noting that W' is Schwartz, we conclude that there exists a constant C depending on W and the norm of q and $\tilde{q}$ in $C_tH^3([-T,T]\times\mathbb{R})$ such that

$$\left| \frac{d}{dt} \|q(t)-\tilde{q}(t)\|_{L^2}^2 \right| \leq C\,\|q(t)-\tilde{q}(t)\|_{L^2}^2 .$$

Grönwall's inequality then yields

$$\|q(t)-\tilde{q}(t)\|_{L^2}^2 \leq \|q(0)-\tilde{q}(0)\|_{L^2}^2\, e^{CT}$$

uniformly for $|t|\leq T$. As the RHS vanishes by premise, we conclude that $\tilde{q}(t)=q(t)$ for all $|t|\leq T$. □

We are now ready to prove our main result. It remains to show that the solution depends continuously upon the initial data in H^s for $s\geq 3$.

Proof of Theorem 3.1.1. Fix an integer $s\geq 3$. We want to show that tidal KdV (3.1.2) is globally well-posed for initial data $q(0)\in H^s(\mathbb{R})$.

Fix $T>0$ and a convergent sequence $q_n(0)$ of initial data in $H^s(\mathbb{R})$. It suffices to show that the corresponding solutions $q_n(t)$ of tidal KdV (3.1.2) constructed in Proposition 3.6.2 are Cauchy in $C_tH^s([-T,T]\times\mathbb{R})$ as $n\to\infty$.

Consider the set $Q := \{q_n(0) : n\in\mathbb{N}\}$ of initial data, which is bounded and equicontinuous in H^s since it is convergent in H^s. Let H_κ^W denote the Hamiltonian for the tidal H_κ flow. We estimate

$$\begin{aligned} \|q_n(t)-q_m(t)\|_{C_tH^s} \leq{}& \|e^{tJ\nabla H_\kappa^W} q_n(0) - e^{tJ\nabla H_\kappa^W} q_m(0)\|_{C_tH^s} \\ &+ 2\sup_{q\in Q}\sup_{\varkappa\geq\kappa} \|e^{tJ\nabla H_\varkappa^W} q - e^{tJ\nabla H_\kappa^W} q\|_{C_tH^s}, \end{aligned} \tag{3.6.9}$$

where the spacetime norms are over the slab $[-T, T] \times \mathbb{R}$. By Proposition 3.6.1, the second term of RHS(3.6.9) can be made arbitrarily small uniformly in n, m by picking κ sufficiently large. The first term of RHS(3.6.9) then converges to zero as $n, m \to \infty$ due to the well-posedness of the tidal H_κ flow (cf. Corollary 3.3.6). $\square$

CHAPTER 4

Well-posedness for $H^{-1}(\mathbb{R})$ perturbations of solutions with exotic spatial asymptotics

4.1 Introduction

In this chapter, we will show that KdV is well-posed for $H^{-1}(\mathbb{R})$ perturbations of the step-like solutions constructed in the previous chapter. More broadly, we aim to develop a framework that applies to any class of initial data that is of physical interest. One important example is initial data that is asymptotically periodic. This includes localized perturbations of a single periodic profile, wave dislocation where the periods as $x \to \pm\infty$ may not align, and waves with altogether different periodic asymptotics as $x \to \pm\infty$. Quasi-periodic spatial asymptotics are also heavily studied in the literature (see, for example, [21, 43, 45, 46, 55]). As we will discuss more thoroughly below, these classes are excluded by traditional analysis on the circle $\mathbb{R}/\mathbb{Z}$.

Specifically, we employ the method of commuting flows that was introduced in [97] to prove well-posedness in $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{R}/\mathbb{Z})$, and developed in several subsequent papers [30, 72, 73, 93–95, 121]. This method relies upon approximating the dynamics of KdV by the flow of the Hamiltonians H_κ defined by (1.4.7) as $\kappa \to \infty$.

Given a solution $V(t, x)$ to KdV we define

$$V_\kappa(t, x) \text{ to be the solution to the } H_\kappa \text{ flow with initial data } V(0, x).$$

We will assume that the background wave V is sufficiently regular in the following sense.

Definition 4.1.1. We call the background wave $V(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *admissible* if for every $T > 0$ it satisfies the following:

(i) V solves KdV (1.1.1) and is bounded in $W^{2,\infty}(\mathbb{R}_x)$ uniformly for $|t| \leq T$,

(ii) The H_κ flows V_κ are bounded in $W^{4,\infty}(\mathbb{R}_x)$ uniformly for $|t| \leq T$ and $\kappa > 0$ sufficiently large,

(iii) $V_\kappa - V \to 0$ in $W^{2,\infty}(\mathbb{R}_x)$ as $\kappa \to \infty$ uniformly for $|t| \leq T$ and initial data in the set $\{V_\varkappa(t) : |t| \leq T,\ \varkappa \geq \kappa\}$.

In this paper we will show that for admissible waves V the KdV equation (1.1.1) is well-posed for $H^{-1}(\mathbb{R})$ perturbations of V (cf. Corollary 4.5.4):

Theorem 4.1.2 (Global well-posedness)**.** *Given V that is admissible in the sense of Definition 4.1.1, the KdV equation* (1.1.1) *with initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ is globally well-posed in the following sense: $u(t) = V(t) + q(t)$ and $q(t)$ is given by a jointly continuous data-to-solution map $\mathbb{R}_t \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ for the equation*

$$\frac{d}{dt} q = -q''' + 6qq' + 6(Vq)' \tag{4.1.1}$$

with initial data $q(0) = u(0) - V(0)$.

Just as $H^{-1}(\mathbb{R})$ is the lowest regularity for which we can hope to have well-posedness in the case $W \equiv 0$ [116], we expect that Corollary 4.1.4 is sharp in the class of $H^s(\mathbb{R})$ spaces. There is a known technique [97, Cor. 5.3] for extending $H^{-1}(\mathbb{R})$ well-posedness to $H^s(\mathbb{R})$, $s > -1$; the key ingredient in the argument is H^{-1}-equicontinuity, which will be further discussed below. In this way, $H^{-1}(\mathbb{R})$ is the key space for establishing well-posedness.

As we cannot make sense of the nonlinearity of KdV for $H^{-1}(\mathbb{R})$ solutions (even in the distributional sense), the solutions in Theorem 4.1.2 will be constructed as limits of the H_κ flows as $\kappa \to \infty$. This is the right notion of solution because it coincides with the classical notion on a dense subset of initial data and Theorem 4.1.2 guarantees continuous dependence of the solution upon the initial data. Specifically, in Section 4.6 we show that for initial data $u(0) \in V(0) + H^3(\mathbb{R})$ our solution $u(t)$ solves KdV and is unique:

Theorem 4.1.3. *Fix V admissible and $T > 0$. Given initial data $u(0) \in V(0) + H^3(\mathbb{R})$, the solution $u(t)$ constructed in Theorem 4.1.2 lies in $V(t) + (C_tH^2 \cap C_t^1H^{-1})\,([-T,T] \times \mathbb{R})$ for all $T > 0$, solves KdV* (1.1.1)*, and is unique in this class.*

There are rich classes of initial data that are admissible according to our definition. Our first application will be to the step-like background waves from Chapter 3. Applying Theorem 3.1.1 to the initial data $q(0) \equiv 0$, we conclude that given W there is a unique global solution $V(t) = W + q(t)$ to KdV (1.1.1) with initial data W, and $t \mapsto V(t) - W$ is a continuous function into $H^s(\mathbb{R})$ for all $s \geq 3$. From our understanding of $V(t)$ at high-regularity (namely Corollary 3.3.6 and Propositions 3.6.1 and 3.6.2), we are able to verify that $V(t)$ is admissible:

Corollary 4.1.4 (Step-like background)**.** *Given initial data $V(0) = W$ of the form* (3.1.1)*, the KdV equation* (1.1.1) *with initial data $u(0) \in W + H^{-1}(\mathbb{R})$ is globally well-posed in the sense of Theorem 4.1.2.*

It is natural to ask whether KdV is also well-posed for $H^{-1}(\mathbb{R})$ perturbations of W. Corollary 4.1.4 and Theorem 3.1.1 provide an affirmative answer to this question. By Corollary 4.1.4, there exists a solution $u(t) = V(t) + q(t)$ to KdV (1.1.1) with initial data $u(0) = W + q(0)$ in $W + H^{-1}(\mathbb{R})$. Together with Theorem 3.1.1, we also obtain that $t \mapsto u(t) - W$ is a continuous function into $H^{-1}(\mathbb{R})$ that depends continuously upon the initial data. For a precise statement of this well-posedness, see Corollary 4.7.1. We do not use this formulation in the statement of Corollary 4.1.4 because it does not reflect the reality of the proof.

Our second application will be to the important case of smooth periodic initial data $V(0,x)$ (cf. Corollary 4.9.4):

Corollary 4.1.5 (Periodic background)**.** *Given* $V(0) \in H^5(\mathbb{R}/\mathbb{Z})$*, the KdV equation* (1.1.1) *is globally well-posed for* $u(0) \in V(0) + H^{-1}(\mathbb{R})$ *in the sense of Theorem 4.1.2.*

In particular, this includes the cnoidal wave solutions (1.2.1) of KdV. Indeed, among all periodic asymptotics appearing in the literature, these are the most common choice for the background wave V. We have already seen in Section 1.2 that there is great interest in perturbations of cnoidal waves from the perspectives of orbital and spectral stability.

More generally, existence for the Cauchy problem with periodic spatial asymptotics was first addressed in the physics literature for the case of cnoidal waves [103], and again for more general periodic backgrounds in [57–59] via the inverse scattering transform. A complete mathematical treatment of the Cauchy problem for highly regular initial data with distinct periodic asymptotics as $x \to \pm\infty$ was later given in [52, 53].

We believe that Theorem 4.1.2 can also be applied to classes of quasi-periodic initial data, or any other class amenable to complete integrability methods. This would require showing that a given class of background waves V are admissible; while such V may be highly studied, their H_κ flows V_κ are not. Nevertheless, the known results on exotic backgrounds use integrable methods like the inverse scattering transform, which are well suited to treat the H_κ flows and verify the admissibility criteria.

In all cases, the presence of the background wave V breaks the macroscopic conservation laws of KdV. If q is a regular solution to (4.1.1) then the momentum functional E_1 (defined in (1.1.3)) is not conserved, but rather evolves according to (1.2.3). Interpreting V as a potential-like function that affects the change in momentum, we will refer to (4.1.1) as *KdV with potential.* Despite this lack of conservation, we are able to adapt the method of commuting flows to KdV with potential (4.1.1) because these conserved quantities do not blow up in finite time.

Just as the Hamiltonian H_κ in (1.4.7) approximates E_2, we define $\widetilde{H}_\kappa$ to approximate the dynamics of KdV with potential (4.1.1); subtracting the background V from u we obtain the $\widetilde{H}_\kappa$ flow of $q = u - V$ from time 0 to t:

$$\widetilde{\Phi}_\kappa(t)q = e^{tJ\nabla H_\kappa}(q + V(0)) - V_\kappa(t).$$

We will show that for admissible V, the $\widetilde{H}_\kappa$ flow is well-posed in $H^{-1}(\mathbb{R})$ and converges to the KdV equation with potential (4.1.1) in $H^{-1}(\mathbb{R})$ uniformly on bounded time intervals as $\kappa \to \infty$. As in [97], one asset of this method is that well-posedness of the $\widetilde{H}_\kappa$ flow in $H^{-1}(\mathbb{R})$ follows from an ODE argument because α is real-analytic on $V + H^{-1}(\mathbb{R})$ and E_1 generates translations.

The major structural difference of our argument from that in [97] is that we cannot assume the existence of regular solutions to (4.1.1). Although some results in this direction do exist (e.g. [52, 53]), we would need to significantly increase our assumptions on the background wave V in order to employ them. Instead of showing that the $\widetilde{H}_\kappa$ flows converge to that of KdV with potential as $\kappa \to \infty$, we show that the $\widetilde{H}_\kappa$ flows are Cauchy and we define the limit to be an H^{-1} solution of (4.1.1). To verify that this is indeed the solution map, we show that it is a jointly continuous map from $\mathbb{R}_t \times H^{-1}(\mathbb{R})$ to $H^{-1}(\mathbb{R})$ and agrees with the classical notion of solution on a dense subset of initial data (cf. Theorem 4.1.3). The proof of Theorem 4.1.3 relies on an energy argument in $H^3(\mathbb{R})$ similar to that of Bona–Smith [24], using the fact that the H_κ flows also conserve the polynomial conservation laws of KdV (as is suggested by the asymptotic expansion (1.4.6) and Poisson commutativity (1.4.1)).

Following the argument of [97], the convergence of the $\widetilde{H}_\kappa$ flows in $H^{-1}(\mathbb{R})$ as $\kappa \to \infty$ is implied by convergence at some lower $H^s(\mathbb{R})$ regularity together with equicontinuity in $H^{-1}(\mathbb{R})$ for all κ large. In the $V \equiv 0$ case [97], equicontinuity quickly followed from the estimates (4.2.19) and the conservation of the functional $\alpha(\varkappa, q)$ under both the H_κ and KdV flows. However, the appearance of the background wave V in the $\widetilde{H}_\kappa$ flow breaks the conservation of $\alpha(\varkappa, q)$. Instead of obtaining full equicontinuity, we introduce a dependence between the energy parameters κ and $\varkappa$ in the proof of convergence (cf. Proposition 4.5.2) that we can match when we estimate the growth of $\alpha(\varkappa, q)$ (cf. Proposition 4.3.1).

This paper is organized as follows. In Section 4.2 we introduce the diagonal Green's function g for perturbations $q \in H^{-1}(\mathbb{R})$ of the background wave V, the key conserved quantity $\alpha(\kappa, q)$ from the case $V \equiv 0$ (cf. (1.4.5)), and the $\widetilde{H}_\kappa$ flow (4.2.22). In Section 4.3 we obtain an *a priori* estimate for the growth of $\alpha(\varkappa, q)$ under the dynamics of $\widetilde{H}_\kappa$ flow (Proposition 4.3.1) with enough independence of the energy parameters κ and $\varkappa$ to facilitate the proof of convergence. In Proposition 4.3.2 we then prove that the $\widetilde{H}_\kappa$ flow is well-posed in $H^{-1}(\mathbb{R})$.

The entirety of Section 4.4 is dedicated to demonstrating that the $\widetilde{H}_\kappa$ flows converge in $H^s(\mathbb{R})$ for some $s < -1$ (Proposition 4.4.1). In Section 4.5 we upgrade this convergence to $H^{-1}(\mathbb{R})$ (Proposition 4.5.2) and then conclude our main result Theorem 4.1.2. The proof of Theorem 4.1.3 is then presented in Section 4.6.

In Section 4.7, we recall the necessary ingredients from Chapter 3 in order to apply Theorem 4.1.2 to step-like initial data.

Lastly, we proceed in Section 4.8 with an application to cnoidal waves (1.2.1), which we present separately because of the significantly shorter length. This is subsumed by the analysis in Section 4.9, where we consider more general smooth periodic backgrounds $V(0) \in H^5(\mathbb{R}/\mathbb{Z})$.

4.2 Diagonal Green's function

In this section, we will continue our study of the diagonal Green's function that we started in Section 3.2.

We begin by taking a closer look at the real-analytic mappings in Proposition 3.2.3. Even for $V \not\equiv 0$, we will need to know that the functionals (3.2.8) are diffeomorphisms in the case $V \equiv 0$. This is because to demonstrate the convergence of the $\widetilde{H}_\kappa$ flows which approximate KdV, it is convenient to make the change of variables $1/g(k, q)$ in place of q, as introduced in [97].

Proposition 4.2.1 (Diffeomorphism property)**.** *Given $A > 0$ there exists $\kappa_0 > 0$ so that the functionals*

$$q \mapsto g(x;\kappa,q) - \frac{1}{2\kappa} \quad \textit{and} \quad q \mapsto \frac{1}{g(x;\kappa,q)} - 2\kappa$$

are real-analytic diffeomorphisms from B_A (defined in (3.2.4)) into $H^1_\kappa(\mathbb{R})$ for all $\kappa \geq \kappa_0$.

Proof. The proof follows that of [97, Prop. 2.2], but now we allow for arbitrary radii $A > 0$ and compensate by taking κ_0 sufficiently large. Using the resolvent identity we calculate the first functional derivative

$$dg|_q(f) = \frac{d}{ds}\bigg|_{s=0} g(x;\kappa,q+sf) = -\langle \delta_x, R(\kappa,q) f R(\kappa,q)\delta_x\rangle. \tag{4.2.1}$$

For $q \equiv 0$, we use the integral kernel formula for R_0 to write

$$dg|_0(f) = -\langle \delta_x, R_0 f R_0 \delta_x\rangle = -\kappa^{-1}[R_0(2\kappa)f](x).$$

Estimating by duality, expanding $R(\kappa,q)$ as the series (3.2.6), and using the operator estimate (3.2.3) we have

$$\|dg|_q(f) - dg|_0(f)\|_{H^1_\kappa} \lesssim \kappa^{-3/2} A \,\|f\|_{H^{-1}_\kappa}$$

uniformly for $q \in B_A$. Taking a supremum over $\|f\|_{H^{-1}_\kappa} \leq 1$, we conclude that there exists $\kappa_0 \gg A^2$ such that

$$\|dg|_q - dg|_0\|_{H^{-1}_\kappa \to H^1_\kappa} \leq \tfrac{1}{2}\kappa^{-1}$$

uniformly for $q \in B_A$ and $\kappa \geq \kappa_0$. Using this and

$$\big\|\big(dg|_0\big)^{-1}\big\|_{H^1_\kappa \to H^{-1}_\kappa} \leq \kappa$$

as input, the standard contraction-mapping proof of the inverse function theorem guarantees that $q \mapsto g - \frac{1}{2\kappa}$ is a diffeomorphism from B_A onto its image for all $\kappa \geq \kappa_0$.

For the second functional $q \mapsto \frac{1}{g} - 2\kappa$ we write

$$\tfrac{1}{g} - 2\kappa = -2\kappa \frac{2\kappa(g - \frac{1}{2\kappa})}{1 + 2\kappa(g - \frac{1}{2\kappa})}.$$

By the embedding $H^1 \hookrightarrow L^\infty$, we note that $f \mapsto \frac{f}{1+f}$ is a diffeomorphism from the ball of radius $\frac{1}{2}$ in H^1 into H^1. The estimates (3.2.9) and (3.2.12) guarantee that we can pick $\kappa_0 \gg A^2$ so that

$$2\kappa \left\| g(x;\kappa,q) - \tfrac{1}{2\kappa} \right\|_{L^\infty} \leq \tfrac{1}{2}$$

for all $q \in B_A$ and $\kappa \geq \kappa_0$. Altogether, we conclude that $q \mapsto \frac{1}{g} - 2\kappa$ is also a diffeomorphism from B_A onto its image for all $\kappa \geq \kappa_0$. □

From the ODE (3.2.18) satisfied by $g(\kappa,q)$, we see that $qg'(\kappa,q)$ is a total derivative. Our next result provides a quantitative estimate of this quantity, which will be useful in obtaining our *a priori* estimate for $\alpha(\varkappa,q)$ under the $\widetilde{H}_\kappa$ flow (Proposition 4.3.1).

Proposition 4.2.2. *Given $A > 0$ there exists $F(x;\kappa,q) \in L^1(\mathbb{R})$ and $\kappa_0 > 0$ so that*

$$q(x)g'(x;\kappa,q) = F'(x;\kappa,q), \qquad \|F\|_{L^1} \lesssim \kappa^{-1} \|q\|_{H_\kappa^{-1}}^2 \tag{4.2.2}$$

for all $q \in B_A(\kappa)$ and $\kappa \geq \kappa_0$.

