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The Picard Group of the Moduli Space of Genus Zero Stable Quotients to Flag Varieties

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The Picard Group of the Moduli Space of Genus Zero Stable Quotients to Flag Varieties

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Perry Robert Strahl is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

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2017

DEDICATION

To my fiancée Kristin, my parents Dana and Tracy, and my brother Kendall

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VITA

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ABSTRACT OF THE DISSERTATION

The Picard Group of the Moduli Space of Genus Zero Stable Quotients to Flag Varieties

by

Perry Robert Strahl
Doctor of Philosophy in Mathematics

University of California San Diego, 2017

Professor Dragos Oprea, Chair

We compute the Picard group of the moduli stack of genus zero stable quasimaps to projective space, Grassmannians, and any flag variety in the case of more than 2 markings. Furthermore, in the case of exactly 2 markings, we calculate the Picard group of the moduli stack of genus zero stable quasimaps to projective space, Grassmannians, and to partial flag varieties where the ranks of the subspaces differ by more than 1. The first two moduli stacks mentioned are the moduli stacks of stable quotients, constructed by Alina Marian, Dragos Oprea, and Rahul Pandharipande. The latter is a generalization of this theory, due to Ionuţ-Ciocan Fontanine, Bumsig Kim, and Davesh Maulik. Projectivity of the coarse moduli space is proved first. The Picard rank is obtained using a torus action on the moduli stack to perform tangent space calculations. When the number of markings is ≥ 3, generators are determined by a geometric analysis of the interior of the moduli stack. When the number of markings is 2, generators and relations are found by intersecting with curves.

0 Introduction

All schemes and stacks in the paper will be over $\mathbb{C}$.

Fix a smooth projective genus g curve C, a smooth projective variety X, and a class $\beta \in A_1(X)$. In enumerative geometry, one considers the space $\mathrm{Hom}_\beta(C, X)$ of morphisms from C to X whose image lies in the rational equivalency class β. Unfortunately, this space is not compact, so intersection theory on the space is not well behaved. However, there are several different compactifications of this space available.

0.1 The Quot scheme

One such compactification of the morphism space, in the case that the target is the Grassmannian, $Gr(r,n)$, is a special instance of Grothendieck's Quot scheme, $Quot_C(Gr(r, n), d)$. Closed points of the Quot scheme parameterize short exact sequences of sheaves

$$0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q} \to 0$$

over C such that the quotient $\mathcal{Q}$ has rank $n - r$ and degree d. Two such exact sequences represent the same point of the Quot scheme if there is a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & \mathcal{Q} \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \mathcal{S}' \arrow[r] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r] & \mathcal{Q}' \arrow[r] & 0
\end{tikzcd}$$

with ϕ an isomorphism.

In analogy with the Grassmannian, over $C \times Quot_C(Gr(r, n), d)$ there exists a universal exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O}_{C\times Quot} \to \mathcal{Q} \to 0$$

with $\mathcal{Q}$ flat over $Quot_C(Gr(r, n), d)$.

Morphisms to the Grassmannian $Gr(r, n)$ yield points in the Quot scheme by pulling back the universal sequence of vector bundles over the Grassmannian. Since the condition that the universal quotient be locally free is open, we see that there is an open subscheme of the Quot scheme which is isomorphic to the Hom space introduced above.

In [Str87], where the curve $C = \mathbb{P}^1$, it was proven that the Quot scheme is a smooth and irreducible variety. [Str87] gave a description of the Chow ring of the Quot scheme and showed that rational equivalence and numerical equivalence coincide.

In [BDW96], the Quot scheme was used to calculate Gromov invariants. Later, in [MO07], the virtual fundamental class of the Quot scheme was constructed, and certain virtual intersection numbers were computed by means of equivariant localization.

[Ven11] returned to the study of the Quot scheme above with $C = \mathbb{P}^1$, calculating the cones of ample and effective divisors, the Mori chambers in the effective cone, and the base locus of the effective divisors.

0.2 The HyperQuot scheme

The HyperQuot scheme is a generalization of the Quot scheme in the same way that flag varieties are a generalization of the Grassmannian. Define

$$\overline{r} := (r_1, \ldots, r_\ell) \in \mathbb{N}_{>0},\ \overline{d} := (d_1, \ldots, d_\ell) \in \mathbb{N}_{\geq 0},$$

where $r_i < r_{i+1}\ \forall\, 1 \leq i \leq \ell$, with $r_{\ell+1} = n$. The case that we will be interested in is the HyperQuot scheme (due to [Lau88])

$$HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}),$$

whose closed points parameterize flags

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ \ldots \to \mathcal{Q}_\ell \to 0$$

where each $\mathcal{S}_i$ is a vector bundle on $\mathbb{P}^1$ of rank r_i and degree $-d_i$. The inclusions $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$ are injective only as morphisms of *sheaves*, not as morphisms of *vector bundles*. Just as with the Grassmannian and the Quot scheme, there is an open subscheme of the HyperQuot scheme isomorphic to $\mathrm{Hom}_{\overline{d}}(\mathbb{P}^1, Fl(\overline{r}, \mathbb{C}^n))$.

The HyperQuot scheme was studied in [Kim] and in [CF95] to calculate the Gromov Witten invariants of flag manifolds and the quantum cohomology ring of flag varieties, respectively. [Lau88], [Kim], and [CF95] proved that the HyperQuot scheme is smooth, irreducible, and projective.

[Che01] determined a generating function for the Poincaré polynomial of the HyperQuot scheme.

In [Ven11], the birational geometry of the HyperQuot scheme was studied. In particular, [Ven11] computed the ample cone of the HyperQuot scheme and the effective cone in certain cases.

0.3 The moduli stack of stable maps

By varying the curve in moduli, one is led to study Kontsevich's moduli stack of stable maps, $\overline{\mathcal{M}}_{g,m}(X, \beta)$.

Fix $(X, \mathcal{O}_X(1))$ a smooth projective variety. Closed points of the moduli space of stable maps consist of the following data:

- a projective, connected, reduced, at worst nodal curve C of arithmetic genus g
- m distinct points $\{p_i\}_{i=1}^m$ on C contained in the smooth locus
- a morphism $f : C \to X$ such that $f_*[C] = \beta$,

subject to the stability condition that the line bundle
$$\omega_C(\sum_{i=1}^m p_i) \otimes f^*\mathcal{O}_X(3)$$
is ample on C.

There is a forgetful map $F : \overline{\mathcal{M}}_{g,m}(X, \beta) \to \mathcal{M}_{g,m}^{pre}$ from the moduli stack of stable maps to the Artin stack of pre-stable curves of arithmetic genus g with m markings. A pre-stable arithmetic genus g curve with m markings is a projective, connected, reduced, at worst nodal curve of arithmetic genus g with the m distinct markings contained in the smooth locus of the curve; see [BM90] *Definition 2.1.*

The moduli stack of stable maps is often not irreducible and contains many components of different dimensions.

However, in the case that the genus is 0, and the target is convex [FP97] ($h^1(\gamma^* TX) = 0$, for all maps $\gamma : \mathbb{P}^1 \to X$), the moduli stack is smooth. [KP01] proved that, in all genera, the coarse moduli space is connected when the target is a homogeneous space.

In a slightly different direction, [Pan99] calculated the Picard group of the moduli space of genus zero stable maps to $\mathbb{P}^n$ ($n \geq 2$), proved that rational and numerical equivalence coincide, and gave an algorithm for computing the top dimensional intersection products of various combinations of the generators of the Picard group. [Pan99] found that the Picard group is generated by the irreducible components of the boundary (the locus of stable maps whose underlying curve is reducible), the evaluation classes $ev_i^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ obtained by pulling back the hyperplane classes under the i^{th} evaluation map from the moduli stack to $\mathbb{P}^n$, and the Cartier divisor $\mathcal{H}$ corresponding to curves whose image meets a fixed codimension 2 subspace of $\mathbb{P}^n$.

[Opr05], [Opr06b] continued this work by considering X an SL flag variety. [Opr05], [Opr06b] proved that the rational cohomology of the moduli stack of genus zero stable maps to flag varieties is tautological, and calculated the Picard group using the torus action and a generalization of a Bialynicki-Birula theorem. To describe the work of [Opr05], [Opr06b], we must describe the moduli stack in a bit more detail.

Fix $\overline{r} := (r_1, \ldots, r_\ell) \in \mathbb{N}^\ell_{>0}$, $\overline{d} := (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell_{\geq 0}$, where $r_i < r_{i+1}$ $\forall\, 1 \leq i \leq \ell$, with $r_{\ell+1} = n$.

The universal curve over the moduli stack $\overline{\mathcal{M}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ is isomorphic to

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,m+1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \arrow[d, shift left, "\pi"] \\ \overline{\mathcal{M}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \arrow[u, shift left, bend left, "\sigma_i"] \end{tikzcd}$$

with universal sections σ_i. There is an evaluation map ev from the universal curve over the moduli stack to the flag variety. The i^{th} evaluation map we mentioned above is the composition $ev_i = ev \circ \sigma_i$. Over the flag variety, there is a universal flag of subbundles and quotients

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0$$

where the $\mathcal{S}_i$ are vector bundles of ranks r_i, and the $\mathcal{Q}_i$ are vector bundles of ranks $n - r_i$.

We can consider the classes $\pi_* ev^* c_1^2(\mathcal{Q}_i)$ for each $1 \leq i \leq \ell$. Then, we can consider the kernels

$$0 \to \mathcal{K}_i \to \mathcal{Q}_i \to \mathcal{Q}_{i+1} \to 0$$

and the classes $\pi_* ev^* c_2(\mathcal{K}_i)$, for rank $\mathcal{K}_i \geq 2$.

Theorem 0.3.1. *([Opr05]) The Picard group of the moduli stack of genus zero stable maps is generated by*

- *the irreducible components of the boundary*
- $\pi_* ev^* c_1^2(\mathcal{Q}_i)$ *for* $1 \leq i \leq \ell$
- $\pi_* ev^* c_2(\mathcal{K}_j)$ *for* $0 \leq j \leq \ell$ *where* $r_{j+1} - r_j \geq 2$
- *if* $m = 1, 2$ *then we include exactly one of the classes* $ev_k^* c_1(\mathcal{Q}_h)$.

Furthermore, there is a relation

$$\sum_j \pi_* ev^* c_2(\mathcal{K}_j) + \sum_i \left(\frac{d_{i-1} + d_{i+1}}{2d_i} - 1\right) \pi_* ev^* c_1^2(\mathcal{Q}_i) = 0$$

modulo the boundary. All other relations are pulled back from $\overline{M}_{0,m}$.

0.4 The moduli stack of stable quotients

On the sheaf theory side, the natural analog of the moduli stack of stable maps is the moduli stack of stable quotients introduced in [MOP11]. Specifically, a closed point of $\overline{\mathcal{Q}}_{g,m}(Gr(r,n), d)$ consists of the following data:

- a projective, connected, reduced, at worst nodal curve C of arithmetic genus g together with m distinct points $p_i \in C^{smooth}$
- a short exact sequence of sheaves on C

  $$0 \to S \to \mathbb{C}^n \otimes \mathcal{O} \to Q \to 0$$

  such that Q is a rank $n - r$, degree d coherent sheaf which is locally free at the markings and nodes.

Stability is the requirement that

$$\omega_C\left(\sum_{i=1}^{m} p_i\right) \otimes det(S^*)^{\epsilon}$$

is ample for every $\epsilon \in \mathbb{Q}_{>0}$.

An isomorphism of two stable quotients

$$\phi : (C, p_1, \dots, p_m, q : \mathbb{C}^n \otimes \mathcal{O} \to Q) \to (C', p'_1, \dots, p'_m, q' : \mathbb{C}^n \otimes \mathcal{O} \to Q')$$

consists of an isomorphism $\phi : C \to C'$ mapping p_i to p'_i for each $1 \le i \le m$, such that we have a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & \phi^* Q' \arrow[r] \arrow[d] & 0 \\ \mathbb{C}^n \otimes \mathcal{O} \arrow[r] & Q \arrow[r] & 0. \end{tikzcd}$$

Returning to the stability condition, we see that the genus zero components must have at least 2 markings or nodes; and if there are exactly 2 markings or nodes, then the degree of the quotient must be strictly positive on this component. From this we derive that the automorphism groups of stable quotients are finite.

Stable quotients have found a number of applications. For instance, in all genera, the stable quotient geometry has been used to prove the Faber-Zagier relations over $\mathcal{M}_g$, see [PP]. In a different direction, stable quotient invariants are also connected to the B-side of mirror symmetry; see [CFK], [CZ14] for the relevant calculations.

[Coo15] studied the geometry of the moduli stack of stable quotients in genus 1 to $Gr(1,n)$ without markings. [Coo15] calculated the Picard group (which was determined to have rank 2), the ample and effective cones, and the canonical class in terms of the generators when n is arbitrary. In addition, when $n = 1$, [Coo15] calculated the Poincaré polynomial. [Coo15] also proved that the coarse moduli space is projective and rationally connected.

We will be considering the genus zero case, but with an arbitrary number of markings. This complicates the calculation of the Picard group, as it is well known that the moduli space of genus zero m-pointed stable curves has a large Picard rank (on the order of 2^{m-1}, see [Kee92]). As in the stable maps case, there is a forgetful morphism

$$\begin{tikzcd} \overline{\mathcal{Q}}_{0,m}(Gr(r,n), d) \arrow[d, "F"] \\ \mathcal{M}^{pre}_{0,m}. \end{tikzcd}$$

Since $h^1(\mathcal{S}^* \otimes \mathcal{Q}) = 0$, the moduli stack of genus zero stable quotients is smooth.

When $m \ge 3$, there is also a stabilization map

$$st : \overline{\mathcal{Q}}_{0,m}(Gr(r,n), d) \to \overline{M}_{0,m}$$

which forgets the quotient sequence and stabilizes the underlying curve.

In this paper we obtain results similar to those already mentioned above by

[Pan99] and [Opr05] for genus zero stable quotients to $Gr(r,n)$.

To start, we prove

Theorem 0.4.1. *The coarse moduli space* $\overline{Q}_{g,m}(Gr(r,n),d)$ *is projective.*

We need this in order to apply the generalization of the work of Bialynicki-Birula from [Opr06b] to calculate the second Betti number of the moduli stack for $g = 0$.

To explain our results on the Picard group, we introduce some notation. Let

$$\begin{tikzcd} \mathcal{C}_{0,m}(Gr(r,n),d) \arrow[d, "\pi"] \\ \overline{\mathcal{Q}}_{0,m}(Gr(r,n),d) \end{tikzcd}$$

be the universal curve over the moduli stack. There exists a universal sequence of sheaves $0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q} \to 0$ over the universal curve, such that $\mathcal{Q}$ is flat over the moduli stack and is locally free at the marked points and nodes when restricted to fibers of π.

This yields a universal *rational* map $ev : \mathcal{C}_{0,m}(Gr(r,n),d) \dashrightarrow Gr(r,n)$.

Notice that the universal curve is not isomorphic to $\overline{\mathcal{Q}}_{0,m+1}(Gr(r,n),d)$; there is not a forgetful morphism $\overline{\mathcal{Q}}_{0,m+1}(Gr(r,n),d) \to \overline{\mathcal{Q}}_{0,m}(Gr(r,n),d)$ because there is not a canonical way to contract the quotient sequence when a component becomes unstable. As above, there are universal sections

$$\begin{tikzcd} \mathcal{C}_{0,m}(Gr(r,n),d) \arrow[d, shift left, "\pi"] \\ \overline{\mathcal{Q}}_{0,m}(Gr(r,n),d) \arrow[u, shift left, "\sigma_i"] \end{tikzcd}$$

along which $\mathcal{Q}$ is locally free. The condition that $\mathcal{Q}$ be locally free at the markings produces evaluation *morphisms*

$$ev_i = \sigma_i \circ ev : \overline{\mathcal{Q}}_{0,m}(Gr(r,n),d) \to \mathcal{C}_{0,m}(Gr(r,n),d) \dashrightarrow G(r,n).$$

Let Δ denote the boundary of the moduli stack (the locus where the underlying curve is reducible).

All of our results on the Picard group are for $d \geq 1$ (if $d = 1$ then the moduli space is isomorphic to $\overline{M}_{0,m}$ or it is empty).

0.4.1 $m \geq 3$

In the case of 3 or more markings, we calculate the Picard group of the moduli space of stable quotients to $Gr(r,n)$ for all r and all n.

Our first result is the following:

Theorem 0.4.2. *For $m \geq 3$, we have the following generators and relations for the Picard group:*

- *For $r = 1$, $r \neq n$, $Pic(\overline{Q}_{0,m}(Gr(r,n), d)) \otimes \mathbb{Q}$ is generated by $\pi_* c_1^2(\mathcal{Q})$ and the irreducible components of Δ. All relations are pulled back from $\overline{M}_{0,m}$.*
- *For $r \geq 2$, $r \neq n$, $Pic(\overline{Q}_{0,m}(Gr(r,n), d)) \otimes \mathbb{Q}$ is generated by $\pi_* c_1^2(\mathcal{Q})$, $\pi_* c_2(\mathcal{Q})$, and the irreducible components of Δ. All relations are pulled back from $\overline{M}_{0,m}$.*

In the next case, when $r = n$, we know that there is an isomorphism of coarse moduli spaces

$$\overline{M}_{0,m|d}/\mathcal{S}_d \cong \overline{Q}_{0,m}(Gr(1,1), d)$$

([MOP11], *Proposition 3*) which induces an isomorphism of Picard groups. The first moduli space is an instance of the moduli space of weighted pointed stable rational curves from [Has03]. [Cey09] has already calculated the Picard group (in fact the Chow groups) of the moduli spaces $\overline{M}_{0,m|d}$ and showed that the Picard group is generated by the pushforwards of the boundary classes under the map which reduces the weight on a marking to $\epsilon << 1$ [Has03]

$$r : \overline{M}_{0,m+d} \to \overline{M}_{0,m|d},$$

and all relations come from pushing forward the relations of [Kee92] on $\overline{M}_{0,m+d}$ under these weight-reducing maps. The Picard group of the moduli space $\overline{M}_{0,m|d}/\mathcal{S}_d$ is the $\mathcal{S}_d$-invariant piece of the Picard group of $\overline{M}_{0,m|d}$, which, as we shall see in Chapter 4, is generated by the boundary divisors.

However, in the stable quotients moduli space, there is an additional, natural class to consider: $\pi_* c_1^2(\mathcal{Q})$. We determine the expression of the class $\pi_* c_1^2(\mathcal{Q})$ in terms of the irreducible components of the boundary.

For each $A \subset [m]$, $0 \leq k \leq d$, define $\Delta_{A,k}$ to be the divisor parametrizing reducible curves with weight one markings labelled by A and a degree k divisor of weight $\epsilon << 1$ on one component, and the rest of the markings and a degree $d - k$

divisor of weight $\epsilon << 1$ on the other component, subject to the stability conditions we mentioned in the definition of stable quotients. Observe that $\Delta_{A,k} = \Delta_{A^c,d-k}$

Lemma 0.4.1. *For $m \geq 2$, under the isomorphism $\overline{M}_{0,m|d}/\mathcal{S}_d \cong \overline{Q}_{0,m}(Gr(1,1), d)$, we have the following relation*

$$\sum_{j=1}^{m} \sum_{k=0}^{[\frac{d}{2}]} \left(\frac{-jd^2(j-1)}{m(m-1)} - \frac{(m-2j)dk}{m} + k(d-k) \right) \sum_{|A|=j} \Delta_{A,k} = \pi_* c_1^2(\mathcal{Q}).$$

The next case we consider is when $r = n \geq 2$. Although considering rational maps to a point may appear trivial, the moduli stack $\overline{\mathcal{Q}}_{0,m}(Gr(n,n), d)$ nonetheless has interesting geometry from a sheaf theory perspective, and it does not have an appropriate analogue for stable maps. There is a determinant map

$$det : \overline{\mathcal{Q}}_{0,m}(Gr(n,n), d) \to \overline{\mathcal{Q}}_{0,m}(Gr(1,1), d) \cong \overline{M}_{0,m|d}/\mathcal{S}_d$$

which takes the determinant of the inclusion of the subsheaf in $\mathbb{C}^n \otimes \mathcal{O}$.

We can pull back the relations among the boundary divisors to obtain relations among the boundary divisors in the case $r = n \geq 2$.

The notation for $\Delta_{A,k}$ is the same as above, where k is the degree of the quotient on the component.

Proposition 0.4.1. *For $m \geq 2$, when $r = n \geq 2$, $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n), d)) \otimes \mathbb{Q}$ is generated by $\pi_* c_2(\mathcal{Q})$, $\pi_* c_1^2(\mathcal{Q})$ and the irreducible components of Δ, with the relations among the boundary divisors coming from $\overline{M}_{0,m|d}/\mathcal{S}_d$ via det, and the following relation*

$$\sum_{j=1}^{m} \sum_{k=0}^{[\frac{d}{2}]} \left(\frac{-jd^2(j-1)}{m(m-1)} - \frac{(m-2j)dk}{m} + k(d-k) \right) \sum_{|A|=j} \Delta_{A,k} = \pi_* c_1^2(\mathcal{Q}).$$

0.4.2 $m = 2$ and $r \neq n$

A different method will apply to the case $m = 2$. We obtain generators and relations by intersecting with curves.

In this case there are two new classes to consider: the evaluation classes $ev_1^* c_1(\mathcal{O}_G(1))$ and $ev_2^* c_1(\mathcal{O}_G(1))$.

As before $\Delta_{1,k} = \Delta_{2,d-k}$ parameterizes reducible curves with the marking 1 on one component such that the restriction of the quotient to this component has degree k, and the marking 2 on the other component such that the restriction of the quotient to this component has degree $d-k$. These are subject to the stability conditions in the definition of stable quotients: $k, d-k > 0$.

