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10
The Work of Stephen Smale in Differential Topology

MORRIS W. HIRSCH

Background

The theme of this conference is "Unity and Diversity in Mathematics." The diversity is evident in the many topics covered. Reviewing Smale's work in differential topology will reveal important themes that pervade much of his work in other topics, and thus exhibit an unexpected unity in seemingly diverse subjects.

Before discussing his work, it is interesting to review the status of differential topology in the middle 1950s, when Smale began his graduate study.

The full history of topology has yet to be written (see, however, Pont [52], Dieudonné [8]). Whereas differentiable manifolds can be traced back to the smooth curves and surfaces studied in ancient Greece, the modern theory of both manifolds and algebraic topology begins with Betti's 1871 paper [4]. Betti defines "spaces" as subsets of Euclidean spaces define by equalities and inequalities on smooth functions.[1] Important improvements in Betti's treatment were made by Poincaré in 1895. His definition of manifold describes what we call a real analytic submanifold of Euclidean space; but it is clear from his examples, such as manifolds obtained by identifying faces of polyhedra, that he had in mind abstract manifolds.[2] Curiously, Poincaré's "homéomorphisme" means a C^1 diffeomorphism. Abstract smooth manifolds in the modern sense—described in terms of coordinate systems—were defined (for the two-dimensional case) by Weyl [73] in his 1913 book on Riemann surfaces.

Despite these well-known works, at mid-century there were few studies of the global geometrical structure of smooth manifolds. The subject had

[1] This is the paper defining, rather imprecisely, what are now called Betti numbers. Pont [52] points out that the same definition is given in unpublished notes of Riemann, who had visited Betti.

[2] It is not obvious that such manifolds imbed in Euclidean space!

not yet been named.[3] Most topologists were not at all interested in smooth maps. The "topology of manifolds" was a central topic, and the name of an important book by Ray Wilder, but it dealt only with algebraic and point-set topology. Steenrod's important book *The Topology of Fibre Bundles* was published in 1956. The de Rham theorems were of more interest to differential geometers than to topologists, and Morse theory was considered part of analysis.

A great deal was known about algebraic topology. Many useful tools had been invented for studying homotopy invariants of CW complexes and their mappings (Eilenberg–MacLane spaces, Serre's spectral sequences, Postnikov invariants, Steenrod's algebras of cohomology operations, etc.) Moreover, there was considerable knowledge of nonsmooth manifolds—or more accurately, manifolds that were not assumed to be smooth, such as combinatorial manifolds, homology manifolds, and so forth. Important results include Moise's theory of triangulations of 3-manifolds, Reidemeister's torsion classification of lens spaces, Bing's work on wild and tame embeddings and decompositions, and "pathology" such as the Alexander horned sphere and Antoine's Necklace (a Cantor set in $\mathbf{R}^3$ whose complement is not simply connected). The deeper significance of many of these theories emerged later, in the light of the h-cobordism theorem and its implications.

A great deal was known about 3-dimensional manifolds, beginning with Poincaré's examples and Heegard's decomposition theory of 1898. The latter is especially important for understanding Smale's work, because it is the origin of the theory of handlebody decompositions.

The deepest results known about manifolds were the duality theorems of Poincaré, Alexander, and Lefschetz; H. Hopf's theorem that the indices of singularities of a vector field on a manifold add up to the Euler characteristic; de Rham's isomorphism between singular real cohomology and the cohomology of exterior differential forms; Chern's generalized Gauss–Bonnet formula; the foliation theories of Reeb and Haefliger; theories of fiber bundles and characteristic classes due to Pontryagin, Stiefel, Whitney, and Chern, with further developments by Steenrod, Weil, Spanier, Hirzebruch, Wu, Thom, and others; Rohlin's index theorem for 4-dimensional manifolds; Henry Whitehead's little-known theory of simple homotopy types; Wilder's work on generalized manifolds; P.A. Smith's theory of fixed points of cyclic group actions. Most relevant to Smale's work was M. Morse's calculus of variations in the large, Thom's theory of cobordism and transversality, and Whitney's studies of immersions, embeddings, and other kinds of smooth maps.

[3] The term "differential topology" seems to have been coined by John Milnor in the late 1950s, but did not become current for some years. The word "diffeomorphism" did not yet exist—it may be due to W. Ambrose. While Smale was at the Institute for Advanced Study, he showed me a letter from an editor objecting to "diffeomorphism," claiming that "differomorphism" was etymologically better!

No one yet knew of any examples of homeomorphic manifolds that were not diffeomorphic, or of topological manifolds not admitting a differentiable structure—Milnor's invention of an exotic 7-sphere was published in 1956. Work on the classification of manifolds, and many other problems, was stuck in dimension 2 by Poincaré's conjecture in dimension 3 (still unsolved).

The transversality methods developed by Pontryagin and Thom were not widely known. The use of manifolds and dynamical systems in mechanics, electrical circuit theory, economics, biology, and other applications is now common[4]; but in the fifties it was quite rare.

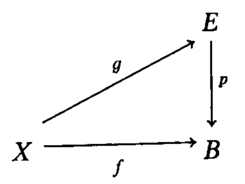
Conversely, few topologists had any interest in applications. The spirit of Bourbaki dominated pure mathematics. Applications were rarely taught or even mentioned; computation was despised; classification of structure was the be-all and end-all. Hardly anyone, pure or applied, used computers (of which there were very few). The term "fractals" had not yet been coined by Mandelbrot; "chaos" was a biblical rather than a mathematical term.

In this milieu, Smale began his graduate studies at Michigan in 1952.[5] The great man in topology at Michigan being Ray Wilder, most topology students chose to work with him. Smale, however, for some reason became the first doctoral student of a young topologist named Raoul Bott. In view of Smale's later work in applications, it is interesting that Bott had a degree in electrical engineering; and the "Bott–Duffin Theorem" in circuit theory is still important.

Immersions

Smale's work in differential topology was preceded by two short papers on the topology of maps [56, 57]. His theorems are still interesting, but not closely related to his later work. Nevertheless, the theme of much subsequent work by Smale, in many fields, is found in these papers: *fibrations*, and more generally, *the topology of spaces of paths*.

