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In a single-channel two-body scattering problem, Levinson's theorem has been generalized[1] to include the resonance poles in the unphysical sheet. It was found that a relationship exists between the total number of composite-particle poles and the phase change of the S matrix along the left-hand cut. In this note we consider further the generalization to the case of many channels each having only two particles.

1. One-Channel Case. For the sake of completeness we summarize here the derivation for the one-channel case. The hypotheses are that the partial-wave S matrix, $S_\ell(s)$, is a real analytic function in the s (energy squared) plane, cut on the real axis from the threshold s_1 to $+\infty$ and from $-\infty$ to s_L, $s_L < s_1$; in the physical region $S_\ell(s)$ satisfies the unitarity condition, and in the asymptotic region as $|s| \to \infty$, $S_\ell(s)$ tends to a constant. The continuation of $S_\ell(s)$ to the unphysical sheet through the unitarity cut leads to the function $S_\ell^u(s)$, which can be established to be $S_\ell^{-1}(s)$. Thus if n_0 and n_p designate respectively the number of zeroes and of poles of $S_\ell(s)$ on the physical sheet, then $n_0 + n_p$ is the total number of poles (elementary and composite particles) in the two-sheeted Riemann surface bounded by the left-hand cut from $-\infty$ to s_L on both sheets.

Consider the integral $I = \int_C ds\, S'(s)/S(s)$, where $S'(s)$ is the first derivative of $S(s)$ (the subscript ℓ having been suppressed) and C is the largest possible closed counterclockwise contour in the cut s plane. Thus C consists of four parts: C_L, a contour tightly around the left-hand cut in a clockwise direction; C_R, tightly around the right-hand cut; and two large semicircles in the upper and lower half planes. By Cauchy theorem, I is identically $(n_0 - n_p)2\pi i$. Since $S(s)$ tends to a constant asymptotically, the integrations along the semicircles contribute nothing. The contribution from C_R is $2i \operatorname{Im} \ln S(s)\big|_{s_1}^{\infty} = 4i[\delta(\infty) - \delta(s_1)]$, where δ is the (real) phase shift. On the basis of analyticity and unitarity, the usual Levinson's theorem can be derived[2]: $\delta(s_1) - \delta(\infty) = (n_p - n_e)\pi$, where n_e is the number of elementary particles or CDD poles. Hence, we obtain $\ln S(s)\big|_{C_L} = (n_0 + n_p - 2n_e)2\pi i$, where the left-hand side implies the change $\ln S(s)$ undergoes as s is taken along C_L. Since there is a zero of $S(s)$ associated with each elementary-particle pole--a property that can be made evident by reducing the strength of interaction between the elementary particle and the scattering system, whereupon the zero approaches the pole position--we have the formula $n_0 + n_p = 2n_e + n_c$; here n_c is the total number of composite-particle poles that are on both sheets. We thus obtain

$$\ln S(s)\big|_{C_L} = 2\pi i n_c . \qquad (1)$$

The meaning of this equation and its application to specific problems in giving an upper bound of n_c are discussed in reference 1.

2. Sheet Structure in the Multichannel Problem. When there are n coupled two-particle channels, we consider the Riemann surface consisting of all the sheets connected by the unitarity cut with n normal thresholds. We derive here the structure of this surface and the total number of sheets. Let $A_{\alpha\beta}(s)$ be the partial-wave scattering amplitude from channel α to channel β, and let $\underset{\sim}{A}(s)$ be an n-by-n matrix in channel space formed by a collection of such amplitudes. Let L_i be an operator that takes any amplitude being operated on along a standard path in the s plane--which is a path leading from the original point s to the neighborhood of the ith threshold, s_i, staying entirely on the sheet which contains s, and then, after a small clockwise rotation around s_i, retreating to s on an adjacent sheet along the same path. We define

$$A_{\alpha\beta}^{(i)}(s) = L_i A_{\alpha\beta}(s) , \qquad A_{\alpha\beta}^{(ij)}(s) = L_j L_i A_{\alpha\beta}(s) ,$$

and so on. Since each channel contains only two particles, the branch points at s_i, $i \in (1,\cdots,n)$, are all of square-root type; thus

$$A_{\alpha\beta}^{(ii)}(s) = A_{\alpha\beta}(s) , \tag{2}$$

or $L_i = L_i^{-1}$. We want to establish that $[L_i, L_j] = 0$, so that

$$A_{\alpha\beta}^{(ij)}(s) = A_{\alpha\beta}^{(ji)}(s) . \tag{3}$$

The unitarity condition may be stated as[3]