Proof. From the ODE (3.2.18) we have

$$qg'(\kappa,q) = \left[\tfrac{1}{2}g''(\kappa,q) - qg(\kappa,q) - 2\kappa^2 g(\kappa,q)\right]'.$$

In [94, Lem. 2.14] it is shown that the potential q may be recovered from the diagonal Green's function via the relation

$$q = \left[\frac{g'(\kappa,q)}{2g(\kappa,q)}\right]' + \left[\frac{g'(\kappa,q)}{2g(\kappa,q)}\right]^2 + \left[\frac{1}{4g(\kappa,q)^2} - \kappa^2\right].$$

Rearranging this identity, we see that the claimed relation (4.2.2) holds for the functional

$$F(\kappa,q) := \tfrac{1}{g(\kappa,q)}\left\{\tfrac{1}{4}g'(\kappa,q)^2 - \kappa^2\left[g(\kappa,q) - \tfrac{1}{2\kappa}\right]^2\right\}. \tag{4.2.3}$$

To see the quadratic dependence on q claimed in the estimate (4.2.2) we will Taylor expand F about $q \equiv 0$. From the series (3.2.11) we note that $g(\kappa,0) \equiv \frac{1}{2\kappa}$, and so we have

$$F(x;\kappa,0) \equiv 0.$$

The Green's function for a translated potential is the translation of the original Green's function, and so

$$g(x;\kappa,q(\cdot+h)) = g(x+h;\kappa,q) \quad \text{for all } h\in\mathbb{R}. \tag{4.2.4}$$

This together with the resolvent identity yields

$$g'(x;\kappa,q) = -\langle \delta_x, R(\kappa,q)q'R(\kappa,q)\delta_x\rangle. \tag{4.2.5}$$

Differentiating (4.2.3) with respect to q we obtain

$$\begin{aligned} dF|_q(f) &= -\tfrac{1}{g(\kappa,q)}F(\kappa,q)\,dg|_q(f) \\ &\quad + \tfrac{1}{g(\kappa,q)}\big\{\tfrac12 g'(\kappa,q)\,d(g')|_q(f) - 2\kappa^2\big[g(\kappa,q)-\tfrac{1}{2\kappa}\big]\,dg|_q(f)\big\}, \end{aligned} \tag{4.2.6}$$

and since F, $g-\frac{1}{2\kappa}$, and g' all vanish for $q\equiv 0$ we conclude

$$dF|_0(f)\equiv 0.$$

Next we turn to the Hessian of $F(q)$. A straightforward computation shows that at $q\equiv 0$ we have

$$d^2F|_0(f,f) = -4\kappa^3\langle\delta_x, R_0fR_0\delta_x\rangle^2 + \kappa\langle\delta_x, R_0f'R_0\delta_x\rangle^2, \tag{4.2.7}$$

and we will estimate both terms on the RHS individually. From the integral kernel formula for R_0 we write

$$\langle\delta_x, R_0fR_0\delta_x\rangle = \kappa^{-1}[R_0(2\kappa)f](x).$$

Using Plancherel's theorem to estimate the first term of RHS(4.2.7) we have

$$4\kappa^3\int |\langle\delta_x, R_0fR_0\delta_x\rangle|^2\,dx = 4\kappa\int\frac{|\hat f(\xi)|^2}{(\xi^2+4\kappa^2)^2}\,d\xi \le \frac1\kappa\int\frac{|\hat f(\xi)|^2}{\xi^2+4\kappa^2}\,d\xi.$$

Similarly, for the second term of RHS(4.2.7) we have

$$\kappa\int |\langle\delta_x, R_0f'R_0\delta_x\rangle|^2\,dx = \frac1\kappa\int\frac{\xi^2|\hat f(\xi)|^2}{(\xi^2+4\kappa^2)^2}\,d\xi \le \frac1\kappa\int\frac{|\hat f(\xi)|^2}{\xi^2+4\kappa^2}\,d\xi.$$

Together we conclude

$$\left\|d^2F|_0(f,f)\right\|_{L^1} \lesssim \kappa^{-1}\|f\|_{H_\kappa^{-1}}^2. \tag{4.2.8}$$

To finish the proof, it suffices to show that the Hessian's modulus of continuity satisfies

$$\left\| d^2F|_q(f,f) - d^2F|_0(f,f) \right\|_{L^1} \lesssim \kappa^{-3/2} A \, \|f\|^2_{H^{-1}_\kappa} \tag{4.2.9}$$

uniformly for $q \in B_A(\kappa)$ and κ large. Indeed, the estimate (4.2.2) then follows by choosing $\kappa_0 \gg A^2$ so that the RHS is smaller than RHS(4.2.8) for $\kappa \geq \kappa_0$. Differentiating the first derivative (4.2.6) we write

$$\begin{aligned}
&d^2F|_q(f,f) - d^2F|_0(f,f) \\
&= \tfrac{2}{g(\kappa,q)^2} F(\kappa,q) \big[dg|_q(f) \big]^2 - \tfrac{1}{g(\kappa,q)} F(\kappa,q)\, d^2 g|_q(f,f) && (4.2.10) \\
&\quad - \tfrac{2}{g(\kappa,q)^2} \big\{ \tfrac{1}{2} g'(\kappa,q)\, d(g')|_q(f) - 2\kappa^2 \big[g(\kappa,q) - \tfrac{1}{2\kappa} \big]\, dg|_q(f) \big\}\, dg|_q(f) && (4.2.11) \\
&\quad + \tfrac{1}{g(\kappa,q)} \big\{ \tfrac{1}{2} g'(\kappa,q)\, d^2(g')|_q(f,f) - 2\kappa^2 \big[g(\kappa,q) - \tfrac{1}{2\kappa} \big]\, d^2 g|_q(f,f) \big\} && (4.2.12) \\
&\quad + \tfrac{1}{g(\kappa,q)} \big\{ \tfrac{1}{2} [d(g')|_q(f)]^2 - 2\kappa^2 [dg|_q(f)]^2 \big\} - d^2F|_0(f,f). && (4.2.13)
\end{aligned}$$

We will prove the estimate (4.2.9) by estimating each of the terms (4.2.10)–(4.2.13) in L^1, but first we record some useful estimates for the functional derivatives of g.

Estimating the first functional derivative (4.2.1) by duality, expanding $R(\kappa, q)$ as the series (3.2.6), and using the operator estimate (3.2.3) we have

$$\| dg|_q(f) \|_{H^1_\kappa} \lesssim \kappa^{-1} \, \|f\|_{H^{-1}_\kappa} \tag{4.2.14}$$

uniformly for $q \in B_A(\kappa)$ and κ large. Similarly, if we remove the leading term of $dg|_q(f)$ we obtain

$$\| dg|_q(f) - dg|_0(f) \|_{H^1_\kappa} \lesssim \kappa^{-3/2} A \, \|f\|_{H^{-1}_\kappa} \tag{4.2.15}$$

uniformly for $q \in B_A(\kappa)$ and κ large. Another application of the resolvent identity shows that

$$d^2 g|_q(f,f) = 2\langle \delta_x, R(\kappa,q) f R(\kappa,q) f R(\kappa,q) \delta_x \rangle,$$

and estimating this by duality we have

$$\left\| d^2 g|_q(f,f) \right\|_{H^1_\kappa} \lesssim \kappa^{-3/2} \, \|f\|^2_{H^{-1}_\kappa} \tag{4.2.16}$$

uniformly for $q \in B_A(\kappa)$ and κ large.

For the first term (4.2.10) we use the estimates (3.2.9), (3.2.12), (4.2.14), and (4.2.16) along with the observation that $\|h\|_{L^2} \lesssim \kappa^{-1} \|h\|_{H^1_\kappa}$ to bound

$$\begin{aligned}\|(4.2.10)\|_{L^1} &\lesssim \left(\|g'(\kappa,q)\|_{L^2}^2 + \kappa^2 \left\|g(\kappa,q) - \tfrac{1}{2\kappa}\right\|_{L^2}^2 \right) \left(\left\| \tfrac{[dg|_q(f)]^2}{g(\kappa,q)^3} \right\|_{L^\infty} + \left\| \tfrac{d^2 g|_q(f,f)}{g(\kappa,q)^2} \right\|_{L^\infty} \right) \\ &\lesssim \kappa^{-2} A^2 \big(\kappa^3 \cdot \kappa^{-3} \|f\|_{H^{-1}_\kappa}^2 + \kappa^2 \cdot \kappa^{-2} \|f\|_{H^{-1}_\kappa}^2 \big) \lesssim \kappa^{-2} A \|f\|_{H^{-1}_\kappa}^2\end{aligned}$$

uniformly for $q \in B_A(\kappa)$ and κ large.

For the second term (4.2.11) we note that $d(g')|_q(f) = [dg|_q(f)]'$ can be bounded in L^2 by (4.2.14), yielding

$$\begin{aligned}\|(4.2.11)\|_{L^1} &\lesssim \left\| \tfrac{dg|_q(f)}{g(\kappa,q)^2} \right\|_{L^\infty} \left(\|g'(\kappa,q)\|_{L^2} \|d(g')|_q(f)\|_{L^2} + \kappa^2 \left\|g(\kappa,q) - \tfrac{1}{2\kappa}\right\|_{L^2} \|dg|_q(f)\|_{L^2} \right) \\ &\lesssim \kappa^{1/2} \|f\|_{H^{-1}_\kappa} \left(\kappa^{-2} A \|f\|_{H^{-1}_\kappa} \right) \lesssim \kappa^{-3/2} A \|f\|_{H^{-1}_\kappa}^2\end{aligned}$$

uniformly for $q \in B_A(\kappa)$ and κ large.

Similarly for the third term (4.2.12) we have $d^2(g')|_q(f,f) = [d^2 g|_q(f,f)]'$, and hence

$$\begin{aligned}\|(4.2.12)\|_{L^1} \leq{} & \tfrac{1}{2} \left\| \tfrac{1}{g(\kappa,q)} \right\|_{L^\infty} \|g'(\kappa,q)\|_{L^2} \left\| d^2(g')|_q(f,f) \right\|_{L^2} \\ & + 2\kappa^2 \left\| \tfrac{1}{g(\kappa,q)} \right\|_{L^\infty} \left\| g(\kappa,q) - \tfrac{1}{2\kappa} \right\|_{L^2} \left\| d^2 g|_q(f,f) \right\|_{L^2} \\ \lesssim{} & \kappa^{-3/2} A \|f\|_{H^{-1}_\kappa}^2 + \kappa^2 \cdot \kappa^{-7/2} A \|f\|_{H^{-1}_\kappa}^2 \lesssim \kappa^{-3/2} A \|f\|_{H^{-1}_\kappa}^2\end{aligned}$$

uniformly for $q \in B_A(\kappa)$ and κ large.

Lastly, to witness the convergence within the fourth term (4.2.13) we estimate $[dg|_q(f)]^2 - [dg|_0(f)]^2$ in L^1 as the difference of squares in L^2 using (4.2.15):

$$\begin{aligned}\|(4.2.13)\|_{L^1} &= \left\| \left[\tfrac{[d(g')|_q(f)]^2}{2g(\kappa,q)} - \kappa [d(g')|_0(f)]^2 \right] - 4\kappa^3 \left[\tfrac{[dg|_q(f)]^2}{2\kappa g(\kappa,q)} - [dg|_0(f)]^2 \right] \right\|_{L^1} \\ &\leq \kappa \left\| [d(g')|_q(f)]^2 - [d(g')|_0(f)]^2 \right\|_{L^1} + \left\| \tfrac{1}{2g(\kappa,q)} - \kappa \right\|_{L^\infty} \|d(g')|_q(f)\|_{L^2}^2 \\ &\quad + 4\kappa^3 \left\| [dg|_q(f)]^2 - [dg|_0(f)]^2 \right\|_{L^1} + 4\kappa^2 \left\| \tfrac{1}{2g(\kappa,q)} - \kappa \right\|_{L^\infty} \|dg|_q(f)\|_{L^2}^2 \\ &\lesssim \kappa^{-3/2} A \|f\|_{H^{-1}_\kappa}^2 + \kappa^{-2} A \|f\|_{H^{-1}_\kappa}^2 \lesssim \kappa^{-3/2} A \|f\|_{H^{-1}_\kappa}^2\end{aligned}$$

uniformly for $q \in B_A(\kappa)$ and κ large. Altogether we have demonstrated the desired inequality (4.2.9), which concludes the proof. $\square$

We now recall the key conserved quantity $\alpha(\kappa, q)$ constructed in [97, Prop. 2.4] to control the H^{-1}-norm of q. The same proof shows that given $A > 0$, if we take the corresponding constant κ_0 from Proposition 3.2.3, then for all $\kappa \geq \kappa_0$ the quantity

$$\alpha(\kappa, q) := \int_{\mathbb{R}} \left\{ -\frac{1}{2g(x;\kappa,q)} + \kappa + 2\kappa[R_0(2\kappa)q](x) \right\} dx$$

exists for all $q \in B_A(\kappa)$, is a real analytic functional of $q \in B_A(\kappa)$, and is conserved under the KdV flow (cf. [97, Prop. 3.1]):

$$\{\alpha, E_2\} = 0. \tag{4.2.17}$$

The quantity $\alpha(\kappa, q)$ is a renormalized logarithm of the transmission coefficient for the Schrödinger operator with potential q (called the perturbation determinant) at energy $-\kappa^2$.

The formula for α is the trace of the integral kernel $-1/2G(x, y; \kappa, q)$ with the first two terms of its functional Taylor series about $q \equiv 0$ canceled, and consequently $\alpha(\kappa, q)$ is a nonnegative, strictly convex, real-analytic functional of $q \in B_A$. Specifically, in [97, Prop. 2.4] it is shown that the first derivative of α is given by

$$\frac{\delta \alpha}{\delta q} = \frac{1}{2\kappa} - g(x; \kappa, q). \tag{4.2.18}$$

This vanishes for $q \equiv 0$, but the nondegenerate second derivative yields

$$\tfrac{1}{4}\kappa^{-1} \|q\|^2_{H^{-1}_\kappa} \leq \alpha(\kappa, q) \leq \kappa^{-1} \|q\|^2_{H^{-1}_\kappa} \tag{4.2.19}$$

uniformly for $q \in B_A(\kappa)$ and $\kappa \geq \kappa_0$. This last statement follows from the original proof of [97, Prop. 2.4] together with the estimate

$$\left| d^2\alpha|_q(f, f) - d^2\alpha|_0(f, f) \right| \lesssim \kappa^{-3/2} A \|f\|^2_{H^{-1}_\kappa}$$

uniformly for $q \in B_A(\kappa)$ and κ large (which is true by (4.2.15) and (4.2.18)).

The quantity α is also used to construct the H_κ flows (cf. [97, Prop. 3.2]) which approximate the KdV flow as $\kappa \to \infty$. For $\kappa \geq 1$ the Hamiltonian evolution induced by

$$H_\kappa := -16\kappa^5 \alpha(\kappa, q) + 2\kappa^2 \int q(x)^2 \, dx \tag{4.2.20}$$

is given by (3.3.1). The flow conserves $\alpha(\varkappa, q(t))$ and commutes with those of KdV and $H_\varkappa$ for all $\varkappa \geq 1$:

$$\{\alpha, H_\kappa\} = 0, \qquad \{H_\kappa, E_2\} = 0, \qquad \{H_\kappa, H_\varkappa\} = 0. \tag{4.2.21}$$

Given a solution $V(t,x)$ to KdV we define

$$V_\kappa(t,x) := e^{tJ\nabla H_\kappa} V(0,x)$$

to be the H_κ evolution of $V(0,x)$. We will always assume that V and V_κ are admissible in the sense of Definition 4.1.1. Just as how we obtained KdV with potential (4.1.1) from KdV, we define the $\widetilde{H}_\kappa$ flow of q from time 0 to t via

$$\widetilde{\Phi}_\kappa(t) q = e^{tJ\nabla H_\kappa}(q + V(0)) - V_\kappa(t).$$

In other words, $q(t,x)$ solves

$$\frac{d}{dt} q(t,x) = 16\kappa^5 \left[g'(x;\kappa, q(t) + V_\kappa(t)) - g'(x;\kappa, V_\kappa(t)) \right] + 4\kappa^2 q'(t,x). \tag{4.2.22}$$

Formally, this flow is induced by the (time-dependent) Hamiltonian

$$\widetilde{H}_\kappa := -16\kappa^5 \left\{ \alpha(\kappa, q + V_\kappa) + \int [g(x;\kappa, V_\kappa) - \tfrac{1}{2\kappa}] q(x) \, dx \right\} + 2\kappa^2 \int q(x)^2 \, dx.$$

We will not need this explicit formula for the Hamiltonian $\widetilde{H}_\kappa$, but we include it so that we are justified in using the Poisson bracket notation for its flow.

Throughout our analysis we will need to know that the first two terms of the series (3.2.11) for $g(\kappa, V_\kappa)$ converge and dominate in the limit $\kappa \to \infty$.

Lemma 4.2.3. *Fix V admissible (in the sense of Definition 4.1.1) and $T > 0$. Then*

$$-4\kappa^3 \left[g(\kappa, V_\kappa) - \tfrac{1}{2\kappa} \right] \to V \quad \text{in } W^{2,\infty} \text{ as } \kappa \to \infty$$

uniformly for $|t| \leq T$.

Proof. First we will examine the leading term of the series (3.2.11) for $g(\kappa, V_\kappa) - \frac{1}{2\kappa}$. From the integral kernel formula for R_0 we note that

$$4\kappa^3 \langle \delta_x, R_0 V_\kappa R_0 \delta_x \rangle = 4\kappa^2 R_0(2\kappa) V_\kappa,$$

and we claim this converges to V in $W^{2,\infty}$ uniformly for $|t| \leq T$. The operator $4\kappa^2 R_0(2\kappa)$ is convolution by the function $\kappa e^{-2\kappa|x|}$, whose integral is 1 for all $\kappa > 0$. Using this and the fundamental theorem of calculus we have

$$\begin{aligned} \left|4\kappa^2 R_0(2\kappa)V_\kappa(x) - V_\kappa(x)\right| &= \left| \int \kappa e^{-2\kappa|y|} [V_\kappa(x-y) - V_\kappa(x)] \, dy \right| \\ &\leq \kappa \left\| V_\kappa' \right\|_{L^\infty} \int e^{-2\kappa|y|} |y| \, dy = 2\kappa^{-1} \left\| V_\kappa' \right\|_{L^\infty} \end{aligned}$$

for all x. As $V_\kappa' \in L^\infty$ and $V_\kappa \to V$ in L^∞ uniformly for $|t| \leq T$ (by Definition 4.1.1), we conclude that $4\kappa^2 R_0(2\kappa)V_\kappa \to V$ in L^∞ uniformly for $|t| \leq T$. Differentiation commutes with $R_0(2\kappa)$, and so replacing V_κ with V_κ', V_κ'' and recalling that $V_\kappa \in W^{3,\infty}$ uniformly for $|t| \leq T$, we conclude that $4\kappa^2 R_0(2\kappa)V_\kappa \to V$ in $W^{2,\infty}$ uniformly for $|t| \leq T$.

It remains to show that

$$-4\kappa^3 \left[g(\kappa, V_\kappa) - \tfrac{1}{2\kappa} \right] - 4\kappa^2 R_0(2\kappa) V_\kappa \to 0 \quad \text{in } W^{2,\infty} \text{ as } \kappa \to \infty$$

uniformly for $|t| \leq T$. Using the series (3.2.11) we estimate

$$\begin{aligned} \left| 4\kappa^3 \left[g(x;\kappa, V_\kappa) - \tfrac{1}{2\kappa} \right] + 4\kappa^2 R_0(2\kappa) V_\kappa(x) \right| &\leq 4\kappa^3 \sum_{\ell=2}^\infty \left\| \sqrt{R_0} \delta_x \right\|_{L^2}^2 \left\| \sqrt{R_0} V_\kappa \sqrt{R_0} \right\|_{\text{op}}^\ell \\ &\lesssim 4\kappa^2 \sum_{\ell=2}^\infty \left(\kappa^{-2} \left\| V_\kappa \right\|_{L^\infty} \right)^\ell \lesssim_V \kappa^{-2}, \end{aligned}$$

where we noted that $\|\sqrt{R_0}\delta_x\|_{L^2}^2 \lesssim \kappa^{-1}$ in Fourier variables. This demonstrates the desired convergence in L^∞. Differentiating the translation identity (4.2.4) with respect to h at $h = 0$ yields

$$g'(\kappa, V_\kappa) = \sum_{\ell=1}^\infty (-1)^\ell \sum_{j=0}^{\ell-1} \left\langle \delta_x, R_0 (V_\kappa R_0)^j V_\kappa' R_0 (V_\kappa R_0)^{\ell-1-j} \delta_x \right\rangle.$$

Computing the second derivative similarly and using the same estimates, we also conclude

$$\|4\kappa^3 g'(\kappa, V_\kappa) + 4\kappa^2 R_0(2\kappa)V_\kappa'\|_{L^\infty} + \|4\kappa^3 g''(\kappa, V_\kappa) + 4\kappa^2 R_0(2\kappa)V_\kappa''\|_{L^\infty} \lesssim_V \kappa^{-2}$$

because $V_\kappa \in W^{2,\infty}$ uniformly for $|t| \le T$. This demonstrates that the first and second derivatives converge in L^∞ uniformly for $|t| \le T$ as well. □

4.3 The $\widetilde{H}_\kappa$ flow

To eventually show that the $\widetilde{H}_\kappa$ flow (4.2.22) converges to KdV with potential (4.1.1) we will need to control the H^{-1}-norm of $q(t)$ under the $\widetilde{H}_\kappa$ flow. As the $\widetilde{H}_\kappa$ flow already has the associated energy parameter κ, our tool for controlling q in H^{-1} is $\alpha(\varkappa, q(t))$ at an independent energy parameter $\varkappa$. Both the $\widetilde{H}_\kappa$ flow and α involve the diagonal Green's function, and so we will be led to an integral involving $g(x;\kappa,q)$ and $g(x;\varkappa,q)$. Expanding both into series, the resulting summands are no longer simply traces and so we will need to develop a new technique in order efficiently estimate such an integral.

To introduce the technique that we later use, we will first prove the commutativity relation

$$\int g(x;\varkappa,q)g'(x;\kappa,q)\,dx = 0 \tag{4.3.1}$$

for Schwartz functions q, which expresses that $\alpha(\kappa,q)$ and $\alpha(\varkappa,q)$ Poisson commute (cf. [97, Prop. 3.2]). When $\varkappa = \kappa$ the integrand is a total derivative and the vanishing of the integral is immediate, so assume $\varkappa \ne \kappa$. First, we use the ODE (3.2.18) for $g(\kappa,q) = g(x;\kappa,q)$ to write

$$4(\kappa^2 - \varkappa^2)g'(\kappa,q) = g'''(\kappa,q) - 2qg'(\kappa,q) - 2\,[qg(\kappa,q)]' - 4\varkappa^2 g'(\kappa,q). \tag{4.3.2}$$

Substituting this for $g'(\kappa,q)$ in (4.3.1) and integrating by parts we obtain

$$\begin{aligned}&\int g(\varkappa,q)g'(\kappa,q)\,dx\\ &= \tfrac{1}{4(\kappa^2-\varkappa^2)}\int g(\varkappa,q)\left\{g'''(\kappa,q) - 2qg'(\kappa,q) - 2\,[qg(\kappa,q)]' - 4\varkappa^2 g'(\kappa,q)\right\}\,dx\end{aligned}$$

$$= -\tfrac{1}{4(\kappa^2-\varkappa^2)} \int \big\{ g'''(\varkappa, q) - 2qg'(\varkappa, q) - 2\,[qg(\varkappa, q)]' - 4\varkappa^2 g'(\varkappa, q) \big\}\, g(\kappa, q)\, dx.$$

Now we see that this last integral vanishes due to the ODE (3.2.18) for $g(\varkappa, q)$, thus proving (4.3.1). We will refer to this procedure of using the ODE for one term, integrating by parts, and using the ODE for the other term as the *commutativity relation trick*.