Given a (continuous) map $p: E \to B$, a *lift* of a map $f: X \to B$ is a map $g: X \to E$ such that the following diagram commutes:



That is, $p \circ g = f$. We say (p, E, B) is a *fibration* if every path $f{:}\to B$ can be

[4] Thanks largely to Smale's pioneering efforts in these fields.

[5] Smale's autobiographical memoir in this volume recounts some of his experiences in Michigan.

lifted, the lift depending continuously on specified initial values in E.[6] Fibrations, the subject of intense research in the fifties, are the maps for which the tools of algebraic topology are best suited.

In his doctoral thesis [65], Smale introduced the use of fibrations of spaces of differentiable maps as a tool for classifying immersions. This novel technique proved to be of great importance in many fields of geometric topology, as will be discussed below.

Smale's first work in differential topology was about immersions. An *immersion* $f\colon M \to N$ is a smooth map between manifolds M, N such that at every $x \in M$, the tangent map $T_x f\colon T_x M \to T_{f(x)} N$ is injective. Here TM denotes the tangent vector bundle of M, with fiber Tf_x over $x \in M$. A *regular homotopy* is a homotopy f_t, $0 \leq t \leq 1$ of immersions such that Tf_t is a homotopy of bundle maps.[7] An immersion is an *embedding* if it is a homeomorphism onto its image, which is necessarily a locally closed smooth submanifold. A regular homotopy of embeddings is an *isotopy.*

Here is virtually everything known about immersions in the early fifties: In a *tour de force* of differential and algebraic topology and geometric intuition in 1944, Hassler Whitney [77, 78] had proved that every (smooth) n-dimensional manifold could be embedded in $\mathbf{R}^{2n}$ for $n \geq 1$, and immersed in $\mathbf{R}^{2n-1}$ for $n \geq 2$. On the other hand, it was known that the projective plane and other nonorientable surfaces could not be embedded in $\mathbf{R}^3$, and Whitney had proved other impossibility results using characteristic classes. Steenrod had a typewritten proof that the Klein bottle does not embed in real projective 3-space. The Whitney–Graustein theorem [76] showed that immersions of the circle in the plane are classified by their winding numbers. As a student working with Ed Spanier I proved the complex projective plane, which could be embedded in $\mathbf{R}^7$, could not be immersed in $\mathbf{R}^6$.[8]

Immersions of Circles

The problem Smale solved in his thesis is that of classifying regular homotopy classes of immersions of the circle into an arbitrary manifold N. More generally, he classified immersions $f\colon I \to N$ of the closed unit interval $I =$

[6] Precisely: Given a compact polyhedron P and maps $F\colon P \times I \to B$, $g\colon P \times 0 \to E$ such that $p \circ g = F|P \times 0 \to E$, there is an extension of g to a map $G\colon P \times I \to E$ such that $p \circ G = F$.

[7] What is important and subtle here is joint continuity in (t, x) of $\partial f_t(x)/\partial x$. Without it, "regular homotopy" would be the same as "homotopy of immersions." In the plane, for example, the identity immersion of the unit circle is not regularly homotopic to its reflection in a line, but these two immersions *are* homotopic through immersions, as can be seen by deforming a figure-eight immersion into each of them.

[8] The proof consisted of computing the secondary obstruction to a normal vector field on an embedding in $\mathbf{R}^7$, using a formula of S.D. Liao [38], another student of Spanier. This calculation was immediately made trivial by a general result of W.S. Massey [40].

$[0, 1]$ having fixed boundary data, i.e., fixed initial and terminal tangent vectors $f'(0)$ and $f'(1)$.

Smale's approach was to study the map $p: E \to B$, where

- E is the space of immersions[9] $f: I \to N$ having fixed initial value $f(0)$ and fixed initial tangent $f'(0)$;
- B is the space of nonzero tangent vectors to N;
- p assigns to f the terminal tangent vector $f'(1)$.

The classification problem is equivalent to enumerating the path components of the fibers because it can be seen that such a path component is a regular homotopy class for fixed boundary data.

Bott asked Smale an extraordinarily fruitful question: *Is (p, E, B) a fibration?* This amounts to asking for a *Regular Homotopy Extension Theorem.* In his thesis [65], Smale proved the following:

Theorem. *Let $\{u_t, 0 \le t \le 1\}$ be a deformation of $f'(1)$ in B, i.e., a path of nonzero tangent vectors beginning with $f'(1)$. Then there is a regular homotopy $F: S^1 \times I \to N$, $F(x, t) = f_t(x)$ such that $f_0 = f$, all f_t have the same initial tangent, and the terminal tangent of f_t is u_t. Moreover, F can be chosen to depend continuously on the data f and the deformation $\{u_t\}$.*

This result is nontrivial, as can be seen by observing that it is false if N is 1-dimensional (exercise!).

It is not hard to see that the total space E is contractible. Therefore the homotopy theory of fibrations implies that the kth homotopy group of any fiber F is naturally isomorphic to the $(k + 1)$st homotopy group of the base space B. Now B has a deformation retraction onto the space $T_1 N$ of unit tangent vectors. By unwinding the homotopies involved, Smale proved the following result theorem for Riemannian manifolds N of dimension $B \ge 2$:

Theorem. *Assume N is a manifold of dimension $n \ge 2$. Fix a base point x_0 in the circle, and a nonzero "base vector" v_0 of length 1 tangent to N. Let F denote the space of immersions $f: S^1 \to N$ having tangent v_0 at x_0. To f assign the loop $f_\#: S^1 \to T_1 N$, where $f_\#$ sends $x \in S^1$ to the normalized tangent vector to f at x, namely $f'(x)/\|f'(x)\|$. Then $f_\#$ induces a bijection between the set of path components of F and the fundamental group $\pi_1(T_1 N, v_0)$.*

For the special case where N is the plane, this result specializes to the Whitney–Graustein theorem [76] stated above.

[9] A space of immersions is given the C^1 topology. This means that two immersions are close if at each point their values are close and their tangents are close. It turns out that the homotopy type of a space of immersions is the same for all C^r topologies, $1 \le r \le \infty$.

