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The arithmetic of the Hitchin component

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The arithmetic of the Hitchin component

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by

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The Dissertation of Michael Zshornack is approved.

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The arithmetic of the Hitchin component

To my parents.

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Abstract

The arithmetic of the Hitchin component

by

Michael Zshornack

This thesis presents a number of results surrounding the arithmetic properties of the Hitchin component, a space of representations of the fundamental group of a closed surface into split real forms of simple Lie groups such as $\mathrm{SL}(n, \mathbf{R})$. The first main result is when the existence of a Hitchin representation defined over certain prescribed subrings of $\mathbf{R}$ may be deformed to a Zariski-dense one defined over the same ring. The second, produced in joint work with Jacques Audibert, provides a topological characterization for the collection of Hitchin representations defined over $\mathbf{Q}$. Central to establishing these results is further developing the arithmetic nature of bending deformations on the Hitchin component. This further develops a perspective first taken in work of Long and Thistlethwaite who studied these deformations in an arithmetic context in order to produce examples of thin surface subgroups of $\mathrm{SL}(2k+1, \mathbf{Z})$.

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Chapter 1

Motivation

The results contained in this thesis concern the arithmetic properties of the Hitchin component, a particular space of representations of surface groups that forms an example of a *higher-rank Teichmüller space.* These results share themes with much work in the last decades of low-dimensional geometry and topology, particularly in seeking to understand how surface groups embed into other groups in interesting algebraic, arithmetic and dynamical manners. We begin with some discussion of the surrounding motivation and history of the subject, and a brief introduction to the main ideas used. Most details and definitions have been deferred to Chapter 2.

Let G be a semisimple (real or complex) Lie group and $\Lambda < G$ a lattice, that is, a finitely generated discrete subgroup of finite covolume. By a *surface subgroup* of Λ, we mean a subgroup isomorphic to the fundamental group of a closed, connected, orientable surface of genus 2 or more. Much of this work is motivated by the following question.

Question 1. Does Λ contain a surface subgroup?

Lattices in Lie groups have been extensively studied throughout mathematics from a number of different perspectives, but our collective understanding of which finitely generated, infinite groups arise as their subgroups is relatively poor. The simplest possible class of such subgroups (e.g. in the sense of cohomological dimension) are free subgroups, and the Tits alternative (see [Tit72]) provides an answer to the analog of the above question for these. Armed with this knowledge for free subgroups, surface subgroups are a natural next step, amidst the hierarchy of isomorphism types of finitely generated groups to consider. When Λ is cocompact and G has real-rank one, this question is also a special instance of one originally asked by Gromov in [Gro87], on whether every one-ended word hyperbolic group contains a surface subgroup. Thus the above can also be thought of as a natural generalization of Gromov's question to the more general non-positively curved setting.

While Question 1 remains unknown in full generality, its resolution in various special cases constitutes a number of celebrated results in mathematics. For instance, when $G = \mathrm{SO}(3,1)$, the answer to Question 1 is "yes" for all lattices by work of Cooper–Long–Reid [CLR97] and Kahn–Marković [KM12] establishing the result for non-cocompact and cocompact lattices respectively. Notably, this latter work was instrumental in Agol's resolution to the virtual Haken conjecture of Waldhausen [Ago13] and illustrates how even just understanding the surface subgroups of a lattice can provide deep information about the overall geometry of the lattice itself. A positive answer to Question 1 is also known to be true for many other classes of cocompact lattices, for instance, ones in other rank one Lie groups not locally isomorphic to $\mathrm{SO}(2k,1)$ by work of Hamenstädt [Ham15] and ones in complex Lie groups by work of Kahn–Labourie–Mozes [KLM18].

Amidst the search for surface subgroups of lattices, it is also natural to further refine our parameters and ask for such subgroups possessing particular qualities of interest to other areas of mathematics. For instance, the following is a natural algebraic refinement of Question 1.

Question 2. Does Λ contain a *Zariski-dense* surface subgroup?

Such Zariski-dense subgroups often form examples of *thin groups* (see Definition 2.1.5) and are of independent interest within number theory for the many properties they share with lattices, despite having infinite covolume. For instance, thin groups satisfy forms of the strong approximation theorem (see Theorem 2.1.3). Exhibiting surface subgroups of lattices which are thin is also of note, as such examples are rare in an appropriate sense. More generally, geometric techniques similar to the ones used in this work lie behind a significant bulk of the known examples of non-free isomorphism types of thin subgroups of lattices in higher-rank Lie groups (see Question 4 and the discussion following).

The following is another refinement of Question 1, now of a more dynamical flavor.

Question 3. Does Λ contain an *Anosov* surface subgroup?

Anosov subgroups (see Definition 2.3.4) are of interest in dynamics and geometry, due to the rich properties seen by the actions of these groups on the symmetric spaces associated to G. In addition to mathematicians' independent interest in them, understanding the Anosov subgroups of Λ also presents a particularly promising means of understanding the Zariski-dense ones, due to the deformation theory accompanying much of the study of the former. In fact, this idea of using Anosov representations of surface groups and the rich deformation theory thereof to answer forms of Question 2 originates with work of (various combinations of) Long, Reid and Thistlethwaite first resolving this question for various higher-rank lattices (e.g. see [LRT11, LT18, LT22] for ones of the form $\mathrm{SL}(n, \mathbf{Z})$). Continuing this line of thought, a major theme explored in this thesis is the utility in understanding Question 3 as a means of understanding Question 2.

Remark. Though Question 3 focuses on one class of surface subgroups which are interesting from a geometric perspective, it also is an interesting question to ask whether a lattice possesses a surface subgroup which is *not* Anosov. For instance, it is a deep result of Agol that cocompact lattices of $\mathrm{SO}(3, 1)$ also contain surface subgroups which virtually fiber [Ago13], and these are never Anosov. Though perhaps notably, this work crucially relies on the results of [KM12], and so for other Lie groups, understanding the existence of Anosov surface subgroups may also be a helpful step in understanding the existence of non-Anosov ones.

It is of note that while many of the above mentioned results, particularly for rank one Lie groups, do not discuss the further refinements of Questions 1 mentioned, the examples produced in the works [KM12, Ham15, KLM18] are all Anosov and can easily be made Zariski-dense with small modifications. Additionally, Cooper–Futer and Kahn–Wright independently provided Zariski-dense and Anosov examples for non-cocompact lattices in $\mathrm{SO}(3,1)$ [CF19, KW21].

This thesis primarily focuses on when G is a *higher-rank* Lie group, where most of the presented results are novel. This is also the context where the notion of a subgroup being *Anosov* is really of use, whereas in rank one, this condition is equivalent to being convex cocompact which is, in practice, easier to work with. Unlike in the rank one setting where much of the previous work resolving Question 1 has simultaneously answered Questions 2 and 3, in higher-rank, the three questions appear to be much more distinct. For example, at the time of writing, it is currently unknown whether the lattice $\mathrm{SL}(6,\mathbf{Z}) < \mathrm{SL}(6,\mathbf{R})$ possesses a Zariski-dense surface subgroup, despite it possessing many Anosov ones (see Example 3.3.1).

On the other hand, in the higher-rank context, the importance of arithmetic tools takes a spotlight. This is exemplified by work of Margulis who roughly showed that for higher-rank G, all irreducible lattices arise through "number-theoretic constructions" (see Definition 2.1.4 and Theorem 2.1.2). This is in contrast with rank one where "most lattices" can be non-arithmetic (see Theorem 2.2.3). Thus, where higher-rank poses new difficulties not present in rank one, some of the pain is mitigated by the niceties provided by its more arithmetic nature.

The approach taken in this thesis towards understanding the above questions is representation theoretic. That is, rather than seeking to understand how surface groups embed into lattices of G directly, we consider $\mathrm{Hom}(\pi_1(S), G)$, the space of all group homomorphisms from a surface group, $\pi_1(S)$, into the target Lie group G. From this perspective the representations we would like to understand, i.e. faithful ones with image contained in a lattice, now live inside a larger space possessing the natural structure of an algebraic variety. Margulis's arithmeticity theorem suggests a loose connection between the surface subgroups of lattices and the number-theoretic properties of $\mathrm{Hom}(\pi_1(S), G)$, in an arithmetic geometric sense. The main results of this thesis (Theorems 2.4.1, 2.4.2 and 2.4.3) illustrate how one may understand aspects of the arithmetic of representation varieties by leveraging their underlying geometry, with a view towards understanding the above questions. These results also provide a large class of examples of surface subgroups which, while not necessarily contained in lattices of G, are contained in lattices of larger groups containing G (see Example 2.1.4).

A means of understanding faithfulness of representations will be crucial in the passage from surface group representations to surface subgroups as in the initial questions. Thus much of the discussion of these results will primarily be focused on representations lying on the *Hitchin component* (see Definition 2.3.2) where, like the holonomies associated to hyperbolic structures on surfaces, all representations are discrete and faithful. In general, the theory of higher-rank Teichmüller spaces and other spaces of Anosov representations provide a particularly fruitful geometric means of understanding when representations of surface groups and other word hyperbolic groups are discrete and faithful. The study of these representations constitutes an incredibly deep amount of research today, spanning various areas of mathematics such as geometry, dynamics and group theory (e.g. see [Wie18]) and various aspects of this theory will be crucial in establishing our main results. Some indications on how our results may be generalized to other spaces of Anosov representations are discussed as well.

Finally, central to the proofs of the main results is a deformation of representations that has appeared in many contexts throughout the years, known as *bending* (see §3.1). In the context of surfaces, these deformations were first discussed by Thurston in the context of quasi-Fuchsian subgroups of $\mathrm{PSL}(2, \mathbf{C})$ [Thu79, §8.7]. Similar deformations can be performed on any representation of a surface group, and their geometric properties have been extensively studied by many. The utility of applying these deformations towards arithmetic questions first came to light when Long and Thistlethwaite used bending to construct Zariski-dense surface subgroups of $\mathrm{SL}(2k + 1, \mathbf{Z})$ [LT22]. One of the major novel contributions of this thesis is further developing and understanding the arithmetic perspective behind these deformations first arising with their work. In particular, the construction performed in [LT22] is extended to a more general context, outlining how understanding the arithmetic of the whole Hitchin component can be reduced to understanding the arithmetic of a certain family of algebraic groups. In addition to these methods featuring throughout this work, this is also a similar perspective taken by Audibert in [Aud22, Aud23], highlighting how these specific deformations can be particularly fruitful for arithmetic purposes.

1.1 Organization

The remainder of this thesis is organized into three chapters. Chapter 2 discusses the necessary preliminary background on classical and higher-rank Teichmüller spaces and on thin groups needed to state the main results. Most of the content here is due to the work of previous authors and, when omitted, appropriate references to documented proofs in the literature are given. This is also where the statement of this thesis's main results are contained (see §2.4). Chapter 3 focuses on the proofs of Theorems 2.4.1

and 2.4.2, by investigating the underlying *integral* structure of the Hitchin component, one particularly nice space of surface group representations. Constructions of Zariski-dense surface subgroups defined over certain rings of integers are discussed, in addition to the *bending* construction being first developed in detail. Most of the content of this chapter appears in work of the author in [Zsh22]. Chapter 4 focuses on the *rational* structure of the Hitchin component, proving Theorem 2.4.3 which provides a topological characterization of Hitchin representations of surface groups which are defined over $\mathbf{Q}$. The interactions between bending and the underlying symplectic geometry of the Hitchin component are further developed here as well. Much of the content of this chapter appears in joint work of the author with Jacques Audibert in [AZ23] and has been reproduced here with the latter's permission.

Chapter 2

Preliminaries

The majority of the content contained in this chapter consists of the necessary background needed in order to frame the discussion of this thesis's main goals in §2.4. For most results reviewed, citations are given to proofs in the literature.

2.1 Lie groups and their discrete subgroups

The standing assumptions for this section will be that G is a semisimple, real Lie group with finitely many connected components and finite center. It would not hurt to just take $G = \mathrm{SL}(n, \mathbf{R})$ in most of what follows, though the use of other Lie groups will come to play at various points, and so we keep the discussion relatively general. We first recount some key notions associated to G before focusing on its discrete subgroups, mostly following [Mor15].

Definition 2.1.1. The *symmetric space* of G is the homogeneous space G/K where $K \leqslant G$ is a (unique up to conjugacy) maximal compact subgroup of G.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ an associated Cartan decomposition. The restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{p}$ is positive-definite and $\mathfrak{p}$ can be identified with the tangent space to the identity coset $T_K(G/K)$. Thus, the Killing form induces a G-invariant Riemannian metric on G/K, the volume form on which agrees with the one induced by the Haar measure on G.

Example 2.1.1 (The symmetric space of $\mathrm{SL}(n,\mathbf{R})$)**.** $\mathrm{SO}(n) < \mathrm{SL}(n,\mathbf{R})$ is a maximal compact subgroup. The following is an explicit description of the symmetric space $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$, given in more detail, for instance, in [Sch13]. Let X_n denote the set of $n \times n$ real, symmetric, positive-definite matrices of determinant 1. Define an action $\mathrm{SL}(n,\mathbf{R}) \curvearrowright X_n$ by

$$A \cdot M := AMA^T.$$

Observe that this action is transitive since by elementary linear algebra, any $M \in X_n$ is of the form $M = AA^T$ for some $A \in \mathrm{SL}(n,\mathbf{R})$. The tangent space at the identity, $T_{I_n}(X_n)$ is the collection of traceless, symmetric matrices, on which the form $\langle X, Y\rangle = \mathrm{Tr}(XY)$ is an inner product. Transitivity of the action allows one to extend this form to an $\mathrm{SL}(n,\mathbf{R})$-invariant Riemannian metric on X_n. The map

$$\begin{aligned} \mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n) &\to X_n \\ A\,\mathrm{SO}(n) &\mapsto AA^T \end{aligned}$$

is an $\mathrm{SL}(n,\mathbf{R})$-equivariant Riemannian isometry, giving an identification of the symmetric space $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$ with X_n.

Definition 2.1.2. The $\mathbf{R}$-*rank* (or sometimes just *rank*) of G, $\mathrm{rank}_{\mathbf{R}}(G)$, is the maximal dimension k for which there is a totally geodesic flat subspace of G/K homeomorphic to $\mathbf{R}^k$. A G of $\mathbf{R}$-rank 2 or more will be called a *higher-rank* group.

Example 2.1.2 (The rank one simple Lie groups)**.** The rank one simple Lie groups of non-compact type are precisely the ones whose symmetric spaces are hyperbolic, that is, ones of (possibly variable) negative sectional curvature. Up to local isomorphism, they are indexed by the three infinite families $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$ (the isometry groups of the real, complex and quaternionic hyperbolic spaces respectively) and one exceptional group F_4^{-20} (the isometry group of the Cayley plane). Note that there are local isomorphisms $\mathrm{PSL}(2,\mathbf{R}) \cong \mathrm{SO}(2,1) \cong \mathrm{SU}(1,1)$ and $\mathrm{PSL}(2,\mathbf{C}) \cong \mathrm{SO}(3,1) \cong \mathrm{Sp}(1,1)$.

Example 2.1.3 ($\mathrm{rank}_{\mathbf{R}}(\mathrm{SL}(n,\mathbf{R})) = n-1$)**.** Let $\Delta = \{(t_1,\ldots,t_n) \in \mathbf{R}^n \mid \sum_i t_i = 0\}$ and equip it with the metric induced by the standard one on $\mathbf{R}^n$. Note that Δ is homeomorphic to $\mathbf{R}^{n-1}$ and, identifying the symmetric space of $\mathrm{SL}(n,\mathbf{R})$ with X_n as in Example 2.1.1, the map

$$\Delta \to X_n$$
$$(t_1,\ldots,t_n) \mapsto \mathrm{diag}(e^{t_1},\ldots,e^{t_n})$$

is a geodesically embedded flat in X_n of maximal dimension.

2.1.1 Lattices

We next turn to discussing the discrete subgroups of Lie groups. The prototypical examples of which are lattices, or, roughly, the subgroups of G which are neither too small nor too big. In other words:

Definition 2.1.3. A *lattice* is a discrete subgroup $\Lambda < G$ of finite covolume, that is $\mu(\Lambda\backslash G) < \infty$ where μ denotes the Haar measure. A lattice is *cocompact* if $\Lambda\backslash G$ is, in fact, compact.

The study of properties of lattices has a deep history in mathematics, involving much mathematics spanning geometry, group theory, number theory, dynamics and more. Some of the simplest constructions of lattices arise via the following result of Borel and Harish-Chandra.

Theorem 2.1.1 ([BHC62]). *Let* $\mathbf{G}$ *be a linear, semisimple, algebraic group, defined over* $\mathbf{Q}$*. Then* $\mathbf{G}(\mathbf{Z})$ *is a lattice in* $\mathbf{G}(\mathbf{R})$*.*

Thus, for example, subgroups such as $\mathrm{SL}(n, \mathbf{Z}) < \mathrm{SL}(n, \mathbf{R})$ are lattices. Via Weil's *restriction of scalars* map, one may construct more examples of lattices using the above theorem such as in the following.

Example 2.1.4. Let $\sigma : \mathbf{Q}(\sqrt{2}) \to \mathbf{R}$ be the field embedding induced by sending $\sqrt{2} \mapsto -\sqrt{2}$. Embed $\mathrm{SL}(n, \mathbf{Z}[\sqrt{2}]) \to \mathrm{SL}(n, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ via the map

$$A \mapsto (A, A^\sigma)$$

where A^σ denotes applying the field embedding σ entry-wise to A. Then the image of $\mathrm{SL}(n, \mathbf{Z}[\sqrt{2}])$ is a lattice in $\mathrm{SL}(n, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$. More specifically, we may think of $\mathrm{SL}(n, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ as the $\mathbf{R}$-points of a $\mathbf{Q}(\sqrt{2})$-algebraic group, $\mathbf{G}$, whose group of rational points, $\mathbf{G}(\mathbf{Q}(\sqrt{2}))$, is precisely the image of $\mathrm{SL}(n, \mathbf{Q}(\sqrt{2}))$ under the above map. Restriction of scalars yields a new algebraic group, $\mathbf{H} = \mathrm{Res}_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}} \mathbf{G}$, so that $\mathbf{H}(\mathbf{Q}) = \mathbf{G}(\mathbf{Q}(\sqrt{2}))$ and $\mathbf{H}(\mathbf{Z})$ is precisely the image of $\mathrm{SL}(n, \mathbf{Z}[\sqrt{2}])$, and thus, a lattice in $\mathrm{SL}(n, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ by Theorem 2.1.1.

Being a lattice is also preserved under passage to quotient with compact kernel, providing even more sources of lattices. More generally, the following definition encompasses all those that arise via these constructions.

Definition 2.1.4. A lattice $\Lambda < G$ is *arithmetic* if there exists a semisimple $\mathbf{Q}$-algebraic group $\mathbf{H}$ and a surjective Lie group homomorphism with compact kernel $\psi : \mathbf{H}(\mathbf{R})^\circ \to G$ so that $\psi(\mathbf{H}(\mathbf{Z}) \cap \mathbf{H}(\mathbf{R})^\circ)$ is commensurable to Λ.

Not all lattices are arithmetic. For example, the above construction only ever produces, at most, countably many conjugacy classes of lattices in a fixed Lie group G. On the other hand, $\mathrm{PSL}(2, \mathbf{R})$ possesses uncountably many such lattices (see Theorem 2.2.3), hence "most" must be non-arithmetic. Moreover, while Mostow–Prasad rigidity also implies that the other rank one Lie groups $\mathrm{SO}(n, 1)$ for $n \geq 3$ only possess countably many conjugacy classes of lattices, non-arithmetic examples exist in these groups for every $n \geq 3$, for instance, ones due to Gromov–Piatetski-Shapiro [GPS88]. Thus, with these examples in mind, the following result of Margulis [Mar91] may come as a surprise, saying that essentially, such examples are constrained to rank one (see, e.g., [Mor15, Theorem 5.2.1] for a proof)

Theorem 2.1.2 (Margulis's arithmeticity theorem)**.** *Suppose* $\mathrm{rank}_{\mathbf{R}}(G) \geq 2$ *and that* $\Lambda < G$ *is an irreducible lattice, i.e.* Λ *does not decompose as a non-trivial product of the form* $\Lambda_1 \times \Lambda_2$*. Then* Λ *is arithmetic.*

Remark. The same conclusion as above also happens to be true for the rank one groups $\mathrm{Sp}(n, 1)$ and F_4^{-20} by work of Corlette [Cor92], thus non-arithmeticity can be viewed as a phenomenon constrained to $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$. In fact, at the time of writing, among the $\mathrm{SU}(n, 1)$ for $n \geq 2$, all non-arithmetic examples known are covered by 22 commensurability classes of examples in $\mathrm{SU}(2, 1)$ and 2 commensurability classes in $\mathrm{SU}(3, 1)$ (see [Der20, DPP21]).

2.1.2 Thin groups

In what follows, we will further assume that our Lie group G arises as the $\mathbf{R}$-points of an algebraic group. Thus, in addition to the *Euclidean topology* coming from its manifold structure, G also possesses a *Zariski topology*, coming from its structure as an algebraic variety. Much of this thesis concerns objects (G and otherwise) equipped with both of a Euclidean and Zariski topology. We shall typically preface qualities which are true in the latter topology with "Zariski-" and use no prefix when dealing with the former. Therefore, a subset is *Zariski-dense* if it is dense in the Zariski-topology and it is just *dense* if it is dense in the Euclidean one.

Thin groups are an interesting class of subgroups of G whose traits appear drastically different as the topology they are viewed in changes: they are "very sparse" in one yet dense in the other. Following [Sar14], we make the following definition.

Definition 2.1.5. Let $\Lambda < G$ be a lattice. A subgroup $\Gamma < \Lambda$ is *thin* if it is infinite-index in Λ and Zariski-dense in G.

Remark. More generally, one may define thin groups without mention of an ambient lattice, instead stating that $\Gamma < G$ is thin if it is both Zariski-dense and infinite covolume. We will only refer to groups as being thin if they satisfy the more narrow definition.

Much more recently than the previous section's results on lattices, the study of thin groups has attracted much attention in mathematics as properties of lattices have been extended to this larger class of groups. In short, mathematicians began to note that proofs of many properties of lattices did not need the full strength of the assumption that these groups had finite covolume. Rather, for many proofs, the assumption that the lattices were Zariski-dense (a result due to Borel in [Bor60]) was sufficient. Thus, a number of results that were previously only known for lattices could be expanded to much broader classes of groups.