Now we are prepared to prove our main estimate for controlling the H^{-1}-norm of $q(t)$ under the $\widetilde{H}_\kappa$ flow:

Proposition 4.3.1. *Fix $T > 0$ and V admissible. There exists a constant $C > 0$ so that the following holds: given $A > 0$ there exists $\varkappa_0 > 0$ so that solutions $q(t) \in B_A(\varkappa)$ to the $\widetilde{H}_\kappa$ flow* (4.2.22) *obey*

$$\left| \frac{d}{dt} \alpha(\varkappa, q(t)) \right| \leq C\alpha(\varkappa, q(t))$$

uniformly for $|t| \leq T$, $\kappa \geq 2\varkappa$, and $\varkappa \geq \varkappa_0$.

Proof. We initialize $\varkappa_0$ so that the results from Section 4.2 hold for the balls $B_A(\varkappa)$ for all $\varkappa \geq \varkappa_0$. First we compute the time derivative of $\alpha(\varkappa, q(t))$. We will show that the $\widetilde{H}_\kappa$ flow is locally well-posed in $H^{-1}(\mathbb{R})$ in Proposition 4.3.2, and so we may assume that q is Schwartz by approximation. Using the functional derivative (4.2.18) of α and the $\widetilde{H}_\kappa$ flow (4.2.22), we compute

$$\begin{aligned} \frac{d}{dt}\alpha(\varkappa, q(t)) &= \{\alpha, \widetilde{H}_\kappa\} \\ &= -\int \big(g(x; \varkappa, q) - \tfrac{1}{2\varkappa} \big) \big\{ 16\kappa^5 \,[g'(x; \kappa, q + V_\kappa) - g'(x; \kappa, V_\kappa)] + 4\kappa^2 q'(x) \big\}\, dx, \end{aligned}$$

where we have suppressed the time dependence of q and V_κ. The contribution from q' vanishes because the H_κ flow conserves momentum; this can be seen by integrating by parts and noting from (4.2.2) that $qg'(\kappa, q)$ is a total derivative. We are left with the expression

$$\frac{d}{dt}\alpha(\varkappa, q(t)) = -16\kappa^5 \int \big(g(x; \varkappa, q) - \tfrac{1}{2\varkappa} \big)\, [g'(x; \kappa, q + V_\kappa) - g'(x; \kappa, V_\kappa)]\, dx.$$

We expect this to remain bounded in the limit $\kappa \to \infty$ from the convergence of H_κ to E_2, but the factor of κ^5 obscures this bound. To circumvent this, we use the commutativity relation trick (4.3.2) introduced at the beginning of this section. Using the ODEs (3.2.18) for $g(\kappa, q+V_\kappa)$ and $g(\kappa, V_\kappa)$, integrating by parts, and then using the ODE for $g(\varkappa, q)$, we obtain

$$\frac{d}{dt}\alpha(\varkappa, q(t))$$

$$= -\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int \big\{ \big[q(g(\varkappa,q)-\tfrac{1}{2\varkappa})\big]' + qg'(\varkappa,q)\big\} g(\kappa,V_\kappa)\,dx \tag{4.3.3}$$

$$+ \tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int \tfrac{1}{2\varkappa}q'\left[g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)\right]\,dx \tag{4.3.4}$$

$$- \tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int \big\{ \big[V_\kappa(g(\varkappa,q)-\tfrac{1}{2\varkappa})\big]' + V_\kappa g'(\varkappa,q)\big\}\left[g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)\right]\,dx. \tag{4.3.5}$$

We have suppressed the spatial integration variable x for all integrands. We will show that (4.3.3) and (4.3.4) are acceptable contributions, and then we will manipulate (4.3.5) further.

For the term (4.3.3) we insert $\frac{1}{2\kappa} - \frac{1}{4\kappa^3}V$ in place of $g(\kappa, V_\kappa)$:

$$\begin{aligned}(4.3.3) = &-\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int \big[q(g(\varkappa,q)-\tfrac{1}{2\varkappa})\big]'\big(g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big)\,dx \\ &-\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int qg'(\varkappa,q)\big(g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big)\,dx \\ &-\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\int \big\{\big[q(g(\varkappa,q)-\tfrac{1}{2\varkappa})\big]'+qg'(\varkappa,q)\big\}\big(\tfrac{1}{2\kappa}-\tfrac{1}{4\kappa^3}V\big)\,dx.\end{aligned}$$

To estimate the first integral on the RHS we integrate by parts, use $H^1_\varkappa$-$H^{-1}_\varkappa$ duality, and use the estimates (3.2.2), (3.2.9), and (4.2.19):

$$\begin{aligned}&\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\left|\int (g(\varkappa,q)-\tfrac{1}{2\varkappa})q\big(g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big)'\,dx\right| \\ &\lesssim \tfrac{\kappa^5}{\kappa^2-\varkappa^2}\big\|g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big\|_{W^{2,\infty}}\big\|g(\varkappa,q)-\tfrac{1}{2\varkappa}\big\|_{H^1_\varkappa}\|q\|_{H^{-1}_\varkappa} \\ &\lesssim \tfrac{\kappa^5}{\kappa^2-\varkappa^2}\big\|g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big\|_{W^{2,\infty}}\alpha(\varkappa,q).\end{aligned}$$

The prefactor of $\alpha(\varkappa, q)$ here is bounded uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large by the convergence of Lemma 4.2.3. For the second integral we use the identity (4.2.2) for $qg'(\varkappa, q)$ and the α estimate (4.2.19):

$$\tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\left|\int (g(\varkappa,q)-\tfrac{1}{2\varkappa})\big[q\big(g(\kappa,V_\kappa)-\tfrac{1}{2\kappa}+\tfrac{1}{4\kappa^3}V\big)\big]'\,dx\right|$$

$$\lesssim \tfrac{\kappa^5}{\kappa^2-\varkappa^2} \left\| g(\kappa, V_\kappa) - \tfrac{1}{2\kappa} + \tfrac{1}{4\kappa^3} V \right\|_{W^{1,\infty}} \alpha(\varkappa, q),$$

and the prefactor is again bounded uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large. For the third integral we integrate by parts to obtain

$$\tfrac{8\kappa^5}{\kappa^2-\varkappa^2} \int (g(\varkappa, q) - \tfrac{1}{2\varkappa}) \big\{ \big[(\tfrac{1}{2\kappa} - \tfrac{1}{4\kappa^3} V) q \big]' - \tfrac{1}{4\kappa^3} V' q \big\}\, dx.$$

Note that the contribution from the term $\frac{1}{2\kappa}$ vanishes since $qg'(\varkappa, q)$ is a total derivative (cf. (4.2.2)). This leaves

$$-\tfrac{2\kappa^2}{\kappa^2-\varkappa^2} \int (g(\varkappa, q) - \tfrac{1}{2\varkappa}) \left[V'q + (Vq)' \right] dx.$$

The prefactor $\frac{2\kappa^2}{\kappa^2-\varkappa^2}$ is now bounded uniformly for $\kappa \geq 2\varkappa$. As before the contribution of $V'q$ is estimated by $H^1_\varkappa$-$H^{-1}_\varkappa$ duality and the contribution of $(Vq)'$ by (4.2.2), yielding

$$\tfrac{2\kappa^2}{\kappa^2-\varkappa^2} \left| \int (g(\varkappa, q) - \tfrac{1}{2\varkappa}) \left[V'q + (Vq)' \right] dx \right| \lesssim \|V\|_{W^{2,\infty}} \alpha(\varkappa, q).$$

To estimate the second term (4.3.4) we expand

$$(4.3.4) = \tfrac{4\kappa^5}{(\kappa^2-\varkappa^2)\varkappa} \sum_{\ell=1}^{\infty} (-1)^\ell \operatorname{tr}\{ (R(\kappa, V_\kappa) q)^\ell R(\kappa, V_\kappa) q' \}.$$

Next, we write

$$R(\kappa, V_\kappa) q' = [\partial, R(\kappa, V_\kappa) q] - [\partial, R(\kappa, V_\kappa)] q.$$

Note that contribution from $[\partial, R(\kappa, V_\kappa) q]$ vanishes by cycling the trace, and the contribution from $[\partial, R(\kappa, V_\kappa)] = -R(\kappa, V_\kappa) V_\kappa' R(\kappa, V_\kappa)$ is acceptable using the estimate (3.2.7):

$$\begin{aligned}
|(4.3.4)| &\leq \tfrac{4\kappa^5}{(\kappa^2-\varkappa^2)\varkappa} \sum_{\ell=1}^{\infty} \left| \operatorname{tr}\{ (R(\kappa, V_\kappa) q)^\ell R(\kappa, V_\kappa) V_\kappa' R(\kappa, V_\kappa) q \} \right| \\
&\leq \tfrac{4\kappa^5}{(\kappa^2-\varkappa^2)\varkappa} \sum_{\ell=1}^{\infty} \left\| \sqrt{R(\kappa, V_\kappa)}\, q \sqrt{R(\kappa, V_\kappa)} \right\|_{\mathfrak{I}_2}^{\ell+1} \left\| \sqrt{R(\kappa, V_\kappa)} V_\kappa' \sqrt{R(\kappa, V_\kappa)} \right\|_{\mathrm{op}} \\
&\lesssim \tfrac{4\kappa^5}{(\kappa^2-\varkappa^2)\varkappa} \sum_{\ell=1}^{\infty} \left(\kappa^{-1/2} \|q\|_{H^{-1}_\kappa} \right)^{\ell+1} \kappa^{-2} \|V_\kappa'\|_{L^\infty} \lesssim \|V_\kappa'\|_{L^\infty} \alpha(\varkappa, q).
\end{aligned}$$

In the last step we noted that $\|q\|_{H^{-1}_\kappa} \leq \|q\|_{H^{-1}_\varkappa}$ for $\kappa \geq 2\varkappa$.

It remains to estimate the third term (4.3.5), which will require more manipulation because the leading term in the expansion of $g(\kappa, q+V_\kappa) - g(\kappa, V_\kappa)$ is only $O(\kappa)$. First, we integrate by parts to move the derivative back onto $g(\kappa, q+V_\kappa) - g(\kappa, V_\kappa)$:

$$(4.3.5) = \tfrac{8\kappa^5}{\kappa^2-\varkappa^2} \int V_\kappa'(g(\varkappa,q) - \tfrac{1}{2\varkappa})\,[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)]\,dx \tag{4.3.6}$$

$$+ \tfrac{16\kappa^5}{\kappa^2-\varkappa^2} \int V_\kappa(g(\varkappa,q) - \tfrac{1}{2\varkappa})\,[g'(\kappa,q+V_\kappa) - g'(\kappa,V_\kappa)]\,dx. \tag{4.3.7}$$

Using H^1_κ-H^{-1}_κ duality, the diagonal Green's function estimate (3.2.9), and the observation that $\|f\|_{H^{-1}_\kappa} \leq \kappa^{-2}\|f\|_{H^1_\kappa}$, we have

$$\begin{aligned} |(4.3.6)| &\leq \tfrac{8\kappa^5}{\kappa^2-\varkappa^2} \|V_\kappa'\|_{W^{1,\infty}} \|g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)\|_{H^{-1}_\kappa} \left\|g(\varkappa,q) - \tfrac{1}{2\varkappa}\right\|_{H^1_\kappa} \\ &\lesssim \tfrac{8\kappa^5}{\kappa^2-\varkappa^2} \|V_\kappa'\|_{W^{1,\infty}} \kappa^{-3} \|q\|_{H^{-1}_\kappa} \varkappa^{-1} \|q\|_{H^{-1}_\varkappa} \lesssim \|V_\kappa'\|_{W^{1,\infty}}\, \alpha(\varkappa,q) \end{aligned}$$

since $\|q\|_{H^{-1}_\kappa} \leq \|q\|_{H^{-1}_\varkappa}$ for $\kappa \geq 2\varkappa$.

The remaining term (4.3.7) resembles our original expression for $\frac{d}{dt}\alpha(\varkappa, q(t))$, except we have gained κ^{-2} in decay and have introduced an extra factor of V_κ. Consequently, we repeat the commutativity relation trick (4.3.2); pushing derivatives past the factor of V_κ introduces extra terms, but they are relatively harmless. After this manipulation, we regroup terms to arrive at

$$(4.3.7)$$

$$= \int \tfrac{8\kappa^5 V_\kappa}{(\kappa^2-\varkappa^2)^2}\big\{\big[q(g(\varkappa,q) - \tfrac{1}{2\varkappa})\big]' + qg'(\varkappa,q)\big\}g(\kappa,V_\kappa) \tag{4.3.8}$$

$$- \int \tfrac{8\kappa^5 V_\kappa q'}{2\varkappa(\kappa^2-\varkappa^2)^2}[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)] \tag{4.3.9}$$

$$+ \int \tfrac{8\kappa^5 V_\kappa}{(\kappa^2-\varkappa^2)^2}\big\{\big[V_\kappa(g(\varkappa,q) - \tfrac{1}{2\varkappa})\big]' + V_\kappa g'(\varkappa,q)\big\}[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)] \tag{4.3.10}$$

$$- \int \tfrac{4\kappa^5}{(\kappa^2-\varkappa^2)^2}(V_\kappa''' - 4\varkappa^2 V_\kappa' - 2V_\kappa')(g(\varkappa,q) - \tfrac{1}{2\varkappa})[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)] \tag{4.3.11}$$

$$- \int \tfrac{12\kappa^5}{(\kappa^2-\varkappa^2)^2}\big[V_\kappa' g'(\varkappa,q)\big]'[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)] \tag{4.3.12}$$

$$- \int \tfrac{8\kappa^5 V_\kappa'}{(\kappa^2-\varkappa^2)^2}(g(\varkappa,q) - \tfrac{1}{2\varkappa})\big\{qg(\kappa,q+V_\kappa) + V_\kappa[g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)]\big\}. \tag{4.3.13}$$

The first three terms (4.3.8)–(4.3.10) are analogous to (4.3.3)–(4.3.5) respectively, and the new terms (4.3.11)–(4.3.13) are the result of derivatives falling on the new factor of V_κ.

Estimating (4.3.8) as we did (4.3.3), we obtain

$$|(4.3.8)| \lesssim \big(\|V_\kappa\|_{W^{2,\infty}} + \|V_\kappa\|_{W^{2,\infty}}^2 \big) \alpha(\varkappa, q).$$

Conversely, the extra factor of V_κ prohibits us from treating the term (4.3.9) as we did (4.3.4). Instead, we must maneuver the derivative onto V_κ. Expanding

$$(4.3.9) = -\frac{4\kappa^5}{(\kappa^2 - \varkappa^2)^2} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\varkappa} \operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell R(\kappa, V_\kappa) V_\kappa [\partial, q]\},$$

we write

$$\begin{aligned} &\operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell R(\kappa, V_\kappa) V_\kappa [\partial, q]\} \\ &= \operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell [R(\kappa, V_\kappa), V_\kappa] \partial q\} - \operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell V_\kappa [\partial, R(\kappa, V_\kappa)] q\} \end{aligned}$$

by linearity and cycling the trace. For the contribution of $[\partial, R(\kappa, V_\kappa)] = -R(\kappa, V_\kappa) V_\kappa' R(\kappa, V_\kappa)$, we use the operator estimate (3.2.7) to bound

$$\begin{aligned} &|\operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell V_\kappa [\partial, R(\kappa, V_\kappa)] q\}| \\ &\le \Big\| \sqrt{R(\kappa, V_\kappa)} V_\kappa' \sqrt{R(\kappa, V_\kappa)} \Big\|_{\mathrm{op}} \Big\| \sqrt{R(\kappa, V_\kappa)}\, q \sqrt{R(\kappa, V_\kappa)} \Big\|_{\mathfrak{I}_2}^{\ell+1} \\ &\lesssim \kappa^{-2} \|V_\kappa'\|_{L^\infty} \big(\kappa^{-1/2} \|q\|_{H_\kappa^{-1}} \big)^{\ell+1} \end{aligned}$$

for $\ell \ge 1$. For the contribution of first commutator

$$[R(\kappa, V_\kappa), V_\kappa]\partial = R(\kappa, V_\kappa) V_\kappa'' R(\kappa, V_\kappa) \partial + R(\kappa, V_\kappa) 2V_\kappa' \partial R(\kappa, V_\kappa) \partial,$$

we pair each q with two copies of $\sqrt{R(\kappa, V_\kappa)}$ in $\mathfrak{I}_2$ and each ∂ with one copy of $\sqrt{R(\kappa, V_\kappa)}$ in operator norm; the first term contributes

$$\begin{aligned} &|\operatorname{tr}\{(R(\kappa, V_\kappa)q)^\ell R(\kappa, V_\kappa) V_\kappa'' R(\kappa, V_\kappa) \partial q\}| \\ &\le \Big\| \sqrt{R(\kappa, V_\kappa)} V_\kappa'' \partial \sqrt{R(\kappa, V_\kappa)} \Big\|_{\mathrm{op}} \Big\| \sqrt{R(\kappa, V_\kappa)}\, q \sqrt{R(\kappa, V_\kappa)} \Big\|_{\mathfrak{I}_2}^{\ell+1} \end{aligned}$$

$$
\begin{aligned}
&\quad + |\operatorname{tr}\{(R(\kappa,V_\kappa)q)^\ell R(\kappa,V_\kappa)V_\kappa''[\partial,R(\kappa,V_\kappa)]q\}| \\
&\lesssim \kappa^{-1}\big(\, \|V_\kappa\|_{W^{2,\infty}} + \|V_\kappa\|_{W^{2,\infty}}^2 \,\big)\big(\kappa^{-1/2}\,\|q\|_{H_\kappa^{-1}}\,\big)^{\ell+1},
\end{aligned}
$$

and the second term contributes

$$
\begin{aligned}
&|\operatorname{tr}\{(R(\kappa,V_\kappa)q)^\ell R(\kappa,V_\kappa)V_\kappa'\partial R(\kappa,V_\kappa)\partial q\}| \\
&\leq \Big\|\sqrt{R(\kappa,V_\kappa)}\big(\partial V_\kappa' - V_\kappa''\big)\partial\sqrt{R(\kappa,V_\kappa)}\Big\|_{\mathrm{op}}\Big\|\sqrt{R(\kappa,V_\kappa)}\,q\sqrt{R(\kappa,V_\kappa)}\Big\|_{\mathfrak{I}_2}^{\ell+1} \\
&\quad + |\operatorname{tr}\{(R(\kappa,V_\kappa)q)^\ell R(\kappa,V_\kappa)V_\kappa'\partial[\partial,R(\kappa,V_\kappa)]q\}| \\
&\lesssim \|V_\kappa'\|_{L^\infty}\big(\kappa^{-1/2}\,\|q\|_{H_\kappa^{-1}}\,\big)^{\ell+1}.
\end{aligned}
$$

Summing over $\ell \geq 1$ we gain a factor of κ^{-1} to counteract the prefactor $\kappa^5(\kappa^2 - \varkappa^2)^{-2}$, and the remaining factor of $\varkappa^{-1}$ is paired with $\|q\|_{H_\kappa^{-1}}^2 \leq \|q\|_{H_\varkappa^{-1}}^2$ to conclude

$$
|(4.3.9)| \lesssim \big(\, \|V_\kappa\|_{W^{2,\infty}} + \|V_\kappa\|_{W^{2,\infty}}^2 \,\big)\alpha(\varkappa,q) \tag{4.3.14}
$$

uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large.