Our result is the following:

Theorem 0.4.3. *When $m = 2$, $r \neq n$:*

- *For $r = 1$, $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n), d)) \otimes \mathbb{Q}$ has a basis given by*
$$\{\{\Delta_{1,k}\}_{k=1}^{d-1}, ev_1^* c_1(\mathcal{O}_G(1)), ev_2^* c_1(\mathcal{O}_G(1))\}.$$
Furthermore, there is a relation
$$d(ev_1^* c_1(\mathcal{O}_G(1)) + ev_2^* c_1(\mathcal{O}_G(1))) + \sum_{k=1}^{d-1} k(d-k)\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q}).$$
- *For $r \geq 2$, $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n), d)) \otimes \mathbb{Q}$ has a basis given by*
$$\{\{\Delta_{1,k}\}_{k=1}^{d-1}, \pi_* c_2(\mathcal{Q}), ev_1^* c_1(\mathcal{O}_G(1)), ev_2^* c_1(\mathcal{O}_G(1))\}.$$
Furthermore, there is a relation
$$d(ev_1^* c_1(\mathcal{O}_G(1)) + ev_2^* c_1(\mathcal{O}_G(1))) + \sum_{k=1}^{d-1} k(d-k)\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q}).$$

0.5 The moduli stack of stable quasimaps to GIT quotients

[CFKM14] introduced the moduli stack of stable quasimaps $\mathrm{Qmap}_{0,m}(V/\!/G, \beta)$ to certain GIT quotients $V/\!/G$. The setup of the moduli stack is more general than what we need for the purposes of this paper, as can be seen from the exposition below. However, the moduli stack we will be interested in is the special case of the moduli stack of genus zero stable quasimaps to partial flag varieties. This moduli stack is a generalization of the moduli stack of stable quotients, the underlying philosophy being that, in the compactification of the morphism space, we should not only allow the curve to vary in moduli but also allow the morphism to the target to degenerate to a rational map.

We will explain what points of the moduli stack are. Define
$$\overline{r} := (r_1, \ldots, r_\ell) \in \mathbb{N}_{>0}^\ell, \ \overline{d} := (d_1, \ldots, d_\ell) \in \mathbb{N}_{\geq 0}^\ell,$$
where $r_i < r_{i+1}$ $\forall\, 1 \leq i \leq \ell$, with $r_{\ell+1} = n$. To set up the definition (in our specific case), let
$$V \cong \bigoplus_{i=1}^{\ell} \mathrm{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r_{i+1}})$$
$$G \cong \prod_{i=1}^{\ell} GL(r_i, \mathbb{C}).$$
G acts on V as
$$(g_1, \ldots, g_\ell) \cdot (A_1, \ldots, A_\ell) := (g_2 \circ A_1 \circ g_1^{-1}, \ldots, A_\ell \circ g_\ell^{-1}).$$

Taking the GIT quotient $V/\!\!/G$ with the linearization coming from the trivial line bundle on V endowed with the nontrivial representation $\prod_{i=1}^{\ell} det_i$ yields the (partial) flag variety $Fl(r_1, \ldots, r_\ell; \mathbb{C}^n)$ with the linearization $\bigotimes_{i=1}^{\ell} det(\mathcal{E}_i^*)$, where $\mathcal{E}_i$ is the i^{th} universal subbundle over the flag variety. See the Appendix for details.

A map from a quasi-stable curve to the stack quotient $[V/G]$ consists of the following data:

- a projective, connected, reduced, at worst nodal curve of arithmetic genus g C with m distinct marked points in the smooth locus of C
- a principal G-bundle $\rho : \mathcal{P} \to C$ with a G-equivariant morphism to V,

  $$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\rho"'] & V \,. \\ C \end{tikzcd}$$

  This is equivalent to giving the data of the fiber bundle

  $$\begin{tikzcd} \mathcal{P} \times_G V \arrow[d, shift left] \\ C \arrow[u, bend left, "u"] \end{tikzcd}$$

  where $\mathcal{P} \times_G V$ is the quotient of $\mathcal{P} \times V$ with G acting diagonally ($(p{\cdot}g, v) \sim (p, g{\cdot}v)$), and whose fibers are isomorphic to V. Here u is a section of the fiber (vector) bundle such that $(\mathcal{P}, u)$ is of class $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\chi(G), \mathbb{Z})$.

We follow the explanation of β in [CFKM14]: since V is a vector space, its Picard group is trivial. Thus, the group of G-equivariant line bundles on V is equivalent to the character group of G. Given a character $\chi \in \chi(G)$, we get a G-equivariant line bundle on V, $L_\chi := \mathbb{C} \times_\chi V$. In turn, this yields a line bundle

$$\begin{tikzcd} \mathcal{P} \times_G L_\chi \arrow[d] \\ \mathcal{P} \times_G V. \end{tikzcd}$$

We can pull back the line bundle $\mathcal{P} \times_G L_\chi$ under the section u to get a line bundle $\mathcal{P} \times_G \mathbb{C}_\chi$ over C, and take the degree of this line bundle.

This data yields a homomorphism $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\chi(G), \mathbb{Z})$. Given that $\chi(G) \cong \bigoplus_{i=1}^{\ell} \mathbb{Z}\, det_i$, which we prove in the Appendix, we see that the choice of β relevant for our setting is $\beta = (d_1, \ldots, d_\ell)$ under the above isomorphism.

The map from the quasi-stable curve C to the stack quotient $[V/G]$ above yields a *quasimap* to the GIT quotient $V/\!\!/G$ if there exist finitely many points

$t \in C^{smooth} \backslash \{p_i\}_{i=1}^m$ such that $u(t) \in V^{us}$, and for all other $x \in C$, $u(x) \in V^s$. Here, V^s, V^{us} are the stable and unstable points of the G action on V with respect to the linearization χ; see [MF82] for the definitions.

In [CFKM14], the moduli stack of stable quasimaps to $V//G$ was proven to be a Deligne Mumford stack of finite type over $Spec(\mathbb{C})$ which is proper over the affine quotient $V/_{aff}G$.

In order to explain our results, we must introduce some notation. The notation comes from the isomorphism we will produce in the next chapter, where we will show that the moduli stack is isomorphic to a moduli stack $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ of generalized stable quotients to the flag variety.

Define $\overline{r} := (r_1, \ldots, r_\ell) \in \mathbb{N}^\ell_{>0}$, $\overline{d} := (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell_{\geq 0}$, where $r_i < r_{i+1}$ $\forall\, 1 \leq i \leq \ell$, with $r_{\ell+1} = n$.

Unravelling the definitions, we see that the moduli stack of generalized stable quotients to the flag variety parameterizes:

- a projective, connected, reduced, at worst nodal curve C of arithmetic genus g
- $\{p_j\}_{j=1}^m$ distinct markings contained in the smooth locus of the curve
- a flag sequence $0 \hookrightarrow S_1 \hookrightarrow \ldots \hookrightarrow S_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to Q_1 \to \ldots \to Q_\ell \to 0$ such that Q_i is a coherent sheaf of rank $n - r_i$ and degree d_i
- each inclusion of sheaves $S_i \hookrightarrow S_{i+1}$ only fails to be an inclusion of vector bundles at finitely many points which are necessarily away from the nodes and markings - this is the nondegeneracy condition imposed on the section u
- subject to the stability condition that
$$\omega_C(\sum_{j=1}^m p_j) \otimes (\bigotimes_{i=1}^\ell det(S_i)^*)^\epsilon$$
is ample for any $\epsilon \in \mathbb{Q}_{>0}$.

An isomorphism of generalized stable quotients
$$\begin{array}{c} (C,\, \{p_j\}_{j=1}^m,\, \mathbb{C}^n \otimes \mathcal{O}_C \to Q_1 \to \ldots \to Q_\ell \to 0) \\ \big\downarrow \phi \\ (C',\, \{p'_j\}_{j=1}^m, \mathbb{C}^n \otimes \mathcal{O}_{C'} \to Q'_1 \to \ldots \to Q'_\ell \to 0) \end{array}$$

consists of the following data:

- an isomorphism of curves $\phi : C \to C'$ which maps p_j to p'_j for all $1 \leq j \leq m$
- such that we have a commutative diagram
$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_C \arrow[r] \arrow[d, equal] & \phi^* Q'_1 \arrow[r] \arrow[d, "\phi_1"] & \dots \arrow[r] & \phi^* Q'_\ell \arrow[r] \arrow[d, "\phi_\ell"] & 0 \\ \mathbb{C}^n \otimes \mathcal{O}_C \arrow[r] & Q_1 \arrow[r] & \dots \arrow[r] & Q_\ell \arrow[r] & 0 \end{tikzcd}$$
with the vertical arrows isomorphisms.

We construct the moduli space of generalized stable quotients as a quotient stack in Chapter 1.

Proposition 0.5.1. *The moduli stack of generalized stable quotients is a global quotient*
$$\overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \cong [X \,/\, \mathbf{PGL}_N]$$
for a quasiprojective scheme X with an action of $\mathbf{PGL}_N$, for some N.

We use this construction to prove the following theorem:

Theorem 0.5.1. *The coarse moduli space $\overline{Q}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ is projective.*

We describe the structures over the moduli stack.

There is a universal curve with m universal disjoint sections and a universal *rational* map ev
$$\begin{tikzcd} \mathcal{C}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \arrow[r, dashed, "ev"] \arrow[d, "\pi"'] & Fl(\overline{r}, \mathbb{C}^n) \\ \overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \arrow[u, bend left, "\sigma_j"] & \end{tikzcd}$$
together with a universal flag sequence
$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \dots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \dots \to \mathcal{Q}_\ell \to 0$$
such that for all $1 \leq i \leq \ell$,

- the i^{th} quotient $\mathcal{Q}_i$ is flat over the moduli stack and has fiberwise degree d_i and rank $n - r_i$
- fiberwise, the inclusion of sheaves $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$ only fails to be an inclusion of vector bundles at finitely many points which are necessarily away from the nodes and markings of fibers of π

- $ev \circ \sigma_i := ev_i$ is a genuine morphism to the flag variety.

We can pull back classes from the flag variety along each ev_i. Notice that if $d_i = 0$, then the maps

$$\rho_i \circ ev_j : \overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \to Fl(\overline{r}, \mathbb{C}^n) \to Gr(r_i, n)$$

all agree since the fibers of π are collapsed by the universal evaluation map followed by the projection ρ_i. Let ϕ_j denote this morphism.

Let $\mathcal{F}_k$ denote the k^{th} universal quotient over the flag variety.

In the case that the number of markings is ≥ 3, as in the stable quotients case, there is a stabilization morphism

$$st : \overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \to \overline{M}_{0,m}$$

which forgets the flag sequence and stabilizes the underlying curve. When $m \geq 3$, we obtain generators and relations for the Picard group of genus zero stable quasimaps to the flag variety of any degree type $\overline{d}$ and rank type $\overline{r}$.

Theorem 0.5.2. *For $m \geq 3$, $Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q}$ is generated by*

- $\phi_j^* c_1(\mathcal{F}_j)$ *for each $1 \leq j \leq \ell$ such that $d_j = 0$*
- $\pi_* c_1^2(\mathcal{Q}_k)$ *for each $1 \leq k \leq \ell$ such that $d_k > 0$*
- $\pi_* c_2(\mathcal{Q}_i)$ *for each $1 \leq i \leq \ell$ such that $r_i - r_{i-1} > 1$ and $d_i > 0$*
- $\pi_* c_2(\mathcal{Q}_h)$ *for each $1 < h \leq \ell$ such that $r_h - r_{h-1} = 1$ and $d_h, d_{h-1} > 0$*
- *the irreducible components of the boundary.*

All relations among the boundary divisors are pulled back from $\overline{M}_{0,m}$, and there are no other relations.

When $m = 2$, we obtain the result for partial flag varieties where the ranks of the subspaces differ by at least 2 in each pair of consecutive positions in the rank type $\overline{r}$, and all entries in the degree type are > 0. The method we use when $m = 2$ is intersection with curves together with an induction argument to relate to the Grassmannian case.

Following with our notation for the Grassmannian case, if we fix a tuple

$$\overline{e} := (e_1, \ldots, e_\ell) \in \mathbb{N}_{\geq 0}$$

subject to the stability condition that $\exists i$ such that $e_i > 0$, and $\exists k$ such that $d_k - e_k >$

0, the divisor $\Delta_{1,\overline{e}}$ parameterizes reducible curves with the marking p_1 on one component such that the degree of each Q_i restricted to this component is e_i.

Theorem 0.5.3. *When $r_i - r_{i-1} > 1$, $d_i > 0$ for all $1 \le i \le \ell$, the rational Picard group of $\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ is spanned by the following classes:*

- *the boundary divisors*
- *$\pi_* c_2(\mathcal{Q}_i)$ for each $1 \le i \le \ell$*
- *$\pi_* c_1^2(\mathcal{Q}_j)$ for each $1 \le j \le \ell$*
- *$ev_u^* c_1(\mathcal{F}_k)$ for each $u = 1, 2$ and $1 \le k \le \ell$*

with the following relations:

- *for each $1 \le j \le \ell$,*
  $\sum_{\overline{e}} e_j(d_j - e_j)\Delta_{1,\overline{e}} + d_j(ev_1^* c_1(\mathcal{F}_j) + ev_2^* c_1(\mathcal{F}_j)) = \pi_* c_1^2(\mathcal{Q}_j)$
- *for each pair (j,k), $1 \le j \ne k \le \ell$,*
  $-\frac{1}{d_j} ev_1^* c_1(\mathcal{F}_j) + \frac{1}{d_k} ev_1^* c_1(\mathcal{F}_k) + \frac{1}{d_j} ev_2^* c_1(\mathcal{F}_j) - \frac{1}{d_k} ev_2^* c_1(\mathcal{F}_k)$
  $+ \sum_{\overline{e}} (\frac{e_k}{d_k} - \frac{e_j}{d_j})\Delta_{1,\overline{e}} = 0.$

0.6 The canonical class and future study

We make a conjecture on the expression of the canonical class of the moduli stack of genus zero stable quotients (to the Grassmannian $Gr(r,n)$). This formula is obtained via intersecting with test curves. The result hinges on a few technical details on the cotangent complex for the Artin stack of semistable curves $\mathcal{M}_{0,m}^{ss}$ which we could not find in the literature.

Conjecture 0.6.1. *For $m \ge 2$, $d > 1$, the canonical class of the moduli stack $\overline{\mathcal{Q}}_{0,m}(Gr(r,n), d)$ can be expressed in terms of the generators we found above as*

$$K_{\overline{\mathcal{Q}}_{0,m}(Gr(r,n),d)} = \sum_{j=1}^{m-1} \sum_{k=0}^{[\frac{d}{2}]} \left(\frac{nk(d-k)}{2d} - \frac{j(j-1)}{m-1} + j - 2 \right) \sum_{|A|=j} \Delta_{A,k} +$$
$$\left(\frac{(n-2r-2)d-n}{2d} \right) \pi_* c_1^2(\mathcal{Q}) + (-n+2r)\pi_* c_2(\mathcal{Q}).$$

It would be interesting to determine the canonical class of the moduli space of genus zero generalized stable quotients to the flag variety as well. The case of flag varieties of type B, C, and D, as well as toric targets deserves further study.

Given the calculation of the Picard group, this allows us to study the birational geometry of the moduli space, similar to the work of [Ven11] on the Quot and HyperQuot scheme. We plan to pursue this avenue in future work.

In a slightly different vein, a logical next step is to calculate the cohomology ring (or Chow ring) of the moduli space of generalized stable quotients. Along similar lines, it would be interesting to calculate the Poincaré polynomial, as [Che01] has done for the HyperQuot scheme.

0.7 Outline of the Dissertation

- In the first chapter we define the moduli stack of generalized stable quotients to the flag variety and show that it is isomorphic to the moduli stack of stable quasimaps to the flag variety as defined in [CFKM14]. We construct the moduli stack as a stack quotient. We also prove smoothness of the moduli stack when $g = 0$.

- In the second chapter we prove projectivity of the coarse moduli space. First, we produce a semipositive vector bundle on the moduli stack. From here we construct an ample line bundle which descends to the coarse moduli space, a priori an algebraic space.

- In the third chapter we use the fact that the coarse moduli space is projective to allow us to use the Bialynicki Birula stratification for smooth DM stacks with the action of a torus. We calculate the number of relevant fixed loci which contribute to the calculation of the second Betti number of the moduli stack. Then we calculate the second Betti numbers of the relevant fixed loci. The latter are finite group quotients of moduli spaces of weighted pointed stable rational curves of [Has03].

- In the fourth chapter we analyze the interior of the moduli stack in the case that the number of markings is greater than 2, and we intersect with curves when the number of markings is 2. This will be combined with the Picard rank calculation to describe the Picard group completely.

- In the fifth chapter we repeat the steps in the third chapter for the case of the flag variety. The analysis is of course more involved.

- In the sixth chapter we use the calculation of the Picard rank in the previous chapter, combined with an analysis of the interior of the moduli stack when $m \geq 3$, to produce generators and relations for the Picard group of the moduli stack of genus zero generalized stable quotients to the flag variety of any rank and degree type. When $m = 2$, we use intersections with test curves to determine generators and relations for the Picard group of the moduli stack of genus zero generalized stable quotients to a partial flag variety.

1 Quotient Construction and Foundational Results

Recall from the introduction that the flag variety $Fl(\overline{r}, \mathbb{C}^n)$ can be constructed as a GIT quotient $V/\!/G$, for V a suitable vector space and G an algebraic group acting linearly on V.

In this chapter we prove that the moduli stack of stable quasimaps to $V/\!/G$ and the moduli stack of generalized stable quotients to the flag variety are isomorphic

$$\mathrm{Qmap}_{g,m}(V/\!/G, \overline{d}) \cong \overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}; \mathbb{C}^n), \overline{d}).$$

We first define the category fibered in groupoids of generalized stable quotients to the flag variety. Next, we briefly describe its construction as a stack quotient, in parallel with the construction of the moduli stack of stable quotients to the Grassmannian in [MOP11].

Finally, we prove that the moduli stack of genus zero generalized stable quotients to the flag variety is smooth.

We will use our alternate construction of the moduli stack of generalized stable quotients in our proof of projectivity of the coarse moduli space in the next chapter.

1.1 The category of generalized stable quotients

Given a scheme T, the category fibered in groupoids $\overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ associates to T the collection of all families over T consisting of the following data which we label $(\star)$:

- A proper flat family $\pi : \mathcal{C} \to T$ of connected, reduced, at worst nodal curves of arithmetic genus g with m distinct sections $\sigma_j : T \to \mathcal{C}$

- A flag sequence of subsheaves of $\mathbb{C}^n \otimes \mathcal{O}$ over $\mathcal{C}$
  $0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0$
  subject to the conditions
  - $\mathcal{S}_i$ has rank r_i
  - the associated quotients $\mathcal{Q}_i$ are T flat with fiberwise degree d_i
  - the inclusion of sheaves $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$ only fails to be an inclusion of vector bundles at finitely many points away from the nodes and markings of the fibers of π
- the line bundle $\omega_\pi(\sum_{j=1}^{m} \sigma_j) \otimes (\bigotimes_{i=1}^{\ell} det(\mathcal{S}_i^*))^\epsilon$ is π relatively ample $\forall \epsilon \in \mathbb{Q}_{>0}$.

An isomorphism of two families of generalized stable quotients consists of

- an isomorphism of families of curves with sections over T:
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, shift left, "\pi"] & \mathcal{C}' \arrow[d, shift right, "\pi'"'] \\ T \arrow[u, bend left, "\sigma_j"] \arrow[r, equal] & T \arrow[u, bend right, "\sigma'_j"'] \end{tikzcd} \quad \text{such that } \sigma'_j = \phi \circ \sigma_j$$
- a commutative diagram whose columns are isomorphisms
$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & \phi^* \mathcal{Q}'_1 \arrow[r] \arrow[d] & \ldots \arrow[r] & \phi^* \mathcal{Q}'_\ell \arrow[r] \arrow[d] & 0 \\ \mathbb{C}^n \otimes \mathcal{O} \arrow[r] & \mathcal{Q}_1 \arrow[r] & \ldots \arrow[r] & \mathcal{Q}_\ell \arrow[r] & 0. \end{tikzcd}$$

.

When $\ell = 1$, this recovers the definition of the moduli stack of stable quotients to the Grassmannian from [MOP11].

1.2 Construction of the moduli stack

We give a quotient construction of the moduli stack.

Consider a generalized stable quotient $(\star)$ over $T = Spec(\mathbb{C})$.

Let $d = \sum_{i=1}^{\ell} d_i$.

By $(\star)$ and the same proof of *Lemma 5* in [MOP11], $\forall\, k \geq 5$,
$$\omega_C^{k(d+1)}(\sum_{j=1}^{\ell} k(d+1)p_j) \otimes (\bigotimes_{i=1}^{\ell} det(\mathcal{S}_i^*))^k := \mathcal{L}_k$$
is very ample without higher cohomology.

Let W be a complex vector space with a fixed isomorphism

$$W \cong \mathbb{C}^{1-g+k(d+1)(2g-2+m)+kd}.$$

Then, given an isomorphism $H^0(C, \mathcal{L}_k) \cong W^*$, $\mathcal{L}_k$ gives an embedding of the curve C (up to the $\mathbf{PGL}(W)$ action) into $\mathbb{P}(W)$.

We consider the Hilbert scheme of curves *Hilb* of genus g and degree $k(d+1)(2g-2+m)+kd$ in $\mathbb{P}(W)$. The data of the marked curve yields a point in $\mathcal{H} = \mathit{Hilb} \times \prod_{j=1}^{m} \mathbb{P}(W)$. There is a closed subscheme of $\mathcal{H}$ where the markings are on the curve.

By *Proposition 5*, page 193 of [ACG11], there exists an open subscheme corresponding to connected, reduced, at worst nodal curves with m distinct markings contained in the smooth locus. Call this subscheme $\mathcal{H}'$. Let $\pi : \mathcal{C}' \to \mathcal{H}'$ be the universal curve with the m sections $\sigma_j : \mathcal{H}' \to \mathcal{C}'$.

Now, we will construct an open subscheme of the π relative HyperQuot scheme. We do this inductively. Start by forming the π relative Quot scheme

$$\mathit{Quot}_\pi(\mathbb{C}^n \otimes \mathcal{O}, n - r_1, d_1)$$

parameterizing rank $n - r_1$ degree d_1 coherent quotient sheaves of the trivial rank n vector bundle on the fibers of π. We have the diagram

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{H}'} \mathit{Quot} \arrow[r] \arrow[d, shift left, "\pi_1"] & \mathcal{C}' \arrow[d, shift left, "\pi"] \\
\mathit{Quot} \arrow[r] \arrow[u, bend left, "\sigma_j^1"] & \mathcal{H}' \arrow[u, bend left, "\sigma_j"]
\end{tikzcd}$$

We can find an open subscheme corresponding to quotients which are locally free at the nodes and marked sections on the fibers of π_1. Call this subscheme Y_1.