One of the earlier results extending facts previously known about lattices to the class of thin groups is the following "strong approximation" result of Matthews–Vaserstein–Weisfeiler.

Theorem 2.1.3 ([MVW84]). *If* $\mathbf{G}$ *is a connected, simply-connected, absolutely almost simple* $\mathbf{Q}$*-algebraic group and* $\Gamma < \mathbf{G}(\mathbf{Q})$ *a finitely generated, Zariski-dense subgroup,* *then for all but finitely many primes p, the reduction map*

$$\pi_p : \Gamma \to \mathbf{G}(\mathbf{F}_p)$$

is surjective.

Even more recently, this theorem for thin groups has been strengthened even further and studied from the point of view of spectral gap results and expansion properties associated to the Cayley graphs of the groups $\pi_p(\Gamma)$. This has led to the theory of "superstrong approximation" drawing much recent attention in combinatorics and number theory (see [KLLR19] for a survey).

But despite the deep investigation into the rich properties of thin groups, constructions of such examples are comparatively sparse. There is belief that "generically," subgroups of lattices should be thin (c.f. [Fuc14]), suggesting that randomly selecting elements in a lattice and taking the group generated by them should yield a thin group with high probability. However, a significant flaw of such "random" considerations is that these probabilistic constructions also almost surely produce groups which are isomorphic to free groups (see [Aou11]), and thus producing non-free examples requires a more subtle strategy. Even moreso, given generators which one may happen to know generate a non-free subgroup of a lattice, determining thinness is much more difficult due to the fact that computing the index of a subgroup given only generators is hard. This raises the following (c.f. Question 2).

Question 4. Which isomorphism types of groups arise as thin subgroups of lattices?

There is recent interest, even just among low-dimensional topologists, in such a question. For instance, there have been a number of large developments in recent years on whether or not various finitely generated, residually finite groups are *profinitely rigid.* Using geometric methods, Bridson, McReynolds, Reid and Spitler establish profinite rigidity for various rank one lattices in [BMRS20, BMRS21]. In the former of these papers, they highlight that a major obstacle in extending their methods to higher-rank lattices, like $\mathrm{SL}(n, \mathbf{Z})$, is the considerable lack in our understanding of their thin subgroups.

Question 4 was also raised in [BL20] prior to the authors showing that fundamental groups of many arithmetic hyperbolic manifolds can arise as thin subgroups in $\mathrm{SL}(n, \mathbf{R})$. In addition to these examples and the examples of surface groups listed in Chapter 1, see [Bal20] for examples isomorphic to Gromov–Piatetski-Shapiro lattices and [Dou22] for examples constructed using reflection groups. Thus, with regards to producing thin subgroups *of known isomorphism type*, geometric methods have proven particularly useful in showing that what is predicted to be a rare phenomenon can in fact occur.

The following proposition illustrates how geometry can circumvent the difficulties presented earlier.

Proposition 2.1.4. *Let Λ be an irreducible lattice in a higher-rank G. If Λ contains a word hyperbolic subgroup $\Gamma \leqslant \Lambda$, then $[\Lambda : \Gamma] = \infty$.*

Proof. One may regard this as a manifestation of the idea that higher-rank lattices should exhibit *strictly* non-positively curved phenomena. Indeed, if $[\Lambda : \Gamma] < \infty$, then Γ would itself be an irreducible lattice in G. However, higher-rank lattices are known to contain free abelian subgroups of rank equal to $\mathrm{rank}_{\mathbf{R}}(G)$ (c.f. [PR72, Corollary 2.9]), whereas word hyperbolic groups can only contain free abelian subgroups of rank at most 1 (c.f. [Löh17, Corollary 7.5.15]). □

Remark. Our interest in this proof is that, in some sense, it is more elementary, yet more general, than other proofs of this same fact that the author was aware of. For example, a similar fact can be established for certain arithmetic hyperbolic n-manifolds using Kazhdan's property (T) or Margulis's normal subgroups theorem, in tandem with facts about the virtual first Betti numbers of such manifolds (c.f. [BL20, Proposition 4.1]). On the other hand, the relevant facts of [PR72] can be established without use of property (T) and predates much of Margulis's work (and in some cases, can even be proven via direct means c.f. [Pra94]), while the nonexistence of higher-rank free abelian subgroups of hyperbolic groups is a standard fact.

Consequently, Zariski-dense hyperbolic subgroups of higher-rank lattices are automatically thin, thus understanding how hyperbolic groups embed into lattices *in general* presents as a natural goal. It is especially convenient then that parallel to the development of such goals, the representation theory of hyperbolic groups has seen an explosion in recent activity through the interest in Anosov representations (see §2.3.1). Paired with the fact that Zariski-dense representations of any finitely generated group into G form a Zariski-open (though possibly empty) set in the space of all representations, it should then be of no surprise that a representation-theoretic approach can yield fruit.

We thus turn to discussing the representation theory for surface groups, a particularly nice class of non-free hyperbolic groups where aspects of the theory are especially nice and motivates much of the discussion around general Anosov representations.

2.2 Hyperbolic structures on surfaces

Much of the geometric properties of the Hitchin components that are central to this work resemble analogous results established many years earlier for classical Teichmüller spaces. We recall the relevant aspects of the theory of the latter topic here, mostly following the development in [FM12].

Definition 2.2.1. A *surface* will always denote a closed (i.e. compact and without boundary), connected, oriented, 2-dimensional manifold. A *surface group* will refer to the fundamental group of such an object.

Surfaces are classified up to homeomorphism by a single nonnegative integer invariant: their *genus*. S_g will denote a surface of genus g so that, for example, S_0 and S_1 correspond to the 2-dimensional sphere and torus respectively.

Classically, the complex analytic viewpoint on Teichmüller theory studies the deformation space of marked Riemann surface structures on a closed surface. When the genus is at least 2, uniformization allows one to replace any such Riemann surface structure with an equivalent (isomorphic) structure locally modeled on the hyperbolic plane, a simply connected, 2-dimensional Riemannian manifold of constant sectional curvature -1, leading to a more geometric point of view. Any two models of the hyperbolic plane are isometric by the Cartan–Hadamard theorem, and so, to formally discuss the geometric point of view, it will be convenient to fix one such model.

Definition 2.2.2. The *upper half-space model* for the hyperbolic plane is the subset of the complex plane:

$$\mathbf{H}^2 = \{x + iy \in \mathbf{C} \,|\, x, y \in \mathbf{R}, y > 0\},$$

equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Given a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{R})$, the map $z \mapsto \frac{az+b}{cz+d}$ is an orientation-preserving isometry of $\mathbf{H}^2$. This induces a surjective group homomorphism $\mathrm{SL}(2, \mathbf{R}) \to \mathrm{Isom}^+(\mathbf{H}^2)$ whose kernel is precisely $\{\pm I_2\}$. Thus we may identify the group of orientation-preserving isometries of $\mathbf{H}^2$, $\mathrm{Isom}^+(\mathbf{H}^2)$, with $\mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\{\pm I_2\}$. Similarly, the full isometry group $\mathrm{Isom}(\mathbf{H}^2)$ can be identified with the group $\mathrm{PGL}(2, \mathbf{R})$.

Roughly speaking, to endow a surface with a hyperbolic structure is to topologically realize the surface by gluing together pieces of $\mathbf{H}^2$ by isometries. More formally, we have the following.

Definition 2.2.3. A *hyperbolic structure* on a surface S is a manifold M, homeomorphic to S, equipped with an atlas of charts $\{\varphi_\alpha : U_\alpha \to \mathbf{H}^2\}$ where the U_α are open sets covering M such that every transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is the restriction of an orientation-preserving isometry of $\mathbf{H}^2$.

If M is a hyperbolic structure on a surface, the compatibility condition on transition maps in the above definition ensures that the Riemannian metric on $\mathbf{H}^2$ descends to one on M. In particular, a hyperbolic structure induces a metric structure.

For a group G acting transitively on a manifold X, there is a notion of a (G, X)-structure on a manifold M that, as in the above, sees M covered by an X-valued atlas of charts so that transition maps are given by (restrictions of) maps coming from the action $G \curvearrowright X$. This point of view on geometries was first taken by Klein in his Erlangen program (see [Kle93]), and from this framework a hyperbolic structure on a surface is simply a $(\mathrm{PSL}(2, \mathbf{R}), \mathbf{H}^2)$-structure.

Example 2.2.1 (A hyperbolic structure on a genus two surface)**.** A simple continuity argument shows that there exists a regular geodesic octagon P in $\mathbf{H}^2$ with an interior angle sum of 2π (see Figure 2.1 for one such polygon shown in the Poincaré disk model).

For each side of P, there is a unique orientation-preserving isometry of $\mathbf{H}^2$ taking that side onto its matching one in accordance with the labels of Figure 2.1. If $\Gamma < \mathrm{Isom}^+(\mathbf{H}^2)$ is the group generated by these isometries, then the quotient $\Gamma\backslash\mathbf{H}^2$ is homeomorphic to a surface of genus 2. Local inverses of the quotient map provide natural charts equipping $\Gamma\backslash\mathbf{H}^2$ with a hyperbolic structure, where the condition that the internal angle sum is 2π

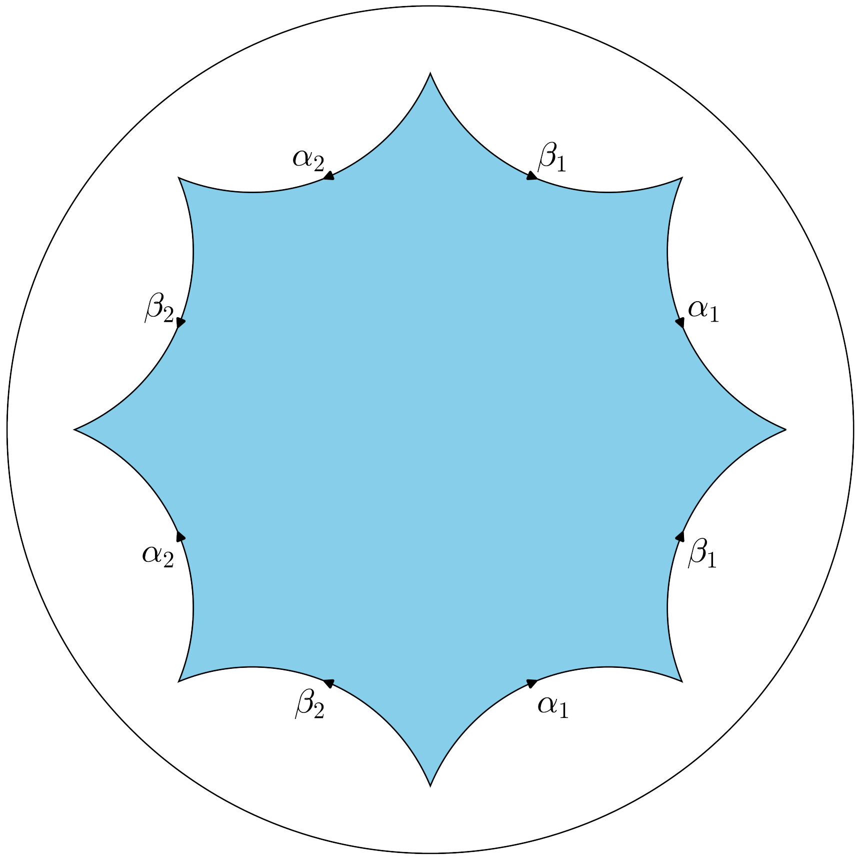


Figure 2.1: A geodesic regular octagon in $\mathbf{H}^2$ with internal angle sum 2π

guarantees that there is a chart defined in a neighborhood of the image of the vertices.

In fact, a closed surface admits a hyperbolic structure if and only if its genus is 2 or more. The forward implication being an immediate consequence of the Gauss–Bonnet theorem since, for instance, $\chi(S_g) = 2 - 2g$ and this is negative precisely when $g \geq 2$. For the reverse implication, one may perform a similar construction as in the above example, instead on a regular geodesic $4g$-gon in $\mathbf{H}^2$.

There is enough flexibility in the above constructions to indicate that a single surface S_g should admit "many" inequivalent hyperbolic structures. For instance, one need not require that the above polygon be regular, only that pairs of identified sides have equal length and that the interior angle sum equal 2π. One can therefore in fact demonstrate that there should be a continuum of hyperbolic structures since one may construct geodesic $4g$-gons in $\mathbf{H}^2$ where the lengths of a pair of identified sides continuously varies, and make identifications according to the same pattern.

We will make this heuristic precise by defining an appropriate deformation space of equivalence classes of hyperbolic structures, but first, for this deformation space to have a nice topology (in particular, that of a smooth manifold), it will be necessary to introduce the notion of a marking.

Definition 2.2.4. A *marked hyperbolic structure* on a surface S is a pair (M, f) where M is a hyperbolic structure on S and $f : S \to M$ is an orientation-preserving homeomorphism. Two marked hyperbolic structures, (M, f) and (N, g), are said to be *equivalent* if there exists an isometry $\phi : M \to N$ so that $\phi \circ f$ is isotopic to g.

The above definition of equivalence of marked hyperbolic structures is easily checked to be an equivalence relation, and thus we may finally define:

Definition 2.2.5. The *Teichmüller space* of S, $\mathcal{T}(S)$, is the set of equivalence classes of marked hyperbolic structures on S.

2.2.1 The character variety perspective

For the moment, this defines $\mathcal{T}(S)$ simply as a set. One manner of topologizing $\mathcal{T}(S)$ is by interpreting it as a subset of a particular character variety. This will also be particularly crucial in discussing the higher-rank generalizations of Teichmüller spaces.

Definition 2.2.6. The $\mathrm{PSL}(2, \mathbf{R})$-*character variety of* $\pi_1(S)$ is the space

$$\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R})) := \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R})) / \mathrm{PSL}(2, \mathbf{R})$$

of all conjugacy classes of group homomorphisms from $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbf{R})$.

If S has genus g, fixing a set of $2g$-generators for $\pi_1(S)$ gives a manner of identifying $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$ with a subset of $\mathrm{PSL}(2, \mathbf{R})^{2g}$ cut out by polynomial equations determined by the relation of the group. Endowing $\mathrm{PSL}(2, \mathbf{R})$ with its Euclidean topology therefore induces a topology on $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$.

Given a marked hyperbolic structure on S, (M, f), the hyperbolic structure on M induces a deck action of $\pi_1(M)$ by isometries on its universal cover, $\widetilde{M}$. The *developing map* associated to a hyperbolic structure induces an identification $\widetilde{M} \cong \mathbf{H}^2$, and in particular an identification of $\mathrm{Isom}^+(\widetilde{M})$ with $\mathrm{PSL}(2, \mathbf{R})$. The other datum associated to a marked hyperbolic structure, namely, the marking, similarly induces an isomorphism at the level of fundamental groups, hence an identification $\pi_1(S) \cong \pi_1(M)$. All together, the data of (M, f) yields a group homomorphism $\rho_{(M,f)} : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{R})$, known as the *holonomy representation*. Equivalent marked hyperbolic structures induce conjugate homomorphisms, and thus there is a well-defined *holonomy* map

$$\mathrm{hol} : \mathcal{T}(S) \to \mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$$

associating to an equivalence class $[(M, f)]$ the conjugacy class $[\rho_{(M,f)}]$. The following then is an immediate consequence of the above set up and some basics in covering space theory (e.g. see [FM12, Theorem 10.2]).

Proposition 2.2.1. $\mathrm{hol} : \mathcal{T}(S) \to \mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$ *is injective.*

Topologize $\mathcal{T}(S)$ by pulling back the topology on $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$ (that is, equip $\mathcal{T}(S)$ with the coarsest topology so that hol is continuous). Moving forward, we will identify $\mathcal{T}(S)$ with its image inside $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$, denoting specific points in $\mathcal{T}(S)$ by $[\rho]$ where $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{R})$ is a representation and $[\rho]$ its conjugacy class.

One can in fact say more about the image of the map hol. Immediate from the fact that all the holonomy representations are induced by deck actions is that the image is contained entirely within the set

$$\mathrm{DF}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R})) / \mathrm{PSL}(2, \mathbf{R})$$

of conjugacy classes of *discrete and faithful* representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbf{R})$. A result of Weil [Wei60] states that the set $\mathrm{DF}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R})) / \mathrm{PSL}(2, \mathbf{R})$ is open in $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$ and a result of Chuckrow [Chu68] shows that it is closed. Therefore it is a union of connected components of $\mathfrak{X}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$. In fact, it consists of precisely two connected components: one corresponding to the image $\mathrm{hol}(\mathcal{T}(S))$ and the other to $\mathrm{hol}(\mathcal{T}(\overline{S}))$ where $\overline{S}$ denotes S equipped with the opposite orientation.

Example 2.2.2. Fix the presentation $\pi_1(S_2) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \,|\, [\alpha_1, \beta_1][\alpha_2, \beta_2] \rangle$ of the fundamental group of a genus 2 surface. The holonomy associated to the hyperbolic structure on a genus 2 surface given in Example 2.2.1 (under some fixed marking) is (approximately) given by

$$\begin{aligned}
\alpha_1 &\mapsto \pm \begin{pmatrix} -0.322996749069888 & 0.866210365932833 \\ -2.54800319644026 & 3.73721031144298 \end{pmatrix} \\
\beta_1 &\mapsto \pm \begin{pmatrix} 3.73721031144298 & -2.54800319644026 \\ 0.866210365932835 & -0.322996749069890 \end{pmatrix} \\
\alpha_2 &\mapsto \pm \begin{pmatrix} 3.73721031144298 & 2.54800319644026 \\ -0.866210365932833 & -0.322996749069888 \end{pmatrix} \\
\beta_2 &\mapsto \pm \begin{pmatrix} -0.322996749069889 & -0.866210365932833 \\ 2.54800319644027 & 3.73721031144298 \end{pmatrix}.
\end{aligned}$$

One readily checks that these matrices satisfy the required relation and this does indeed define a discrete and faithful representation $\rho : \pi_1(S_2) \to \mathrm{PSL}(2, \mathbf{R})$.

For notational reasons, it will often be convenient to work with $\mathrm{SL}(2, \mathbf{R})$ representations, rather than $\mathrm{PSL}(2, \mathbf{R})$ ones. This is always possible, in light of the following.

Proposition 2.2.2. *Let $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{R})$ be a discrete and faithful representation. Then ρ admits a lift $\widetilde{\rho} : \pi_1(S) \to \mathrm{SL}(2, \mathbf{R})$.*

Proof. Suppose S has genus g and we work with the following presentation of $\pi_1(S)$:

$$\pi_1(S) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] \rangle$$

where $[-, -]$ is the commutator. Begin by fixing lifts $\widetilde{\rho}(\alpha_i), \widetilde{\rho}(\beta_i) \in \mathrm{SL}(2, \mathbf{R})$ of each $\rho(\alpha_i)$ and $\rho(\beta_i)$. Whether or not these choices of lifts define a representation depends on the value of the element

$$[\widetilde{\rho}(\alpha_1), \widetilde{\rho}(\beta_1)] \ldots [\widetilde{\rho}(\alpha_g), \widetilde{\rho}(\beta_g)] \in \{\pm I_2\}.$$

Note that this element does not depend on the choice of lifts, due to the relation on $\pi_1(S)$ being a product of commutators and $\pm I_2$ being central in $\mathrm{SL}(2, \mathbf{R})$. Thus, the obstruction to lifting $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R}))$ to $\widetilde{\rho} \in \mathrm{Hom}(\pi_1(S), \mathrm{SL}(2, \mathbf{R}))$ can be expressed a map

$$o : \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbf{R})) \to \{\pm 1\}.$$

One can show, e.g. following [Gol88], that this obstruction map is given by

$$o(\rho) = (-1)^{e(\rho)}$$

where $e(\rho)$ is the *Euler number* of the S^1-bundle over S given by the representation

$\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{R})$. Moreover, e.g. by [Gol88, Corollary C], as ρ is discrete and faithful, $e(\rho) = \pm\chi(S)$. In particular, $e(\rho)$ is even, so $o(\rho) = +1$ and hence ρ lifts. □

In light of this result, we will often identify $\mathcal{T}(S)$ with a connected component of $\mathrm{DF}(\pi_1(S), \mathrm{SL}(2, \mathbf{R}))/\mathrm{SL}(2, \mathbf{R})$. There is some ambiguity in how one makes such an identification: there are two choices for how each generator of the group lifts so in total, there are 2^{2g} connected components of this space corresponding to lifts of $\mathcal{T}(S)$ (and another 2^{2g} connected components coming from lifts of $\mathcal{T}(\overline{S})$). This ambiguity will not matter in this work, so we just select one such lift.

2.2.2 The geometry and topology of Teichmüller space

Until this point, much about the specific topology of $\mathcal{T}(S)$ has yet to be discussed, but many of the results in the later chapters in fact rely on the deep results regarding the smooth geometry of (higher) Teichmüller spaces. In general, arbitrary character varieties of finitely generated groups can be highly singular algebraic varieties. In fact, as defined in the previous section by means of taking a topological quotient, the character varieties we work with are in general often not even Hausdorff. Thus the following result, attributed to Fricke and first appearing in [FK65], may come as a surprise.

Theorem 2.2.3 ([FM12, Theorem 10.6])**.** *Let S be a surface of genus $g \geq 2$. Then $\mathcal{T}(S)$ is a smooth manifold diffeomorphic to $\mathbf{R}^{6g-6}$.*

Thus while the topology of character varieties can often be quite complicated, the topology of $\mathcal{T}(S)$ is quite nice: it is a smooth, contractible, open manifold. Moreover, while it is simple, from a topological point of view, it carries a rich amount of geometric structure, too much to state all such features here, but a few key aspects of which are particularly relevant in higher-rank as they pertain to the results of this thesis. We state these relevant parts next.