Immersions of Spheres in Euclidean Spaces

Smale soon generalized the classification to immersions of the k-sphere S^k in Euclidean n-space $\mathbf{R}^n$. Again the key was a fibration theorem. Let E now denote the space of immersions of the closed unit k-disk D^k into $\mathbf{R}^n$, and B the space of immersions of S^{k-1} into $\mathbf{R}^n$. The map $p: E \to B$ assigns to an immersion $f: D^k \to \mathbf{R}^n$ its restriction to the boundary. Smale proved that (p, E, B) is a fibration *provided* $k < n$. Geometrically, this says regular homotopies of $f|S^{k-1}$ can be extended over D^k to get a regular homotopy of f, and similarly for k-parameter families of immersions.

Using this and similar fibration theorems, Smale obtained the following result [58, 59]:

Theorem. *The set of regular homotopy classes of immersions $S^k \to \mathbf{R}^n$ corresponds bijectively to $\pi_k(V_{n,k})$, the kth homotopy group of the Stiefel manifold of k-frames in $\mathbf{R}^n$, provided $k < n$.*

To an immersion $f: S^k \to \mathbf{R}^n$ Smale assigned the homotopy class of a map $\theta: S^k \to V_{n,k}$ as follows. By a regular homotopy, we can assume f coincides with the standard inclusion $S^k \to \mathbf{R}^n$ on a small open k-disk in S^k, whose complement is a closed k-disk B. Let $e(x)$ denote a field of k-frames tangent to B. Form a k-sphere Σ by gluing two copies B_0 and B_1 of B along the boundary. Define a map $\sigma(f): \Sigma \to V_{n,k}$ by mapping x to $f_* e(x)$ if $x \in B_0$, and to $e(x)$ if $x \in B_1$. Here f_* denotes the map of frames induced by Tf. The homotopy class of $\sigma(f)$ is called the *Smale invariant* of the immersion f.

The calculation of homotopy groups is a standard task for algebraic topology. While it is by no means trivial, in any particular case a lot can usually be calculated. The Stiefel manifold $V_{n,k}$ has the homotopy type of the homogeneous space $O(n)/O(n-k)$, where $O(m)$ denotes the Lie group of real orthogonal $m \times m$ matrices. Therefore explicit classifications of immersions were possible for particular values of k and n, thanks to Smale's theorem.[10]

A surprising application of Smale's classification is his theorem that *all immersions of the* 2*-sphere in* 3*-sphere are regularly homotopic*, the reason being that $\pi_2(O(3)) = 0$.[11] In particular *the identity map of S^2, considered as an immersion into $\mathbf{R}^3$, is regularly homotopic to the antipodal map.* The analogous statement is false for immersions of the circle in the plane.

When Smale submitted his paper on immersions of spheres for publication, one reviewer claimed it could not be correct, since the identity and

[10] Even where the homotopy group $\pi_k(V_{n,k})$ has been calculated, there still remains the largely unsolved geometric problem of finding an explicit immersion $f: S^k \to \mathbf{R}^n$ representing a given homotopy class. Some results for $k = 3$, $n = 4$ were obtained by J. Hass and J. Hughes [18].

[11] Always remember: π_2 of *any* Lie group is 0.

antipodal maps of S^2 have Gauss maps of different degrees![12] Exercise: Find the reviewer's mistake!

It is not easy to visualize such a regular homotopy, now called an *eversion* of the 2-sphere. After Smale announced his result, verbal descriptions of the eversion were made by Arnold Shapiro (whom I could not understand), and later by Bernard Morin (whom I could).[13]

One way to construct an eversion is to first regularly homotop the identity map of the sphere into the composition of the double covering of the projective plane followed by *Boy's surface*, an immersion of the projective plane into 3-space pictured in *Geometry and the Imagination* [24]. Since this identifies antipodal points, the antipodal can also be regularly homotoped to this same composition.

Tony Phillips' *Scientific American* article [49] presents pictures of an eversion. Charles Pugh made prizewinning wire models of the eversion through Boy's surface, unfortunately stolen from Evans Hall on the Berkeley campus. There is also an interesting film by Nelson Max giving many visualizations of eversions. Even with such visual aids, it is a challenging task to understand the deformation of the identity map of S^2 to the antipodal map through immersions.

Smale's proof of the Regular Homotopy Extension Theorem (for spheres and disks of all dimensions) is based on integration of certain vector fields, foreshadowing his later work in dynamics.

There is no problem in extending a regular homotopy of the boundary restriction of f to a smooth homotopy of f; the difficulty is to make the extension a regular homotopy. Smale proceeded as follows.

Since D^k is contractible, the normal bundle to an immersion $f: D^k \to \mathbf{R}^n$ is trivial. Therefore, to each $x \in D^k$, we can continuously assign a nonzero vector $w(x)$ normal to the tangent plane to $f(D^k)$ at $f(x)$.[14] (Note the use of the hypothesis $k < n$.) Now $f(D^k)$ is not an embedded submanifold, and w is not a well-defined vector field on $f(D^k)$, but f is locally an embedding, and w extends locally to a vector field in $\mathbf{R}^n$. This is good enough to use integral curves of w to push most of $f(D^k)$ along these integral curves, out of the way of the given deformation of f along S^{k-1}. Because of the extra dimension, Smale was able to use this device to achieve regularity of the extension. Of course, the details, containing the heart of the proof, are formidable. But the concept is basically simple.

In his theory of immersions of spheres in Euclidean spaces, Smale introduced two powerful new methods for attacking geometrical problems:

[12] The *Gauss map*, of an embedding f of a closed surface S into 3-space, maps the surface to the unit 2-sphere by sending each point $x \in S$ to the unit vector outwardly normal to $f(S)$ at $f(x)$.

[13] Morin is blind.

[14] More precisely, $w(x)$ is normal to the image of $Df(x)$, the derivative of f at x.

Dynamical systems theory (i.e., integration of vector fields) was used to construct deformations in order to prove that certain restriction maps on function spaces are fibrations; and then algebraic topology was used to obtain isomorphisms between homotopy groups. These techniques were soon used in successful attacks on a variety of problems.

Further Development of Immersion Theory

I first learned of Smale's thesis at the 1956 Symposium on Algebraic Topology in Mexico City. I was a rather ignorant graduate student at the University of Chicago; Smale was a new Ph.D. from Michigan.[15] While I understood very little of the talks on Pontryagin classes, Postnikov invariants and other arcane subjects, I thought I could understand the deceptively simple geometric problem Smale addressed: *Classify immersed curves in a Riemannian manifold.*

In the fall of 1956, Smale was appointed Instructor at the Universtiy of Chicago. Having learned of Smale's work in Mexico City, I began talking with him about it, and reading his immersion papers. I soon found much simpler proofs of his results. Every day I would present them to Smale, who would patiently explain to me why my proofs were so simple as to be wrong. By this process I gradually learned the real difficulties, and eventually I understood Smale's proofs.