We now have enough decay in κ to treat the term (4.3.10) directly. Using Cauchy–Schwarz, the diagonal Green's function estimate (3.2.9), and the observation that $\|f\|_{L^2} \lesssim \kappa^{-1}\,\|f\|_{H_\kappa^1}$ we have

$$
\begin{aligned}
|(4.3.10)| &\lesssim \tfrac{\kappa^5}{(\kappa^2-\varkappa^2)^2}\,\|V_\kappa\|_{W^{1,\infty}}^2\,\big\|g(\varkappa,q) - \tfrac{1}{2\varkappa}\big\|_{H^1}\,\|g(\kappa,q+V_\kappa) - g(\kappa,V_\kappa)\|_{L^2} \\
&\lesssim \tfrac{\kappa^5}{(\kappa^2-\varkappa^2)^2}\,\|V_\kappa\|_{W^{1,\infty}}^2\,\varkappa^{-1}\,\|q\|_{H_\varkappa^{-1}}\,\kappa^{-2}\,\|q\|_{H_\kappa^{-1}} \lesssim \|V_\kappa\|_{W^{1,\infty}}^2\,\alpha(\varkappa,q)
\end{aligned}
$$

uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large. This same technique also works for (4.3.11) and (4.3.12) after integrating by parts once:

$$
\begin{aligned}
|(4.3.11)| &\lesssim \tfrac{\kappa^5}{(\kappa^2-\varkappa^2)^2}\big(\,\|V_\kappa''\|_{L^\infty}\,\varkappa^{-2}\kappa^{-1} + \|V_\kappa'\|_{L^\infty}\,\kappa^{-2}\big)\,\|q\|_{H_\varkappa^{-1}}\,\|q\|_{H_\kappa^{-1}} \lesssim \|V_\kappa'\|_{W^{1,\infty}}\,\alpha(\varkappa,q), \\
|(4.3.12)| &\lesssim \tfrac{\kappa^5}{(\kappa^2-\varkappa^2)^2}\,\|V_\kappa'\|_{L^\infty}\,\varkappa^{-1}\,\|q\|_{H_\varkappa^{-1}}\,\kappa^{-1}\,\|q\|_{H_\kappa^{-1}} \lesssim \|V_\kappa'\|_{L^\infty}\,\alpha(\varkappa,q)
\end{aligned}
$$

uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large. For the last term (4.3.13) we note that the extra factor of $g(\kappa, q+V_\kappa)$ can be put in H^1 by the second inequality of (3.2.2), and so by $H_\varkappa^1$-$H_\varkappa^{-1}$ duality we have

$$
\begin{aligned}
|(4.3.13)| \lesssim \tfrac{\kappa^5}{(\kappa^2-\varkappa^2)^2} \|V_\kappa'\|_{L^\infty} \big\{ &\|g(\kappa, q+V_\kappa)\|_{H^1} \|q\|_{H^{-1}_\varkappa} \big\|g(\varkappa, q) - \tfrac{1}{2\varkappa}\big\|_{H^1_\varkappa} \\
&+ \|V_\kappa\|_{L^\infty} \big\|g(\varkappa, q) - \tfrac{1}{2\varkappa}\big\|_{L^2} \|g(\kappa, q+V_\kappa) - g(\kappa, V_\kappa)\|_{L^2} \big\} \\
\lesssim \|V_\kappa'\|_{L^\infty} (1 + \|V_\kappa\|_{L^\infty})\, \alpha(\varkappa, q)&
\end{aligned}
$$

uniformly for $\kappa \geq 2\varkappa$ and $\varkappa$ large. This concludes the estimate of (4.3.7) and hence the proof of Proposition 4.3.1. □

From Proposition 4.3.1 we are able to conclude that the H^{-1}-norm of $q(t)$ is controlled by that of $q(0)$. We use this to show that the approximate flows $\widetilde{H}_\kappa$ are globally well-posed in $H^{-1}(\mathbb{R})$:

Proposition 4.3.2. *Fix V admissible. Given $A, T > 0$ there exists $\kappa_0 > 0$ so that for $\kappa \geq \kappa_0$ the $\widetilde{H}_\kappa$ flows (4.2.22) with initial data $q(0) \in B_A$ have solutions $q_\kappa(t)$ which are unique in $C_tH^{-1}([-T,T] \times \mathbb{R})$, depend continuously on the initial data, and are bounded in $C_tH^{-1}([-T,T] \times \mathbb{R})$ uniformly for $\kappa \geq \kappa_0$.*

Moreover, for all $\varkappa$ sufficiently large the diagonal Green's function $g(\varkappa, q) = g(x; \varkappa, q(t))$ evolves according to

$$
\begin{aligned}
\tfrac{d}{dt}\tfrac{1}{2g(\varkappa,q)} &= \Big\{ \tfrac{4\kappa^5}{\kappa^2-\varkappa^2} \Big[\tfrac{\varkappa}{\kappa} - \tfrac{g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)}{g(\varkappa,q)} \Big] + \Big(2\kappa^2 + \tfrac{\kappa^4}{\kappa^2-\varkappa^2} \Big) \Big[\tfrac{1}{g(\varkappa,q)} - 2\varkappa \Big] \Big\}' \\
-\tfrac{4\kappa^5}{\kappa^2-\varkappa^2} \tfrac{1}{g(\varkappa,q)^2} &\int G(x,y) \big\{ \big[q(g(\kappa, V_\kappa) - \tfrac{1}{2\kappa}) \big]' + q g'(\kappa, V_\kappa) + \big[V_\kappa (g(\kappa, q+V_\kappa) - g(\kappa, V_\kappa)) \big]' \\
&\qquad + V_\kappa [g'(\kappa, q+V_\kappa) - g'(\kappa, V_\kappa)] \big\}(y) G(y,x)\, dy,
\end{aligned}
$$

where $G(x,y) = G(x,y;\varkappa,q)$ and the dependence on (t,x) is suppressed.

Proof. The solution $q(t)$ of the $\widetilde{H}_\kappa$ flow satisfies the integral equation

$$
q(t) = e^{t4\kappa^2\partial_x} q(0) + 16\kappa^5 \int_0^t e^{(t-s)4\kappa^2\partial_x} \left[g'(\kappa, q(s) + V_\kappa(s)) - g'(\kappa, V_\kappa(s)) \right] ds.
$$

Local well-posedness is proved by contraction mapping, provided we have the Lipschitz estimate

$$\begin{aligned}&\|g'(\kappa, q+V_\kappa) - g'(\kappa, \tilde{q}+V_\kappa)\|_{H^{-1}}\\&\lesssim \|[g(\kappa, q+V_\kappa) - g(\kappa, V_\kappa)] - [g(\kappa, \tilde{q}+V_\kappa) - g(\kappa, V_\kappa)]\|_{H^1} \lesssim \|q-\tilde{q}\|_{H^{-1}}.\end{aligned}$$

To prove this Lipschitz estimate, it suffices to show that $f \mapsto d[g(\kappa, \cdot + V_\kappa)]|_q(f)$ is bounded as an operator $H^{-1} \to H^1$ uniformly for $q \in B_A$. Using the resolvent identity we calculate

$$d[g(\kappa, \cdot + V_\kappa)]|_q(f) = -\langle \delta_x, R(\kappa, q+V_\kappa) f R(\kappa, q+V_\kappa)\delta_x\rangle.$$

Estimating by duality, expanding the series (3.2.6), and using the estimate (3.2.7) we obtain

$$\|d[g(\kappa, \cdot + V_\kappa)]|_q(f)\|_{H^1} \lesssim \|f\|_{H^{-1}}$$

uniformly for $q \in B_A$ and $\kappa \geq \kappa_0$. Here, κ_0 is chosen so that the results of Section 4.2 and the above estimate apply to the ball of radius A.

For global well-posedness, we will need to choose κ_0 even larger. Let C denote the constant from Proposition 4.3.1, which depends only on the background wave V and $T > 0$. Then Grönwall's inequality and the α estimate (4.2.19) tell us that the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ obey

$$\|q_\kappa(t)\|^2_{H^{-1}_\varkappa} \leq 4\varkappa\alpha(\varkappa, q_\kappa(t)) \leq 4e^{CT}\varkappa\alpha(\varkappa, q(0)) \leq 4e^{CT}A^2 \tag{4.3.15}$$

for $|t| \leq T$, $\kappa \geq 2\varkappa$, and $\varkappa$ sufficiently large.

Fix $\varkappa = \varkappa_0$ sufficiently large so that Proposition 4.3.1 and the α estimate (4.2.19) apply throughout the ball $B_R(\varkappa)$ in $H^{-1}_\varkappa$ with radius $R = 2e^{CT/2}A$. The estimate (4.3.15) then applies, and so the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ remain in the ball $B_R(\varkappa)$ as long as $|t| \leq T$ and $\kappa \geq 2\varkappa$. For each $\kappa \geq 2\varkappa$, the $\widetilde{H}_\kappa$ flow is locally well-posed on the ball B_R in $H^{-1}_\varkappa$ by the elementary estimate $\|f\|_{H^{-1}_\varkappa} \approx_\varkappa \|f\|_{H^{-1}}$. Therefore we may iterate the local well-posedness result to the whole time interval $[-T, T]$. Moreover, using the estimate $\|f\|_{H^{-1}} \lesssim \varkappa\|f\|_{H^{-1}_\varkappa}$ again, we conclude that the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ are bounded in $H^{-1}(\mathbb{R})$ uniformly for $|t| \leq T$ and $\kappa \geq 2\varkappa$.

Next we turn to the second statement. From the expression (4.2.1) for the functional derivative of g we have

$$\begin{aligned}
&\frac{d}{dt}g(x;\varkappa,q(t)) = \big\{g(\varkappa,q),\widetilde{H}_\kappa\big\} = -\int G(x,y)\frac{dq}{dt}(y)G(y,x)\,dy \\
&= -\int G(x,y)\big\{16\kappa^5[g'(\kappa,q+V_\kappa)-g'(\kappa,V_\kappa)]+4\kappa^2q'\big\}(y)G(y,x)\,dy \\
&= -4\kappa^2g'(x;\varkappa,q)+16\kappa^5\int G(x,y)[g'(\kappa,q+V_\kappa)-g'(\kappa,V_\kappa)](y)G(y,x)\,dy,
\end{aligned}$$

where $G(x,y)=G(x,y;\varkappa,q)$. Using the ODEs (3.2.18) for $g(\kappa,q+V_\kappa)$ and $g(\kappa,V_\kappa)$ and then the identity (3.2.17), we obtain

$$\begin{aligned}
&16\kappa^5\int G(x,y)[g'(\kappa,q+V_\kappa)-g'(\kappa,V_\kappa)](y)G(y,x)\,dy = \\
&= \tfrac{8\kappa^5}{\kappa^2-\varkappa^2}\big\{g'(\varkappa,q)[g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)]-g(\varkappa,q)\big[g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)\big]'\big\} \\
&\quad -\tfrac{4\kappa^5}{\kappa^2-\varkappa^2}\int G(x,y)\big\{\big(V_\kappa[g(\kappa,q+V_\kappa)-g(\kappa,V_\kappa)]\big)' \\
&\qquad + V_\kappa[g'(\kappa,q+V_\kappa)-g'(\kappa,V_\kappa)]+\big[qg(\kappa,V_\kappa)\big]'+qg'(\kappa,V_\kappa)\big\}(y)G(y,x)\,dy.
\end{aligned}$$

Lastly, replacing g by $g-\frac{1}{2\kappa}$ in the term $[qg(\kappa,V_\kappa)]'$ and using the formula (4.2.5) for g' we write

$$\begin{aligned}
&-\tfrac{4\kappa^5}{\kappa^2-\varkappa^2}\int G(x,y)\big[qg(\kappa,V_\kappa)\big]'(y)G(y,x)\,dy \\
&= -\tfrac{4\kappa^5}{\kappa^2-\varkappa^2}\int G(x,y)\big[q(g(\kappa,V_\kappa)-\tfrac{1}{2\kappa})\big]'(y)G(y,x)\,dy+\tfrac{2\kappa^4}{\kappa^2-\varkappa^2}g'(x;\varkappa,q).
\end{aligned}$$

Differentiating $1/2g(\varkappa,q)$ using the chain rule and regrouping terms yields the desired expression. □

4.4 Convergence at low regularity

Ultimately we will show that the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ are convergent in $H^{-1}(\mathbb{R})$ as $\kappa\to\infty$. To this end, we will show the difference $q_\kappa-q_\varkappa$ for $\varkappa\geq\kappa$ converges to zero as $\kappa\to\infty$. This is a difficult task, as it involves estimating two different functions that solve separate nonlinear equations. To circumvent this, we will use that the H_κ and $H_\varkappa$ flows commute (cf. (4.2.21)). This allows us to write the $H_\varkappa$ flow of u by time t as

$$e^{tJ\nabla H_\varkappa}u = e^{tJ\nabla(H_\varkappa - H_\kappa)}e^{tJ\nabla H_\kappa}u.$$

We apply this identity to $u = q + V$ and $u = V$. Then $q_\varkappa(t)$, the $\widetilde{H}_\varkappa$ flow of $q(0)$ by time t, is the solution to

$$\begin{aligned} \frac{d}{dt}q = 16\varkappa^5[g'(\varkappa, q + W(t)) - g'(\varkappa, W(t))] + 4\varkappa^2 q' \\ - 16\kappa^5[g'(\kappa, q + W(t)) - g'(\kappa, W(t))] - 4\kappa^2 q' \end{aligned} \tag{4.4.1}$$

at time t with initial data $q_\kappa(t)$. Here, the background wave $W(t) \equiv W_{\varkappa,\kappa}(t)$ is the solution to

$$\frac{d}{dt}W = 16\varkappa^5 g'(\varkappa, W) + 4\varkappa^2 W' - 16\kappa^5 g'(\kappa, W) - 4\kappa^2 W' \tag{4.4.2}$$

at time t with initial data $V_\kappa(t)$. The upshot of this manipulation is that we may now write the difference $q_\varkappa(t) - q_\kappa(t)$ as the solution to the *single* equation (4.4.1) minus its initial data.

The purpose of this section is to first demonstrate convergence at some lower H^s regularity. As was introduced in [97], the change of variables $1/g(k, q)$ in place of q is convenient in witnessing this convergence.

Proposition 4.4.1. *Fix V admissible, $T > 0$, and $k > 0$ sufficiently large. Given a bounded and equicontinuous set $Q \subset H^{-1}(\mathbb{R})$ of initial data, define the set of H_κ and $\widetilde{H}_\kappa$ flows*

$$V_T^*(\kappa) := \{e^{tJ\nabla H_\varkappa}V(0) : |t| \le T,\ \varkappa \ge \kappa\}, \qquad Q_T^*(\kappa) := \{\widetilde{\Phi}_\kappa(t)q : q \in Q,\ |t| \le T\}$$

for $\kappa > 0$ sufficiently large. Then the solutions $q(t)$ to the difference flows (4.4.1) *with background waves $W(t)$ and initial data $q(0)$ obey*

$$\lim_{\kappa\to\infty} \sup_{\substack{q(0)\in Q_T^*(\kappa)\\ W(0)\in V_T^*(\kappa)}} \sup_{\varkappa\ge\kappa} \left\| \frac{1}{g(k, q(t))} - \frac{1}{g(k, q(0))} \right\|_{C_t H^{-2}([-T,T]\times\mathbb{R})} = 0.$$

Throughout the proof of Proposition 4.4.1 all spacetime norms will be over the slab $[-T, T] \times \mathbb{R}$. As V is admissible, there exists a constant κ_0 so that the H_κ flows $V_\kappa(t)$ exist and are bounded in $C_tW^{4,\infty}$ uniformly for $\kappa \geq \kappa_0$. By Proposition 4.3.2 the difference flows $q(t)$ for $q(0) \in Q_T^*(\kappa)$ are bounded in C_tH^{-1} uniformly for κ large, and hence are contained in a ball B_A for some $A > 0$. In particular, the functional $g(k,q) - \frac{1}{2k}$ for $q(t)$ exists for all k sufficiently large.

By the fundamental theorem of calculus we have

$$\left\| \frac{1}{2g(k,q(t))} - \frac{1}{2g(k,q(0))} \right\|_{C_tH^{-2}} \leq T \left\| \frac{d}{dt}\left(\frac{1}{2g(k,q(t))} - k \right) \right\|_{C_tH^{-2}},$$

and so it suffices to show that

$$\lim_{\kappa\to\infty} \sup_{\substack{q(0)\in Q_T^*(\kappa) \\ W(0)\in V_T^*(\kappa)}} \sup_{\varkappa\geq\kappa} \left\| \frac{d}{dt}\left(\frac{1}{2g(k,q(t))} - k \right) \right\|_{C_tH^{-2}} = 0.$$

The equation (4.4.1) for the evolution of q is the difference of the equations for the $\widetilde{H}_\varkappa$ and $\widetilde{H}_\kappa$ flows with the same background wave $W(t)$. In fact, for a general function $F(q)$ evaluated at $q(t)$ we have

$$\frac{d}{dt}F(q(t)) = \{F, \widetilde{H}_\varkappa\} - \{F, \widetilde{H}_\kappa\}.$$

We will apply this to the quantity $1/2g(k,q(t))$, for which the Poisson brackets above were computed in Proposition 4.3.2. After regrouping terms, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2g(k,q(t))} \\
&= \Big\{ \tfrac{1}{g(k,q)} \Big(q + \tfrac{4\kappa^5}{\kappa^2-k^2}\left[g(\kappa,q+W) - g(\kappa,W)\right] - \tfrac{4k^5}{\kappa^2-k^2}\left[g(k,q) - \tfrac{1}{2k}\right] \Big) \Big\}' && (4.4.3)\\
&\quad + \tfrac{1}{g(k,q)^2} \int G(x,y)\big\{V'q + \tfrac{4\kappa^5}{\kappa^2-k^2} g'(\kappa,W)q\big\}(y)G(y,x)\,dy && (4.4.4)\\
&\quad + \tfrac{1}{g(k,q)^2} \int G(x,y)\big\{V'q + \tfrac{4\kappa^5}{\kappa^2-k^2} W'[g(\kappa,q+W) - g(\kappa,W)]\big\}(y)G(y,x)\,dy && (4.4.5)\\
&\quad + \tfrac{1}{g(k,q)^2} \int G(x,y)\big\{(Vq)' + \tfrac{4\kappa^5}{\kappa^2-k^2} ([g(\kappa,W) - \tfrac{1}{2\kappa}]q)'\big\}(y)G(y,x)\,dy && (4.4.6)\\
&\quad + \tfrac{1}{g(k,q)^2} \int G(x,y)\big\{2Vq' + \tfrac{8\kappa^5}{\kappa^2-k^2} W[g(\kappa,q+W) - g(\kappa,W)]'\big\}(y)G(y,x)\,dy && (4.4.7)\\
&\quad - \{(4.4.3)\text{–}(4.4.7) \text{ with } \kappa \text{ replaced by } \varkappa\},
\end{aligned}$$

where $G(x,y) = G(x,y;k,q)$. Note that for each term we have subtracted the limiting expression as $\kappa \to \infty$ (e.g. inserting $(q/g(k,q))'$ in (4.4.3) and $V'q$ in the integrand of (4.4.4)) which is canceled by its counterpart in the corresponding $\varkappa$ terms.

To prove Proposition 4.4.1 we must show that all of the terms above converge to zero in C_tH^{-2} as $\kappa \to \infty$ uniformly for $\varkappa \geq \kappa$, $q(0) \in Q_T^*(\kappa)$, and $W(0) \in V_T^*(\kappa)$. To simplify the notation, we will only show that the terms (4.4.3)–(4.4.7) converge to zero as $\kappa \to \infty$; the upper bound we will obtain for each κ term will also hold for the corresponding $\varkappa$ term uniformly for $\varkappa \geq \kappa$.

First, we claim that the admissibility of V implies that the background waves $W(t) = e^{tJ\nabla(H_\varkappa - H_\kappa)}W$ obey

$$\lim_{\kappa\to\infty} \sup_{W\in V_T^*(\kappa)} \sup_{\varkappa\geq\kappa} \|e^{tJ\nabla(H_\varkappa-H_\kappa)}W - W\|_{C_tW^{2,\infty}} = 0. \tag{4.4.8}$$

For $\varkappa \geq \kappa$ and $W \in V_T^*(\kappa)$, we use the commutativity of the KdV and H_κ flows (cf. (4.2.21)) to write

$$\begin{aligned}
&\|e^{tJ\nabla(H_\varkappa-H_\kappa)}W - W\|_{C_tW^{2,\infty}} \\
&\leq \|e^{tJ\nabla H_\varkappa}e^{-tJ\nabla H_\kappa}W - e^{tJ\nabla E_2}e^{-tJ\nabla H_\kappa}W\|_{C_tW^{2,\infty}} + \|e^{-tJ\nabla H_\kappa}e^{tJ\nabla E_2}W - W\|_{C_tW^{2,\infty}} \\
&\leq 2 \sup_{W\in V_{2T}^*(\kappa)} \sup_{\varkappa\geq\kappa} \|e^{tJ\nabla H_\varkappa}W - e^{tJ\nabla E_2}W\|_{C_tW^{2,\infty}}.
\end{aligned}$$

The RHS converges to zero as $\kappa \to \infty$ by condition (iii) of Definition 4.1.1.

Now we turn to the first term (4.4.3), which arises in the case $V \equiv 0$ and is handled as in [97]. Using the second estimate of (3.2.2) we can put the factor of $1/g(k,q)$ in H^1 and bound

$$\begin{aligned}
\|(4.4.3)\|_{H^{-2}} &\lesssim \left\|\tfrac{1}{g(k,q)}\right\|_{H^1} \left\|q + \tfrac{4\kappa^5}{\kappa^2-k^2}\left[g(\kappa,q+W) - g(\kappa,W)\right] - \tfrac{4k^5}{\kappa^2-k^2}\left[g(k,q) - \tfrac{1}{2k}\right]\right\|_{H^{-1}} \\
&\lesssim \left\|q + 4\kappa^3\left[g(\kappa,q+W) - g(\kappa,W)\right]\right\|_{H^{-1}} \\
&\quad + \kappa\left\|g(\kappa,q+W) - g(\kappa,W)\right\|_{H^{-1}} + \kappa^{-2}\left\|g(k,q) - \tfrac{1}{2k}\right\|_{H^{-1}}.
\end{aligned}$$

We allow implicit constants to depend on the fixed constant $k > 0$. The second and third terms converge to zero as $\kappa \to 0$ by the estimate (3.2.9). In the following lemma we check that the first term also converges to zero:

Lemma 4.4.2. *We have*

$$4\kappa^3[g(\kappa, q+W) - g(\kappa, W)] + q \to 0 \quad \text{in } H^{-1} \text{ as } \kappa \to \infty$$

uniformly for $q \in Q_T^*(\kappa)$.

Proof. We claim that the first term $4\kappa^2 R_0(2\kappa)q$ of the series for $4\kappa^3[g(\kappa, q+W) - g(\kappa, W)]$ converges to q in H^{-1}. We compute

$$\left\|4\kappa^2 R_0(2\kappa)q - q\right\|_{H^{-1}}^2 = \int \frac{\xi^4 |\hat{q}(\xi)|^2}{(\xi^2+4\kappa^2)^2(\xi^2+4)}\, d\xi \lesssim \int \frac{|\hat{q}(\xi)|^2}{\xi^2+4\kappa^2}\, d\xi = \|q\|_{H_\kappa^{-1}}^2 .$$

Note that for $q \in Q_T^*(\kappa)$, Proposition 4.3.1 only gives us control over $\alpha(\varkappa, q)$ for $\kappa \geq 2\varkappa$ and not $\varkappa = \kappa$. To circumvent this, we simply take $\varkappa = \kappa/2$ and note that trivially $\|q\|_{H_\kappa^{-1}}^2 \leq \|q\|_{H_{\kappa/2}^{-1}}^2$. If we let C denote the constant from Proposition 4.3.1, then Grönwall's inequality and the α estimate (4.2.19) yield

$$\|q\|_{H_\kappa^{-1}}^2 \leq \|q\|_{H_{\kappa/2}^{-1}}^2 \lesssim \tfrac{\kappa}{2}\alpha(\tfrac{\kappa}{2}, q) \leq e^{CT}\tfrac{\kappa}{2}\alpha(\tfrac{\kappa}{2}, q(0)),$$

where $q(0) \in Q$ is the initial data for $q \in Q_T^*(\kappa)$. As Q is equicontinuous, we know $\frac{\kappa}{2}\alpha(\frac{\kappa}{2}, q(0))$ converges to zero as $\kappa \to \infty$ uniformly for $q(0) \in Q$ by Lemma 3.5.1 and the α estimate (4.2.19). This completes the claim.