One system of coordinates on $\mathcal{T}(S)$ of geometric significance are the *Fenchel–Nielsen* coordinates: start by fixing a *pants decomposition* of S, that is, a collection of disjoint simple closed curves $\{c_1, \ldots, c_{3g-3}\}$ so that every component of $S - \{c_1, \ldots, c_{3g-3}\}$ is topologically a sphere with three boundary components. There are associated functions $\ell_i : \mathcal{T}(S) \to \mathbf{R}_+$ and $\theta_i : \mathcal{T}(S) \to \mathbf{R}$ for $i = 1, \ldots, 3g-3$ so that the map

$$[\rho] \mapsto (\ell_1([\rho]), \ldots, \ell_{3g-3}([\rho]), \theta_1([\rho]), \ldots, \theta_{3g-3}([\rho]))$$

is a diffeomorphism from $\mathcal{T}(S)$ to $(\mathbf{R}_+)^{3g-3} \times \mathbf{R}^{3g-3}$ (this is the perspective taken by Farb–Margalit in their proof of Theorem 2.2.3). The *length functions* $\ell_i([\rho])$ record the length, in the hyperbolic structure determined by $[\rho]$, of the (unique) geodesic in the free homotopy class of c_i while the *twist functions* $\theta_i([\rho])$, roughly, record the amount of "twisting" on either side of each seam c_i.

Note that if a curve c occurs in two topologically distinct pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ of S, then the twist coordinate associated to c in the former pants decomposition may differ from the twist coordinate associated to c in the latter. The same is not true of the length functions, as the length of a geodesic in the homotopy class of c only depends on the hyperbolic structure c is viewed in. Thus more generally, one may make the following definition.

Definition 2.2.7. Let $\gamma \in \pi_1(S)$. The *length function* associated to γ is the function $\ell_\gamma : \mathcal{T}(S) \to \mathbf{R}_+$ where $\ell_\gamma([\rho])$ is the length of a geodesic freely homotopic to γ in the hyperbolic structure determined by $[\rho]$.

Remark. To hopefully avoid confusion in notation, we will typically use Greek letters, e.g. α, β, γ, to denote *based* homotopy classes of loops, i.e. elements of the fundamental group $\pi_1(S)$. We will use corresponding Roman letters, e.g. a, b, c, to denote their *free* homotopy classes.

As the length functions don't depend on choices of pants decompositions, it may be desirable to seek a system of coordinates on $\mathcal{T}(S)$ using only length functions associated to simple closed curves. This turns out to not be possible, there are no *global* coordinates on $\mathcal{T}(S)$ expressed solely in terms of lengths of simple closed curves. A partial result towards this parameterization is the following result, known colloquially as "the $9g-9$ theorem" (see, for instance, [FM12, Theorem 10.7] for a proof).

Theorem 2.2.4 (The $9g-9$ theorem)**.** *There exists a collection* $\{\gamma_1, \dots, \gamma_{9g-9}\} \subset \pi_1(S)$*, representable by simple closed curves, so that the map* $\mathcal{T}(S) \to \mathbf{R}_+^{9g-9}$

$$[\rho] \mapsto (\ell_{\gamma_1}([\rho]), \dots, \ell_{\gamma_{9g-9}}([\rho]))$$

is injective.

In short, though there are no $6g-6$ length functions which provide a *homeomorphism* from $\mathcal{T}(S)$ onto $\mathbf{R}^{6g-6}$, there are finitely many length functions which *embed* $\mathcal{T}(S)$ into some $\mathbf{R}^N$.

In addition to the various natural coordinates on $\mathcal{T}(S)$ corresponding to information on the hyperbolic geometry of S, $\mathcal{T}(S)$ also carries a wealth of metric structures underlying much of its rich geometry. Three of note include the Teichmüller metric, defined using dilations of quasi-conformal maps, the Weil–Petersson metric, defined using Beltrami differentials and the Thurston (asymmetric) metric, defined using Lipschitz maps. The second of these metrics is, in fact, Riemannian and a result of Ahlfors showed that it is actually Kähler [Ahl61]. In particular, there is an induced closed non-degenerate 2-form, ω_{WP}, making $\mathcal{T}(S)$ into a symplectic manifold.

The use of Beltrami differentials in the definition of ω_{WP} makes its original definition too complicated to state for these purposes, but work of Wolpert on the symplectic geometry of $\mathcal{T}(S)$ resulted in the following simpler expression (see [Wol10] for a reference).

Theorem 2.2.5 ("Wolpert's magic formula"). *Fixing a pants decomposition of S, ω_{WP} is given, in Fenchel–Nielsen coordinates, by the formula*

$$\omega_{WP} = \frac{1}{2}\sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i.$$

In particular, the right-hand side is independent of the choice of pants decomposition.

The above formula expresses a duality between the length and twist functions, higher-rank analogs of which are leveraged to establish the results of Chapter 4.

Example 2.2.3 (The Fenchel–Nielsen twist flow on $\mathcal{T}(S)$). Let $\gamma \in \pi_1(S)$ be a (based) homotopy class of loops that is freely isotopic to a non-separating simple closed curve on S. Start by defining a flow Φ_γ^t on $\mathrm{DF}(\pi_1(S), \mathrm{SL}(2,\mathbf{R}))$ as follows. Let $\gamma' \in \pi_1(S)$ denote a curve that intersects γ precisely once in the positive orientation (i.e. if $i(\gamma', \gamma) = +1$ where i is the algebraic intersection). For any $\rho \in \mathrm{DF}(\pi_1(S), \mathrm{SL}(2,\mathbf{R}))$ and $t \in \mathbf{R}$, define

$$\Phi_\gamma^t(\rho)(\alpha) := \begin{cases} \rho(\alpha) & \text{if } \alpha \in \pi_1(S \backslash \gamma) \\ P_{\rho(\gamma)}\,\mathrm{diag}(e^t, e^{-t}) P_{\rho(\gamma)}^{-1}\rho(\alpha) & \text{if } \alpha = \gamma', \end{cases} \tag{2.1}$$

where $P_{\rho(\gamma)} \in \mathrm{SL}(2,\mathbf{R})$ is so that $P_{\rho(\gamma)}^{-1}\rho(\gamma)P_{\rho(\gamma)} = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ for some $\lambda > 1$ (where diagonalizability of $\rho(\gamma)$ is ensured by the discrete and faithfulness of ρ). Because γ can be represented by a non-separating simple closed curve, $\pi_1(S)$ can be expressed as an HNN extension of the group $\pi_1(S \backslash \gamma)$ and Equation 2.1 gives a well defined homomorphism $\Phi_\gamma^t(\rho) : \pi_1(S) \to \mathrm{SL}(2,\mathbf{R})$ (c.f. §3.1). Continuity in t of the above formula and continuity of the map $\rho \mapsto P_{\rho(\gamma)}$ yields a resulting continuous flow on $\mathrm{DF}(\pi_1(S), \mathrm{SL}(2,\mathbf{R}))$. Setting $\varphi_\gamma^t([\rho]) := [\Phi_\gamma^t(\rho)]$ and restricting yields a resulting flow on $\mathcal{T}(S)$.

Up to rescaling in time, the resulting flow φ_γ^t on $\mathcal{T}(S)$ is known as the *Fenchel–Nielsen twist flow* about γ (see Figure 2.2). That is, geometrically, the above deformations corre-

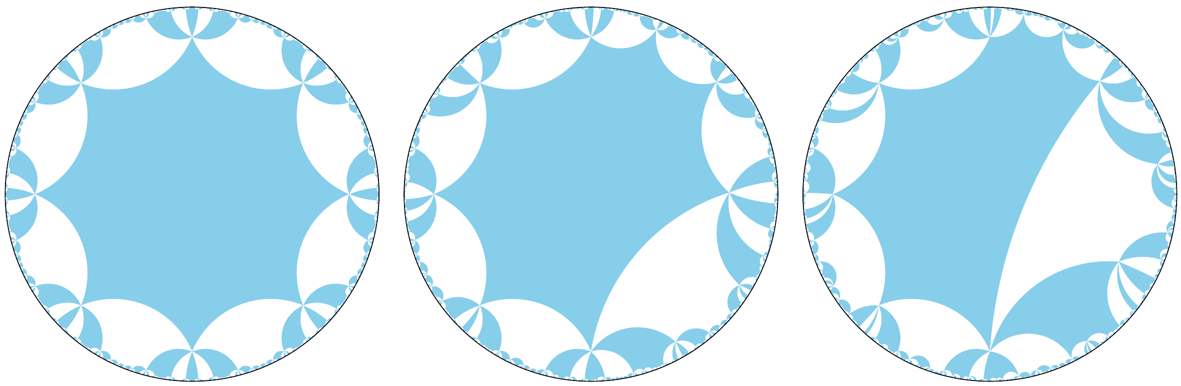


Figure 2.2: The evolution of the Fenchel–Nielsen twist flow on the hyperbolic surface of Example 2.2.1.

spond to cutting a hyperbolic structure on S about a simple closed geodesic representative of γ, twisting along the geodesic by some amount determined by t, then regluing along the boundary pieces to get a new hyperbolic structure. Theorem 2.2.5 expresses a duality between the length and twist coordinates which in turn implies that these flows are Hamiltonian with respect to the length function ℓ_γ. In particular, the local behavior of the flow φ_γ^t is controlled by the length functions ℓ_γ while its global behavior can be understood via Equation 2.1.

2.2.3 Hyperbolic structures on 2-orbifolds

Finally, for practical reasons, it will be useful to extend much of the above discussion for closed surfaces to the context of hyperbolic orbifolds. This can be done with relatively few modifications via the character variety perspective, despite the (G, X)-structure perspective being more subtle to state, as the underlying spaces are no longer manifolds.

Definition 2.2.8. A *closed orientable hyperbolic* 2*-orbifold* X is a quotient of the form $X = \mathbf{H}^2/\Gamma$ where $\Gamma < \mathrm{PSL}(2, \mathbf{R})$ is a cocompact lattice. The group Γ is the *orbifold fundamental group* of X, which we may also denote as $\pi_1(X)$.

Every point on X as above is contained in a neighborhood locally modelled either by an open subset of $\mathbf{H}^2$ or by the quotient of an open subset of $\mathbf{H}^2$ by a finite group of isometries. Any finite subgroup of $\mathrm{PSL}(2, \mathbf{R})$ is cyclic and so neighborhoods of the singular points on X are always modeled by open subsets of $\mathbf{H}^2$ modulo the action of a cyclic group acting by rotations. Consequently, by compactness, the collection of singular points on X is a finite set and its fundamental group Γ is determined, up to isomorphism, by its *signature* $(g; n_1, \ldots, n_\ell)$ where g is the genus of the underlying (punctured) surface of X and $n_1 \leq \ldots \leq n_\ell$ are the orders of the cyclic groups corresponding to the ℓ distinct singular, or cone, points on X.

More generally, one may make the same definition as above for cocompact lattices in $\mathrm{PGL}(2, \mathbf{R})$, allowing for possibly *non-orientable* closed hyperbolic 2-orbifolds. There is a similar classification of these, though slightly more complex to state as now, in addition to a finite set of cone points, the singular set of X may contain *mirrors* (where nontrivial point-stabilizers correspond to a $\mathbf{Z}/2$ acting by reflection about an axis) and *corner reflectors* (where point-stabilizers correspond to dihedral groups acting by isometries on an embedded regular geodesic polygon).

Definition 2.2.9. Let $X = \mathbf{H}^2/\Gamma$ be a closed orientable hyperbolic 2-orbifold. The *Teichmüller space* of X, $\mathcal{T}(X)$, is the connected component of the space of conjugacy classes of all discrete and faithful representations of Γ into $\mathrm{PSL}(2, \mathbf{R})$:

$$\mathcal{T}(X) \subseteq \mathrm{DF}(\Gamma, \mathrm{PSL}(2, \mathbf{R})) / \mathrm{PSL}(2, \mathbf{R})$$

containing the homomorphism induced by the inclusion $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbf{R})$.

Many of the results of the previous section are no longer true for Teichmüller spaces of orbifolds. For instance, some of these spaces will have odd dimension, and thus certainly cannot carry a symplectic structure. Nonetheless, working with orbifolds can be particularly beneficial for various practical reasons. For instance, the spaces one works with are often of much smaller dimension, hence making computation much easier to perform. Moreover, by Selberg's lemma, such orbifolds contain finite-index surface subgroups, inducing embeddings $\mathcal{T}(X) \hookrightarrow \mathcal{T}(S)$ coming from restriction. This allows one to recover much about closed surfaces from working with orbifolds, so little is lost.

The practicality, for purposes of computation, of working with orbifolds is perhaps best illustrated via the following rigid example.

Proposition 2.2.6. *Let $p, q, r \geq 2$ be integers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let X be the closed orientable hyperbolic 2-orbifold with signature $(0; p, q, r)$. Then $\mathcal{T}(X)$ is a single point.*

Proof. The existence of such an orbifold is a simple exercise in hyperbolic geometry: the condition that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$ ensures that one can construct an embedded geodesic triangle in $\mathbf{H}^2$ whose interior angle sum is $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r}$. If $\Gamma_0 < \mathrm{PGL}(2, \mathbf{R})$ is the group generated by reflections about the sides of this triangle, then taking $\Gamma < \Gamma_0$ the index 2 subgroup consisting of orientation-preserving isometries yields the desired orbifold. [*Note*: For the rest of this work, we shall refer to a group Γ constructed in the above manner as the (p, q, r)-*triangle group*, even though it is perhaps more common, e.g. for those working with Coxeter groups, to refer to Γ_0 as the (p, q, r)-triangle group.]

To show rigidity, we note that the group Γ admits a presentation of the form $\Gamma \cong \langle \alpha, \beta \,|\, \alpha^p = \beta^q = (\alpha\beta)^r \rangle$. Given a discrete and faithful representation $\rho : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$, if Δ is the geodesic triangle with vertices the fixed points of the (finite-order) isometries $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\alpha\beta)$ and $\overline{\Delta}$ is the reflection of Δ across the edge opposite the fixed point of $\rho(\alpha)$, then $\Delta \cup \overline{\Delta}$ is a fundamental domain for the action of $\rho(\Gamma)$ on $\mathbf{H}^2$. If $\rho' : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ is another discrete and faithful representation, there is another fundamental domain $\Delta' \cup \overline{\Delta'}$ constructed in the same manner. The triangles Δ and Δ' both have interior angles $\frac{\pi}{p}, \frac{\pi}{q}$ and $\frac{\pi}{r}$ which, by basic hyperbolic trigonometry, implies that the triangles Δ and Δ' differ by an isometry of $\mathbf{H}^2$ (possibly reversing orientation depending on the cyclic ordering of the vertices). Such an isometry will take the fundamental domain $\Delta \cup \overline{\Delta}$ to $\Delta' \cup \overline{\Delta'}$ and conjugate ρ to ρ'. Hence there is a single PGL$(2, \mathbf{R})$-conjugacy class of discrete and faithful representations of Γ into PSL$(2, \mathbf{R})$ and two isolated PSL$(2, \mathbf{R})$-conjugacy classes, corresponding to the Teichmüller spaces on each orientation of X. □

The following two examples will be particularly relevant starting points for the discussion in Chapters 3 and 4.

Example 2.2.4. Let X be the orbifold of signature $(0; 3, 4, 4)$. Its fundamental group is the triangle group $\Delta(3, 4, 4) = \langle \alpha, \beta \,|\, \alpha^3 = \beta^4 = (\alpha\beta)^4 \rangle$ and the unique point in $\mathcal{T}(X)$ may be represented by the homomorphism:

$$\rho(\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \; \rho(\beta) = \begin{pmatrix} 0 & -1-\sqrt{2} \\ -1+\sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Notice that $\rho(\Delta(3, 4, 4)) < \mathrm{PSL}(2, \mathbf{Z}[\sqrt{2}])$.

Example 2.2.5. Let X be the orbifold of signature $(1; 2)$. Its deformation space, $\mathcal{T}(X)$, is 2-dimensional and was computed explicitly in [Mag73] and shown to be parameterized by rational functions. Using the presentation $\pi_1(X) = \langle \alpha, \beta \,|\, [\alpha, \beta]^2 \rangle$, one point in $\mathcal{T}(X)$ is represented by the homomorphism:

$$\rho(\alpha) = \begin{pmatrix} 3 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \; \rho(\beta) = \begin{pmatrix} 0 & -2 \\ \frac{1}{2} & \frac{83}{8} \end{pmatrix}.$$

This representation has come to be known as the Long–Reid representation and first drew attention due to computational evidence suggesting that an action of a surface group on a product of p-adic trees associated to the above representation may be free (cf. [BFMvL23] and the discussion in Example 2.4.2). Its inclusion here is due to the simple observation that $\rho(\pi_1(X)) < \mathrm{PSL}(2, \mathbf{Q})$ and that $\pi_1(X)$ contains finite-index surface subgroups of every genus, which is *much* less interesting than what first drew attention to this example in the first place.

2.3 The Hitchin component

The theory of the Hitchin component, and more generally of higher Teichmüller theory, generalizes much of the content of the previous section to other character varieties. We begin with a discussion of the main spaces of focus.

Definition 2.3.1. Let G be a reductive real Lie group. The *G-character variety of S* is

$$\mathfrak{X}(\pi_1(S), G) := \mathrm{Hom}(\pi_1(S), G)/G$$

where G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation.

Remark. Similar to comments made earlier, as written, the above definition is slightly inaccurate. For instance, the orbits of the G-action on $\mathrm{Hom}(\pi_1(S), G)$ are, in general, not closed and so the resulting "naive" quotient possesses undesirable topological qualities. One can fix these issues by taking a quotient in the sense of *invariant theory* or instead restricting to the *completely reducible* representations: $\mathrm{Hom}_{\mathrm{red}}(\pi_1(S), G)$, where the topological quotient by G arises as the Hausdorffification of the above space. We will not worry about either of these remedies here as instead, our attention will be restricted to a subset of $\mathfrak{X}(\pi_1(S), G)$ with none of these "bad" features.

Basic considerations in Lie theory show that for every $n \geq 2$, there is a unique up to conjugacy, irreducible, representation $\iota_n : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{SL}(n, \mathbf{R})$. For instance, ι_n arises as the n-th symmetric power of the standard representation $\iota_2 : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{SL}(2,\mathbf{R})$ or, equivalently, as the action on homogeneous polynomials of degree $n-1$. Identifying $\mathcal{T}(S)$ with a subset of $\mathrm{DF}(\pi_1(S), \mathrm{SL}(2,\mathbf{R}))/\mathrm{SL}(2,\mathbf{R})$, post-composition with ι_n gives an embedding:

$$\mathcal{T}(S) \to \mathfrak{X}(\pi_1(S), \mathrm{SL}(n,\mathbf{R}))$$
$$[\rho] \mapsto [\iota_n \circ \rho].$$

As $\mathcal{T}(S)$ is connected by Theorem 2.2.3, its image is contained entirely within a single component of $\mathfrak{X}(\pi_1(S), \mathrm{SL}(n,\mathbf{R}))$.

Definition 2.3.2. The *Hitchin component* of $\mathfrak{X}(\pi_1(S), \mathrm{SL}(n,\mathbf{R}))$ is the connected component, $\mathcal{H}_n(S)$, containing the image of $\mathcal{T}(S)$. Representations whose conjugacy classes are in the image of $\mathcal{T}(S)$ are called *n-Fuchsian representations*. In general, representations whose conjugacy class are in $\mathcal{H}_n(S)$ are called *Hitchin representations*.

Remark. Note that $\iota_2 = \mathrm{id}$ and so $\mathcal{H}_2(S) = \mathcal{T}(S)$.

If G is a split real form of a complex simple Lie group, then G contains *principally embedded subgroups* isomorphic to $\mathrm{SL}(2,\mathbf{R})$ (c.f. [Kos59]), and one may define a Hitchin component, $\mathcal{H}_G(S)$, as the connected component of $\mathfrak{X}(\pi_1(S), G)$ containing the embedded copy of $\mathcal{T}(S)$ induced by the principal subgroup. Some results in this thesis will only deal with the $\mathrm{SL}(n,\mathbf{R})$-Hitchin component $\mathcal{H}_n(S)$ but will admit natural generalizations to the Hitchin components associated to the split groups $\mathrm{SO}(k+1,k)$, $\mathrm{Sp}(2k,\mathbf{R})$ and G_2, which are all contained in $\mathcal{H}_n(S)$ (for appropriate choices of n). These generalizations will be indicated as they arise.

In addition to changing the target group, one may also replace S with a closed hyperbolic 2-orbifold X and get a notion of a Hitchin component for X, $\mathcal{H}_n(X)$ (c.f. §2.2.3).

Many of the results following will be stated for closed surfaces, but can be generalized to orbifolds as well (see [ALS23]).

Hitchin components were first detected by Hitchin (as the name may have indicated) in [Hit92]. Using Higgs bundles techniques, he showed the following result about these topological components of the highly-singular character varieties:

Theorem 2.3.1 ([Hit92, Theorem A]). *$\mathcal{H}_n(S)$ is homeomorphic to an open ball of dimension $(2g-2)(n^2-1)$. In particular, $\mathcal{H}_n(S)$ is smooth.*

The above result was the first indication of other connected components of character varieties with similar properties to $\mathcal{T}(S)$, but at the time this result first arose, this was the extent of the resemblance. Shortly after, Goldman and Choi established that every representation in $\mathcal{H}_3(S)$, like in $\mathcal{T}(S)$, was discrete and faithful and parameterized exactly the convex $\mathbf{RP}^2$-structures on S [CG93], illustrating that the resemblance extended further. Nearly a decade later, Labourie (and independently, Fock and Goncharov) extended discrete and faithfulness to all n.

Theorem 2.3.2 ([Lab06, FG06]). *Every representation in $\mathcal{H}_n(S)$ is discrete and faithful. Moreover, for any Hitchin representation ρ and every $\gamma \in \pi_1(S)$, the matrix $\rho(\gamma)$ is purely loxodromic, i.e. diagonalizable with distinct real eigenvalues.*

Remark. When working with Hitchin representations into $\mathrm{SL}(n, \mathbf{R})$, which are all lifts of ones into $\mathrm{PSL}(n, \mathbf{R})$, one can in fact show that one may always choose a lift so that the element $\rho(\gamma)$ has distinct, *positive*, real eigenvalues (c.f. Theorem 2.2.2).