Pull back the universal curve, the sections, and the universal quotient to Y_1 (the $'$ is used to denote the restriction to the subscheme Y_1 of the Quot scheme and will be used throughout the construction to make this distinction)

$$\begin{tikzcd}
\mathcal{C} \times_{\mathcal{H}'} Y_1 \arrow[d, shift left, "\pi_1'"] & , \; 0 \to \mathcal{S}_1 \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to 0. \\
Y_1 \arrow[u, bend left, "\sigma_j^{1\prime}"] &
\end{tikzcd}$$

Next, form the π_1' relative Quot scheme $\mathit{Quot}_{\pi_1'}(\mathcal{Q}_1, n-r_2, d_2)$ parameterizing rank $n - r_2$ degree d_2 coherent sheaf quotients of $\mathcal{Q}_1$ on the fibers of π_1'. As before, we can find an open subscheme corresponding to quotients which are locally free at the nodes and markings on the fibers of π_1'. Call this subscheme Y_2. Pulling back the universal curve and quotient sequences

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{H}'} Y_2 \arrow[d, "\pi'_2"] & , 0 \to \mathcal{S}_1 \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to 0,\ 0 \to \mathcal{K}_1 \to \mathcal{Q}_1 \to \mathcal{Q}_2 \to 0 \\ Y_2 \arrow[u, bend left, "\sigma_j^{2\prime}"] \end{tikzcd}$$

we see that we have a universal sequence

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \mathcal{S}_2 \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \mathcal{Q}_2 \to 0$$

where $\mathcal{S}_2$ is the kernel of the composition $\mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \mathcal{Q}_2$.

Notice that the cokernel of $\mathcal{S}_1 \hookrightarrow \mathcal{S}_2$ is $\mathcal{K}_1$, so we see that this inclusion of sheaves only fails to be an inclusion of vector bundles at finitely many points of each fiber of π'_2 away from the nodes and markings.

Iterating this process, we end up with a scheme Y_ℓ with the following data:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi'_\ell"] & , 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0 \\ Y_\ell \arrow[u, bend left, "\sigma_j^{\ell\prime}"] \end{tikzcd}$$

such that

- for $1 \le i \le \ell$, each $\mathcal{Q}_i$ has rank $n - r_i$ and degree d_i on the fibers of π'_ℓ
- for $1 \le i \le \ell$, each inclusion of subsheaves $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$ fails to be an inclusion of vector bundles only at finitely many points of each fiber of π'_ℓ, away from the nodes and markings (here if $i = \ell$ then $\mathcal{S}_{\ell+1} = \mathbb{C}^n \otimes \mathcal{O}$).

As in *Proposition 5.1* of [MF82], we can construct a locally closed subscheme of Y_ℓ where the line bundles

- $\mathcal{L}'_k := \omega_{\pi'_\ell}^{k(d+1)}(\sum_{j=1}^{m} k(d+1)\sigma_j^{\ell\prime}) \otimes (\bigotimes_{i=1}^{\ell} det(\mathcal{S}_i^*))^k$
- $\mathcal{O}_{\mathbb{P}(W)\times Y_\ell}(1)$

agree on the fibers of π'_ℓ as described below.

By [MB66], *Item b*, the relative Picard functor of the universal family of curves over Y_ℓ is representable. The two line bundles $\mathcal{L}'_k$, $\mathcal{O}_{\mathbb{P}(W)\times Y_\ell}(1)$ yield a morphism $(\epsilon, \eta) : Y_\ell \to \mathcal{P}ic_{\pi'_\ell} \times \mathcal{P}ic_{\pi'_\ell}$, where ϵ is induced by $\mathcal{L}'_k$, and η is induced by $\mathcal{O}_{\mathbb{P}(W)\times Y_\ell}(1)$. We can consider the fiber product

$$\begin{tikzcd} Q' \arrow[r] \arrow[d] & Y_\ell \arrow[d, "(\epsilon,\eta)"] \\ \mathcal{P}ic_{\pi'_\ell} \arrow[r, "\Delta"'] & \mathcal{P}ic_{\pi'_\ell} \times_{Y_\ell} \mathcal{P}ic_{\pi'_\ell}. \end{tikzcd}$$

Notice that Δ is always a locally closed immersion, since affine locally the corresponding diagonal map of rings is surjective. Thus, Q' is the universal locally closed subscheme of Y_ℓ with the property that the restrictions of $\mathcal{L}'_k$ and $\mathcal{O}_{\mathbb{P}(W)\times Y_\ell}$ to $\mathcal{C}\times_{Y_\ell} Q'$ differ by a line bundle pulled back from the base.

The map of sheaves

$$\rho_*(\mathcal{O}_{\mathbb{P}(W)\times Q'}(1)) \to \pi'_*\mathcal{O}_{\mathbb{P}(W)\times Q'}(1)|_{\mathcal{C}\times_{Q_\ell} Q'}$$

has a cokernel, call it $\mathcal{R}$. We can consider the open subscheme where $\mathcal{R}$ is zero on stalks, call it $\mathcal{Q}'$. This is the locus of flag sequences whose underlying curves are embedded via the global sections of $\mathcal{L}'_k$.

There is a natural $\mathbf{PGL}(W)$ action on $\mathcal{H}'$, which induces an action of $\mathbf{PGL}(W)$ on $\mathcal{Q}'$. We take the stack quotient $[\mathcal{Q}'/\mathbf{PGL}(W)]$.

Notice that the universal sequence of sheaves on the universal curve over $\mathcal{Q}'$ can be endowed with a $\mathbf{PGL}(W)$-equivariant structure.

We check that this stack quotient $[\mathcal{Q}'/\mathbf{PGL}(W)]$ is equivalent to the category fibered in groupoids $\overline{\mathcal{Q}}_{g,m}(Fl(\overline{r};\mathbb{C}^n),\overline{d})$ (Proposition 0.5.1 from the Introduction).

Proof. By *Lemma 5.1*, page 282 of [ACG11], it suffices to give a functor A between the two categories fibered in groupoids such that for any scheme T, we have an equivalence of categories

$$A_T : \overline{\mathcal{Q}}_{g,m}(Fl(\overline{r};\mathbb{C}^n),\overline{d})(T) \to [\mathcal{Q}'/\mathbf{PGL}(W)](T).$$

Our argument will follow the argument given in *Theorem 5.6*, in the same reference, as well as [Vis05] *Theorem 4.38*, with some modifications for the data of the flag sequence.

Define the functor A as follows. Consider a family of generalized stable quotients $(\star)$ and the vector bundle $(\pi_*\mathcal{L}_k)^*$ on T. We can projectivize this vector bundle, and consider the associated $\mathbf{PGL}(W)$ bundle $\rho : \mathcal{P}\to T$.

We have an embedding of the family of curves and sections as a family of curves in $\mathcal{C}\hookrightarrow\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ over T. When we pull back $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ along ρ, we get a canonical isomorphism

$$\rho^*\mathbb{P}((\pi_*\mathcal{L}_k)^*)\cong\mathbb{P}(W)\times\mathcal{P}.$$

Pulling back the family under ρ yields a family

$$\begin{tikzcd} \mathcal{C}\times_T\mathcal{P} \arrow[r] \arrow[d, shift left, "\pi'"] & \mathbb{P}(W)\times\mathcal{P} \arrow[dl] \\ \mathcal{P} \arrow[u, shift left, "\sigma'_j"] & \end{tikzcd}, \quad 0\hookrightarrow\mathcal{S}'_1\hookrightarrow\ldots\hookrightarrow\mathcal{S}'_\ell\hookrightarrow\mathbb{C}^n\otimes\mathcal{O}\to\mathcal{Q}'_1\to\ldots\to\mathcal{Q}'_\ell\to 0.$$

This data yields a $\mathbf{PGL}(W)$-equivariant morphism $\phi : \mathcal{P} \to \mathcal{Q}'$ by construction. Since $\mathcal{L}_k$ restricted to fibers of π is very ample and without higher cohomology, cohomology and base change ([Oss]) yields that $\pi_* \mathcal{L}_k$ commutes with base change. Also, the formation of the associated principal bundle to a vector bundle commutes with base change, so this defines a functor.

To check that A is an equivalence of categories fibered in groupoids, by *Lemma 5.1* in *Chapter XII* of [ACG11], we must prove that A_T is fully faithful, and that A_T is essentially surjective.

Let $(\pi, \{\sigma_j\}_{j=1}^m, \{q_i\}_{i=1}^\ell)$ denote the family of generalized stable quotients over T, and let (ρ, ϕ) denote the family of principal $\mathbf{PGL}(W)$ bundles over T together with the $\mathbf{PGL}(W)$ equivariant map to $\mathcal{Q}'$, as above.

Then, we must show that we have a natural isomorphism

$$A_T : Hom_{\overline{\mathcal{Q}}(T)}((\pi, \{\sigma_j\}_{j=1}^m, \{q_i\}_{i=1}^\ell), (\pi, \{\sigma_j\}_{j=1}^m, \{q_i\}_{i=1}^\ell)) \cong Hom_{[\mathcal{Q}'/\mathbf{PGL}(W)](T)}((\rho, \phi), (\rho, \phi)).$$

To see this, we must prove that every automorphism of the family of generalized stable quotients is induced by a unique automorphism of
$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\rho"] & \mathcal{Q}' \\ T & \end{tikzcd}$$
, and vice versa.

Notice that curves embedded in $\mathbb{P}(W)$ via the choice of an isomorphism

$$H^0(C, \mathcal{L}_k) \to W^*$$

are not contained in any hyperplane inside $\mathbb{P}(W)$.

Suppose we are given an automorphism ψ

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "\phi"] \arrow[d, "\rho"'] & & \mathcal{Q}' \\ T \arrow[dr, equal] & \mathcal{P} \arrow[ul, "\psi"] \arrow[ur, "\phi"'] \arrow[d, "\rho"] & \\ & T & \end{tikzcd}$$

of the principal bundle and the $\mathbf{PGL}(W)$ equivariant morphism associated to the family of generalized stable quotients over T.

Then, this induces an automorphism $\tilde{\psi} : \mathbb{P}((\pi_* \mathcal{L}_k)^*) \to \mathbb{P}((\pi_* \mathcal{L}_k)^*)$. I claim that this automorphism is actually an automorphism of

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\pi"] & \mathbb{P}((\pi_* \mathcal{L}_k)^*) \arrow[dl] \\ T \arrow[u, bend left, "\sigma_j"] & \end{tikzcd}, \quad 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0.$$

This is a consequence of the condition that the automorphism comes from an automorphism not only of the principal bundle but also of the equivariant map to $\mathcal{Q}'$.

First, we check that the induced automorphism $\tilde{\psi} : \mathbb{P}((\pi_* \mathcal{L}_k)^*) \to \mathbb{P}((\pi_* \mathcal{L}_k)^*)$ yields an automorphism of the family of curves $\pi : \mathcal{C} \to T$. We can check this at the level of fibers. Let $\alpha_t : \mathbb{P}((\pi_* \mathcal{L}_k)^*)_t \to \mathbb{P}(W)$ be an element of $\mathcal{P}$. Suppose that $\psi(\alpha_t) = \beta_t$. Then, we have a diagram

$$\begin{tikzcd}
\mathcal{C}_t \arrow[r, "i_t"] \arrow[d] & \mathbb{P}((\pi_* \mathcal{L}_k)^*)_t \arrow[r, "\alpha_t"] \arrow[d, "\tilde{\psi}_t"] & \mathbb{P}(W) \arrow[d, "g_{\psi_t}"] \\
\mathcal{C}_t \arrow[r, "i_t"'] & \mathbb{P}((\pi_* \mathcal{L}_k)^*)_t \arrow[r, "\beta_t"] & \mathbb{P}(W)
\end{tikzcd}$$

in which the rightmost square commutes, and we claim the left square is commutative. The composition of the horizontal arrows in either row defines the embedding of the fiber of $\mathcal{C} \times_T \mathcal{P} \to \mathcal{P}$ over α_t, β_t, respectively, into $\mathbb{P}(W)$. Commutativity of the outer square follows by pulling back the universal curve under ϕ and $\psi \circ \phi$. It follows that the left square commutes. The same argument shows that the automorphism $\tilde{\psi}$ yields an automorphism of the sections σ_j.

We must check that the automorphism $\tilde{\psi}|_{\mathcal{C}}$ yields a commutative diagram

$$\begin{tikzcd}
\tilde{\psi}|_{\mathcal{C}}^* \mathcal{S}_1 \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & \tilde{\psi}|_{\mathcal{C}}^* \mathcal{S}_\ell \arrow[r, hook] \arrow[d] & \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \arrow[d, equal] \\
\mathcal{S}_1 \arrow[r, hook] & \dots \arrow[r, hook] & \mathcal{S}_\ell \arrow[r, hook] & \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}
\end{tikzcd}$$

in which all columns are isomorphisms. We use fpqc descent to prove this. Let $\mathcal{C}^U \to \mathcal{Q}'$ denote the universal curve, $\overline{\rho} : \mathcal{C} \times_T \mathcal{P} \to \mathcal{C}$ denote the natural projection, $\overline{\phi} : \mathcal{C}^U \times_{\mathcal{Q}'} \mathcal{P} \to \mathcal{C}^U$ denote the natural projection, and $\overline{\psi}$ denote the induced automorphism of $\mathcal{C}^U \times_{\mathcal{Q}'} \mathcal{P}$. That we have such a commutative diagram when we pull both sequences back to $\mathcal{C} \times_T \mathcal{P}$ follows from the condition that the automorphism ψ preserve the equivariant map ϕ: the pullbacks of the sequences to $\mathcal{C} \times_T \mathcal{P}$ are canonically isomorphic to the pullbacks of the universal sequence on $\mathcal{C}^U$ along $\overline{\phi} \circ \overline{\psi}$, $\overline{\phi}$, respectively. We have a commutative diagram whose columns are isomorphisms

$$
\begin{tikzcd}
\overline{\rho}^*\tilde{\psi}|_{\mathcal{C}}^*\mathcal{S}_1 \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & \overline{\rho}^*\tilde{\psi}|_{\mathcal{C}}^*\mathcal{S}_\ell \arrow[r, hook] \arrow[d] & \mathbb{C}^n \otimes \overline{\rho}^*\tilde{\psi}|_{\mathcal{C}}^*\mathcal{O}_{\mathcal{C}} \arrow[d] \\
\overline{\psi}^*\overline{\phi}^*\mathcal{S}_1^U \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & \overline{\psi}^*\overline{\phi}^*\mathcal{S}_\ell^U \arrow[r, hook] \arrow[d] & \mathbb{C}^n \otimes \overline{\psi}^*\overline{\phi}^*\mathcal{O}_{\mathcal{C}^U} \arrow[d] \\
\overline{\phi}^*\mathcal{S}_1^U \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & \overline{\phi}^*\mathcal{S}_\ell^U \arrow[r, hook] \arrow[d] & \mathbb{C}^n \otimes \overline{\phi}^*\mathcal{O}_{\mathcal{C}^U} \arrow[d] \\
\overline{\rho}^*\mathcal{S}_1 \arrow[r, hook] & \dots \arrow[r, hook] & \overline{\rho}^*\mathcal{S}_\ell \arrow[r, hook] & \mathbb{C}^n \otimes \overline{\rho}^*\mathcal{O}_{\mathcal{C}}
\end{tikzcd}
$$

where the ^{U} denotes the universal object. The isomorphisms in the middle diagram are equivariant, as are the isomorphisms in the top and bottom diagrams. By *Theorem 4.23*, *Theorem 4.46* of [Vis05], the isomorphisms given by the compositions of the isomorphisms in the columns descend to isomorphisms of sheaves on $\mathcal{C}$ which yield a commutative diagram as desired.

Now, we claim that the induced automorphism of the family of generalized stable quotients determines the automorphism of the principal bundle and the $\mathbf{PGL}(W)$-equivariant morphism.

Over open subschemes of T where the projective bundle is trivial, the automorphism of the underlying curve in each fiber of the family of generalized stable quotients determines the automorphism of the projective space since the curves span $\mathbb{P}(W)$. Since each automorphism on a fiber comes from a global automorphism of the family of generalized stable quotients, these automorphisms of the projective bundle glue. This determines the automorphism of the projective bundle, which in turn determines the automorphism of the principal bundle. This proves that the functor A_T is full.

Suppose we have two automorphisms γ, χ, of the family of generalized stable quotients over T which induce the same automorphism of the principal $\mathbf{PGL}(W)$ bundle and the $\mathbf{PGL}(W)$-equivariant morphism to $\mathcal{Q}'$. Then, we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[d, "\pi"'] \arrow[dr, bend left, "\gamma"] \arrow[dr, bend right, "\chi"'] & & \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[dll] \arrow[dr] & \\
T \arrow[dr, equal] & C \arrow[rr] \arrow[d, "\pi"] & & \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[dll] \\
& T & &
\end{tikzcd}
$$

where the horizontal arrows are the fiberwise embedding of C from before, and the arrow from $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ to itself is the composition

$$\begin{tikzcd} \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[r] \arrow[d, equal] & \mathbb{P}((\pi_*\gamma^*\mathcal{L}_k)^*) \arrow[r] & \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[d, equal] \\ \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[r] & \mathbb{P}((\pi_*\chi^*\mathcal{L}_k)^*) \arrow[r] & \mathbb{P}((\pi_*\mathcal{L}_k)^*). \end{tikzcd}$$

Over open subschemes of T where the associated principal bundle (and hence the projective bundle itself) is trivial, we see that the automorphisms γ, χ are induced by the corresponding automorphism of the projective space on each fiber (since the curve in each fiber of π spans the projective space, specifying the automorphism on the curve is the same as specifying the automorphism of the projective space). This forces them to be equal on the fibers, and thus equal on all of $\mathcal{C}$. Automatically they induce the same isomorphism of the pulled back flag sequence with the original flag sequence. Therefore A_T is also faithful.

We have shown that A_T is fully faithful.

We now show that A_T is essentially surjective.

Given an object in the fiber $[\mathcal{Q}'/\mathbf{PGL}(W)](T)$, we have the following data:

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\rho"] & \mathcal{Q}' \\ T \end{tikzcd}$$

where $\mathcal{P}$ is a principal $\mathbf{PGL}(W)$ bundle and ϕ is a $\mathbf{PGL}(W)$-equivariant morphism.

Now, we can pullback the universal curve together with its sections and the universal flag sequence to yield a family of generalized stable quotients over $\mathcal{P}$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\pi", shift left] & \mathbb{P}(W)\times\mathcal{P} \\ \mathcal{P} \arrow[u, "\sigma_j", bend left] \end{tikzcd}, \quad 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n\otimes\mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0.$$

By definition of $\mathcal{Q}'$, there is an isomorphism of line bundles on $\mathcal{C}$:

$$\gamma : \mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}} \cong \mathcal{L}_k \otimes \pi^*\pi_*(\mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}} \otimes \mathcal{L}_k^*).$$

The embedding $\mathcal{C} \to \mathbb{P}(W)\times\mathcal{P}$ is a $\mathbf{PGL}(W)$-equivariant embedding, since ϕ is $\mathbf{PGL}(W)$-equivariant.

However, this does not imply that $\mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}}$ is a $\mathbf{PGL}(W)$-equivariant line bundle on $\mathcal{C}$, since $\mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)$ does not admit a $\mathbf{PGL}(W)$ linearization ([MF82] pg 33). Instead, we know that $\mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(N)$ admits a $\mathbf{PGL}(W)$ linearization, where $N := dim(W)$ ([Dol03], [MF82]).

Since the embedding of the family of curves $\mathcal{C} \to \mathbb{P}(W) \times \mathcal{P}$ is nondegenerate, we know that we have an exact sequence of sheaves

$$\mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1) \to \mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}} \to 0$$

such that, when we push forward along the map $\psi : \mathbb{P}(W) \times \mathcal{P} \to \mathcal{P}$, we obtain an isomorphism

$$W^* \otimes \mathcal{O} = \psi_* \mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1) \to \pi_* \mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}} \cong \pi_* \mathcal{L}_k \otimes \pi_*(\mathcal{L}_k^* \otimes \mathcal{O}_{\mathbb{P}(W)\times\mathcal{P}}(1)|_{\mathcal{C}}).$$

This yields an isomorphism of projective bundles over $\mathcal{P}$,

$$\overline{\gamma} : \mathbb{P}(W) \times \mathcal{P} \to \mathbb{P}((\pi_* \mathcal{L}_k)^*).$$

We have the family of curves embedded equivariantly in $\mathbb{P}(Sym^N W) \times \mathcal{P}$ together with the $\mathbf{PGL}(W)$ linearization of $\mathcal{O}_{\mathbb{P}(Sym^N W)\times\mathcal{P}}(1)|_{\mathcal{C}}$ on $\mathcal{C}$, along with the $\mathbf{PGL}(W)$ equivariant locally free sheaf $Sym^N W^* \otimes \mathcal{O}_{\mathcal{P}}$ on $\mathcal{P}$. This means that we have an isomorphism

$$\theta : pr_2^* Sym^N W^* \otimes \mathcal{O} \to \sigma^* Sym^N W^* \otimes \mathcal{O}$$

where $\sigma : \mathbf{PGL}(W) \times \mathcal{P} \to \mathcal{P}$ is the action, and $pr_2 : \mathbf{PGL}(W) \times \mathcal{P} \to \mathcal{P}$ is the projection onto the second factor, such that the following diagram ([Vis05] *Proposition 3.49*) commutes

$$\begin{tikzcd}
(pr_2 \circ (\mu \times id_{\mathcal{P}}))^* \mathcal{A} \arrow[rrr, "(\mu \times id_{\mathcal{P}})^* \theta"] \arrow[d, equal] & & & (\sigma \circ (\mu \times id_{\mathcal{P}}))^* \mathcal{A} \arrow[d, equal] \\
(pr_2 \circ pr_{23})^* \mathcal{A} \arrow[dr, "pr_{23}^* \theta"'] & & & (\sigma \circ (id_{\mathbf{PGL}(W)} \times \sigma))^* \mathcal{A} \\
& (\sigma \circ pr_{23})^* \mathcal{A} \arrow[r, equal] & (pr_2 \circ (id_{\mathbf{PGL}(W)} \times \sigma))^* \mathcal{A} \arrow[ur, "(id_{\mathbf{PGL}(W)} \times \sigma)^* \theta"'] &
\end{tikzcd}$$

where $\mathcal{A} := Sym^N W^* \otimes \mathcal{O}$, μ is the group operation, and $pr_3 : \mathbf{PGL}(W) \times \mathbf{PGL}(W) \times \mathcal{P} \to \mathcal{P}$, $pr_{23} : \mathbf{PGL}(W) \times \mathbf{PGL}(W) \times \mathcal{P} \to \mathbf{PGL}(W) \times \mathcal{P}$ are the projections onto the third, second and third factors, respectively.