In my own thesis [25] directed by Ed Spanier, I extended Smale's theory to the classification of immersions $f: M \to N$ between arbitrary manifolds, provided $\dim N > \dim M$. In this I received a great deal of help from both Smale and Spanier. The main tool was again a fibration theorem: the restriction map, going from immersions of M to germs of immersions of neighborhoods of a subcomplex of a smooth triangulation of M, is a fibration.

The proof of this fibration theorem used Smale's fibration theorem for disks as a local result; the globalization was accomplished by means of a smooth triangulation of M, the simplices of which are approximately disks.

The classification took the following form. Consider the assignment to f of its tangent map $TF: TM \to TN$ between tangent vector bundles. This defines a map Φ going from the space of immersions of M in N to the space of (linear) bundle maps from TM to TN that are injective on each fiber. The homotopy class of this map (among such bundle maps) generalizes the Smale invariant. Using the fibration theorem and Smale's theorems, I showed Φ induces isomorphisms on homotopy groups. By results of Milnor and J.H.C. Whitehead, this implies Φ is a homotopy equivalence. The proof of the classification is a bootstrapping induction on the dimension of M; the inductive step uses the fibration theorem.

[15] For an account of the atmosphere in Chicago in the fifties, see my memoir [28].

Thus immersions are classified by certain kinds of bundle maps, whose classification is a standard task for algebraic topology. A striking corollary of the classification is that every parallelizable manifold is immersible in Euclidean space of one dimension higher.[16]

Several topologists[17] reformulated the classification of immersions of an m-dimensional manifold M into a Euclidean space $\mathbf{R}^{n+k}$ as follows. Let Ψ_M: $M \to BO$ be the classifying map (unique up to homotopy) for the stable normal bundle of M. An immersion $f: M \to \mathbf{R}^{n+k}$ determines a lift of Ψ_M over the natural map $BO(k) \to BO$. Using homotopy theory, it can be deduced from the classification theorem that regular homotopy classes of immersions correspond bijectively in this way to homotopy classes of lifts of Ψ_M.

Subsequently many other classification problems were solved by showing them to be equivalent to the homotopy classification of certain lifts, or what is the same thing, crosssections of a certain fibration. The starting point for this approach to geometric topology was the extraordinarily illuminating talk of R. Thom at the International Congress of 1958 [70], in which he stated that smoothings of a piecewise linear manifold correspond to sections of a certain fibration.[18]

Other proofs of the general immersion classification theorem were obtained by R. Thom [69], A. Phillips [48], V. Poenaru [50], and M. Gromov and Ja. Eliasberg [14, 15] (see also A. Haefliger [16]). Each of these different approaches gave new insights into the geometry of immersions.

Many geometrically minded topologists were struck by the power of the fibration theorem and attacked a variety of mapping and structure problems with fibration methods.

The method of fibrations of function spaces was applied to *submersions* (smooth maps $f: M \to N$ of rank equal to dim N) by A. Phillips [48]. Again the key was a fibration theorem, and the classification was by induced maps between tangent bundles. This was generalized to *k-mersions* (maps of rank $k >$ dim N) by S. Feit [10]. General immersion theory was made applicable to immersions between manifolds of the *same* dimension, provided the domain manifold has no closed component, by V. Poenaru [50] and myself [26].

Fibration methods were used to classify piecewise linear immersions by Haefliger and Poenaru [17]. Topological immersions were classified by J.A. Lees [37] and R. Lashof [36].

M. Gromov [11–13] made a profound study of mapping problems amenable to fibration methods, and successfully attacked many geometric problems of the most diverse types. The article by D. Spring in this volume discusses

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[16] Exercise (unsolved): Describe an explicit immersion of real projective 7-space in $\mathbf{R}^8$!

[17] The first may have been M. Atiyah [2].

[18] This may have been only a conjecture—Thom was not guilty of excessive clarity.

Gromov's far-reaching extensions of immersion theory to other mapping problems such as immersions which are symplectic, holomorphic, or isometric; see Chapters 2 and 3 of Gromov's book [13].

R. Thom [69] gave a new, more conceptual proof of Smale's theorem. R.S. Palais [47] proved an isotopy extension theorem, showing that the restriction map for *embeddings* is not merely a fibration, it is a locally trivial fiber bundle (see also E. Lima [39]). R. Edwards and R. Kirby [9] proved an isotopy extension theorem for topological manifolds.

The 1977 book [35] by R. Kirby and L. Siebenmann contains a unified treatment of many classification theories for structures on topological, piecewise linear, and smooth manifolds. Besides many new ideas, it presents developments and analogues of Smale's fibration theories, Gromov's ideas, and Smale's later theory of handlebodies. See in particular, Siebenmann's articles [54] and [55], a reprinting of [53].

Diffeomorphisms of Spheres

In 1956 Milnor astounded topologists with his construction of an exotic differentiable structure on the 7-sphere, that is, *a smooth manifold homeomorphic but not diffeomorphic to* S^7. This wholly unexpected phenomenon triggered intense research into the classification of differentiable structures, and the relation between smooth, piecewise linear, and topological manifolds.

Milnor's construction was based on a diffeomorphism of the 6-sphere which, he proved, could not be extended to a diffeomorphism of the 7-ball; it was, therefore, not isotopic to any element of the orthogonal group $O(7)$ considered as acting on the 6-sphere. His exotic 7-sphere was constructed by gluing together two 7-balls by this diffeomorphism of their boundaries. These ideas stimulated investigation into diffeomorphism groups.

Two-spheres

In 1958 Smale [59] published the following result:

Theorem. *The space* $\mathrm{Diff}(S^2)$ *of diffeomorphism of the 2-sphere admits the orthogonal group* $O(3)$ *as a deformation retract.*[19]

Again a key role in the proof was played by dynamical systems. I recall Smale discussing his proof of this at Chicago. At one stage he did not see

[19] Around this time an outline of a proof attributed to Kneser was circulating by word of mouth; it was based on an alleged version of the Riemann mapping theorem which gives smoothness at the boundary of smooth Jordan domains, and smooth dependence on parameters. I do not know if such a proof was ever published.

why the proof did not go through for spheres of all dimensions, except that he knew that the analogous result for the 6-sphere would contradict Milnor's constructions of an exotic differentiable structure on the 7-sphere! It turned out that the Poincaré–Bendixson Theorem, which is valid only in dimension 2, played a key role in his proof.

