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Solutions to the 2D Euler Equations Satisfying the Serfati Condition With Relaxed Constraints on the Initial Vorticity

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Solutions to the 2D Euler Equations Satisfying the Serfati Condition With Relaxed Constraints on the Initial Vorticity

A Dissertation submitted in partial satisfaction
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ABSTRACT OF THE DISSERTATION

Solutions to the 2D Euler Equations Satisfying the Serfati Condition
With Relaxed Constraints on the Initial Vorticity

by

Taylor Gunter Baldwin

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2020
Dr. James Kelliher, Chairperson

We prove finite-time existence of solutions to the 2D Euler equations where the velocity grows slower than the square root of the distance from the origin and with vorticity up to linear growth provided the initial vorticity is quasibounded, or with growth in the vorticity related to the growth in the velocity provided the initial vorticity is stable. In the quasibounded case, we provide an example with unbounded vorticity and velocity. We also prove uniqueness for solutions with stable initial vorticity and with a certain bound on the modulus of continuity of the initial velocity. This thesis expands on the recent work of Cozzi and Kelliher, which substitutes Serfati's identity in place of the Biot-Savart law to demonstrate uniqueness and short-time existence of bounded vorticity solutions with slow growing velocity.

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1. Preliminaries

1.1. Introduction

In 1757, Leonhard Euler introduced his namesake system of partial differential equations which models the evolution of an inviscid fluid [7]. The velocity formulation of the two-dimensional incompressible Euler equations is

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u(t,x)$ is the divergence-free velocity field and $p(t,x)$ is the scalar pressure.

By taking the scalar curl of (1.1), we obtain the vorticity-stream formulation of the two-dimensional Euler equations, which is the formulation we will work with:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{1.2}$$

Here, the divergence-free velocity field u is determined from the vorticity $\omega = \nabla \times u = \partial_1 u_2 - \partial_2 u_1$ by the Biot-Savart law

$$u = K * \omega, \qquad K(x) := \frac{x^\perp}{2\pi |x|^2},$$

where $x^\perp := (-x_2, x_1)$.

The standard weak solution result to the two-dimensional Euler equations was established in 1963 by Yudovich in [19], where he demonstrated the global-in-time well-posedness of the equations for bounded vorticity on a bounded domain in $\mathbb{R}^2$. Yudovich's work was subsequently extended to the full plane for vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$; the proof to the following theorem can be found, for example, in [12].

Theorem 1.1.1 ([12]). *If $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then for any $T > 0$ there exists a unique solution with initial vorticity ω_0 in the sense that the velocity field u and vorticity $\omega = \nabla \times u$ satisfy the following criteria:*

- $\omega \in L^\infty([0, T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$,
- $u = K * \omega$,
- *The vorticity equation $\omega_t + u \cdot \nabla\omega = 0$ holds in the sense of distributions.*

In recent years, efforts have been made to demonstrate existence of solutions with velocity and/or vorticity which does not decay at infinity or which is unbounded. In [20], Yudovich extended his own result to allow for slightly unbounded vorticities, but still on a bounded domain. In [18], Vishik, while working in the full plane, was able to establish uniqueness for a slightly larger class of unbounded vorticities than Yudovich, but still assumed the vorticity vanished at infinity. In [16], Taniuchi built upon [15] to prove global existence for a class of solutions with bounded unbounded vorticity and with bounded velocity that is not required to decay.

Elgindi and Jeong, in [6], were able to establish the well-posedness of solutions to the Euler equations with merely bounded initial vorticity and with velocity with up to linear growth provided the initial vorticity was m-fold symmetric, $m \geq 3$. Generally, the Biot-Savart law requires some decay of the vorticity. To get around this, the authors use the observation made in [5] that the Biot-Savart kernel exhibits integral decay in when convoluted with a function with vanishing first and second Fourier modes.

Serfati took a different approach in [15], where he presents the outlines of proofs for the well-posedness of solutions in $\mathbb{R}^2$ requiring only that the velocity and vorticity both be bounded. Serfati replaced the Biot-Savart law $u = K * \omega$ with the identity

$$u^j(t) = u_0^j + (aK^j) * (\omega(t) - \omega_0) + \int_0^t (\nabla\nabla^\perp[(1-a)K^j]) *\cdot (u \otimes u)(s)\, ds, \quad (1.3)$$

where a is any radially symmetric cutoff function with $a = 1$ in a neighborhood of the origin, and where $A *\cdot B := \sum_{i,j} A^{ij} * B^{ij}$ for any matrix-valued functions A, B. We call identity (1.3) *Serfati's identity.*

With this substitution, Serfati was able to prove the following.

Theorem 1.1.2 (Serfati [15])**.** *We say a divergence-free vector field $u \in S$ if $\|u\|_S := \|u\|_{L^\infty(\mathbb{R}^2)} + \|\omega(u)\|_{L^\infty(\mathbb{R}^2)} < \infty$, where $\omega(u) = \nabla \times u$. If $u_0 \in S$, then for any $T > 0$ there exists a unique solution to the Euler equations with initial data u_0 in the sense that the velocity field u and vorticity $\omega = \nabla \times u$ satisfy the following criteria:*

- $u \in L^\infty([0,T]; S) \cap C([0,T] \times \mathbb{R}^2)$,
- *Serfati's identity in* (1.3) *holds for any radially symmetric cutoff function a with $a = 1$ in a neighborhood of the origin,*
- *The vorticity equation $\omega_t + u \cdot \nabla\omega = 0$ holds in the sense of distributions.*

Serfati presents in [15] only the outline of the proof to Theorem 1.1.2. A more fleshed-out proof of Serfati's result, along with an extension of the result to exterior domains, is provided in [1]; see also [11] for a fuller characterization of these Serfati solutions. Serfati's result was improved upon in [2] to show the existence of solutions with bounded vorticity and with locally bounded velocity that grows slower than the power of a logarithmic function.

In [4], Cozzi and Kelliher extended Serfati's result more generally (using different techniques than were used in [2]) to establish the well-posedness of solutions with velocity with slow growth at infinity and bounded vorticity. To measure the growth of the velocity field, Cozzi and Kelliher introduced multiple classes of single-variable scalar functions collectively called *growth bounds*.

Definition 1.1.3 (Growth Bounds)**.**

1. We say a function $h : [0, \infty) \to (0, \infty)$ is a *pre-growth bound* if it is increasing, concave, differentiable on $[0, \infty)$, and twice continuously differentiable on $(0, \infty)$.

2. A pre-growth bound h is a *growth bound* if $\int_1^\infty h(s)s^{-2}\, ds < \infty$.

3. A growth bound h is a *well-posedness growth bound* if h^2 is also a growth bound.

We also define $H[h](r) := \int_r^\infty h(s)s^{-2}\, ds$. We note that $H[h](r) < \infty$ for all $r > 0$ if h is a growth bound.

Remark 1.1.4. For simplicity of notation, given a pre-growth bound h and any $x \in \mathbb{R}^2$, we will often write $h(x)$ in place of $h(|x|)$, $h'(x)$ in place of $h'(|x|)$, etc.

Given a function u and a growth bound h, we can compare the growth of u and h by considering $\|u/h\|_{L^\infty}$. We will generally require that $\|u/h\|_{L^\infty} < \infty$, so that in no place on its domain does u grow much faster than h.

A elementary example of a well-posedness growth bound is $h_1(r) = 1$; in this case, only if a function u is bounded will $u/h \in L^\infty$. Examples of growth bounds which allow growth in u when $u/h \in L^\infty$ include $h_2(r) = 1 + r^\alpha$, where h_2 is a growth bound when $\alpha \in [0, 1)$ and a well-posedness growth bound when $\alpha \in [0, 1/2)$, and $h_3(r) = \log^{\frac{1}{4}}(e + r)$.

Cozzi and Kelliher's result in [4] follows.

Theorem 1.1.5 (Cozzi and Kelliher [4]). *We say a divergence-free vector field* $u \in S_h$*, where* h *is a pre-growth bound, if* $\|u\|_{S_h} := \|u/h\|_{L^\infty(\mathbb{R}^2)} + \|\omega(u)\|_{L^\infty(\mathbb{R}^2)} < \infty$. *If* $u_0 \in S_h$*, then for some* $T > 0$ *there exists a unique solution to the Euler equations with initial data* u_0 *in the sense that the velocity field* u*, vorticity* $\omega = \nabla \times u$*, and unique flow map* X *satisfy the following criteria:*

- $u \in C([0,T]; S_h)$,
- *Serfati's identity in* (1.3) *holds for any radially symmetric cutoff function* a *with* $a = 1$ *in a neighborhood of the origin,*
- $\omega(t,x) = \omega^0(X^{-t}(x))$ *on* $[0,T] \times \mathbb{R}^2$*, where* X^{-t} *is the inverse of* X *at time* t.

None of the above results allow initial data where both the velocity and the vorticity are allowed to grow as $|x| \to \infty$. The purpose of this dissertation is to investigate to what extent we can relax the boundedness condition on the vorticity in Theorem 1.1.5 so as to allow for the existence of solutions with growth in both the initial velocity and the initial vorticity. We also explore uniqueness of a class of solutions.

1.2. Main Results

To state our main results, we first must establish several definitions.

Definition 1.2.1. Let h, g be pre-growth bounds. We define $S_{h,g}$ to be the Banach space of divergence-free vector fields u over $\mathbb{R}^2$ such that

$$\|u\|_{S_{h,g}} := \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \frac{\omega(u)}{g} \right\|_{L^\infty(\mathbb{R}^2)} < \infty,$$

where $\omega(u) = \nabla \times u$.

Remark 1.2.2. Note that the assumptions on the initial data u_0 in Theorem 1.1.2 are equivalent to requiring $u_0 \in S_{1,1}$. Similarly, the assumptions on u_0 in Theorem 1.1.5 are equivalent to requiring $u_0 \in S_{h,1}$ for a well-posedness growth bound h.

Definition 1.2.3 (Radial Cutoff Function and Cutoff Biot-Savart Kernel)**.** Choose a radially symmetric function $a \in C_c^\infty(\mathbb{R}^2)$ so that $a : \mathbb{R}^2 \to [0,1]$ is supported in $\overline{B_1(0)}$ and $a = 1$ on $\overline{B_{1/2}(0)}$. For any $\lambda > 0$, we define the *scaled radial cutoff function*

$$a_\lambda(x) := a\left(\frac{x}{\lambda}\right).$$

We also define the *cutoff Biot-Savart kernel*

$$K_\lambda := \mathbb{1}_{B_\lambda(0)} K,$$

where $\mathbb{1}_A$ is the characteristic function of a set A.

We are now prepared to define what we mean by a solution to the 2D Euler equations.

Definition 1.2.4 (Lagrangian Solution)**.** Let h, g be pre-growth bounds and let $T > 0$. Assume $u \in L^\infty([0,T]; S_{h,g})$ with vorticity $\omega = \nabla \times u$ and unique flow-map X. Let $u_0 = u|_{t=0}$ and $\omega_0 = \nabla \times u_0$. Then we say u is a solution to the Euler equations on $[0,T]$ if

(1) $\omega(t,x) = \omega_0(X^{-t}(x))$ for each $(t,x) \in [0,T] \times \mathbb{R}^2$, where X^{-t} is the inverse of X at time t, and

(2) Serfati's identity (1.3) holds for each $(t,x) \in [0,T] \times \mathbb{R}^2$ for each scaled radial cutoff function a_λ.

We note that Definition 1.2.4 is the Lagrangian formulation of (1.2) with the Biot-Savart law replaced by Serfati's identity. As in [4], we use Lagrangian solutions, as opposed to Eulerian solutions, as we need the fact that the initial vorticity is transported by a unique flow map in our uniqueness proof. Well-posedness of Eulerian solutions to the two-dimensional Euler equations is most often established by first constructing Lagrangian solutions (which are automatically Eulerian), then proving that Eulerian solutions are unique. This works for bounded velocity, bounded vorticity solutions; see, for example, [3] and [17]. Whether this approach can be extended to the solutions studied here is unknown and is a subject for future work.

In [4], the authors establish the well-posedness of solutions with initial data in $S_{h,1}$, where h is a well-posedness growth bound, for at least short time; thus, they assume the initial voriticity is bounded. As we shall show, one way to relax this condition is to assume the initial vorticity is merely "almost" bounded.

Definition 1.2.5 (Quasibounded Functions)**.** For $f : \mathbb{R}^2 \to \mathbb{R}$ and a pre-growth bound h, define

$$\|f\|_{\mathcal{T}_h} = \sup_{\substack{x\in\mathbb{R}^2,\lambda\geq h(x)\\ \xi\in MPH(\mathbb{R}^2)}} \frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|f(y)|}{|x-\xi(y)|}\,dy,$$

where we define $MPH(\mathbb{R}^2)$ as the set of all measure-preserving homeomorphisms on $\mathbb{R}^2$. If $\|f\|_{\mathcal{T}_h} < \infty$, we say f is *h-quasibounded*, or merely quasibounded when h is understood.

We call $\mathcal{T}_h = \{f : \|f\|_{\mathcal{T}_h} < \infty\}$ the *space of h-quasibounded functions.*

The uniform norm takes the supremum over all individual values of a function. In contrast, the quasibounded norm takes into account the points surrounding each individual value, weighting the points differently, but with no preference as to which point is weighted most heavily.

Another possible way to relax the boundedness requirement on the initial vorticity in the existence argument is to require the initial vorticity to be stable in some sense.

Definition 1.2.6 (Stabilizers)**.** Let h be a pre-growth bound. We say that a differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an *h-stabilizer*, or simply a *stabilizer* when h is understood, if $\|h\nabla\phi\|_{L^\infty(\mathbb{R}^2)} < \infty$. We say that a function $\omega : \mathbb{R}^2 \to \mathbb{R}$ is *stable relative to ϕ* if $\|\omega - \phi\|_{L^\infty(\mathbb{R}^2)} < \infty$.

Definition 1.2.7 (Induced Growth Bounds)**.** Let h be a pre-growth bound. Define $\hbar(r) := 1 + \int_0^r \frac{ds}{h(s)}$. We show in Lemma 1.3.5 that $\hbar$ is a pre-growth bound. We call $\hbar$ the *pre-growth bound induced by h.*

Theorems 1.2.8 to 1.2.10 comprise our main results.

Theorem 1.2.8 (Existence with Quasibounded Initial Vorticity)**.** *Let $u_0 \in S_{h,g}$ for some well-posedness growth bound h and some pre-growth bound g. If $\omega_0 \in \mathcal{T}_h$, then there exists a solution to the Euler equations $u \in L^\infty([0,T]; S_{h,g})$ with initial data u_0 for some $T > 0$, where $\|\omega(t)\|_{\mathcal{T}_h} = \|\nabla \times u(t)\|_{\mathcal{T}_h}$ is uniformly bounded for $t \in [0,T]$.*

Theorem 1.2.9 (Existence with Stable Initial Vorticity)**.** *Let h be a well-posedness growth bound, let $g \leq C\hbar$ be a pre-growth bound, and let $u_0 \in S_{h,g}$. If ω_0 is stable relative to some h-stabilizer ϕ, then there exists a solution to the Euler equations $u \in L^\infty([0,T]; S_{h,g})$ with initial data u_0 for some $T > 0$, where $\omega(t) = \nabla \times u(t)$ is uniformly stable relative to ϕ; that is, $\|\omega(t) - \phi\|_{L^\infty(\mathbb{R}^2)} < C$ for some C independent of $t \in [0,T]$.*

In the case of stable initial vorticity, we are also able to prove uniqueness of solutions in some situations.

Theorem 1.2.10 (Uniqueness with Stable Initial Vorticity)**.** *Let h be a well-posedness growth bound, let $g \leq C\hbar$ be a pre-growth bound, and let $u_0 \in S_{h,g}$. Let* *the initial vorticity ω_0 be stable with respect to some h-stabilizer ϕ. Suppose*

$$|u_0(x+y) - u_0(x)| \le Ch(x)\mu\left(\frac{|y|}{h(x)}\right)$$

for all $x, y \in \mathbb{R}^2$ such that $|y| \le C(1+|x|)$. Then if there is a solution $u \in L^\infty([0,T]; S_{h,g})$ with initial data u_0 for some $T > 0$, the solution is unique.

We prove Theorem 1.2.8 in Section 2.4. We also provide an example of a solution that has initial data with quasibounded, yet unbounded, initial vorticity in Section 2.4.

We prove Theorem 1.2.9 in Section 2.5. We also discuss the challenges of finding initial data satisfying the assumptions of Theorem 1.2.10 that do not also satisfy the assumptions of Theorem 1.1.5; it is possible no such initial data exists.

Finally, we prove theorem 1.2.10 in Section 3.2.

1.3. Properties of Growth Bounds

In this section, we investigate the properties of growth bounds. We begin with Lemmas 1.3.1 to 1.3.4, in which we prove important properties of growth bounds first observed in [4].

Lemma 1.3.1. *Let $h : [0,\infty) \to [0,\infty)$ be an increasing, concave function. Then h is* subadditive*; that is, for all $r, s \ge 0$,*

$$h(r+s) \le h(r) + h(s).$$

Proof. Since h is concave with $h(0) \ge 0$, for any constant $a \in [0,1]$ we have

$$ah(r) \le ah(r) + (1-a)h(0) \le h(ar + (1-a)0) = h(ar).$$

Then setting $a = r/(r+s)$ (so that $(1-a) = s/(r+s)$), we have

$$\begin{aligned} h(r+s) &= ah(r+s) + (1-a)h(r+s) \\ &\leq h(a(r+s)) + h((1-a)(r+s)) \\ &= h(r) + h(s). \end{aligned}$$ ∎

Lemma 1.3.2. *Let $h : [0, \infty) \to [0, \infty)$ be an increasing, concave function. Then for any $a \geq 1$ and $r \geq 1$,*

$$h(ar) \leq 2ah(r). \tag{1.4}$$

Proof. We first consider $h(nr)$ for $n \in \mathbb{N}$. Assuming $h((n-1)r) \leq (n-1)h(r)$, the concavity of h implies

$$h(nr) = h((n-1)r + r) \leq (n-1)h(r) + h(r) = nh(r).$$

Thus, $h(nr) \leq nh(r)$ for all n by induction. Then for $a \geq 1$, choose $n \in \mathbb{N}$ and $\alpha \in [0, 1)$ so that $a = n + \alpha$. The subadditivity of h implies

$$\begin{aligned} h(ar) &= h(nr + \alpha r) \\ &\leq h(nr) + h(\alpha r) \\ &\leq nh(r) + h(r) \\ &\leq 2nh(r) \\ &\leq 2ah(r). \end{aligned}$$ ∎

Lemma 1.3.3. *Let h be a pre-growth bound. Then for all $r \geq 0$,*

$$h(r) \leq cr + d,$$

where $c = h'(0)$ *and* $d = h(0)$*.*

Proof. Since h is concave, $h'(r) \leq h'(0)$ for all $r \geq 0$. Then by the mean-value theorem,

$$h(r) = \frac{h(r) - h(0)}{r} r + h(0) \leq h'(0)r + h(0). \qquad \blacksquare$$

Lemma 1.3.4. *Let* h *be a pre-growth bound. Then for any* $a \geq 1$ *and* $r \geq 1$*,*

$$h(h(r)) \leq C(h)h(r). \tag{1.5}$$

Proof. Use Lemma 1.3.3 together with the fact that h is an increasing function to conclude

$$h(h(r)) \leq h'(0)h(r) + h(0) \leq h'(0)h(r) + h(r) \leq C(h)h(r). \qquad \blacksquare$$

In Lemmas 1.3.5 and 1.3.6, we establish important properties of the relationship between a growth bound and its conjugate growth bound.