Labourie's result starkly contrasts what can happen for other character varieties of surface groups containing embedded copies of $\mathcal{T}(S)$. For instance, the $\mathrm{SL}(2, \mathbf{C})$-character variety $\mathfrak{X}(\pi_1(S), \mathrm{SL}(2, \mathbf{C}))$ contains one such copy induced by the inclusion $\mathrm{SL}(2, \mathbf{R}) \hookrightarrow \mathrm{SL}(2, \mathbf{C})$, and here, small deformations of Fuchsian representations remain discrete and faithful, the so-called *quasi-Fuchsian* representations. However, it is possible to continuously deform a quasi-Fuchsian representation "too much" and eventually lose discrete and faithfulness (e.g. see [Thu79, §8]). In particular, the above theorem is certainly false for the component of $\mathfrak{X}(\pi_1(S), \mathrm{SL}(2, \mathbf{C}))$ containing $\mathcal{T}(S)$.

Labourie established Theorem 2.3.2 by showing representations in $\mathcal{H}_n(S)$ were *Anosov* (see §2.3.1), and extensive amounts of research have since been dedicated in the years since to further exploring the analogies between representations in $\mathcal{H}_n(S)$, representations in $\mathcal{T}(S)$ and Anosov representations of hyperbolic groups.

The foremost importance of Labourie's result, as to the work of this thesis, is that if one begins with an n-Fuchsian representation, ρ_0, and deforms it in a continuous manner to some representation ρ_1, then ρ_1 will still be Hitchin, and hence discrete and faithful. The main results of this thesis are, in brief, achieved by starting with particular choices of ρ_0 and continuously deforming in a manner that "tracks" the arithmetic of the resulting representations. Labourie's theorem ensures that the results of these deformations still yield surface subgroups.

There are a number of other key resemblances between the Hitchin component and $\mathcal{T}(S)$ which are used throughout this work. For one, Hitchin representations also satisfy a simple marked length rigidity, analogous to Theorem 2.2.4. First, we make the following definitions.

Definition 2.3.3. For any $\gamma \in \pi_1(S)$, define $\mathrm{Tr}_\gamma : \mathcal{H}_n(S) \to \mathbf{R}$ as

$$\mathrm{Tr}_\gamma([\rho]) := \mathrm{Tr}(\rho(\gamma)).$$

This is well-defined by conjugation-invariance of the trace.

One may regard Tr_γ as a notion of a "length" associated to the curve γ in higher-rank.

Indeed, for $\mathcal{H}_2(S) = \mathcal{T}(S)$,

$$\ell_\gamma([\rho]) = 2\cosh^{-1}\left(\frac{\mathrm{Tr}_\gamma([\rho])}{2}\right),$$

and so, Tr_γ on $\mathcal{T}(S)$ records "essentially" the same information as the length of the geodesic representative of γ. This is a notion of length only in analogy, however, as representations in $\mathcal{H}_n(S)$ do not necessarily correspond to geometric structures on the surface S. Nonetheless, the following result is due to Bridgeman–Canary–Labourie, serving as an analog of Theorem 2.2.4.

Theorem 2.3.3 ([BCL20, Theorem 1.2]). *Fix an $n \geq 2$ and a surface S of genus 3 or more. There is a finite subset $\mathcal{S} \subset \pi_1(S)$ of curves representable by non-separating simple closed curves so that the map $\mathcal{H}_n(S) \to \mathbf{R}^{\mathcal{S}}$*

$$[\rho] \mapsto (\mathrm{Tr}_\gamma([\rho]))_{\gamma \in \mathcal{S}}$$

is injective.

In short, two Hitchin representations are conjugate if and only if their simple marked trace spectra coincide, just as is true for Fuchsian representations.

Finally, the Hitchin component also possesses an underlying symplectic structure. Predating much of the above work on $\mathcal{H}_n(S)$, Goldman showed that other smooth components of $\mathfrak{X}(\pi_1(S), G)$ can be equipped with a symplectic form that exactly agrees with the Weil–Petersson symplectic structure when $G = \mathrm{PSL}(2, \mathbf{R})$ [Gol84]. We shall refer to this form as the *Goldman symplectic form*, and its nature is key to the results in Chapter 4, but for the moment, the reader need only know of its existence.

2.3.1 Anosov representations

The more general notion of an Anosov representation was developed by Guichard and Wienhard in [GW12] and allows much of the above discussion to be widened to other word hyperbolic groups. We shortly discuss this more general theory here as we indicate some connections to other spaces of Anosov representations in our work.

Guichard and Wienhard's original definition resembles Labourie's original dynamical one for surface groups in [Lab06], but the one we give is a simplification due to a number of authors [GGKW17, KLP18, BPS19]. To first set some notation, given a matrix $A \in \mathrm{SL}(n, \mathbf{R})$ let $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$ denote its singular values, i.e., the square roots of the eigenvalues of AA^T.

Definition 2.3.4. Let $\rho : \Gamma \to \mathrm{SL}(n, \mathbf{R})$ be a representation and fix some word metric $\|-\|$ on Γ (with respect to a fixed finite generating set). ρ is *k-Anosov* for $1 \leq k \leq n-1$ if there exist constants $C, K > 0$ such that

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq C\|\gamma\| - K.$$

ρ is just *Anosov* if it is k-Anosov for some k.

While the definitions make no mention of hyperbolicity, it turns out that if a group Γ admits an Anosov representation into some $\mathrm{SL}(n, \mathbf{R})$, then Γ is word hyperbolic. Some of the utility of Anosov representations is encapsulated in the following result.

Theorem 2.3.4 ([GW12, Theorems 5.3 and 5.13])**.** *The collection of Anosov representations is open in* $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{R}))$ *and consists only of discrete representations with finite kernel. In particular, if* Γ *is torsion-free, all Anosov representations are discrete and faithful.*

This theorem provides geometric tools for understanding discrete embeddings of hyperbolic groups and thus presents a promising connection to the discussion at the end of §2.1.2. In particular, the above states that being Anosov is stable under small deformations, and so some discussions on how our proofs can make use of this fact for other groups are discussed as well.

2.4 The main results

After now developing the relevant objects, we may begin to state our main results and discuss the context surrounding them.

Definition 2.4.1. For a subring $R \subseteq \mathbf{R}$, set

$$\mathcal{H}_n(S)_R := \mathcal{H}_n(S) \cap \operatorname{Hom}(\pi_1(S), \mathrm{SL}(n, R)) / \mathrm{SL}(n, \mathbf{R}).$$

That is, $\mathcal{H}_n(S)_R$ consists of the representations which are conjugate into subgroups of $\mathrm{SL}(n, R)$.

The above sets are "almost" the R-points of $\mathcal{H}_n(S)$, but this is not quite true for a number of reasons. For one, $\mathcal{H}_n(S)$ itself is not an algebraic variety, it is instead *semi-algebraic* in that it is defined by polynomial equalities *and* inequalities. Moreover, while the invariant theory quotient description of $\mathfrak{X}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ is an algebraic variety, $\mathcal{H}_n(S)_R$ is not quite the intersection of the R-points of $\mathfrak{X}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ with $\mathcal{H}_n(S)$ either. These instead would be the representations with R-valued trace which is a strictly weaker condition, but contains the subsets of representations we consider here. These sets being the R-points is not far from the truth though. Indeed, one may identify a component $\widetilde{\mathcal{H}}_n(S) \subset \operatorname{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ for which the quotient map $\widetilde{\mathcal{H}}_n(S) \to \mathcal{H}_n(S)$ is a principal $\mathrm{SL}(n, \mathbf{R})$-bundle, and $\mathcal{H}_n(S)_R$ is the image under this fibration, of the R-

points of the affine algebraic variety $\mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ contained in $\widetilde{H}_n(S)$. Thus, we will think of these sets $\mathcal{H}_n(S)_R$ as encoding the "arithmetic" of the Hitchin component.

Two questions one must seek to understand, motivated by the discussion of Chapter 1, are the following.

Question 5. For a given ring $R \subseteq \mathbf{R}$, can one decide if $\mathcal{H}_n(S)_R \neq \emptyset$?

Question 6. If $\mathcal{H}_n(S)_R \neq \emptyset$, can one qualitatively describe "how many" such representations there are? For instance, are there only finitely many, modulo the action of the mapping class group?

Answering forms of these questions is essentially how the previous work of Long, Reid and Thistlethwaite in [LRT11, LT18, LT22] established the existence of thin surface subgroups in $\mathrm{SL}(n, \mathbf{Z})$ whenever n is odd or 4. Briefly, they show, through direct computation, $\mathcal{H}_n(S)_{\mathbf{Z}} \neq \emptyset$ for these values of n and various S, using examples coming from triangle groups, giving an answer to Question 5, and give a meaningful sense in which for some S, there are "many enough" representations in $\mathcal{H}_n(S)_{\mathbf{Z}}$ to guarantee that a Zariski-dense one exists, answering Question 6. Establishing some notion of abundancy for surface subgroups contained in fixed lattices often is relevant to establishing Zariski-density. For instance, the works [KM12, CF19, KW21] are also able to yield thin surface subgroups in $\mathrm{SO}(3,1)$ because they show such examples are appropriately "ubiquitous."

In light of these questions, the main results of this thesis describe how the theory of Anosov representations can provide answers to Question 6 for various rings R. They are as follows.

Theorem 2.4.1 ([Zsh22, Corollary 1.1.1])**.** *If $\mathcal{H}_n(S)_{\mathbf{Z}} \neq \emptyset$, then there is a finite-sheeted cover $S' \to S$ for which $\mathcal{H}_n(S')_{\mathbf{Z}}$ contains Zariski-dense representations. In particular, the existence of an integral Hitchin surface subgroup of $\mathrm{SL}(n, \mathbf{Z})$ implies the existence of a thin one.*

Theorem 2.4.2 ([Zsh22, Theorem 1.1]). *Let $K \neq \mathbf{Q}$ be a number field. Suppose that K is not totally imaginary and has class number one. Let $\mathcal{O}_K$ be its ring of integers. If $\mathcal{H}_n(S)_{\mathcal{O}_K} \neq \emptyset$, then it contains Zariski-dense representations.*

Theorem 2.4.3 ([AZ23, Theorem 1.1]). *For any S of genus at least 3, the set $\mathcal{H}_n(S)_{\mathbf{Q}}$ is dense in $\mathcal{H}_n(S)$.*

In addition to the above questions, following arithmeticity (Theorem 2.1.2), one can also regard these results as coarse steps towards understanding how Question 3 can provide answers to Question 2. Finally, while the connection to lattices in Theorem 2.4.1 is clear, we close this chapter with two examples illustrating how the latter two results are still directly related to questions surrounding lattices in products.

Example 2.4.1 ($R = \mathbf{Z}[\sqrt{2}]$). First, consider in rank one, a $[\rho] \in \mathcal{H}_2(S)_{\mathbf{Z}[\sqrt{2}]}$. Applying an appropriate conjugation, we may assume such an example comes from a Fuchsian $\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbf{Z}[\sqrt{2}])$. As in Example 2.1.4, the latter group is a lattice in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$. Consequently, there is an immersed surface

$$\rho(\pi_1(S))\backslash\mathbf{H}^2 \to \mathrm{SL}(2, \mathbf{Z}[\sqrt{2}])\backslash(\mathbf{H}^2 \times \mathbf{H}^2).$$

The right-hand side of the above is an example of a *Hilbert modular variety*, and the geometry associated to such immersed surfaces has been investigated by a number of others. For example, McMullen constructs ones which are geodesic with respect to the *Kobayashi metric* using representations of this form coming from triangle groups [McM23].

The same is true replacing $\mathbf{Z}[\sqrt{2}]$ with other rings of integers of number fields. Instead, one performs a construction similar to the above but with a factor in the product for each *place* of the number field, using $\mathbf{H}^3$ instead of $\mathbf{H}^2$ if the place is complex. This also remains true in higher-rank where the representations in $\mathcal{H}_n(S)_{\mathcal{O}_K}$ yield immersed surfaces in locally symmetric spaces which are quotients of products of symmetric spaces and can be thought of as natural generalizations of Hilbert modular varieties to higher-rank.

Example 2.4.2 ($R = \mathbf{Q}$)**.** Given a Hitchin $\rho(\pi_1(S)) < \mathrm{SL}(n, \mathbf{Q})$, there is, in fact, an N (depending on ρ) so that $\rho(\pi_1(S)) < \mathrm{SL}(n, \mathbf{Z}[\frac{1}{N}])$. This latter group is not a lattice in $\mathrm{SL}(n, \mathbf{R})$, but it is a lattice in a broader sense. One may embed $\mathrm{SL}(n, \mathbf{Z}[\frac{1}{N}])$ into the product of real and p-adic Lie groups:

$$\mathrm{SL}(n, \mathbf{R}) \times \prod_{p|N} \mathrm{SL}(n, \mathbf{Q}_p)$$

and under this embedding, $\mathrm{SL}(n, \mathbf{Z}[\frac{1}{N}])$ is now a lattice (a so-called $\mathcal{S}$*-arithmetic lattice*). Based on these observations, the surface subgroup $\rho(\pi_1(S))$ will act discretely on an analog of a symmetric space associated to the above product of groups (defined by taking a product of symmetric spaces and *buildings* associated to each non-archimedean factor). The dynamics of the *real* part of this action are comparatively well understood, by virtue of the surface group coming from a Hitchin representation, and so a natural question is to investigate the associated non-archimedean dynamics. That is, what properties can be gleaned about $\rho(\pi_1(S))$ if we view it instead as a subgroup of $\prod_{p|N} \mathrm{SL}(n, \mathbf{Q}_p)$? Much about such actions is not known at the time of this writing (c.f. [FLSS18, BFMvL23]), but unlike in the previous example, the embedding of $\mathrm{SL}(n, \mathbf{Z}[\frac{1}{N}])$ is symmetric: there is no Galois conjugation used, and so it lives inside the copy of $\mathrm{SL}(n, \mathbf{R})$ "in the same manner" as it does in each $\mathrm{SL}(n, \mathbf{Q}_p)$. Thus it seems plausible that the symmetry of the embeddings and the dynamics of Hitchin representations (some of which is unique to higher-rank) can inform some aspects around these questions that were previously difficult to understand in rank one.

Chapter 3

Integrality

The following chapter investigates various integral structures associated to representations on the Hitchin component to prove Theorems 2.4.1 and 2.4.2. Much of the content here is a reproduction of the results of the author's appearing in the paper *Integral Zariski dense surface groups in* $\mathrm{SL}(n, \mathbf{R})$ [Zsh22]. The work of this paper originates with the following result of Long and Thistlethwaite's.

Theorem 3.0.1 ([LT22, Theorem 1.1])**.** $\mathrm{SL}(2k+1, \mathbf{Z})$ *contains thin surface subgroups for every* $k \geq 1$*.*

The starting point of their work was the observation that the composition of the discrete and faithful representation of the $(3, 4, 4)$-triangle group into $\mathrm{SL}(2, \mathbf{R})$ (see Example 2.2.4) with the irreducible representation $\iota_n : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{SL}(n, \mathbf{R})$ could be conjugated into $\mathrm{SL}(n, \mathbf{Z})$ precisely when n was odd [LT22, Theorem 2.1]. Thus, if X denotes the orbifold of signature $(0; 3, 4, 4)$, $\mathcal{H}_{2k+1}(X)_{\mathbf{Z}} \neq \emptyset$. A priori, it was not initially evident whether Zariski-dense representations in $\mathcal{H}_{2k+1}(X)_{\mathbf{Z}}$ existed as this computation only produces a single such representation, and its image lies within a principally embedded $\mathrm{SL}(2, \mathbf{R})$ inside $\mathrm{SL}(2k+1, \mathbf{R})$. However, Long and Thistlethwaite use an argument suited to the topology of X to show that after passing to a finite-sheeted cover $Y \to X$, the restricted representation of $\pi_1(Y)$ was sufficiently flexible in that it admitted many continuous deformations whose images still remained in $\mathrm{SL}(2k+1, \mathbf{Z})$. Moreover, using recent classifications of Zariski closures of Hitchin representations (see Theorem 3.0.2) and tools in the theory of algebraic groups, they could guarantee that among the many representations in $\mathcal{H}_n(Y)_{\mathbf{Z}}$, some were Zariski-dense. Passing to a finite-index surface subgroup yielded their result (recall that the image would automatically have infinite-index by Proposition 2.1.4) which provided the first examples of freely indecomposable isomorphism types of thin subgroups of $\mathrm{SL}(n, \mathbf{Z})$ for infinitely many n.

Theorems 2.4.1 and 2.4.2 develop the methods behind Long and Thistlethwaite's proof into a more general framework, without resting on facts specific to any particular orbifold's topology, in addition to discussing the case when n is even. As a result, we reduce the problem of finding integral (with respect to a number field), thin Hitchin representations to just merely finding integral ones.

Definition 3.0.1. Let $K/\mathbf{Q}$ be a number field. Call a representation $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{R})$ *K-integral* if $\rho(\pi_1(S)) \leqslant \mathrm{SL}(n, \mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of K.

For the remainder of this chapter we make the standing assumptions that $K \subset \mathbf{R}$ (possibly with $K = \mathbf{Q}$) is a number field of class number one and $\mathcal{O}_K$ its ring of integers (so that $\mathcal{O}_K = \mathbf{Z}$ if $K = \mathbf{Q}$). Given a Hitchin, K-integral $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathcal{O}_K)$, we will perform a series of deformations which alter ρ so that after the process (and possibly passing to a finite-sheeted cover if $K = \mathbf{Q}$), the resulting deformed representation is Zariski-dense. Guaranteeing Zariski-density will come from the following classification originally proven by Guichard, with a full proof appearing in later work by Sambarino.

Theorem 3.0.2 ([Sam20]). *Let $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{R})$ be Hitchin and let G denote the Zariski closure of $\rho(\pi_1(S))$. Then G is conjugate to one of the following groups:*

(i) $\iota_n(\mathrm{SL}(2, \mathbf{R}))$.

(ii) $\mathrm{Sp}(2k, \mathbf{R})$ *if* $n = 2k$ *for some* $k \in \mathbf{Z}$.

(iii) $\mathrm{SO}(k+1, k)$ *if* $n = 2k + 1$ *for some* $k \in \mathbf{Z}$.

(iv) *The image of the fundamental representation of the short root of* G_2 *in* $\mathrm{SL}(7, \mathbf{R})$ *if* $n = 7$.

(v) $\mathrm{SL}(n, \mathbf{R})$.

Remark. Note that for all $k \geq 1$, there are inclusions

$$\iota_{2k}(\mathrm{SL}(2, \mathbf{R})) < \mathrm{Sp}(2k, \mathbf{R}) < \mathrm{SL}(2k, \mathbf{R})$$
$$\iota_{2k+1}(\mathrm{SL}(2, \mathbf{R})) < \mathrm{SO}(k+1, k) < \mathrm{SL}(2k+1, \mathbf{R}) \text{ for } k \neq 3$$
$$\iota_7(\mathrm{SL}(2, \mathbf{R})) < \mathrm{G}_2 < \mathrm{SO}(4,3) < \mathrm{SL}(7, \mathbf{R})$$

so in a sense, this classification provides a totally ordered hierarchy of possibilities for what the Zariski closures of Hitchin representations might be.

In the proof proceeding, we will further assume $n \neq 7$ for simplicity in order to rule out possibility (iv) above. This does not come at much of a loss as [LT22] includes a computation explicitly producing Zariski-dense surface groups in $\mathrm{SL}(7, \mathbf{Z})$, handling the main case of interest when $n = 7$. As a result of this classification, if $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathcal{O}_K)$ is not Zariski-dense, then its Zariski closure must be conjugate to one of the groups listed in options (i)–(iii). We proceed by then showing that for each of these cases, there is a continuous deformation of ρ that remains K-integral but with strictly larger Zariski closure. By performing this deformation at most twice, the groups listed in (i)–(iii) will be ruled out, and thus at the final stage, the Zariski closure will be all of $\mathrm{SL}(n, \mathbf{R})$, hence Zariski-dense.

3.1 Bending on the Hitchin component

The continuous deformation process described in the previous section is known as *bending*, a construction that has appeared in numerous contexts throughout geometry and topology, but used here towards a more arithmetically oriented goal. We also note that as the work in [Zsh22] was finalized, Audibert independently developed similar techniques in [Aud22, Aud23] using these same deformations to produce thin surface groups in various other lattices of $\mathrm{SL}(n, \mathbf{R})$. His work further highlights that the arithmetic side of these deformations should be further explored.

Let $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{R})$ be a representation of a surface group and let $\gamma \in \pi_1(S)$ be representable by a non-separating simple closed curve. Let $S \backslash \gamma$ denote the compact surface with boundary one gets by cutting S along a simple representative of γ. Let γ_1 and γ_2 denote the two boundary components of $S \backslash \gamma$ so that, up to some homeomorphism of S, our setup is of the form shown in Figure 3.1.

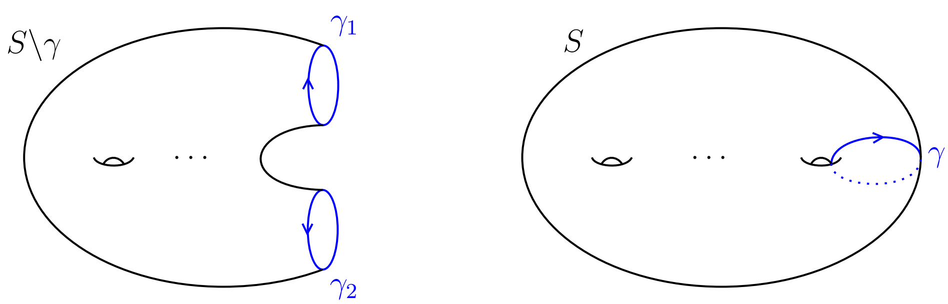


Figure 3.1: S and the resulting cut subsurface.

In S, the curves γ, γ_1 and γ_2 will all be freely homotopic. Thus, by a minor abuse of notation, we will also write γ_1 and γ_2 to denote elements of $\pi_1(S \backslash \gamma)$ and identify γ with γ_1 in $\pi_1(S \backslash \gamma)$. Under this setup, $\pi_1(S)$ arises as an *HNN extension* of $\pi_1(S \backslash \gamma)$. That is, it admits the presentation

$$\pi_1(S) = \pi_1(S \backslash \gamma) *_{\gamma'} := \langle \pi_1(S \backslash \gamma), \gamma' \mid \gamma' \gamma_2 \gamma'^{-1} = \gamma_1 \rangle. \tag{3.1}$$

The stable letter, γ', is an element of $\pi_1(S)$ which, geometrically, corresponds to a simple closed curve on S with $i(\gamma', \gamma) = +1$. Next, let $Z_{\rho(\gamma)}(\mathbf{R})$ denote the $\mathrm{SL}(n, \mathbf{R})$-centralizer of the matrix $\rho(\gamma)$

$$Z_{\rho(\gamma)}(\mathbf{R}) = \{A \in \mathrm{SL}(n, \mathbf{R}) \mid A\rho(\gamma) = \rho(\gamma)A\}.$$

The deformation we will use in our construction is the following.