$$\underset{\sim}{A}^{-1}(s + i\epsilon) = \underset{\sim}{R}(s + i\epsilon) - i\underset{\sim}{\rho}(s + i\epsilon)\,\underset{\sim}{\theta}(s) , \qquad s > s_1 ,$$

where $\underset{\sim}{R}$ is a real, symmetric matrix; $\rho_{\alpha\beta}(s) = \rho_\alpha(s)\delta_{\alpha\beta}$, the phase-space factor $\rho_\alpha(s)$ has a square-root branch point at s_α; and $\theta_{\alpha\beta} = \theta(s - s_\alpha)\delta_{\alpha\beta}$. It then follows that

$$(1 - L_i)\, \underset{\sim}{A}^{-1}(s) = -2i\rho_i(s)\underset{\sim}{\Delta}^{i}_{\alpha\beta}, \qquad \Delta^{i}_{\alpha\beta} = \delta_{\alpha i}\,\delta_{\beta i}, \qquad (4)$$

for any s in the cut complex plane where $\underset{\sim}{A}^{-1}(s)$ exists. From this, clearly $L_j L_i \underset{\sim}{A}^{-1} = L_i L_j \underset{\sim}{A}^{-1}$, so $[L_i, L_j]\underset{\sim}{A} = 0$, and (3) therefore follows.

On the basis of (2) and (3) we can now label the sheets as follows [with the physical sheet denoted by (0)]: (i), $i = 0,1,\cdots,n$; (ij), $1 \leqslant i < j \leqslant n$; (ijk), $1 \leqslant i < j < k \leqslant n$; $\cdots$; $(1, 2,\cdots,n)$. There are altogether 2^n sheets.

3. <u>Condition for Poles in the Unphysical Sheets.</u> From (4) we see that on sheet $(i_1, i_2,\cdots,i_m)$, $m \leqslant n$, the inverse matrix is[4]

$$\underset{\sim}{A}^{-1(i_1,\cdots,i_m)}(s) = \underset{\sim}{A}^{-1}(s) + 2i \sum_{r=1}^{m} \rho_{i_r}(s)\, \underset{\sim}{\Delta}^{i_r}. \qquad (5)$$

Since any pole of $\underset{\sim}{A}(s)$ is in every element of the matrix, and since the residues at the pole are factorizable--i.e., the residue matrix is of rank one--we find firstly that $\det \underset{\sim}{A}(s)$ has only simple poles and secondly that each element of $\underset{\sim}{A}^{-1}(s)$ is regular at the position of any pole of $\underset{\sim}{A}(s)$. Because the second term on the right side of (5) is a kinematical quantity only, whereas $\underset{\sim}{A}^{-1}$ has dynamical content, $\det \underset{\sim}{A}^{-1[m]}(s)$, $[m] \equiv (i_1,\cdots,i_m)$, is in general nonvanishing at the value of s where $\det A^{-1}(s)$ vanishes. Thus, by inverting (5),

$$\underset{\sim}{A}^{[m]}(s) = \underset{\sim}{A}(s)\underset{\sim}{D}^{-1}_{[m]}(s),$$

$$\underset{\sim}{D}_{[m]}(s) = \underset{\sim}{1} + 2i \sum_{r=1}^{m} \rho_{i_r}(s)\, \underset{\sim}{\Delta}^{i_r} \underset{\sim}{A}(s), \qquad (6)$$

we see that the only places where $\underset{\sim}{A}^{[m]}(s)$ can have poles are where $\det \underset{\sim}{D}_{[m]}(s) = 0$. Here, $\underset{\sim}{D}_{[m]}$ is an n-by-n matrix. It is easy to establish that $\det \underset{\sim}{D}_{[m]} = \det \underset{\sim}{S}_{[m]}$, where $\underset{\sim}{S}_{[m]}$ is the S matrix in an m-dimensional channel space spanned by the channels $i_1,\cdots,i_m$. That is,

$$\underset{\sim}{S}_{(i)} = S_{ii} , \qquad \underset{\sim}{S}_{(ij)} = \begin{pmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{pmatrix} , \cdots ,$$

where

$$S_{ij}(s) = \delta_{ij} + 2i[\rho_i(s)\,\rho_j(s)]^{\frac{1}{2}} A_{ij}(s) .$$

Note that all elements of $\underset{\sim}{A}^{[m]}$ in the unphysical sheet $[m]$ must have poles at the same position. There are 2^n-1 equations of the form

$$\det \underset{\sim}{S}_{[m]}(s) = 0 , \qquad [m] = (i_1,\cdots,i_m) , \qquad 1 \leqslant m \leqslant n , \tag{7}$$

which determine the positions of poles in the 2^n-1 unphysical sheets in terms of amplitudes on the physical sheet.