Lemma 1.3.5. *Let* h *be a pre-growth bound. Then* $\hbar$ *is also a pre-growth bound, and* $1/h$ *is a convex function.*

Proof. Since h is a pre-growth bound, it follows that $h(r) > 0$, that $h'(r) \geq 0$, and that $h''(r) \leq 0$ for all $r > 0$.

Then recalling that

$$\hbar(r) := 1 + \int_0^r \frac{1}{h(s)}\, ds,$$

we note that for all $r \geq 0$,

$$\begin{aligned} \hbar(0) &= 1 > 0, \\ \hbar'(r) &= \frac{1}{h(r)} > 0, \\ \hbar''(r) &= \frac{d}{dr}\frac{1}{h(r)} = -\frac{h'(r)}{(h(r))^2} \leq 0, \end{aligned}$$

so that $\hbar$ is a pre-growth bound.

Differentiating once more, we find that for $r > 0$,

$$\frac{d^2}{dr^2}\frac{1}{h(r)} = \frac{2(h'(r))^2 - h(r)h''(r)}{(h(r))^3} \geq 0,$$

which establishes the convexity of $1/h(r)$. ■

Lemma 1.3.6. *If h is a well-posedness growth bound and $g \leq C\hbar$ is a pre-growth bound, then, in asymptotic notation, $g(r)h(r) = O(r)$.*

Proof. Since $g(r)h(r) \leq Ch(r)\hbar(r)$, it is sufficient to show that $h(r)\hbar(r) = O(r)$.

First consider the case where $h'(r_0) = 0$ for some $r_0 \geq 0$. Since h' is a decreasing non-negative function, if $h'(r_0) = 0$, then $h(r) = 0$ for all $r \geq r_0$. Thus, $h'(r)\hbar(r) \leq h'(0)\hbar(r_0)$ for all $r \geq 0$. Hence,

$$\begin{aligned} \frac{d}{dr}\big(h(r)\hbar(r)\big) &= h'(r)\hbar(r) + h(r)\hbar'(r) \\ &= h'(r)\hbar(r) + 1 \\ &\leq h'(0)\hbar(r_0) + 1 \\ &\leq C. \end{aligned}$$

So $h(r)\hbar(r) = O(r)$.

To get this bound when $h'(r) > 0$ for all $r > 0$, first note that since h^2 is a pre-growth bound, it is concave. Then

$$\frac{d^2}{dr^2}\left(\frac{1}{2}h^2(r)\right) = h''(r)h(r) + h'(r)^2 \leq 0.$$

Rearranging the inequality yields

$$\frac{1}{h(r)} \leq -\frac{h''(r)}{h'(r)^2}.$$

Integrating both sides of the inequality gives us

$$\int_0^r \frac{1}{h(s)}\, ds \leq \int_0^r -\frac{h''(s)}{h'(s)^2}\, ds = \left.\frac{1}{h'(s)}\right|_0^r \leq \frac{1}{h'(r)}.$$

Returning once more to the derivative of $h(r)\hbar(r)$, we see that

$$\begin{aligned}\frac{d}{dr}\big(h(r)\hbar(r)\big) &= h'(r)\hbar(r) + 1 \\ &= h'(r)\left(1 + \int_0^r \frac{1}{h(s)}\, ds\right) + 1 \\ &\leq h'(r)\left(1 + \frac{1}{h'(r)}\right) + 1 \\ &\leq h'(0) + 2 \\ &\leq C.\end{aligned}$$

So $h(r)\hbar(r) = O(r)$ in this case, too. ■

Remark 1.3.7. Note that since $(h\hbar)' = h'\hbar + 1 \geq 1$, the bound in Lemma 1.3.6 is, in fact, tight.

Recalling that we seek to bound the velocity with a growth bound h and the vorticity with the conjugate growth bound g, Lemma 1.3.6 is significant in that it bounds the combined growth of the velocity and vorticity. This suggests that a more restrictive bound on the velocity would allow for a more permissive bound on the vorticity, and vice-versa.

Lemma 1.3.8. *Let h be a pre-growth bound and let u be a time-dependent vector field with associated flow map X. Let $\zeta : \mathbb{R}^2 \to \mathbb{R}$ such that $|\nabla\zeta(x)| \leq C/h(|x|)$. Then for $t_1 \leq t_2$, we have*

$$|\zeta(X(t_2, x)) - \zeta(X(t_1, x))| \leq C \int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds. \tag{1.6}$$

Proof. Observe that,

$$\begin{aligned}
|\zeta(X(t_2,x)) - \zeta(X(t_1,x))| &\leq \int_{t_1}^{t_2} |\nabla\zeta(X(s,x))||u(s,X(s,x))|\ ds \\
&\leq \int_{t_1}^{t_2} \frac{C}{h(X(s,x))}|u(s,X(s,x))|\ ds \\
&\leq C\int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds.
\end{aligned}$$

■

Corollary 1.3.9. *Let h, g be pre-growth bounds. Given a time-dependent vector field u with flow map X, we have for $t_1 \leq t_2$,*

$$\exp\left(-C\int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds\right) \leq \frac{g(X(t_2,x))}{g(X(t_1,x))} \leq \exp\left(C\int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds\right). \tag{1.7}$$

Furthermore, if $g \leq C\hbar$, then

$$|g(X(t_2,x)) - g(X(t_1,x))| \leq C\int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds, \tag{1.8}$$

And if h is a well-posedness growth bound, then

$$|h(X(t_2,x)) - h(X(t_1,x))| \leq C\int_{t_1}^{t_2} \left\|\frac{u(s)}{h}\right\|_{L^\infty} ds. \tag{1.9}$$

Proof. To obtain bound (1.7), first observe that

$$\left(\frac{g}{g'}\right)' = 1 - \frac{gg''}{(g')^2} \geq 1 \geq Ch',$$

It follows that $g/g' \geq Ch$. Thus,

$$\frac{d}{dr}\log g(r) = \frac{g'}{g} \leq \frac{C}{h}.$$

Then we can apply Lemma 1.3.8 with $\zeta(x) = \log g(x)$ to obtain

$$\log \frac{g(X(t_1,x))}{g(X(t_2,x))} \leq C \int_{t_1}^{t_2} \left\| \frac{u(s)}{h} \right\| \, ds$$

and

$$\log \frac{g(X(t_2,x))}{g(X(t_1,x))} \leq C \int_{t_1}^{t_2} \left\| \frac{u(s)}{h} \right\| \, ds.$$

Take the exponential of both sides of both these inequalities to obtain (1.7).

Next, observe that since $g \leq C\hbar$, it follows that $g' \leq C\hbar' = C/h$, so that we obtain bound (1.8) by applying Lemma 1.3.8 with $\zeta(x) = g(x)$.

Finally, inequality (1.9) is obtained by observing that, because h^2 is a growth bound, the derivative $\frac{d}{dr}h^2(r) = 2h'(r)h(r)$ is bounded. Hence, $h' \leq C/h$. Then Lemma 1.3.8 gives the desired result. ■

1.4. The Biot-Savart Kernel

In this section, we establish several important bounds relating to the Biot-Savart kernel. Generally, the results in this section were first proved elsewhere; we include the results and their proofs here for convenience. We first define the cutoff Biot-Savart Kernel, which will prove useful later. We continue with Lemmas 1.4.2 to 1.4.4, which are proved in [1], and Lemmas 1.4.5 and 1.4.6, which are proved in [4]. We conclude with Lemmas 1.4.7 and 1.4.8

Definition 1.4.1 (Cutoff Biot-Savart Kernel)**.** For any $\lambda > 0$, we define the *cutoff Biot-Savart kernel* as

$$K_\lambda = \mathbb{1}_{B_\lambda(0)} K,$$

where $\mathbb{1}_A$ is the characteristic function of a set A.

Note that $|a_\lambda K| \leq |K_\lambda|$, so that the following bound on $\|K_\lambda\|_{L^1(\mathbb{R}^2)}$ also holds for $\|a_\lambda K\|_{L^1(\mathbb{R}^2)}$.

Lemma 1.4.2. *There exists a constant $C > 0$ so that, for each $\lambda > 0$,*

$$\|K_\lambda\|_{L^1(\mathbb{R}^2)} \leq C\lambda.$$

Proof. We calculate

$$\begin{aligned}\|K_\lambda\|_{L^1(\mathbb{R}^2)} &= C\int_{B_\lambda(0)} \frac{1}{|x|}\,dx \\ &\leq C\int_0^\lambda dr \\ &= C\lambda.\end{aligned}$$

■

Lemma 1.4.3. *Let $U \subset \mathbb{R}^2$ have finite measure, and choose $p \in [1, 2)$. Then*

$$\|K\|_{L^p(U)}^p \leq \frac{2}{(4\pi)^{\frac{p}{2}}(2-p)}|U|^{1-\frac{p}{2}}.$$

Proof. Let $R = (\pi^{-1}|U|)^{1/2}$ (so that $|U| = \pi R^2$). We note that $|K(x)|$ is a radially symmetric function and decreases with increasing $|x|$. As such, the value of $\|K\|_{L^p(U)}$ is maximized when $U = B_R(0)$. Thus,

$$\begin{aligned}\|K\|_{L^p(U)}^p &\leq \|K\|_{L^p(B_R(0))}^p \\ &= \frac{1}{(2\pi)^p}\int_{B_R(0)} \frac{1}{|x|^p}\,dx \\ &= \frac{2\pi}{(2\pi)^p}\int_0^R \frac{r}{r^p}\,dr \\ &= \frac{1}{(2\pi)^{p-1}(2-p)}R^{2-p}.\end{aligned}$$

Plugging in our choice for R gives the result. ■

Lemma 1.4.4. *For any $z_1, z_2 \in \mathbb{R}^2$ and any conjugate exponents $p, q \in (1, \infty)$,*

$$|K(z_1) - K(z_2)| \leq \frac{2^{\frac{1}{p}}|z_1 - z_2|^{\frac{1}{q}}}{2\pi \min\{|z_1|, |z_2|\}^{2-\frac{1}{p}}}.$$

Proof. We first observe that for any $z_1, z_2 \in \mathbb{R}^2$, direct calculation yields

$$|K(z_1) - K(z_2)| = \frac{1}{2\pi} \frac{|z_1 - z_2|}{|z_1||z_2|}.$$

Now, to be specific, assume without loss of generality that $|z_1| \leq |z_2|$. Then it follows that $|z_1 - z_2| \leq 2|z_2|$ and that $|z_1||z_2| \geq |z_1|^{2-\frac{1}{p}}|z_2|^{\frac{1}{p}}$. Combining these observations gives us

$$|K(z_1) - K(z_2)| \leq \frac{1}{2\pi} \frac{|z_1 - z_2|}{|z_1|^{2-\frac{1}{p}}|z_2|^{\frac{1}{p}}} \leq \frac{1}{2\pi} \frac{|z_1 - z_2|}{|z_1|^{2-\frac{1}{p}}(2^{-1}|z_1 - z_2|)^{\frac{1}{p}}},$$

from which we obtain the result. ■

Lemma 1.4.5. *Let $X_1, X_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be measure-preserving homomorphisms, and let $\delta := \|X_1 - X_2\|_{L^\infty(\mathbb{R}^2)} < \infty$. Let $U \subset \mathbb{R}^2$ have finite measure. Then*

$$\|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} \leq C\sqrt{|U|}\mu\left(\frac{\delta}{\sqrt{|U|}}\right),$$

where

$$\mu(r) := \begin{cases} -r\log r, & \text{if } r < e^{-1}, \\ e^{-1}, & \text{if } r \geq e^{-1}. \end{cases} \tag{1.10}$$

Proof. For any $p, q > 1$ such that $p^{-1}+q^{-1} = 1$, we utilize Lemma 1.4.4 to compute

$$\begin{aligned}
&\|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} \\
&\quad \le C \left\| \frac{|X_1(z) - X_2(z)|^{\frac{1}{q}}}{\min\{|x - X_1(z)|, |x - X_2(z)|\}^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} \\
&\quad \le C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \left\| \frac{1}{|x - X_j(z)|^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} \\
&\quad \le C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \left\| \frac{1}{|x - y|^{2-\frac{1}{p}}} \right\|_{L^1_y(X_j(U))} \\
&\quad \le C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \|K(x - y)\|^{2-\frac{1}{p}}_{L_y^{2-\frac{1}{p}}(X_j(U))} \\
&\quad \le C\delta^{\frac{1}{q}} p|U|^{\frac{1}{2p}},
\end{aligned}$$

where we used Lemma 1.4.3 for the last inequality.

This bound is minimized when $p = -\log(\delta/\sqrt{|U|})$ so long as $\delta/\sqrt{|U|} < e^{-1}$ (which condition ensures that $p > 1$). Thus, when $\delta/\sqrt{|U|} < e^{-1}$, we obtain

$$\begin{aligned}
&\|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} \\
&\quad \le C\delta^{1+\frac{1}{\log(\delta/\sqrt{|U|})}} \left(-\log \frac{\delta}{\sqrt{|U|}} \right) |U|^{\frac{1}{-2\log(\delta/\sqrt{|U|})}} \\
&\quad \le -C\delta \left(\log \frac{\delta}{\sqrt{|U|}} \right) \left(\frac{\delta}{\sqrt{|U|}} \right)^{\frac{1}{\log(\delta/\sqrt{|U|})}} \\
&\quad \le C\sqrt{|U|}\mu \left(\frac{\delta}{\sqrt{|U|}} \right),
\end{aligned}$$

since $x^{1/ln(x)} = e$. So we obtain the result for $\delta/\sqrt{|U|} < e^{-1}$.

When $\delta/\sqrt{|U|} \ge e^{-1}$, Lemma 1.4.3 with $p = 1$ immediately gives us the desired bound. ■

Lemma 1.4.6. *Let $X_1, X_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be measure-preserving homeomorphisms and fix $x \in \mathbb{R}^2$ and $\lambda > 0$. Let $V = \operatorname{supp} a_\lambda(X_1(x) - X_1(\cdot)) \cup \operatorname{supp} a_\lambda(X_1(x) - X_2(\cdot))$*

and assume $\delta := \|X_1(\cdot) - X_2(\cdot)\|_{L^\infty(V)} < \infty$. *Then*

$$\int |(a_\lambda K(X_1(x) - X_1(y)) - a_\lambda K(X_1(x) - X_2(y)))| \, dy \le C\lambda\mu\left(\frac{\delta}{\lambda}\right),$$

where μ *is as in* (1.10).

Proof. First, we write

$$\begin{aligned}
&\int |(a_\lambda K(X_1(x) - X_1(y)) - a_\lambda K(X_1(x) - X_2(y)))| \, dy \\
&\quad \le \int |a_\lambda(X_1(x) - X_1(y))(K(X_1(x) - X_1(y)) - K(X_1(x) - X_2(y)))| \, dy \\
&\qquad + \int |(a_\lambda(X_1(x) - X_1(y)) - a_\lambda(X_1(x) - X_2(y)))K(X_1(x) - X_2(y))| \, dy \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Now let $U = \operatorname{supp} a_\lambda(X_1(x) - X_1(\cdot))$. Then noting that $|U| = \pi\lambda^2$, Lemma 1.4.5 implies

$$\begin{aligned}
I_1 \le \|K(X_1(x) - X_1(y)) - K(X_1(x) - X_2(y))\|_{L^1_y(U)} &\le C\lambda\mu\left(\frac{\delta}{\sqrt{\pi}\lambda}\right) \\
&\le C\lambda\mu\left(\frac{\delta}{\lambda}\right).
\end{aligned}$$

Next, let $W = \{X_1(y) - X_2(y) \mid y \in V\}$. Then for I_2, we use the fact that the Lipschitz constant of a_λ is $C\lambda^{-1}$ to calculate

$$\begin{aligned}
I_2 &\le \frac{C}{\lambda}\int_V |X_1(y) - X_2(y)||K(X_1(y) - X_2(y))| \, dy \\
&\le \frac{C}{\lambda}\delta \int_V |K(X_1(y) - X_2(y))| \, dy \\
&\le \frac{C}{\lambda}\delta \int_W |K(y)| \, dy \\
&\le C\delta,
\end{aligned}$$

where we used Lemma 1.4.3 together with the observation that

$$|W| \leq |V| \leq |\operatorname{supp} a_\lambda(X_1(x) - X_1(\cdot))| + |\operatorname{supp} a_\lambda(X_1(x) - X_2(\cdot))| = 2\pi\lambda^2.$$

Then observe that $\delta \leq \lambda\mu(\delta/\lambda)$ so long as $\delta \leq e^{-1}\lambda$.

On the other hand, since $|a_\lambda| \leq 1$ and is supported in V, we can again use Lemma 1.4.3 to calculate

$$I_2 \leq 2\int_V |K(X_1(y) - X_2(y))|\, dy \leq 2\int_W |K(y)|\, dy \leq C\lambda.$$

But $\mu(\delta/\lambda) = e^{-1}$ whenever $\delta > e^{-1}\lambda$. ∎

Lemma 1.4.7. *For each $n \in \mathbb{N}$ and $\lambda > 0$, there exists a constant $C > 0$, depending only on n and λ, so that*

$$|\nabla^n K(x)| = C|x|^{-(n+1)} \tag{1.11}$$

$$|\nabla^n (a_\lambda K(x))| \leq C|x|^{-(n+1)}, \tag{1.12}$$

for every $x \in \mathbb{R}^2$, where $\nabla^n = \overbrace{\nabla\nabla\cdots\nabla}^{n \text{ times}}$.

Proof. We first note that for any smooth function f,

$$\big|\nabla|f|\big| = \left|\nabla f \frac{f}{|f|}\right| = |\nabla f|.$$

Then assuming $|\nabla^{n-1}K(x)| = C|x|^{-n}$, we observe that

$$\begin{aligned} |\nabla^n K| &= \big|\nabla|\nabla^{n-1}K|\big| \\ &= |\nabla(C|x|^{-n})| \\ &= Cn|x|^{-(n+1)}. \end{aligned}$$

Since $|K| = C|x|^{-1}$, equation (1.11) holds by induction.

And since $\nabla^n a_\lambda \in C_c^\infty(\mathbb{R}^2)$, we can, of course, find a constant C dependent on n and λ so that $|\nabla^n a_\lambda(x)| \le C|x|^{-(n+1)}$. Inequality (1.12) immediately follows. ■

Lemma 1.4.8. *Let h be a well-posedness growth bound and let u, v be vector fields on $\mathbb{R}^2$. Then*

$$\left|(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u)\right| \le C \left\|\frac{u}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2 \left(H[h^2](\lambda/2) + \frac{h^2}{\lambda}\right), \tag{1.13}$$

$$\begin{aligned}&\left|(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u - v \otimes v)\right| \\ &\qquad \le C \left(\left\|\frac{u}{h}\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\frac{v}{h}\right\|_{L^\infty(\mathbb{R}^2)}\right) \left\|\frac{u-v}{h}\right\|_{L^\infty(\mathbb{R}^2)} \left(H[h^2](\lambda/2) + \frac{h^2}{\lambda}\right)\end{aligned} \tag{1.14}$$

Proof. To obtain bound (1.13), we use Lemma 1.4.7 to calculate

$$\begin{aligned}\left|\big((\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u)\big)(x)\right| &\le C \int_{B_{\lambda/2}(x)} \frac{1}{|x-y|^3} |u(y)|^2 \, dy \\ &\le C \left\|\frac{u}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2 \int_{B_{\lambda/2}(x)} \frac{h^2(y)}{|x-y|^3} \, dy.\end{aligned}$$

Then utilizing the subadditivity of h^2, we have

$$\begin{aligned}\int_{B_{\lambda/2}(x)} \frac{h^2(y)}{|x-y|^3} \, dy &\le \int_{B_{\lambda/2}(x)} \frac{h^2(x-y)}{|x-y|^3} \, dy + h^2(x) \int_{B_{\lambda/2}(x)} \frac{1}{|x-y|^3} \, dy \\ &\le C \int_{\lambda/2}^\infty \frac{h^2(s)}{s^2} \, ds + Ch^2(x) \int_{\lambda/2}^\infty \frac{1}{s^2} \, ds \\ &= CH[h^2](\lambda/2) + C\frac{h^2(x)}{\lambda/2},\end{aligned}$$

from which (1.13) follows.