It remains to show that

$$4\kappa^3[g(\kappa, q+W) - g(\kappa, W)] + 4\kappa^2 R_0(2\kappa)q \to 0 \quad \text{in } H^{-1} \text{ as } \kappa \to \infty$$

uniformly for $q \in B_A$. We expand $g(\kappa, q+W) - g(\kappa, W)$ as a series in powers of q, and then expand each resolvent $R(\kappa, W)$ in powers of W. We estimate by duality; for $f \in H^1$ and $R_0 = R_0(\kappa)$ we have

$$\left| \int f(x) \big\{ 4\kappa^3[g(\kappa, q+W) - g(\kappa, W)] + \tfrac{1}{\kappa} R_0(2\kappa) q \big\}(x)\, dx \right|$$

$$\le 4\kappa^3 \sum_{\substack{m_0,m_1\ge 0\\ m_0+m_1\ge 1}} \big|\operatorname{tr}\{fR_0(WR_0)^{m_0}qR_0(WR_0)^{m_1}\}\big|$$

$$+ 4\kappa^3 \sum_{\ell\ge 2,\ m_0,\dots,m_\ell\ge 0} \big|\operatorname{tr}\{fR_0(WR_0)^{m_0}qR_0(WR_0)^{m_1}qR_0\cdots qR_0(WR_0)^{m_\ell}\}\big|.$$

In the first sum, we put $\sqrt{R_0}q\sqrt{R_0}$ and $\sqrt{R_0}f\sqrt{R_0}$ in $\mathfrak{I}_2$ and measure the rest in operator norm. For the second sum there are always at least two factors of $\sqrt{R_0}q\sqrt{R_0}$, and so we put $\sqrt{R_0}q\sqrt{R_0}$ in $\mathfrak{I}_2$ and the rest in operator norm:

$$\lesssim \kappa^3 \sum_{\substack{m_0,m_1\ge 0\\ m_0+m_1\ge 1}} \frac{\|f\|_{L^2}}{\kappa^{3/2}}\frac{\|q\|_{H^{-1}_\kappa}}{\kappa^{1/2}}\left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m_0+m_1}$$

$$+ \kappa^3 \sum_{\ell\ge 2,\ m_0,\dots,m_\ell\ge 0} \frac{\|f\|_{L^\infty}}{\kappa^2}\left(\frac{\|q\|_{H^{-1}_\kappa}}{\kappa^{1/2}}\right)^{\ell}\left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m_0+\dots+m_\ell}.$$

Re-indexing $m = m_0 + \dots + m_\ell$, we compute

$$\begin{aligned}\sum_{m_0,\dots,m_\ell\ge 0}\left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m_0+\dots+m_\ell} &= \sum_{m=0}^{\infty}\frac{(\ell+m)!}{\ell!\,m!}\left(\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{m}\\ &= \left(1-\frac{\|W\|_{L^\infty}}{\kappa^2}\right)^{\ell+1} \le 1\end{aligned} \tag{4.4.9}$$

uniformly for $\ell \ge 1$ and κ large. Altogether we obtain

$$\left|\int f\big\{4\kappa^3[g(\kappa,q+W)-g(\kappa,W)] + \tfrac{1}{\kappa}R_0(2\kappa)q\big\}\,dx\right| \lesssim \|f\|_{H^1}\,\|q\|_{H^{-1}_\kappa}.$$

Taking a supremum over $\|f\|_{H^1}\le 1$, we conclude

$$\big\|4\kappa^3[g(\kappa,q+W)-g(\kappa,W)] + \tfrac{1}{\kappa}R_0(2\kappa)q\big\|_{H^{-1}} \lesssim \|q\|_{H^{-1}_\kappa}.$$

We have already shown that the RHS converges to zero as $\kappa\to\infty$ uniformly for $q(0)\in Q_T^*(\kappa)$, and so the claim follows. □

For the terms (4.4.4) and (4.4.5) we note that the expressions inside the curly brackets converge to zero in H^{-1} by (4.4.8) and Lemmas 4.2.3 and 4.4.2. In fact, this is enough to show that the contributions of (4.4.4) and (4.4.5) converge to zero in H^{-2} because the integral operator is bounded on H^{-1}:

Lemma 4.4.3. *There exists $k > 0$ sufficiently large so that the operator*

$$h(x) \mapsto \frac{1}{g(x;k,q)^2} \int G(x,y;k,q)h(y)G(y,x;k,q)\,dy$$

is bounded $H^{-1} \to H^1$ uniformly for $q \in B_A$.

Proof. First, we use the second estimate of (3.2.2) to put the factors of $1/g(k,q)$ in H^1. The remaining operator is easily estimated by duality, expanding $G(k,q)$ in a series, and using the estimate (3.2.3). $\square$

The remaining two terms (4.4.6) and (4.4.7) are more delicate, because the derivative falling on q obstructs convergence of curly-bracketed terms in H^{-1}. First we consider (4.4.6). Write

$$\|(4.4.6)\|_{H^{-2}} \lesssim \left\| \int G(x,y;k,q)(F_\kappa q)'(y)G(y,x;k,q)\,dy \right\|_{H^{-1}}$$

for a function F_κ which we know converges to 0 in $W^{2,\infty}$ by (4.4.8) and Lemma 4.2.3. To show that the RHS converges to zero as $\kappa \to \infty$, we exploit that the integrand and the Green's functions all contain q:

Lemma 4.4.4. *If $F_\kappa \to 0$ in $W^{2,\infty}$ as $\kappa \to \infty$, then there exists $k > 0$ sufficiently large so that*

$$\int G(x,y;k,q)(F_\kappa q)'(y)G(y,x;k,q)\,dy \to 0 \quad \text{in } H^{-1} \text{ as } \kappa \to \infty$$

uniformly for $q \in B_A$.

Proof. We estimate the integral by duality and maneuver the derivative onto F_κ and the test function. For $f \in H^1$ we have

$$\begin{aligned}
&\int f(x) \int G(x,y;k,q)(F_\kappa q)'(y)G(y,x;k,q)\,dy \\
&= \sum_{\ell,m=0}^{\infty} (-1)^{\ell+m} \operatorname{tr}\{f(R_0q)^\ell R_0[\partial, F_\kappa q]R_0(qR_0)^m\} \\
&= \sum_{\ell,m=0}^{\infty} (-1)^{\ell+m} \big(\operatorname{tr}\{f(R_0q)^\ell R_0 \partial F_\kappa (qR_0)^{m+1}\} - \operatorname{tr}\{f(R_0q)^{\ell+1} F_\kappa R_0 \partial (qR_0)^m\} \big),
\end{aligned}$$

where $R_0 = R_0(k)$. The first factor of fR_0 prevents us from combining the two terms in the summand to create a commutator. Instead, we pair the first term of the (ℓ, m) summand with the second term of the $(\ell - 1, m + 1)$ summand to create a commutator, and leave the first term of the $(0, m)$ summand and the second term of the $(\ell, 0)$ summand:

$$\left| \int f(x) \int G(x, y; k, q)(F_\kappa q)'(y) G(y, x; k, q)\, dy \right|$$

$$\leq \left| \sum_{n=1}^{\infty} (-1)^n \sum_{j=1}^{n} \operatorname{tr}\{f(R_0 q)^j [R_0 \partial, F_\kappa](qR_0)^{n+1-j}\} \right| \tag{4.4.10}$$

$$+ \left| \sum_{m=1}^{\infty} \operatorname{tr}\{fR_0 \partial F_\kappa (qR_0)^{m+1}\} \right| + \left| \sum_{\ell=1}^{\infty} \operatorname{tr}\{f(R_0 q)^{\ell+1} F_\kappa R_0 \partial\} \right| \tag{4.4.11}$$

$$+ \left| \operatorname{tr}\{fR_0 \partial F_\kappa q R_0\} - \operatorname{tr}\{fR_0 q F_\kappa R_0 \partial\} \right|. \tag{4.4.12}$$

For the first term (4.4.10), we write

$$[R_0 \partial, F_\kappa] = R_0[\partial, F_\kappa] + [R_0, F_\kappa]\partial = R_0 F_\kappa' + R_0 F_\kappa'' \partial R_0 + R_0 2F_\kappa' \partial^2 R_0.$$

Putting each factor of $\sqrt{R_0} q \sqrt{R_0}$ in $\mathfrak{I}_2$ and pairing each ∂ with one copy of $\sqrt{R_0}$ in operator norm (cf. (4.3.14)) we estimate

$$(4.4.10) \lesssim \sum_{n=1}^{\infty} nk^{-2} \|f\|_{L^\infty} \|F_\kappa\|_{W^{2,\infty}} \left(k^{-1/2} \|q\|_{H^{-1}}\right)^{n+1} \lesssim \|f\|_{H^1} A^2 \|F_\kappa\|_{W^{2,\infty}}$$

uniformly for $q \in B_A$ and k sufficiently large. The RHS converges to zero because $F_\kappa \to 0$ in $W^{2,\infty}$.

Similarly, for the second term (4.4.11) we have

$$(4.4.11) \leq 2 \sum_{m=1}^{\infty} \left\| f\sqrt{R_0} \right\|_{\mathrm{op}} \left\| \partial \sqrt{R_0} \right\|_{\mathrm{op}} \left\| \sqrt{R_0} F_\kappa q \sqrt{R_0} \right\|_{\mathfrak{I}_2} \left\| \sqrt{R_0}\, q \sqrt{R_0} \right\|_{\mathfrak{I}_2}^{m}$$

$$\lesssim \|f\|_{H^1} A^2 \|F_\kappa\|_{W^{2,\infty}}$$

uniformly for $q \in B_A$ and k sufficiently large, and again the RHS converges to zero.

For the last term (4.4.12), we recombine the traces as a commutator to obtain

$$(4.4.11) \lesssim \left\| \sqrt{R_0} f' \sqrt{R_0} \right\|_{\mathfrak{I}_2} \left\| \sqrt{R_0} F_\kappa q \sqrt{R_0} \right\|_{\mathfrak{I}_2} \lesssim \|f\|_{H^1} A \|F_\kappa\|_{W^{2,\infty}}$$

uniformly for $q \in B_A$ and k sufficiently large. Again, the RHS converges to zero since $F_\kappa \to 0$ in $W^{2,\infty}$. $\square$

To finish the proof of Proposition 4.4.1, we must show that the last term (4.4.7) converges to zero in H^{-2} as $\kappa \to \infty$. As previously mentioned the convergence within the curly brackets occurs in H^{-2}, and now q is concealed within another Green's function. To overcome this, we use the commutativity relation trick (4.3.2) used in Proposition 4.3.1. Using the ODEs (3.2.18) for $g(\kappa, q+W)$ and $g(\kappa, W)$, applying the identity (3.2.17), and regrouping terms, we have

$$
\begin{aligned}
&(4.4.7)\\
&= -\tfrac{4\kappa^5}{(\kappa^2-k^2)^2}\left\{\tfrac{W[g(\kappa,q+W)-g(\kappa,W)]}{g(k,q)}\right\}' && (4.4.13)\\
&\quad + \tfrac{2\kappa^5}{(\kappa^2-k^2)^2}\tfrac{1}{g(k,q)^2}\int G(x,y)\big\{(-W''' + 4k^2W')[g(\kappa,q+W)-g(\kappa,W)] && (4.4.14)\\
&\qquad - 3W''\big[g(\kappa,q+W)-g(\kappa,W)\big]' - 3W'\big[g(\kappa,q+W)-g(\kappa,W)\big]'' && (4.4.15)\\
&\qquad - (W^2)'[g(\kappa,q+W)-g(\kappa,W)] - 4W^2\big[g(\kappa,q+W)-g(\kappa,W)\big]'\big\}G(y,x) && (4.4.16)\\
&\quad + \tfrac{4\kappa^5}{(\kappa^2-k^2)^2}\tfrac{1}{g(k,q)^2}\int G(x,y)\big\{W'q[g(\kappa,q+W)-g(\kappa,W)]\big\}(y)G(y,x)\,dy && (4.4.17)\\
&\quad + \tfrac{1}{g(k,q)^2}\int G(x,y)\big\{2Vq' - \tfrac{4\kappa^5}{(\kappa^2-k^2)^2}W\big[qg(\kappa,W)\big]'\big\}(y)G(y,x)\,dy, && (4.4.18)
\end{aligned}
$$

where $G(x,y) = G(x,y;k,q)$. Note that in (4.4.18) we have isolated the term which cancels $2Vq'$. We will show that each of the terms (4.4.13)–(4.4.18) converge to zero.

The first term (4.4.13) is easily estimated using the estimates (3.2.2):

$$
\|(4.4.13)\|_{H^{-2}} \lesssim \|W\|_{W^{1,\infty}} \left\|\tfrac{1}{g(k,q)}\right\|_{H^1} \tfrac{4\kappa^5}{(\kappa^2-k^2)^2}\, \|g(\kappa,q+W)-g(\kappa,W)\|_{H^{-1}}.
$$

The RHS converges to zero as $\kappa \to \infty$ by Lemma 4.4.2.

For the contribution from (4.4.14), we first use Lemma 4.4.3 to put the curly bracketed terms in H^{-1}:

$$
\|(4.4.14)\|_{H^{-2}} \lesssim \|W'\|_{W^{3,\infty}} \tfrac{2\kappa^5}{(\kappa^2-k^2)^2}\, \|g(\kappa,q+W)-g(\kappa,W)\|_{H^{-1}}.
$$

Again, the RHS converges to zero by Lemma 4.4.2. It was exactly to estimate this term that we required that V_κ be in $W^{4,\infty}$ in Definition 4.1.1.

For the contributions (4.4.15) and (4.4.16) we again use Lemma 4.4.3 to obtain

$$\begin{aligned}\|(4.4.15)\|_{H^{-2}} &\lesssim \|W'\|_{W^{2,\infty}} \tfrac{2\kappa^5}{(\kappa^2-k^2)^2} \|g(\kappa,q+W)-g(\kappa,W)\|_{H^1}\,,\\ \|(4.4.16)\|_{H^{-2}} &\lesssim \|W\|_{W^{2,\infty}}^2 \tfrac{2\kappa^5}{(\kappa^2-k^2)^2} \|g(\kappa,q+W)-g(\kappa,W)\|_{H^1}\,.\end{aligned}$$

These still converge to zero as $\kappa\to\infty$ by the equicontinuity of Q:

Lemma 4.4.5. *We have*

$$\kappa[g(\kappa,q+W)-g(\kappa,W)]\to 0 \quad \textit{in } H^1 \textit{ as } \kappa\to\infty$$

uniformly for $q\in Q_T^*(\kappa)$.

Proof. By the diagonal Green's function estimate (3.2.9) we have

$$\kappa\,\|g(\kappa,q+W)-g(\kappa,W)\|_{H^1} \lesssim \|q\|_{H_\kappa^{-1}}\,.$$

In the proof of Lemma 4.4.2 we used Grönwall's inequality and equicontinuity to show that the RHS converges to zero uniformly for $q\in Q_T^*(\kappa)$. □

For the term (4.4.17) we use Lemma 4.4.3 and the estimates (3.2.2) to put q in H^{-1}:

$$\|(4.4.17)\|_{H^{-2}} \lesssim \|W'\|_{W^{1,\infty}} A\tfrac{2\kappa^5}{(\kappa^2-k^2)^2} \|g(\kappa,q+W)-g(\kappa,W)\|_{H^1}\,,$$

and the RHS converges to zero by Lemma 4.4.5.

Finally, for the last term (4.4.18) we write

$$\begin{aligned}\|(4.4.18)\|_{H^{-2}} \lesssim &\left\| \int G(x,y)\big\{2Vq' - \tfrac{4\kappa^5}{(\kappa^2-k^2)^2}Wg(\kappa,W)q'\big\}(y)G(y,x)\,dy\right\|_{H^{-1}}\\ &+\left\| \int G(x,y)\big\{\tfrac{4\kappa^5}{(\kappa^2-k^2)^2}Wg'(\kappa,W)q\big\}(y)G(y,x)\,dy\right\|_{H^{-1}}.\end{aligned}$$

The first term converges to zero by Lemmas 4.2.3 and 4.4.4 due to the leading term $\frac{1}{2\kappa}$ of $g(\kappa,W)$. The second term converges to zero by Lemmas 4.2.3 and 4.4.3. This completes the estimate of (4.4.7), and hence concludes the proof of Proposition 4.4.1.

4.5 Well-posedness

We are now equipped to prove that KdV with potential (4.1.1) is globally well-posed in $H^{-1}(\mathbb{R})$. We begin by constructing solutions as the limit of the $\widetilde{H}_\kappa$ flows as $\kappa \to \infty$.

Theorem 4.5.1. *Fix V admissible and $T > 0$. Given initial data $q(0) \in H^{-1}(\mathbb{R})$, the corresponding solutions $q_\kappa(t)$ to the $\widetilde{H}_\kappa$ flows* (4.2.22) *are Cauchy in $C_t H^{-1}([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$.*

We define the limit $q(t) := \lim_{\kappa\to\infty} q_\kappa(t)$ in $C_t H^{-1}$ $([-T,T] \times \mathbb{R})$ to be the H^{-1} solution of (4.1.1) *with initial data $q(0)$.*

Proof. In the following all spacetime norms will be taken over the slab $[-T,T]\times\mathbb{R}$. Proposition 4.3.2 guarantees that there exists a constant κ_0 so that the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ are bounded in $H^{-1}(\mathbb{R})$ uniformly for $|t| \le T$ and $\kappa \ge \kappa_0$.

We want to show that the difference $q_\kappa - q_\varkappa$ for $\varkappa \ge \kappa$ converges to zero as $\kappa \to \infty$. As the H_κ and $H_\varkappa$ flows commute (cf. (4.2.21)), we may write the $H_\varkappa$ flow of u by time t as

$$e^{tJ\nabla H_\varkappa} u = e^{tJ\nabla(H_\varkappa - H_\kappa)} e^{tJ\nabla H_\kappa} u.$$

We apply this identity to $u = q + V$ and $u = V$. This allows us to write

$$q_\varkappa(t) = \widetilde{\Phi}_{\varkappa,\kappa,W}(t) q_\kappa(t),$$

where $\widetilde{\Phi}_{\varkappa,\kappa,W}(t)$ denotes the flow of (4.4.1) by time t for the background wave obeying (4.4.2) with initial data $W(0) = V_\kappa(t)$. We estimate

$$\|q_\kappa - q_\varkappa\|_{C_tH^{-1}} \le \sup_{\substack{q\in Q_T^*(\kappa)\\ W(0)\in V_T^*(\kappa)}} \sup_{\varkappa\ge\kappa} \|\widetilde{\Phi}_{\varkappa,\kappa,W}(t)q - q\|_{C_tH^{-1}}, \tag{4.5.1}$$

for the sets

$$Q_T^*(\kappa) := \{\widetilde{\Phi}_\kappa(t)q : q \in Q,\ |t| \le T\}, \quad V_T^*(\kappa) := \{e^{tJ\nabla H_\varkappa} V(0) : |t| \le T,\ \varkappa \ge \kappa\} \tag{4.5.2}$$

with $Q = \{q(0)\}$. As $Q \subset H^{-1}(\mathbb{R})$ is trivially bounded and equicontinuous, then the following more general fact will conclude the proof. We allow Q to be an arbitrary bounded and equicontinuous set so that we may reuse this fact in Theorem 4.5.3. □

Proposition 4.5.2. *Fix V admissible, $T > 0$, and a bounded and equicontinuous set $Q \subset H^{-1}(\mathbb{R})$ of initial data. Then the solutions $\widetilde{\Phi}_{\varkappa,\kappa,W}(t)q$ of the difference flow* (4.4.1) *with background wave $W(t)$ and initial data q obey*

$$\lim_{\kappa\to\infty} \sup_{\substack{q\in Q_T^*(\kappa)\\ W(0)\in V_T^*(\kappa)}} \sup_{\varkappa\geq\kappa} \|\widetilde{\Phi}_{\varkappa,\kappa,W}(t)q - q\|_{C_tH^{-1}([-T,T]\times\mathbb{R})} = 0, \tag{4.5.3}$$

where $Q_T^*(\kappa)$ and $V_T^*(\kappa)$ are defined in (4.5.2).

Proof. For $q \in Q_T^*(\kappa)$ and $W(0) \in V_T^*(\kappa)$, let $q(t)$ denote the solution to the difference flow (4.4.1) with initial data q and background wave $W(t)$. As was introduced in [97], the change variables $1/2g(k,q)$ in place of q is convenient in witnessing this convergence. Indeed, it suffices to show that under difference flow (4.4.1) we have

$$\lim_{\kappa\to\infty} \sup_{\substack{q\in Q_T^*(\kappa)\\ W(0)\in V_T^*(\kappa)}} \left\| \frac{1}{2g(k,q(t))} - \frac{1}{2g(k,q)} \right\|_{C_tH^1} = 0$$

for $k > 0$ fixed sufficiently large, because then an application of the diffeomorphism property (Proposition 4.2.1) shows that this implies (4.5.3).