In 1958 Smale went to the Institute for Advanced Study. This was a very fertile period for topology, and a remarkable group of geometers and topologists were assembled in Princeton. These included Shiing–Shen Chern, Ed Spanier, Armand Borel, Ed Floyd, Dean Montgomery, Lester Dubins, Andy Gleason, John Moore, Ralph Fox, Glenn Bredon, John Milnor, Richard Palais, Jim Munkres, André Weil, Henry Whitehead, Norman Steenrod, Bob Williams, Frank Raymond, S. Kinoshita, Lee Neuwirth, Stewart Cairns, John Stallings, Barry Mazur, Papakyriakopoulos, and many others.

I shared an office with Smale and benefited by discussing many of his ideas at an early stage in their development. Among the many questions that interested him was a famous problem of P.A. Smith: Can an involution (a map of period 2) of S^3 have a knotted circle of fixed points? He did not solve it, but we published a joint paper [29] on smooth involutions having only two fixed points. Unfortunately it contains an elementary blunder, and is totally wrong.[20]

Three-spheres

Smale worked on showing that the space Diff(S^3) of diffeomorphism of the 3-sphere admits the orthogonal group $O(4)$ as a deformation retract. Using several fibrations, such as the restriction map going from diffeomorphisms of S^3 to embeddigs of D^3 in S^3, and from the latter to embeddings of S^2 in S^3, and so forth, he reduced this to the same problem for the space of embeddings of S^2 in $\mathbf{R}^3$. Although it failed, his approach was important, and stimulated much further research. Hatcher [20] proved Smale's conjecture in 1975.

In a manuscript for this work Smale analyzed an embedding in Euclidean space by considering a *height function*, i.e., the composition of the embedding with a nonzero linear function.[21]

Smale tried to find a height function which, for a given compact set of embeddings of S^2 in $\mathbf{R}^3$, would look like a Morse function for each embedding, exhibiting it as obtained from the unit sphere by extruding pseudopods in a manageable way. These could then all be pushed back, following the height function, until they all became diffeomorphisms of S^2.

[20] I am glad to report that other people have also made mistakes in this problem.

[21] This idea goes back to Möbius [45], who used it in an attempt to classify surfaces; see Hirsch [27] for a discussion. J. Alexander [1] had used a similar method to study piecewise linear embeddings of surfaces.

At that point, appeal to his theorem on diffeomorphisms of S^2 would finish the proof.

He had a complicated inductive proof; but Robert Williams, Henry Whitehead and I (all at the Institute then) found that the induction failed at the first step!

Nevertheless the idea was fruitful. J. Cerf [6] succeeded in proving that Diff(S^3) has just two path components. Cerf used a more subtle development of Smale's height function: He showed that for a one-parameter family of embeddings, there is a function having at worst cubic singularities, but behaving topologically like a Morse function for each embedding in the family. Cerf's ideas were to prove useful in other deformation problems in topology and dynamics, and surprisingly, in algebraic K-theory. See Hatcher [19], Hatcher and Wagoner [21], and Cerf [5, 7].

Smale would return to the use of height functions as tools for dissecting manifolds in his spectacular attack on the generalized Poincaré conjecture.

In using height functions to analyze embedded 2-spheres, Smale was grappling with a basic problem peculiar to the topology of manifolds: *There is no easy way to decompose a manifold.* Unlike a simplicial complex, which come equipped with a decomposition into the simplest spaces, a smooth manifold —without any additional structure such as a Riemannian metric—is a homogeneous global object. If it is "closed"—compact, connected and without boundary—it contains no proper closed submanifold of the same dimension, is not a union of a countable family of closed submanifolds of lower dimension. This is a serious problem if we need to analyze a closed manifold because it means *we cannot decompose it into simpler objects of the same kind.*

Before 1960 the traditional tool for studying the geometric topology of manifolds was a smooth triangulation. Cairns and Whitehead had shown such triangulations exist and are unique up to isomorphic subdivisions. Thus to every smooth manifold there is associated a combinatorial manifold. In this way simplicial complexes, for which combinatorial techniques and induction on dimension are convenient tools, are introduced into differential topology. But useful as they are for algebraic purposes, they are not well-suited for studying differentiable maps.[22]

Smale would shortly return to the use of Morse functions to analyze manifolds. His theory of theory of handlebodies was soon to supply topologists with a highly succussful technique for decomposing smooth manifolds.

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[22] Simplicial complexes were introduced, as were so many other topological ideas, by Poincaré. Using them he gave a new and much more satisfactory definition of Betti numbers, which had originally been defined in terms of boundaries of smooth submanifolds. It is interesting that while the old definition was obviously invariant under Poincaré's equivalence relation of "homéomorphisme," which meant what we call "C^1 diffeomorphism," invariance of simplicially defined Betti numbers is not at all obvious. (It was later proved by J. Alexander.) Thus the gap between simplicial and differentiable techniques has plagued topology from its beginnings.

The Generalized Poincaré Conjecture and the h-Cobordism Theorem

In January of 1960 Smale arrived in Rio de Janeiro to spend six months at the Instituto de Matematica Pura e Aplicada (IMPA). Early in 1960, he submitted a research announcement: *The generalized Poincaré conjecture in higher dimensions* [60], along with a handwritten manuscript outlining the proof. The editors of the *Bulletin of the American Mathematical Society* asked topologists in Princeton to look over the manuscript. I remember Henry Whitehead, who had once published his own (incorrect) proof, struggling with Smale's new techniques.[23]

The theorem Smale announced in his 1960 *Bulletin* paper is, verbatim:

Theorem (Theorem A). *If M^n is a closed differentiable (C^∞) manifold which is a homotopy sphere, and if $n \neq 3, 4$, then M^n is homeomorphic to S^n.*

The notation implies M^n has dimension n. "Closed" means compact without boundary. Such a manifold is a *homotopy sphere* if it is simply connected and has the same homology groups as the n-sphere (which implies it has the same homotopy type as the n-sphere).