To obtain bound (1.14), we first note that

$$
\begin{aligned}
|(u \otimes u - v \otimes v)(y)| &= |(u \otimes (u-v) - (u-v) \otimes v)(y)| \\
&\le |u(y)||(u-v)(y)| + |(u-v)(y)||v(y)| \\
&\le \left(\left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \frac{v}{h} \right\|_{L^\infty(\mathbb{R}^2)} \right) \left\| \frac{u-v}{h} \right\|_{L^\infty(\mathbb{R}^2)} h^2(y),
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left| (\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u - v \otimes v) \right| \\
&\qquad \le C \left(\left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \frac{v}{h} \right\|_{L^\infty(\mathbb{R}^2)} \right) \left\| \frac{u-v}{h} \right\|_{L^\infty(\mathbb{R}^2)} \int_{B_{\lambda/2}(x)} \frac{h^2(y)}{|x-y|^3}\, dy.
\end{aligned}
$$

The bound on $\int_{B_{\lambda/2}(x)} h^2(y)|x-y|^{-3}\, dy$ above then yields (1.14). ■

1.5. Locally Log-Lipschitz Velocity Fields

In [1], the authors obtain a bound on the modulus of continuity of a bounded velocity field with bounded vorticity. The authors of [4] adapt the technique to obtain a similar bound in the case when $u \in S_{h,1}$, where h is a pre-growth bound. In Proposition 1.5.2, we tweak the argument to show that all $u \in S_{h,g}$ are locally log-Lipschitz for any pre-growth bounds h, g.

First, though, we establish a simple result that we will allow us to bound the stream function of a velocity field.

Lemma 1.5.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\psi(z) = 0$ for some fixed $z \in \mathbb{R}^2$. Set $u = \nabla^\perp \psi$. Then for any pre-growth bound h and any $x \in \mathbb{R}^2$,*

$$
|\psi(x)| \le |x-z| h(|x| + |z|) \left\| \frac{u}{h} \right\|_{L^\infty}.
$$

Proof. Let $\gamma(\alpha) = \alpha x + (1-\alpha)z$, with $0 \le \alpha \le 1$. Then

$$\begin{aligned}
|\psi(x)| &= \left| \int_\gamma \nabla\psi \cdot ds \right| \\
&= \left| \int_0^1 u^\perp(\alpha x + (1-\alpha)z) \cdot (x-z) d\alpha \right| \\
&\le \int_0^1 \left| \frac{u(\alpha x + (1-\alpha)z)}{h(|\alpha x + (1-\alpha)z|)} \right| |x-z| h(|\alpha x + (1-\alpha)z|)\, d\alpha \\
&\le |x-z| h(|x|+|z|) \left\| \frac{u}{h} \right\|_{L^\infty}. \qquad \blacksquare
\end{aligned}$$

Proposition 1.5.2. *Let h, g be pre-growth bounds and let $u \in S_{h,g}$. For all $x, y \in \mathbb{R}^2$ such that $|y| \le C(1+|x|)$ for some $C > 0$, we have*

$$|u(x+y) - u(x)| \le C\|u\|_{S_{h,g}} g(x)h(x)\mu\left(\frac{|y|}{h(x)}\right),$$

where μ is as in (1.10).

Proof. Fix $x \in \mathbb{R}^2$ and choose a stream function ψ of u so that $\psi(x) = 0$. Define $\overline{u} = \nabla^\perp(a_{2R}(x-\cdot)\psi(\cdot))$ for arbitrary $R > 0$, and let $\overline{\omega} = \nabla \times \overline{u}$. Then $u(x+y) = \overline{u}(x+y)$ whenever $|y| \le R$. Thus, for $|y| \le R$ and $p > 2$, Morrey's inequality implies

$$|u(x+y) - u(x)| = |\overline{u}(x+y) - \overline{u}(x)| \le C\left\|\nabla\overline{u}\right\|_{L^p(\mathbb{R}^2)} |y|^{1-\frac{2}{p}}. \tag{1.15}$$

Now note that since $\overline{\omega}$ is compactly supported, $\overline{u} = K * \overline{\omega}$. Then applying the

Calderon-Zygmund inequality to (1.15) yields

$$\begin{aligned}|u(x+y)-u(y)| &\leq Cp\|\overline{\omega}\|_{L^p(\mathbb{R}^2)}|y|^{1-\frac{2}{p}}\\ &\leq Cp\|\overline{\omega}\|_{L^p(B_{2R}(x))}|y|^{1-\frac{2}{p}}\\ &\leq C\left\|\frac{\overline{\omega}}{g}\right\|_{L^\infty(\mathbb{R}^2)} g(|x|+2R)pR^{2/p}|y|^{1-\frac{2}{p}}\\ &\leq C|y|\left\|\frac{\overline{\omega}}{g}\right\|_{L^\infty(\mathbb{R}^2)} g(|x|+2R)p|R^{-1}y|^{-\frac{2}{p}}\end{aligned}$$

for all $p > 2$.

It is easy to show the infimum of $p|R^{-1}y|^{-\frac{2}{p}}$ over all p occurs when $p = -2\log(|R^{-1}y|)$, which satisfies $p > 2$ so long as $|y| < R/e$. Thus, as long as $R^{-1}|y| < e^{-1}$, we have

$$|u(x+y)-u(y)| \leq -C|y|\left\|\frac{\overline{\omega}}{g}\right\|_{L^\infty} g(|x|+2R)\log(|R^{-1}y|).$$

We also observe that for $|x-\xi| \leq 2R$ we have

$$\begin{aligned}\left|\frac{\overline{\omega}(\xi)}{g(\xi)}\right| &= \left|\frac{\Delta(a_{2R}(x-\xi)\psi(\xi))}{g(\xi)}\right|\\ &\leq \frac{|\Delta a_{2R}(x-\xi)||\psi(\xi)| + 2|\nabla a_{2R}(x-\xi)||\nabla\psi(\xi)| + |a_{2R}(x-\xi)||\Delta\psi(\xi)|}{g(\xi)}\\ &\leq \frac{CR^{-2}|x-\xi|h(|x|+|\xi|)\left\|\frac{u}{h}\right\|_{L^\infty} + CR^{-1}h(\xi)\left\|\frac{u}{h}\right\|_{L^\infty} + |\omega(\xi)|}{g(\xi)}\\ &\leq \frac{CR^{-1}h(2|x|+2R)\left\|\frac{u}{h}\right\|_{L^\infty} + CR^{-1}h(|x|+2R)\left\|\frac{u}{h}\right\|_{L^\infty} + |\omega(\xi)|}{g(\xi)}\\ &\leq CR^{-1}h(2|x|+2R)\left\|\frac{u}{h}\right\|_{L^\infty} + \left\|\frac{\omega}{g}\right\|_{L^\infty},\end{aligned}$$

where we used Lemma 1.5.1 to bound $|\psi(\xi)|$. So

$$\begin{aligned}&|u(x+y)-u(y)|\\ &\quad\leq -C|y|\left(CR^{-1}h(2|x|+2R)\left\|\frac{u}{h}\right\|_{L^\infty} + \left\|\frac{\omega}{g}\right\|_{L^\infty}\right) g(|x|+2R)\log(|R^{-1}y|).\end{aligned}$$

Now choose $R = h(x)$. We note that for any pre-growth bound ζ and any constants $a, b \geq 1$, we have

$$\begin{aligned}
\zeta(a|x| + bh(x)) &\leq \zeta(a|x| + b(c|x| + d)) \\
&= \zeta((a + bc)|x| + bd) \\
&\leq C\zeta(x) + \zeta(bd) \\
&\leq C\zeta(x).
\end{aligned}$$

Thus, as long as $|y| < e^{-1}h(x)$,

$$|u(x + y) - u(y)| \leq C\|u\|_{S_{h,g}} g(x)h(x)\mu\left(\frac{|y|}{h(x)}\right),$$

which establishes the bound for this case.

Now assume $e^{-1}h(x) \leq |y| \leq C(1+|x|)$. Observe that this bound on $|y|$ implies both that $\mu(|y|/h(x)) = e^{-1}$ and that $h(x + y) \leq Ch(x)$. Then

$$\begin{aligned}
|u(x + y) - u(x)| &\leq \frac{|u(x + y)|}{h(x + y)} h(x + y) + \frac{|u(x)|}{h(x)} h(x) \\
&\leq \left\|\frac{u}{h}\right\|_{L^\infty} (h(x + y) + h(x)) \\
&\leq C\left\|\frac{u}{h}\right\|_{L^\infty} h(x) \\
&\leq C\left\|\frac{u}{h}\right\|_{L^\infty} h(x)\mu\left(\frac{|y|}{h(x)}\right),
\end{aligned}$$

Since this bound when $|y| \geq e^{-1}h(x)$ is stronger than the bound desired, the proof is complete. ■

1.6. Flow Map Bounds and Properties

The following lemma represents a slight adjustment to lemma 4.2 in [4] to allow for simpler notation.

Lemma 1.6.1. *Let g, h be pre-growth bound and let $u_1, u_2 \in L^\infty([0,T], S_{h,g})$ with associated flow maps X_1, X_2. Set $M_j = \|u_j\|_{L^\infty([0,T],S_{h,g})}$. Then assuming for concreteness that $M_1 \le M_2$, we have for each $x \in \mathbb{R}^2$ and $t \in [0,T]$,*

$$\frac{|X_j(t,x) - x|}{h(x)} \le M_j t e^{M_j t}, \qquad j = 1, 2, \tag{1.16}$$

$$\frac{|X_1(t,x) - X_2(t,x)|}{h(x)} \le C M_2 t e^{M_2 t}, \tag{1.17}$$

$$\frac{|X_j^{-t}(x) - x|}{h(x)} \le C M_j t e^{M_j t}, \qquad j = 1, 2, \tag{1.18}$$

$$\frac{|X_1^{-t}(x) - X_2^{-t}(x)|}{h(x)} \le C M_2 t e^{M_2 t}, \tag{1.19}$$

where C is independent of n.

Proof. We use the observation that $|u_j(x)| \le M_j h(x)$ together with (1.7) to calculate

$$\begin{aligned}
|X_j(t,x) - x| &\le \int_0^t |u_j(s, X_j(s,x))| \, ds \\
&\le M_j \int_0^t |h(X_j(s,x))| \, ds \\
&\le M_j e^{M_j t} \int_0^t |h(x)| \, ds \\
&\le M_j t e^{M_j t} h(x),
\end{aligned}$$

which proves (1.16).

Then observing that

$$|X_1(t,x) - X_2(t,x)| \le |X_1(t,x) - x| + |X_2(t,x) - x|$$

immedately yields (1.17).

To prove (1.18) and (1.19), observe $x = X_j(t, X_j^{-t}(x))$, then proceed to bound $|X_j^{-t}(x) - x|$ as above. ∎

The following result is analogous to lemma 8.2 in [12]; the arguments are similar, though adjustments had to be made to account for the particulars of our problem.

Lemma 1.6.2. *Let* $u \in L^\infty([0,T]; S_{h,g})$ *with associated flow map* X*, where* $T > 0$ *and* h, g *are pre-growth bounds. Then for any* $x_1, x_2 \in \mathbb{R}^2$*, there is a non-negative function* $\beta(x)$*, dependent on* $\|u\|_{L^\infty([0,T];S_{h,g})}$ *and* T *and which increases as* $|x|$ *increases, so that whenever* $|x_1 - x_2| < e^{-\exp(\beta(x_1))}$*, we have*

$$|X(t, x_1) - X(t, x_2)| \leq |x_1 - x_2|^{\exp(-\beta(x_1))}, \tag{1.20}$$

$$|X^{-t}(x_1) - X^{-t}(x_2)| \leq |x_1 - x_2|^{\exp(-\beta(x_1))}, \tag{1.21}$$

for all $t \in [0, T]$.

Similarly, for each $t_1, t_2 \in [0, T]$*, there exists a constant* C *dependent only on* $\|u\|_{L^\infty([0,T];S_{h,g})}$*, so that, given any* $x \in \mathbb{R}^2$*, we have*

$$|X(t_1, x) - X(t_2, x)| \leq (Ch(x)|t_1 - t_2|)^{\exp(-\beta(x))}\,, \tag{1.22}$$

$$|X^{-t_1}(x) - X^{-t_2}(x)| \leq (Ch(x)|t_1 - t_2|)^{\exp(-\beta(x))} \tag{1.23}$$

whenever $|t_1 - t_2| < (Ch(x))^{-1} e^{-\exp(\beta(x_1))}$.

Proof. We prove only estimates (1.21) and (1.23) for the inverse flow maps. Estimates (1.20) and (1.22) for the flow maps in the forward direction are proved similarly.

We begin by proving (1.21). Start by setting

$$Y_t(\tau, x) = X(t - \tau, X^{-t}(x)),$$

where $\tau \in [0,t]$. Note $Y_t(0,x) = x$ and $Y_t(t,x) = X^{-t}(x)$. Also observe that

$$\begin{aligned}\frac{d}{d\tau}Y_t(\tau,x) &= \frac{d}{d\tau}X(t-\tau, X^{-t}(x)) \\ &= -u(t-\tau, X(t-\tau, X^{-t}(x))) \\ &= -u(t-\tau, Y_t(\tau,x)),\end{aligned}$$

so that

$$Y_t(\tau,x) = x - \int_0^\tau u(t-s, Y_t(s,x))\ ds.$$

Then setting $\rho(\tau) = |Y_t(\tau,x_1) - Y_t(\tau,x_2)|$, we have

$$\rho(\tau) \leq |x_1 - x_2| + \int_0^\tau |u(t-s, Y_t(s,x_1)) - u(t-s, Y_t(s,x_2))|\ ds.$$

But by Proposition 1.5.2, we have

$$\begin{aligned}&|u(t-s, Y_t(s,x_1)) - u(t-s, Y_t(s,x_2))| \\ &\qquad \leq C\|u\|_{L^\infty([0,T];S_{h,g})} g(Y_t(s,x_1))h(Y_t(s,x_1))\mu\left(\frac{\rho(s)}{h(Y_t(s,x_1))}\right) \\ &\qquad \leq C\|u\|_{L^\infty([0,T];S_{h,g})} g(Y_t(s,x_1))h(Y_t(s,x_1))\mu\left(\frac{\rho(s)}{h(0)}\right) \\ &\qquad \leq C\|u\|_{L^\infty([0,T];S_{h,g})} g(Y_t(s,x_1))h(Y_t(s,x_1))\mu(\rho(s)),\end{aligned}$$

where we used Lemma 1.3.2 in the last inequality. Thus, by (1.7),

$$\rho(\tau) \leq \rho(0) + C(\|u\|_{L^\infty([0,T];S_h)})g(x_1)h(x_1)\int_0^\tau \mu\left(\rho(s)\right)\ ds.$$

Now we apply Osgood's lemma to the inequality. (A statement and short proof of Osgood's lemma can be found, for example, in [10].) Doing so,we obtain

$$\begin{aligned}\int_{\rho(0)}^{\rho(\tau)} \frac{ds}{\mu(s)} &\leq C(\|u(s)\|_{L^\infty([0,T];S_h)})g(x_1)h(x_1)\tau \\ &\leq C(\|u(s)\|_{L^\infty([0,T];S_h)})g(x_1)h(x_1)T \\ &=: \beta(x_1).\end{aligned} \tag{1.24}$$

Recall that $\mu(r) = -r\log r$ when $r < e^{-1}$ and observe that

$$\int_{\alpha}^{e^{-1}} \frac{ds}{-s\log(s)} > \beta(x_1)$$

whenever $0 \leq \alpha < e^{-\exp(\beta(x_1))}$. Therefore, as long as $\rho(0) < e^{-\exp(\beta(x_1))}$, it follows that $\rho(\tau) < e^{-1}$, in which case

$$\begin{aligned}\int_{\rho(0)}^{\rho(\tau)} \frac{ds}{\mu(s)} &= \int_{\rho(0)}^{\rho(\tau)} \frac{ds}{-s\log s} \\ &= -\log|\log s|\Big|_{\rho(0)}^{\rho(\tau)}.\end{aligned}$$

Substituting this into (1.24) and solving for $\rho(\tau)$, we calculate

$$\begin{aligned}\log\left|\frac{\log\rho(0)}{\log\rho(\tau)}\right| &\leq \beta(x_1), \\ \log\left|\frac{\log\rho(\tau)}{\log\rho(0)}\right| &\geq -\beta(x_1), \\ \frac{\log\rho(\tau)}{\log\rho(0)} &\geq e^{-\beta(x_1)}, \\ \log\rho(\tau) &\leq e^{-\beta(x_1)}\log\rho(0).\end{aligned}$$

Thus, we obtain

$$\rho(\tau) \leq \rho(0)^{\exp(-\beta(x_1))}$$

for all $\tau \in [0,t]$. Setting $\tau = t$ yields (1.21).

Next, we prove (1.23). By (1.21) we have

$$\begin{aligned}|X^{-t_2}(x) - X^{-t_1}(x)| &= |X^{-t_1}(X(t_1, X^{-t_2}(x))) - X^{-t_1}(x)| \\ &\le |X(t_1, X^{-t_2}(x)) - x|^{\exp(-\beta(x))}\end{aligned}$$

provided that $|X(t_1, X^{-t_2}(x)) - x| < e^{-\exp(\beta(x))}$. But

$$\begin{aligned}|X(t_1, X^{-t_2}(x)) - x| &= |X(t_1, X^{-t_2}(x)) - X(t_2, X^{-t_2}(x))| \\ &= \left|\int_{t_1}^{t_2} u(s, X(s, X^{-t_2}(x)))\, ds\right| \\ &\le \|u\|_{L^\infty([0,T];S_{h,g})} \left|\int_{t_1}^{t_2} h(X(s, X^{-t_2}(x)))\, ds\right| \\ &\le Ch(x)|t_1 - t_2|,\end{aligned}$$

where we once again used inequality (1.7), and where constant C depends only on $\|u\|_{L^\infty([0,T];S_{h,g})}$. But this implies that

$$|X^{-t_2}(x) - X^{-t_1}(x)| \le (Ch(x)|t_1 - t_2|)^{\exp(-\beta(x))}$$

so long as $|t_1 - t_2| \le (Ch(x))^{-1}e^{-\exp(\beta(x))}$, as desired. ■

Lemma 1.6.3. *Let $u_1, u_2 \in L^\infty([0,T], S_{h,g})$ with associated flow maps X_1, X_2, where h, g are pre-growth bounds and $T > 0$. Fix $x \in \mathbb{R}^2$ and $t \in [0,T]$. Let $V(t) = U_1(t) \cup U_2(t)$, where $U_j(t) = \{y : |X_1(t,x) - X_j(t,y)| < h(x)\}$. Then*

$$\|X_1(t,y) - X_2(t,y)\|_{L^\infty_y(V(t))} \le Ch(x) \left\|\frac{X_1(t,y) - X_2(t,y)}{h(y)}\right\|_{L^\infty_y(\mathbb{R}^2)},$$

where C depends on T, $\|u_1\|_{L^\infty([0,T],S_{h,g})}$, and $\|u_2\|_{L^\infty([0,T],S_{h,g})}$.