Definition 3.1.1. Under the setup as above, let $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{R})$ be Hitchin and let $A \in Z_{\rho(\gamma)}(\mathbf{R})$. The *bend of* ρ *about* γ *by* A is the new representation $\rho^A : \pi_1(S) \to \mathrm{SL}(n, \mathbf{R})$ defined by setting:

$$\rho^A|_{\pi_1(S\setminus\gamma)} := \rho|_{\pi_1(S)} \quad \text{and} \quad \rho^A(\gamma') := A\rho(\gamma').$$

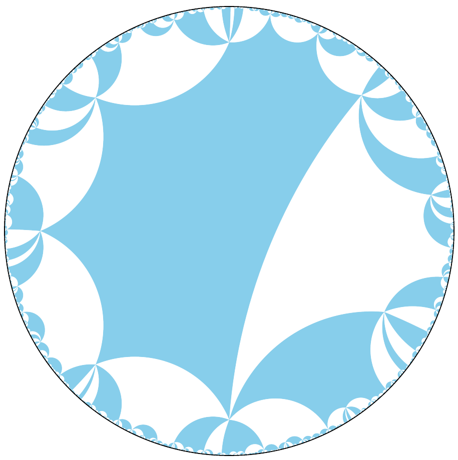
Note that by the presentation given in 3.1 and the fact that A centralizes $\rho(\gamma)$, ρ^A does indeed define a group homomorphism.

Remark. There is an analogous description using free products with amalgamation when γ separates S into two components. We will not treat this version here as it is unnecessary for our applications.

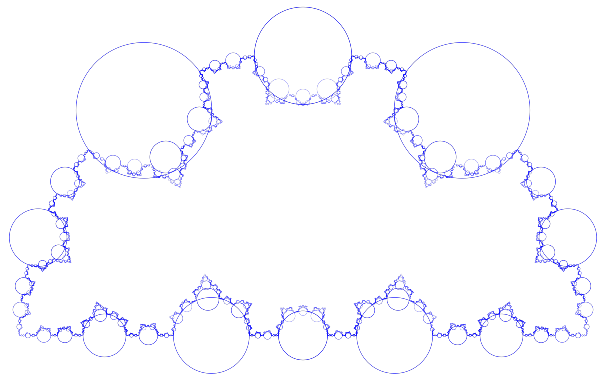
We have already seen deformations of this form: the Fenchel–Nielsen twist deformation $\Phi^t_\gamma(\rho)$ of Example 2.2.3 is a bend of ρ about γ by the matrix $P_{\rho(\gamma)} \operatorname{diag}(e^t, e^{-t}) P^{-1}_{\rho(\gamma)}$ (see Figure 3.2a). Similar deformations have appeared in a number of other low-dimensional contexts as well. The name *bending* comes from the geometry associated to these deformations when the target group is $\mathrm{SL}(2, \mathbf{C})$. In this case, bends of Fuchsian representations "in the universal cover" correspond to a geodesic copy of $\mathbf{H}^2$ inside of $\mathbf{H}^3$ being bent by fixed angles equivariantly along lifts of a geodesic. For small choices of angles, the resulting piecewise-isometrically embedded plane is quasi-isometrically embedded and the result is a quasi-Fuchsian representation (see Figure 3.2b for an approximation of the resulting limit set). Goldman studied analogous deformations for convex projective structures (or equivalently the Hitchin component $\mathcal{H}_3(S)$) and referred to these as *bulging* [Gol13] (see Figure 3.2c). Despite the markedly different geometry associated to these deformations in their varying contexts, we will not make the distinction between the names twisting, bending or bulging as, at their core, all these deformations are of the form of Definition 3.1.1. We also note that a similar construction can be done replacing S with a higher-dimensional hyperbolic n-manifold and γ with a totally geodesic submanifold of codimension 1 [JM87].

As for the proofs of the main theorems, the following observations illustrate how bending can be used to preserve desired features of the original representation. First, if the centralizing matrix, A, is instead chosen to be in the identity component of the centralizer, $Z^\circ_{\rho(\gamma)}(\mathbf{R})$, then the bent representation ρ^A lies on the same path component as ρ. This is immediate from the observation that A lies on the same path component as the identity I_n and $\rho^{I_n} = \rho$. In particular, since the conjugacy class $[\rho]$ is on the Hitchin component, so is $[\rho^A]$, hence the bent representation remains discrete and faithful. Secondly, if the initial representation ρ were K-integral and the centralizing matrix A were in the $\mathcal{O}_K$-points of the centralizer, $Z_{\rho(\gamma)}(\mathcal{O}_K)$, then the bent representation ρ^A will be K-integral as well. This is immediate from the formula defining the bend. In tandem, if ρ is a K-integral Hitchin representation, then for any $A \in Z^\circ_{\rho(\gamma)}(\mathcal{O}_K)$, ρ^A is K-integral and Hitchin.

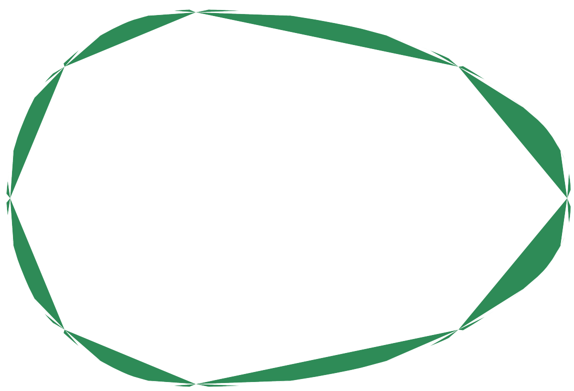
Thus the bending construction can be done so as to preserve the original dynamical and arithmetic properties of the representation. Its utility in producing representations which are Zariski-dense comes from the work of the following sections which show that, for appropriate choices of $A \in Z^\circ_{\rho(\gamma)}(\mathcal{O}_K)$, the bent representations ρ^A will have "larger" Zariski-closure.



(a) Twisting as a hyperbolic metric.



(b) Bending as a quasi-Fuchsian representation.



(c) Bulging as a convex projective structure.

Figure 3.2: The various guises of "bending" deformations applied to Example 2.2.1.

3.2 Constructing the bending matrices

Suppose now that $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathcal{O}_K)$ is Hitchin but not Zariski-dense. Then by Theorem 3.0.2, its Zariski closure is conjugate to one of three possibilities (assuming $n \neq 7$). To achieve the proof of Theorems 2.4.1 and 2.4.2, we outline how one may construct appropriate choices of non-separating simple $\gamma \in \pi_1(S)$ (after possibly passing to a finite-sheeted cover) and matrices $A \in Z^{\circ}_{\rho(\gamma)}(\mathcal{O}_K)$ so that the resulting bend ρ^A is closer to being Zariski dense than ρ. These constructions slightly differ depending on which of the three possibilities give the Zariski-closure of $\rho(\pi_1(S))$, but we observe that the principally embedded $\iota_n(\mathrm{SL}(2,\mathbf{R}))$ has image contained in either $\mathrm{Sp}(2k,\mathbf{R})$ or $\mathrm{SO}(k+1,k)$ depending on the parity of n. Thus, in all of the non-Zariski-dense possibilities, there is always a non-degenerate bilinear form that is preserved by the image of ρ. We denote such a form by J. Moreover, if $n = 2k$, J is alternating and equivalent over $\mathbf{R}$ to the standard symplectic form on $\mathbf{R}^{2k}$ and if $n = 2k+1$, then J is symmetric and equivalent over $\mathbf{R}$ to the standard symmetric form of signature $(k+1,k)$. Before discussing the constructions in each case individually, we will need the following lemmas throughout.

Lemma 3.2.1. *If $A \in \mathrm{SL}(n,K)$ has characteristic polynomial in $\mathcal{O}_K[t]$, there is some $j > 0$ so that $A^j \in \mathrm{SL}(n,\mathcal{O}_K)$.*

Proof. Using the rational canonical form of A, there is a $P \in \mathrm{GL}(n,K)$ so that $A = PRP^{-1}$ for some $R \in \mathrm{SL}(n,\mathcal{O}_K)$. Note that this already uses the hypothesis that K has class number one. That is, if $f \in \mathcal{O}_K[t]$ is the characteristic polynomial of A, then a factorization of f into irreducibles over $\mathcal{O}_K$ will be a factorization into irreducibles over K by Gauss's lemma as $\mathcal{O}_K$ is a unique factorization domain (recall that each $\mathcal{O}_K$ is a Dedekind domain, for which being a UFD is equivalent to being a PID). Thus, the invariant factors of f can be guaranteed to all be in $\mathcal{O}_K[t]$ and hence, the rational canonical form of f will have entries in $\mathcal{O}_K$.

For any $N \in \mathcal{O}_K$, we consider the reduction map $\mathrm{SL}(n,\mathcal{O}_K) \to \mathrm{SL}(n,\mathcal{O}_K/(N))$. The latter is a finite group, hence there is some j such that $R^j \equiv I_n \pmod{(N)}$. In other words, we may write $R^j = I + NX$ for some $n \times n$ matrix $X \in M_n(\mathcal{O}_K)$. In this case, we have that

$$A^j = P(I+NX)P^{-1} = I + NPXP^{-1}.$$

In the above expression, the only term depending on N (other than N itself) is X, which already has entries in $\mathcal{O}_K$. Thus, if we pick a $N \in \mathcal{O}_K$ so that it clears all denominators of entries in P and P^{-1}, we get that A^j has entries in $\mathcal{O}_K$ and hence $A^j \in \mathrm{SL}(n,\mathcal{O}_K)$. □

Lemma 3.2.2. *Suppose $\gamma \in \pi_1(S)$ is so that $\rho(\gamma) \in \mathrm{SL}(n, \mathcal{O}_K)$ has characteristic polynomial in $\mathcal{O}_K[t]$ which is irreducible over $\mathcal{O}_K$. Then there is some $A \in Z^\circ_{\rho(\gamma)}(\mathcal{O}_K)$ which does not preserve the form J.*

Proof. Let $f \in \mathcal{O}_K[t]$ be the characteristic polynomial of $\rho(\gamma)$. By assumption, it is irreducible over the UFD $\mathcal{O}_K$, hence irreducible over K. Let $\alpha \in \overline{K}$ denote some root of f in an algebraic closure and consider the extension $K(\alpha)/K$. Note that $[K(\alpha) : K] = n$ as f must be the minimal polynomial of α due to irreducible. All the roots of f must be distinct and real as ρ is on the Hitchin component (c.f. Theorem 2.3.2) and so $K(\alpha)$ has at least n real embeddings as we may always extend the real embedding of K (from the assumption that $K \subset \mathbf{R}$) to one of $K(\alpha)$ by sending α to any of the other real roots of f. As a consequence, by Dirichlet's unit theorem (see, e.g., [Neu99, Theorem 7.4]) the unit group $\mathcal{O}^\times_{K(\alpha)}$ has rank $\geq n - 1$.

By comparison, by diagonalizing $\rho(\gamma)$ over $\mathbf{R}$ and using the fact that this matrix has distinct real eigenvalues, one sees that the centralizer of $\rho(\gamma)$ in $\mathrm{SO}(J; \mathbf{R})$ has rank $\frac{n}{2}$ (where $\mathrm{SO}(J; \mathbf{R})$ is the group of matrices in $\mathrm{SL}(n, \mathbf{R})$ preserving the alternating or symmetric form J). Notice then that as long as $n > 2$, $n - 1 > \frac{n}{2}$.

Note now that there is a K-algebra isomorphism $K(\alpha) \cong K[\rho(\gamma)]$ induced by the map $\alpha \mapsto \rho(\gamma)$, as both of these K-algebras are isomorphic to $K[t]/(f)$. Consider now, the image of $\mathcal{O}^\times_{K(\alpha)} \subset K(\alpha)$ under this map. By the above rank considerations, there is an infinite order $u \in \mathcal{O}^\times_{K(\alpha)}$ whose image in $K[\rho(\gamma)]$, A', satisfies the property that no power of A' preserves the form J. We may further assume that $u > 0$ by replacing u with $-u$ if necessary. Now, as $A' \in K[\rho(\gamma)]$, it is a polynomial in powers of $\rho(\gamma)$ hence will centralize $\rho(\gamma)$.

At the moment, this matrix A' still has some possibly undesirable properties, but note that its characteristic polynomial is in $\mathcal{O}_K[t]$. Indeed, its characteristic polynomial factors as $\prod_{i=1}^{n}(t-u^{\sigma_i})$ where each $\sigma_i : K(\alpha) \to \mathbf{C}$ are the distinct K-embeddings of $K(\alpha)$. The determinant of A' at the moment is given by the unit $v = \mathrm{Norm}_{K(\alpha)/K}(u) \in \mathcal{O}_K^{\times}$, but we may pass to a higher power of A' and rescale by some power of v^{-1} so that the resulting matrix has determinant 1. That is, there is some i_1, i_2 so that $A'' := v^{-i_1}(A')^{i_2} \in \mathrm{SL}(n, K)$. A'' will still have characteristic polynomial in $\mathcal{O}_K[t]$, hence by Lemma 3.2.1, some further power $A = (A'')^{i_3}$ has that $A \in \mathrm{SL}(n, \mathcal{O}_K)$.

Notice that A does not preserve the bilinear form J by the assumptions on the unit u. A still centralizes $\rho(\gamma)$ as A' did (it was a polynomial in $\rho(\gamma)$), and so $A \in Z_{\rho(\gamma)}(\mathcal{O}_K)$. What remains is to show why A is on the same connected component as the identity matrix. For this, notice that we chose $u > 0$. Hence by this fact, and the fact that $\rho(\gamma)$ is the exponential of some matrix in $\mathfrak{sl}(n, \mathbf{R})$ (as its eigenvalues can be assumed to be distinct, *positive* and real), then A will also be the exponential of some matrix in $\mathfrak{sl}(n, \mathbf{R})$. Thus $A \in Z^{\circ}_{\rho(\gamma)}(\mathcal{O}_K)$ and $A \notin \mathrm{SO}(J; \mathbf{R})$ as desired. □

Remark. If n is odd, the characteristic polynomials of $\rho(\gamma)$, when ρ is not Zariski dense, are never irreducible over $\mathcal{O}_K$ as they always have an eigenvalue 1. Nonetheless, the same conclusions of this lemma can be adopted in an identical manner, to the situation where f in the above proof is replaced with the polynomial g so that $(t-1)g(t)$ is the characteristic polynomial of $\rho(\gamma)$ and $g \in \mathcal{O}_K[t]$ is irreducible over $\mathcal{O}_K$. That is, in this case, we instead conjugate $\rho(\gamma)$ over $\mathcal{O}_K$ so that it is block-diagonal with a single $(n-1) \times (n-1)$ block along the diagonal, and apply the above argument to this block, yielding the same result.

Lemma 3.2.2 provides sufficient conditions that allow one to construct a Zariski-dense K-integral representation, as follows.

Lemma 3.2.3. *Suppose $\gamma \in \pi_1(S)$ is a non-separating simple closed curve and $A \in Z^{\circ}_{\rho(\gamma)}(\mathcal{O}_K)$ does not preserve the bilinear form J. Then the bent representation ρ^A has*

Zariski-dense image.

Proof. If ρ^A did not have Zariski-dense image, then again by the classification of Zariski closures in Theorem 3.0.2, the image of ρ^A must preserve some (alternating or symmetric) bilinear form J'.

But from the definition of the bend (Definition 3.1.1), ρ and ρ^A agree on the subgroup $\pi_1(S\backslash\gamma)$, on which the representations are irreducible (by generalizations of Labourie's work to the case of surfaces with boundary, e.g., in [ALS23]). This implies that for any element $\beta \in \pi_1(S\backslash\gamma)$,

$$J^{-1}J'\rho(\beta) = \rho(\beta)J^{-1}J'. \tag{3.2}$$

But $\rho(\gamma)$ by Theorem 2.3.2, after applying some appropriate change of basis, the above implies that some conjugate $PJ^{-1}J'P^{-1}$ centralizes a diagonal matrix with distinct, real eigenvalues and hence $PJ^{-1}J'P^{-1}$ is diagonal itself.

We claim then that $PJ^{-1}J'P^{-1}$ must be a homothety. If not, there is some λ so that E_λ, the λ-eigenspace of $J^{-1}J'$, is a proper non-trivial subspace of $\mathbf{R}^n$ which, by Equation 3.2 is invariant under the action of $\pi_1(S\backslash\gamma)$. But this contradicts irreducibility of $\rho|_{\pi_1(S\backslash\gamma)}$, so $PJ^{-1}J'P^{-1}$ must be a homothety. Thus $J' = \lambda J$ and these forms define the same symplectic or orthogonal groups, which in turn implies that ρ^A preserves the form J. But this contradicts our assumption on A, so this cannot in fact occur.

Thus the image of ρ^A cannot preserve any non-degenerate bilinear forms hence, by Theorem 3.0.2, is a Zariski-dense representation in $\mathcal{H}_n(S)_{\mathcal{O}_K}$. □

3.2.1 Bending when $K \neq \mathbf{Q}$

The philosophically simplest case is when the extension $K/\mathbf{Q}$ has degree larger than one, i.e. the case of Theorem 2.4.2. In this case, we show that *any* choice of simple closed curve suffices.

Proof of Theorem 2.4.2. Recall our assumptions that $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathcal{O}_K)$ is Hitchin and not Zariski-dense and J denotes whichever bilinear form is preserved by the image of ρ. Fix *any* nontrivial, non-separating, simple $\gamma \in \pi_1(S)$ and let $f \in \mathcal{O}_K[t]$ be the characteristic polynomial of $\rho(\gamma)$. If f is already irreducible over $\mathcal{O}_K$, then we can take A to be the matrix guaranteed by Lemma 3.2.2 which, by Lemma 3.2.3, gives the Zariski-dense, Hitchin and K-integral representation ρ^A needed for the desired result.

Thus, we need only consider the case where f is reducible over $\mathcal{O}_K$. For this, it suffices to assume that $f = f_1 f_2$ for $f_i \in \mathcal{O}_K[t]$ of degree $n_1, n_2 \geq 1$ respectively. Replacing f_1 and f_2 with uf_1 and $u^{-1}f_2$ for some $u \in \mathcal{O}_K^\times$ if necessary, we may further assume that f_1 and f_2 are monic and have constant terms $(-1)^{n_1}$ and $(-1)^{n_2}$ respectively. From the factorization of our characteristic polynomial and the rational canonical form (again, using that K has class number one), there is some $P \in \mathrm{GL}(n, K)$ so that

$$P\rho(\gamma)P^{-1} = \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix}$$

in block-diagonal form where, for $i = 1, 2$, $C_i \in \mathrm{SL}(n_i, \mathcal{O}_K)$ and has characteristic polynomial f_i.

Now as $K \neq \mathbf{Q}$, $[K : \mathbf{Q}] > 1$. Since we assume K is not totally imaginary, K has at least one real embedding which forces $\mathcal{O}_K^\times$ to have rank ≥ 1 by Dirichlet's unit theorem. We may thus fix an infinite order unit $u \in \mathcal{O}_K^\times$ and, replacing u with $-u$ if necessary, assume that $u > 0$. In this case, let A' be the matrix, in block-diagonal form, given by

$$A' := P^{-1} \begin{pmatrix} u^{n_2} I_{n_1} & \\ & u^{-n_1} C_2 \end{pmatrix} P.$$

$A' \in \mathrm{SL}(n, K)$ has characteristic polynomial still in $\mathcal{O}_K[t]$ and centralizes $\rho(\gamma)$ as $PA'P^{-1}$

has the same block-diagonal structure as $P\rho(\gamma)P^{-1}$. As $u > 0$, C_2 is diagonalizable and hence $A' = \exp(X)$ for some diagonalizable $X \in \mathfrak{sl}(n, \mathbf{R})$ as well. Moreover, note that A' (nor any power of it) preserves the form J as its eigenvalues are u^{n_2} (with multiplicity n_1) and $u^{-n_1}\lambda$ as λ varies among any of the (distinct, real) eigenvalues of C_2, and these eigenvalues do not satisfy the symmetry needed in order to preserve a (alternating or symmetric) bilinear form J.

Passing to a higher power of A' as in Lemma 3.2.1 yields a matrix $A \in \mathrm{SL}(n, \mathcal{O}_K)$ still centralizing $\rho(\gamma)$, in the image of the exponential map and not preserving the form J (as u was an infinite order unit). Thus, by Lemma 3.2.3, the bend ρ^A has Zariski-dense image. □

The simplicity of the proof when $K \neq \mathbf{Q}$ is reflected in the construction above in the case of reducible characteristic polynomial. The presence of infinite order units in the ring of integers $\mathcal{O}_K$ provides enough flexibility that one may essentially manually construct matrices that produce Zariski-dense representations after a single bend. The lack of such units when $K = \mathbf{Q}$ requires a more subtle strategy in the remaining case.

3.2.2 Bending when $K = \mathbf{Q}$

The construction in the $K = \mathbf{Q}$ case is the result of Theorem 2.4.1. As the previous section handled the case where $K \neq \mathbf{Q}$, our standing assumptions from here on are that $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{Z})$ is Hitchin, but not Zariski-dense. In this case, the same construction as in the previous section's proof for Theorem 2.4.2 does not immediately generalize as $\mathbf{Z}$ does not possess any infinite order units. This lack of apparent flexibility in the centralizers of elements in $\rho(\pi_1(S))$ mean we must construct the bending matrices with a more nuanced approach.

Assuming the Zariski-closure of $\rho(\pi_1(S))$ is "as small as possible," we will first establish that a relatively direct argument shows a single bend can guarantee a Zariski-closure larger than a principal $\mathrm{SL}(2,\mathbf{R})$. Thus leaving the cases of $\mathrm{SO}(k+1,k)$ and $\mathrm{Sp}(2k,\mathbf{R})$ to rule out. We will also rule these cases out with a second bend, requiring more sophisticated tools.