4. Boundaries of the Riemann Surface. Let the Riemann surface, which consists of the 2^n sheets connected to one another by the unitarity cut, be bounded by the left-hand cuts on each sheet. By left-hand cuts we mean all branch cuts beside the unitarity one. In the multichannel problem there are scattering amplitudes whose left-hand cut overlaps with the unitarity cut on the real axis below the physical region. Thus, in order to construct the Riemann surface as we have defined, we must first introduce small imaginary parts to the masses of the particles so that the left- and right-hand cuts are disentangled. After the standard paths (discussed in

Sec. 2) have been constructed, we then take the limit of real masses.

Another complication that arises in the multichannel problem is that some reaction amplitudes may have anomalous thresholds, or have complex singularities in the case of unstable external particles. In such situations we distort the right-hand cuts so as to avoid intersection with the left-hand cuts. The structure of the Riemann surface is, of course, unaltered by this distortion. The amplitudes on the unphysical sheets are still related to the amplitudes on the physical sheet in the same way, except that the domains of definition of the sheets are deformed according to the new locations of the distorted right-hand cuts.

Let a contour C_L in the complex s plane be defined in such a way that it encircles tightly a clockwise direction around the left-hand branch cuts of det $\underset{\sim}{S}(s)$, which generally has all the left-hand singularities of every element of $\underset{\sim}{S}(s)$. Since $\underset{\sim}{S}_{[m]}$ are submatrices of $\underset{\sim}{S}$, it is clear that C_L must also enclose all the left-hand branch cuts of any det $\underset{\sim}{S}_{[m]}(s)$. We may therefore regard the Riemann surface as being bounded by C_L on every sheet.

5. Total Number of Composite-Particle Poles. We now generalize (1) to the multichannel case, and obtain

$$\sum_{m=1}^{n} \sum_{i_1<\cdots<i_m} \ln \det S_{(i_1,\cdots,i_m)}(s) \Big|_{C_L} = 2\pi i n_c ,$$

where n_c is now the total number of composite-particle poles on the 2^n-sheeted surface. Equivalently we have

$$\ln \Sigma(s) \Big|_{C_L} = 2\pi i n_c , \qquad (8)$$

where

$$\Sigma(s) = \left[\prod_i S_{(i)}(s)\right]\left[\prod_{i<j} \det \underset{\sim}{S}_{(ij)}(s)\right]\cdots\left[\det \underset{\sim}{S}(s)\right].$$

This result can be understood along the line of argument given in reference 1, where the movements of the composite-particle poles are considered as functions of the interaction strength. If the dynamical system is not coupled to any elementary particles, and if the effective interaction strength is sufficiently weak, $\Sigma(s)$ can be made arbitrarily close to unity for any s in the complex plane bounded by C_L, assuming that C_L is at a small, but finite, distance away from any left-hand singularity. Thus, initially there are no poles in the Riemann surface in agreement with the fact that the phase change of $\Sigma(s)$ along C_L vanishes. As the effective interaction strength is increased, poles can enter into the surface only through C_L in the unphysical sheets, corresponding to zeroes of $\Sigma(s)$ passing through C_L. By conformal mapping, each time a zero of $\Sigma(s)$ enters (or leave) the s plane through C_L, $\Sigma(s)\big|_{C_L}$ gains (or loses) phase by 2π. Since the movements of poles between sheets through the unitarity cut do not change n_c, (8) is verified. Application of (8) to obtain an upper bound of n_c for any given set of discontinuities across the left-hand cuts can presumably be made following an analysis similar to the one outlined earlier.[1]

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FOOTNOTES AND REFERENCES

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3. P. Matthews and A. Salam, Nuovo Cimento 13, 381 (1959).

4. The amplitudes on the right-hand side of (5) are evaluated on the physical sheet, for which the superscript (0) shall always be suppressed.