Poincaré [51] had raised the question of whether a simply connected 3-manifold having the homology of the 3-sphere is homeomorphic to the 3-sphere S^3.[24] Some form of the generalized conjecture (i.e., the result proved by Smale without any dimension restriction) had been known for many years; it may be have been due originally to Henry Whitehead.

Very little progress had been made since Poincaré on his conjecture.[25] Because natural approaches to the generalized conjecture seemed to require knowledge of manifolds of lower dimension, Smale's announcement was

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[23] Whitehead was very good about what he called "doing his homework," that is, reading other people's papers. "I would no more use someone's theorem without reading the proof," he once remarked, "than I would use his wallet without permission." He once published a paper relying on an announcent by Pontryagin, without proof, of the formula $\pi_4(S^2) = 0$, which was later shown (also by Pontryagin) to have order 2. Whitehead was quite proud of his footnote stating that he had not seen the proof. Smale, on the other hand, told me that if he respected the author, he would take a theorem on trust.

[24] As Smale points out in his *Mathematical Intelligencer* article [67], Poincaré does not hazard a guess as the the answer. He had earlier mistakenly announced that a 3-manifold is a 3-sphere provided it has the same homology. In correcting his mistake, by constructing the dodecahedral counterexample, he invented the fundamental group. Thus we should really call it Poincaré's *question*, not conjecture.

[25] There is still no good reason to believe in it, except a lack of counterexamples; and some topologists think the opposite conjecture is more likely. *Maybe* it is undecidable!

astonishing. Up to then, no one had dreamed of proving things only for manifolds of higher dimension, three dimensions already being too many to handle.

Nice Functions, Handles, and Cell Structures

Smale's approach is intimately tied to his work, both later and earlier, on dynamical systems. At the beginning of his stay in Princeton, he had been introduced to Mauricio Peixoto, who got Smale interested in dynamical systems.[26]

Smale's proof of Theorem A begins by decomposing the manifold M (dropping the superscript) by a special kind of Morse function $f: M \to \mathbf{R}$, which he called by the rather dull name of "nice function." He wrote:

> The first step in the proof is the construction of a nice cellular type structure on any closed C^∞ manifold M. More precisely, define a real-valued f on M to be a *nice* function if it possesses only nondegenerate critical points and for each critical point β, $f(\beta) = \lambda(\beta)$, the index of β.

It had long been known (due to M. Morse) that any Morse function gives a homotopical reconstruction of M as a union of cells, with one s-cell for each critical point of index s.

Smale observed that the s-cell can be "thickened" in M to a set which is diffeomorphic to $D^s \times D^{n-s}$. Such a set he calls a *handle of type s*; the type of a handle is the dimension of its *core* $D^s \times 0$. Thus from a Morse function he derived a description of M as a union of handles with disjoint interiors.

But Smale wanted the handles to be successively adjoined in the order of their types: First 0-handles (n-disks), then 1-handles, and so on. For this, he needed a "nice" Morse function: The value of the function at a critical point equals the index of the critical point. A little experimentation shows that most Morse functions are not nice. Smale stated:

Theorem (Theorem B). *On every closed C^∞ manifold there exist nice functions.*

To get a nice function, Smale had to rearrange the k-cell handle cores, and to do this he first needed to make the stable and unstable manifolds of all the critical points to meet each other transversely.[27]

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[26] See Peixoto's article on Smale's early work, in this volume; and also Smale's autobiographical article [66].

[27] If p is a singular point of a vector field, its *stable manifold* is the set of points whose trajectories approach p as $t \to \infty$. The *unstable manifold* is the stable manifold for $-f$, comprising trajectories going to p in negative time.

Smale referred to his article "Morse Inequalities for a Dynamical System" [61] for the proof that a gradient vector field on a Riemannian manifold can be C^1 approximated by a gradient vector field for which the stable and unstable manifolds of singular points meet each other transversely. From this he was able to construct a nice function. The usefulness of this will be seen shortly.

In his *Bulletin* announcement [60] Smale then made a prescient observation:

> The stable manifolds of the critical points of a nice function can be thought of as the cells of a complex while the unstable manifolds are the dual cells. This structure has the advantage over previous structures that both the cells and the duals are differentiably imbedded in M. We believe that nice functions will replace much of the use of C^1 triangulations and combinatorial methods in differential topology.

With nice functions at his disposal, Smale could decompose any closed manifold into a union of handles, successively adjoined in the same order as their type. This is a far-reaching generalization of the work of Möbius [45], who used what we call Morse functions in a similar way to decomposed surfaces.

Eliminating Superfluous Handles

The results about nice functions stated so far apply to all manifolds. To prove Theorem A required use of the hypothesis that M is a homotopy sphere of dimension at least five. What Smale proved was that in this case there is a Morse function with *exactly* two critical points—necessarily a maximum and a minimum. It then follows easily, using the grid of level surfaces and gradient lines, that M is the union of two smooth n-dimensional submanifolds with boundary, meeting along their common boundary, such that each is diffeomorphic to D^n.

From this it is simple to show that M is homeomorphic to S^n.[28] Actually more is true. In the first place, it is not hard to show from the decomposition of M into two n-balls that *the complement of point in M is diffeomorphic to* $\mathbf{R}^n$. Second, it follows from the theory of smooth triangulations that *the piecewise linear (PL) manifold*[29] *which smoothly triangulates M is PL isomorphic to the standard PL n-sphere.*

How did Smale get a Morse function with only two critical points? He used the homotopical hypothesis to eliminate all other critical points. To

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[28] In fact, it takes some thought to see why one cannot immediately deduce that M is *diffeomorphic* to S^n; but recall Milnor's celebrated 7-dimensional counterexample [42].

[29] A *piecewise linear manifold* has a triangulation in which the closed star of every vertex is isomorphic to a rectilinear subdivision of a simplex.

visualize the idea behind his proof, imagine a sphere embedded in 3-space in the form of a U-shaped surface. Letting the height be the nice Morse function, we see that there are two maxima, one minimum, and one saddle. The stable manifold of the saddle is a curve, the two ends of which limit at the two maxima. Now change the embedding by pushing down on one of the maxima until the part of the U capped by that maximum has been mashed down to just below the level of the saddle. This can be done so that on the final surface the saddle point has become noncritical, and no new saddle has been introduced. Thus on the new surface, which is diffeomorphic to the original, there is a Morse function with only two critical points. *If we had not known that the original surface is diffeomorphic to the 2-sphere, we would realize it now.*

The point to observe in this process is that *we canceled the extra maximum against the saddle point*; both disappeared at the same time.