Proof. We first note that

$$\begin{aligned}
\|X_1(t,y) - X_2(t,y)\|_{L^\infty_y(V(t))} &= \left\| \frac{X_1(t,y) - X_2(t,y)}{h(y)} h(y) \right\|_{L^\infty_y(V(t))} \\
&\le \left\| \frac{X_1(t,y) - X_2(t,y)}{h(y)} \right\|_{L^\infty_y(\mathbb{R}^2)} \|h\|_{L^\infty(V(t))} \\
&\le \left\| \frac{X_1(y) - X_2(y)}{h(y)} \right\|_{L^\infty_y(\mathbb{R}^2)} \sum_{j=1}^{2} \|h\|_{L^\infty(U_j(t))}.
\end{aligned}$$

But note that $|X_j(t,y)| \le h(x) + |X_1(t,x)|$ for all $y \in U_j$. Thus, by (1.7) and (1.16) and Lemmas 1.3.1, 1.3.2 and 1.3.4, for all $y \in U_j$,

$$\begin{aligned}
h(y) &\le Ch(X_j(t,y)) \\
&\le Ch(h(x) + |X_1(t,x)|) \\
&\le Ch(h(x) + |x| + Ch(x)) \\
&\le Ch(x),
\end{aligned}$$

which completes the proof. ∎

Proposition 1.6.4. *Let h be a well-posedness growth bound and let u be a time-dependent vector field with associated flow map X. Then for any $x, y \in \mathbb{R}^2$,*

$$|x - X^{-t}(y)| \le |x - y| + Ch(y) \left(1 + \int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds \right)^2.$$

Proof. Using (1.9), we calculate

$$
\begin{aligned}
|x - X^{-t}(y)| &\leq |x-y| + |y - X^{-t}(y)| \\
&= |x-y| + |X(t, X^{-t}(y)) - X^{-t}(y)| \\
&\leq |x-y| + \int_0^t |u(s, X(s, X^{-t}(y)))| \; dy \\
&\leq |x-y| + \int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} h(X(s, X^{-t}(y))) \; dy \\
&\leq |x-y| + \int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} \left(h(y) + C \int_s^t \left\| \frac{u(\sigma)}{h} \right\|_{L^\infty(\mathbb{R}^2)} d\sigma \right) ds \\
&\leq |x-y| + Ch(y) \left(\int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds + \left(\int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds \right)^2 \right) \\
&\leq |x-y| + Ch(y) \left(1 + \int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds \right)^2 .
\end{aligned}
$$

■

2. Existence

Our proof for existence follows the same general outline as the existence proofs in [1] and [4], while bringing in elements of [12]:

Construct a Sequence of Approximating Solutions. Given satisfactory initial data, we construct a sequence of smooth, compactly supported functions which approximates the initial data and for which a classical solution to the Euler equations may be obtained. Our goal is to show that the resulting sequence of solutions to the approximate initial data converges in some sense to a solution satisfying the original initial data.

Obtain a Uniform Bound on the Sequence of Solutions. Once we construct a sequence of approximating solutions, we will show the sequence is bounded uniformly in an appropriate sense. As we shall see, this bound is essential to showing convergence of our sequence. This step is also the one in which [1], [4], [12], and our approach here differ most significantly from one another. It is this step that is primarily responsible for the conditions we are required to impose on initial data to guarantee the existence of a solution.

Show a Subsequence of Flow Maps Converges. We find a subsequence of approximating flow maps that converges locally uniformly. We take the limit as the flow map of our candidate solution and show it is measure-preserving.

Show the Sequence of Vorticities Converges We use the limit flow map to define a candidate vorticity function by transporting the initial vorticity with the flow. We then show that a subsequence of approximating vorticities in fact converges to this candidate vorticity in an appropriate sense.

Show the Sequence of Velocities Converges. We use Serfati's identity to show that a subsequence of approximating velocities is Cauchy in an appropriate Banach space. We take the limit of this subsequence as the velocity field of our candidate solution.

Prove the Limit is a Solution. Once we have obtained a candidate solution, we confirm that it actually is a solution to the Euler equations in the sense of Definition 1.2.4: we show the limit velocity is in $L^\infty([0,T]; S_{h,g})$ (where h, g are suitable pre-growth bounds), that the limit flow map actually is the flow map of the limit velocity, and that the candidate solution satisfies Serfati's identity.

2.1. Constructing an Approximating Sequence

Definition 2.1.1 (Sequence of Approximating Initial Data)**.** Let $u_0 \in S_{h,g}$, where h, g are pre-growth bounds. Let ψ_0 be the stream function associated with u_0 so that $\psi_0(0) = 0$, and let $\omega_0 = \nabla \times u_0 = \Delta\psi_0$. Set

$$\begin{aligned} \psi_n^0 &= a_n(\psi_0 * \nu_n), \\ u_n^0 &= \nabla^\perp \psi_n^0, \\ \omega_n^0 &= \Delta\psi_n^0 = \nabla \times u_n^0, \end{aligned} \tag{2.1}$$

where

$$\nu_n(x) := n^2\nu(nx)$$

for some mollifier ν.

We say (u_n^0) is the *sequence of approximating initial data generated by* u_0.

Lemma 2.1.2. *Let* $\psi : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\psi(0) = 0$. *Let* $u = \nabla^\perp \psi$ *and* $\omega = \Delta\psi = \nabla \times u$. *Let* h, g *be pre-growth bounds. Then for* $|x| \le n$,

$$|(\psi * \nu_n)(x)| \le C \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} nh(x), \tag{2.2}$$

$$|(u * \nu_n)(x)| \le C \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} h(x), \tag{2.3}$$

$$|(\omega * \nu_n)(x)| \le C \left\| \frac{\omega}{g} \right\|_{L^\infty(\mathbb{R}^2)} g(x), \tag{2.4}$$

where C *is independent of* x, n.

Proof. For $|x| \le n$, we use the fact that ν has support within some ball $B_R(0)$ to calculate

$$\begin{aligned}
|u * \nu_n(x)| &\le \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} h(x-y)\nu_n(y)\ dy \\
&\le \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} h(|x| + R/n) \|\nu_n\|_{L^1(\mathbb{R}^2)} \\
&\le \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} (h(x) + h(R)) \\
&\le \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} (h(x) + Ch(0)) \\
&\le \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} (h(x) + Ch(x)) \\
&\le C \left\| \frac{u}{h} \right\|_{L^\infty(\mathbb{R}^2)} h(x).
\end{aligned}$$

We calculate (2.4) analogously.

And by Lemma 1.5.1, we have for $|x| \le n$,

$$\begin{aligned}
|\psi_0 * \nu_n(x)| &\le \left\| \frac{u_0}{h} \right\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |x-y| h(x-y)\nu_n(y)\ dy \\
&\le \left\| \frac{u_0}{h} \right\|_{L^\infty(\mathbb{R}^2)} (|x| + R/n) h(|x| + R/n) \|\nu_n\|_{L^1(\mathbb{R}^2)} \\
&\le C \left\| \frac{u_0}{h} \right\|_{L^\infty(\mathbb{R}^2)} nh(x).
\end{aligned}$$

■

Proposition 2.1.3. *Let $u_0 \in S_{h,g}$, where h, g are pre-growth bounds. Let (u_n^0) be the sequence of approximating initial data generated by u_0. Then for each n,*

$$\|u_n^0\|_{S_{h,g}} \leq C,$$

where C depends only on $\|u_0\|_{S_{h,g}}$.

Proof. Observing that u_n^0 is supported in $\overline{B_n(0)}$, we calculate

$$\begin{aligned}
\left\|\frac{u_n^0}{h}\right\|_{L^\infty(\mathbb{R}^2)} &= \left\|\frac{u_n^0}{h}\right\|_{L^\infty(B_n(0))} \\
&= \left\|\frac{\nabla^\perp a_n(\psi_0 * \nu_n) + a_n(u_0 * \nu_n)}{h}\right\|_{L^\infty(B_n(0))} \\
&\leq C\left\|\frac{\psi_0 * \nu_n}{nh}\right\|_{L^\infty(B_n(0))} + \left\|\frac{u_0 * \nu_n}{h}\right\|_{L^\infty(B_n(0))}.
\end{aligned}$$

Thus, by (2.2) and (2.3),

$$\left\|\frac{u_n^0}{h}\right\|_{L^\infty(\mathbb{R}^2)} \leq C\left\|\frac{u_0}{h}\right\|_{L^\infty(\mathbb{R}^2)}.$$

Similarly,

$$\begin{aligned}
\left\|\frac{\omega_n^0}{g}\right\|_{L^\infty(\mathbb{R}^2)} &= \left\|\frac{\omega_n^0}{g}\right\|_{L^\infty(B_n(0))} \\
&= \left\|\frac{\Delta a_n(\psi_0 * \nu_n) + 2\nabla^\perp a_n \cdot (u_0 * \nu_n) + a_n(\omega_0 * \nu_n)}{g}\right\|_{L^\infty(B_n(0))} \\
&\leq C\left\|\frac{\psi_0 * \nu_n}{n^2 g}\right\|_{L^\infty(B_n(0))} + C\left\|\frac{u_0 * \nu_n}{ng}\right\|_{L^\infty(B_n(0)} + \left\|\frac{\omega_0 * \nu_n}{g}\right\|_{L^\infty(B_n(0))}.
\end{aligned}$$

So Lemmas 1.3.3 and 2.1.2 yield

$$\left\|\frac{\omega_n^0}{g}\right\|_{L^\infty(\mathbb{R}^2)} \leq C\|u_0\|_{S_{h,g}}.$$

■

Definition 2.1.4 (Approximating Sequence)**.** Let $u_0 \in S_{h,g}$, where h, g are pre-growth bounds, and let (u_n^0) be the sequence of approximating initial data generated by u_0. Given any smooth, compactly supported initial velocity, it is well-known that there exists a unique classical solution to the Euler equations; see, for example, [14]. In particular, observe that, for each n, there exists a solution u_n to the Euler equations with initial data u_n^0. We call (u_n) the *approximating sequence generated by* u_0.

2.2. Obtaining a Uniform Bound on the Approximating Sequence

To prove convergence, we need to establish a uniform bound for the approximating sequence generated by given initial data. The following lemma from [4] will prove essential to establishing a uniform bound on the approximating sequence.

Lemma 2.2.1. *Assume* $\Lambda : [0, \infty) \to [0, \infty)$ *satisfies*

$$\Lambda(t) \leq \Lambda_0 + \eta \left(\int_0^t \Lambda(s)\ ds \right)$$

for some $\Lambda_0 \geq 0$*, where* $\eta : [0, \infty) \to [0, \infty)$ *is a continuous, non-decreasing convex function with* $\eta(0) = 0$*, and* $C(t) \geq 0$*. Then for all* $t \in [0, 1]$*,*

$$\int_{\Lambda_0}^{\Lambda(t)} \frac{ds}{\eta(s)} \leq t, \tag{2.5}$$

and for a fixed $T > 1$*, we have*

$$\int_{\Lambda_0}^{\Lambda(t)} \frac{ds}{\eta(Ts)} \leq \frac{t}{T} \tag{2.6}$$

for all $t \in [0, T]$*.*

Proof. Utilizing the convexity of η, we can apply Jensen's inequality to obtain

$$\Lambda(t) \leq \Lambda_0 + \eta\left(\frac{1}{t}\int_0^t t\Lambda(s)\ ds\right) \leq \Lambda_0 + \frac{1}{t}\int_0^t \eta(t\Lambda(s))\ ds.$$

Observe for any $a \in [0,1]$ and $r \geq 0$,

$$\eta(ar) = \eta(ar + (1-a)0) \leq a\eta(r) + (1-a)\eta(0) = a\eta(r).$$

Then for all $t \in [0,1]$ we have

$$\Lambda(t) \leq \Lambda_0 + \int_0^t \eta(\Lambda(s))\ ds.$$

We immediately obtain (2.5) by applying Osgood's Lemma (see Lemma A.1.1).

Now for $T > 1$, we have for all $t \leq T$

$$\eta(t\Lambda(s)) = \eta\left(\frac{t}{T}T\Lambda(s)\right) \leq \frac{t}{T}\eta(T\Lambda(s)).$$

Then

$$\Lambda(t) \leq \Lambda_0 + \frac{t}{T}\int_0^t \eta(T\Lambda(s))\ ds.$$

Applying Osgood's Lemma here yields (2.6). ■

We are now prepared to establish a uniform bound on the approximating sequence.

Proposition 2.2.2. *Let $u_0 \in S_{h,g}$ for some well-posedness growth bound h and pre-growth bound g. Let (u_n) be the approximating sequence of velocities generated by u_0, with corresponding vorticities (ω_n). If there exists a continuous, non-decreasing convex function $\eta_0 : [0,\infty) \to [0,\infty)$ with $\eta_0(0) = 0$ such that*

$$\frac{1}{h(x)}|K_{h(x)}| * |\omega_n(t) - \omega_n^0|(x) \leq C + \sqrt{\eta_0\left(\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2 ds\right)} \qquad (2.7)$$

for all n, $x \in \mathbb{R}^2$, and $t \geq 0$ sufficiently small, then there exists a $T > 0$ so that the sequence (u_n) is bounded uniformly in $L^\infty([0,T], S_{h,g})$.

Proof. It is shown in [1] that Serfati's identity holds for a classical weak solution to the Euler equations with compactly supported initial data. Then dividing Serfati's identity by $h(x)$, each solution in our approximating sequence satisfies the inequality

$$\begin{aligned}\left|\frac{u_n(t,x)}{h(x)}\right| \leq \left|\frac{u_n^0(x)}{h(x)}\right| + \frac{1}{h(x)}|(a_\lambda K) * (\omega_n(t) - \omega_n^0)(x)| \\ + \frac{1}{h(x)}\int_0^t \left|\left(\nabla\nabla^\perp[(1-a_\lambda)K]\right) *\cdot (u_n \otimes u_n)(s,x)\right| \, ds.\end{aligned}$$

Recall that by Lemma 1.4.8, we have the bound

$$\begin{aligned}\frac{1}{h(x)}\left|\left(\nabla\nabla^\perp[(1-a_\lambda)K]\right) *\cdot (u_n \otimes u_n)(s,x)\right| \\ \leq C\left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 \left[\frac{H(\lambda/2)}{h(x)} + C\frac{h(x)}{\lambda}\right], \quad (2.8)\end{aligned}$$

where $H = H[h^2]$. We desire (2.8) to be a uniform bound over x; therefore, we must choose $\lambda(x)$ so that $h(x) = O(\lambda(x))$ to control the term. But we wish to similarly bound

$$\frac{1}{h(x)}\left|(a_\lambda K) * (\omega_n(t) - \omega_n^0)(x)\right| \leq \frac{1}{h(x)}|K_\lambda| * |\omega_n(t) - \omega_n^0|(x),$$

where control of the term tends to improve with smaller order λ. As such, we choose $\lambda(x) = h(x)$ in an attempt to satisfactorily control both terms.

Thus, using assumption (2.7), we have for each n,

$$\left|\frac{u_n(t,x)}{h(x)}\right| \leq \left|\frac{u_n^0(x)}{h(x)}\right| + C + \sqrt{\eta_0\left(\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2 ds\right)}$$
$$+ C\left[1 + \frac{H(h(x)/2)}{h(x)}\right] \int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2 ds.$$

But by Proposition 2.1.3, we can bound $\|u_n^0\|_{S_{h,g}}$ independent of n. And since H is a decreasing function and h is a well-posedness growth bound,

$$H(h(x)/2) \leq H(h(0)/2) < \infty.$$

Therefore

$$\left|\frac{u_n(t,x)}{h(x)}\right| \leq C + \left(\sqrt{\eta_0\left(\int_0^t \Lambda_n(s)\ ds\right)} + C\int_0^t \Lambda_n(s)\ ds\right),$$

where

$$\Lambda(s) = \left\|\frac{u_n^0(s)}{h}\right\|_{L^\infty(\mathbb{R}^2)}^2.$$

Taking the supremum over x and squaring both sides of the inequality yields

$$\Lambda_n(t) \leq C + \left(\sqrt{\eta_0\left(\int_0^t \Lambda_n(s)\ ds\right)} + C\int_0^t \Lambda_n(s)\ ds\right)^2, \tag{2.9}$$

Then applying Lemma 2.2.1 and (2.9) with $\eta(r) = (\sqrt{\eta_0(r)} + Cr)^2$ immediately gives us a uniform bound on Λ_n for at least finite time. To see why this is so for $T < 1$, set

$$D = \int_C^\infty \frac{ds}{\eta(s)}.$$

Choose $T \in (0,1)$ so that $T < D$ and (2.7) holds for all $t \leq T$. Note that there exists some $M < \infty$ so that

$$\int_C^M \frac{ds}{\eta(s)} = T.$$

Then since

$$\int_C^{\Lambda_n(t)} \frac{ds}{\eta(s)} \le t \le T,$$

it follows that $\Lambda_n(t) \le M$ for all $t \in [0, T]$ and every n.

Similar reasoning can be used to obtain a uniform bound on $\Lambda_n(t)$ in $[0, T]$ when $T \ge 1$, provided D is sufficiently large.

And now we can deduce from (1.7) that

$$\begin{aligned}
|\omega_n(t,x)| &= |\omega_n^0(X_n^{-t}(x))| \\
&\le Cg(X_n^{-t}(x)) \\
&\le Cg(x) \exp\left(C \int_0^t \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds \right) \\
&\le Cg(x) \exp\left(CT \left\| \frac{u_n}{h} \right\|_{L^\infty([0,T]\times\mathbb{R}^2)} \right).
\end{aligned}$$

Then, by the just-derived uniform bound over n on $\|u_n/h\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we see that $\|\omega_n/g\|_{L^\infty([0,T]\times\mathbb{R}^2)}$ is also uniformly bounded over n.

Thus, (u_n) is bounded uniformly on $L^\infty([0,T]; S_{h,g})$ for any sufficiently small $T > 0$. ■

2.3. Proof of Existence

Theorem 2.3.1. *Let h be a well-posedness growth bound, let g be a pre-growth bound, and let $u_0 \in S_{h,g}$. If the approximating sequence generated by u_0 is uniformly bounded in $L^\infty([0,T]; S_{h,g})$ for some $T > 0$, then there exists a solution to the Euler equations (in the sense of Definition 1.2.4) with initial data u_0 in $S_{h,g}$ on $[0,T]$.*

In particular, note that if the conditions of Proposition 2.2.2 are satisfied for some initial data $u_0 \in S_{h,g}$, then Theorem 2.3.1 implies that there exists a short-time solution to the Euler equations with initial data u_0. Indeed, the proofs of

Theorems 1.2.8 and 1.2.9 use this framework to prove short-time existence of solution given satisfactory initial conditions by demonstrating Proposition 2.2.2 holds for such initial data.

Proof.

Convergence of Flow Maps Let (X_n^{-t}) be the approximating sequence of inverse flow maps generated by u_0. We first note that, by Lemma 1.6.2, there exists some positive function $\beta(x)$ which increases with $|x|$ so that for all $x_1, x_2 \in \mathbb{R}^2$ and $t_1, t_2 \in [0,T]$ with $|x_1 - x_2|, |t_1 - t_2|$ sufficiently small, we have

$$\begin{aligned} |X_n^{-t_1}(x_1) - X_n^{-t_2}(x_2)| &\leq |X_n^{-t_1}(x_1) - X_n^{-t_2}(x_1)| + |X_n^{-t_2}(x_1) - X_n^{-t_2}(x_2)| \\ &\leq (Ch(x_1)|t_1 - t_2|)^{\exp(-\beta(x_1))} + |x_1 - x_2|^{\exp(-\beta(x_1))} \end{aligned}$$

for each n, where C is independent of n. Thus, the family (X_n^{-t}) is uniformly equicontinuous on $[0,T] \times B_R(0)$ for any $R > 0$. But by (1.18), (X_n^{-t}) is also uniformly bounded on $[0,T] \times B_R(0)$ for any $R > 0$.