Fix $\varepsilon > 0$. We aim to show that

$$\left\| \frac{1}{2g(k,q(t))} - \frac{1}{2g(k,q)} \right\|_{C_tH^1} \leq \varepsilon \tag{4.5.4}$$

for all κ sufficiently large, uniformly for $\varkappa \geq \kappa$, $q \in Q_T^*(\kappa)$, and $W(0) \in V_T^*(\kappa)$. In Proposition 4.4.1 we already saw this convergence in H^{-2}, and now we will upgrade this convergence to H^1. We do not rely on equicontinuity as in [97] because the presence of the background wave V breaks the conservation of α. Instead, we will choose the parameters κ and $\varkappa$ dependently.

For $h \in \mathbb{R}$, we estimate the translation of q by h in H^{-1} by truncating in Fourier variables at a large radius K:

$$\begin{aligned}
&\|q(t,x+h)-q(t,x)\|_{H_x^{-1}}^2 \lesssim \int_{\mathbb{R}} |e^{ih\xi}-1|^2 \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4}\,d\xi \\
&\leq K^2h^2 \int_{|\xi|\leq K} \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4}\,d\xi + \int_{|\xi|\geq K} \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4}\,d\xi \\
&\lesssim K^2h^2\,\|q(t)\|_{H^{-1}}^2 + \int_{\mathbb{R}} \frac{|\hat{q}(t,\xi)|^2}{\xi^2+4K^2}\,d\xi \lesssim K^2h^2A^2 + K\alpha(K,q(t)),
\end{aligned} \tag{4.5.5}$$

where in the last inequality we used the α estimate (4.2.19). If we let C denote the constant from Proposition 4.3.1 then Grönwall's inequality yields

$$K\alpha(K,q(t)) \leq e^{2CT}K\alpha(K,q) \leq e^{3CT}K\alpha(K,q(0))$$

for all $|t| \leq T$ and $\kappa \geq 2K$, where $q(0) \in Q$ is the initial data of the $\widetilde{H}_\kappa$ flow $q \in Q_T^*(\kappa)$. As the set Q is equicontinuous, then $K\alpha(K,q(0)) \to 0$ as $K \to \infty$ uniformly for $q(0) \in Q$ by Lemma 3.5.1 and the α estimate (4.2.19). Therefore, given $\eta = \eta(\varepsilon) > 0$ small (to be chosen later) there exists K sufficiently large so that

$$K\alpha(K,q(t)) \lesssim \eta \quad \text{uniformly for } |t| \leq T,\ q \in Q_T^*(\kappa),\ \text{and } \kappa \geq 2K.$$

Combining this with the estimate (4.5.5) and optimizing in h, there is a value $h_0 \sim \eta^{1/2}K^{-1}$ such that

$$\|q(t,x+h)-q(t,x)\|_{C_tH^{-1}}^2 \lesssim \eta \quad \text{for } |h| \leq h_0. \tag{4.5.6}$$

We now will turn this control over the translates of q into control of the Fourier tails of $1/g(k,q)$. For $r \in \mathbb{R}$ we have

$$\int |e^{i\xi h}-1|^2 re^{-2r|h|}\,dh = \frac{2\xi^2}{\xi^2+4r^2} \geq \begin{cases} \frac{2}{5} & |\xi| \geq r, \\ 0 & |\xi| < r. \end{cases}$$

Writing $\mathcal{F}$ for the Fourier transform, this yields

$$\sup_{|t|\leq T} \int_{|\xi|\geq r} \left|\mathcal{F}\left[\frac{1}{2g(k,q(t))}\right](\xi)\right|^2 (\xi^2+1)\,d\xi$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}} \left\| \frac{1}{2g(x+h;k,q(t))} - \frac{1}{2g(x;k,q(t))} \right\|_{C_t H^1_x}^2 re^{-2r|h|}\,dh \\
&= \int_{\mathbb{R}} \left\| \frac{1}{2g(x;k,q(t,\cdot+h))} - \frac{1}{2g(x;k,q(t,\cdot))} \right\|_{C_t H^1_x}^2 re^{-2r|h|}\,dh \\
&\lesssim \int_{\mathbb{R}} \|q(t,x+h)-q(t,x)\|_{C_t H^{-1}_x}^2\, re^{-2r|h|}\,dh
\end{aligned}$$

by the diffeomorphism property (Proposition 4.2.1). Splitting this integral into $|h| \leq h_0$ and $|h| \geq h_0$ and using (4.5.6), we obtain the upper bound

$$\lesssim \eta h_0 r + e^{-2rh_0}.$$

Optimizing in r, we pick $r = r_0 := \frac{1}{2h_0}\log\frac{2}{\eta}$ to arrive at

$$\lesssim \eta\big(1+\log\tfrac{2}{\eta}\big).$$

Finally, we employ this uniform control over the Fourier tails of $1/q(k,q)$ to upgrade our H^{-2} convergence to H^1. Separating $|\xi| \leq r_0$ and $|\xi| \geq r_0$, we have

$$\begin{aligned}
\left\| \frac{1}{2g(k,q(t))} - \frac{1}{2g(k,q(0))} \right\|_{H^1}^2 \lesssim (r_0^2+1)^6 \left\| \frac{1}{2g(k,q(t))} - \frac{1}{2g(k,q(0))} \right\|_{H^{-2}}^2 \\
+ \sup_{|t|\leq T} \left\| \frac{1}{2g(k,q(t))} \right\|_{H^1(|\xi|\geq r_0)}^2 .
\end{aligned} \tag{4.5.7}$$

We just saw that the second term of RHS(4.5.7) is $\lesssim \eta(1+\log\frac{2}{\eta})$, and picking $\eta = \eta(\varepsilon) > 0$ sufficiently small we can make this upper bound $\leq \frac{1}{2}\varepsilon$. With all other parameters determined, we then use Proposition 4.4.1 to make the first term of RHS(4.5.7) less than $\frac{1}{2}\varepsilon$ for all κ sufficiently large. This demonstrates (4.5.4), and hence concludes Proposition 4.5.2 and Theorem 4.5.1. □

Applying the previous result to a different set Q, we also obtain uniform control over the limits $q(t)$ as we vary the initial data:

Theorem 4.5.3. *Fix V admissible and $T > 0$. Given a convergent sequence $q_n(0) \in H^{-1}(\mathbb{R})$ of initial data, the corresponding solutions $q_n(t)$ of KdV with potential* (4.1.1) *constructed in Theorem 4.5.1 are Cauchy in $C_t H^{-1}([-T,T]\times\mathbb{R})$ as $n \to \infty$.*

Proof. Consider the set $Q := \{q_n(0) : n \in \mathbb{N}\}$ of initial data, which is bounded and equicontinuous in H^{-1} since it is convergent in H^{-1}. We estimate the difference $q_n(t) - q_m(t)$ using the triangle inequality, by first mediating via $\widetilde{H}_\varkappa$ flows and then estimating the difference between $\widetilde{H}_\kappa$ and $\widetilde{H}_\varkappa$ flows using (4.5.1). This yields

$$\begin{aligned} \|q_n(t) - q_m(t)\|_{C_tH^{-1}} \leq \|\widetilde{\Phi}_\kappa(t)q_n(0) - \widetilde{\Phi}_\kappa(t)q_m(0)\|_{C_tH^{-1}} \\ + 2 \sup_{\substack{q\in Q_T^*(\kappa) \\ W(0)\in V_T^*(\kappa)}} \sup_{\varkappa\geq\kappa} \|\widetilde{\Phi}_{\varkappa,\kappa,W}(t)q - q\|_{C_tH^{-1}}, \end{aligned} \tag{4.5.8}$$

where $Q_T^*(\kappa)$ and $V_T^*(\kappa)$ are defined in (4.5.2). Fix $\varepsilon > 0$. By Proposition 4.5.2 there exists κ_0 sufficiently large so that the second term of RHS(4.5.8) is $\leq \frac{1}{2}\varepsilon$ for all $n, m \in \mathbb{N}$. With $\kappa = \kappa_0$ fixed, we then know that the first term of RHS(4.5.8) is $\leq \frac{1}{2}\varepsilon$ for all n, m sufficiently large due to the well-posedness of the $\widetilde{H}_\kappa$ flow (cf. Proposition 4.3.2). □

Finally, we use Theorem 4.5.3 to conclude our main result Theorem 4.1.2:

Corollary 4.5.4. *Given V admissible, the KdV equation with potential* (4.1.1) *with initial data $q(0) \in H^{-1}(\mathbb{R})$ is globally well-posed, in the sense that the solution map $\Phi : \mathbb{R} \times H^{-1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$ obtained in Theorem 4.5.1 is jointly continuous.*

Proof. Given $q \in H^{-1}(\mathbb{R})$, we define $\Phi(t, q(0))$ to be the limit

$$\Phi(t, q(0)) = \lim_{\kappa\to\infty} q_\kappa(t)$$

guaranteed by Theorem 4.5.1. The limit exists in $H^{-1}(\mathbb{R})$ and the convergence is uniform on bounded time intervals. Fix $T > 0$ and a sequence $q_n(0) \to q(0)$ in $H^{-1}(\mathbb{R})$. From Theorem 4.5.3 we obtain

$$\sup_{|t|\leq T} \|\Phi(t, q_n(0)) - \Phi(t, q(0))\|_{H^{-1}} \to 0 \quad \text{as } n \to \infty,$$

and so we conclude that Φ is jointly continuous. □

4.6 Uniqueness for regular initial data

We will now demonstrate that for more regular initial data the solutions constructed in Theorem 4.5.1 solve KdV and are unique.

First, we use a well-known L^2-energy argument to show that we have uniqueness for H^2 initial data.

Lemma 4.6.1. *Fix V admissible and $T > 0$. Given initial data $q(0) \in H^2$, there is at most one corresponding solution to KdV with potential* (4.1.1) *in* $(C_t H^2 \cap C_t^1 H^{-1})([-T, T] \times \mathbb{R})$.

Proof. Suppose $q(t)$ and $\tilde{q}(t)$ are both in $(C_t H^2 \cap C_t^1 H^{-1})([-T, T] \times \mathbb{R})$, solve KdV with potential, and have the same initial data $q(0) = \tilde{q}(0)$. From the differential equation (4.1.1) we see that the L^2-norm of the difference grows according to

$$\left| \frac{d}{dt} \int \tfrac{1}{2}(q - \tilde{q})^2 \, dx \right| = \left| \int (q - \tilde{q})\{-(q - \tilde{q})''' + 3(q^2 - \tilde{q}^2)' + [6V(q - \tilde{q})]'\} \, dx \right| .$$

The first term $(q-\tilde{q})'''$ contributes a total derivative and vanishes, while the remaining terms can be integrated by parts to obtain

$$\begin{aligned} &= \left| \int (q - \tilde{q})^2 \{ \tfrac{3}{2}(q + \tilde{q})' + 3V' \}(t, x) \, dx \right| \\ &\leq \left(\tfrac{3}{2} \|q'\|_{L^\infty} + \tfrac{3}{2} \|\tilde{q}'\|_{L^\infty} + 3 \|V'\|_{L^\infty} \right) \|q - \tilde{q}\|_{L^2}^2 . \end{aligned}$$

Estimating $\|q'\|_{L^\infty} \lesssim \|q\|_{H^2}$, $\|\tilde{q}'\|_{L^\infty} \lesssim \|\tilde{q}\|_{H^2}$, and recalling that $V \in W^{2,\infty}$ uniformly for $|t| \leq T$ by Definition 4.1.1, we conclude that there exists a constant C (depending on V and the norms of q and $\tilde{q}$ in $C_t H^2([-T, T] \times \mathbb{R})$) such that

$$\left| \frac{d}{dt} \|q(t) - \tilde{q}(t)\|_{L^2}^2 \right| \leq C \|q(t) - \tilde{q}(t)\|_{L^2}^2 .$$

Grönwall's inequality then yields

$$\|q(t) - \tilde{q}(t)\|_{L^2}^2 \leq e^{CT} \|q(0) - \tilde{q}(0)\|_{L^2}^2$$

for $|t| \leq T$. The RHS vanishes by premise, and so we conclude $\tilde{q}(t) = q(t)$ for all $|t| \leq T$. □

In order to employ the uniqueness of Lemma 4.6.1, we need to first show that the limits of Theorem 4.5.1 are in $C_tH^2 \cap C_t^1 H^{-1}$ and solve KdV with potential. The following proposition shows that it suffices to know that the sequence $q_\kappa(t)$ of $\widetilde{H}_\kappa$ flows converges in C_tH^2:

Proposition 4.6.2. *Fix V admissible and $T > 0$. If the sequence $q_\kappa(t)$ of solutions to the $\widetilde{H}_\kappa$ flow* (4.2.22) *converges in $C_tH^2([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$, then the limit $q(t)$ is in $(C_tH^2 \cap C_t^1 H^{-1})([-T,T] \times \mathbb{R})$ and solves KdV with potential* (4.1.1).

Proof. In the following all spacetime norms will be taken over the slab $[-T,T] \times \mathbb{R}$. We will extract the linear and quadratic terms of the $\widetilde{H}_\kappa$ flow to witness its convergence to KdV with potential. Differentiating the translation identity (4.2.4) for $g(\kappa, q_\kappa + V_\kappa) - g(\kappa, V_\kappa)$ at $h = 0$, expanding it as a series in powers of q, and then expanding each resolvent $R(\kappa, V_\kappa)$ in powers of V_κ, we write

$$
\begin{aligned}
&\frac{d}{dt} q_\kappa \\
&= -16\kappa^5 \langle \delta_x, R_0 q_\kappa' R_0 \delta_x \rangle + 4\kappa^2 q_\kappa' && (4.6.1)\\
&\quad + 16\kappa^5 \langle \delta_x, [\partial, R_0 q_\kappa R_0 q_\kappa R_0] \delta_x \rangle && (4.6.2)\\
&\quad + 16\kappa^5 \big\{ \langle \delta_x, [\partial, R_0 V_\kappa R_0 q_\kappa R_0] \delta_x \rangle + \langle \delta_x, [\partial, R_0 q_\kappa R_0 V_\kappa R_0] \delta_x \rangle \big\} && (4.6.3)\\
&\quad + 16\kappa^5 \sum (\text{terms with 3 or more } q_\kappa \text{ or } V_\kappa). && (4.6.4)
\end{aligned}
$$

We will show that the first three terms (4.6.1)–(4.6.3) converge to the three terms of KdV with potential (4.1.1) respectively, and the tail (4.6.4) converges to zero as $\kappa \to \infty$.

We begin with the linear term (4.6.1). Using the operator identity (3.2.19) we write

$$(4.6.1) = -q_\kappa''' - R_0(2\kappa)\partial^2 (q_\kappa - q)''' - R_0(2\kappa)\partial^2 q'''.$$

As $q_\kappa \to q$ in C_tH^2 by premise, then the first term of the RHS above converges to $-q'''$ in C_tH^{-1} and the second term converges to zero in C_tH^{-1} because the operators $R_0(2\kappa)\partial^2$ are bounded uniformly in κ. The last term converges to zero since the operator $R_0(2\kappa)\partial^2$

is readily seen in Fourier variables to converge strongly to zero as $\kappa \to \infty$. Altogether we conclude

$$(4.6.1) \to -q''' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$

Next, we turn to the first quadratic term (4.6.2). First we write

$$(4.6.2) = 6q_\kappa q_\kappa' + \big\{16\kappa^5 \langle \delta_x, [\partial, R_0 q_\kappa R_0 q_\kappa R_0]\delta_x\rangle - 6q_\kappa q_\kappa'\big\}.$$

As $q_\kappa \to q$ in $C_t H^2$ by premise, then the first term of the RHS above converges to $6qq'$ in $C_t H^1$ and hence in $C_t H^{-1}$ as well. For the second term we distribute the derivative $[\partial, \cdot]$, use the operator identity (3.2.20), and estimate in H^{-1} by duality. For $\phi \in H^1$, the identity (3.2.20) yields

$$\left| \int \big\{16\kappa^5 \langle \delta_x, R_0 q_\kappa R_0 q_\kappa' R_0 \delta_x\rangle - 3q_\kappa q_\kappa'\big\}\phi \, dx \right| =$$
$$\bigg| \int \big\{-3[R_0(2\kappa)q_\kappa''][R_0(2\kappa)q_\kappa''']\phi + 4\kappa^2[R_0(2\kappa)q_\kappa'][R_0(2\kappa)q_\kappa''](-5\phi + R_0(2\kappa)\phi'')$$
$$+4\kappa^2[R_0(2\kappa)q_\kappa][R_0(2\kappa)q_\kappa'](5\phi'' + 2R_0(2\kappa)\partial^2\phi'')\big\}\, dx \bigg|.$$

For those terms with ϕ'' we integrate by parts once to obtain ϕ', which can be put in L^2. Putting the highest order q_κ term in L^2, putting one term in $L^\infty \supset H^1$, and using $\|R_0(2\kappa)\partial^j\|_{\mathrm{op}} \lesssim \kappa^{j-2}$ for $j = 0, 1, 2$ (the estimate for $j = 0$ is also true as an operator on L^∞ by the explicit kernel formula for R_0 and Young's inequality), we obtain

$$\left| \int \big\{16\kappa^5 \langle \delta_x, R_0 q_\kappa R_0 q_\kappa' R_0 \delta_x\rangle - 3q_\kappa q_\kappa'\big\}\phi \, dx \right| \lesssim \kappa^{-2} \|\phi\|_{H^1} \|q_\kappa\|_{H^2}^2 .$$

Taking a supremum over $\|\phi\|_{H^1} \le 1$, and noting that the other term from the product rule is handled analogously (indeed, the identity (3.2.20) is symmetric in f and h), we conclude

$$(4.6.2) \to 6qq' \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty.$$

The second quadratic term (4.6.3) is similar, but now we must put V_κ in L^∞. First we write

$$(4.6.3) = 6(V_\kappa q_\kappa)' + \big\{16\kappa^5 \langle \delta_x, [\partial, R_0 V_\kappa R_0 q_\kappa R_0]\delta_x\rangle$$

$$+ 16\kappa^5\langle \delta_x, [\partial, R_0 q_\kappa R_0 V_\kappa R_0]\delta_x\rangle - 6(V_\kappa q_\kappa)'\big\}.$$

As $V_\kappa \to V$ in $W^{2,\infty}$, the first term of the RHS above converges to $6(Vq)'$ in C_tH^1 and hence in C_tH^{-1} as well. For the second term we distribute the two derivatives $[\partial, \cdot]$ to get four terms, use the operator identity (3.2.20), and then estimate in H^{-1} by duality. For example, for $\phi \in H^1$ we have

$$\left| \int \big\{ 16\kappa^5\langle \delta_x, R_0 V_\kappa R_0 q_\kappa' R_0 \delta_x\rangle - 3V_\kappa q_\kappa' \big\}\phi \, dx \right| =$$

$$\bigg| \int \big\{ -3[R_0(2\kappa)V_\kappa''][R_0(2\kappa)q_\kappa'''] \phi + 4\kappa^2[R_0(2\kappa)V_\kappa'][R_0(2\kappa)q_\kappa''](-5\phi + R_0(2\kappa)\phi'')$$

$$+4\kappa^2[R_0(2\kappa)V_\kappa][R_0(2\kappa)q_\kappa'](5\phi'' + 2R_0(2\kappa)\partial^2\phi'')\big\} \, dx \bigg|.$$

For those terms with ϕ'' we integrate by parts once to obtain ϕ', which can be put in L^2. Putting all V_κ terms in L^∞ and the remaining terms in L^2, we obtain

$$\left| \int \big\{ 16\kappa^5\langle \delta_x, R_0 V_\kappa R_0 q_\kappa' R_0 \delta_x\rangle - 3q_\kappa q_\kappa' \big\}\phi \, dx \right| \lesssim \kappa^{-2} \|\phi\|_{H^1} \|V_\kappa\|_{W^{2,\infty}} \|q_\kappa\|_{H^2}.$$

The other three terms obtained from the product rule are handled analogously; replacing q_κ' by q_κ and V_κ by V_κ' is harmless because we know that $V_\kappa \in W^{4,\infty}$ uniformly for $|t| \leq T$ and κ large. Taking a supremum over $\|\phi\|_{H^1} \leq 1$, we conclude

$$(4.6.3) \to 6(Vq)' \quad \text{in } C_tH^{-1} \text{ as } \kappa \to \infty.$$

Lastly, we show that the series tail (4.6.4) converges to zero in C_tH^{-1}. We estimate by duality; for $\phi \in H^1$ we write

$$\left| \int \phi \cdot (4.6.4) \, dx \right| \leq 16\kappa^5 \sum_{\substack{\ell \geq 1, \; m_0, \dots, m_\ell \geq 0 \\ \ell + m_0 + \cdots + m_\ell \geq 3}} \left| \operatorname{tr}\big\{ \phi [\partial, R_0 (V_\kappa R_0)^{m_0} q_\kappa R_0 \cdots q_\kappa R_0 (V_\kappa R_0)^{m_\ell}] \big\} \right|.$$

Recall that we first expanded $g(\kappa, q_\kappa + V_\kappa)$ in powers of q_κ, the ℓth term having ℓ-many factors of $q_\kappa R(\kappa, V_\kappa)$, and then expanded each $R(\kappa, V_\kappa)$ into a series in V_κ indexed by m_i. The condition $\ell+m_0+\cdots+m_\ell \geq 3$ reflects that we have already accounted for all of the summands with one and two q_κ or V_κ. We distribute the derivative $[\partial, \cdot]$, use the estimate (3.2.3) and the observation $\|f\|_{H^{-1}_\kappa} \lesssim \kappa^{-1} \|f\|_{L^2}$ to put ϕ and all copies of q_κ in L^2, and then estimate V_κ in operator norm to obtain

$$\lesssim \kappa^5 \sum_{\substack{\ell\geq 1,\ m_0,\ldots,m_\ell\geq 0\\ \ell+m_0+\cdots+m_\ell\geq 3}} \frac{\|\phi\|_{L^2}}{\kappa^{3/2}} \left(\frac{\|q_\kappa\|_{H^1}}{\kappa^{3/2}}\right)^{\ell} \left(\frac{\|V_\kappa\|_{W^{1,\infty}}}{\kappa^2}\right)^{m_0+\cdots+m_\ell}.$$

We first sum over the indices $m_0, \ldots, m_\ell \geq 0$ as we did in (4.4.9) using that $V_\kappa \in C_t W^{4,\infty}$ uniformly for κ large. Then we sum over $\ell \geq 1$ and use that q_κ is bounded in $C_t H^2$ for κ sufficiently large. The condition $\ell + m_0 + \cdots + m_\ell \geq 3$ guarantees that summing over the two pararenthetical terms yields a gain $\lesssim (\kappa^{-3/2})^3$, from which we obtain

$$\lesssim \kappa^{-1} \|\phi\|_{H^1}.$$

Taking a supremum over $\|\phi\|_{H^1} \leq 1$, we conclude

$$(4.6.4) \to 0 \quad \text{in } C_t H^{-1} \text{ as } \kappa \to \infty. \qquad \square$$

It only remains to show that the sequence $q_\kappa(t)$ converges in H^2 as $\kappa \to \infty$. To accomplish this task, we will use that the sequence $q_\kappa(t)$ is uniformly bounded in H^3, which follows from an elementary *a priori* estimate; this suffices by interpolating with the convergence in H^{-1}.