By *Theorem 4.46* and *Theorem 4.23* of [Vis05], since $QCoh/Spec(\mathbb{C})$ is a stack in the fpqc topology, $\mathcal{P} \to T$ is a $\mathbf{PGL}(W)$ torsor (and thus an fpqc torsor), $\mathcal{A}$ is a $\mathbf{PGL}(W)$-equivariant locally free sheaf, there exists a unique locally free sheaf $\mathcal{M}$ on T together with an isomorphism $\nu : \rho^* \mathcal{M} \to \mathcal{A}$. This yields a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] & \mathbb{P}(\mathcal{A}^*) \arrow[r] \arrow[d] & \mathcal{P} \arrow[d, "\rho"] \\
& \mathbb{P}(\mathcal{M}^*) \arrow[r] & T
\end{tikzcd}$$

where the right square is a fiber diagram.

Since $\mathcal{C} \to \mathbb{P}(\mathcal{A}^*)$ is a $\mathbf{PGL}(W)$-equivariant embedding, $Aff/Spec(\mathbb{C})$ is a stack in the fpqc topology ([Vis05] *Theorem 4.33*), $\mathbb{P}(\mathcal{A}^*) \to \mathbb{P}(\mathcal{M}^*)$ is a $\mathbf{PGL}(W)$ torsor (by the cartesian property of the diagram above), then by *Theorem 4.46* of [Vis05], we see that there exists a unique affine scheme $\mathcal{C}'$ over $\mathbb{P}(\mathcal{M}^*)$ together with an isomorphism $\mathcal{C}' \times_{\mathbb{P}(\mathcal{M}^*)} \mathbb{P}(\mathcal{A}^*) \cong \mathcal{C}$. By [Sta17, Tag 02YJ], being a closed immersion is local in the fpqc topology, so $\mathcal{C}'$ is a closed subscheme of $\mathbb{P}(\mathcal{M}^*)$. By [Sta17, Tag 0C58], being an at worst nodal curve is fpqc local on the target, so $\mathcal{C}'$ is a family of nodal curves. It is clear that the genus of all fibers is zero, since we can consider any fiber of π mapping to the fiber of $\mathcal{C}' \to T$: this will be an isomorphism of curves over $Spec(\mathbb{C})$. Using the same arguments applied to the sections σ_j, together with [Sta17, Tag 02YJ], we see that we have disjoint sections of $\pi' : \mathcal{C}' \to T$ which do not pass through the nodes of the family by the same argument as above (looking at a fiber of π mapping to the fiber of π').

Since the morphism $\mathcal{P} \to \mathcal{Q}'$ is $\mathbf{PGL}(W)$ equivariant, and the universal sequence of sheaves on $\mathcal{C}^U$ is $\mathbf{PGL}(W)$-equivariant, this implies that the flag sequence

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0$$

is a $\mathbf{PGL}(W)$-equivariant sequence of equivariant sheaves.

By *Theorem 4.23, Theorem 4.46* of [Vis05], the sequence over $\mathcal{C} \to \mathcal{P}$ descends to a flag sequence

$$0 \hookrightarrow \mathcal{S}'_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}'_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}'_1 \to \ldots \to \mathcal{Q}'_\ell \to 0$$

over $\mathcal{C}'$, such that the pullback of this flag sequence to $\mathcal{C}$ is equivalent to the original flag sequence on $\mathcal{C}$. The flag sequence has all the desired properties since $\mathcal{C} \to \mathcal{C}'$ is fpqc.

Stability on each fiber of π' follows from considering stability on a fiber of π mapping to the corresponding fiber of π' (the relative dualizing sheaf is functorial, the sections pull back to the corresponding sections, and the flag sequence pulls back to an equivalent flag sequence).

We would now like to show that, given this data of a family of generalized stable quotients over T, we can recover the principal bundle $\rho : \mathcal{P} \to T$ and the equivariant map $\phi : \mathcal{P} \to \mathcal{Q}'$. Since pulling back $\mathbb{P}(({\pi'}_*{\mathcal{L}'}_k)^*)$ along $\mathcal{P} \to T$ yields a projective bundle isomorphic to $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$, which in turn is isomorphic to

$\mathbb{P}(W) \times \mathcal{P}$, then by the universal property of the associated principal $\mathbf{PGL}(W)$ bundle to $\mathbb{P}((\pi'_*\mathcal{L}'_k)^*)$, which we call

$$\mathcal{P}' := Isom(\mathbb{P}((\pi'_*\mathcal{L}_k)^*), \mathbb{P}(W) \times T)$$

we get a unique morphism

$$\epsilon : \mathcal{P} \to \mathcal{P}'$$

such that the isomorphism

$$\mathbb{P}((\pi'_*\mathcal{L}'_k)^*) \times_T \mathcal{P} \to \mathbb{P}(W) \times \mathcal{P}$$

is a pull back of the universal isomorphism

$$\mathbb{P}((\pi'_*\mathcal{L}'_k)^*) \times_T \mathcal{P}' \to \mathbb{P}(W) \times \mathcal{P}'$$

over $\mathcal{P}'$. We seek to prove that ϵ is equivariant, from which it follows that ϵ is an isomorphism.

Notice that we can factor ϵ as

$$\mathcal{P} \to Isom(\mathbb{P}((\pi_*\mathcal{L}_k)^*), \mathbb{P}(W) \times \mathcal{P}) \cong Isom(\mathbb{P}((\pi'_*\mathcal{L}'_k)^*), \mathbb{P}(W) \times T) \times_T \mathcal{P} \to \mathcal{P}'$$

where the first arrow comes from the trivialization given as part of the data of the equivariant morphism $\phi : \mathcal{P} \to \mathcal{Q}'$, the second arrow is the induced isomorphism by the canonical isomorphism $\rho^*\pi'_*\mathcal{L}'_k \cong \pi_*\mathcal{L}_k$, and the last morphism is the projection of the fiber product onto the first factor. The last two morphisms are $\mathbf{PGL}(W)$ equivariant, where the action on $\mathcal{P}' \times_T \mathcal{P}$ is the diagonal action. We just need to see that the arrow $\mathcal{P} \to \mathcal{P}' \times_T \mathcal{P}$ is $\mathbf{PGL}(W)$ equivariant. We have a fiber diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\rho'"] \arrow[d, "\pi"] & \mathcal{C}' \arrow[d, "\pi'"] \\ \mathcal{P} \arrow[r, "\rho"] & T \end{tikzcd}$$

and $\rho^*\pi'_*\mathcal{L}'_k \cong \mathcal{L}_k$. The family of curves $\mathcal{C}$ is embedded equivariantly in $\mathbb{P}(W)$. Via the map $\mathbb{P}((\pi_*\mathcal{L}_k)^*) \cong \mathbb{P}((\pi'_*\mathcal{L}'_k)^*) \times_T \mathcal{P}$, we see that for each collection of fibers of $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ which lie over points in the same fiber of ρ, we get a canonical identification $\mathbb{P}((\pi_*\mathcal{L}_k)^*)_p \cong \mathbb{P}((\pi'_*\mathcal{L}'_k)^*)_t$, where $\rho(p) = t$. From the diagram above, the corresponding fiber of π embedded in $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ is isomorphic to the fiber of π' over t embedded in $\mathbb{P}((\pi'_*\mathcal{L}'_k)^*)_t$. We have a commutative diagram

$$
\begin{tikzcd}
 & \mathcal{C}_p \arrow[rr] \arrow[dl] & & \mathbb{P}((\pi_*\mathcal{L}_k)^*)_p \arrow[rr] \arrow[dl] & & \mathbb{P}(W) \arrow[dl, equal] \arrow[dd, "g\cdot"] \\
\mathcal{C}'_t \arrow[rr] \arrow[dd, equal] & & \mathbb{P}((\pi'_*\mathcal{L}'_k)^*)_t \arrow[rr] \arrow[dd, equal] & & \mathbb{P}(W) \arrow[dd] & \\
 & \mathcal{C}_{g\cdot p} \arrow[rr] \arrow[dl] & & \mathbb{P}((\pi_*\mathcal{L}_k)^*)_{g\cdot p} \arrow[rr] \arrow[dl] & & \mathbb{P}(W) \arrow[dl, equal] \\
\mathcal{C}'_t \arrow[rr] & & \mathbb{P}((\pi'_*\mathcal{L}'_k)^*)_t \arrow[rr] & & \mathbb{P}(W) &
\end{tikzcd}
$$

from which it follows that the leftmost vertical arrow from $\mathbb{P}(W)$ to itself must be multiplication by g. This shows that the arrow

$$\mathcal{P} \to Isom(\mathbb{P}((\pi'_*\mathcal{L}'_k)^*), \mathbb{P}(W)) \times_T \mathcal{P}$$

is $\mathbf{PGL}(W)$-equivariant. Thus, the composition from $\mathcal{P}$ to $\mathcal{P}'$ is $\mathbf{PGL}(W)$-equivariant. It follows that $\mathcal{P} \cong Isom(\mathbb{P}((\pi'_*\mathcal{L}'_k)^*), \mathbb{P}(W) \times T)$.

Therefore, if we form the associated principal bundle to $\mathbb{P}((\pi'_*\mathcal{L}'_k)^*)$ we recover a principal bundle isomorphic to $\mathcal{P}$, and we recover a family isomorphic to the original family over $\mathcal{P}$, which yields the equivariant map to $\mathcal{Q}'$. This proves that A_T is essentially surjective. □

This proves Proposition 0.5.1.

1.3 The equivalence of categories $\mathbf{Qmap}_{g,m}(V/\!/G, \overline{d}) \cong \overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$

We begin by defining a functor

$$F : \mathrm{Qmap}_{g,m}(V/\!/G, \overline{d}) \to \overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}).$$

Suppose we are given a family of stable quasimaps to $V/\!/G$ of class $\overline{d} \in \mathrm{Hom}_{\mathbb{Z}}(\chi(G), \mathbb{Z})$, which amounts to:

- A proper, flat family of connected, reduced, at worst nodal curves of arithmetic genus g with m distinct sections σ_j $\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ T \arrow[u, bend left, "\sigma_j"] \end{tikzcd}$
- a principal G bundle over $\mathcal{C}$ together with a G-equivariant morphism to V

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\rho"] & V . \\ \mathcal{C} & \end{tikzcd}$$

Equivalently, this yields a section $u : \mathcal{C} \to \mathcal{P} \times_G V$ of the induced vector bundle on $\mathcal{C}$ such that when restricted to a fiber C_t, several conditions are satisfied:

- $(\mathcal{P}, u)$ has class $\overline{d}$
- u maps the generic point of each fiber into the stable locus of V
- if $u(x) \in V^{us}$, then x is not one of the nodes or markings of the fiber.

- Furthermore, the line bundle $\omega_\pi(\sum_{j=1}^{m} \sigma_j) \otimes u^*(\mathcal{P} \times_G L_\chi)^\epsilon$ is π relatively ample $\forall \epsilon \in \mathbb{Q}_{>0}$.

The data of the principal G bundle allows us to construct a vector bundle $\mathcal{P} \times_G \bigoplus_{i=1}^{\ell} \mathbb{C}^{r_i} \cong \bigoplus_{i=1}^{\ell} \mathcal{S}_i$ (it splits into a direct sum of rank r_i vector bundles since G is the product $\prod_{i=1}^{\ell} GL(r_i, \mathbb{C})$).

The fiber bundle $\mathcal{P} \times_G V \to \mathcal{C}$ splits into a direct sum $\bigoplus_{j=1}^{\ell} \mathcal{W}_i$, where each $\mathcal{W}_i$ has rank $r_i \cdot r_{i+1}$, based on the fact that the G action preserves the splitting of V. Notice that the $\mathcal{W}_i$ have transition functions given by the inverse of the transition functions for $\mathcal{S}_i$ tensored with the transition functions for $\mathcal{S}_{i+1}$. From this description, it is clear that each $\mathcal{W}_i \cong \mathcal{H}om(\mathcal{S}_i, \mathcal{S}_{i+1})$. The section $u \in H^0(\bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i, \mathcal{S}_{i+1})) = \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathcal{S}_i, \mathcal{S}_{i+1})$ defines morphisms

$$f_i : \mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$$

such that, when restricted to fibers of π, each f_i drops rank at only finitely many points of the fiber which are away from the nodes and markings of C (by the conditions imposed on u, see the Appendix for details).

Therefore we get a flag of subsheaves of $\mathbb{C}^n \otimes \mathcal{O}$ on $\mathcal{C}$:

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$

such that the $\mathcal{S}_i$ fail to be subbundles of each other at only finitely many points of each fiber, away from the nodes and markings.

Recall that $\beta = (d_1, \ldots, d_\ell)$, and the character χ used in the GIT linearization is the product of the determinants $\prod_{i=1}^{\ell} det_i$. Returning to the construction of the

GIT quotient, we have that $L_\chi := \mathbb{C}_\chi \times V \cong \bigotimes_{i=1}^{\ell} det((\mathbb{C}^{r_i}_{GL(r_i,\mathbb{C})} \times V)^*)$. Thus, the line bundle $\mathcal{P} \times_G L_\chi$ over $\mathcal{P} \times_G V$ is isomorphic to $\bigotimes_{i=1}^{\ell} det(\mathcal{P} \times_G (\mathbb{C}^{r_i}_{GL(r_i,\mathbb{C})} \times V)^*)$. Then, pulling this back to C under u, we see that this is isomorphic to $\bigotimes_{i=1}^{\ell} det(\mathcal{S}_i^*)$, and the condition $\beta = (d_1, \dots, d_\ell)$ translates into $deg(\mathcal{S}_i^*) = d_i$ on the fibers of π. Thus, $\omega_\pi(\sum_{j=1}^{m} \sigma_j) \otimes u^*(\mathcal{P} \times_G L_\chi)^\epsilon \cong \omega_\pi(\sum_{j=1}^{m} \sigma_j) \otimes (\bigotimes_{i=1}^{\ell} det(\mathcal{S}_i^*))^\epsilon$ is ample when restricted to the fibers of π.

This defines a functor since the formation of the associated vector bundle to a principal G bundle and taking determinants commute with base change.

Proposition 1.3.1. *F yields an equivalence of categories*

$$Qmap_{g,m}(V /\!/ G, \overline{d}) \cong \overline{\mathcal{Q}}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}).$$

Proof. By *Lemma 5.1* in *Chapter XII* of [ACG11], we must show that F_T is an equivalence of categories, for any scheme T.

We must first show that there is a bijection

$$F_T : Hom_{\mathrm{Qmap}(T)}((\pi, \sigma_j, \rho, \phi, u), (\pi, \sigma_j, \rho, \phi, u)) \cong Hom_{\overline{\mathcal{Q}}(T)}((\pi, \sigma_j, q_i), (\pi, \sigma_j, q_i)).$$

Suppose we are given a family of stable quasimaps (†):

$$\begin{tikzcd} & \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\rho"] & V \end{tikzcd}, \quad \begin{tikzcd} \mathcal{P} \times_G V \arrow[d, shift left] \\ \mathcal{C} \arrow[u, bend left, "u"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\sigma_j"] & \mathcal{C} \arrow[l, bend left, "\pi"] \end{tikzcd}$$

Then, form the associated family of stable quotients. Fix an automorphism ψ of the associated family of generalized stable quotients

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ T \arrow[u, bend left, "\sigma_j"] \end{tikzcd}, \quad 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \dots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \dots \to \mathcal{Q}_\ell \to 0$$

This consists of an automorphism of the family of curves

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathcal{C} \arrow[d, "\pi"] \\ T \arrow[u, bend left, "\sigma_j"] \arrow[r, equal] & T \arrow[u, bend left, "\sigma_j"] \end{tikzcd}$$

such that we have a commutative diagram

$$\begin{tikzcd} \psi^*\mathcal{S}_1 \arrow[r, hook] \arrow[d, "\psi_1"] & \dots \arrow[r, hook] & \psi^*\mathcal{S}_\ell \arrow[r, hook] \arrow[d, "\psi_\ell"] & \mathbb{C}^n \otimes \psi^*\mathcal{O} \arrow[d] \\ \mathcal{S}_1 \arrow[r, hook] & \dots \arrow[r, hook] & \mathcal{S}_\ell \arrow[r, hook] & \mathbb{C}^n \otimes \mathcal{O} \end{tikzcd}$$

where the last vertical arrow is the canonical isomorphism.

We can take the direct sum of the subsheaves in both rows and we get an isomorphism

$$\psi^* \bigoplus_{i=1}^{\ell} \mathcal{S}_i \to \bigoplus_{i=1}^{\ell} \mathcal{S}_i.$$

Passing to the GL bundles of frames, we obtain an isomorphism $\overline{\psi} : \psi^*\mathcal{P} \to \mathcal{P}$. We can consider the inverse of this map

$$\overline{\psi}^{-1} : \mathcal{P} \to \psi^*\mathcal{P}$$

(via the isomorphisms ψ_i^{-1}).

The following diagram

$$\begin{tikzcd} \mathcal{P} \times_G V \arrow[r, "\overline{\psi}^{-1}"] & \mathcal{P} \times_G V \\ \mathcal{C} \arrow[u, "u"] \arrow[ur, "\psi^*(u)"'] & \end{tikzcd}$$

is commutative (where u, $\psi^* u$ are induced by the flag sequence and its pullback). Thus we get an automorphism of the associated stable quasimap. Given the isomorphism of principal bundles and the commutative diagram as above, we see that this data is equivalent to giving an automorphism of the family of curves and sections which yields an isomorphism between the flag sequence (coming from u) and the pullback of the flag sequence along ψ (coming from $\psi^*(u)$). This proves that F_T is full.

To see that it is faithful, suppose we are given an automorphism ψ of the family of stable quasimaps over T (†). The automorphism consists of the following data

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\psi"] \arrow[d, shift left, "\pi"] & \mathcal{C} \arrow[d, shift left, "\pi"] \\ T \arrow[u, bend left, "\sigma_j"] \arrow[r, equal] & T \arrow[u, bend left, "\sigma_j"] \end{tikzcd}, \ \overline{\gamma} : \mathcal{P} \to \psi^*\mathcal{P} \text{ which induces } \begin{tikzcd} \mathcal{P} \times_G V \arrow[r, "\overline{\gamma}"] & \mathcal{P} \times_G V \\ \mathcal{C} \arrow[u, "u"] \arrow[ur, "\psi^*(u)"'] & \end{tikzcd}$$

We form the associated vector bundle to $\mathcal{P}$. This splits as a direct sum $\bigoplus_{i=1}^{\ell} \mathcal{S}_i$.

$\overline{\gamma}$ yields isomorphisms between each of the direct summands $\gamma_i : \mathcal{S}_i \to \psi^*\mathcal{S}_i$. The condition that the second diagram commutes implies that we have a commutative diagram

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, hook] \arrow[d, "\gamma_1"] & \ldots \arrow[r, hook] & \mathcal{S}_\ell \arrow[r, hook] \arrow[d, "\gamma_\ell"] & \mathbb{C}^n \otimes \mathcal{O} \arrow[d, equal] \\ \psi^*\mathcal{S}_1 \arrow[r, hook] & \ldots \arrow[r, hook] & \psi^*\mathcal{S}_\ell \arrow[r, hook] & \mathbb{C}^n \otimes \mathcal{O} \end{tikzcd}$$

where the horizontal maps between the sheaves are induced by the sections u, $\psi^* u$, and the vertical maps are the isomorphisms induced by $\overline{\gamma}$. By taking the inverse of the γ_i's, call them μ_i, together with the automorphism of the family of curves

ψ we get an automorphism of the family of generalized stable quotients.

Given this automorphism, we get an isomorphism $\mu_i^{-1} : \mathcal{S}_i \to \psi^* \mathcal{S}_i$ for each $1 \le i \le \ell$. This induces an isomorphism $\overline{\mu}^{-1} : \bigoplus_{i=1}^{\ell} \mathcal{S}_i \to \psi^* \bigoplus_{i=1}^{\ell} \mathcal{S}_i$, which in turn induces an isomorphism of the associated principal bundles $\overline{\mu}^{-1} : \mathcal{P} \to \psi^* \mathcal{P}$. From the equivalence of categories between the category of vector bundles over $\mathcal{C}$ and the category of principal GL bundles over $\mathcal{C}$, we recover $\overline{\psi}$ as $\overline{\mu}^{-1}$. We recover the section u via the homomorphisms $\mathcal{S}_i \to \mathcal{S}_{i+1}$. The commutative diagram of sheaves above yields the commutativity of the diagram

$$\begin{tikzcd} \mathcal{P} \times_G V \arrow[r, "\overline{\gamma}"] & \mathcal{P} \times_G V . \\ \mathcal{C} \arrow[u, "u"] \arrow[ur, "\psi^*(u)"] & \end{tikzcd}$$

Therefore the functor F_T is faithful.

To see that the functor is essentially surjective, fix a family of generalized stable quotients. Since the functor F_T preserves the family of curves, all we have to see is that given a flag sequence over $\mathcal{C}$, we can produce a principal bundle over $\mathcal{C}$ with a G-equivariant morphism to V and a section of the vector bundle $\mathcal{P} \times_G V$ such that when we form the associated bundle to $\mathcal{P}$ and we consider the induced morphisms between its direct summands (by the section u), we recover the flag sequence. This can be done as follows.

Given a family of generalized stable quotients, (we omit the family of curves)

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0$$

we define $\mathcal{P}$ to be the associated principal bundle to $\bigoplus_{i=1}^{\ell} \mathcal{S}_i$. Notice that $\mathcal{P} \times_G V \cong \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i, \mathcal{S}_{i+1})$, and this vector bundle has a global section u induced by the flag sequence on $\mathcal{C}$. Now, if we take $\mathcal{P}$ and form the associated vector bundle, we get back a bundle isomorphic to $\bigoplus_{i=1}^{\ell} \mathcal{S}_i$. The section $u : \mathcal{C} \to \mathcal{P} \times_G V \cong \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i, \mathcal{S}_{i+1})$ yields homomorphisms between the direct summands, and since u gave the data of the homomorphisms of the original flag sequence, we recover the flag sequence on the family $\mathcal{C}$. $\square$

1.4 Smoothness for $g = 0$

We will prove that the moduli stack is smooth over $Spec(\mathbb{C})$ when $g = 0$. Notice that, as in [MOP11], the moduli stack is equipped with a morphism to the Artin stack of prestable curves,

$$\nu : \overline{Q}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \to \mathcal{M}_{0,m}^{pre}$$

the latter of which is known to be smooth [Beh97]. Therefore we just need to show that the morphism ν is smooth. We will use an analog of [MOP11] *Theorem 2*.