Bending out of a principal $\mathrm{SL}(2,\mathbf{R})$

The simplest step is the first one, where $\rho : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Z})$ has Zariski-closure a principal $\mathrm{SL}(2,\mathbf{R})$ inside of $\mathrm{SL}(n,\mathbf{R})$. In this case, we will see that one can still perform a bend in a fairly direct manner.

Lemma 3.2.4. *Let $\rho : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Z})$ be as above and fix any non-separating simple $\gamma \in \pi_1(S)$. Then there is an $A \in Z^\circ_{\rho(\gamma)}(\mathbf{Z})$ so that the Zariski-closure of ρ^A is not a principal* $\mathrm{SL}(2,\mathbf{R})$.

Proof. In this case, if n is even, let $f \in \mathbf{Z}[t]$ denote the characteristic polynomial of $\rho(\gamma)$. In the case that n is odd, as in the remark following the proof of Lemma 3.2.2, we instead let $f \in \mathbf{Z}[t]$ be so that $(t-1)f(t)$ is the characteristic polynomial of $\rho(\gamma)$. If f is irreducible over $\mathbf{Z}$ in either of these cases, we again apply Lemma 3.2.2 to construct the bending matrix A. In particular, this matrix does not preserve the form preserved by the image of ρ hence, by Lemma 3.2.3 is automatically Zariski-dense and the claim is shown.

What remains is the case where f is reducible over $\mathbf{Z}$, thus we assume $f(t)$ admits a factorization of the form

$$f_1(t)f_2(t) \quad \text{or} \quad (t-1)f_1(t)f_2(t)$$

(depending on the parity of n) where $f_1, f_2 \in \mathbf{Z}[t]$ are polynomials of degree $n_1 \geq n_2 \geq 1$ respectively. If n is even, assume further without loss of generality that $n_1 > 1$. From this factorization of the characteristic polynomials, there are matrices $P \in \mathrm{GL}(n, \mathbf{Q})$ so that $P\rho(\gamma)P^{-1}$ is of the form

$$\begin{pmatrix} 1 & & \\ & C_1 & \\ & & C_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix},$$

for $C_i \in \mathrm{SL}(n_i, \mathbf{Z})$ with characteristic polynomial f_i (again, depending on the parity of n). We then take $A' \in \mathrm{SL}(n, \mathbf{Z})$ to be the matrix:

$$A' = \begin{cases} P^{-1} \begin{pmatrix} I_{n_1+1} & \\ & C_2 \end{pmatrix} P & \text{if } n \text{ is odd} \\ P^{-1} \begin{pmatrix} I_{n_1} & \\ & C_2 \end{pmatrix} P & \text{if } n \text{ is even.} \end{cases}$$

In both cases, $\det(A') = \det(C_2) = 1$ and A' is in the $\mathbf{Q}$-centralizer of $\rho(\gamma)$. By Lemma 3.2.1, there is some $j > 0$ so that $A = (A')^j$ has entries in $\mathbf{Z}$. By construction $A \in \mathrm{SL}(n, \mathbf{Z})$ centralizes $\rho(\gamma)$. Moreover, A is not contained in any principal $\mathrm{SL}(2, \mathbf{R})$. This is because the irreducible representation ι_n sends the diagonal matrix with eigenvalues λ, λ^{-1} to a diagonal matrix whose eigenvalues are of the form λ^{n-2i-1} as i ranges from 0 to $n-1$. In particular, any diagonalizable matrix in any principally embedded $\mathrm{SL}(2, \mathbf{R})$ has eigenvalues of this form. But the constructed matrix A' clearly does not since it has an eigenvalue 1 with multiplicity $n_1 + 1$ when n is odd and n_1 when n is even, and both of these are greater than 1 by our choice of factorization of f. As A is a power of A', this same eigenvalue argument applies to see that A does not lie in a principal $\mathrm{SL}(2, \mathbf{R})$ either.

What remains is why A lies in the same connected component as the identity matrix in the centralizer. For this, it again suffices to show that A' lies in the image of the exponential map, as A is a power of A'. We note that since $\rho(\gamma)$ has distinct *positive* real eigenvalues, then the block matrix C_2 will have distinct, positive real eigenvalues, hence will lie in the image of the exponential map. Namely, A' will be conjugate to the matrix $\exp(\mathrm{diag}(0,\ldots,0,\log\lambda_1,\ldots,\log\lambda_{n_2}))$ where $\lambda_1,\ldots,\lambda_{n_2}$ are the positive eigenvalues of C_2. Thus A' and A lie in the image of the exponential map. □

The proof of this lemma in fact establishes the partial result towards Theorem 2.4.1 that if $\mathcal{H}_n(S)_{\mathbf{Z}} \neq \emptyset$, then $\mathcal{H}_n(S)_{\mathbf{Z}}$ contains representations whose Zariski-closures are conjugate to either $\mathrm{SO}(k+1,k)$ or $\mathrm{Sp}(2k,\mathbf{R})$. Moreover, such representations can be constructed by bending elements in $\mathcal{H}_n(S)_{\mathbf{Z}}$ about *any* $\gamma \in \pi_1(S)$ represented by a non-separating simple closed curve.

Bending out of a symplectic or orthogonal group

What remains is the case that $\rho : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Z})$ has Zariski-closure conjugate to either $\mathrm{SO}(k+1,k)$ or $\mathrm{Sp}(2k,\mathbf{R})$. This is also the most complicated case in establishing Theorem 2.4.1 and in [LT22]. The starting point is in establishing the following.

Lemma 3.2.5. *Let* $\rho : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Z})$ *have Zariski-closure conjugate to* $\mathrm{Sp}(2k,\mathbf{R})$ *if* $n = 2k$ *or* $\mathrm{SO}(k+1,k)$ *if* $n = 2k+1$. *Then there is a* $\gamma \in \pi_1(S)$ *(which may not be simple) so that either* n *is even and* $\rho(\gamma)$ *has* $\mathbf{Z}$*-irreducible characteristic polynomial or* n *is odd and* $\rho(\gamma)$ *has characteristic polynomial* $(t-1)f(t)$ *where* $f \in \mathbf{Z}[t]$ *is* $\mathbf{Z}$*-irreducible.*

Proof. We let G denote the Zariski-closure of $\rho(\pi_1(S))$. First, we handle the case $n = 2k$, in which case $G \cong \mathrm{Sp}(2k,\mathbf{R})$. In this case, we may apply Theorem 2.1.3 for G and fix a prime p so that the reduction map $\pi_p(\rho(\pi_1(S))$ is all of $\mathrm{Sp}(2k,\mathbf{F}_p)$. If $R_p(2k)$ denotes the set of $2k \times 2k$ matrices in $\mathrm{Sp}(2k,\mathbf{F}_p)$ whose characteristic polynomials are reducible over

$\mathbf{F}_p$, then a bound, attributed to Borel and proven in [Cha97, Corollary 3.6], establishes that

$$|R_p(2k)| \leq \left(1 - \frac{1}{3k}\right) |\operatorname{Sp}(2k, \mathbf{F}_p)|.$$

In particular, $\operatorname{Sp}(2k, \mathbf{F}_p)$ contains matrices with $\mathbf{F}_p$-irreducible characteristic polynomials. In particular, there exists some $\gamma \in \pi_1(S)$ so that the characteristic polynomial of $\pi_p(\rho(\gamma))$ is irreducible modulo p. Consequently, $\rho(\gamma)$ must have irreducible characteristic polynomial over $\mathbf{Z}$ (as otherwise, any factorization would induce a factorization over $\mathbf{F}_p$).

Now, assume $n = 2k + 1$, in which case $G = \operatorname{SO}(J; \mathbf{R})$ where J is a symmetric bilinear form of signature $(k + 1, k)$. In this case, we cannot simply apply the strong approximation theorem as stated in Theorem 2.1.3 simply because the algebraic group $\operatorname{SO}(k + 1, k)$ is not simply connected. But for a prime p, we still consider the reduction maps $\pi_p : \operatorname{SO}(J; \mathbf{Z}) \to \operatorname{SO}(J; \mathbf{F}_p)$. Here, it is a classical result that there is a unique equivalence class of symmetric bilinear forms over odd-dimensional vector spaces over finite fields (c.f. [Suz82, Theorem 5.8]), thus we will denote the reduction map by $\pi_p : \operatorname{SO}(J; \mathbf{Z}) \to \operatorname{SO}(n; \mathbf{F}_p)$. In this case, while strong approximation does not directly apply, a corollary of it following from work of Weisfeiler in [Wei84] establishes that for all but finitely many primes, $\pi_p(\rho(\pi_1(S)))$ contains $\Omega(n; \mathbf{F}_p)$, the commutator subgroup of $\operatorname{SO}(n; \mathbf{F}_p)$ (which has index 4 in $\operatorname{SO}(n; \mathbf{F}_p)$). Fix one such prime p. There exist matrices in $\operatorname{SO}(n; \mathbf{F}_p)$ of the form $(t - 1)\overline{f}(t)$ where $\overline{f}$ is irreducible over $\mathbf{F}_p$ (e.g. there is a direct construction of such a matrix in [LT22, Proposition 3.8]), and thus a $\gamma \in \pi_1(S)$ so that $\pi_p(\rho(\gamma))$ has characteristic polynomial $(t - 1)\overline{f}(t)$ where $\overline{f}$ is irreducible over $\mathbf{F}_p$. Thus, as in the even n case, $\rho(\gamma)$ will have characteristic polynomial $(t-1)f(t)$ for f irreducible over $\mathbf{Z}$. □

The previous lemma produces $\gamma \in \pi_1(S)$ whose characteristic polynomial is irreducible over $\mathbf{Z}$, but in order to bend, we need a curve γ which is simple. Long and

Thistlethwaite reached this same point, after which, they constructed a tower of finite-sheeted covers over their orbifold in which their curve would lift simply. We do the same, but as we work over closed surface groups in general, we may apply the following result of Scott who showed that subgroups of surface groups are always *geometric* in some finite-sheeted cover.

Theorem 3.2.6 ([Sco78]). *For any $\gamma \in \pi_1(S)$, there is a finite-sheeted cover $S' \to S$ in which γ lifts to a simple closed curve.*

Using Scott's theorem to replace the tower of covers, we may now prove our result.

Proof of Theorem 2.4.1. Suppose $\rho : \pi_1(S) \to \mathrm{SL}(n, \mathbf{Z})$ is not Zariski-dense. By Lemma 3.2.4, after possibly performing one bend, we may assume the Zariski-closure of $\rho(\pi_1(S))$, G, is conjugate to either $\mathrm{Sp}(2k, \mathbf{R})$ or $\mathrm{SO}(k+1, k)$. By Lemma 3.2.5, there is a $\gamma \in \pi_1(S)$ whose characteristic polynomial is of the form indicated in the statement of the Lemma and by Scott's theorem, there is some finite-sheeted cover $S' \to S$ in which γ lifts simply. Since every separating curve lifts to a non-separating one in a finite-sheeted cover as well, we may further assume that γ lifts to a non-separating simple closed curve.

Let H denote the Zariski-closure of the restriction $\rho|_{\pi_1(S')}$ (after identifying $\pi_1(S')$ with its image in $\pi_1(S)$ under the covering map). First, we claim that $H = G$. To see this, note that $\rho|_{\pi_1(S')}$ lies on the Hitchin component of S', hence Theorem 3.0.2 also implies that either $H = G$ or H is a principal $\mathrm{SL}(2, \mathbf{R})$. As $\pi_1(S')$ has finite-index in $\pi_1(S)$, H has finite-index in G and so, if $g_1, \ldots, g_j$ were finitely many coset representatives for G/H, then $g_1 H \cup \ldots \cup g_j H$ is a Zariski-closed subset containing G. If H were a principal $\mathrm{SL}(2, \mathbf{R})$, this would either make $\mathrm{Sp}(2k, \mathbf{R})$ or $\mathrm{SO}(k+1, k)$ a finite union of principal $\mathrm{SL}(2, \mathbf{R})$'s and simply for dimension reasons, this is not the case. Thus, in fact, $H = G$.

Thus, the restriction $\rho|_{\pi_1(S')} : \pi_1(S') \to \mathrm{SL}(n, \mathbf{Z})$ lies on the Hitchin component for S', has Zariski-closure conjugate to either $\mathrm{Sp}(2k, \mathbf{R})$ or $\mathrm{SO}(k+1, k)$ and possesses a simple non-separating $\gamma \in \pi_1(S')$ whose characteristic polynomial is either $f(t)$ or $(t-1)f(t)$ for $\mathbf{Z}$-irreducible f. Under this setup, we may then apply Lemmas 3.2.2 and 3.2.3, to conclude the existence of a matrix $A \in Z^{\circ}_{\rho(\gamma)}(\mathbf{Z})$ so that the bend ρ^A has Zariski-dense image. Thus $\mathcal{H}_n(S')_{\mathbf{Z}}$ contains Zariski-dense representations. □

3.3 Further applications

There are a number of further avenues left to explore following the methods developed in proving Theorems 2.4.1 and 2.4.2.

The first begins with the observation that there is quite a lot of flexibility in the construction of §3.2.1 used in the proof of Theorem 2.4.2. Namely, these same methods indicate that in fact, when $\mathcal{H}_n(S)_{\mathcal{O}_K}$ is nonempty, it possesses infinitely many representations which are all distinct, even modulo the action of the mapping class group. This is because in the proof given, one is able to choose *any* non-separating simple $\gamma \in \pi_1(S)$ to bend about and pass to further powers of the bending matrix used, different choices of which all result in distinct representations. In particular, there is an even more meaningful means in which one may say that there are "many" representations in $\mathcal{H}_n(S)_{\mathcal{O}_K}$.

The covering space argument used in the proof of Theorem 2.4.1 precludes us from saying the same about $\mathcal{H}_n(S)_{\mathbf{Z}}$, and so further tools are required to better quantify the amount of such representations. In fact, doing as much through developing integral point counting tools is listed as "Task 25" in [Wie18]. Even moreso, at the time of writing, the sets $\mathcal{H}_n(S)_{\mathbf{Z}}$ are only known to be nonempty when n is odd or $n = 4$ by [LT18, LT22], and so even developing further tools (computational and not) to determine obstructions to integral points even existing at all merits further exploration.

On the other hand, there are other n for which $\mathrm{SL}(n, \mathbf{Z})$ does contain Anosov surface subgroups which are not necessarily Hitchin, and bending deformations can be defined for these as well, there is just a considerable lack in our understanding of what these deformations correspond to geometrically. For instance, when using bending to construct quasi-Fuchsian subgroups of $\mathrm{SL}(2, \mathbf{C})$, there is a descriptive geometric picture (c.f. Figure 3.2b) which is compatible with the algebraic description of the theory. One can in fact use this geometric perspective to understand when "too much bending" can result in representations which are not discrete nor faithful. Developing an analogous understanding for other Anosov, but possibly non-Hitchin, representations of surface groups presents one path towards finding thin surface subgroups of $\mathrm{SL}(2k, \mathbf{Z})$ whenever $k \geq 3$. We record one such example where such an application would be of note here.

Example 3.3.1. Take a Zariski-dense Hitchin representation $\rho : \pi_1(S) \to \mathrm{SL}(2k - 1, \mathbf{Z})$ and let $\tau : \mathrm{SL}(2k-1, \mathbf{R}) \hookrightarrow \mathrm{SL}(2k, \mathbf{R})$ be the inclusion induced by embedding $\mathrm{SL}(2k-1, \mathbf{R})$ in the upper-left corner of a $2k \times 2k$ matrix. The representation $\tau \circ \rho : \pi_1(S) \to \mathrm{SL}(2k, \mathbf{Z})$ is Anosov (c.f. [Can20, Corollary 32.5] and [Bar10]) and therefore, yields a surface subgroup of $\mathrm{SL}(2k, \mathbf{Z})$ whose Zariski closure has codimension $2n - 1$. Small deformations of $\tau \circ \rho$ remain discrete and faithful by Theorem 2.3.4, but in many cases, one can practically compute "large" integral bends of $\tau \circ \rho$ which result in Zariski-dense representations of surface groups into $\mathrm{SL}(2k, \mathbf{Z})$. However, because preserving the integrality property often imposes less choice in size of centralizing matrices, it is currently unknown whether any of these remain faithful. Nonetheless, developing an understanding for how the Anosov property interacts with bending deformations specifically presents one possible path forward.

Even more generally, any word hyperbolic group Γ which decomposes either as a nontrivial free product with amalgamation or an HNN extension admits suitable deformations on representations of Γ which one may also refer to as a "bending." Johnson and Millson study aspects related to the geometry of such deformations when Γ is the fundamental group of a hyperbolic n-manifold [JM87], and how these specific deformations interact with the geometry of general Anosov representations of Γ may present a means towards further understanding aspects around Question 4. To this end, Dey and Kapovich very recently provided some criteria towards understanding when bends of Anosov representations remain Anosov [DK23a, DK23b] and even more recently, Maloni et al. studied bending deformations for *quasi-Hitchin* representations into $\mathrm{SL}(n, \mathbf{C})$ [MMMZ23]. In future work, we plan to investigate how these works interact with the arithmetic perspective presented here.

Chapter 4

Rationality

The following chapter investigates the rational structure of representations on the Hitchin component. The content of this chapter up to and including §4.3 appears in the paper *Rational approximation for Hitchin representations* [AZ23], produced by the author in joint work with Jacques Audibert and reproduced here with the latter's permission. The main result presented is the proof of Theorem 2.4.3.

This result provides a topological characterization for Hitchin representations which are conjugate into $\mathrm{SL}(n, \mathbf{Q})$. Such subgroups have a number of interesting properties. For instance, these are all forms of subgroups for which the strong approximation theorem applies (Theorem 2.1.3). Moreover, they are contained in lattices of products of Lie groups and p-adic Lie groups, and as such, admit natural actions on products of symmetric spaces and Bruhat–Tits buildings, where much about the properties of these actions are unknown (c.f. Example 2.4.2).

Before we discuss the proof, we recall a previously known case of this result in rank one. For closed hyperbolic 2-orbifolds, X, there is one obvious obstruction on the signature that must be satisfied in order for $\pi_1(X)$ to admit a discrete and faithful representation into $\mathrm{SL}(2, \mathbf{Q})$. Namely, finite order elements of $\mathrm{PSL}(2, \mathbf{Q})$ can only have order 2 and 3 and hence the orders of cone points of X can only be 2 or 3. In [Tak71], Takeuchi shows that if the signature of X satisfies this obvious condition, then $\mathcal{H}_2(X)_{\mathbf{Q}}$ is dense in $\mathcal{H}_2(X)$ (recall that $\mathcal{H}_2(X) = \mathcal{T}(X)$). Thus the torsion-free and genus 3 or more case of Takeuchi's result is the $n = 2$ case of Theorem 2.4.3 (our proof in fact does work when $g = 2$ and $n = 2$, but we do not include this case in the statement since this was previously known anyways). His proof, however, is essentially by a direct argument, and we can illustrate the idea for closed surface groups here. Suppose S has genus g and fix the following presentation of its fundamental group

$$\pi_1(S) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \,|\, [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] \rangle.$$

Using the above presentation, the space of all homomorphisms of $\pi_1(S)$ into $\mathrm{SL}(2, \mathbf{R})$ can be viewed as a subset of $\mathrm{SL}(2, \mathbf{R})^{2g}$ cut out by the polynomial equations determined by the requirement that $[\rho(\alpha_1), \rho(\beta_1)] \ldots [\rho(\alpha_g), \rho(\beta_g)] = I_2$. In total there are four such polynomials, one for each entry of the matrix. But when restricting to only representations in $\mathcal{H}_2(X)$, one can show that it suffices to instead consider the simpler equation $\mathrm{Tr}([\rho(\alpha_1), \rho(\beta_1)] \ldots [\rho(\alpha_g), \rho(\beta_g)]) = 2$ by ruling out other parabolics using discrete and faithfulness. This reduction, from four polynomial equations to a single one, and a (then relatively recent) rigidity result of Weil [Wei60] allowed Takeuchi to conclude his density result.

We take care to note Takeuchi's proof because the method also makes apparent why a similar line of thinking has little hope of generalizing to higher-rank. For one, some of the tools used simply do not apply in the slightly new setting. Weil's rigidity result, for example, relies on the fact that the discrete and faithful representations into $\mathrm{SL}(2, \mathbf{R})$ had cocompact image, whereas for $n > 2$, the images of representations in $\mathcal{H}_n(S)$ are always of infinite covolume. More subtly, in considering representations on the Hitchin component, the required relation the matrices must satisfy imposes n^2 polynomial equations. The behavior of parabolics in higher-rank is much more complicated as well and so no similar "tricks" allow one to reduce the question of density to the study of rational points on a *single* polynomial equation. Heuristically, the difficulty of extending the methods of [Tak71] grows about quadratically in n.

Reflecting these difficulties, it should come of little surprise that our proof of Theorem 2.4.3 is of a completely different nature and is much more geometric. Moreover, while Theorem 2.4.3 concerns conjugacy classes of representations, the same result will hold at the level of individual representations. Namely, if we let

$$\widetilde{\mathcal{H}}_n(S) \subset \mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$$

denote the connected component of representations for which the quotient map $\widetilde{\mathcal{H}}_n(S) \to \mathcal{H}_n(S)$ is a principal $\mathrm{SL}(n, \mathbf{R})$-bundle, and let $\widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ denote the set of representations with image in $\mathrm{SL}(n, \mathbf{Q})$, then Theorem 2.4.3 along with the well-known fact that $\mathrm{SL}(n, \mathbf{Q})$ is dense in $\mathrm{SL}(n, \mathbf{R})$ gives rise to the following immediate corollary.

Corollary 4.0.0.1. *$\widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ is dense in $\widetilde{\mathcal{H}}_n(S)$.*

Remark. We also note that prior to this work, other rationality properties for $\mathrm{SL}(n, \mathbf{R})$-character varieties of surface groups have also been studied in [RBKC96].

At this point, before moving on to the proof, we make the standing assumption that S is a closed surface of genus at least 3. We still believe the result is likely true in genus 2, but there is one key point where the current methods fail for this remaining case, which we will note when it arises.