Then by the Arzelá-Ascoli theorem, on any compact subset of $[0,T] \times \mathbb{R}^2$ there is a uniformly convergent subsequence of (X_n^{-t}). In particular, there is a subsequence $(X_{n_1}^{-t})$ of (X_n^{-t}) that converges uniformly on $[0,T] \times B_1(0)$. But applying the Arzelá-Ascoli theorem a second time, we find a subsequence $(X_{n_2}^{-t})$ of $(X_{n_1}^{-t})$ that converges uniformly on $[0,T] \times B_2(0)$. Continuing this process, we obtain by induction an infinite family of sequences

$$(X_n^{-t}) \supseteq (X_{n_1}^{-t}) \supseteq (X_{n_2}^{-t}) \supseteq \cdots$$

such that $(X_{n_j}^{-t})$ converges uniformly on $[0,T] \times B_j(0)$ for each j.

Using a diagonalization argument, we construct a new sequence (X_k^{-t}) by taking the first term of $(X_{n_1}^{-t})$, the second term of $(X_{n_2}^{-t})$, and so on. Then by construction,

(X_k^{-t}) is a subsequence of (X_n^{-t}) that converges uniformly on all compact subsets of $[0, T] \times \mathbb{R}^2$; say $X^{-t} = \lim X_k^{-t}$.

We wish to show that X^{-t} is measure-preserving. To do so, let $f \in C_c(\mathbb{R}^2)$ and choose $R > 0$ so that $\operatorname{supp} f \subseteq B_R(0)$. Observe that for every x, k such that $|X_k^{-t}(x)| < R$, bound (1.16) implies that

$$|x| = |X_k(t, X_k^{-t}(x))| \leq |X_k^{-t}(x)| + C(T)h(X_k^{-t}(x)) \leq R + C(T)h(R) =: R'.$$

Thus, $\operatorname{supp} f(X_k^{-t}(\cdot)) \subseteq B_{R'}(0)$. But then defining $F : \mathbb{R}^2 \to \mathbb{R}$ as

$$F(x) = \begin{cases} \sup |f|, & |x| \leq R' \\ 0, & \text{otherwise} \end{cases},$$

we have $|f(X_k^{-t}(\cdot))| \leq F$ for every k. So we can apply the Lebesgue dominated convergence theorem together with the fact that X_k^{-t} is measure preserving for every k to conclude that

$$\int_{\mathbb{R}^2} f(X^{-t}(x))\, dx = \lim \int_{\mathbb{R}^2} f(X_k^{-t}(x))\, dx = \int_{\mathbb{R}^2} f(x)\, dx. \tag{2.10}$$

But since $C_c(\mathbb{R}^2)$ is dense in $L^1(\mathbb{R}^2)$, identity (2.10) holds for all $f \in L^1(\mathbb{R}^2)$.

Now let $A \subset\subset \mathbb{R}^2$ be a Borel set, let $\mathbb{1}_A$ be the the corresponding characteristic function, and let m be Lebesgue measure. Noting that $\mathbb{1}_A \in L^1(\mathbb{R}^2)$, we observe

$$m(A) = \int_A dx = \int \mathbb{1}_A(x)\, dx = \int \mathbb{1}_A(X^{-t}(x))\, dx = \int_{X(t,A)} dx = m(X(t, A)),$$

so that X^{-t} is a measure preserving transformation.

Convergence of Vorticities Let (ω_k) be the subsequence of the approximating sequence of vorticities (ω_n) corresponding to (X_k^{-t}). Set $\omega(t, \cdot) = \omega_0(X^{-t})$. Choose a compact $L \subset \mathbb{R}^2$ and an $\varepsilon > 0$. Observe

$$\begin{aligned}
\|\omega_k(t) - \omega(t)\|_{L^p(L)} &= \|\omega_k^0(X_k^{-t}) - \omega_0(X^{-t})\|_{L^p(L)} \\
&\leq \|\omega_k^0(X_k^{-t}) - \omega_0(X_k^{-t})\|_{L^p(L)} + \|\omega_0(X_k^{-t}) - \omega_0(X^{-t})\|_{L^p(L)} \\
&\leq \|\omega_k^0 - \omega_0\|_{L^p(X_k^{-t}(L))} + \|\omega_0(X_k^{-t}) - \omega_0(X^{-t})\|_{L^p(L)} \\
&\leq \|\omega_k^0 - \omega_0\|_{L^p(B_\alpha(0))} + \|\omega_0(X_k^{-t}) - \omega_0(X^{-t})\|_{L^p(L)},
\end{aligned}$$

where α is chosen so that $X_k^{-t}(L) \subset B_\alpha(0)$ for all k and all $t \in [0,T]$, a fact guaranteed by (1.18). But

$$\begin{aligned}
\|\omega_k^0 - \omega_0\|_{L^p(B_\alpha(0))} &\leq \|\Delta a_k(\psi_0 * \nu_k)\|_{L^p(B_\alpha(0))} \\
&\quad + 2\|\nabla^\perp a_k \cdot (u_0 * \nu_k)\|_{L^p(B_\alpha(0))} + \|a_k(\omega_0 * \nu_k) - \omega_0\|_{L^p(B_\alpha(0))}.
\end{aligned}$$

And since $\operatorname{supp}|\Delta a_k| = \operatorname{supp}|\nabla a_k| = \{x : \frac{1}{2}k \leq |x| \leq k\}$, it follows that

$$B_\alpha(0) \cap \operatorname{supp}|\Delta a_k| = B_\alpha(0) \cap \operatorname{supp}|\nabla a_k| = \emptyset$$

for all sufficiently large k, meaning

$$\|\Delta a_k(\psi_0 * \nu_k)\|_{L^p(B_\alpha(0))} + 2\|\nabla^\perp a_k \cdot (u_0 * \nu_k)\|_{L^p(B_\alpha(0))} \to 0$$

as $k \to \infty$. And it is well known that $a_k(\omega_0 * \nu_k) \to \omega_0$ in L^p_{loc}; for example, see theorem C.7 in [8]. Then $\|\omega_0^k - \omega_0\|_{L^p(B_\alpha(0))} < \varepsilon$ for sufficiently large k.

In fact, choose a particular m so large that we in fact do have $\|\omega_m^0 - \omega_0\|_{L^p(B_\alpha(0))} <$

ε. Then

$$\begin{aligned}
&\|\omega_0(X_k^{-t}) - \omega_0(X^{-t})\|_{L^p(L)} \\
&\quad \le \|\omega_0(X_k^{-t}) - \omega_m^0(X_k^{-t})\|_{L^p(L)} + \|\omega_m^0(X_k^{-t}) - \omega_m^0(X^{-t})\|_{L^p(L)} \\
&\qquad\qquad + \|\omega_m^0(X^{-t}) - \omega_0(X^{-t})\|_{L^p(L)} \\
&\quad \le \|\omega_0 - \omega_m^0\|_{L^p(B_\alpha(0))} + \|\omega_m^0(X_n^{-t}) - \omega_m^0(X^{-t})\|_{L^p(L)} + \|\omega_m^0 - \omega_0\|_{L^p(B_\alpha(0))} \\
&\quad \le 2\varepsilon + \|\omega_m^0(X_k^{-t}) - \omega_m^0(X^{-t})\|_{L^p(L)}.
\end{aligned}$$

To bound $\|\omega_m^0(X_k^{-t}) - \omega_m^0(X^{-t})\|_{L^p(L)}$, we first note that because ω_m^0 is continuous and $X_k^{-t} \to X^{-t}$ locally uniformly, we have $\omega_m^0(X_k^{-t}) \to \omega_m^0(X^{-t})$ pointwise as $k \to \infty$. And for each k and each $x \in L$,

$$|\omega_m^0(X_k^{-t}(x)) - \omega_m^0(X^{-t}(x))|^p \le (2\|\omega_m^0\|_{L^\infty(B_\alpha(0))})^p.$$

But $2\|\omega_m^0\|_{L^\infty(B_\alpha(0))}\mathbb{1}_L(x) \in L^1(L)$. Hence, Lebesgue dominated convergence implies that $\omega_m^0(X_k^{-t}) \to \omega_m^0(X^{-t})$ in $L^p(L)$ as $k \to \infty$

Combining the above results, we see that $\omega_k \to \omega$ in $L^p(L)$. Since L was an arbitrary compact set, and since all the bounds we obtained were independent of $t \in [0,T]$, we have that $\omega_k \to \omega$ in $L^\infty([0,T]; L^p_{loc}(\mathbb{R}^2))$.

Convergence of Velocities Let (u_k) be the subsequence of the approximating sequence of velocities (u_n) corresponding to (X_k^{-t}). Again take L as an arbitrary compact subset of $\mathbb{R}^2$. Our goal is to show (u_k) is Cauchy in $L^\infty([0,T] \times L)$. First observe that Serfati's identity implies that

$$|u_i(t,x) - u_j(t,x)| \le |u_i^0(x) - u_j^0(x)| + I_1 + I_2 + I_3$$

for any $u_i, u_j \in (u_k)$, where

$$I_1 = |(a_\lambda K) * (\omega_i(t) - \omega_j(t))|,$$
$$I_2 = |(a_\lambda K) * (\omega_i^0 - \omega_j^0)|,$$
$$I_3 = \left| \int_0^t (\nabla\nabla^\perp[(1 - a_\lambda)K] *\cdot (u_i \otimes u_i - u_j \otimes u_j)(s) \, ds \right|.$$

Now choose $\varepsilon > 0$. By construction, $u_k^0 \to u_0$ locally uniformly. Thus, there exists an $N > 0$ so that $\|u_i^0 - u_j^0\|_{L^\infty(L)} \le \varepsilon$ for all $i, j \ge N$.

To bound I_1 for $x \in L$, set $p = 3/2$ and $q = 3$. (Actually, any conjugate exponents p, q such that $p \in (1, 2)$ will work; we merely choose $p = 3/2$ to be concrete.) Set $L_\lambda = L + B_\lambda(0)$ for some fixed λ to be determined later. Then by Lemma 1.4.3 and Hölder's inequality,

$$I_1 \le C|L_\lambda|^{\frac{1}{6}} \|\omega_i(t) - \omega_j(t)\|_{L^3(L_\lambda)}.$$

Similarly,

$$I_2 \le C|L_\lambda|^{\frac{1}{6}} \|\omega_i^0 - \omega_j^0\|_{L^3(L_\lambda)}.$$

But for all i, j sufficiently large

$$\|\omega_i - \omega_j\|_{L^\infty([0,T];L^3(L_\lambda))} < \frac{\varepsilon}{|L_\lambda|^{\frac{1}{6}}},$$

in which case

$$I_1, I_2 \le C\varepsilon.$$

And by Lemma 1.4.8,

$$I_3(t, x) \le C(t) \left(H[h^2](\lambda/2) + \frac{h^2(x)}{\lambda} \right).$$

Since $H[h^2](r) \to 0$ as $r \to \infty$, we can find a δ with $0 < \delta < \varepsilon$ so that

$$H[h^2]\left(\frac{h^2(x)}{2\delta}\right) < \varepsilon.$$

Then setting $\lambda = h^2(x)/\delta$, we obtain the bound $I_3 \leq C(T)\varepsilon$.

Thus, for N sufficiently large,

$$\|u_i - u_j\|_{L^\infty([0,T]\times L)} \leq C(T)\varepsilon$$

whenever $i, j \geq N$. Then since L was arbitrary, u_k converges to some function u in $L^\infty_{loc}([0,T] \times \mathbb{R}^2)$.

Convergence to a Solution First, we demonstrate that $u \in L^\infty([0,T]; S_{h,g})$. First note that for any $(t,x) \in [0,T] \times L$, where $L \subset\subset \mathbb{R}^2$ is arbitrary, we have by Proposition 1.5.2 that

$$\begin{aligned}
&|u_k(t, X_k(t,x)) - u(t, X(t,x))| \\
&\qquad \leq |u_k(t, X_k(t,x)) - u_k(t, X(t,x))| + |u_k(t, X(t,x)) - u(t, X(t,x))| \\
&\qquad \leq C(L)\mu(C|X_k(t,x) - X(t,x)|) + \|u_k - u\|_{L^\infty([0,T]\times L)} \\
&\qquad \leq C(L)\mu(C\|X_k - X\|_{L^\infty([0,T]\times L)}) + \|u_k - u\|_{L^\infty([0,T]\times L)},
\end{aligned}$$

implying $u_k(t, X_k(t,x)) \to u(t, X(t,x))$ locally uniformly.

But then since $X_k(t,x) \to X(t,x)$ in $L^\infty_{loc}([0,T] \times \mathbb{R}^2)$, it follows that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left[(X_k(t,x) - x) - \int_0^t u_k(s, X_k(s,x))\, ds \right] \\
&= (X(t,x) - x) - \int_0^t u(s, X(s,x))\, ds,
\end{aligned}$$

so that X is a flow map of u. Specifically, note that this means that Corollary 1.3.9 holds for u, X.

Therefore, for any $(t, x) \in [0, T] \times \mathbb{R}^2$,

$$\begin{aligned}\left|\frac{\omega(t,x)}{g(x)}\right| &\leq \left|\frac{\omega_0(X^{-t}(x))}{g(x)}\right| \\ &\leq C\left|\frac{\omega_0(X^{-t}(x)}{g(X^{-t}(x))}\right| \\ &\leq C\|u_0\|_{S_h}.\end{aligned}$$

Thus, $\|\omega/g\|_{L^\infty([0,T]\times\mathbb{R}^2)} < \infty$.

Finally, observe that

$$\left\|\frac{u-u_k}{h}\right\|_{L^\infty([0,T]\times L)} \leq C\|u-u_k\|_{L^\infty([0,T]\times L)} \to 0,$$

so

$$\left\|\frac{u}{h}\right\|_{L^\infty([0,T]\times L)} \leq \sup_k \left\|\frac{u_k}{h}\right\|_{L^\infty([0,T]\times\mathbb{R}^2)} < \infty.$$

Since L was arbitrary, we in fact have

$$\left\|\frac{u}{h}\right\|_{L^\infty([0,T]\times\mathbb{R}^2)} < \infty.$$

Thus, $u \in L^\infty([0,T]; S_{h,g})$.

Finally, we prove that u satisfies Serfati's identity. Fix $(t, x) \in [0, T] \times \mathbb{R}^2$ and $\lambda > 0$. First, by construction $u_k^0 \to u_0$ pointwise, and we demonstrated that $u_k \to u$ pointwise above.

Next, observe that by Hölder's inequality and Lemma 1.4.3,

$$|[(a_\lambda K^j) * (\omega(t) - \omega_k(t))](x)| \leq C\sqrt[3]{\lambda}\|\omega - \omega_k\|_{L^\infty([0,T];L^3(B_\lambda(x)))} \to 0,$$

so that

$$[(a_\lambda K^j) * \omega_k(t)](x) \to [(a_\lambda K^j) * \omega(t)](x).$$

pointwise as $k \to \infty$. We can similarly show that

$$[(a_\lambda K^j) * \omega_k^0](x) \to [(a_\lambda K^j) * \omega_0](x).$$

Now let $\varepsilon > 0$ be arbitrary. To show convergence in the final term of Serfati's identity, choose $R \geq \max\{\lambda/2, h^2(x)\varepsilon^{-1}\}$ large enough so that $H[h^2](R) < \varepsilon$. Let $G = B_{\lambda/2}(x)^c \cap B_R(x)$. Observe

$$\begin{aligned}
&\left| \int_0^t \left[(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u - u_k \otimes u_k)(s) \right](x)\, ds \right| \\
&\quad \leq C(T) \int_0^t \int_{B_{\lambda/2}(x)^c} \frac{h(y)}{|x-y|^3} |u(s,y) - u_k(s,y)|\, dy\, ds \\
&\quad \leq C(T) \Bigg(\int_0^t \int_G \frac{h(y)}{|x-y|^3} |u(s,y) - u_k(s,y)|\, dy\, ds \\
&\qquad\quad + \int_0^t \int_{B_R(x)^c} \frac{h(y)}{|x-y|^3} |u(s,y) - u_k(s,y)|\, dy\, ds \Bigg) \\
&\quad =: C(T)(A(x,t) + B(x,t)).
\end{aligned}$$

Now

$$\begin{aligned}
A(x,t) &\leq t\|u - u_k\|_{L^\infty([0,T]\times G)} \int_G \frac{h(y)}{|x-y|^3}\, dy \\
&\leq t\|u - u_k\|_{L^\infty([0,T]\times G)} \left(\int_G \frac{h(x-y)}{|x-y|^3}\, dy + h(x) \int_G \frac{1}{|x-y|^3}\, dy \right) \\
&\leq Ct\|u - u_k\|_{L^\infty([0,T]\times G)} \left(\int_{\frac{\lambda}{2}}^R \frac{h(r)}{r^2}\, dy + h(x) \int_{\frac{\lambda}{2}}^R \frac{1}{r^2}\, dy \right) \\
&\leq Ct\|u - u_k\|_{L^\infty([0,T]\times G)} \left(H[h](\lambda/2) + h(x) \left(\frac{1}{\lambda/2} - \frac{1}{R} \right) \right) \\
&\leq C(T,\lambda)\|u - u_k\|_{L^\infty([0,T]\times G)} h(x)
\end{aligned}$$

We choose k large enough so that

$$\|u - u_k\|_{L^\infty([0,T]\times G)} \leq \frac{\varepsilon}{h(x)},$$

meaning that $A(x,t) \le C\varepsilon$.

Furthermore,

$$\begin{aligned} B(x,t) &= \int_0^t \int_{B_R(x)^c} \frac{h^2(y)}{|x-y|^3} \frac{|u(s,y)-u_k(s,y)|}{h(y)}\, ds\, dy \\ &\le C(T) \int_{B_R(x)^c} \frac{h^2(y)}{|x-y|^3}\, dy \\ &\le C(T) \left(H[h^2](R) + \frac{h^2(x)}{R} \right). \end{aligned}$$

Due to our choice of R, we have

$$B(x,t) \le C\varepsilon.$$

Since ε was arbitrary, it follows that

$$\begin{aligned} &\int_0^t \left[(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u_k \otimes u_k)(s) \right](x)\, ds \\ &\qquad \to \int_0^t \left[(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u \otimes u)(s) \right](x)\, ds \end{aligned}$$

pointwise as $k \to \infty$.

Combining the above results, we see that as we let $k \to \infty$ in Serfati's identity

$$\begin{aligned} u_k^j(x) = (u_k^0)^j(x) + [(a_\lambda K^j) * (\omega_k(t) - \omega_k^0)](x) \\ + \int_0^t [(\nabla\nabla^\perp[(1-a_\lambda)K^j]) *\cdot (u_k \otimes u_k)(s)](x)\, ds, \end{aligned}$$

we obtain

$$u^j(x) = u_0^j(x) + [(a_\lambda K^j) * (\omega(t) - \omega_0)](x) + \int_0^t [(\nabla\nabla^\perp[(1-a_\lambda)K^j]) * (u \otimes u)(s)](x)\, ds$$

for each $x \in \mathbb{R}^2$, as desired.