Let C be the underlying curve to any genus 0 generalized stable quotient. Notice that the fibers of ν are open subschemes of HyperQuot schemes over the curves corresponding to the points in $\mathcal{M}_{0,m}^{pre}$. In [Lau88] *Proposition 2.5*, [CF99], *Proposition E*, it is shown that

- the tangent space to the HyperQuot scheme $HQuot_C(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ at a point
$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0$$
is given by the global sections of the kernel $\mathcal{K}$ in the following exact sequence
$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i,\, \mathcal{Q}_i) \to \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\mathcal{S}_j,\, \mathcal{Q}_{j+1}) \to 0,$$
where the last map precomposes a section ϕ_i of $\mathcal{H}om(\mathcal{S}_i, \mathcal{Q}_i)$ with the map $\mathcal{S}_{i-1} \hookrightarrow \mathcal{S}_i$ and takes the difference of this section with the section obtained by post-composing ϕ_{i-1} with the map $\mathcal{Q}_{i-1} \to \mathcal{Q}_i$

- the HyperQuot scheme is smooth at this point if $\mathrm{Ext}^1(\mathcal{S}_i,\, \mathcal{Q}_i) = 0$ for all i and $H^1(\mathcal{K}) = 0$.

We will show that both of these are satisfied. For the first, we have the exact sequences

$$0 \to \mathcal{S}_i \otimes \mathcal{S}_i^* \to \mathbb{C}^n \otimes \mathcal{S}_i^* \to \mathcal{S}_i^* \otimes \mathcal{Q}_i \to 0 \qquad .$$

We see that we have another exact sequence obtained by dualizing the original quotient sequences

$$0 \to \mathcal{H}om(\mathcal{Q}_i, \mathcal{O}) \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{S}_i^* \to \mathcal{E}xt^1(\mathcal{Q}_i, \mathcal{O}) \to 0.$$

Since the arithmetic genus is 0, $\mathcal{O}$ has no higher cohomology. We can split this up into two exact sequences, which, after taking cohomology, yield $\implies$

$$H^1(\mathbb{C}^n \otimes \mathcal{O}) \to H^1(im) \to 0$$
$$H^1(im) \to H^1(\mathcal{S}_i^*) \to H^1(\mathcal{E}xt^1(\mathcal{Q}_i, \mathcal{O})) \to 0.$$

The first term in the first sequence is 0 as we saw above, which forces the second term to be 0 as well. Since $\mathcal{Q}_i$ only fails to be locally free at finitely many points of C, the last term in the second sequence is also zero. Thus, $H^1(\mathcal{S}_i^*) = 0$ for all i which forces $H^1(\mathcal{S}_i^* \otimes \mathcal{Q}_i) = 0$ by the first sequence. Since $\mathcal{S}_i$ is locally free, $\operatorname{Ext}^1(\mathcal{S}_i, \mathcal{Q}_i) \cong H^1(\mathcal{S}_i^* \otimes \mathcal{Q}_i) = 0$.

To prove the next statement, we use the method of [CF99] : there is a morphism $\rho : \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O} \to \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i, \mathcal{Q}_i)$ given by mapping ϕ to the precomposition of ϕ with $\mathcal{S}_i \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$ followed by the projection onto $\mathcal{Q}_i$. This yields a commutative diagram

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[r, hook] \arrow[d, hook] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r, "\phi"] \arrow[d, equal] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & \mathcal{Q}_1 \arrow[d] \\
\vdots \arrow[d, hook] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d] \\
\mathcal{S}_\ell \arrow[r, hook] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r, "\phi"] & \mathbb{C}^n \otimes \mathcal{O} \arrow[r] & \mathcal{Q}_\ell
\end{tikzcd}$$

from which it immediately follows that $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O} \to \bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i, \mathcal{Q}_i)$ factors through $\mathcal{K}$. We claim that this map only fails to be surjective at finitely many points of C, and these points are exactly the points where $\mathcal{Q}_i$ fails to be locally free. Away from this locus, the inclusions $\mathcal{S}_i \hookrightarrow \mathcal{S}_{i+1}$ are inclusions of vector bundles. Choose such a point where all $\mathcal{Q}_i$ are locally free.

Suppose we are given a collection of homomorphisms of sheaves $(\{\phi\}_{i=1}^{\ell})$ which are sections of the kernel $\mathcal{K}$ over some open subset $U \subset C$. We will show that their images in the stalk $\mathcal{K}_p$ are in the image of $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O}_{C,p}$. Since ϕ is a section of $\mathcal{K}$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{S}_{1p} \arrow[r, hook] \arrow[d, hook] \arrow[rrr, bend left, "\phi_1"] & \mathbb{C}^n \otimes \mathcal{O}_{C,p} \arrow[d, equal] & \mathbb{C}^n \otimes \mathcal{O}_{C,p} \arrow[r, two heads] \arrow[d, equal] & \mathcal{Q}_{1p} \arrow[d, two heads] \\
\vdots \arrow[d, hook] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d, two heads] \\
\mathcal{S}_{\ell p} \arrow[r, hook] \arrow[rrr, bend right, "\phi_\ell"'] & \mathbb{C}^n \otimes \mathcal{O}_{C,p} & \mathbb{C}^n \otimes \mathcal{O}_{C,p} \arrow[r, two heads] & \mathcal{Q}_{\ell p}
\end{tikzcd}$$

We would like to fill in this diagram with a single morphism $\mathbb{C}^n \otimes \mathcal{O}_{C,p} \to \mathbb{C}^n \otimes \mathcal{O}_{C,p}$.

Choose generators for $\mathcal{S}_{1p}$, $\{\alpha_j\}_{j=1}^{r_1}$ as an $\mathcal{O}_{C,p}$ module (it is free). Since the vertical arrows on the left side have free cokernels, inductively we can choose generators for each subsequent $\mathcal{S}_{i\,p}$ such that the first $i-1$ generators are given by the images of $\{\alpha_j\}_{j=1}^{r_{i-1}}$, which we will also denote by α_j. Also, since the vertical arrows on the right side have free kernels, we can choose generators $\{\beta_k\}_{k=1}^{n-r_1}$ for $\mathcal{Q}_{1\,p}$ such that the images of $\{\beta_k\}_{k=r_i-r_1}^{n-r_1}$ in $\mathcal{Q}_{i\,p}$ generate it as a $\mathcal{O}_{C,p}$ module, which we will also denote by β_k.

Next, we pick preimages of β_k in $\mathbb{C}^n \otimes \mathcal{O}_{C_p}$, call them γ_k. Now, we can write the ϕ_i images of $\{\alpha_j\}_{j=1}^{r_i}$ in $\mathcal{Q}_i$ in terms of $\{\beta_k\}_{k=r_i-r_1}^{n-r_1}$, say $\phi(\alpha_j) = \sum\limits_{k=r_i-r_1}^{n-r_1} r_{j,k}\beta_k$. Then, we define a map from $\mathcal{S}_{i\,p}$ to $\mathbb{C}^n \otimes \mathcal{O}_{C,p}$ by $\psi_i(\alpha_j) = \sum\limits_{k=r_i-r_1}^{n-r_1} r_{j,k}\gamma_k$. Notice that these maps are compatible with ϕ, the injections $\mathcal{S}_{i\,p} \subset \mathcal{S}_{i+1\,p}$, and the surjections $\mathcal{Q}_{i\,p} \to \mathcal{Q}_{i+1\,p}$.

We can pick generators of the orthogonal complement of $\mathcal{S}_{\ell p}/m_p\mathcal{S}_{\ell p} \subset \mathbb{C}^n \otimes \mathbb{C}_p$ and lift these to elements of $\mathbb{C}^n \otimes \mathcal{O}_{C,p}$. By Nakayama's lemma, the collection of generators for the submodule $\mathcal{S}_{\ell p}$ and the elements we lifted from $\mathbb{C}^n \otimes \mathbb{C}_p$ generate $\mathbb{C}^n \otimes \mathcal{O}_{C,p}$. We can define the map from $\mathbb{C}^n \otimes \mathcal{O}_{C,p}$ to $\mathbb{C}^n \otimes \mathcal{O}_{C,p}$ by $\psi|_{\mathcal{S}_{i\,p}} = \psi_i$, and $\psi|_{\mathcal{S}^c_{\ell p}} = 0$, where $\mathcal{S}^c_{\ell p}$ denotes the submodule generated by the lifts of the generators of the orthogonal complement. It is clear that this map makes the diagram commutative, so it is a lift of ϕ.

Therefore we have an exact sequence

$$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O} \to \mathcal{K} \to \tau \to 0$$

where the cokernel τ is a torsion sheaf supported on finitely many points of C. Since $H^1(C, \mathcal{O}) = 0$, then $H^1(C, im(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O} \to \mathcal{K})) = 0$. Since τ is a torsion sheaf supported on finitely many points, it too has no higher cohomology. Thus, $H^1(\mathcal{K}) = 0$.

This shows that the moduli stack is smooth over $\mathcal{M}^{pre}_{0,m}$.

2 Projectivity

In this chapter, we will prove that the coarse moduli space of generalized stable quotients to the flag variety is projective, using the construction of the moduli stack as a quotient stack from the previous chapter. This will enable us to use the homology basis theorem from [Opr06a] in the next chapter to compute the second cohomology group of the stack (with coefficients in $\mathbb{Q}$), which we use to determine the rank of the rational Picard group.

2.1 Semipositivity

We will produce a semipositive vector bundle, functorial under base change, on the base of every family of generalized stable quotients, using the methods [Kol90], [FP97]. In the next section we will use this to show that we have a set theoretic classifying map to a product of Grassmannians coming from taking frames of quotients of vector bundles, where the quotients are semipositive. Then, we will use this to produce an ample line bundle.

Definition 2.1.1. *A vector bundle $\mathcal{V}$ on a scheme S is said to be semipositive if $\forall$ smooth complete curve C and $\forall f : C \to S$, every quotient bundle of $f^*\mathcal{V}$ has nonnegative degree.*

The following are *Proposition 3.3* , *Corollary 3.4i*, and *Proposition 4.7* of [Kol90]

Proposition 2.1.1. *A locally free sheaf $\mathcal{V}$ on a scheme T is semipositive if any one of the following conditions is satisfied:*

- $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ *is nef on* $\mathbb{P}_T(\mathcal{V})$*.*

- $\forall$ *map from a proper curve* $\phi : C \to T$*, every quotient bundle of* $\phi^*\mathcal{V}$ *has nonnegative degree.*

- $\forall$ *map from a proper curve* $\phi : C \to T$*, every quotient line bundle of* $\phi^*\mathcal{V}$ *has nonnegative degree.*

- $\forall$ *map from a proper curve* $\phi : C \to \mathcal{T}$ *and* $\forall$ *ample line bundle* $\mathcal{L}$ *on* C *the bundle* $\mathcal{L} \otimes \phi^*\mathcal{V}$ *is ample.*

Lemma 2.1.1. *Quotients and extensions of semipositive vector bundles are semipositive.*

Proposition 2.1.2. *Let* $f : S \to C$ *be a map from a smooth complete surface to a smooth curve. Assume that the general fiber of* f *is smooth. Let* $\{C_i\}$ *be a set of distinct sections of* f*. Then*

$$f_*(\omega^k_{S/C}(\sum a_i C_i))$$

is semipositive provided that $k \geq 2$ *and* $0 < a_i \leq k$ *for every* i*.*

Now, we have the main result of this section

Proposition 2.1.3. *Let* $\pi : C \to T$ *be a proper, flat family of genus* g *quasi-stable curves over a smooth curve* T *,* $p_i : T \to C$ *be disjoint sections of* π *for* $i = 1, \cdots, m$*, and*

$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_C \to \mathcal{Q}_1 \to \ldots \mathcal{Q}_\ell \to 0$$

be a family of degree $(d_1, \ldots, d_\ell)$ *stable quotients to the partial flag variety* $Fl(\overline{r}, \mathbb{C}^n)$*. Let* P_i *be the images of the sections* p_i*. Define*

$$\mathcal{L}_k := \omega^f_\pi(\sum_{i=1}^m f\, P_i) \otimes (\bigotimes_{h=1}^\ell det(\mathcal{S}^*_h))^k$$

where $f = k(d+1)$*, and* $d = \sum_{h=1}^\ell d_h$*. Then the pushforward* $\pi_*\mathcal{L}_k$ *is a semipositive vector bundle on* T*,* $\forall k \geq 5$*.*

- The basic strategy is to express $\pi_*\mathcal{L}_k$ as an extension of semipositive vector bundles then apply *Corollary 3.4* of [Kol90].

- We start by reducing to the case where the family of stable quotients is glued from families of stable quotients each of whose total space and general fiber are smooth.

- From here we can apply *Proposition 4.7* of [Kol90].
- Then, using a relative version of the residue sequence for families of curves, we will be able to express our vector bundle as an extension of two semipositive vector bundles, allowing us to apply *Corollary 3.4i* of [Kol90].

Proof. First, I claim that $\pi_*\mathcal{L}_k$ commutes with base change.

We know by *Lemma 5* from [MOP11] that if $k \geq 5$ and $f = k(d+1)$, then $\mathcal{L}_k|_{C_t}$ is very ample with no higher cohomology, where

$$\mathcal{L}_k|_{C_t} \cong \omega_{C_t}^f\left(\sum_{i=1}^m f\, p_i\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*)\right)^k.$$

Then $R^1\pi_*\mathcal{L}_k$ vanishes and $\pi_*\mathcal{L}_k$ is locally free by *Corollary 1.5* of [Oss]. By *Theorem 1.1* in [Oss], $R^i\pi_*\mathcal{L}_k$ are flat over T $\forall i$, so cohomology and base change commute for $\mathcal{L}_k$ in all degrees.

We start with some reductions to bring us to the case where our family is glued from a collection of generically smooth families of curves. These follow the reduction steps in the proof of semipositivity in [FP97], as well as the proof of *Lemma 5.5* in *Chapter XIV* of [ACG11], pg 427.

Let Z be the union of the one dimensional components of the locus of nodes in the fibers of π. Since the fibers are at worst nodal, Z is unramified over T [Sta17, Tag 0C58]. We can base change under a finite map $g : V \to T$ so that Z splits up into a union of sections of π.

Next, we can perform a base change, possibly ramified at certain points of the base, so as to kill off the monodromy in the branches of the singular locus. We normalize the base to obtain a smooth curve as our base. Thus the 1 dimensional components of the nodal locus are smooth sections and the resulting family, call it $C' \to V$, has the property that there are two distinct everywhere defined branches of each nodal section.

This way, when we normalize the total space of the family along these sections, the family splits up into a disjoint union of components, and the preimages of the nodal sections remain sections.

Let P_j^n be the sections of $C' \to V$ which are the 1 dimensional components of the nodal locus. We already know that the bundle $\pi_*\mathcal{L}_k$ commutes with base change, where we have the fiber diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ V \arrow[r] & T \end{tikzcd}$$

I claim that it suffices to prove that $\pi'_*\mathcal{L}_k$ is semipositive on V.

Suppose $\pi'_*\mathcal{L}_k$ is semipositive on V. Now, let $f : T' \to T$ be a map from a projective curve T' to T. Suppose we have a quotient $f^*\pi_*\mathcal{L}_k \to \mathcal{N}$. Form the fiber diagram

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "f'"] & T' \arrow[d, "f"] \\ V \arrow[r] & T \end{tikzcd}$$

The base change V' will also be a complete curve, although it may be singular, so we can normalize it. Now, we can pull back $\mathcal{N}$ to V'^{ν} (where the ν refers to the normalization) and, by semipositivity of $\pi'_*\mathcal{L}_k$ on V, we obtain that the pullback of $\mathcal{N}$ to V'^{ν} has nonnegative degree. Since the map $V'^{\nu} \to T'$ has finite degree, using the push-pull formula we see that the degree of $\mathcal{N}$ on T' is nonnegative. Thus, we can prove the result after finite base change, so we replace $C \to T$ with $C' \to V$.

Now, we normalize C along $\bigcup\limits_j P_j^n$

$$\begin{tikzcd} C^{\nu} \arrow[r, "\nu"] \arrow[dr, "\pi_{\coprod\limits_v}"'] & C \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

to obtain $C^{\nu} = \coprod\limits_v C_v$, which is a disjoint union of families of curves whose general fibers are irreducible nonsingular curves.

Notice that the preimages of P_j^n are disjoint sections of the projection map $\pi_{\coprod\limits_v}$, call them $P_{v,j,b}^n$. Here v denotes which component of C^{ν} the section lies on, j denotes which section the section corresponds to in the original family π, and b denotes which of the two preimages of P_j^n we are referring to.

After pulling back the quotient sequence and the marked sections to each component C_v, as well as marking the $P_{v,j,b}^n$'s, we have a family of stable quotients on each component.

Thus, for each v, we have the data of a proper, flat family of genus g_v curves over T, $\pi_v : C_v \to T$, with the general fiber irreducible and nonsingular, and

- $P_{i,v} : T \to C_v$ are the pullbacks of the natural sections which lie on C_v

- $P^n_{v,j,b}$ are the preimages of the nodal divisors P^n_j in the fibers of π which lie on C_v

- $0 \hookrightarrow \mathcal{S}_{1,v} \hookrightarrow \ldots \hookrightarrow \mathcal{S}_{\ell,v} \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_{C_v} \to \ \mathcal{Q}_{1\,v} \to \ \ldots \to \mathcal{Q}_{\ell\,v} \to 0$
  is the pullback of the quotient sequence to the component C_v

We would like to apply *Proposition 4.7* of [Kol90] to
$$\pi_{v\,*} i_v^* \mathcal{L}_k \cong \pi_{v\,*}\left(\omega^f_{\pi_v}\left(\sum f\, P_{i,v} + \sum (f-1)\, P^n_{v,j,b}\right) \otimes \left(\otimes_{h=1}^{\ell} det(\mathcal{S}_{h\,v}{}^*)\right)^k\right)$$
where $i_v : C_v \to C^\nu$ is the inclusion, but the surfaces C_v are not smooth as a result of the base change, and the sections $P_{i,v}$, $P^n_{v,j,b}$ may not be distinct from sections of $det(\mathcal{S}^*_{h\,v})$. We will show that both of these problems can be remedied. We will use the following lemma now and later in the proof.

Lemma 2.1.0.1. *$\mathcal{Q}_h|_P$ is locally free, for all $1 \le h \le \ell$.*

Proof. Suppose q lies in the fiber C_t. Then, by definition of stable quasimaps in families, the restriction of $\mathcal{Q}_h$ to fibers of π is locally free at the nodes and markings of the fiber. Therefore $(i_{C_t}{}^* \mathcal{Q}_h)_q$ is a free $\mathcal{O}_{C_t,q}$ module of rank $n-1$. By definition
$$(i_{C_t}{}^* \mathcal{Q}_h)_q \cong \mathcal{Q}_{h\,q} \otimes_{\mathcal{O}_{C,q}} \mathcal{O}_{C_t,q}$$
Since this is a free $\mathcal{O}_{C_t,q}$ module,
$$projdim_{\mathcal{O}_{C_t,q}}((i_{C_t}{}^* \mathcal{Q}_h)_q) = 0$$
I claim that this implies $projdim_{(\mathcal{O}_C)_q}(\mathcal{Q}_{h\,q}) = 0$. The Auslander-Buchsbaum formula [Eis95] tells us that
$$pd(\mathcal{Q}_{h\,q}) = depth(\mathcal{O}_{C,q}) - depth(\mathcal{Q}_{h\,q})$$
since $pd(\mathcal{Q}_{h\,q}) < \infty$ (localizing the original quotient sequence
$$\mathcal{S}_h \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_h \to 0$$
at q provides a finite free resolution of $\mathcal{Q}_{h\,q}$) and $\mathcal{Q}_q$ is finitely generated as an $\mathcal{O}_{C,q}$ module. Then restricting $\mathcal{O}_C$ to a fiber corresponds to modding out by a non-zerodivisor (since the base T is smooth, the generator of the maximal ideal is not a zero divisor in the local ring $\mathcal{O}_{T,t}$, and by flatness it cannot be a zero divisor in $\mathcal{O}_{\mathcal{C},q}$ either). Therefore passing from $\mathcal{O}_{C,q}$ to $\mathcal{O}_{C_t,q}$ corresponds to taking the quotient by an ideal generated by a non-zerodivisor in the local ring.

Similarly, passing from $\mathcal{Q}_{h\,q}$ to $(\mathcal{Q}_{h\,t})_q$ corresponds to taking the quotient by a submodule generated by a non-zerodivisor : if z is the generator of the maximal ideal of $\mathcal{O}_{T,t}$, then if z was a zero divisor on $\mathcal{Q}_{h,q}$, this would imply that

$\mathcal{Q}_{h,q} \otimes (z) \to \mathcal{Q}_{h,q}$ is not injective, contradicting flatness of $\mathcal{Q}_{h,q}$ over $\mathcal{O}_{T,t}$. Since modding out by a non-zerodivisor drops the depth by 1 [Sta17, Tag 00LE] we have the following equalities:

$$depth(\mathcal{O}_{C_t,q}) = depth(\mathcal{O}_{C,q}) - 1$$
$$depth((\mathcal{Q}_{h\,t})_q) = depth(\mathcal{Q}_{h\,q}) - 1$$

Thus,

$$\begin{gathered} pd(\mathcal{Q}_{h\,q}) \\ = depth(\mathcal{O}_{C,q}) - depth(\mathcal{Q}_{h\,q}) = depth(\mathcal{O}_{C_t,q}) + 1 - (depth((\mathcal{Q}_{h\,t})_q) + 1) \\ = 0 \end{gathered}$$

since

$$depth(\mathcal{O}_{C_t,q}) - depth((\mathcal{Q}_{h\,t})_q) = pd((\mathcal{Q}_{h\,t})_q) = 0$$

because $(\mathcal{Q}_{h\,t})_q$ is a free $\mathcal{O}_{C_t,q}$ module by the definition of stable quotients in families. Therefore $\mathcal{Q}_{h\,q}$ is a free $\mathcal{O}_{C,q}$ module. Let $q \in P$

$$(\mathcal{Q}_h|_P)_q \cong \mathcal{Q}_{h\,q} \otimes_{\mathcal{O}_{C,q}} \mathcal{O}_{P,q} \cong \bigoplus^{n-r_h} \mathcal{O}_{C,q} \otimes_{\mathcal{O}_{C,q}} \mathcal{O}_{P,q} \cong \bigoplus^{n-r_h} \mathcal{O}_{P,q}$$

using the fact that $\mathcal{Q}_{h\,q}$ is a free $\mathcal{O}_{C,q}$ module. Thus, $\mathcal{Q}_h|_P$ is a locally free $\mathcal{O}_P$ module. □

Lemma 2.1.0.2. *We can pick* $s \in H^0(\,C_v, \bigotimes_{h=1}^{\ell} det(\mathcal{S}^*_{h\,v})\,)$ *so that* s *does not vanish identically on any of the* $P_{i,v}, P^n_{v,j,b}$.