Smale's task was to do this in a general way. Because M is connected, there is no topological reason for the existence of more than one maximum and one minimum. If there are two maxima, the Morse inequalities, plus some topology, require the existence of a saddle whose stable manifold is one-dimensional and limits at two maxima, just as in the U-shaped example earlier. Smale redefined the Morse function on the level surfaces above this saddle, and just below it, to obtain a new nice function having one fewer saddle and one fewer maximum. In this way, he proved [62] there exists, on any connected manifold, a nice function with only one maximum and one minimum (and possibly other critical points).

The foregoing had already been proved by M. Morse [46]. Smale went further. Assuming that M is simply connected and of dimension at least five, he used a similar cancellation of critical points to eliminate all critical points of index 1 and $n - 1$.

Each handle corresponds both to a critical point and to a generator in a certain relative singular chain group. Under the assumptions that homology groups vanish, it follows that these generators must cancel algebraically in a certain sense. The essence of Smale's proof of Poincaré's conjecture was to show how to imitate this algebraic calculation vith a geometric one: By isotopically rearranging the handles, he showed that a pair of handles of successive dimensions fit together to form an n-disk. By absorbing this disk into previously added handles, he produced a new handle decompostion with two fewer handles, together with a new Morse function having two fewer critical points. To make the algebra work and to perform the isotopies, Smale had to assume the manifold is simply connected and of dimension at least five.

In this way he proved the following important result. Recall that M is r-connected if every map of an i-sphere into M is contractible to a point for $0 \leq i \leq r$. The ith *type number* μ_i of a Morse function is the number of critical points of index i.

Theorem (Theorem D). *Let M^n be a closed $(m-1)$-connected C^∞ manifold, with $n \geq 2m$ and $(n,m) \neq (4,2)$. Then there is a nice function on M whose type numbers satisfy $\mu_0 = \mu_n = 1$ and $\mu_i = 0$ for $0 < i < m$, $n - m < i < n$.*

This is a kind of converse to the theorem on Morse inequalities.

Smale applied Theorem D to obtain structure theorems for certain simply connected manifolds manifold having no homology except in the bottom, top, and middle dimensions. To state them, we need Smale's definition of a *handlebody of type* (n, k, s): This is an n-dimensional manifold H obtained "by attaching s-disks, k in number, to the n-disk and 'thickening' them."[30] The class of such handlebodies Smale denoted by $\mathscr{H}(n,k,s)$. Notice that a manifold in $\mathscr{H}(n,0,s)$ is a homotopy n-sphere that is the union of two n-disks glued along their boundaries.

For odd-dimensional manifolds, Smale generalized the classical Heegard decomposition of a closed 3-manifold [22, 23]:

Theorem (Theorem E). *Let M be a closed $(m-1)$-connected C^∞ manifold of dimension $2m+1$, $m \neq 2$. Then M is the union of two handlebodies $H, H' \in \mathscr{H}(2m+1,k,m)$.*

For highly connected even-dimensional manifolds, Smale proved the following result which generalizes the classification of closed surfaces:

Theorem (Theorem F). *Let M be a closed $(m-1)$-connected C^∞ $2m$-manifold, $m \neq 2$. Then there is a nice function on M whose type numbers equal the Betti numbers of M.*

For a surface $m = 1$, and one should additionally assume M is orientable (otherwise the projective plane is a counterexample). Suppose M is a connected compact orientable surface of genus g. If $g = 0$ then the first Betti number is zero, and Theorem F says there is a Morse function with only two critical points, which implies M is a sphere. For higher genus, one can derive from Theorem F the usual picture of sphere with g hollow handles.[31]

[30] "Handlebody" is from the German "henkelkörper," a term common in the fifties (but J. Eells always said "Besselhagen"). Although it sounds innocuous today, at the time "handlebody" struck many people as a clumsy neologism—which only made Smale use it more.

[31] Both Jordan [31] and Möbius [45] published proofs of the classification of compact surfaces in the 1860s. From a modern standpoint these are failures. The authors lacked even the language to define what we mean by homeomorphic spaces. It is striking that the following "definition" was used by both of them: Two surfaces are equivalent if each can be decomposed into infinitely small pieces so that contiguous pieces of one correspond to contiguous pieces of the other. While we find it hard to make sense of this, apparently none of their readers was disturbed by it!

Theorem (Theorem H). *There exists a triangulated manifold with no differentiable structure.*

In fact he proved a significantly stronger result: There is a closed PL manifold which does not have the homotopy type of any smooth manifold.

Smale started with a certain 12-dimensional handlebody $H \in \mathscr{H}(12; 8, 6)$ previously constructed by Milnor in 1959 [43]. Milnor had shown that the boundary is a homotopy sphere which could not be diffeomorphic to a standard sphere because H has the wrong index. Smale's results showed that the boundary homeomorphic to S^{11} and a smooth triangulation makes the boundary PL homeomorphic to S^{11}. By gluing a 12-disk to H along the boundary, Smale constructed a closed PL 12-manifold M. Milnor's index argument implied that M did not have the homotopy type of any smooth closed manifold.

An entirely different example of this kind was independently constructed by M. Kervaire [33].[32]

The h-Cobordism Theorem

In his address to the Mexico City symposium in 1956 [71], Thom introduced a new equivalence relation between manifolds, which he called "J-equivalence." This was renamed "h-cobordism" by Kervaire and Milnor [34]. Two closed smooth n-manifolds M_0, M_1 are *h-cobordant* if there is a smooth compact manifold W of dimension $n + 1$ whose boundary is diffeomorphic to the disjoint union of submanifolds V_i, $i = 0, 1$, such that M_i and N_i are diffeomorphic, and each N_i is a deformation retract of W. Such a W is an *h-cobordism* between M_0 and $M - 1$.