Therefore, u is a solution to the Euler equations (in the sense of Definition 1.2.4) with initial data u_0. ■

2.4. Existence of Solutions with Quasibounded Initial Vorticity

Before proving Theorem 1.2.8, we first must show that that the sequence of approximating initial vorticities, generated by some quasibounded initial vorticity, is uniformly quasibounded. To do so, we need the following lemma.

Lemma 2.4.1. *Let $\xi \in MPH(\mathbb{R}^2)$, let $\lambda > 0$, and let $x \in \mathbb{R}^2$. Then*

$$\int_{B_\lambda(x)} \frac{1}{|x - \xi(y)|}\, dy \le 2\pi\lambda.$$

Proof. As in our argument to Lemma 1.4.3, we note that since $|x-z|^{-1}$ is radially symmetric about $z = x$ and decreases as $|x - z|$ increases, we calculate

$$\begin{aligned}\int_{B_\lambda(x)} \frac{1}{|x - \xi(y)|}\, dy &\le \int_{\xi(B_\lambda(x))} \frac{1}{|x - z|}\, dz \\ &\le \int_{B_\lambda(x)} \frac{1}{|x - z|}\, dz \\ &= 2\pi\lambda.\end{aligned}$$

■

Proposition 2.4.2. *Let h, g be pre-growth bounds and let $u_0 \in S_{h,g}$. Let (ω_n^0) be the sequence of approximating initial vorticities generated by u_0. If $\omega_0 \in \mathcal{T}_h$, then*

$$\|\omega_n^0\|_{\mathcal{T}_h} \le C,$$

where C is independent of n.

Proof. First, recall that

$$
\begin{aligned}
|\omega_n^0| &= \big|\Delta\big(a_n(\psi_0 * \nu_n)\big)\big| \\
&\leq |\Delta a_n(\psi_0 * \nu_n)| + 2|\nabla^\perp a_n \cdot (u_0 * \nu_n)| + |a_n(\omega_0 * \nu_n)|.
\end{aligned}
$$

Recall also that since ν has support within some ball $B_R(0)$ for some $R > 0$, ν_n has support within $B_{R/n}(0)$. Now using Lemmas 1.4.3 and 1.5.1, we have for any $x \in \mathbb{R}^2, \lambda \geq h(x), \xi \in MPH(\mathbb{R}^2)$,

$$
\begin{aligned}
&\frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\Delta a_n(y)(\psi_0 * \nu_n)(y)|}{|x-\xi(y)|}\, dy \\
&\quad\leq \frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\Delta a_n(y)|}{|x-\xi(y)|}\left(\int_{\mathbb{R}^2} |\psi_0(z)|\nu_n(y-z)\, dz\right) dy \\
&\quad\leq \frac{1}{2\pi\lambda}\int_{B_\lambda(x)\cap B_n(0)} \frac{Cn^{-2}}{|x-\xi(y)|}\left(\int_{B_{R/n}(y)} |z|h(z)\nu_n(y-z)\, dz\right) dy \\
&\quad\leq \frac{C}{2\pi\lambda}\int_{B_\lambda(x)\cap B_n(0)} \frac{1}{|x-\xi(y)|}\left(\int_{B_{R/n}(y)} \nu_n(y-z)\, dz\right) dy \\
&\quad\leq \frac{C\|\nu_n\|_{L^1(\mathbb{R}^2)}}{2\pi\lambda}\int_{B_\lambda(x)\cap B_n(0)} \frac{1}{|x-\xi(y)|}\, dy \\
&\quad\leq C,
\end{aligned}
$$

where we used Lemma 1.3.3 and Lemma 2.4.1 together with the observation that $|z| \leq n + R/n \leq Cn$ whenever $|y-z| \leq R/n$ and $|y| \leq n$. Similarly,

$$
\begin{aligned}
&\frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\nabla^\perp a_n(y)\cdot(u_0 * \nu_n)(y)|}{|x-\xi(y)|}\, dy \\
&\quad\leq \frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\nabla a_n(y)|}{|x-\xi(y)|}\left(\int_{\mathbb{R}^2} |u_0(z)|\nu_n(y-z)\, dz\right) dy \\
&\quad\leq \frac{\|u_0/h\|_{L^\infty(\mathbb{R}^2)}}{2\pi\lambda}\int_{B_\lambda(x)} \frac{Cn^{-1}\mathbb{1}_{B_n(0)}(y)}{|x-\xi(y)|}\left(\int_{B_{R/n}(y)} h(z)\nu_n(y-z)\, dz\right) dy \\
&\quad\leq C.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{2\pi\lambda} \int_{B_\lambda(x)} & \frac{|\omega_0 * \nu_n(y)|}{|x-\xi(y)|}\, dy \\
&\leq \frac{1}{2\pi\lambda} \int_{B_{R/n}(0)} \int_{B_\lambda(x)} \frac{|\omega_0(y-z)|}{|x-\xi(y)|} \nu_n(z)\, dy\, dz \\
&= \frac{1}{2\pi\lambda} \int_{B_{R/n}(0)} \nu_n(z) \int_{B_\lambda(x-z)} \frac{|\omega_0(\alpha)|}{|x-\xi(\alpha-z)|}\, d\alpha\, dz,
\end{aligned}
$$

where we obtained the last step by substituting $\alpha = y - z$. But noting that $\xi_z := \xi(\cdot - z) \in MPH(\mathbb{R}^2)$, we calculate for all $|z| \leq R/n \leq R$,

$$
\int_{B_\lambda(x-z)} \frac{|\omega_0(\alpha)|}{|x-\xi_z(\alpha)|}\, d\alpha \leq \int_{B_{\lambda+R}(x)} \frac{|\omega_0(\alpha)|}{|x-\xi_z(\alpha)|}\, d\alpha \leq 2\pi(\lambda+R)\|\omega_0\|_{\mathcal{T}_h}.
$$

Thus,

$$
\frac{1}{2\pi\lambda} \int_{B_\lambda(x)} \frac{|\omega_0 * \nu_n(y)|}{|x-\xi(y)|}\, dy \leq \frac{\lambda+R}{\lambda} \|\nu_n(z)\|_{L^1(\mathbb{R}^2)} \|\omega_0\|_{\mathcal{T}_h} \leq C. \qquad \blacksquare
$$

Now we are prepared to use Proposition 2.2.2 and theorem 2.3.1 to prove Theorem 1.2.8.

Proof of Theorem 1.2.8. Let (u_n) be the approximating sequence generated by u_0. Observe that by Proposition 1.6.4 that $X_n^{-t}(B_\lambda(x)) \subset B_{\hat{\lambda}_n}(x)$ for any $x \in \mathbb{R}^2$, where

$$
\hat{\lambda}_n = \hat{\lambda}_n(t,x) = \lambda + Ch(|x|+\lambda)\left(1 + \int_0^t \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds\right)^2.
$$

Then by simple substitution, for $\lambda \geq h(x)$ we calculate

$$\begin{aligned}
\big(|K_\lambda| * |\omega_n(t) - \omega_n^0|\big)(x) &\leq \frac{1}{2\pi} \int_{B_\lambda(x)} \frac{|\omega_n^0(X_n^{-t}(y)) - \omega_n^0(y)|}{|x-y|}\, dy \\
&\leq \frac{1}{2\pi} \int_{B_{\hat{\lambda}_n}(x)} \frac{|\omega_n^0(z)|}{|x - X_n(t,z)|}\, dz + \frac{1}{2\pi} \int_{B_\lambda(x)} \frac{|\omega_n^0(y)|}{|x-y|}\, dy \\
&\leq \|\omega_n^0\|_{\mathcal{T}_h} \hat{\lambda}_n + \|\omega_n^0\|_{\mathcal{T}_h} \lambda \\
&\leq C(\hat{\lambda}_n + \lambda),
\end{aligned}$$

where, by Proposition 2.4.2, C is independent of x, t, n.

In particular, if we set $\lambda = h(x)$, then

$$\begin{aligned}
\hat{\lambda}_n &\leq h(x) + Ch(x) \left(1 + \int_0^t \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)}\right)^2 \\
&\leq h(x) \left(C + Ct \int_0^t \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)}^2\right),
\end{aligned}$$

so that

$$\left(|K_{h(x)}| * |\omega_n(t) - \omega_n^0|\right)(x) \leq h(x) \left(C + Ct \int_0^t \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)}^2\right).$$

Existence of a solution on some interval $[0, T]$ follows from Proposition 2.2.2 and Theorem 2.3.1.

To show that $\|\omega(t)\|_{\mathcal{T}_h}$ is uniformly bounded on $[0, T]$, we proceed similar to above. Since $X_n^{-t}(B_\lambda(x)) \subset B_{\hat{\lambda}}(x))$, where

$$\hat{\lambda}(t, x) = \lambda + Ch(|x| + \lambda) \left(1 + \int_0^t \left\| \frac{u(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds\right)^2,$$

we calculate for any $t \in [0,T]$, $x \in \mathbb{R}^2$, $\lambda \geq h(x)$, and $\xi \in MPH(\mathbb{R}^2)$ that

$$\begin{aligned}
\frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\omega(t,y)|}{|x-\xi(y)|}\,dy &= \frac{1}{2\pi\lambda}\int_{B_\lambda(x)} \frac{|\omega_0(X^{-t}(y))|}{|x-\xi(y)|}\,dy \\
&\leq \frac{1}{2\pi\lambda}\int_{B_{\hat{\lambda}}(x)} \frac{|\omega_0(z)|}{|x-\xi(X(t,z))|}\,dy \\
&\leq \frac{\hat{\lambda}}{\lambda}\|\omega_0\|_{\mathcal{T}_h} \\
&\leq C(T).
\end{aligned}$$

■

In the following example, we use confined eddies (see Appendix A.2) to construct initial data with quasibounded initial vorticity. By Theorem 1.2.8, there exists a short-time solution to the Euler equations with this initial data.

Example 2.4.3. Let h be a well-posedness growth bound and let g be a pre-growth bound. For each n, construct a confined eddy centered at $x_n = (n,0)$ with velocity $\hat{u}_n$ and vorticity $\hat{\omega}_n$ so that

- $\|\hat{\omega}_n\|_{L^\infty(\mathbb{R}^2)} = g(x_n)$,
- the eddy has support on $B_{r_n}(x_n)$, where $r_n < (g(x_n))^{-1}$ is small enough so that $\|\hat{u}_n\|_{L^\infty(\mathbb{R}^2)} \leq h(x_n)$, and so the supports of the eddies are pairwise disjoint.

Note that by Lemma 1.4.3, for any $x \in \mathbb{R}^2$, $\lambda > 1$, $\xi \in MPH(\mathbb{R}^2)$ we have

$$\begin{aligned}
\sum_n \int_{B_\lambda(x)} \frac{|\hat{\omega}_n(y)|}{|x - \xi(y)|}\, dy &\leq \sum_{\{n : B_{r_n}(x_n) \cap B_\lambda(x) \neq \emptyset\}} \int_{B_{r_n}(x_n)} \frac{|\hat{\omega}_n(y)|}{|x - \xi(y)|}\, dy \\
&\leq \sum_{\{n : B_{r_n}(x_n) \cap B_\lambda(x) \neq \emptyset\}} C r_n \|\hat{\omega}_n\|_{L^\infty(B_{r_n}(x_n))} \\
&\leq \sum_{\{n : B_{r_n}(x_n) \cap B_\lambda(x) \neq \emptyset\}} C r_n g(x_n) \\
&\leq \sum_{\{n : B_{r_n}(x_n) \cap B_\lambda(x) \neq \emptyset\}} C \\
&\leq C\lambda.
\end{aligned}$$

Now choose $\overline{u}_0 \in S_{h,1}$ with bounded vorticity $\overline{\omega}_0$. Define

$$\begin{aligned}
u_0 &= \overline{u}_0 + \sum_n \hat{u}_n, \\
\omega_0 &= \overline{\omega}_0 + \sum_n \hat{\omega}_n.
\end{aligned}$$

By construction and the observations above, it follows that $u_0 \in S_{h,g}$ with $\omega_0 \in \mathcal{T}_h$. Thus, by Theorem 1.2.8, there is a short-time solution to the Euler equations with initial data u_0. ■

2.5. Existence of Solutions with Stable Initial Vorticity

We begin with several important results regarding stabilizers, including their relationship with approximating initial data generated by stable initial data.

Proposition 2.5.1. *Let h be a pre-growth bound and let ϕ be an h-stabilizer. Then*

$$\|\phi/\hbar\|_{L^\infty(\mathbb{R}^2)} < \infty.$$

Proof. Observe,

$$\begin{aligned}
|\phi(x) - \phi(0)| &= \left| \int_0^1 \nabla \phi(\alpha x) \cdot x \, d\alpha \right| \\
&\leq \int_0^1 |\nabla \phi(\alpha x)||x| \, d\alpha \\
&\leq C \int_0^1 \frac{|x|}{h(\alpha |x|)} \, d\alpha \\
&= C \int_0^{|x|} \frac{1}{h(\alpha)} \, d\alpha \\
&\leq C \hbar(x).
\end{aligned}$$

■

Proposition 2.5.2. *Let h be a pre-growth bound and let ϕ be an h-stabilizer. Then for any $x_1, x_2 \in \mathbb{R}^2$,*

$$|\phi(x_2) - \phi(x_1)| \leq C \frac{|x_1 - x_2|}{h(\min\{|x_1|, |x_2|\})}.$$

Proof. Recall that by Lemma 1.3.5, $1/h$ is convex. Thus,

$$\begin{aligned}
|\phi(x_1) - \phi(x_2)| &= \left| \int_0^1 \nabla \phi(\alpha x_1 + (1-\alpha) x_2) \cdot (x_1 - x_2) \, d\alpha \right| \\
&\leq C |x_1 - x_2| \int_0^1 \frac{1}{h(\alpha x_1 + (1-\alpha) x_2)} \, d\alpha \\
&\leq C |x_1 - x_2| \int_0^1 \frac{\alpha}{h(x_1)} + \frac{1-\alpha}{h(x_2)} \, d\alpha \\
&\leq C \frac{|x_1 - x_2|}{h(\min\{|x_1|, |x_2|\})}.
\end{aligned}$$

■

Definition 2.5.3. Let h be a pre-growth bound and let ϕ be an h-stabilizer. For each n, let

$$\phi_n = a_n (\phi * \nu_n).$$

We call (ϕ_n) is the *sequence of approximating stabilizers generated by ϕ.*

Proposition 2.5.4. *Let h, g be pre-growth bounds, and let $u_0 \in S_{h,g}$ with vor-*

ticity ω_0 being stable with respect to some h-stabilizer ϕ. Let (u_n^0) and (ω_n^0) be, respectively, the sequences of approximating initial velocities and initial vorticities generated by u_0, and let (ϕ_n) be the sequence of approximating stabilizers generated by ϕ. Then for each n, ϕ_n is an h-stabilizer, and ω_n^0 is stable with respect to ϕ_n. In particular,

$$\|\omega_n^0 - \phi_n\|_{L^\infty(\mathbb{R}^2)} \leq C,$$

where C does not depend on n.

Proof. First note that since ϕ_n is smooth and compactly supported, there exists a constant $C = C(n)$ so that $|\nabla \phi_n(x)| \leq C/h(x)$. Thus, ϕ_n is an h-stabilizer.

And we can use Young's convolution inequality together with Lemma 2.1.2 and Lemma 1.5.1 to calculate

$$\begin{aligned}
|\omega_n^0 - \phi_n| &= |(\psi_0 * \nu_n)\Delta a_n + 2\nabla^\perp a_n \cdot (u_0 * \nu_n) + a_n(\omega_0 * \nu_n) - a_n(\phi_0 * \nu_n)| \\
&\leq C \mathbb{1}_{B_n(0)} \left| \frac{\psi_0 * \nu_n}{n^2} \right| + C \mathbb{1}_{B_n(0)} \left| \frac{u_0 * \nu_n}{n} \right| + |(\omega_0 - \phi_0) * \nu_n| \\
&\leq C \left\| \frac{u_0}{h} \right\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0 - \phi\|_{L^\infty(\mathbb{R}^2)} \|\nu_n\|_{L^1(\mathbb{R}^2)} \\
&\leq C \left\| \frac{u_0}{h} \right\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0 - \phi\|_{L^\infty(\mathbb{R}^2)}. \qquad \blacksquare
\end{aligned}$$

Proposition 2.5.5. *Let h be a pre-growth bound and ϕ an h-stabilizer. Let (ϕ_n) be the sequence of approximating stabilizers generated by ϕ, and let (u_n) be a sequence of time-dependent vector fields with associated flow maps (X_n). Then there exists a constant C independent of n so that*

$$\|\phi_n(X_n(t_2)) - \phi_n(X_n(t_1))\|_{L^\infty(\mathbb{R}^2)} \leq C \int_{t_1}^{t_2} \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)} ds.$$

Proof. We calculate

$$\begin{aligned}
|\nabla\phi_n| &= |\nabla(a_n(\phi * \nu_n))| \\
&\le |\nabla a_n||\phi * \nu_n| + |a_n||\nabla\phi * \nu_n| \\
&\le \frac{C}{n}\mathbb{1}_{B_n(0)}|\phi * \nu_n| + |\nabla\phi * \nu_n|.
\end{aligned}$$

But by Lemma 1.3.6 and proposition 2.5.1,

$$\begin{aligned}
\frac{C}{n}\mathbb{1}_{B_n(0)}(x)|(\phi * \nu_n)(x)| &\le \frac{C}{n}\mathbb{1}_{B_n(0)}(x)\int_{\mathbb{R}^2}|\phi(x-y)|\nu_n(y)\ dy \\
&\le \frac{C}{n}\mathbb{1}_{B_n(0)}(x)\int_{\mathbb{R}^2}\frac{C|x-y|+C}{h(x-y)}\nu_n(y)\ dy \\
&\le \frac{C}{n}\int_{\mathbb{R}^2}\frac{C\left(n+\frac{R}{n}\right)+C}{h(x-y)}\nu_n(y)\ dy \\
&\le C\int_{\mathbb{R}^2}\frac{1}{h(x-y)}\nu_n(y)\ dy.
\end{aligned}$$

Also,

$$\begin{aligned}
|(\nabla\phi * \nu_n)(x)| &\le \int_{\mathbb{R}^2}|\nabla\phi(x-y)|\nu_n(y)\ dy \\
&\le C\int_{\mathbb{R}^2}\frac{1}{h(x-y)}\nu_n(y)\ dy.
\end{aligned}$$

Then recalling that ν is supported in $B_R(0)$ for some $R > 0$, we have

$$|\nabla\phi_n(x)| \le C\int_{B_{R/n}(0)}\frac{1}{h(x-y)}\nu_n(y)\ dy.$$

Now for $|y| \le R/n$ and any $x \in \mathbb{R}^2$,

$$h(x) \le h(x-y) + h(y) \le h(x-y) + h(R) \le h(x-y) + Ch(0) \le Ch(x-y).$$

But then

$$|\nabla\phi_n(x)| \le \frac{C}{h(x)}\int_{\mathbb{R}^2}\nu_n(y)\ dy \le \frac{C}{h(x)},$$

where C is independent of n.