Proof. We just need to show that, for each $P = P_{i,v}$, $P^n_{v,j,b}$,

$$H^0(det(\mathcal{S}^*_{h\,v})) \neq H^0(det(\mathcal{S}^*_{h\,v}) \otimes \mathcal{I}_P).$$

Given this, then we see that since each of the subspaces $H^0(det(\mathcal{S}^*_{h\,v}) \otimes \mathcal{I}_P)$ are closed in $H^0(det(\mathcal{S}^*_{h\,v}))$, their complements are dense in $H^0(det(\mathcal{S}^*_{h\,v}))$. Then we can consider the intersection of their complements, which will be nonempty since $H^0(det(\mathcal{S}^*_{h\,v}))$ is irreducible.

Notice that since $\mathcal{Q}_{h\,v}$ is locally free at the nodes and markings of the family of curves, the first map in the exact sequence

$$0 \to \mathcal{S}_{h\,v} \to \mathbb{C}^n \otimes \mathcal{O}_{C_v} \to \mathcal{Q}_{h\,v} \to 0$$

cannot drop rank along P, otherwise $\mathcal{Q}_{h\,v}$ would have torsion supported on the node or marking, which is not allowed.

The map $\mathcal{S}_{h\,v} \to \mathbb{C}^n \otimes \mathcal{O}$ is given by n sections of $\mathcal{S}^*_{h\,v}$. Since the map has full rank along P, not all $\binom{n}{r_h}$ tuples of r_h sections of the above n sections of $\mathcal{S}^*_{h\,v}$ can be linearly dependent when restricted to every point of P. Therefore there exist r_h

of these sections which become linearly dependent only when restricted to finitely many points of P. Taking the wedge of these sections yields a section of $det(\mathcal{S}^*_{h\,v})$ which does not vanish identically on P by construction, although it may vanish at finitely many points in P.

This proves the nonemptiness of the complements, and thus there exists an open dense subscheme of $H^0(det(\mathcal{S}^*_{h\,v}))$ consisting of sections which do not vanish identically on any of the P's above. □

We can use a finite base change (following the same method we used before to separate the nodal locus into sections) to split up $div(s)$ into a union of (not necessarily disjoint) distinct sections of π'_v, where we have the diagram

$$\begin{tikzcd} C'_v \arrow[r, "h'_v"] \arrow[d, "\pi'_v"'] & C_v \arrow[d, "\pi_v"] \\ T_v \arrow[r, "h_v"'] & T \end{tikzcd}$$

and we pull back the sections $P_{i,v}, P^n_{v,j,b}$ along with the flag sequences.

Now we address the fact that the base changes introduced singularities of the total space of the family. The resulting surface C'_v has finitely many singularities of the form $x_1x_2 - {x_3}^\alpha, \alpha > 1$.

We can use the change of coordinates

$$x_1 \mapsto x + iy, x_2 \mapsto x - iy, x_3 \mapsto (-1)^{\frac{1}{\alpha}} z$$

to see that the singularity is analytically isomorphic to the singularity at the origin of $x^2 + y^2 + z^\alpha$, the $A_{\alpha-1}$ singularity. To resolve the singularity, we must blow up $\alpha - 1$ times. This will result in a chain of $\alpha - 1$ E_i's ($\mathbb{P}^1$'s), each of which has self intersection -2. We also know that $E_i.E_j = 1$ if $i = j - 1$ or vice versa, and 0 otherwise.

Call each individual blowup morphism $Bl_i : \widetilde{C'_{v,i}} \to \widetilde{C'_{v,i-1}}$, and call the composition of all of them $Bl : \widetilde{C'_v} \to C'_v$. Since $P'_{i,v}$, ${P^n}'_{v,j,b}$, and the divisor of zeros of the chosen section of $\bigotimes_{h=1}^{\ell} det({\mathcal{S}'}^*_{h\,v})$ are contained in the smooth locus of C'_v,

$$Bl^*\left(\mathcal{O}_{C'_v}(\sum f P'_{i,v} + \sum(f-1){P^n}'_{v,j,b}) \otimes \left(\bigotimes_{h=1}^{\ell} det({\mathcal{S}'}^*_{h\,v})\right)^k\right)$$
$$\cong$$
$$\mathcal{O}_{\widetilde{C'_v}}\left(\sum f \widetilde{P'_{i,v}} + \sum(f-1)\widetilde{{P^n}'_{v,j,b}}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\widetilde{{\mathcal{S}'}^*_{h\,v}})\right)^k.$$

Now we have that the family

$$\begin{tikzcd} \widetilde{C'_v} \arrow[d, shift left, "\widetilde{\pi'_v}"] \\ T_v \arrow[u, bend left=60, "\widetilde{P'_{i,v}}"] \arrow[u, bend right=60, "\widetilde{P'_{v,j,b}}"'] \end{tikzcd} \qquad 0 \hookrightarrow \widetilde{\mathcal{S}'_{1\,v}} \hookrightarrow \ldots \hookrightarrow \widetilde{\mathcal{S}'_{\ell\,v}} \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \widetilde{\mathcal{Q}'_{1\,v}} \to \ldots \to \widetilde{\mathcal{Q}'_{\ell\,v}} \to 0$$

$$\omega^f_{\widetilde{\pi'_v}}\left(\sum f\widetilde{P'_{i,v}} + \sum(f-1)\widetilde{P^{n\,\prime}_{v,j,b}}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\widetilde{\mathcal{S}'^{\,*}_{h\,v}})\right)^k$$

is a proper flat family of curves with smooth general fiber whose total space is smooth together with a family of generalized stable quotients and a line bundle of the form $\omega^f_{\widetilde{\pi'_v}}\left(\sum w_\alpha \widetilde{W_{v,\alpha}}\right)$, where $f = k(d+1)$, $w_\alpha \le f$ and $\widetilde{W_{v,\alpha}}$ are distinct sections of $\widetilde{\pi'_v}$.

This allows us to apply *Proposition 4.7* of [Kol90] to obtain:

$$\widetilde{\pi'_v}_* \left(\omega^f_{\widetilde{\pi'_v}}\left(\sum f\widetilde{P'_{i,v}} + \sum(f-1)\widetilde{P^{n\,\prime}_{v,j,b}}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\widetilde{\mathcal{S}'^{\,*}_{h\,v}})\right)^k\right)$$

is a semipositive vector bundle on T_v, for each v.

We would like to show that

$$\pi'_{v\,*}\left(\omega^f_{\pi'_v}\left(\sum f\, P'_{i,v} + \sum(f-1)\, P^{n\,\prime}_{v,j,b}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\mathcal{S}'^{\,*}_{h\,v})\right)^k\right)$$

is semipositive. We claim that

$$\widetilde{\pi'_v}_* \left(\omega^f_{\widetilde{\pi'_v}}\left(\sum f\widetilde{P'_{i,v}} + \sum(f-1)\widetilde{P^{n\,\prime}_{v,j,b}}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\widetilde{\mathcal{S}'^{\,*}_{h\,v}})\right)^k\right)$$
$$\cong$$
$$\pi'_{v\,*}\left(\omega^f_{\pi'_v}\left(\sum f\, P'_{i,v} + \sum(f-1)\, P^{n\,\prime}_{v,j,b}\right) \otimes \left(\bigotimes_{h=1}^{\ell} det(\mathcal{S}'^{\,*}_{h\,v})\right)^k\right)$$

It suffices to prove the following lemma.

Lemma 2.1.0.3. $Bl^*\omega_{\pi'_v} \cong \omega_{\widetilde{\pi'_v}}$

Proof. We already know that $Bl^*\omega_{\pi'_v}$ and $\omega_{\widetilde{\pi'_v}}$ agree when restricted to $\widetilde{C'_v}\backslash \bigcup E_i$ We have the following sequence of Chow groups

$$A_1(\bigcup E_i) \to\; A_1(\widetilde{C'_v}) \to\; A_1(\widetilde{C'_v}\backslash \bigcup E_i) \to\; 0.$$

Therefore $Bl^*\omega_{\pi'_v}$ and $\omega_{\widetilde{\pi'_v}}$ differ by $\mathcal{O}(\sum n_i E_i)$, for some n_i's.

I claim that all $n_i = 0$.

Notice that $Bl^*\omega_{\pi'_v}.E_i = 0$ by push-pull. Therefore the restriction of $Bl^*\omega_{\pi'_v}$ to E_i is trivial. Also, the restriction of $\omega_{\widetilde{\pi'_v}}$ to E_i is $\mathcal{O}_{\mathbb{P}^1}(-2)\otimes\mathcal{O}_{\mathbb{P}^1}(p_1+p_2) \cong \mathcal{O}_{\mathbb{P}^1}$ where $p_{i,1}$ and $p_{i,2}$ are the two nodes on E_i. Therefore $\mathcal{O}(n_i E_i)$ restricted to each E_j is trivial. [Mum61] has shown that the associated quadratic form to the intersection form $(E_i.E_j)$ is negative definite. Thus the intersection form is nondegenerate, so all $n_i = 0$. $\square$

Using the projection formula, we see that the above lemma together with our earlier observations yield the desired isomorphism.

Thus,

$$\pi'_{v\,*}\left(\omega^{f}_{\pi'_v}\left(\sum f\,P'_{i,v}+\sum(f-1)\,P'^{\,n}_{v,j,b}\right)\otimes\left(\bigotimes_{h=1}^{\ell}(det(\mathcal{S}'^{\,*}_{h\,v})\right)^{k}\right)$$

is a semipositive vector bundle on T_v, $\forall k\geq 5$, where $f=k(d+1)$.

We would like to produce semipositive vector bundles on T which fit into an exact sequence with $\pi_*\mathcal{L}_k$. We have a semipositive vector bundle on each T_v. We will show that each of these is pulled back from a semipositive vector bundle on T coming from each family $\pi_v : C_v \to T$.

To prove this, we check that

$$R^1\pi_{v\,*}\left(\omega^{f}_{\pi_v}(\sum f\,P_{i,v}+\sum(f-1)\,P^{n}_{v,j,b})\otimes(\bigotimes_{h=1}^{\ell}(det(\mathcal{S}^{*}_{h\,v}))^{k}\right)=0,$$

which will allow us to conclude that the semipositive vector bundle

$$\pi'_{v\,*}\left(\omega^{f}_{\pi'_v}(\sum f P'_{i,v}+\sum(f-1)P^{n\,\prime}_{v,j,b})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}'^{\,*}_{h\,v}))^{k}\right)$$

is the pullback of

$$\pi_{v\,*}\left(\omega^{f}_{\pi_v}(\sum f\,P_{i,v}+\sum(f-1)\,P^{n}_{v,j,b})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v}))^{k}\right)$$

along $T_v \to T$. Using our argument from before will allow us to conclude that the latter is also semipositive on T.

We can check that there is no higher cohomology on the fibers of π_v. Observe that

$$\omega^{f}_{C_{v,t}}(\sum f\,p_{i,v,t}+\sum(f-1)\,p^{n}_{v,j,b,t})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v,t}))^{k}$$

$$\cong$$

$$\omega_{C_{v,t}}\otimes\omega^{f-1}_{C_{v,t}}(\sum(f-1)\,p_{i,v,t}+\sum(f-1)\,p^{n}_{v,j,b,t})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v,t}))^{k}\otimes\mathcal{O}_{C_{v,t}}(\sum p_{i,v,t})$$

where

$$\omega^{f-1}_{C_{v,t}}(\sum(f-1)\,p_{i,v,t}+\sum(f-1)\,p^{n}_{v,j,b,t})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v,t}))^{k}$$

is ample by the stability condition (let $\epsilon=k/(f-1)$) , thus it must have positive degree when restricted to each component.

By Serre duality,

$$h^1(\omega_{C_{v,t}}\otimes\omega^{f-1}_{C_{v,t}}(\sum(f-1)\,p_{i,v,t}+\sum(f-1)\,p^{n}_{v,j,b,t})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v,t}))^{k}\otimes\mathcal{O}_{C_{v,t}}(\sum p_{i,v,t}))$$

$$\cong$$

$$h^0(\,(\omega^{f-1}_{C_{v,t}}(\sum(f-1)\,p_{i,v,t}+\sum(f-1)\,p^{n}_{v,j,b,t})\otimes(\bigotimes_{h=1}^{\ell}det(\mathcal{S}^{*}_{h\,v,t}))^{k})^{*}\otimes\mathcal{O}_{C_{v,t}}(-\sum p_{i,v,t})).$$

As we noted above, the first term in the second expression has negative degree on each component, and the last term has degree ≤ 0. Therefore h^1 vanishes and thus the higher direct images vanish. Thus

$$\pi_{v\,*}\left(\omega_{\pi_v}^{f}(\sum f\,P_{i,v} + \sum(f-1)\,P_{v,j,b}^{n}) \otimes (\bigotimes_{h=1}^{\ell} det(\mathcal{S}_{h\,v}^{*}))^{k}\right)$$

is a semipositive vector bundle on T, $\forall v$.

Now, we would like to relate these semipositive vector bundles to $\pi_*\mathcal{L}_k$. We have the map $\nu : \coprod_v C_v \to C$. As in [Vak01] *Lemma 2.3*, the map ν is a clutching morphism, and we have an associated exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{C^\nu} \to \bigoplus_j i_{j\,*}\mathcal{O}_{P_j^n} \to 0$$

Tensor by ω_π and observe that (as in [Vak01])

$$\nu^*\omega_\pi \cong \omega_{\pi_{\coprod_v}}(\sum_{b=1}^{2} P_{v,j,b}^{n}) :$$

$$0 \to \omega_\pi \to \nu_*\omega_{\pi_{\coprod_v}}(\sum_{b=1}^{2} P_{v,j,b}^{n}) \to \bigoplus_j i_{j*}i_j^*\omega_\pi \to 0$$

In the last map in this sequence we take the residue of the section along the two preimages of each nodal section, and then take the difference of these values, for each j.

Using this, we have another exact sequence

$$0 \to \nu_*\omega_{\pi_{\coprod_v}} \to \nu_*\omega_{\pi_{\coprod_v}}(\sum P_{v,j,b}^{n}) \to \bigoplus_{b=1}^{2} \alpha_{j,b\,*}i_{j,b}^{*}\nu^*\omega_\pi \to 0$$

where $\alpha_{j,b} : P_{v,j,b}^{n} \to P_j^n$ are the restrictions of ν to the preimages of the nodal sections, which are isomorphisms and $i_{j,b} : P_{v,j,b}^{n} \to C^\nu$ are the inclusions. We also have an exact sequence

$$0 \to \omega_\pi \otimes \mathcal{I} \to \omega_\pi \to \bigoplus_j i_{j\,*}i_j^*\omega_\pi \to 0$$

where $\mathcal{I}$ is the ideal sheaf of the P_j^n's obtained by tensoring the ideal sheaf exact sequence with $\omega - \pi$. Now we will put these together into a commutative diagram as follows :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \omega_\pi \otimes \mathcal{I} \arrow[r] \arrow[d] & \omega_\pi \arrow[r] \arrow[d] & \bigoplus_j i_{j\,*} i_j^* \omega_\pi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nu_* \omega_{\pi_{\coprod_v}} \arrow[r] \arrow[d] & \nu_* \omega_{\pi_{\coprod_v}}(\sum P^n_{v,j,b}) \arrow[r] \arrow[d] & \bigoplus_{b=1}^{2} \alpha_{j,b\,*} i^*_{j,b} \nu^* \omega_\pi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_j i_{j*} i_j^* \omega_\pi \arrow[r] \arrow[d] & \bigoplus_j i_{j*} i_j^* \omega_\pi \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It is clear that all three rows are exact. We already know that the middle column is exact. The last column is the anti diagonal embedding $z \mapsto (z, -z)$ (since $\alpha_{j,b\,*} i^*_{j,b} \nu^* \omega_\pi \cong i_{j\,*} i_j^* \omega_\pi$ by the projection formula) followed by the sum map, which is exact. The map $\omega_\pi \otimes \mathcal{I} \to \nu_* \omega_{\pi_{\coprod_v}}$ is induced by the commutativity of the top right square, which is clear. Commutativity of the bottom right square is also clear from our discussion above. The bottom left square is also commutative since sections of $\nu_* \omega_{\pi_{\coprod_v}}$ are holomorphic along the preimages of the nodal sections, so taking the residue yields 0. Thus the diagram commutes, all the rows are exact and the last two columns are exact. A diagram chase yields an isomorphism $\omega_\pi \otimes \mathcal{I} \cong \nu_* \omega_{\pi_{\coprod_v}}$.

Thus, we have an exact sequence

$$0 \to \nu_* \omega_{\pi_{\coprod_v}} \to \omega_\pi \to \bigoplus_j i_{j\,*} i_j^* \omega_\pi \to 0$$

We would like to show that the quotient sheaf is actually $\bigoplus_j i_{j\,*} \mathcal{O}_{P^n_j}$. Using the projection formula together with the isomorphism

$$\nu^* \omega_\pi \cong \omega_{\pi_{\coprod_v}}(\sum P^n_{v,j,b}),$$

it suffices to prove the following lemma.

Lemma 2.1.0.4. *$\omega_{\pi_v}(\sum P^n_{v,j,b})|_{P^n_{v,j',b'}} \cong \mathcal{O}_{P^n_{v,j',b'}}$ $\forall j', b'$ such that $P^n_{v,j',b'} \subset C_v$, $\forall v$.*

Proof. Since the nodal sections are disjoint, it is clear that the restriction of $\mathcal{O}(P^n_{v,j,b})$ to $P^n_{v,j',b'}$ is trivial unless $j = j'$ and $b = b'$. Now, the total space of C_v may have singularities, but these are not on the marked nodal sections $P^n_{v,j,b}$. Therefore we can perform a resolution of singularities $Bl : \tilde{C}_v \to C_v$. The new

family will have smooth total space and smooth general fiber. Notice that
$$\omega_{\tilde{\pi}_v}(\widetilde{P^n_{v,j',b'}})|_{\widetilde{P^n_{v,j',b'}}} \cong Bl^* \omega_{\pi_v}(P^n_{v,j',b'})|_{\widetilde{P^n_{v,j',b'}}}$$
As in [HM98], since the total space of $\tilde{C}_v$ is smooth, the relative dualizing sheaf is isomorphic to $K_{\tilde{C}_v} \otimes \tilde{\pi}_v^* K_T^*$. Thus, using adjunction
$$\omega_{\tilde{\pi}_v}(\widetilde{P^n}_{v,j',b'})|_{\tilde{P^n}_{v,j',b'}} \cong \mathcal{O}_{\tilde{P^n}_{v,j',b'}}.$$
Using the projection formula for Bl, since $P^n_{v,j',b'}$ is isomorphic to its strict transform, we find
$$\omega_{\pi_v}(P^n_{v,j',b'})|_{P^n_{v,j',b'}} \cong \mathcal{O}_{P^n_{v,j',b'}}$$
$\square$

Therefore we have an exact sequence :
$$0 \to \nu_* \omega_{\pi_{\coprod_v}} \to \omega_\pi \to \bigoplus_j i_{j\,*} \mathcal{O}_{P^n_j} \to 0$$
Now, to yield an exact sequence where $\mathcal{L}_k$ is the middle term, we can tensor this with
$$\omega_\pi^{f-1}(\sum f P_i) \otimes (\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*))^k$$
and the result is
$$0 \to \nu_* \left(\omega^f_{\pi_{\coprod_v}}(\sum f P_{i,v} + \sum (f-1)\, P^n_{v,j,b}) \otimes (\bigotimes_{h=1}^{\ell} det(\mathcal{S}^*_{h\, \coprod_v}))^k \right) \to \mathcal{L}_k$$
$$\to \bigoplus_j i_{j*} i_j^* (\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*))^k) \to 0$$
using what we showed above together with the fact that the marked sections on C are in the smooth locus.

Now, notice that if we restrict the first sheaf to fibers of $\pi_{\coprod_v}$, it has no higher cohomology (from what we have already shown) and the pushforward via π of
$$\nu_* \left(\omega^f_{\pi_{\coprod_v}}(\sum f P_{i,v} + \sum (f-1)\, P^n_{v,j,b}) \otimes (\bigotimes_{h=1}^{\ell} det(\mathcal{S}^*_{h\, \coprod_v}))^k \right).$$
coincides with the semipositive vector bundle
$$\pi_{\coprod_v *} \omega^f_{\pi_{\coprod_v}}(\sum f P_{i,v} + \sum (f-1)\, P^n_{v,j,b}) \otimes (\bigotimes_{h=1}^{\ell} det(\mathcal{S}^*_{h\, \coprod_v}))^k$$
Using our lemma from earlier, when we restrict each quotient sequence to each marked section P_i
$$0 \to \mathcal{S}_h|_{P_i} \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_h|_{P_i} \to 0$$
the quotient is locally free on P_i. When we dualize, we obtain a surjection
$$\mathbb{C}^n \otimes \mathcal{O} \to \mathcal{S}_h^*|_{P_i} \to 0.$$
Taking r_h^{th} exterior powers, we find that we have a surjection
$$\mathbb{C}^{\binom{n}{r_h}} \otimes \mathcal{O} \to det(\mathcal{S}_h^*|_{P_i}) \to 0.$$
The trivial bundle is clearly semipositive from the definitons, so by *Corollary 3.4i* of

[Kol90] the quotient $det(\mathcal{S}_h^*|_{P_i})$ is semipositive on P_i. This holds for each $1 \le i \le m$, and for each $1 \le h \le \ell$.

By *Corollary 3.5* of [Kol90], tensor products of semipositive vector bundles are semipositive, so we see that, for each $1 \le i \le m$, $(\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*))^k|_{P_i}$ is semipositive on P_i. Since $\pi \circ i_j : P_j \to T$ is an isomorphism for all i, we see that the pushforward $\pi_* i_{j*} i_j^* \left((\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*))^k \right)$ is semipositive on T.