4.1 The symplectic nature of bending

The proof of Theorem 2.4.3 utilizes bending deformations, as in the previous Chapter, but reinterpreted into a more flow-theoretic perspective. Thus, while they are of the same form as those in Definition 3.1.1, the flow-theoretic perspective will more resemble the Fenchel–Nielsen twist flows of Example 2.2.3. In line with the themes of much of this thesis's work, the core of the proof we present rests on key interactions between the analytic, geometric and arithmetic perspectives behind these deformations.

Recall from Definition 2.3.3 that $\mathrm{Tr}_\gamma : \mathcal{H}_n(S) \to \mathbf{R}$ is the function $\mathrm{Tr}_\gamma([\rho]) := \mathrm{Tr}(\rho(\gamma))$. Let $F : \mathrm{SL}(n, \mathbf{R}) \to \mathfrak{sl}(n, \mathbf{R})$ be the projection

$$F(A) := A - \frac{\mathrm{Tr}(A)}{n} I_n.$$

We first begin by defining a flow on $\mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ as follows. For any nontrivial $\gamma \in \pi_1(S)$ which is freely homotopic to a non-separating simple closed curve, recall that $\pi_1(S)$ arises as an HNN-extension of $\pi_1(S\backslash\gamma)$ (c.f. Equation 3.1). Therefore, fixing one such $\gamma' \in \pi_1(S)$ such that $i(\gamma', \gamma) = +1$, the map

$$\Xi^t_\gamma(\rho)(\alpha) = \begin{cases} \rho(\alpha) & \text{if } \alpha \in \pi_1(S\backslash\gamma) \\ \exp(tF(\rho(\gamma)))\rho(\alpha) & \text{if } \alpha = \gamma' \end{cases} \tag{4.1}$$

defines a continuous flow on $\mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ (c.f. Equation 2.1). Note that indeed, this is a representation of $\pi_1(S)$ as $\exp(tF(\rho(\gamma))) \in Z^\circ_{\rho(\gamma)}(\mathbf{R})$ and that $\Xi^t_\gamma(\rho)$ is a bend of the representation ρ about γ, by the notation of Definition 3.1.1.

Definition 4.1.1. The flow Ξ^t_γ on $\mathrm{Hom}(\pi_1(S), \mathrm{SL}(n, \mathbf{R}))$ defined by Equation 4.1 is called a *generalized twist flow* about γ.

The main result of this section, and a key step in proving Theorem 2.4.3, is the following lemma which states that two Hitchin representations may be connected via a path which is a piecewise concatenation of orbits of twist flows of this form.

Lemma 4.1.1. *Let ρ_1 and ρ_2 be Hitchin representations. Then there exist non-separating simple closed curves $\gamma_1, \ldots, \gamma_k$ and real numbers $t_1, \ldots, t_k$ so that the representation*

$$\rho_2' := \Xi_{\gamma_k}^{t_k}(\ldots(\Xi_{\gamma_1}^{t_1}(\rho_1))\ldots)$$

is conjugate to ρ_2. In other words, $[\rho_2'] = [\rho_2]$ on $\mathcal{H}_n(S)$.

Remark. Analogs of this lemma in the context of other Lie groups have been known before (e.g. [GX11, Lemma 3.2] proves this for SU(2)-character varieties), but to our knowledge, a proof in the context of the $\mathrm{SL}(n, \mathbf{R})$-Hitchin component has never been recorded before and may be of independent interest in itself.

To establish this result, we first exploit a connection between these flows and the underlying geometry of the Hitchin component that was studied when they were first defined by Goldman. Recall that in [Gol84], Goldman defines a symplectic form on smooth components of character varieties of surface groups which gives $\mathcal{H}_n(S)$ the structure of a connected symplectic manifold. Recall that in general, a smooth function $f : M \to \mathbf{R}$ on a symplectic manifold (M, ω) induces a vector field, X_f, the *Hamiltonian vector field associated to* f, characterized by the property that for any smooth vector field Y, $\omega(X_f, Y) = df(Y)$. In particular, the Goldman symplectic form associates to the function $\mathrm{Tr}_\gamma : \mathcal{H}_n(S) \to \mathbf{R}$ its Hamiltonian vector field which, in turn, generates a flow on $\mathcal{H}_n(S)$ which we denote by ξ_γ^t. The similarity in notation between the flows Ξ_γ^t and ξ_γ^t is not coincidence as these two are related via the following result of Goldman's.

Theorem 4.1.2 ([Gol86, Theorem 4.7]). *The flow Ξ_γ^t restricted to $\widetilde{\mathcal{H}}_n(S)$ covers the flow ξ_γ^t on $\mathcal{H}_n(S)$. In other words, $\xi_\gamma^t([\rho]) = [\Xi_\gamma^t(\rho)]$ for all $\rho \in \widetilde{\mathcal{H}}_n(S)$.*

If one regards Tr_γ as a higher-rank analog of the "length" associated to γ (as in the discussion following Definition 2.3.3), then this result is exactly an analog of the duality between twist flows and length functions seen in Wolpert's magic formula (Theorem 2.2.5). This observation allows us to establish the Lemma 4.1.1 by reducing the transitivity in the statement to a purely local condition. We do so via an application of the following infinitesimal version of Theorem 2.3.3, proven in Bridgeman, Canary and Labourie's same paper.

Theorem 4.1.3 ([BCL20, Proposition 10.1]). *For any $[\rho] \in \mathcal{H}_n(S)$, the collection of differentials*

$$\{(d\,\mathrm{Tr}_\gamma)_{[\rho]} : \gamma \textit{ is a non-separating simple closed curve}\}$$

spans the cotangent space to $\mathcal{H}_n(S)$ at $[\rho]$.

The use of this theorem in proving Lemma 4.1.1 is the *only* step in our proof of Theorem 2.4.3 where we require that the genus of S is at least 3. The authors in [BCL20] deduce this claim by showing that certain infinitesimal deformations which locally preserve simple traces are trivial along certain 3-generator subgroups of $\pi_1(S)$. These subgroups are defined via an arrangement of curves which can only exist when the genus of S is at least 3. If one can remove this assumption in their work, the conclusions of Theorem 2.4.3 hold for genus 2 as well.

Proof of Lemma 4.1.1. Let $\mathfrak{G}$ denote the group generated by the flows ξ_γ^t for all non-separating simple closed curves γ and all t. We first wish to establish that $\mathfrak{G}$ acts transitively on $\mathcal{H}_n(S)$ (c.f. the strategy in [GX11, Lemma 3.2]). First, by Theorem 4.1.3, the differentials $d\,\mathrm{Tr}_\gamma$ associated to all non-separating, simple γ span the cotangent spaces

$T^*_{[\rho]}(\mathcal{H}_n(S))$ at every $[\rho] \in \mathcal{H}_n(S)$. The non-degeneracy of the Goldman symplectic form induces an isomorphism between cotangent and tangent spaces of the Hitchin component. This isomorphism sends $d\,\mathrm{Tr}_\gamma$ to the derivative of ξ^t_γ at $t = 0$ as the latter are the Hamiltonian flows associated to the former. In particular, the $d\,\mathrm{Tr}_\gamma$ spanning all cotangent spaces implies that the derivatives of the ξ^t_γ span the tangent spaces at every point on the Hitchin component. By the inverse function theorem, this implies that the $\mathfrak{G}$-orbit of any point is an open subset of $\mathcal{H}_n(S)$. The orbits also partition $\mathcal{H}_n(S)$, yet $\mathcal{H}_n(S)$ is connected, hence cannot be partitioned into proper disjoint open subsets. Therefore, the action of $\mathfrak{G}$ has only one orbit and it acts transitively on $\mathcal{H}_n(S)$. In other words, for Hitchin ρ_1 and ρ_2, there exist simple closed curves $\gamma_1, \ldots, \gamma_k$ and $t_1, \ldots, t_k \in \mathbf{R}$ so that

$$[\rho_2] = \xi^{t_k}_{\gamma_k}(\ldots(\xi^{t_1}_{\gamma_1}([\rho_1]))\ldots).$$

The claim then follows from this equality, viewed at the level of representations and Theorem 4.1.2. □

Independent to its use in establishing results about the arithmetic of $\mathcal{H}_n(S)$, Lemma 4.1.1 seems of independent interest worth further investigation for Hitchin representations. This result establishes that any two Hitchin representations may be connected by a finite sequence of bends about non-separating simple closed curves, but is there, in fact, a uniform bound on the number of bends one must perform to deform any one Hitchin representation into another? Perhaps simpler, is there a uniform bound on the number of bends one must perform to deform an n-Fuchsian representation into an arbitrary Hitchin representation? It seems possible, though maybe not plausible, that 2 may even be such an upper bound and that one can get from one Hitchin representation to another by bending about two curves (which generically fill S).

4.2 Density

To establish density of $\mathcal{H}_n(S)_{\mathbf{Q}}$, we use the results of the previous section to first build approximations to Hitchin representations via generalized twist flows. From Equation 4.1, one observes that the flow lines of Ξ_γ^t are essentially controlled by the matrix $\exp(tF(\rho(\gamma))) \in Z_{\rho(\gamma)}(\mathbf{R})$. In turn, one can then build a rational approximation to an arbitrary Hitchin representation by combining this observation with enough control over the rational points of the algebraic groups $Z_{\rho(\gamma)}$. This control comes via the following "weak approximation" property for these groups.

Theorem 4.2.1 ([PR94, Theorem 7.7]). *Let* $\mathbf{G}$ *be a connected algebraic group defined over* $\mathbf{Q}$*. Then* $\mathbf{G}(\mathbf{Q})$ *is dense in* $\mathbf{G}(\mathbf{R})$ *in the Euclidean topology.*

Observe then that $\mathbf{G} = Z_{\rho(\gamma)}$ is a connected algebraic group. In fact, it is an algebraic torus as Theorem 2.3.2 implies that $Z_{\rho(\gamma)}(\mathbf{C}) \cong (\mathbf{C}^\times)^{n-1}$. Moreover, if $\rho \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$, then this group is in fact defined over $\mathbf{Q}$. Thus, as a consequence of weak approximation, we arrive at the following.

Corollary 4.2.1.1. *If* $\rho \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$*,* $Z_{\rho(\gamma)}(\mathbf{Q})$ *is dense in* $Z_{\rho(\gamma)}(\mathbf{R})$*.*

This corollary is of use for building rational approximations to representations because of the observation that whenever $\rho \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ and ρ' is a bend of ρ about any non-separating $\gamma \in \pi_1(S)$ by a *rational* $A \in Z^\circ_{\rho(\gamma)}(\mathbf{Q})$, then $\rho' \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ (this is immediate from the formula of a bend in Definition 3.1.1). As the Ξ_γ^t are themselves a form of a bend (c.f. Equation 4.1), these flows may be "perturbed" using appropriate choice of rational centralizing matrices. Combining all the work of this chapter, we then arrive at the proof of our main result.

Proof of Theorem 2.4.3. First observe that $\widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ is nonempty. To see this, notice that the irreducible representation $\iota_n : \mathrm{SL}(2, \mathbf{R}) \to \mathrm{SL}(n, \mathbf{R})$ can be defined over $\mathbf{Z}$, hence takes $\mathrm{SL}(2,\mathbf{Q})$ to $\mathrm{SL}(n,\mathbf{Q})$. Thus, as there exist representations in $\widetilde{\mathcal{H}}_2(S)_{\mathbf{Q}}$ (e.g. by Takeuchi's result or by Example 2.2.5), the composition of one such representation with ι_n produces a representation in $\widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$.

To prove Theorem 2.4.3, first fix an arbitrary Hitchin representation $\rho_0 : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Q})$. For each $k \geq 1$, introduce the set

$$\mathcal{H}_n^k(S,[\rho_0]) = \left\{ [\rho] \in \mathcal{H}_n(S) \; : \; \begin{array}{c} \text{there exist simple, non-separating } \gamma_1,\ldots,\gamma_k \in \pi_1(S) \\ \text{and } t_1,\ldots,t_k \in \mathbf{R} \text{ so that } [\rho] = \xi_{\gamma_k}^{t_k}(\ldots(\xi_{\gamma_1}^{t_1}([\rho_0]))\ldots) \end{array} \right\}.$$

In other words, $\mathcal{H}_n^k(S,[\rho_0])$ consists of the points on the Hitchin component which can be reached by performing at most k generalized twists about non-separating simple closed curves beginning with $[\rho_0]$. Lemma 4.1.1 implies that

$$\mathcal{H}_n(S) = \bigcup_{k=1}^{\infty} \mathcal{H}_n^k(S,[\rho_0]),$$

so to establish density, we show that the closure of $\mathcal{H}_n(S)_{\mathbf{Q}}$ in $\mathcal{H}_n(S)$ contains $\mathcal{H}_n^k(S,[\rho_0])$ for all k.

First, for any $[\rho] \in \mathcal{H}_n^1(S,[\rho_0])$, ρ is conjugate to $\Xi_{\gamma_1}^{t_1}(\rho_0)$ for some non-separating $\gamma_1 \in \pi_1(S)$ and $t_1 \in \mathbf{R}$. By Corollary 4.2.1.1, there is a sequence $\{A_j\}_j$ of elements in $Z_{\rho_0(\gamma_1)}(\mathbf{Q})$ converging to $\exp(t_1 F(\rho_0(\gamma_1)))$. Let ρ_j be the bend of ρ_0 about γ_j by A_j (c.f. Definition 3.1.1). Notice that $\rho_j \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ as $\rho_0 \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ and $A_j \in \mathrm{SL}(n,\mathbf{Q})$. Moreover, ρ_j converges pointwise to $\Xi_{\gamma_1}^{t_1}(\rho_0)$, hence $[\rho_j] \to [\rho]$ and the closure of $\mathcal{H}_n(S)_{\mathbf{Q}}$ contains all of $\mathcal{H}_n^1(S,[\rho_0])$.

Inductively, suppose that the closure of $\mathcal{H}_n(S)_{\mathbf{Q}}$ contains $\mathcal{H}_n^{k-1}(S,[\rho_0])$ for some $k \geq 2$ and let $[\rho] \in \mathcal{H}_n^k(S,[\rho_0])$. Then there exist non-separating simple $\gamma_k \in \pi_1(S)$, $t_k \in \mathbf{R}$ and $[\sigma] \in \mathcal{H}_n^{k-1}(S,[\rho_0])$ so that ρ is conjugate to $\Xi_{\gamma_k}^{t_k}(\sigma)$. As $[\sigma] \in \mathcal{H}_n^{k-1}(S,[\rho_0])$, by induction, there exists a sequence $\{\sigma_i : \pi_1(S) \to \mathrm{SL}(n,\mathbf{Q})\}_i$ in $\widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ such that $[\sigma_i] \to [\sigma]$. Conjugating by appropriate elements of $\mathrm{SL}(n,\mathbf{Q})$ if necessary, we may further assume that $\sigma_i \to \sigma$ pointwise.

For each i, let $\{A_{i,j}\}_j$ denote a sequence in $Z_{\sigma_i(\gamma_k)}(\mathbf{Q})$ converging to $\exp(t_k F(\sigma_i(\gamma_k)))$ in $Z_{\sigma_i(\gamma_k)}(\mathbf{R})$, as given by Corollary 4.2.1.1. Fix some distance d on $\mathrm{SL}(n,\mathbf{R})$ inducing its usual topology and for each $m \geq 1$, let $\phi(m)$ denote the smallest i such that

$$d(\exp(t_k F(\sigma_i(\gamma_k))), \exp(t_k F(\sigma(\gamma_k)))) < \frac{1}{m}.$$

Similarly, let $\psi(m)$ denote the smallest j such that

$$d(A_{\phi(m),j}, \exp(t_k F(\sigma_{\phi(m)}(\gamma_k)))) < \frac{1}{m}.$$

By construction of ϕ and ψ, the sequence $A_{\phi(m),\psi(m)}$ converges to $\exp(t_k F(\sigma(\gamma_k)))$ as $m \to \infty$. Then let ρ_m be the bend of $\sigma_{\phi(m)}$ about γ_k by $A_{\phi(m),\psi(m)}$. Again, $\rho_m \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ as $\sigma_{\phi(m)} \in \widetilde{\mathcal{H}}_n(S)_{\mathbf{Q}}$ and $A_{\phi(m),\psi(m)} \in \mathrm{SL}(n,\mathbf{Q})$ and ρ_m converges pointwise to $\Xi_{\gamma_k}^{t_k}(\sigma)$. Therefore $[\rho_m] \to [\rho]$ and the closure of $\mathcal{H}_n(S)_{\mathbf{Q}}$ contains all of $\mathcal{H}_n^k(S,[\rho_0])$, establishing the result. □

4.3 Density beyond $\mathcal{H}_n(S)$

The methods used in this proof also admit generalizations beyond the specific example of $\mathcal{H}_n(S)$. The most immediate one is the analog of this result for Hitchin components associated to other split real forms. If $G = \mathrm{Sp}(2k,\mathbf{R})$, $\mathrm{SO}(k+1,k)$ or G_2, and $\mathcal{H}_G(S)_{\mathbf{Q}}$ denotes the intersection of $\mathcal{H}_G(S)$ with $\mathcal{H}_n(S)_{\mathbf{Q}}$ (identifying the G-Hitchin component with its image in the $\mathrm{SL}(n,\mathbf{R})$-Hitchin component for $n = 2k, 2k+1$ or 7, depending on G), then an immediate consequence of the above proof is the same result for the Hitchin components associated to these groups.

Corollary 4.3.0.1. *$\mathcal{H}_G(S)_{\mathbf{Q}}$ is dense in $\mathcal{H}_G(S)$.*

This follows from the fact that the restrictions of the trace functions Tr_γ to the submanifold $\mathcal{H}_G(S)$ of $\mathcal{H}_n(S)$ (induced by the appropriate inclusions $G \hookrightarrow \mathrm{SL}(n, \mathbf{R})$) will still span the cotangent spaces of $\mathcal{H}_G(S)$ and allow one to conclude the obvious analog of Lemma 4.1.1 for these Hitchin components. Again, the weak approximation result still applies to G-centralizers of matrices in these groups and the examples of n-Fuchsian rational Hitchin representations in $\mathcal{H}_n(S)_{\mathbf{Q}}$ also establish that all of these $\mathcal{H}_G(S)_{\mathbf{Q}}$ are nonempty.

One key difference, however, is that in this setting, one must be more careful in stating the analog of Corollary 4.0.0.1, at the level of representations rather than conjugacy classes. For each of these groups, lifts of the Hitchin component to $\mathrm{Hom}(\pi_1(S), G)$ all preserve an ambient non-degenerate alternating or symmetric bilinear form, depending on the parity of n, and which specific form it preserves depends on how one lifts. While each of these forms are equivalent over $\mathbf{R}$, some have no solutions over $\mathbf{Q}$ and the corresponding lifts of the Hitchin component will have no rational representations whatsoever. See, for instance, [Aud22] for some additional examples of quadratic forms whose associated indefinite orthogonal groups contain rational Hitchin representations.

These methods also indicate possible ways one may establish similar results for other components of the character variety. That is, let $\mathbf{G}$ be a reductive $\mathbf{Q}$-algebraic group and $G = \mathbf{G}(\mathbf{R})$ the corresponding real Lie group. Again, the G-character variety of S is possibly a highly-singular space, but there is a Zariski-open subset $\Omega \subset \mathrm{Hom}(\pi_1(S), G)$ so that the quotient by conjugation Ω/G is a (possibly disconnected) smooth manifold. Goldman's symplectic form in [Gol84] may still be defined on any component of Ω/G, thus allowing much of our discussion to take place in this setting.

That is, let $X \subset \Omega/G$ be one connected component and denote by $X_{\mathbf{Q}}$ the set of representations in X which are conjugate into a subgroup of $\mathbf{G}(\mathbf{Q})$. In order to establish an analog of Theorem 2.4.3 for this component X, one needs an infinitesimal condition to hold establishing a similar result as 4.1.3. In fact, one can get away with a much weaker statement than what we use for the Hitchin component. To any *conjugation invariant* function $f : G \to \mathbf{R}$ and $\gamma \in \pi_1(S)$, there is an induced $f_\gamma : X \to \mathbf{R}$ given by setting $f_\gamma([\rho]) := f(\rho(\gamma))$ (for instance, Tr_γ). If $\mathcal{S} \subset \pi_1(S)$ denotes the collection of elements which are freely homotopic to simple closed curves (separating or not), then one needs to ask the following question of X.

Question 7. Is there a collection of conjugation invariant functions $\mathcal{F} = \{f : G \to \mathbf{R}\}$ so that the differentials

$$\{df_\gamma \,:\, f \in \mathcal{F}, \gamma \in \mathcal{S}\}$$

span the cotangent space $T^*_{[\rho]}X$ at every $[\rho] \in X$?

One may regard this question as asking whether general representations of surface groups satisfy *any* sort of infinitesimal rigidity associated to simple closed curves. If so, then one may understand analogs of Theorem 2.4.3 in general as our same methods show the following.

Theorem 4.3.1. *If $X \subset \Omega/G$ as above has a positive answer to Question 7, then the presence of a single representation in $X_{\mathbf{Q}}$ implies its own density in X.*

For instance, Theorem 4.1.3 shows that when $X = \mathcal{H}_n(S)$ and the genus of S is 3 or more, then $\mathcal{F} = \{\mathrm{Tr}\}$ yields a positive answer to Question 7, but in reality, one may get away with a much larger (and even possibly infinite) class of conjugation invariant functions. If such a collection can be found, the Hamiltonian flows associated to them will still act transitively on X. These flows still admit descriptions in the form of generalized twist flows by Goldman's work [Gol86], allowing one to build rational approximations in an identical manner as in the proof of Theorem 2.4.3 and establishing Theorem 4.3.1 for X.

4.4 Point counting and some experiments

Theorem 2.4.3 provides a qualitative understanding of the distribution of $\mathcal{H}_n(S)_{\mathbf{Q}}$ within $\mathcal{H}_n(S)$, but it is natural to seek a more quantitative picture. Pursuing such a goal may, in turn, help lead to integral point counting results for representations in $\mathcal{H}_n(S)_{\mathbf{Z}}$ (c.f. [Wie18, Task 25]). We lightly discuss how one may tackle such questions in the presence of more computational work done previously and present candidate objects to study towards such goals.