We now proceed as in Lemma 1.3.8. Observe that for each $x \in \mathbb{R}^2$,

$$\begin{aligned}
|\phi_n(X_n(t_2,x)) - \phi_n(X_n(t_1,x))| &\le \int_{t_1}^{t_2} |\nabla\phi_n(X_n(s,x))||u_n(x, X_n(s,x))|\, ds \\
&\le \int_{t_1}^{t_2} \frac{C}{h(X_n(s,x))}|u_n(x, X_n(s,x))|\, ds \\
&\le C \int_{t_1}^{t_2} \left\| \frac{u_n(s)}{h} \right\|_{L^\infty(\mathbb{R}^2)}. \qquad \blacksquare
\end{aligned}$$

Proposition 2.5.6. *Let h, g be pre-growth bounds, and let $u_0 \in S_{h,g}$. Assume ω_0 is stable relative to some h-stabilizer ϕ. If there exists a solution to the Euler equations $u \in L^\infty([0,T]; S_{h,g})$ with initial data u_0 for some $T > 0$, then*

$$\begin{aligned}
\|\omega(t) - \phi\|_{L^\infty(\mathbb{R}^2)} &< C(T), \\
\|\omega(t) - \omega_0\|_{L^\infty(\mathbb{R}^2)} &< C(T),
\end{aligned}$$

where $\omega = \nabla \times u$, for all $t \in [0,T]$.

Proof. Let X be the flow map associated with u. For any $t \in [0,T]$, $x \in \mathbb{R}^2$, we calculate

$$\|\omega(t) - \phi\|_{L^\infty(\mathbb{R}^2)} \le \|\omega_0(X^{-t}) - \phi(X^{-t})\|_{L^\infty(\mathbb{R}^2)} + \|\phi(X^{-t}) - \phi\|_{L^\infty(\mathbb{R}^2)} \le C(T),$$

where we used the stability of ω_0 and Proposition 2.5.5. But then

$$\|\omega(t) - \omega_0\|_{L^\infty(\mathbb{R}^2)} \le \|\omega(t) - \phi\|_{L^\infty(\mathbb{R}^2)} + \|\phi - \omega_0\|_{L^\infty(\mathbb{R}^2)} \le C(T). \qquad \blacksquare$$

We are now prepared to prove Theorem 1.2.9. As in our argument to Theorem 1.2.8, we shall do so by applying Proposition 2.2.2 and theorem 2.3.1.

Proof of Theorem 1.2.9. Let (u_n) be the approximating sequence generated by u_0. Observe that for each n,

$$\begin{aligned}|\omega_n(t,x) - \omega_n^0(x)| &= |\omega_n^0(X_n^{-t}(x)) - \omega_n^0(x)| \\ &\le |\omega_n^0(X_n^{-t}(x)) - \phi_n(X_n^{-t}(x))| \\ &\quad + |\phi_n(X_n^{-t}(x)) - \phi_n(x)| + |\phi_n(x) - \omega_n^0(x)|.\end{aligned}$$

But then, noting that $X_n^{-t}(x) = X_n(0, X^{-t}(x))$ and $x = X_n(t, X_n^{-t}(x))$, by Propositions 2.5.4 and 2.5.5, we have

$$\left\|\omega_n(t) - \omega_n^0\right\|_{L^\infty(\mathbb{R}^2)} \le C + C\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty(\mathbb{R}^2)} ds,$$

where C is independent of n. Applying Hölder's inequality gives

$$\|\omega_n(t) - \omega_n^0\|_{L^\infty(\mathbb{R}^2)} \le C + C\sqrt{t}\left(\int_0^t \left\|\frac{u_n(s)}{h}\right\|^2_{L^\infty(\mathbb{R}^2)}\right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned}|K_\lambda| * |\omega_n(t) - \omega_n^0| &\le \|K_\lambda\|_{L^1(\mathbb{R}^2)}\|\omega_n(t) - \omega_n^0\|_{L^\infty(\mathbb{R}^2)} \\ &\le C\lambda\left(1 + \sqrt{t}\left(\int_0^t \left\|\frac{u_n(s)}{h}\right\|^2_{L^\infty(\mathbb{R}^2)} ds\right)^{\frac{1}{2}}\right).\end{aligned}$$

Then Proposition 2.2.2 and Theorem 2.3.1 give existence. And Proposition 2.5.6 guarantees that $\omega(t)$ is uniformly stable relative to ϕ over $[0,T]$. ■

Observe that any bounded initial vorticity is also stable with respect to the stabilizer $\phi \equiv 0$. Such initial data is, in some sense, trivial: while it does satisfy the conditions of Theorem 1.2.9, since the initial vorticity is bounded, existence is already guaranteed by Theorem 1.1.5.

Discovering non-trivial examples of initial data—that is, initial data with unbounded vorticity—satisfying the conditions of Theorem 1.2.9 has proven difficult.

A natural way to go about trying to construct such initial data is to attempt to find a smooth, unbounded initial vorticity ω_0 so that ω_0 can be its own stabilizer; that is, so $\|h\nabla\omega_0\|_{L^\infty} < \infty$.

While this approach has not yet yielded an example of non-trivial initial data, it has allowed us to approach the threshold. In fact, in Example 2.5.7 below, we have

$$\begin{aligned}\left\|\frac{u_0}{h}\right\|_{L^\infty(\mathbb{R}^2)} &< \infty,\\ \|\omega_0\|_{L^\infty(\mathbb{R}^2)} &< \infty,\\ \|h\nabla\omega_0\|_{L^\infty(\mathbb{R}^2)} &< \infty.\end{aligned}$$

The first and third bounds are required to satisfy the conditions of Theorem 1.2.9, and the bounds are tight. As such, any attempt to modify the second bound to allow some growth in the vorticity breaks the other two bounds.

Example 2.5.7 also demonstrates that there exists initial data with the velocity growing like a well-posedness growth bound and non-decaying vorticity. Cozzi and Kelliher in [4] prove the short-time existence and uniqueness of solutions with such initial data, but they did not provide an example with growing initial velocity and non-decaying initial vorticity.

Example 2.5.7. Let $\hat{u} \neq 0$ be the velocity field of a confined eddy with support in $B_R(0)$, and let h be a well-posedness growth bound. Choose a sequence of points (y_n) in $\mathbb{R}^2$ so that the family of sets $(B_{h(y_n)R}(y_n))$ is pairwise disjoint. (For example, we could choose y_n so that $|y_n| = (2n-1)h(y_n)R$ for each n.) For each n, define

$$\hat{u}_n(x) := h(y_n)\hat{u}\left(\frac{1}{h(y_n)}(x - y_n)\right).$$

Note that, by construction, the supports of the v_n are pairwise disjoint. Also, letting $\hat{\omega}_n = \nabla \times \hat{u}_n$ for each n, observe that

$$\|\hat{u}_n/h\|_{L^\infty(\mathbb{R}^2)} \leq C,$$
$$\|\hat{\omega}_n\|_{L^\infty(\mathbb{R}^2)} \leq C,$$
$$\|h\nabla\hat{\omega}_n\|_{L^\infty(\mathbb{R}^2)} \leq C,$$

where C is independent of n.

Now choose $\overline{u}_0 \in S_{h,1}$ with bounded vorticity $\overline{\omega}_0$. Define

$$u_0 = \overline{u}_0 + \sum_n \hat{u}_n,$$
$$\omega_0 = \overline{\omega}_0 + \sum_n \hat{\omega}_n,$$
$$\phi = \sum_n \hat{\omega}_n.$$

By construction, ϕ is an h-stabilizer, and $u_0 \in S_{h,g}$ with ω_0 stable relative to ϕ. Thus, there is a short time solution to the Euler equations with initial data u_0. ■

3. Uniqueness for Stable Initial Vorticity

3.1. A Tighter Bound on the Modulus of Continuity of the Velocity

In Section 1.5, we established a log-Lipschitz bound on the modulus of continuity for functions $u \in S_{h,g}$, where h, g are pre-growth bounds. However, we will need a slightly tighter bound on the modulus of continuity to establish uniqueness. To do so, we will assume that h is a well-posedness growth bound and $u \in L^\infty([0,T]; S_{h,g})$ for some $T > 0$ is a solution to the Euler equations with stable vorticity. Then we will use Serfati's identity.

Our approach here is an adaptation of Lemma 2.3.1 in [13] and Lemma 2.10 in [6]. However, we substitute Serfati's identity in place of the Biot-Savart law, and, unlike [13], we consider data supported in all of $\mathbb{R}^2$ instead of just in a bounded domain.

Proposition 3.1.1. *Let $u \in L^\infty([0,T]; S_{h,g})$ be a solution to the Euler equations with stable initial vorticity, where h is a well-posedness growth bound and g is a pre-growth bound. Then for any $x, y \in \mathbb{R}^2$ such that $|y| \leq C(1 + |x|)$ and all*

$t \in [0, T]$,

$$|u(t, x + y) - u(t, x)| \leq |u_0(x + y) - u_0(x)| + C(T)h(x)\mu\left(\frac{|y|}{h(x)}\right).$$

Proof. Taking the difference of Serfati's identity evaluated at $(t, x + y)$ and (t, x) gives us

$$\begin{aligned}
&|u(t, x + y) - u(t, x)| \\
&\qquad \leq |u_0(x + y) - u_0(x)| \\
&\qquad\qquad + \left|a_\lambda K * \big((\omega(t, x + y) - \omega_0(x + y)) - (\omega(t, x) - \omega_0(x))\big)\right| \\
&\qquad\qquad + \int_0^t \left|[\nabla\nabla^\perp(1 - a_\lambda)K] * \cdot\big(u \otimes u(s, x + y) - u \otimes u(s, x)\big)\right| \, ds \\
&\qquad =: |u_0(x + y) - u_0(x)| + I_1 + \int_0^t I_2 \, ds.
\end{aligned}$$

To bound I_1, we proceed as in lemma 2.3.1 of [13]. First set $A = \{z : |x - z| \leq 2|y|\}$ and $B = \{z : |x - z| \leq |y| + \lambda\}$, where $\lambda > |y|$. (See fig. 3.1.) We note that $\operatorname{supp} a_\lambda((x + y) - \cdot) \cup \operatorname{supp} a_\lambda(x - \cdot) \subset B$. Then, using Proposition 2.5.6, we have

$$\begin{aligned}
I_1 &\leq \int_B \left|a_\lambda K(x + y - z) - a_\lambda K(x - z)\right|\left|\omega(t, z) - \omega_0(z)\right| \, dz \\
&\leq C(T) \int_B \left|a_\lambda K(x + y - z) - a_\lambda K(x - z)\right| \, dz \\
&\leq C(T) \int_A \left|a_\lambda K(x + y - z) - a_\lambda K(x - z)\right| \, dz \\
&\qquad + C(T) \int_{A^c \cap B} \left|a_\lambda K(x + y - z) - a_\lambda K(x - z)\right| \, dz \\
&=: C(T)(I_1^1 + I_1^2).
\end{aligned}$$

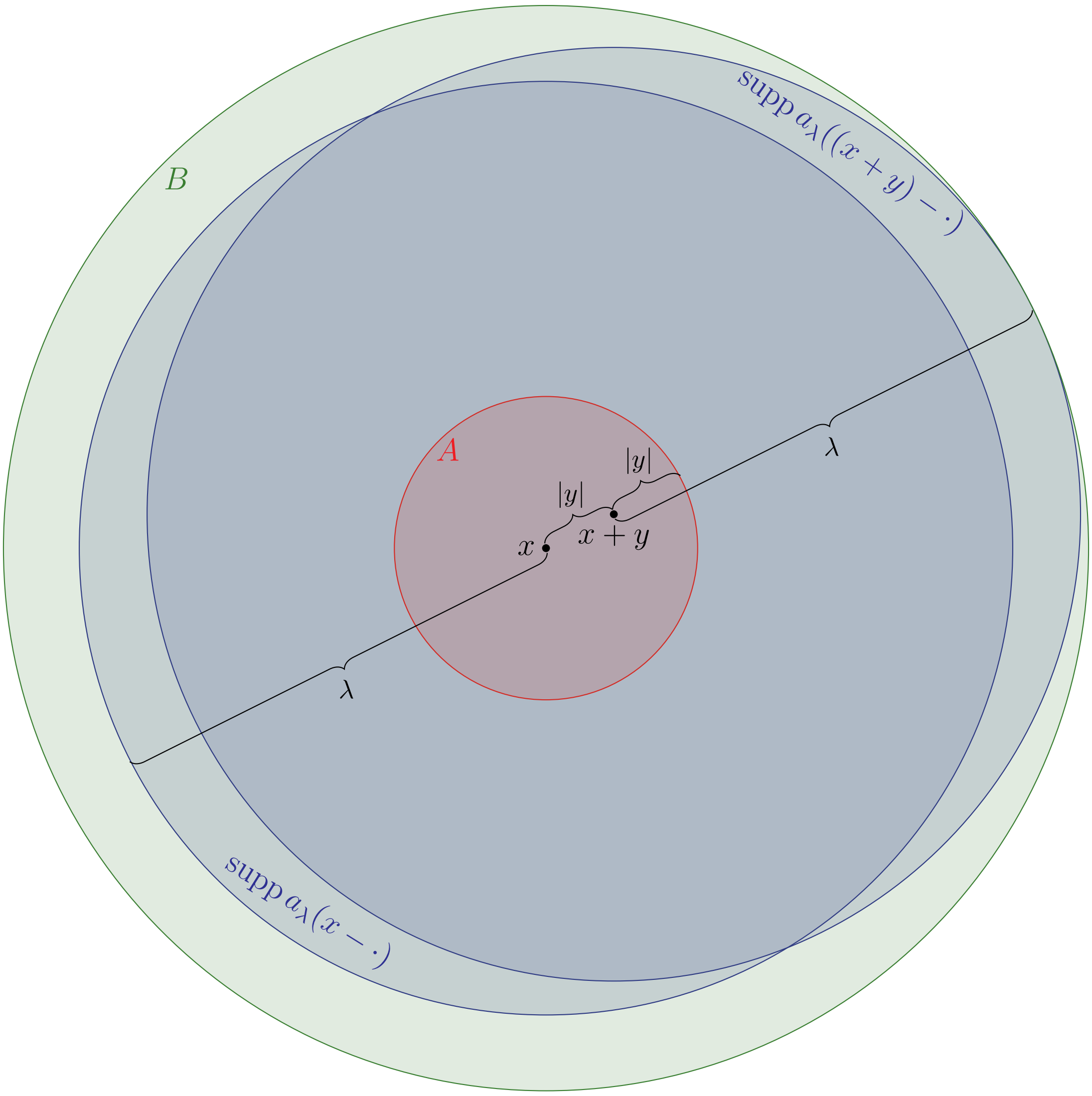

Figure 3.1. Regions of integration involved in bounding I_1

A brute force approach to I_1^1 gives

$$\begin{aligned}
I_1^1 &\leq \frac{1}{2\pi}\int_A \frac{1}{|x+y-z|} + \frac{1}{|x-z|}\, dz \\
&\leq \frac{1}{2\pi}\int_{|x-z|\leq 3|y|} \frac{1}{|x-z|}\, dz + \frac{1}{2\pi}\int_A \frac{1}{|x-z|}\, dz \\
&\leq C\int_{|x-z|\leq 3|y|} \frac{1}{|x-z|}\, dz \\
&\leq C|y|.
\end{aligned}$$

To bound I_1^2, we apply the mean value theorem. Specifically, for each fixed $z \in \mathbb{R}^2$ there exists some x_z on the line segment joining x and $x+y$ so that

$$\big|a_\lambda K(x+y-z) - a_\lambda K(x-z)\big| \leq |y|\big|\nabla(a_\lambda K)(x_z - z)\big|.$$

Then noting that $|x_z - z| \geq \frac{1}{2}|x-z|$ for all $z \in A^c$, we have

$$\begin{aligned}
I_1^2 &\leq |y|\int_{A^c\cap B} \big|\nabla(a_\lambda K)(x_z - z)\big|\, dz \\
&\leq |y|\int_{A^c\cap B} \frac{1}{|x_z - z|^2}\, dz \\
&\leq C|y|\int_{A^c\cap B} \frac{1}{|x-z|^2}\, dz \\
&\leq C|y|\int_{2|y|}^{|y|+\lambda} \frac{1}{r}\, dr \\
&\leq C|y| \log\frac{|y|+\lambda}{2|y|},
\end{aligned}$$

Note that to this point, we have not specified the value of λ, requiring only that $\lambda > |y|$. To get a favorable bound on I_2, we now choose $\lambda = 6|y| + 2h(x)$. Since $\big[\operatorname{supp}(1 - a_\lambda(\xi - \cdot))\big]^c$ is a ball of radius $\lambda/2$ centered at ξ, this implies that $\operatorname{supp}(1 - a_\lambda(x - \cdot)) \cup \operatorname{supp}(1 - a_\lambda(x + y - \cdot)) \subset D$, where $D = \{z : |x - z| > 2|y| + h(x)\}$. We note also that $D \subset A^c$. (See fig. 3.2.) We proceed to bound I_2 in a manner similar to our approach to I_1^2 above, but using Lemmas 1.3.1 and 1.3.3:

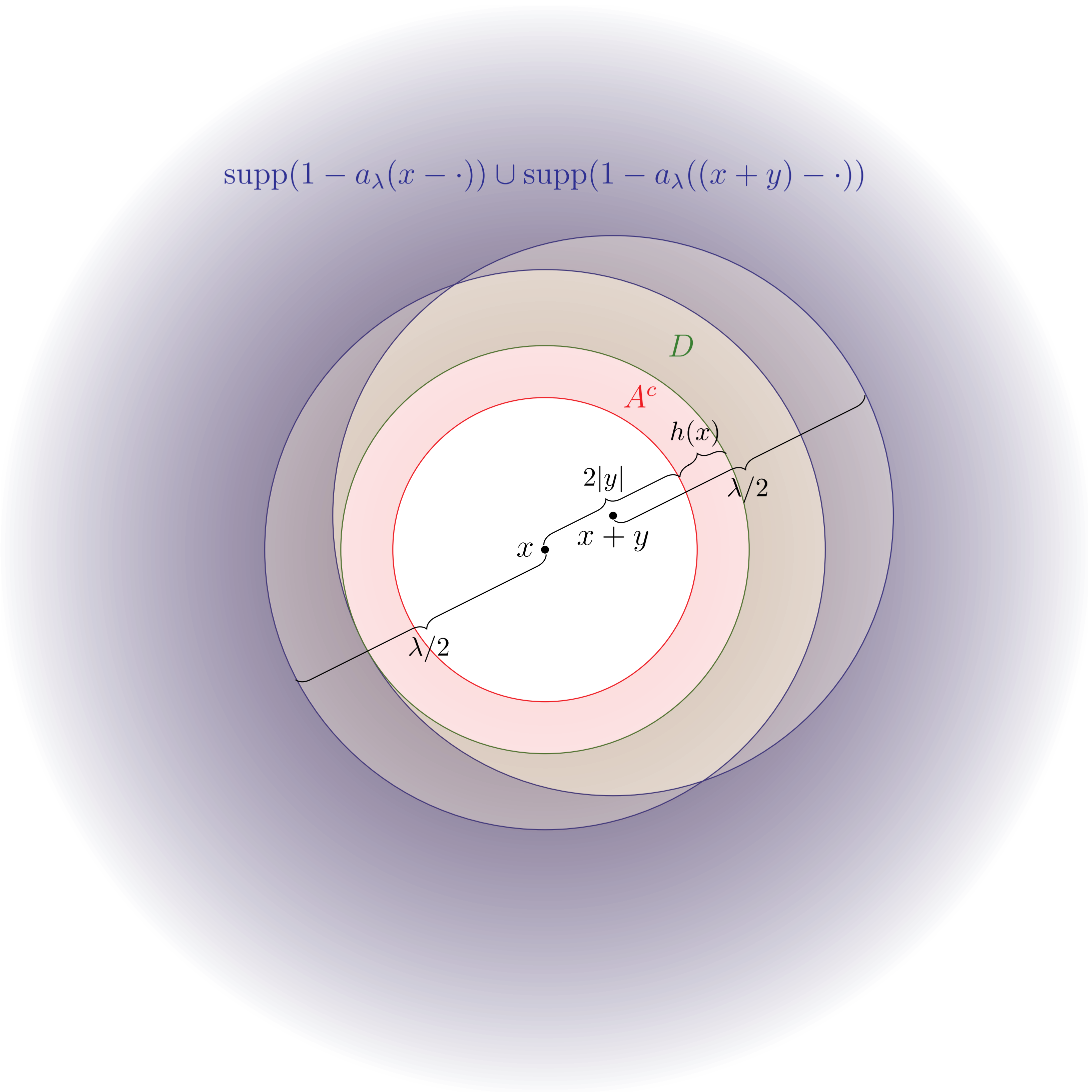

Figure 3.2. Regions of integration involved in bounding I_2

$$
\begin{aligned}
I_2 &\le \int_D \left|\nabla\nabla^\perp[(1-a_\lambda)K(x+y-z)-(1-a_\lambda)K(x-z)]\right||u\otimes u(s,z)|\ dz \\
&\le C(T)\int_D \left|\nabla\nabla^\perp[(1-a_\lambda)K(x+y-z)-(1-a_\lambda)K(x-z)]\right|h^2(z)\ dz \\
&\le C(T)|y|\int_D \left|\nabla\nabla\nabla^\perp[(1-a_\lambda)K(x_z-z)]\right|h^2(z)\ dz \\
&\le C(T)|y|\int_D \frac{h^2(z)}{|x_z-z|^4}\ dz \\
&\le C(T)|y|\int_D \frac{h^2(z)}{|x-z|^4}\ dz \\
&\le C(T)|y|\int_D \frac{h^2(x-z)}{|x-z|^4}+\frac{h^2(x)}{|x-z|^4}\ dz \\
&\le C(T)|y|\int_D \frac{c|x-z|+d}{|x-z|^4}+\frac{h^2(x)}{|x-z|^4}\ dz \\
&\le C(T)|y|\int_D \frac{1}{|x-z|^3}+\frac{h^2(x)}{|x-z|^4}\ dz \\
&\le C(T)|y|\int_{2|y|+h(x)}^\infty \frac{1}{r^2}+\frac{h^2(x)}{r^3}\ dr \\
&\le C(T)|y|\left(\frac{1}{2|y|+h(x)}+\frac{h^2(x)}{(2|y|+h(x))^2}\right) \\
&\le C(T)|y|.
\end{aligned}
$$