Thus, when we pushforward, we get an exact sequence of vector bundles

$$0 \to \pi_{\coprod_v *} \left(\omega^f_{\pi_{\coprod_v}} (\sum f P_{i,v} + \sum (f-1)\, P^n_{v,j,b}) \otimes \otimes_{h=1}^{\ell} (det(\mathcal{S}^*_{h\, \coprod_v}))^k \right) \to \pi_* \mathcal{L}_k$$
$$\to \bigoplus_{j=1}^{m} \pi_* i_{j*} i_j^* \left((\bigotimes_{h=1}^{\ell} det(\mathcal{S}_h^*))^k \right) \to 0$$

The last bundle is a direct sum of semipositive vector bundles, so by *Corollary 3.5* of [Kol90], it is semipositive. Since $\pi_* \mathcal{L}_k$ is an extension of semipositive vector bundles, by *Corollary 3.4i* of [Kol90], $\pi_* \mathcal{L}_k$ is semipositive.

□

Notice that this actually shows that $\pi_* \mathcal{L}_k$ is semipositive for any family of generalized stable quotients over any base: We already showed that the bundle $\pi_* \mathcal{L}_k$ commutes with arbitrary base change. Now, suppose we are given a family of stable quasimaps over a base T.

Let $f : C \to T$ be any map from a projective curve to T. Let $f^* \pi_* \mathcal{L}_k \to \mathcal{M}$ be a quotient line bundle on C. We can pull everything back to the normalization of C, where we have already showed that $\nu^* f^* \mathcal{L}_k$ is semipositive, thus forcing $\nu^* \mathcal{M}$ to have nonnegative degree on C^ν. Since the normalization is finite, it will not affect the sign of $deg(\mathcal{M})$. Thus $\mathcal{M}$ has nonnegative degree on C. By definition, $\pi_* \mathcal{L}_k$ is semipositive on T.

2.2 The ample line bundle

Now, given a family of generalized stable quotients over a scheme T, we would like to construct a collection of tautological semipositive vector bundles and quotients on the base T which are functorial under base change. It will be shown that these yield a set theoretic map to the $GL(W)$ orbits of a product of Grassmannians which factors through the map to moduli.

We work with the family:

$$(*)\qquad \begin{tikzcd} C \arrow[d, "\pi"] & 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_C \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0 \\ T \arrow[u, bend left, "P_i"] \end{tikzcd}$$

The $\mathbb{Q}$ line bundle

$$\omega_\pi(\sum_{j=1}^m P_j) \otimes (\bigotimes_{i=1}(det(\mathcal{S}_i)^*))^\epsilon$$

is π-relatively ample for any $\epsilon \in \mathbb{Q}_{>0}$. Just as in [MOP11] we will fix $\epsilon = \frac{1}{d+1}$. We use our construction of the moduli stack as a stack quotient $[\mathcal{Q}'/\mathbf{PGL}(W)]$ from the first chapter to construct the factorization.

The following is *Definition 3.8* of [Kol90].

Definition 2.2.1. *Suppose T is a scheme and $\mathcal{V}$ is a vector bundle of rank v with structure group G. Let $\mathcal{V} \to \mathcal{Q} \to 0$ be a quotient vector bundle of rank q. The natural map from the closed points of T to the set of G orbits of $Gr(q, v)$ is called the classifying map. This map is said to be finite if every fiber is finite and for each $t \in T$ only finitely many elements of G leave the kernel of $\mathcal{V}_t \to \mathcal{Q}_t \to 0$ invariant.*

Here, $Gr(q, v)$ refers to the Grassmannian of q dimensional *quotients* of $\mathbb{C}^v$ as opposed to subspaces, as it has been used throughout the rest of the paper. This terminology will only be applied in this chapter.

2.2.1 The tautological quotients associated to a family

Consider a family of generalized stable quotients $(*)$.

The bundle $\mathcal{L}_k$ gives an embedding μ of the fibers of C over T into the fibers of $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ over T, which yields a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[dr, "\pi"] & \mathbb{P}((\pi_*\mathcal{L}_k)^*) \arrow[d, "\rho"] \\ & T \arrow[ul, bend left, "p_i"] \arrow[u, bend right, "t_i"'] \end{tikzcd}$$

We obtain sections $t_i : T \to \mathbb{P}((\pi_*\mathcal{L}_k)^*)$ via $\mu \circ p_i$. Call T_i the subscheme of $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$ defined by the image of t_i. We have the natural exact sequence of sheaves on $\mathbb{P}((\pi_*\mathcal{L}_k)^*)$

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}((\pi_*\mathcal{L}_k)^*)} \to \mu_*\mathcal{O}_C \to 0$$

By [Nit05], *Theorem 2.3*, there exists an M such that $\mathcal{I}_{C_t}$ is M regular for all $t \in T$. It follows that $\mu_*\mathcal{O}_{C_t}$ is also M regular for all $t \in T$ by *Lemma 2.1* of [Nit05].

This implies that $\forall m' \geq M$,

- $h^i(\mathcal{I}_{C_t}(m')) = h^i(\mu_{t*}\mathcal{O}_{C_t}(m')) = 0$ for all $i \geq 1$
- $\mathcal{I}_{C_t}(m')$ and $\mu_{t*}\mathcal{O}_{C_t}(m')$ are generated by global sections

for all $t \in T$. By *Theorem 3.7* of [Nit05], we have that, for all $m' \geq M$,

$$0 \to \rho_*\mathcal{I}_C(m') \to Sym^{m'}\pi_*\mathcal{L}_k \to \pi_*(\mathcal{L}_k^{m'}) \to 0$$

is an exact sequence of vector bundles (where all higher direct images vanish) on T such that

- $\rho^*\rho_*\mathcal{I}_C(m') \to \mathcal{I}_C(m')$
- $\rho^* Sym^{m'}\pi_*\mathcal{L}_k \to \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m')$
- $\rho^*\pi_*\mathcal{L}_k^{m'} \to \mu_*\mathcal{L}_k^{m'}$

are surjective $\forall\, m' \geq M$. Therefore we have a commutative diagram of sheaves on C ([Nit05])

$$\begin{tikzcd}
0 \arrow[r] & \rho^*\rho_*\mathcal{I}_C(m') \arrow[r] \arrow[d] & \rho^* Sym^{m'}\pi_*\mathcal{L}_k \arrow[r] \arrow[d] & \rho_*\pi_*\mathcal{L}_k^{m'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_C(m') \arrow[r] & \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m') \arrow[r] & \mu_*\mathcal{L}_k^{m'} \arrow[r] & 0
\end{tikzcd}$$

with the columns surjective and the rows exact.

Notice that, given the data

$$Sym^{m'}\pi_*\mathcal{L}_k \to \pi_*(\mathcal{L}_k^{m'}) \to 0$$

we can recover the original sequence as follows ([Nit05]).

Pulling back along ρ we have an exact sequence of vector bundles

$$\rho^* Sym^{m'}\pi_*\mathcal{L}_k \to \rho^*\pi_*\mathcal{L}_k^{m'} \to 0$$

whose kernel is naturally isomorphic to $\rho^*\rho_*\mathcal{I}_C(m')$. Consider the composition

$$\rho^*\rho_*\mathcal{I}_C(m') \to \rho^* Sym^{m'}\pi_*\mathcal{L}_k \to \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m').$$

Based on the diagram above, we see that the image of this map of sheaves is the kernel of $\mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m') \to \mu_*\mathcal{L}_k^{m'}$. Taking the cokernel of $\rho^*\rho_*\mathcal{I}_C(m') \to \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m')$ we recover $\mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m') \to \mu_*\mathcal{L}_k^{m'}$. Twisting by $\mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(-m')$ allows us to recover the original quotient sequence.

The sections t_i yield surjections

$$\pi_*\mathcal{L}_k \to t_i^*\mathcal{L}_k \to 0.$$

Taking m'^{th} symmetric powers, we obtain surjections

$$Sym^{m'}\pi_*\mathcal{L}_k \to t_i^*\mathcal{L}_k^{m'}.$$

which correspond to the m'-uple embedding of the sections, so we can recover the sections from this data.

We can push forward each of the exact sequences from the flag sequence:

- $0 \to \mu_*\mathcal{S}_1 \to \mathbb{C}^n \otimes \mu_*\mathcal{O}_C \to \mu_*\mathcal{Q}_1 \to 0$
- $0 \to \mu_*\mathcal{K}_j \to \mu_*\mathcal{Q}_j \to \mu_*\mathcal{Q}_{j+1} \to 0$, for $1 \leq j \leq \ell - 1$.

Notice that we have the following commutative diagrams of sheaves on C

$$(\star) \quad \begin{tikzcd} & \mu^*\mu_*\mathcal{S}_1 \arrow[r] \arrow[d] & \mathbb{C}^n \otimes \mu^*\mu_*\mathcal{O}_C \arrow[r] \arrow[d] & \mu^*\mu_*\mathcal{Q}_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{S}_1 \arrow[r] & \mathbb{C}^n \otimes \mathcal{O}_C \arrow[r] & \mathcal{Q}_1 \arrow[r] & 0 \end{tikzcd}$$

$$(\star)_j \quad \begin{tikzcd} & \mu^*\mu_*\mathcal{K}_j \arrow[r] \arrow[d] & \mu^*\mu_*\mathcal{Q}_j \arrow[r] \arrow[d] & \mu^*\mu_*\mathcal{Q}_{j+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}_j \arrow[r] & \mathcal{Q}_j \arrow[r] & \mathcal{Q}_{j+1} \arrow[r] & 0 \end{tikzcd}.$$

with the rows exact and the columns surjections. These diagrams will play a role in what follows.

Considering the kernel $\mathcal{G}_1$ of the composition

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)} \to \mathbb{C}^n \otimes \mu_*\mathcal{O}_C \to \mu_*\mathcal{Q}_1,$$

by [Nit05] *Theorem 2.3* we can find an N such that $\mathcal{G}_1$ is N regular on the fibers of ρ. By [Nit05] *Lemma 2.1*, this implies that $\mu_*\mathcal{Q}_1$ is N regular on the fibers of ρ. Choosing $M' \geq max\{M, N\} + 1$, we see that $\mu_*\mathcal{S}_1$ is M' regular when restricted to fibers of ρ. Thus we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_1(m') \arrow[r] \arrow[d] & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m') \arrow[r] \arrow[d] & \mu_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mu_*(\mathcal{S}_1 \otimes \mathcal{L}_k^{m'}) \arrow[r] & \mathbb{C}^n \otimes \mu_*\mathcal{L}_k^{m'} \arrow[r] & \mu_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[r] & 0 \end{tikzcd}.$$

where the rows are exact, the first two columns are surjections, and all the sheaves in the above sequence are both globally generated and without higher cohomology when restricted to the fibers of ρ, for any $m' \geq M'$. If we pick $m' \geq M + 1$, then pushing forward under ρ will preserve the surjectivity of the columns (the kernels are $M' + 1$ regular). Replace M' by $M' + 1$. Pushing forward under ρ yields a diagram where the rows are exact sequences of vector bundles and there are no higher direct images. By M' regularity, all the sheaves $\mathcal{F}$ in the diagram satisfy

$$\rho^*\rho_*\mathcal{F} \to \mathcal{F}$$

is surjective. This implies that we have a commutative diagram as follows $(\star\star)$

$$\begin{tikzcd}
0 \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[dd] & & 0 \arrow[dd] \\
\rho^*\rho_*\mathcal{G}_1(m') \arrow[rr] \arrow[dr] \arrow[dd] & & \rho^*\pi_*(\mathcal{S}_1 \otimes \mathcal{L}_k^{m'}) \arrow[dr] \arrow[dd] & \\
& \mathcal{G}_1(m') \arrow[rr] \arrow[dd] & & \mu_*(\mathcal{S}_1 \otimes \mathcal{L}_k^{m'}) \arrow[dd] \\
\mathbb{C}^n \otimes \rho^* Sym^{m'} \pi_*\mathcal{L}_k \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{C}^n \otimes \rho^*\pi_*\mathcal{L}_k^{m'} \arrow[dr] \arrow[dd] & \\
& \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(m') \arrow[rr] \arrow[dd] & & \mathbb{C}^n \otimes \mu_*\mathcal{L}_k^{m'} \arrow[dd] \\
\rho^*\pi_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[rr] \arrow[dr] \arrow[dd] & & \rho^*\pi_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[dr] \arrow[dd] & \\
& \mu_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[rr] \arrow[dd] & & \mu_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^{m'}) \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}$$

where all the horizontal and down-right arrows are surjections, with the columns exact.

Inductively we will show that an analogous regularity statement to those we found for the curve, the sections, and the first quotient sequence holds for

$$0 \to \mu_*\mathcal{K}_j \to \mu_*\mathcal{Q}_j \to \mu_*\mathcal{Q}_{j+1} \to 0, \text{ for } 1 \leq j \leq \ell - 1.$$

First, we consider the kernel of $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)} \to \mu_*\mathcal{Q}_1 \to \mu_*\mathcal{Q}_2 \to 0$. By *Theorem 2.3* of [Nit05], there exists an M_2 such that the kernel is M_2 regular when restricted to fibers of ρ. This implies that $\mu_*\mathcal{Q}_2$ is M_2 regular when restricted to fibers of ρ. Using the sequence

$$0 \to \mu_*\mathcal{K}_1 \to \mu_*\mathcal{Q}_1 \to \mu_*\mathcal{Q}_2 \to 0$$

we see that for $M_2' := max\{M', M_2\} + 1$, all sheaves in this exact sequence are M_2' regular when restricted to fibers of ρ.

Suppose that we have found an M_k such that $\forall j \leq k$, $\mu_*\mathcal{Q}_j$, $\mu_*\mathcal{K}_{j-1}$, $\mu_*\mathcal{S}_j$, $\mu_*\mathcal{I}_C$, and $\mu_*\mathcal{O}_C$ are M_k regular when restricted to fibers of ρ. Considering the kernel of

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)} \to \mu_*\mathcal{Q}_1 \to \ldots \to \mu_*\mathcal{Q}_k \to \mu_*\mathcal{Q}_{k+1} \to 0,$$

we see that by, *Theorem 2.3* of [Nit05], there exists M'_k such that this sheaf $\mathcal{G}_{k+1}$ is M'_k regular when restricted to fibers of ρ. Then, letting $M_{k+1} := max\{M'_k, M_k\}$, we see that all of the above sheaves as well as $\mu_*\mathcal{Q}_{k+1}$ and $\mu_*\mathcal{K}_k$ are M_{k+1} regular when restricted to fibers of ρ. The induction is complete.

By M_ℓ regularity, for any $h \geq M_\ell$, we have the following diagram, for each $1 \leq j \leq \ell - 1$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}_{j+1}(h) \arrow[r] \arrow[d] & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(h) \arrow[r] \arrow[d] & \mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[r] \arrow[d, equal] & 0 . \\
0 \arrow[r] & \mu_*(\mathcal{K}_j \otimes \mathcal{L}_k^h) \arrow[r] & \mu_*(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \arrow[r] & \mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[r] & 0
\end{tikzcd}$$

such that all the sheaves have no higher direct images, their direct images are locally free, and the restriction of any sheaf in the diagram to a fiber of ρ is a sheaf generated by its global sections.

If we replace M_ℓ with $M_\ell + 1$, then surjectivity of the columns is preserved after pushing forward along ρ. This yields a diagram $(\star\star)_j$

$$\begin{tikzcd}
0 \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[dd] & & 0 \arrow[dd] \\
\rho^*\rho_*\mathcal{G}_{j+1}(h) \arrow[rr] \arrow[dr] \arrow[dd] & & \rho^*\pi_*(\mathcal{K}_j \otimes \mathcal{L}_k^h) \arrow[dr] \arrow[dd] & \\
& \mathcal{G}_{j+1}(h) \arrow[rr] \arrow[dd] & & \mu_*(\mathcal{K}_j \otimes \mathcal{L}_k^h) \arrow[dd] \\
\mathbb{C}^n \otimes \rho^* Sym^h \pi_*\mathcal{L}_k \arrow[rr] \arrow[dr] \arrow[dd] & & \rho^*\pi_*(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \arrow[dr] \arrow[dd] & \\
& \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(h) \arrow[rr] \arrow[dd] & & \mu_*(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \arrow[dd] \\
\rho^*\pi_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[rr, equal] \arrow[dr] \arrow[dd] & & \rho^*\pi_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[dr] \arrow[dd] & \\
& \mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[rr, equal] \arrow[dd] & & \mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}$$

where the columns are exact and all right and downward-right arrows are surjections.

We can associate to the family of generalized stable quotients the following exact sequences of vector bundles on T

- $0 \to \rho_* \mathcal{I}_C(h) \to Sym^h \pi_* \mathcal{L}_k \to \pi_* \mathcal{L}_k^h \to 0$
- $Sym^h \pi_* \mathcal{L}_k \to t_i^* \mathcal{L}_k^h \to 0$, for $1 \le i \le m$
- $0 \to \pi_*(\mathcal{S}_1 \otimes \mathcal{L}_k^h) \to \mathbb{C}^n \otimes \pi_* \mathcal{L}_k^h \to \pi_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \to 0$
- $0 \to \pi_*(\mathcal{K}_j \otimes \mathcal{L}_k^h) \to \pi_*(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \to \pi_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \to 0$, for $1 \le j \le \ell - 1$.

Using the first exact sequence, we can replace the last ℓ sequences so we have the following collection of exact sequences of vector bundles (for $h \ge M_\ell$)

- $0 \to \rho_* \mathcal{I}_C(h) \to Sym^h \pi_* \mathcal{L}_k \to \pi_* \mathcal{L}_k^h \to 0$
- $Sym^h \pi_* \mathcal{L}_k \to t_i^* \mathcal{L}_k^h \to 0$, for $1 \le i \le m$
- $\mathbb{C}^n \otimes Sym^h \pi_* \mathcal{L}_k \to \pi_*(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \to 0$ for $1 \le j \le \ell$

Call each of these quotients $\mathcal{W}_e$, for $1 \le e \le m + 1 + \ell$. We have already seen that, from the first $m+1$ surjections, we are able to recover the curve C and the sections.

We claim that we can recover the flag sequence as well from the data of the collection of quotients $\{\mathcal{W}_e\}_{e=1}^{m+1+\ell}$ (along with the quotient maps). We describe this below (using [Nit05]).

By pulling back the $m + 2^{nd}$ quotient sequence and taking its kernel we recover

$$0 \to \rho^* \rho_* \mathcal{G}_1(h) \to \mathbb{C}^n \otimes \rho^* Sym^h \pi_* \mathcal{L}_k \to \rho^* \pi_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \to 0.$$

Consider the composition

$$\rho^* \rho_* \mathcal{G}_1(h) \to \mathbb{C}^n \otimes \rho^* Sym^h \pi_* \mathcal{L}_k \to \mathbb{C}^n \otimes \rho^* \pi_* \mathcal{L}_k^h \to \mathbb{C}^n \otimes \mu_* \mathcal{L}_k^h$$

where the first map is the same as in the exact sequence above, the second map is the pullback of the direct sum of n copies of the first map we were given as part of the starting data, and the third map is the direct sum of n copies of the surjection we obtained from M_ℓ regularity. By considering the diagram $(\star\star)$, if we look at the cokernel of the above composition $\rho^* \rho_* \mathcal{G}_1(h) \to \mathbb{C}^n \otimes \mu_* \mathcal{L}_k^h$, then we recover $\mathbb{C}^n \otimes \mu_* \mathcal{L}_k^h \to \mu_*(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \to 0$. We can twist this by $\mathcal{O}_{\mathbb{P}_T((\pi_* \mathcal{L}_k)^*)}(-h)$ and we recover the μ pushforward of the first quotient sequence

$$0 \to \mu_* \mathcal{S}_1 \to \mathbb{C}^n \otimes \mu_* \mathcal{O}_C \to \mu_* \mathcal{Q}_1 \to 0.$$

By the commutativity of $(\star)$, if we consider the cokernel of

$$\mu^*\mu_*\mathcal{S}_1 \longrightarrow \mathbb{C}^n \otimes \mu^*\mu_*\mathcal{O}_C \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_C$$

then we recover the quotient sequence $\mathbb{C}^n \otimes \mathcal{O}_C \to \mathcal{Q}_1 \to 0$.

We will now show inductively that we can recover each of the sequences

$$0 \to \mathcal{K}_j \to \mathcal{Q}_j \to \mathcal{Q}_{j+1} \to 0$$

given that we can recover the curve, its marked sections, and the first quotient sequence above.

Suppose for some j that we can recover all the quotient sequences in the line above for any $j' \leq j$.

Then, using the diagram $(\star\star)_{j+1}$, we see that if we consider the kernel of

$$\mathbb{C}^n \otimes \rho^* Sym^h \pi_* \mathcal{L}_k \to \rho^* \pi_*(\mathcal{Q}_{j+2} \otimes \mathcal{L}_k^h)$$

then we recover the exact sequence

$$0 \to \rho^*\rho_*\mathcal{G}_{j+2}(h) \to \mathbb{C}^n \otimes \rho^* Sym^h \pi_* \mathcal{L}_k \to \rho^* \pi_*(\mathcal{Q}_{j+2} \otimes \mathcal{L}_k^h) \to 0.$$

By the same diagram, if we consider the cokernel of the composition

$$\rho^*\rho_*\mathcal{G}_{j+2}(h) \to \mathbb{C}^n \otimes \rho^* Sym^h \pi_* \mathcal{L}_k \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(h) \to \mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h)$$

(the last two maps we have by the inductive hypothesis) then we recover

$$\mu_*(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \to \mu_*(\mathcal{Q}_{j+2} \otimes \mathcal{L}_k^h) \to 0.$$

Taking the kernel and twisting by $\mathcal{O}_{\mathbb{P}_T((\pi_*\mathcal{L}_k)^*)}(-h)$, we recover the exact sequence

$$0 \to \mu_*\mathcal{K}_{j+1} \to \mu_*\mathcal{Q}_{j+1} \to \mu_*\mathcal{Q}_{j+2} \to 0.$$

By commutativity of $(\star)_{j+1}$, taking the cokernel of

$$\mu^*\mu_*\mathcal{K}_{j+1} \to \mu^*\mu_*\mathcal{Q}_{j+1} \to \mathcal{Q}_{j+1}$$

we recover $\mathcal{Q}_{j+1} \to \mathcal{Q}_{j+2} \to 0$, and the induction is complete.