Proposition 4.6.3. *Given V admissible and $A, T > 0$, there exist constants C and κ_0 such that solutions $q_\kappa(t)$ to the $\widetilde{H}_\kappa$ flow* (4.2.22) *obey*

$$\|q(0)\|_{H^3} \leq A \quad \Longrightarrow \quad \|q_\kappa(t)\|_{H^3} \leq C \text{ for all } |t| \leq T \text{ and } \kappa \geq \kappa_0.$$

The proof is a repetition of the energy arguments that yield the *a priori* estimates in H^s necessary for the Bona–Smith theorem [24] applied to the $\widetilde{H}_\kappa$ flow. It is based on the fact that the H_κ flow preserves the polynomial conservation laws of KdV. For $s = 0, 1, 2$ we control the growth of the first three conserved quantities in time (which are no longer exactly conserved for the $\widetilde{H}_\kappa$ flow), and then for $s = 3$ we directly control the growth of q_κ''' in L^2. See Section 3.3 for details, where for the tidal H_κ flow (3.3.3) we obtained *a priori* estimates in H^s spaces for all integers $s \geq 0$.

It is natural to ask if for initial data in H^3 we have convergence in H^3 and not merely H^2. This is also true, but the argument is more subtle. In Chapter 3 we presented a more thorough argument for the tidal H_κ flow (3.3.3), which can be adapted to this context to directly show convergence in H^3.

Altogether, we can now conclude our main result Theorem 4.1.3:

Corollary 4.6.4. *Fix V admissible and $T > 0$. Given initial data $q(0) \in H^3$, the solution constructed in Theorem 4.5.1 is the unique solution to KdV with potential* (4.1.1) *in $(C_tH^2 \cap C_t^1H^{-1})([-T, T] \times \mathbb{R})$.*

Proof. We know from Theorem 4.5.1 that the $\widetilde{H}_\kappa$ flows $q_\kappa(t)$ converge in H^{-1} as $\kappa \to \infty$, and from Proposition 4.6.3 we know they are bounded in $C_tH^3([-T, T] \times \mathbb{R})$ uniformly for κ large. From the inequality

$$\|f\|_{H^2} \leq \|f\|_{H^{-1}}^{1/4} \|f\|_{H^3}^{3/4}$$

(which can be obtained using Hölder's inequality in Fourier variables), we deduce that q_κ converges in $C_tH^2([-T, T] \times \mathbb{R})$ as well. Proposition 4.6.2 then tells us that the limit $q(t)$ is in $(C_tH^2 \cap C_t^1H^{-1})([-T, T] \times \mathbb{R})$ and solves KdV with potential. Finally, Lemma 4.6.1 guarantees that this is the unique solution in this class. □

4.7 Example: step-like initial data

From our study of tidal KdV at high-regularity in Chapter 3, we are now able to conclude that KdV is well-posed for $H^{-1}(\mathbb{R})$ perturbations of step-like solutions:

Proof of Corollary 4.1.4. Let $V(t) = W + q(t)$ be the solution to KdV (1.1.1) corresponding to the tidal KdV solution with initial data $q(0) \equiv 0$ (and W defined in (3.1.1)). We want to show that KdV (1.1.1) is globally well-posed for initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$. By Theorem 4.1.2, it suffices to show that for every $T > 0$ the conditions (i)–(iii) of Definition 4.1.1 are satisfied.

Fix $T > 0$. As $q(0) \equiv 0$ is in H^5, the *a priori* estimate of Proposition 3.3.5 guarantees that the tidal H_κ flows $q_\kappa(t)$ are bounded in $C_tH^5([-T,T] \times \mathbb{R})$ uniformly for κ large. By definition of the tidal H_κ flow we have that $V_\kappa(t) = W + q_\kappa(t)$ solves the H_κ flow. Combined with the embedding $H^1 \hookrightarrow L^\infty$, this shows that (ii) is satisfied.

By Proposition 3.5.4, we know that the sets $Q(\kappa) := \{q_\varkappa(t) : |t| \leq T,\ \varkappa \geq \kappa\}$ obey (3.5.9). Therefore, by Proposition 3.6.1 we know that $q_\kappa \to q$ in $C_tH^5([-T,T] \times \mathbb{R})$ as $\kappa \to \infty$ uniformly for initial data in $Q(\kappa)$. Consequently $V_\kappa(t)$ converges to $V(t) = W + q(t)$ in $C_tW^{4,\infty}([-T,T] \times \mathbb{R})$, which shows that (iii) is satisfied.

Finally, by Proposition 3.6.2 we know $q(t)$ is in $C_tH^5([-T,T] \times \mathbb{R})$ and solves tidal KdV. Therefore $V(t)$ solves KdV and is in $C_tW^{4,\infty}([-T,T] \times \mathbb{R})$, which shows that (i) is satisfied. □

Lastly, we record the following reformulation of well-posedness for $H^{-1}(\mathbb{R})$ perturbations of W (defined in (3.1.1)):

Corollary 4.7.1. *Fix a sequence of initial data $u_n(0) \in W + H^3(\mathbb{R})$ with $u_n(0) - W$ convergent in $H^{-1}(\mathbb{R})$ as $n \to \infty$, and let $u_n(t)$ denote the corresponding solutions to KdV* (1.1.1) *guaranteed by Theorem 3.1.1. Then there exists a continuous function $u : \mathbb{R}_t \to W + H^{-1}(\mathbb{R})$ so that $u_n(t) - u(t) \to 0$ in $H^{-1}(\mathbb{R})$ as $n \to \infty$ uniformly on bounded time intervals.*

4.8 Example: cnoidal waves

Next, we will see that the periodic traveling wave solutions (cnoidal waves) of KdV are admissible background waves V in the sense of Definition 4.1.1. In fact, we will see that cnoidal wave profiles $V(0, x)$ are also traveling waves for the H_κ flow (3.3.1) (with a different propagation speed), which makes the analysis particularly straightforward. The H_κ flow possessing the same traveling wave profile as KdV is not surprising, since the H_κ flow preserves the polynomial conserved quantities of KdV and cnoidal waves are minimizers of the KdV energy with constrained momentum (cf. [108, §3]).

Rather than working with the Jacobian elliptic functions, it is much easier to perform calculus on the cnoidal waves (1.2.1) when expressed in terms of Weierstrass elliptic functions:

$$V(t, x) = 2\wp\left(x + 6\wp(\omega_1)t + \omega_3; \omega_1, \omega_3\right) + \wp(\omega_1). \tag{4.8.1}$$

Here, $\wp(z; \omega_1, \omega_3) =: \wp(z)$ is the Weierstrass p-function with lattice generators $2\omega_1, 2\omega_3 \in \mathbb{C}$ (see [48, §23.2] for its definition). We must choose ω_1 purely real and ω_3 purely imaginary for the wave (4.8.1) to solve KdV, and to avoid redundancy we insist that ω_1 and ω_3/i are positive. Note that the argument z of $\wp(z)$ in (4.8.1) is not on the real axis but is translated vertically by the imaginary half-period ω_3 and thus runs halfway between two rows of poles for $\wp(z)$; this guarantees that the profile (4.8.1) is regular and real-valued.

Proposition 4.8.1. *The cnoidal wave profile admits the traveling wave solution*

$$V_\kappa(t, x) = V(0, x + \nu t), \qquad \nu = \nu(\kappa)$$

to the H_κ flow (3.3.1).

Proof. Let $V(x) = V(0, x)$ denote the initial data. In order to see that $V(x + \nu t)$ solves the H_κ flow (3.3.1) we need $g'(x; \kappa, V)$ to be proportional to $V(x)$. To compute the diagonal Green's function $g(x; \kappa, V)$, we will use the representation

$$g(x) = \psi_+(x)\psi_-(x) \tag{4.8.2}$$

in terms of normalized Floquet solutions $\psi_\pm$. Recall from Floquet theory that there exist solutions $\psi_\pm(x)$ to

$$-\psi'' + V\psi = -\kappa^2\psi \tag{4.8.3}$$

who decay exponentially (along with their derivatives) as $x \to \pm\infty$ and grow exponentially as $x \to \mp\infty$. Constancy of the Wronskian guarantees that these solutions are unique up to scalar multiples. For the expression (4.8.2) to hold, we partially normalize the solutions $\psi_\pm$ by enforcing the Wronskian relation

$$\psi_+(x)\psi_-'(x) - \psi_+'(x)\psi_-(x) = 1 \tag{4.8.4}$$

and requiring that both $\psi_\pm$ are positive.

Consider the ansatz

$$\psi_\pm(x) = a_\pm \frac{\sigma(x+\omega_3 \pm b)}{\sigma(x+\omega_3)\sigma(\pm b)} e^{\mp\zeta(b)x}, \tag{4.8.5}$$

where $\sigma(z)$ and $\zeta(z)$ are the other two Weierstrass elliptic functions with the same lattice generators ω_1, ω_3 as V (see [48, §23.2(ii)] for their definition and relations), and $a_\pm$ and b are parameters to be chosen depending on κ. Substituting the ansatz (4.8.5) into the eigenvalue equation (4.8.3) and using the additive identities [48, §23.10(i)], we see that (4.8.5) solves (4.8.3) provided that $b = b(\kappa)$ satisfies

$$\kappa^2 = \wp(b) - \wp(\omega_1). \tag{4.8.6}$$

As $\wp(x)$ is real, positive, and symmetrically U-shaped for $x \in (0, 2\omega_1)$, we see that in order to have $\kappa \in (0,\infty)$ we can take $b \in (0,\omega_1)$, with $b(\kappa) \downarrow 0$ as $\kappa \to \infty$. To ensure the ansatz (4.8.5) satisfies the Wronskian relation (4.8.4) and the condition $\psi_\pm(x) > 0$, we set

$$a_\pm = \pm\left[-\wp'(b)\right]^{-\frac{1}{2}}.$$

As $\wp(b)$ is real, positive, and strictly decreasing for $b \in (0,\omega_1)$, then $-\wp'(b)$ is positive and we may take the positive square-root. Although it is incidental to the proof, we note that the Floquet exponents for $\psi_\pm$ are

$$\frac{\psi_\pm(x+2\omega_1)}{\psi_\pm(x)} = e^{\mp 2\omega_1\zeta(b)},$$

and they are multiplicative inverses of each other (as expected from Floquet theory).

Now that we have determined the Floquet solutions (4.8.5), the representation (4.8.2) determines the diagonal Green's function:

$$g(x;\kappa,V) = \frac{\wp(b(\kappa)) - \wp(x+\omega_3)}{-\wp'(b(\kappa))} = \frac{\wp(b(\kappa)) + \frac{1}{2}\wp(\omega_1)}{-\wp'(b(\kappa))} + \frac{1}{2\wp'(b(\kappa))}V(x). \tag{4.8.7}$$

We notice in particular that $g'(x;\kappa,V)$ is proportional to $V'(x)$. Recalling the translation property (4.2.4), we conclude that the solution $V_\kappa(t,x)$ to the H_κ flow (3.3.1) with initial data $V(0,x)$ is the traveling wave $V(0,x+\nu t)$. Moreover, the propagation speed is given by

$$\nu(\kappa) = \frac{8\kappa^5}{\wp'(b(\kappa))} + 4\kappa^2 \tag{4.8.8}$$

for all κ sufficiently large. □

To see the convergence of V_κ to V, we will first need to take a slightly closer look at the exact form of the coefficients.

Lemma 4.8.2. *The diagonal Green's function for the traveling waves V_κ takes the form*

$$g(x;\kappa,V_\kappa(t)) = c_1(\kappa) + c_2(\kappa)V_\kappa(t,x),$$

where the coefficients have the asymptotics

$$c_1(\kappa) = \tfrac{1}{2\kappa} + O(\kappa^{-5}), \quad c_2(\kappa) = -\tfrac{1}{4\kappa^3} + O(\kappa^{-5}) \quad \textit{as } \kappa \to \infty. \tag{4.8.9}$$

The asymptotics (4.8.9) are consistent with the convergence found in Lemma 4.2.3. In fact, for cnoidal waves, Lemma 4.2.3 follows immediately from (4.8.9) and the fundamental theorem of calculus.

Proof. From the expression (4.8.7) for the diagonal Green's function $g(\kappa,V_\kappa)$ we have

$$c_1(\kappa) = \frac{\wp(b(\kappa)) + \frac{1}{2}\wp(\omega_1)}{-\wp'(b(\kappa))}, \qquad c_2(\kappa) = \frac{1}{2\wp'(b)}, \tag{4.8.10}$$

where $b = b(\kappa)$ is defined by the relation (4.8.6). As $\wp'(b)$ is nonvanishing for $b \in (0, \omega_1)$ and the p-function possesses the Laurent expansion [48, Eq. 23.9.2]

$$\wp(z; \omega_1, \omega_2) = \tfrac{1}{z^2} + O(z^2) \quad \text{for } 0 < |z| < \min\{|\omega_1|, |\omega_3|\}, \tag{4.8.11}$$

then the inverse function theorem guarantees that $b(\kappa)$ is an analytic function at $\kappa = +\infty$. Combining the Laurent expansion (4.8.11) with the defining relation (4.8.6) for $b(\kappa)$, we can solve for the first few coefficients in the expansion for $b(\kappa)$:

$$b(\kappa) = \tfrac{1}{\kappa} + O(\kappa^{-5}). \tag{4.8.12}$$

This combined with the coefficient formulas (4.8.10) yields the asymptotics (4.8.9). □

Altogether, we conclude that cnoidal waves are admissible:

Corollary 4.8.3. *If V is a periodic traveling wave solution* (1.2.1) *of KdV, then the KdV equation* (1.1.1) *with initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ is globally well-posed.*

Proof. In order to apply Corollary 4.5.4 we must check that V satisfies the criteria of Definition 4.1.1. It only remains to show that $V_\kappa - V \to 0$ in $W^{2,\infty}$ as $\kappa \to \infty$ uniformly for initial data in $\{V_\varkappa(t) : |t| \le T,\ \varkappa \ge \kappa\}$. By the fundamental theorem of calculus it suffices to show that the wave speed $\nu(\kappa)$ converges to that of the KdV traveling waves (4.8.1). Indeed, the expression (4.8.8) for $\nu(\kappa)$ combined with the asymptotics (4.8.11) and (4.8.12) yields $\nu(\kappa) \to 6\wp(\omega_1)$ as $\kappa \to \infty$, which is the propagation speed for the KdV traveling waves (4.8.1). □

4.9 Example: smooth periodic waves

The purpose of this section is to show that any $V(0, x) \in H^5(\mathbb{T})$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the circle) is admissible in the sense of Definition 4.1.1. The proof consists of an energy argument in the spirit of Bona–Smith [24].

Our convention for the Fourier transform of functions on the circle $\mathbb{T}$ is

$$\hat{f}(\xi) = \int_0^1 e^{-i\xi x} f(x)\,dx, \quad \text{so that} \quad f(x) = \sum_{\xi\in 2\pi\mathbb{Z}} \hat{f}(\xi)e^{i\xi x}.$$

As with functions on the line, we also define the norm

$$\|f\|^2_{H^s_\kappa(\mathbb{T})} = \sum_{\xi\in 2\pi\mathbb{Z}} (\xi^2 + 4\kappa^2)^s |\hat{f}(\xi)|^2.$$

The Schrödinger operator $-\partial^2 + q$ from which we built the diagonal Green's function $g(x;\kappa,q)$ acts on $L^2(\mathbb{R})$ and not $L^2(\mathbb{T})$. Consequently, for potentials q on the circle this operator is no longer a relatively Hilbert–Schmidt (or even relatively compact) perturbation of the case $q \equiv 0$. In place of the fundamental estimate (3.2.3), we will use the following two operator estimates from [97, Lem. 6.1]:

$$\left\| \sqrt{R_0}\, q \sqrt{R_0} \right\|_{\mathrm{op}} \lesssim \kappa^{-1/2} \|q\|_{H^{-1}_\kappa(\mathbb{T})}, \tag{4.9.1}$$

$$\left\| \sqrt{R_0} f\psi R_0 q \sqrt{R_0} \right\|_{\mathfrak{I}_1} \lesssim \kappa^{-1} \|f_\kappa\|_{H^{-1}(\mathbb{T})} \|q\|_{H^{-1}_\kappa(\mathbb{T})}, \tag{4.9.2}$$

both uniformly for $\kappa \geq 1$. Here $\psi \in C^\infty_c(\mathbb{R})$ is a fixed function so that $\sum_{k\in\mathbb{Z}} \psi(x-k) \equiv 1$. This guarantees that we have the duality relation

$$\|h\|_{H^1(\mathbb{T})} = \sup\left\{ \int_{\mathbb{R}} h(x)f(x)\psi(x)\,dx : f \in C^\infty(\mathbb{T}),\ \|f\|_{H^{-1}(\mathbb{T})} \leq 1 \right\}. \tag{4.9.3}$$

Here and throughout this section, we are viewing functions on the circle $\mathbb{T}$ as functions on the line $\mathbb{R}$ by periodic extension.

First, we obtain *a priori* estimates for the H_κ flow:

Lemma 4.9.1. *Given an integer $s \geq 0$ and $A, T > 0$, there exist constants C and κ_0 such that solutions $V_\kappa(t)$ to the H_κ flow* (3.3.1) *obey*

$$\|V(0)\|_{H^s(\mathbb{T})} \leq A \quad \implies \quad \|V_\kappa(t)\|_{H^s(\mathbb{T})} \leq C \quad \textit{for all } |t| \leq T \textit{ and } \kappa \geq \kappa_0.$$

Proof. The Hamiltonian H_κ is constructed from $\alpha(\kappa,q)$ and the momentum functional E_1. Momentum is one of the polynomial conserved quantities of KdV and α can be expressed as a series (1.4.6) in terms of these quantities, and so both Poisson commute with every KdV conserved quantity. Consequently, each KdV conserved quantity is also conserved for smooth solutions $V_\kappa(t)$ of the H_κ flow (which can be individually verified using the algebraic identities (3.2.16)–(3.2.18)). Therefore the classical proof of the estimates (2.2.3) for KdV also apply to the H_κ flow; see [108, Th. 3.1] for details. □

Next, we prove existence for the H_κ flows via a contraction mapping argument:

Proposition 4.9.2. *Given $A, T > 0$, there exists a constant κ_0 so that for $\kappa \geq \kappa_0$ the H_κ flows* (3.3.1) *with initial data in the closed ball $B_A \subset H^5(\mathbb{T})$ of radius A are globally well-posed and the corresponding solutions $V_\kappa(t)$ are in $C_t H^5([-T,T] \times \mathbb{T})$.*

Proof. The solution $V_\kappa(t)$ to the H_κ flow satisfies the integral equation

$$V_\kappa(t) = e^{t4\kappa^2\partial_x} V(0) + 16\kappa^5 \int_0^t e^{(t-s)4\kappa^2\partial_x} g'(\kappa, V_\kappa(s))\, ds. \tag{4.9.4}$$

We will ultimately show that if $W, \tilde{W} \in H^4(\mathbb{T})$ then

$$\|g'(\kappa, W) - g'(\kappa, \tilde{W})\|_{H^5(\mathbb{T})} \lesssim \|g(\kappa, W) - g(\kappa, \tilde{W})\|_{H^6(\mathbb{T})} \lesssim \|W - \tilde{W}\|_{H^4(\mathbb{T})} \tag{4.9.5}$$

uniformly for $\kappa \geq 2\,\|W\|^2_{H^{-1}(\mathbb{T})}, 2\|\tilde{W}\|^2_{H^{-1}(\mathbb{T})}$. Assuming this claim, for fixed initial data $V(0) \in H^5(\mathbb{T})$ we see that $W \mapsto g'(\kappa, W)$ is Lipschitz on the closed ball $B_R \subset H^5(\mathbb{T})$ of radius $R := 2A$ for all $\kappa \geq 2R^2$. Consequently, there exists $\varepsilon > 0$ sufficiently small such that the integral operator (4.9.4) is a contraction on $C_t B_R([-\varepsilon, \varepsilon] \times \mathbb{R})$. Then, given an arbitrary $T > 0$, we use the *a priori* estimates of Lemma 4.9.1 to increase $R := R(A)$ if necessary and iterate in order to conclude that the solution exists in $C_t B_R([-T,T] \times \mathbb{R})$ for all $\kappa \geq 2R^2$.