Synthesizing the above results with our choice of λ yields

$$
\begin{aligned}
|u(t,x+y)-u(t,x)| &\le |u_0(x+y)-u_0(x)|+C(T)|y|\left(1+\log\frac{7|y|+2h(x)}{2|y|}\right) \\
&\le |u_0(x+y)-u_0(x)|+C(T)|y|\log\left(c+\frac{h(x)}{|y|}\right).
\end{aligned}
$$

A simple exercise in calculus establishes that there exists some constant C so that

$$
\log\left(c+\frac{h(x)}{|y|}\right)\le C\log\left(\frac{h(x)}{|y|}\right)
$$

so long as $|y|/h(x)\le e^{-1}$, proving the result in this case. The result when

$|y|/h(x) > e^{-1}$ follows as in the proof of Proposition 1.5.2. ■

3.2. Proof of Uniqueness for Stable Initial Vorticity

The following proof is almost identical to theorem 1.7 in [4]; we include it for completeness.

Proof of Theorem 1.2.10. We first introduce several functions to simplify notation. Set

$$\begin{aligned}
\eta(t) &:= \left\| \frac{X_1(t,x) - X_2(t,x)}{h(x)} \right\|_{L^\infty_x(\mathbb{R}^2)}, \\
L(t) &:= \left\| \frac{u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))}{h(x)} \right\|_{L^\infty_x(\mathbb{R}^2)}, \\
M(t) &:= \int_0^t L(s)\, ds, \\
Q(t) &:= \left\| \frac{u_1(t) - u_2(t)}{h} \right\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned}$$

First, note that

$$\eta(t) \le \left\| \int_0^t \frac{u_1(s, X_1(s,x)) - u_2(s, X_2(s,x))}{h(x)}\, ds \right\|_{L^\infty_x(\mathbb{R}^2)} \le \int_0^t L(s)\, ds = M(t). \tag{3.1}$$

Setting

$$\begin{aligned}
A(s,x) &:= \frac{u_2(s, X_1(s,x)) - u_2(s, X_2(s,x))}{h(x)}, \\
B(s,x) &:= \frac{u_2(s, X_1(s,x)) - u_1(s, X_1(s,x))}{h(x)},
\end{aligned}$$

we see that

$$L(s) \le \|A(s,x)\|_{L^\infty_x(\mathbb{R}^2)} + \|B(s,x)\|_{L^\infty_x(\mathbb{R}^2)}.$$

To bound A, we first note that Lemmas 1.3.3 and 1.6.1 imply that

$$|X_1(s,x) - X_2(s,x)| \leq C(T)h(x) \leq C(T)(1+|x|).$$

Then we can apply Proposition 3.1.1 to obtain

$$|A(s,x)| \leq C(T)\mu(\eta(s)). \tag{3.2}$$

To bound B, we will apply Serfati's identity to each velocity field with $\lambda(x) = h(x)$. This yields

$$|B(s,x)| \leq \frac{1}{h(x)}(|B_1(s,x)| + |B_2(s,x)|),$$

where

$$\begin{aligned}
B_1(s,x) &= \left[(a_{h(x)}K) * (\omega_2(s) - \omega_1(s))\right](X_1(s,x)) \\
B_2(s,x) &= \int_0^s \Big(\nabla\nabla^\perp[(1-a_{h(x)})K] *\cdot (u_1 \otimes u_1 - u_2 \otimes u_2)\Big)(r, X_1(s,x))\, dr.
\end{aligned}$$

Now

$$\begin{aligned}
B_1(s,x) &= \int_{\mathbb{R}^2} (a_{h(x)}K(X_1(s,x)-y))(\omega_0(X_2^{-s}(y)) - \omega_0(X_1^{-s}(y)))\, dy \\
&\leq \int_{\mathbb{R}^2} (a_{h(x)}K(X_1(s,x)-y))(\omega_0(X_2^{-s}(y)) - \phi(X_2(s,X_2^{-s}(y))))\, dy \\
&\quad + \int_{\mathbb{R}^2} (a_{h(x)}K(X_1(s,x)-y))(\phi(X_2(s,X_2^{-s}(y))) - \phi(X_2(s,X_1^{-s}(y))))\, dy \\
&\quad + \int_{\mathbb{R}^2} (a_{h(x)}K(X_1(s,x)-y))(\phi(X_2(s,X_1^{-s}(y))) - \omega_0(X_1^{-s}(y)))\, dy.
\end{aligned}$$

Substituting $y = X_2(s,z)$ in the first integral and $y = X_1(s,z)$ in the third integral yields

$$B_1(s,x) \le \int_{\mathbb{R}^2} (a_{h(x)} K(X_1(s,x) - X_2(s,z)) - a_{h(x)} K(X_1(s,x) - X_1(s,z)))\Omega(s,z)\, dz$$
$$+ \int_{\mathbb{R}^2} (a_{h(x)} K(X_1(s,x) - y))(\phi(X_2(s, X_2^{-s}(y))) - \phi(X_2(s, X_1^{-s}(y))))\ dy$$
$$=: B_1^1(s,x) + B_1^2(s,x),$$

where

$$\Omega(s,z) = \omega_0(z) - \phi(X_2(s,z)).$$

But by Proposition 2.5.5,

$$|\Omega(s,z)| \le |\omega_0(z) - \phi(z)| + |\phi(z) - \phi(X_2(s,z))| \le C(T),$$

so that, by Lemmas 1.4.6 and 1.6.3, we have

$$|B_1^1(s,x)| \le C(T) h(x) \mu(\eta(s)). \tag{3.3}$$

Next, using the observation that $X_2(s, X_2^{-s}(y)) = y = X_1(s, X_1^{-s}(y))$ together with (1.7) and Proposition 2.5.2, we have

$$|\phi(X_2(s, X_2^{-s}(y))) - \phi(X_2(s, X_1^{-s}(y)))|$$
$$= |\phi(X_1(s, X_1^{-s}(y))) - \phi(X_2(s, X_1^{-s}(y)))|$$
$$\le \frac{|X_1(s, X_1^{-s}(y)) - X_2(s, X_1^{-s}(y))|}{\min\{h(X_1(s, X_1^{-s}(y))), h(X_2(s, X_1^{-s}(y)))\}}$$
$$\le C(T) \frac{|X_1(s, X_1^{-s}(y)) - X_2(s, X_1^{-s}(y))|}{h(X_1^{-s}(y))}$$
$$\le C(T)\eta(s).$$

Thus, Lemma 1.4.2 implies

$$|B_1^2(s,x)| \le C(T)h(x)\eta(s). \tag{3.4}$$

Combining (3.3) and (3.4) and gives us

$$|B_1(s,x)| \le C(T)h(x)(\mu(\eta(s)) + \eta(s)). \tag{3.5}$$

To bound B_2, we first observe that by Lemma 1.4.8,

$$\begin{aligned} B_2(s,x) &\le C(T)(H[h^2](h(x)/2) + h(x)) \int_0^s Q(r)\ dr \\ &\le C(T)(H[h^2](h(0)/2) + h(x)) \int_0^s Q(r)\ dr \\ &\le C(T)(1 + h(x)) \int_0^s Q(r)\ dr. \end{aligned}$$

But (1.7) and Proposition 3.1.1 imply

$$\begin{aligned} Q(r) &\le \left\| \frac{u_1(r, X_1(r,y)) - u_2(r, X_1(r,y))}{h(X_1(r,y))} \right\|_{L^\infty_y(\mathbb{R}^2)} \\ &\le \left\| \frac{u_1(r, X_1(r,y)) - u_2(r, X_2(r,y))}{h(X_1(r,y))} \right\|_{L^\infty_y(\mathbb{R}^2)} \\ &\quad + \left\| \frac{u_2(r, X_1(r,y)) - u_2(r, X_2(r,y))}{h(X_1(r,y))} \right\|_{L^\infty_y(\mathbb{R}^2)} \\ &\le C(T) \left(L(r) + \left\| \mu \left(\frac{|X_1(r,y) - X_2(r,y)|}{h(X_1(r,y))} \right) \right\|_{L^\infty_y(\mathbb{R}^2)} \right) \\ &\le C(T) \left(L(r) + \mu(C(T)\eta(r)) \right). \end{aligned} \tag{3.6}$$

Thus,

$$|B_2(s,x)| \le C(T)(1 + h(x)) \int_0^s L(r) + \mu(C(T)\eta(r))\ dr. \tag{3.7}$$

Bounds (3.5) and (3.7) and yield

$$|B(s,x)| \le C(T)\Big(\mu(\eta(s)) + \eta(s)\Big) + C(T)\int_0^s L(r) + \mu(C(T)\eta(r))\ dr. \tag{3.8}$$

Thus, combining (3.2) and (3.8), we obtain

$$L(s) \le C(T)\left(\mu(\eta(s)) + \eta(s) + \int_0^s L(r) + \mu(C(T)\eta(r))\ dr\right). \tag{3.9}$$

Then recalling (3.1) and observing that M is an increasing function, we calculate

$$\begin{aligned} M(t) &\le C(T)\int_0^t \mu(\eta(s)) + \eta(s) + \int_0^s L(r) + \mu(C(T)\eta(r))\ dr\, ds \\ &\le C(T)\int_0^t \mu(M(s)) + M(s) + \int_0^s \mu(C(T)M(r))\ dr\, ds \\ &\le C(T)\int_0^t \mu(C(T)M(s)) + M(s)\ ds. \end{aligned}$$

Then by Osgood's lemma,

$$\int_0^{M(t)} \frac{d\alpha}{\mu(C(T)\alpha) + \alpha} \le C(T)t. \tag{3.10}$$

But $\alpha \le -C(T)\alpha\log(C(T)\alpha) = \mu(C(T)\alpha)$ for $\alpha \le 1/(C(T)e)$ and, for any $\beta \in (0, 1/(C(T)e)]$,

$$\begin{aligned} \int_0^\beta \frac{d\alpha}{\mu(C(t)\alpha)} &= \int_0^\beta \frac{d\alpha}{-C(T)\alpha\log(C(T)\alpha)} \\ &= -\frac{1}{C(T)}\log\log\alpha\Big|_0^{C(T)\beta} \\ &= \infty. \end{aligned}$$

Then (3.10) can only be satisfied if $M(t) = 0$ for all $t \in [0,T]$. But then (3.1) and (3.6) imply that $\eta(t) = L(t) = Q(t) = 0$ for all $t \in [0,T]$. So the solution to the Euler equations in $S_{h,g}$ on $[0,T]$ with the stated initial coniditions is unique. ■

A. Appendix

A.1. Osgood's Lemma

Osgood's lemma is a generalization of Grönwall's inequality. The short proof below is attributed to Tehranchi; see [10].

Lemma A.1.1 (Osgood's Lemma)**.** *Let L be a measurable nonnegative function and γ a nonnegative locally integrable function, each defined on $[t_0, t_1]$. Let $\mu : [0, \infty) \to [0, \infty)$ be continuous and nondecreasing. Suppose that for all $t \in [t_0, t_1]$,*

$$L(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(L(s))\ ds. \tag{A.1}$$

If $a > 0$, then for each $t \in [t_0, t_1]$,

$$\int_{a}^{L(t)} \frac{ds}{\mu(s)} \leq \int_{t_0}^{t_1} \gamma(s)\ ds.$$

Proof. Observe that

$$\begin{aligned}
\int_{a}^{L(t)} \frac{dx}{\mu(x)} &\leq \int_{a}^{a+\int_{t_0}^{t} \gamma(u)\mu(L(u))\ du} \frac{dx}{\mu(x)} \\
&\leq \int_{t_0}^{t_1} \frac{\gamma(s)\mu(L(s))}{\mu(a + \int_{t_0}^{s} \gamma(u)\mu(L(u))\ du)}\ ds \\
&\leq \int_{t_0}^{t_1} \gamma(s)\ ds,
\end{aligned}$$

where we used (A.1) together with the assumption that μ is nondecreasing. ■

A.2. Confined Eddies

Let $\omega_0(r)$ be a radially symmetric smooth function, where $r = |x|$. Note that we can find a stream function ψ_0, which is a solution to the equation $\Delta\psi_0 = \omega_0$, which is radial as well since the Laplacian is rotationally invariant. Then it is simple to calculate that

$$u_0 = \nabla^\perp \psi_0 = \frac{x^\perp}{r}\psi_0', \tag{A.2}$$

$$\omega_0 = \Delta\psi_0 = \psi_0'' + \frac{1}{r}\psi_0'. \tag{A.3}$$

But (A.3) implies that

$$\psi_0'(r) = \frac{1}{r}\int_0^r s\omega_0(s)\, ds,$$

so that

$$u_0(x) = \frac{x^\perp}{r^2}\int_0^r s\omega_0(s)\, ds.$$

Solutions to the Euler equations with such radial initial data are steady; see [12]. We call such a solution a *steady radial eddy*.

A *confined eddy* is a steady radial eddy with compactly supported velocity. In [9], the authors establish the rate of convergence of solutions of the Navier-Stokes equations to that of the Euler equations in the vanishing viscosity limit when the initial vorticity is a superposition of confined eddies having disjoint supports; our interest in confined eddies is primarily in their versatility in constructing a rich class of solutions to the Euler equations.

Two properties of confined eddies make them especially useful for our purposes. First, since confined eddies have compact support, we can superimpose confined eddies with disjoint supports without the eddies interacting.

Second, note that we can construct a confined eddy with an arbitrarily large L^∞ bound on the vorticity and yet with an arbitrarily small L^∞ bound on the velocity by requiring the eddy have sufficiently small support. Indeed, observe that if $\operatorname{supp} u_0 \in B_R(0)$, then

$$\|u_0\|_{L^\infty(\mathbb{R}^2)} = \|\psi_0'\|_{L^\infty(\mathbb{R}^2)} \leq \frac{R}{2}\|\omega_0\|_{L^\infty(\mathbb{R}^2)}.$$

Thus, we can guarantee $\|u_0\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$ for any $\varepsilon > 0$—no matter how large $\|\omega_0\|_{L^\infty(\mathbb{R}^2)} < \infty$ is—as long as $0 < R < 2\varepsilon/\|\omega_0\|_{L^\infty(\mathbb{R}^2)}$.

Confined eddies provide a simple yet versatile class of solutions to the Euler equations that we use as a building block in the initial data we construct in Examples 2.4.3 and 2.5.7. While superpositions of confined eddies with disjoint supports only yield stationary solutions to the Euler equations, even small perturbations to the initial data will lead to time evolving solutions.

Bibliography

[1] D. M. Ambrose, J. P. Kelliher, M. C. L. Filho, and H. J. N. Lopes. Serfati solutions to the 2D Euler equations on exterior domains. *J. Differential Equations*, 259(5):4509–4560, 2015.

[2] E. Cozzi. Solutions to the 2D Euler equations with velocity unbounded at infinity. *J. Math. Anal. Appl.*, 423(1):144–161, 2015.

[3] E. Cozzi and J. P. Kelliher. Incompressible Euler equations and the effect of changes at a distance. *J. Math. Fluid Mech.*, 18(4):765–781, 2016.

[4] E. Cozzi and J. P. Kelliher. Well-posedness of the 2D Euler equations when velocity grows at infinity. *Discrete & Continuous Dynamical Systems - A*, 39(5):2361–2392, 2019.

[5] T. M. Elgindi. Remarks on functions with bounded Laplacian. *arXiv:1605.05266*, 2016.

[6] T. M. Elgindi and I.-J. Jeong. Symmetries and critical phenomena in fluids. *Communications on Pure and Applied Mathematics*, 73(2):257–316, 2020.

[7] L. Euler. Principes généraux du mouvement des fluides. *Mémoires de l'académie des sciences de Berlin*, 11:274–315, 1757.

[8] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2 edition, 2010.

[9] M. L. Filho, H. N. Lopes, and Y. Zheng. Convergence of the vanishing viscosity approximation for superpositions of confined eddies. *Comm. Math. Phys*, 201(2):291–304, 1999.

[10] J. P. Kelliher. Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane. *SIAM J. Math. Anal.*, 38(1):210–232, 2006.

[11] J. P. Kelliher. A characterization at infinity of bounded vorticity, bounded velocity solutions to the 2D Euler equations. *Indiana Univ. Math. J.*, 64(6):1643–1666, 2015.

[12] A. J. Majda and A. L. Bertozzi. *Vorticity and Incompressible Flow*, volume 27 of *Cambridge texts in applied mathematics*. Cambridge University Press, Cambridge, 2002.

[13] C. Marchioro and M. Pulvirenti. *Mathematical Theory of Incompressible Non-viscous Fluids*, volume 96 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1994.

[14] F. J. McGrath. Nonstationary plane flow of viscous and ideal fluids. *Arch. Rational Mech. Anal.*, 27:329–348, 1967.

[15] P. Serfati. Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(5):555–558, 1995.

[16] Y. Taniuchi. Uniformly local L^p extimate for the 2-D vorticity equation and its application to Euler equations with initial vorticity in *bmo*. *Commun. Math. Phys.*, 248:169–186, 2004.

[17] Y. Taniuchi, T. Tashiro, and T. Yoneda. On the two-dimensional Euler equations with spatially almost periodic initial data. *J. Math. Fluid Mech.*, 12(4):594–612, 2010.

[18] M. Vishik. Incompressible flows of an ideal fluid with vorticity in the borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)*, 32(6):769–812, 1999.

[19] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.

[20] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2(1):27–38, 1995.