This is a very convenient relation, linking differential and algebraic topology. It defines an equivalence relation between manifolds in terms of another manifold, just as a homotopy between maps is defined as another map, thus allowing knowledge about manifolds to be used in studying the equivalence relation. Whereas the geometric implications of two manifolds being h-cobordant is not clear, nevertheless it is often an easy task to verify that a given manifold W is a cobordism: It suffices to prove that all the relative homotopy groups of (W, N_i) vanish, and for this the machinery of algebraic topology is available. In contrast, there are very few methods available for proving that two manifolds are diffeomorphic; and a diffeomorphism is a very different object from a manifold.

[32] Kervaire's example is constructed by a similar strategy from a 10-dimensional handlebody in $\mathscr{H}(10, 2, 5)$. It has an elegant description: Take two copies of the unit disk bundle of S^5 and "plumb" them together, interchanging fiber disks and base disks in a product representation over the upper hemisphere. In place of the index, Kervaire used an Arf invariant.

For these reasons there was great excitement when, shortly after the announcement of the generalized Poincaré conjecture, Smale proved the following result [64]:

Theorem (The h-Cobordism Theorem). *Let W be an h-cobordism between M_0 and M_1. If W is simply connected and has dimension at least 6, then W is diffeomorphic to $M_0 \times I$. Therefore, M_1 and M_0 are diffeomorphic.*

So important is the h-cobordism theorem that it deserves to be called The Fundamental Theorem of Differential Topology.

Kervaire and Milnor studied oriented homotopy n-spheres under the relation of h-cobordism. Using the operation of connected sum, they made the set of h-cobordism classes of homotopy n-spheres into an Abelian group Θ_n [34]. They proved these groups to be finite for all $n \neq 3$ (the case $n = 3$ is still open), and computed their orders for $1 \leq n \leq 17$, $n \neq 3$. For example, the order is 1 for $n = 1, 2, 4, 5, 612$; it is 2 for $n = 8, 14, 16$; and it is 992 for $n = 11$. In this work, they did not use the h-cobordism theorem. Use of that theorem, however, sharpens their results, as they remark: "For $n \neq 3$, 4, Θ_n can be described as the set of all diffeomorphism classes of differentiable structures on the topological n-sphere," where it should be understood that the diffeomorphisms preserve orientation.

From Milnor and Kervaire's work Smale proved, as a corollary to the h-cobordism theorem, that *every smooth homotopy 6-sphere is diffeomorphic to S^6.*

The Structure of Manifolds

In this paper "On the Structure of 5-Manifolds" [63], Smale puts handle theory to work in classifying certain manifolds more complicated than homotopy spheres, namely, boundaries of handlebodies of type $(2m, k, m)$.

Using Milnor's surgery methods he is able to show, for example, that a smooth, closed 2-connected 5-manifold, whose second Stiefel–Whitney class vanishes, is the boundary of a handlebody of type $(6, k, 3)$. He then shows that such a 5-manifold is completely determined up to diffeomorphism by its second homology group, and he constructs examples in every diffeomorphism class.

Another result Smale states in this paper is that *every smooth, closed 2-connected 6-manifold is homeomorphic either to S^6 or to a connected sum of $S^3 \times S^3$ with copies of itself.*

In the same issue of the *Annals*, C.T.C. Wall has a paper [72] called "Classification of $(n - 1)$-connected $2n$-manifolds" containing a detailed study of the smooth, combinatorial and homotopical structure of such manifolds. The h-cobordism theorem is the main tool (in addition to results of Milnor and Kervaire, plus a lot of algebra). Wall proves:

Theorem. *Let $n \geq 3$ be congruent modulo 8 to 3, 5, 6 or 7.*[33] *Let M, N be smooth, closed $(n-1)$-connected 2n-manifolds of the same homotopy type. Then M is diffeomorphic to the connected sum of N with a homotopy 2n-sphere. If $n = 3$ or 6 then they are diffeomorphic.*

Milnor had shown in 1956 that there are smooth manifolds homeomorphic but not diffeomorphic to S^7. Kervaire and Milnor's work [34], plus the *h*-cobordism theorem, showed that up to orientation-preserving diffeomorphism there are exactly 28 such manifolds. Wall [72] proved a surprising result about the product of such manifolds:

Theorem. *The product of two smooth manifolds, each homeomorphic to S^7, is diffeomorphic to $S^7 \times S^7$.*

The *s*-Cobordism Theorem

There is no room to chronicle the all consequences, generalizations, and applications of the *h*-cobordism theorem and its underlying idea of handle cancellation. But one—the *s-cobordism theorem*—is worth citing here for its remarkable blend of homotopy theory, algebra and differential topology.

As with much of topology, this story starts with J.H.C. Whitehead. In 1939, he published a paper with the mysterious title "Simplicial Spaces, Nuclei and *m*-Groups" [74], followed a year later by "Simple Homotopy Types" [75]. In these works, he introduced the notion of a *simple* homotopy equivalence between simplicial (or CW) complexes. Very roughly, this means a homotopy equivalence which does not overly distort the natural bases for the cellular chain groups. He answered the question of when a given homotopy equivalence is homotopic to a simple one, by inventing an obstruction, lying in what is now called the Whitehead group of the fundamental group, whose vanishing is necessary and sufficient for the existence of such a homotopy. Whitehead proved that his invariant vanishes—because the Whitehead group is trivial—whenever the fundamental group is cyclic of order 1, 2, 3, 4, 5 or ∞. See Milnor's excellent exposition [44].

Several people independently realized that Whitehead's invariant was the key to extending Smale's *h*-cobordism theorem to manifolds whose fundamental groups are nontrivial: D. Barden [3], B. Mazur [41], and J. Stallings [68]. The result is this:

Theorem (The *s*-Cobordism Theorem). *Let W be an h-cobordism between M_0 and M_1. If W has dimension at least* 6, *and the inclusion of M_0 (or equivalently, of M_1) into W is a simple homotopy equivalence, then W is diffeomorphic to $M_0 \times I$. Therefore, M_1 and M_0 are diffeomorphic.*

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[33] These are the dimensions for which $\pi_{n-1}(SO) = 0$.

Using Whitehead's calculation we immediately obtain:

Corollary. *The conclusion of the h-cobordism theorem is true even if W is not simply connected, provided its fundamental group is infinite cyclic or cyclic of order* ≤ 5.

The *s*-cobordism theorem has been expounded by J. Hudson [30] and M. Kervaire [32].

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