It remains to prove the Lipschitz estimate (4.9.5), but first we must show that $g(\kappa, W) - \frac{1}{2\kappa}$ is in $H^6(\mathbb{T})$ for $W \in H^4(\mathbb{T})$. To accomplish this, we will show that $[g(\kappa, W) - \frac{1}{2\kappa}]^{(s)}$ is in H^1 for $s = 0, 1, \ldots, 5$ using the duality relation (4.9.3). For $f \in C^\infty(\mathbb{T})$ we can obtain a series for $g^{(s)}(\kappa, W)$ by differentiating the translation relation (4.2.4) at $h = 0$:

$$\left| \int [g(x;\kappa,W) - \tfrac{1}{2\kappa}]^{(s)} f(x)\psi(x)\, dx \right| \leq \sum_{\ell=1}^\infty \left| \operatorname{tr}\{ f\psi [\partial^s, R_0(WR_0)^\ell]\} \right|.$$

Using the operator estimates (4.9.1) and (4.9.2), we put all copies of W in $H^{-1}(\mathbb{T})$:

$$\left|\int [g(\kappa,W)-\tfrac{1}{2\kappa}]^{(s)} f\psi\,dx\right| \lesssim \kappa^{-1/2}\,\|f\|_{H^{-1}} \sum_{\ell=1}^{\infty} \sum_{\substack{\sigma\in\mathbb{N}^\ell \\ |\sigma|=s}} \binom{s}{\sigma} \prod_{j=1}^{\ell} \kappa^{-1/2}\|W^{(\sigma_j)}\|_{H^{-1}}.$$

Applying Hölder's inequality in Fourier variables we see that

$$\prod_{j=1}^{\ell} \|W^{(\sigma_j)}\|_{H^{-1}(\mathbb{T})} \leq \|W^{(s)}\|_{H^{-1}(\mathbb{T})}\|W\|_{H^{-1}(\mathbb{T})}^{\ell-1},$$

and so

$$\begin{aligned}\left|\int [g(\kappa,W)-\tfrac{1}{2\kappa}]^{(s)} f\psi\,dx\right| &\lesssim \kappa^{-1}\,\|f\|_{H^{-1}}\,\|W^{(s)}\|_{H^{-1}} \sum_{\ell=1}^{\infty} \ell^s \big(\kappa^{-1/2}\|W\|_{H^{-1}}\big)^{\ell-1} \\ &\lesssim \kappa^{-1}\,\|f\|_{H^{-1}}\,\|W\|_{H^{s-1}}\end{aligned}$$

provided that we have $\kappa \geq 2\|W\|_{H^{-1}}^2$. Taking a supremum over $\|f\|_{H^{-1}(\mathbb{T})} \leq 1$ yields the claim.

Lastly, we turn to the Lipschitz inequality (4.9.5). It suffices to show that the linear functional $h \mapsto dg|_W(h) - dg|_0(h)$ is bounded $H^4(\mathbb{T}) \to H^6(\mathbb{T})$ for $\kappa \geq 2\,\|W\|_{H^{-1}(\mathbb{T})}^2$ by the fundamental theorem of calculus. To demonstrate this, we estimate its sth derivative in $H^1(\mathbb{T})$ for $s = 0, \ldots, 5$ using the duality relation (4.9.3) and the previous argument. Expanding the resolvents within the functional derivative expression (4.2.1) into series, we have

$$\left|\int [dg|_W(h) - dg|_0(h)]^{(s)}(x) f(x)\psi(x)\,dx\right| \lesssim \kappa^{-3/2}\,\|f\|_{H^{-1}}\,\|W\|_{H^{s-1}}\|h\|_{H^{s-1}}$$

for $\kappa \geq 2\|W\|_{H^{-1}}^2$. Taking a supremum over $\|f\|_{H^{-1}(\mathbb{T})} \leq 1$ yields the claim. □

We now know that the H_κ flows $V_\kappa(t)$ satisfy the second condition in the definition of admissibility, provided that we take initial data $V(0) \in H^5(\mathbb{T})$. The first condition—that the corresponding solution $V(t)$ of KdV is sufficiently regular—then follows from the well-posedness of KdV in $H^3(\mathbb{T})$. Alternatively, we could reprove this classical well-posedness result by constructing $V(t)$ as the limit of the H_κ flows V_κ, but we will not pursue this here.

Our next objective is to verify the third condition in the definition of admissibility, which says that V_κ converges to V in $W^{2,\infty}(\mathbb{R})$ as $\kappa \to \infty$. To begin, we control the growth of the difference $V_\kappa - V$ in $L^2(\mathbb{T})$:

Proposition 4.9.3. *Given $A, T > 0$, there exists a constant C so that the quantity*

$$P(t) := \tfrac{1}{2} \int_{\mathbb{T}} [V_\kappa(t,x) - V(t,x)]^2 \, dx \quad \text{with} \quad V_\kappa(0,x) = V(0,x) \in H^5(\mathbb{T})$$

obeys

$$\left| \frac{d}{dt} P(t) \right| \leq C\big(P + o(1)\sqrt{P}\big) \quad \text{as } \kappa \to \infty$$

uniformly for $|t| \leq T$ and $\|V(0)\|_{H^5(\mathbb{T})} \leq A$.

Proof. Let $u := V_\kappa - V$ so that $P(t) = \frac{1}{2} \|u\|^2_{L^2(\mathbb{T})}$. Then u obeys the differential equation

$$\begin{aligned} \frac{d}{dt} u &= 16\kappa^5 g'(\kappa, V_\kappa) + 4\kappa^2 V_\kappa' + V''' - 6VV' \\ &= 16\kappa^5 g'(\kappa, V_\kappa) + 4\kappa^2 V_\kappa' + V''' - 6V_\kappa V_\kappa' + 6(V_\kappa u)' - 6uu'. \end{aligned}$$

Multiplying by $u \in C^\infty(\mathbb{T})$ and integrating over $\mathbb{T}$, we obtain an equality for the time derivative of $P(t)$. The contribution from $6uu'$ is a total derivative and hence vanishes. Expanding $g'(\kappa, V_\kappa)$ in a series and extracting the linear and quadratic terms, we write

$$\frac{d}{dt} P(t)$$

$$= 6 \int u(x)(V_\kappa u)'(x) \, dx \tag{4.9.6}$$

$$+ \int u(x)\{-16\kappa^5 \langle \delta_x, R_0 V_\kappa' R_0 \delta_x \rangle + 4\kappa^2 V_\kappa'(x) + V'''(x)\} \, dx \tag{4.9.7}$$

$$+ \int u(x)\{16\kappa^5 \langle \delta_x, [\partial, R_0 V_\kappa R_0 V_\kappa R_0] \delta_x \rangle - 3(V_\kappa^2)'(x)\} \, dx \tag{4.9.8}$$

$$+ \int u(x) \, 16\kappa^5 \sum_{\ell=3}^{\infty} (-1)^\ell \langle \delta_x, [\partial, R_0 (V_\kappa R_0)^\ell] \delta_x \rangle \, dx. \tag{4.9.9}$$

We will estimate the four terms (4.9.6)–(4.9.9) individually.

For the first term (4.9.6), we integrate by parts to move the derivative onto V_κ:

$$|(4.9.6)| = \left| \int 3V_\kappa'(x) u^2(x) \, dx \right| \leq 3 \|V_\kappa'\|_{L^\infty} P.$$

We know $\|V_\kappa'\|_{L^\infty}$ is bounded uniformly for $|t| \leq T$ and κ large by the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and the *a priori* estimates of Lemma 4.9.1.

Next we estimate the linear term (4.9.7). Using the first operator identity of (3.2.19), we have

$$(4.9.7) = \int u\big\{\big[-16\kappa^4 R_0(2\kappa) + 4\kappa^2 + \partial^2\big]V'\big\}\,dx + \int u\big\{\big[-16\kappa^4 R_0(2\kappa) + 4\kappa^2\big]u'\big\}\,dx.$$

As differentiation commutes with the resolvent $R_0(2\kappa)$, the last integrand is a total derivative and the integral vanishes. For the remaining term, we use the rest of the identity (3.2.19) and Cauchy–Schwarz to estimate

$$|(4.9.7)| \leq \|R_0(2\kappa)V^{(5)}\|_{L^2} P^{1/2} \leq \kappa^{-2}\|V^{(5)}\|_{L^2} P^{1/2}.$$

The factor $\|V^{(5)}\|_{L^2}$ is bounded uniformly for $|t| \leq T$ and κ large by the *a priori* estimates of Lemma 4.9.1.

Now we examine to the quadratic contribution (4.9.8). Consider the term when the derivative $[\partial, \cdot]$ hits the second factor of V_κ, and expand in Fourier variables:

$$\int_{\mathbb{T}} u(x)\langle \delta_x, R_0 V_\kappa R_0 V_\kappa' R_0 \delta_x\rangle\,dx = \sum_{\xi_1,\xi_2,\xi_3 \in 2\pi\mathbb{Z}} \frac{\hat{u}(\xi_1 - \xi_3)\widehat{V_\kappa}(\xi_3 - \xi_2)\widehat{V_\kappa'}(\xi_2 - \xi_1)}{(\xi_3^2 + \kappa^2)(\xi_2^2 + \kappa^2)(\xi_1^2 + \kappa^2)}.$$

Re-indexing $\eta_1 = \xi_2 - \xi_1$, $\eta_2 = \xi_3 - \xi_2$, $\eta_3 = \xi_3$, the RHS becomes

$$\sum_{\eta_1,\eta_2,\eta_3 \in 2\pi\mathbb{Z}} \frac{\hat{u}(-\eta_1 - \eta_2)\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)}{(\eta_3^2 + \kappa^2)((\eta_3 - \eta_2)^2 + \kappa^2)((\eta_3 - \eta_1 - \eta_2)^2 + \kappa^2)}.$$

The numerator is now independent of η_3, and so if we approximate the sum over $\eta_3 \in 2\pi\mathbb{Z}$ by an integral over $\eta_3 \in \mathbb{R}$ then we can evaluate the integral using residue calculus and eliminate η_3:

$$\begin{aligned}&\sum_{\eta_1,\eta_2 \in 2\pi\mathbb{Z}} \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{u}(-\eta_1 - \eta_2)\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)}{(\eta_3^2 + \kappa^2)((\eta_3 - \eta_2)^2 + \kappa^2)((\eta_3 - \eta_1 - \eta_2)^2 + \kappa^2)}\,d\eta_3 \\ &= \kappa^{-1}\sum_{\eta_1,\eta_2 \in 2\pi\mathbb{Z}} \frac{\hat{u}(-\eta_1 - \eta_2)\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)(12\kappa^2 + \eta_1^2 + \eta_1\eta_2 + \eta_2^2)}{(\eta_1^2 + 4\kappa^2)(\eta_2^2 + 4\kappa^2)((\eta_1 + \eta_2)^2 + 4\kappa^2)}.\end{aligned}$$

Note that this last summand is symmetric in η_1 and η_2, and so both terms of $[\partial, R_0 V_\kappa R_0 V_\kappa R_0]$ produce the same contribution.

We are now prepared to estimate the term (4.9.8). Changing to Fourier variables and replacing the sum over $\eta_3 \in 2\pi\mathbb{Z}$ with an integral over $\eta_3 \in \mathbb{R}$, we write

$$\begin{aligned}
&(4.9.8)\\
&= \sum_{\eta_1,\eta_2} \hat{u}(-\eta_1-\eta_2)\widehat{V_\kappa}(\eta_1)\widehat{V_\kappa'}(\eta_2)\left[\tfrac{32\kappa^4(12\kappa^2+\eta_1^2+\eta_1\eta_2+\eta_2^2)}{(\eta_1^2+4\kappa^2)(\eta_2^2+4\kappa^2)((\eta_1+\eta_2)^2+4\kappa^2)} - 6\right] \qquad (4.9.10)\\
&\quad + \sum_{\eta_1,\eta_2}\left[\sum_{\eta_3} F(\eta_1,\eta_2,\eta_3) - \frac{1}{2\pi}\int_{\mathbb{R}} F(\eta_1,\eta_2,\eta_3)\,d\eta_3\right]. \qquad (4.9.11)
\end{aligned}$$

Here, all summations are over $2\pi\mathbb{Z}$ and the integrand is given by

$$F(\eta_1,\eta_2,\eta_3) := \frac{16\kappa^5\hat{u}(-\eta_1-\eta_2)\big[\widehat{V_\kappa'}(\eta_2)\widehat{V_\kappa}(\eta_1)+\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)\big]}{(\eta_3^2+\kappa^2)((\eta_3-\eta_2)^2+\kappa^2)((\eta_3-\eta_1-\eta_2)^2+\kappa^2)}.$$

The upshot of our manipulation is that in (4.9.10) the $O(1)$ term as $\kappa\to\infty$ cancels out, and we are left with

$$\left|\frac{32\kappa^4(12\kappa^2+\eta_1^2+\eta_1\eta_2+\eta_2^2)}{(\eta_1^2+4\kappa^2)(\eta_2^2+4\kappa^2)((\eta_1+\eta_2)^2+4\kappa^2)} - 6\right| \lesssim \frac{\eta_1^2+\eta_2^2}{\kappa^2}.$$

Absorbing η_1^2 and η_2^2 as derivatives on V_κ, we put u and the copy of V_κ with the most derivatives in ℓ^2 and estimate

$$\begin{aligned}
|(4.9.10)| &\lesssim \kappa^{-2}\sup_{i,j\in\{0,2\}}\sum_{\eta_1,\eta_2}\left|\hat{u}(-\eta_1-\eta_2)\widehat{V_\kappa^{(i)}}(\eta_1)\widehat{V_\kappa^{(j+1)}}(\eta_2)\right|\\
&\le \kappa^{-2}\sup_{i,j\in\{0,2\}}\|u\|_{L^2(\mathbb{T})}\,\|V_\kappa^{(j+1)}\|_{L^2(\mathbb{T})}\sum_{\eta_2}|\widehat{V_\kappa^{(i)}}(\eta_2)| \lesssim \kappa^{-2}\,\|V_\kappa\|_{H^3(\mathbb{T})}^2\,P^{1/2}.
\end{aligned}$$

In the last inequality, we used Cauchy–Schwarz to estimate

$$\sum_{\eta_2}|\widehat{V_\kappa}(\eta_2)| \le \Big(\sum_{\eta_2}(1+\eta_2^2)^{-1}\Big)^{\frac12}\Big(\sum_{\eta_2}(1+\eta_2^2)|\widehat{V_\kappa}(\eta_2)|^2\Big)^{\frac12} \lesssim \|V_\kappa\|_{H^1(\mathbb{T})}.$$

Next, we must check that the remainder (4.9.11) yields an acceptable contribution, which is due to the smoothness of the integrand F. First, we will bound the trapezoid rule error term

$$E_n(h) := \tfrac{1}{2}h[F(\eta_1,\eta_2,a_n)+F(\eta_1,\eta_2,a_n+h)] - \int_{a_n}^{a_n+h} F(\eta_1,\eta_2,\eta_3)\,d\eta_3$$

for arbitrary $a_n \in \mathbb{R}$, $n \in \mathbb{Z}$. It is easily checked that $E_n(0) = E_n'(0) = 0$, and so

$$\begin{aligned}|E_n(h)| = \left|\int_0^h \int_0^t E_n''(s)\,ds\,dt\right| &= \left|\int_0^h \int_0^t \tfrac{1}{2}s\,\partial_{\eta_3}^2 F(\eta_1,\eta_2,a_n+s)\,ds\,dt\right| \\ &\le \tfrac{1}{12}h^3 \left\|\partial_{\eta_3}^2 F\right\|_{L^\infty_{\eta_3}([a_n,a_n+h])}.\end{aligned}$$

Therefore, setting $a_n = 2\pi n$ and $h = 2\pi$ we have

$$\begin{aligned}&\left|\pi[F(\eta_1,\eta_2,2\pi n)+F(\eta_1,\eta_2,2\pi(n+1))] - \int_{2\pi n}^{2\pi(n+1)} F(\eta_1,\eta_2,\eta_3)\,d\eta_3\right| \\ &\le \tfrac{1}{12}(2\pi)^3 \left\|\partial_{\eta_3}^2 F\right\|_{L^\infty_{\eta_3}([2\pi n,2\pi(n+1)])}\end{aligned}$$

for all $n \in \mathbb{Z}$. Each η_3 derivative applied to F introduces one order of decay in $|\eta_3|$. The term $(\eta_3^2+\kappa^2)^{-1}$ is bounded by the summable sequence $(n^2+1)^{-1}$, and every other order of decay in $|\eta_3|$ yields a factor of κ^{-1}. Altogether we estimate

$$\left\|\partial_{\eta_3}^2 F\right\|_{L^\infty_{\eta_3}([2\pi n,2\pi(n+1)])} \lesssim \frac{\left|\hat{u}(-\eta_1-\eta_2)\left[\widehat{V_\kappa'}(\eta_2)\widehat{V_\kappa}(\eta_1)+\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)\right]\right|}{\kappa(n^2+1)}.$$

Summing over $n \in \mathbb{Z}$, the trapezoid rule error estimate yields

$$|(4.9.11)| \lesssim \kappa^{-1} \sum_{\eta_1,\eta_2 \in 2\pi\mathbb{Z}} \left|\hat{u}(-\eta_1-\eta_2)\left[\widehat{V_\kappa'}(\eta_2)\widehat{V_\kappa}(\eta_1)+\widehat{V_\kappa}(\eta_2)\widehat{V_\kappa'}(\eta_1)\right]\right| \lesssim \kappa^{-1}\left\|V_\kappa\right\|_{H^1}^2 P^{1/2}.$$

This is as acceptable contribution, and thus concludes our estimation of the quadratic term (4.9.8).

Lastly we estimate the contribution (4.9.9) from the tail of the series. Using the operator estimates (4.9.1) and (4.9.2), we have

$$\begin{aligned}|(4.9.9)| &\le \kappa^5 \sum_{\ell=3}^\infty \left|\operatorname{tr}\{u[\partial, R_0(V_\kappa R_0)^\ell]\}\right| \\ &\lesssim \kappa^4 \|u\|_{H_\kappa^{-1}(\mathbb{T})} \|V_\kappa'\|_{H_\kappa^{-1}(\mathbb{T})} \sum_{\ell=3}^\infty \left(\kappa^{-1/2}\|V_\kappa\|_{H_\kappa^{-1}(\mathbb{T})}\right)^{\ell-1} \lesssim \kappa^{-1}\|V_\kappa\|_{L^2(\mathbb{T})}^2 \|V_\kappa'\|_{L^2(\mathbb{T})} P^{1/2}.\end{aligned}$$

This concludes the estimate of $\frac{d}{dt}P(t)$ and hence the proof of Proposition 4.9.3. □

We are now prepared to prove Corollary 4.1.5:

Corollary 4.9.4. *Given a background wave $V(0) \in H^5(\mathbb{T})$, the KdV equation* (1.1.1) *with initial data $u(0) \in V(0) + H^{-1}(\mathbb{R})$ is globally well-posed.*

Proof. In view of Corollary 4.5.4 it suffices to check that V satisfies the three conditions of Definition 4.1.1. Conditions (i) and (ii) are satisfied by the embedding $H^1 \hookrightarrow L^\infty$ and the *a priori* estimates of Lemma 4.9.1, and so it only remains to verify condition (iii).

Fix $T > 0$. By the embedding $H^1 \hookrightarrow L^\infty$, it suffices to show that $V_\kappa - V$ converges to zero in $C_t H^3([-T,T] \times \mathbb{T})$ uniformly for initial data in the fixed set $\{V_\varkappa(t) : |t| \leq T,\ \varkappa \geq \kappa_0\}$. By Proposition 4.9.2, we may pick the constant κ_0 so that the $H_\varkappa$ flows $\{V_\varkappa(t) : |t| \leq T,\ \varkappa \geq \kappa_0\}$ are contained in a ball $B_A \subset H^5(\mathbb{T})$ of radius $A > 0$, and so that the H_κ flows are well-posed on B_A for $\kappa \geq \kappa_0$. From Proposition 4.9.3 and the observation $o(1)\sqrt{P} \leq P + o(1)$, we have

$$\left| \frac{d}{dt} P(t) \right| \leq C(P + o(1)) \quad \text{as } \kappa \to \infty$$

uniformly for $|t| \leq T$ and initial data in B_A. Grönwall's inequality then yields

$$\tfrac{1}{2} \left\| V_\kappa - V \right\|_{L^2(\mathbb{T})}^2 = P(t) \leq e^{CT} P(0) + o(1)(e^{CT} - 1)$$

uniformly for $|t| \leq T$ and initial data in B_A. As $P(0) = 0$ by definition, we conclude

$$\left\| V_\kappa - V \right\|_{C_t L^2([-T,T]\times\mathbb{T})} \to 0 \quad \text{as } \kappa \to \infty$$

uniformly for initial data in B_A.

Using Hölder's inequality in Fourier variables, we have

$$\|f\|_{H^3(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}^{2/5} \|f\|_{H^5(\mathbb{T})}^{3/5}.$$

By the *a priori* estimates of Lemma 4.9.1, V_κ is bounded in $C_t H^5([-T,T] \times \mathbb{T})$ uniformly for κ large and initial data in B_A. Therefore, applying the above inequality to $V_\kappa - V$, we conclude that $V_\kappa - V \to 0$ in $C_t H^3([-T,T] \times \mathbb{T})$ uniformly for initial data in the smaller set $\{V_\varkappa(t) : |t| \leq T,\ \varkappa \geq \kappa_0\} \subset B_A$. □

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