We begin with the following result of Long–Reid–Thistlethwaite which explicitly identifies the Hitchin component of a particular triangle group in $\mathrm{SL}(3,\mathbf{R})$ using the computational methods outlined in [CLT06]. First, we let $X_{3,3,4}$ the orbifold of signature $(0;3,3,4)$ and $\Delta(3,3,4) = \langle \alpha, \beta \,|\, \alpha^3 = \beta^3 = (\alpha\beta)^4 \rangle$ its fundamental group, i.e. the $(3,3,4)$-triangle group.

Theorem 4.4.1 ([LRT11, §2]). *Let $Y \subset \mathbf{R}^3$ denote the triples of points (u, v, D) which are solutions to the polynomial equation*

$$D^2 = -4u^2(5+u) + 4u(8+u)v + (-20+u(4+u))v^2 - 4v^3. \tag{4.2}$$

Let $\rho_{u,v,D} : \Delta(3,3,4) \to \mathrm{SL}(3,\mathbf{R})$ *be the representation defined by mapping*

$$\alpha \mapsto \begin{pmatrix} 1 & 1 & \frac{u(-u^3v+4uv^2-2v^3+u^2(-6v+D))}{2(u^2+v^2)(u^2-uv+v^2)} \\ 0 & -\frac{v}{u} & 1 \\ 0 & \frac{-u^2+uv-v^2}{u^2} & -1+\frac{v}{u} \end{pmatrix}$$
$$\beta \mapsto \begin{pmatrix} -1+\frac{v}{u} & 0 & \frac{u(2u^3+2v^3-u^2v(2+v)+uv(-2v+D))}{2(u^2+v^2)(u^2-uv+v^2)} \\ \frac{-2v^2+u^2(2+v)+u(4v+D)}{2u^2} & 1 & -1 \\ \frac{-2u^3-2v^3+u^2v(2+v)+uv(2v+D)}{2u^3} & 0 & -\frac{v}{u} \end{pmatrix}.$$

The birational map $Y \dashrightarrow \mathfrak{X}(\Delta(3,3,4),\mathrm{SL}(3,\mathbf{R}))$ *defined by* $(u,v,D) \mapsto [\rho_{u,v,D}]$ *takes the points in* Y *such that* $u>0$ *and* $v>0$ *bijectively onto* $\mathcal{H}_3(X_{3,3,4})$.

Let

$$D(u,v) := \sqrt{-4u^2(5+u)+4u(8+u)v+(-20+u(4+u))v^2-4v^3}$$

(c.f. the right-hand side of Equation 4.2). Since a pair $(u,v) \in \mathbf{R}_+^2$ uniquely determines two possible values of D so that $(u,v,D) \in Y$ (namely, $\pm D(u,v)$), the above computation shows that $\mathcal{H}_3(X_{3,3,4})$ is 2-dimensional (which agrees with what had been previously known in [CG05]). The above map not only parameterizes this Hitchin component explicitly, but it does so in a particularly nice way with regards to the arithmetic of the spaces involved. Namely, the map of the above theorem precisely identifies the rational representations in $\mathcal{H}_3(X_{3,3,4})$.

Proposition 4.4.2. *The map of Theorem 4.4.1 restricts to a bijection between the elements of* $Y(\mathbf{Q})$ *with* $u,v>0$ *and the representations in* $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Q}}$.

Proof. Firstly, it's clear why if $(u,v,D) \in Y(\mathbf{Q})$ satisfies $u,v>0$, then $[\rho_{u,v,D}] \in \mathcal{H}_3(X_{3,3,4})_{\mathbf{Q}}$ since the entries in the map of Theorem 4.4.1 are rational functions defined over $\mathbf{Q}$. It is not, however, immediately clear why if $[\rho_{u,v,D}] \in \mathcal{H}_3(X_{3,3,4})_{\mathbf{Q}}$, then $u, v, D \in \mathbf{Q}$. To see this, one can first check that the matrix $\rho_{u,v,D}(\alpha\beta^{-1})$ has characteristic polynomial given by

$$t^3 - (2+v)t^2 + (2+u)t - 1. \tag{4.3}$$

Therefore, if $[\rho_{u,v,D}] \in \mathcal{H}_3(X_{3,3,4})_{\mathbf{Q}}$, the characteristic polynomial of this matrix must have $\mathbf{Q}$-coefficients and so $u, v \in \mathbf{Q}$. Similarly, one can also check that the matrix $\rho_{u,v,D}(\alpha^{-1}\beta^{-1}\alpha\beta)$ has characteristic polynomial

$$t^3 - \left(\frac{2+2u+2v+uv-D}{2}\right)t^2 + \left(\frac{2+2u+2v+uv+D}{2}\right)t - 1 \tag{4.4}$$

hence $D \in \mathbf{Q}$ as well. □

Consequently, one can seek to understand the distributions of rational Hitchin representations in $\mathcal{H}_3(X_{3,3,4})$ by instead studying the algebraic surface defined by Equation 4.2 through number-theoretic means.

For instance, one may then study distributions of rational points by enumerating rational values of u and v of bounded height and studying the asymptotics of this function. Recall that if $x = \frac{p}{q} \in \mathbf{Q}$ for $p, q \in \mathbf{Z}$ coprime with $q \neq 0$, the *height* of x is the value $ht(x) = \max\{|p|, |q|\}$. Thus, for example, one may calculate (by hand or through computer check) that there are 38047 positive rational numbers of height bounded above by 250. This gives 38047^2 possible choices of (u, v) which may yield a representation in $\mathcal{H}_3(X_{3,3,4})$. Among these $(u, v) \in \mathbf{Q}_+^2$ satisfying $D(u, v) \geq 0$, only 1565 satisfy the additional condition that $D(u, v) \in \mathbf{Q}$. The plot in Figure 4.1 shows a portion of $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Q}}$ where a point (u, v) corresponds to the representation $[\rho_{u,v,D(u,v)}]$. The curve plotted is where $D(u, v) = 0$ and points in either shade of blue are one of these 1565 rational points produced in this example computation. The points in dark blue are the representations which are, in fact, integral.

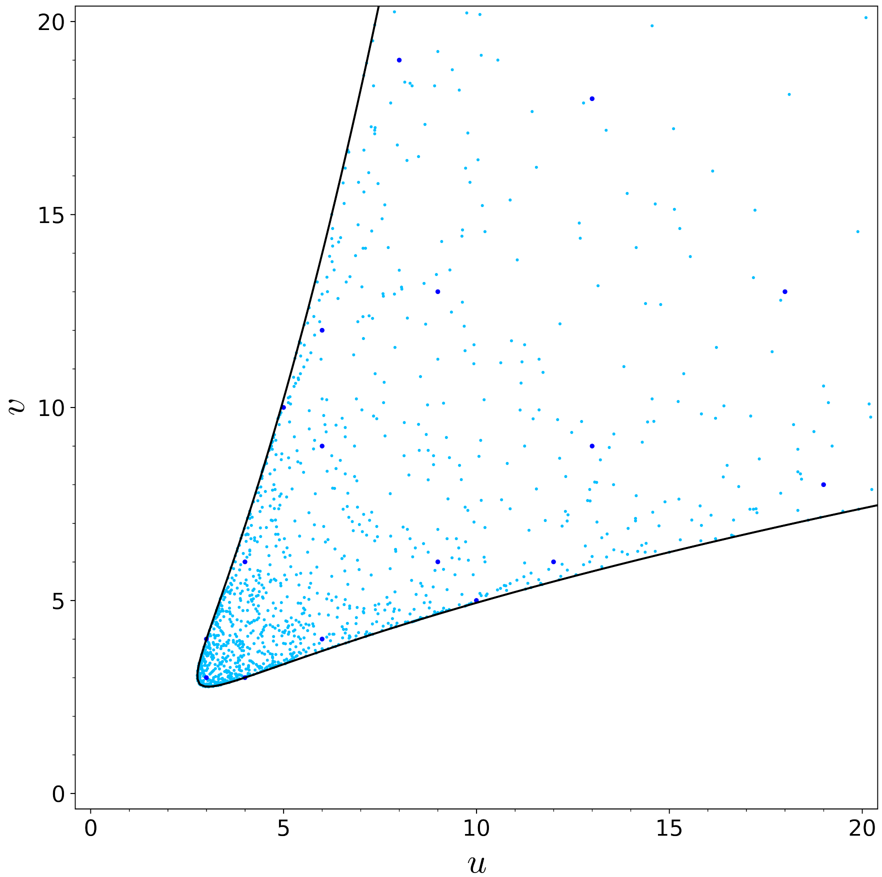


Figure 4.1: Some rational points on $\mathcal{H}_3(X_{3,3,4})$ of height ≤ 250.

Figure 4.1 already seemingly illustrates that there may be interesting rational structure worth exploring. The point at the cusp of the $D(u, v) = 0$ curve is where $(u, v) = (2\sqrt{2}, 2\sqrt{2})$ and is the (unique) discrete and faithful representation of $\Delta(3, 3, 4)$ into $\mathrm{SO}(2, 1)$. Figure 4.1 and further computations such as these seemingly suggest that points of small height are distributed much closer to the Fuchsian representation.

One can further refine these techniques to search for representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$. This requires a slightly more subtle approach though, as the parameterization of Theorem 4.4.1 is one by *rational* functions and hence points in $Y(\mathbf{Z})$ do not necessarily correspond to representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$. Nonetheless, work of Bass in [Bas80] and a similar argument as in Proposition 4.4.2 show that if $(u, v, D) \in \mathbf{Z}_+^3$ satisfies Equation 4.2 and makes the polynomials of Equations 4.3 and 4.4 defined over $\mathbf{Z}$, then $\rho_{u,v,D}(\Delta(3,3,4))$ conjugates into $\mathrm{SL}(3, \mathbf{Z})$ (c.f. [LRT11, Proposition 2.1]). Figure 4.2 contains a plot of all such points with $0 < u, v \leq 10000$ and $D(u, v) \geq 0$. Upon closer inspection of Figure 4.2, there appear to be four infinite "rays" emanating from the origin. Each "ray" is in fact, a half of one of two conics, $6 + u^2 + v^2 - u - v - 2uv = 0$ or $u^2 - 2uv + v^2 - 4u - 4v + 36 = 0$, each of which contain infinitely many representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$. These two families were first constructed in [LRT11] and provided the first examples of infinitely many non-conjugate, Zariski-dense surface subgroups of $\mathrm{SL}(3, \mathbf{Z})$.

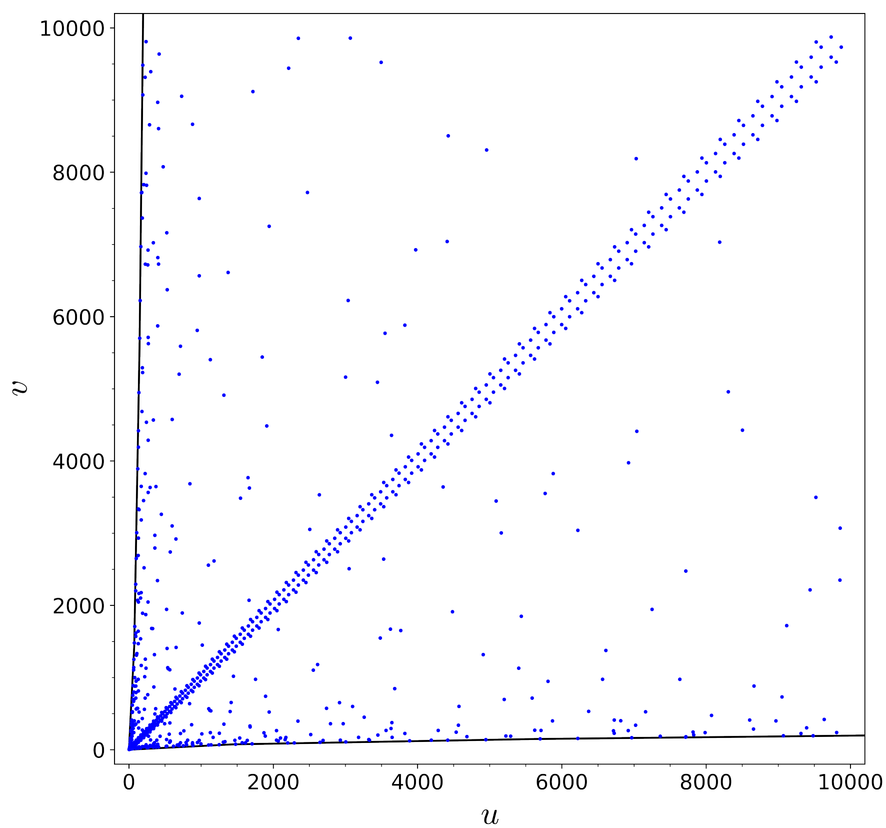


Figure 4.2: Representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$.

Again, comparing Figures 4.1 and 4.2, clustering behavior near the Fuchsian representation seems apparent. This is somewhat artificial in the former of these figures as

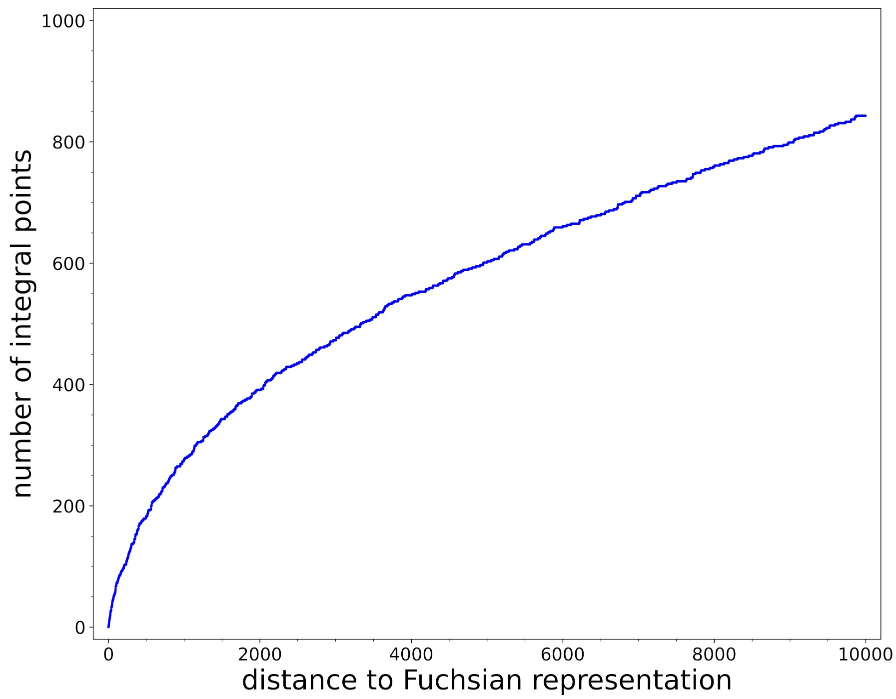


Figure 4.3: Distributions of representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$.

rational numbers of bounded height will tend to be distributed around the smaller values, but is, more curious in the latter figure. One can in fact, more precisely quantify this phenomenon, counting the number of representations in $\mathcal{H}_3(X_{3,3,4})_{\mathbf{Z}}$ with integral trace and bounded distance to the Fuchsian representation (in uv-coordinates). Asymptotically, as the distance to the Fuchsian representation, d, grows, the number of such integral representations grows approximately at a rate of $\sim \sqrt{d}$ (see Figure 4.3).

Of course, the discussion here is highly extrinsic. These computations massively rely on the parameterization of $\mathcal{H}_3(X_{3,3,4})$ given by Theorem 4.4.1 and it is not known, for instance, whether the coordinates u and v correspond to anything geometrically meaningful. An immediate goal would be to perform computations such as the ones presented above instead utilizing more intrinsic geometric features of $\mathcal{H}_n(S)$.

For example, there are a number of natural metrics one might place on the Hitchin component, many of which generalize rank one counterparts for Teichmüller space (see, e.g., [BCLS15]). Inspired by various heights in number theory and much of the above computations, we define one possible candidate height function on the rational Hitchin representations.

Definition 4.4.1. The *denominator* of a rational Hitchin representation is the function den : $\mathcal{H}_n(S)_{\mathbf{Q}} \to \mathbf{R}$ by:

$$\mathrm{den}([\rho]) := \min\{N \ : \ \mathrm{Tr}(\rho(\gamma)) \in \mathbf{Z}[\tfrac{1}{N}] \ \text{ for all } \gamma \in \pi_1(S)\}.$$

Similar to the discussion above, work of Bass allows one to show that in fact, $\mathrm{den}^{-1}(1) = \mathcal{H}_n(S)_{\mathbf{Z}}$ [Bas80], thus understanding the behavior of this "height function" on $\mathcal{H}_n(S)_{\mathbf{Q}}$ may possibly lead to resolving the remaining cases left open by Theorem 3.0.1. In this vein, we ask the following natural question.

Question 8. Can one characterize the collection of representations in $\mathcal{H}_n(S)_{\mathbf{Q}}$ of bounded denominator and bounded distance to the n-Fuchsian locus $\iota_n(\mathcal{T}(S))$ (under any suitable notion of distance on the Hitchin component)?

An answer to such a question would provide a more *intrinsic* picture than the extrinsic one provided by the computations in this section.

4.5 Other approximation-type questions

Theorem 2.4.3 provides a meaningful sense in which one can approximate an arbitrary Hitchin representation by one with "nicer" arithmetic properties, namely, being rational. Naturally, one may prefer other arithmetic properties to being rational, prompting a slew of other questions regarding the extent arithmetic approximations in $\mathcal{H}_n(S)$ can be made. The purpose of this section is mostly to record such questions.

Like much in higher-rank, much of this discussion is motivated by analogous results in rank one and asking to what extent the obvious generalization holds or fails.

Definition 4.5.1. Let $\Gamma < \mathrm{PSL}(2, \mathbf{C})$ be a Kleinian group. Say that Γ admits a *cofinite extension* if there exists a lattice Γ' with $\Gamma < \Gamma'$. A cofinite extension is *cocompact* if the lattice is cocompact, otherwise, it is *strictly cofinite.*

In [Bro86], Brooks used the theory of circle packings on the sphere to establish the following characterization of cofinite extensions in a rank one example.

Theorem 4.5.1 ([Bro86, Theorems 1 and 2])**.** *Suppose* $\Gamma < \mathrm{PSL}(2, \mathbf{C})$ *is a geometrically finite Kleinian group. Then there exist arbitrarily small quasi-conformal deformations,* Γ_ε*, of* Γ*, such that* Γ_ε *admits a cofinite extension. Moreover, if* Γ *has no cusps, one can take* Γ_ε *admitting a cocompact extension.*

We will interpret the above theorem in the case where $\Gamma = \pi_1(S)$ is a surface group for simplicity. Let $\mathrm{QF}(S)$ denote the collection of quasi-Fuchsian representations of $\pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$. In this case, the above result interpreted in the language of character varieties gives that the collection of representations in $\mathrm{QF}(S)$ with image contained in a hyperbolic 3-manifold group is dense in $\mathrm{QF}(S)$. In fact, using Brock–Canary–Minsky's resolution to the Ending Lamination Conjecture [BCM12], representations in $\mathrm{QF}(S)$ with image contained in hyperbolic 3-manifold groups are dense in the space $\mathrm{AH}(S)$ of *all* Kleinian surface groups isomorphic to $\pi_1(S)$.

In rank one, all quasi-Fuchsian representations are Anosov, and so asking the same question for other spaces of Anosov representations is a natural next step.

Definition 4.5.2. A Hitchin representation ρ admits a *cofinite extension* if there exists a lattice $\Lambda < \mathrm{SL}(n, \mathbf{R})$ such that $\rho(\pi_1(S)) < \Lambda$. We denote the collection of conjugacy classes of such representations by $\mathcal{H}_n(S)_{\mathrm{cf}}$.

In light of Brooks's result, we ask the following.

Question 9. Is $\mathcal{H}_n(S)_{\mathrm{cf}}$ dense in $\mathcal{H}_n(S)$?

For instance, the representations pictured in Figure 4.2 are all in $\mathcal{H}_n(S)_{\mathrm{cf}}$ as these all conjugate into $\mathrm{SL}(3, \mathbf{Z})$. Arithmeticity (Theorem 2.1.2), however, suggests that the answer to this Question 9 should be "no," as heuristically, there are much fewer lattices in higher-rank than there are in rank one. Nonetheless, one way to get around this is possibly by allowing passage to finite-index subgroups.

Definition 4.5.3. A Hitchin representation ρ admits a *virtual cofinite extension* if there exists a lattice $\Lambda < \mathrm{SL}(n, \mathbf{R})$ and a finite-index subgroup $\Gamma \leqslant \rho(\pi_1(S))$ so that $\Gamma < \Lambda$. We denote the collection of conjugacy classes of such representations by $\mathcal{H}_n(S)_{\mathrm{vcf}}$.

This notion allows for more representations to be considered, hence perhaps makes it more reasonable that the following may have a positive answer.

Question 10. Is $\mathcal{H}_n(S)_{\mathrm{vcf}}$ dense in $\mathcal{H}_n(S)$?

For $n \geq 3$, the Zariski-dense representations in $\mathcal{H}_n(S)_{\mathrm{vcf}}$ are precisely the ones which are *subarithmetic*, in the sense of [GPS88, §0.4]. There, subarithmeticity is only discussed for subgroups of rank one groups, where the authors raise the question of *which lattices* contain (or in fact, are generated by) subarithmetic subgroups. This same question is trivial in higher-rank, by virtue of Theorem 2.1.2, but the related question of *which subgroups* are subarithmetic remains nontrivial, and Question 10 concerns this.

Using a similar parameterization as in Theorem 4.4.1, Long and Reid show that there is an embedding $\mathbf{R} \hookrightarrow \mathcal{H}_3(X_{3,4,4})$ so that every *totally real Pisot number* maps to a representation in $\mathcal{H}_3(X_{3,4,4})_{\mathrm{vcf}}$ (see [LR16, §4.1]). Nonetheless, this is far from establishing density and in fact, does not even establish whether this set contains a limit point. Even if both Questions 9 and 10 possess a negative answer, providing *any* characterization of the closures of $\mathcal{H}_n(S)_{\mathrm{cf}}$ or $\mathcal{H}_n(S)_{\mathrm{vcf}}$ would be of interest. For instance, at the time of writing, it is unknown (at least to the author) whether or not either of these sets are even discrete.

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