2.2.2 The classifying map

Given $t \in T$, once we pick an identification $H^0(C_t, \mathcal{L}_k) \cong W^*$, we obtain quotient sequences (for $h \geq M_\ell$)

- $Sym^h W^* \cong Sym^h \pi_* \mathcal{L}_k|_t \to \pi_* \mathcal{L}_k^h|_t \to 0$
- $Sym^h W^* \cong Sym^h \pi_* \mathcal{L}_k|_t \to t_j^* \mathcal{L}_k^h|_t \to 0$, for $1 \leq j \leq m$
- $\mathbb{C}^n \otimes Sym^h W^* \cong \mathbb{C}^n \otimes Sym^h \pi_* \mathcal{L}_k|_t \to \pi_*(\mathcal{Q}_i \otimes \mathcal{L}_k^h)|_t \to 0$ for $1 \leq i \leq \ell$

Let the quotients have ranks w_k. Considering the collection of all identifications yields a map from the closed points of T to the $GL(W)$ orbits of

$$\prod_{v=1}^{m+1} Gr(w_v, Sym^h W^*) \times \prod_{i=m+2}^{m+1+\ell} Gr(w_i, \mathbb{C}^n \otimes Sym^h W^*).$$

Notice that this identification is only up to $\mathbb{C}^*$ multiplication since multiplying by $\mathbb{C}^*$ does not change the kernel.

We seek to prove the following proposition.

Proposition 2.2.1. *There exists a set theoretic classifying map*

$$T \to \left(\prod_{v=1}^{m+1} Gr(w_v, Sym^h(W^*)) \times \prod_{i=m+2}^{m+1+\ell} Gr(w_i, \mathbb{C}^n \otimes Sym^h(W^*))\right)/GL(W)$$

where the action is the diagonal action, which factors through the map to moduli via a set theoretic injection

$$\begin{array}{c} \overline{Q}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \\ \big\downarrow \\ \left(\prod_{v=1}^{m+1} Gr(w_v, Sym^h(W^*)) \times \prod_{i=m+2}^{m+\ell+1} Gr(w_i, \mathbb{C}^n \otimes Sym^h(W^*))\right)/ GL(W). \end{array}$$

Proof. We proved the existence of the classifying map above. It suffices to prove the following lemma.

Lemma 2.2.1.1. *There exists an injective map of sets*

$$\begin{array}{c} \overline{Q}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \\ \big\downarrow \\ \left(\prod_{i=1}^{m+1} Gr(w_i, Sym^h(W^*)) \times Gr(w_{m+2}, \mathbb{C}^n \otimes Sym^h(W^*))\right)/ GL(W). \end{array}$$

Proof. Suppose

$$(C, \{p_j\}_{j=1}^m, 0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \ldots \to \mathcal{Q}_\ell \to 0)$$
$$(C', \{p'_j\}_{j=1}^m, 0 \hookrightarrow \mathcal{S}'_1 \hookrightarrow \ldots \hookrightarrow \mathcal{S}'_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}'_1 \to \ldots \to \mathcal{Q}'_\ell \to 0)$$

are two generalized stable quotients which are mapped to the same $GL(W)$ orbit in the product of Grassmannians.

Then, there exist isomorphisms $\phi_\alpha : H^0(C, \mathcal{L}_k) \cong W^*$, $\psi_\beta : H^0(C', \mathcal{L}_k) \cong W^*$ such that we have commutative diagrams $(**)$

- $$\begin{array}{ccccc} Sym^h H^0(C, \mathcal{L}_k) & \xrightarrow{\phi_\alpha} & Sym^h W^* & \xrightarrow{q_1} & H^0(C, \mathcal{L}_k^h) \\ & & \Big\| & & \big\downarrow \\ Sym^h H^0(C', \mathcal{L}_k) & \xrightarrow{\psi_\beta} & Sym^h W^* & \xrightarrow{q'_1} & H^0(C', \mathcal{L}_k^h) \end{array}$$

- $$\begin{tikzcd} Sym^h H^0(C, \mathcal{L}_k) \arrow[r, "\phi_\alpha"] & Sym^h W^* \arrow[r, "q_e"] \arrow[d, equal] & H^0(\mathbb{C}_{p_i} \otimes \mathcal{L}_k^h) \arrow[d] \\ Sym^h H^0(C', \mathcal{L}_k) \arrow[r, "\psi_\beta"] & Sym^h W^* \arrow[r, "q'_e"] & H^0(\mathbb{C}_{p'_i} \otimes \mathcal{L}_k^h) \end{tikzcd}$$

- $$\begin{tikzcd} \mathbb{C}^n \otimes Sym^h H^0(C, \mathcal{L}_k) \arrow[r, "\phi_\alpha"] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q_{m+1+i}"] \arrow[d, equal] & H^0(C, \mathcal{Q}_i \otimes \mathcal{L}_k^h) \arrow[d] \\ \mathbb{C}^n \otimes Sym^h H^0(C', \mathcal{L}_k) \arrow[r, "\psi_\beta"] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q'_{m+1+i}"] & H^0(C', \mathcal{Q}'_i \otimes \mathcal{L}_k^h) \end{tikzcd}$$

where $2 \le e \le m+1$, $1 \le i \le \ell$, and $e = j+1$. The isomorphisms ϕ_α, ψ_β yield embeddings of the curves C, C' and their markings $\{p_i\}_{i=1}^m$, $\{p'_i\}_{i=1}^m$ in $\mathbb{P}(W)$. If we consider the kernels of q_1, q'_1 in the first diagram, we recover

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{I}_C(h)) \arrow[r] \arrow[d] & Sym^h W^* \arrow[r] \arrow[d, equal] & H^0(C, \mathcal{L}_k^h) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(\mathcal{I}_{C'}(h)) \arrow[r] & Sym^h W^* \arrow[r] & H^0(C', \mathcal{L}_k^h) \arrow[r] & 0 \end{tikzcd}$$

by definition of the classifying map. From what we have already seen, if we consider the cokernels of the maps

$$H^0(\mathcal{I}_C(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(h)$$
$$H^0(\mathcal{I}_{C'}(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(h)$$

then we recover

$$\mathcal{O}_{\mathbb{P}(W)}(h) \to \overline{\phi}_{\alpha *}\mathcal{L}_k^h \to 0$$
$$\mathcal{O}_{\mathbb{P}(W)}(h) \to \overline{\psi}_{\beta *}\mathcal{L}_k^h \to 0.$$

Now, putting this together with the data we have already found, we have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[dd] & & 0 \arrow[dd] \\
H^0(\mathcal{I}_C(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[rr] \arrow[dr] \arrow[dd] & & H^0(\mathcal{I}_{C'}(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[dr] \arrow[dd] & \\
& \mathcal{I}_C(h) \arrow[dd] & & \mathcal{I}_{C'}(h) \arrow[dd] \\
Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[rr, equal] \arrow[dr] \arrow[dd] & & Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[dr] \arrow[dd] & \\
& \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[rr, equal] \arrow[dd] & & \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[dd] \\
H^0(C, \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[rr] \arrow[dr] \arrow[dd] & & H^0(C', \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[dr] \arrow[dd] & \\
& \overline{\phi}_{\alpha *}\mathcal{L}_k^h \arrow[dd] & & \overline{\psi}_{\beta *}\mathcal{L}_k^h \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}
$$

where each of the downward-right arrows are surjections, each right arrow is an isomorphism, and the columns are exact. We can fill in the kernels of the leftmost of the two collections of downward-right arrows, and we see that we have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

where $\mathcal{E}_i$ is the kernel of the i^{th} leftmost downward-right arrow. We can do the same for the rightmost downward-right arrows yielding an exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

where $\mathcal{E}_2 \cong \mathcal{F}_2$. Since the composition

$$\mathcal{E}_2 \to \mathcal{F}_2 \to Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(h) \to \overline{\psi}_{\beta *}\mathcal{L}_k^h$$

is zero, and $\mathcal{E}_2 \to \mathcal{E}_3$ is surjective, by commutativity of the diagram we see that the composition

$$\mathcal{E}_3 \to H^0(C, \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to H^0(C', \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \overline{\psi}_{\beta *}\mathcal{L}_k^h$$

is the zero map, which yields a morphism $\mathcal{E}_3 \to \mathcal{F}_3$. In turn, this induces a morphism $\overline{\phi}_{\alpha *}\mathcal{L}_k^h \to \overline{\psi}_{\beta *}\mathcal{L}_k^h$. By commutativity, we see that this must be surjective.

But now since all of the right arrows are isomorphisms we can repeat this argument (by reversing the arrows) to show that there is an induced morphism $\overline{\psi}_{\beta *}\mathcal{L}_k^h \to \overline{\phi}_{\alpha *}\mathcal{L}_k^h$. By the commutativity of the following square (‡) in either direction

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] \arrow[d, equal] & \overline{\phi}_{\alpha *}\mathcal{L}_k^h \arrow[d, shift left] \\ \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \overline{\psi}_{\beta *}\mathcal{L}_k^h \arrow[u, shift left] \end{tikzcd}$$

and the fact that the rows are surjective, we see that the composition of the two opposite direction vertical maps in either order is the identity, and so C and C' represent the same curve embedded in $\mathbb{P}(W)$. Therefore the two generalized stable quotients correspond to the same point in *Hilb*, recalling the notation we used in the construction of the moduli space.

For the markings, we can consider the kernels of q_j, q'_j in the second collection of diagrams, and we recover the diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{I}_{p_i} \otimes \mathcal{L}_k^h) \arrow[r] \arrow[d] & Sym^h W^* \arrow[r] \arrow[d, equal] & H^0(\mathbb{C}_{p_i} \otimes \mathcal{L}_k^h) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(\mathcal{I}_{p'_i} \otimes \mathcal{L}_k^h) \arrow[r] & Sym^h W^* \arrow[r] & H^0(\mathbb{C}_{p'_i} \otimes \mathcal{L}_k^h) \arrow[r] & 0 \end{tikzcd}$$

by definition of the classifying map. Just as for the quotients q_1, q'_1, if we consider the cokernels of the maps

$$H^0(\mathcal{I}_{p_i} \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(h)$$
$$H^0(\mathcal{I}_{p'_i} \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(h)$$

then we recover the quotients

$$\mathcal{O}_{\mathbb{P}(W)} \to \overline{\phi}_{\alpha *}\mathbb{C}_{p_i} \otimes \mathcal{L}_k^h$$
$$\mathcal{O}_{\mathbb{P}(W)} \to \overline{\psi}_{\beta *}\mathbb{C}_{p'_i} \otimes \mathcal{L}_k^h$$

The same diagram chase as before will show that we have a commutative diagram with the right vertical arrow an isomorphism

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] \arrow[d, equal] & \overline{\phi}_{\alpha *}\mathbb{C}_{p_i} \otimes \mathcal{L}_k^h \arrow[d] \\ \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \overline{\phi}_{\alpha *}\mathbb{C}_{p'{}_i} \otimes \mathcal{L}_k^h \end{tikzcd}$$

This tells us that the points represent the same point in $\mathbb{P}(W)$ under the h-uple embedding, and so they represent the same point in $\mathbb{P}(W)$.

Now we show that the flag sequences are isomorphic. By definition of the classifying map, we have unique factorizations

$$\begin{tikzcd}
\mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q_1"] \arrow[d, equal] \arrow[rr, bend left, "q_{m+2}"] & \mathbb{C}^n \otimes H^0(C, \mathcal{L}_k^h) \arrow[r, "q"] \arrow[d] & H^0(C, \mathcal{Q}_1 \otimes \mathcal{L}_k^h) . \arrow[d] \\
\mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q_1'"'] \arrow[rr, bend right, "q_{m+2}'"'] & \mathbb{C}^n \otimes H^0(C', \mathcal{L}_k^h) \arrow[r, "q'"] & H^0(C, \mathcal{Q}_1' \otimes \mathcal{L}_k^h)
\end{tikzcd}$$

Take the kernels of q_{m+2} and q'_{m+2} to yield the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{G}_1(h)) \arrow[r] \arrow[d] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r] \arrow[d, equal] & H^0(C, \mathcal{Q}_1 \otimes \mathcal{L}_k^h) \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & H^0(\mathcal{G}_1'(h)) \arrow[r] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r] & H^0(C', \mathcal{Q}_1' \otimes \mathcal{L}_k^h) \arrow[r] & 0
\end{tikzcd}$$

Recall that we have the following commutative diagrams as a result of M_ℓ regularity

$$\begin{tikzcd}
H^0(\mathcal{G}_1(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & \mathbb{C}^n \otimes Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & \mathbb{C}^n \otimes H^0(C, \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & H^0(C, \mathcal{Q}_1 \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[d] \\
\mathcal{G}_1(h) \arrow[r] & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h \arrow[r] & \overline{\phi}_{\alpha *}(\mathcal{Q}_1 \otimes \mathcal{L}_k^h)
\end{tikzcd}$$

$$\begin{tikzcd}
H^0(\mathcal{G}_1'(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & \mathbb{C}^n \otimes Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & \mathbb{C}^n \otimes H^0(C', \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d] & H^0(C', \mathcal{Q}_1' \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[d] \\
\mathcal{G}_1'(h) \arrow[r] & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h \arrow[r] & \overline{\psi}_{\beta *}(\mathcal{Q}_1' \otimes \mathcal{L}_k^h)
\end{tikzcd}$$

where the columns are surjective. Therefore we can recover

$$\mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h \to \overline{\phi}_{\alpha *}(\mathcal{Q}_1 \otimes \mathcal{L}_k^h)$$
$$\mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h \to \overline{\psi}_{\beta *}(\mathcal{Q}_1' \otimes \mathcal{L}_k^h)$$

as the cokernels of the compositions

$$H^0(\mathcal{G}_1(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes H^0(C, \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h$$
$$H^0(\mathcal{G}_1'(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes H^0(C', \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h$$

using what we have already proved.

Taking the kernels, we obtain a diagram

$$
\begin{tikzcd}[column sep=small]
0 \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[dd] & & 0 \arrow[dd] \\
H^0(\mathcal{S}_1 \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[rr] \arrow[dd] \arrow[dr] & & H^0(\mathcal{S}_1' \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[dd] \arrow[dr] & \\
& \overline{\phi}_{\alpha *}(\mathcal{S}_1 \otimes \mathcal{L}_k^h) \arrow[dd] & & \overline{\psi}_{\beta *}(\mathcal{S}_1' \otimes \mathcal{L}_k^h) \arrow[dd] \\
\mathbb{C}^n \otimes H^0(C, \mathcal{L}_k^h) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathbb{C}^n \otimes H^0(C', \mathcal{L}_k^h) \arrow[dd] \arrow[dr] & \\
& \mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h \arrow[rr] \arrow[dd] & & \mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h \arrow[dd] \\
H^0(C, \mathcal{Q}_1 \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[rr] \arrow[dd] \arrow[dr] & & H^0(C', \mathcal{Q}_1' \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[dd] \arrow[dr] & \\
& \overline{\phi}_{\alpha *}(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \arrow[dd] & & \overline{\psi}_{\beta *}(\mathcal{Q}_1' \otimes \mathcal{L}_k^h) \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}
$$

where the arrow $\mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h \to \mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h$ exists from what we have already shown, and it is an isomorphism. The same diagram chase as before produces an arrow

$$\overline{\phi}_{\alpha *}(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \to \overline{\psi}_{\beta *}(\mathcal{Q}_1' \otimes \mathcal{L}_k^h)$$

and vice versa (by reversing the horizontal arrows in the diagram)

The fact that the composition of these is the identity (in either order) follows from the commutativity in either direction of

$$
\begin{tikzcd}
\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] \arrow[d, equal] & \mathbb{C}^n \otimes \overline{\phi}_{\alpha *}\mathcal{L}_k^h \arrow[r] \arrow[d, shift left] & \overline{\phi}_{\alpha *}(\mathcal{Q}_1 \otimes \mathcal{L}_k^h) \arrow[d, shift left] \\
\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \mathbb{C}^n \otimes \overline{\psi}_{\beta *}\mathcal{L}_k^h \arrow[r] \arrow[u, shift left, bend left] & \overline{\psi}_{\beta *}(\mathcal{Q}_1' \otimes \mathcal{L}_k^h). \arrow[u, shift left, bend left]
\end{tikzcd}
$$

After twisting by $\mathcal{O}_{\mathbb{P}(W)}(-h)$, we have a commutative diagram

$$
\begin{tikzcd}
\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] \arrow[d, equal] & \mathbb{C}^n \otimes i_{C'' *}\tilde{\phi}_{\alpha *}\mathcal{O}_C \arrow[r] \arrow[d] & i_{C'' *}\tilde{\phi}_{\alpha *}\mathcal{Q}_1 \arrow[d] \\
\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)} \arrow[r] & \mathbb{C}^n \otimes i_{C'' *}\tilde{\psi}_{\beta *}\mathcal{O}_{C'} \arrow[r] & i_{C'' *}\tilde{\psi}_{\beta *}\mathcal{Q}_1'
\end{tikzcd}
$$

where here $\tilde{\phi}_\alpha : C \to C''$, $\tilde{\psi}_\beta : C' \to C''$ are the isomorphisms with the embedded curve in $\mathbb{P}(W)$, which we call C''.

Notice that the first two arrows factor uniquely through the map

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes i_{C''*}\mathcal{O}_{C''}.$$

Thus we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^n \otimes i_{C''*}\mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \mathbb{C}^n \otimes i_{C''*}\tilde{\phi}_{\alpha*}\mathcal{O}_C \arrow[r] \arrow[d] & i_{C''*}\tilde{\phi}_{\alpha*}\mathcal{Q}_1 \arrow[d] \\
\mathbb{C}^n \otimes i_{C''*}\mathcal{O}_{C''} \arrow[r] & \mathbb{C}^n \otimes i_{C''*}\tilde{\psi}_{\beta*}\mathcal{O}_{C'} \arrow[r] & i_{C''*}\tilde{\psi}_{\beta*}\mathcal{Q}_1'
\end{tikzcd}.$$

Let $\tilde{\mathcal{S}}_1$, $\tilde{\mathcal{S}}_1{}'$ be the kernels of

$$\mathbb{C}^n \otimes \mathcal{O}_{C''} \to \mathbb{C}^n \otimes \tilde{\phi}_{\alpha*}\mathcal{O}_C \to \tilde{\phi}_{\alpha*}\mathcal{Q}_1$$
$$\mathbb{C}^n \otimes \mathcal{O}_{C''} \to \mathbb{C}^n \otimes \tilde{\psi}_{\beta*}\mathcal{O}_{C'} \to \tilde{\psi}_{\beta*}\mathcal{Q}_1'.$$

Pulling back along $i_{C''}$ and using the fact that, for all sheaves $\mathcal{F}$ on C'', $i^*_{C''}i_{C''*}\mathcal{F} \to \mathcal{F}$ is surjective, we find the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[dd] & & 0 \arrow[dd] \\
i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1 \arrow[rr] \arrow[dr] \arrow[dd] & & i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1{}' \arrow[dr] \arrow[dd] & \\
& \tilde{\mathcal{S}}_1 \arrow[dd] & & \tilde{\mathcal{S}}_1 \arrow[dd] \\
\mathbb{C}^n \otimes i^*_{C''}i_{C''*}\mathcal{O}_{C''} \arrow[rr, equal] \arrow[dr] \arrow[dd] & & \mathbb{C}^n \otimes i^*_{C''}i_{C''*}\mathcal{O}_{C''} \arrow[dr] \arrow[dd] & \\
& \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[rr, equal] \arrow[dd, "\tilde{q}"] & & \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[dd, "\tilde{q}'"] \\
i^*_{C''}i_{C''*}\tilde{\phi}_{\alpha*}\mathcal{Q}_1 \arrow[rr] \arrow[dr] \arrow[dd] & & i^*_{C''}i_{C''*}\tilde{\psi}_{\beta*}\mathcal{Q}_1' \arrow[dr] \arrow[dd] & \\
& \tilde{\phi}_{\alpha*}\mathcal{Q}_1 \arrow[dd] & & \tilde{\psi}_{\beta*}\mathcal{Q}_1' \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}$$

where $\tilde{q}$ and $\tilde{q}'$ are recovered as the cokernels of the compositions

$$i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1 \to \mathbb{C}^n \otimes i^*_{C''}i_{C''*}\mathcal{O}_{C''} \to \mathbb{C}^n \otimes \mathcal{O}_{C''}$$
$$i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1{}' \to \mathbb{C}^n \otimes i^*_{C''}i_{C''*}\mathcal{O}_{C''} \to \mathbb{C}^n \otimes \mathcal{O}_{C''}.$$

We can replace $i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1$, $i^*_{C''}i_{C''*}\tilde{\mathcal{S}}_1{}'$ with their images in $\mathbb{C}^n \otimes i^*_{C''}i_{C''*}\mathcal{O}_{C''}$ and we will still have a commutative diagram except now the rows will be exact, and all downward right arrows will be surjective. The same diagram chase as before (for showing the curves are the same in $\mathbb{P}(W)$) yields a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \tilde{\phi}_{\alpha *}\mathcal{Q}_1 \arrow[d] \\ \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] & \tilde{\psi}_{\beta *}\mathcal{Q}'_1 \end{tikzcd}.$$

The same proof as before shows that the right vertical arrow is an ismorphism.

Since $\tilde{\phi}_{\alpha *}$ and $\tilde{\phi}_{\alpha}^{-1\,*}$ are naturally isomorphic functors, and $\tilde{\psi}_{\beta *}$ and $\tilde{\psi}_{\beta}^{-1\,*}$ are naturally isomorphic functors, we see that the first steps of the flag sequences are equivalent. Thus, we have a commutative diagram with surjective rows and whose columns are isomorphisms

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_1 \arrow[d] \\ \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_1. \end{tikzcd}$$

We will inductively prove that we have a commutative diagram with the right arrows surjections and the downward arrows isomorphisms

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_\ell \arrow[d] \\ \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_1 \arrow[r] & \cdots \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_\ell. \end{tikzcd}$$

Suppose now that for $j \geq 1$, the first $m + j + 1$ diagrams in $(**)$ yield the following diagram

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_j \arrow[d] \\ \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_1 \arrow[r] & \cdots \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_j \end{tikzcd}$$

whose right arrows are surjective and whose columns are isomorphisms.

Now, consider the $m + j + 2^{nd}$ diagram in $(**)$

$$\begin{tikzcd} \mathbb{C}^n \otimes Sym^h H^0(C, \mathcal{L}_k) \arrow[r, "\phi_\alpha"] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q_{m+j+2}"'] \arrow[d, equal] & H^0(C, \mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[d] \\ \mathbb{C}^n \otimes Sym^h H^0(C', \mathcal{L}_k) \arrow[r, "\psi_\beta"] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r, "q'_{m+j+2}"'] & H^0(C', \mathcal{Q}'_{j+1} \otimes \mathcal{L}_k^h). \end{tikzcd}$$

We know that q_{m+j+2} and q'_{m+j+2} factor as

$$\begin{tikzcd}
\mathbb{C}^n \otimes Sym^h W^* \arrow[r] \arrow[d, equal] \arrow[rrr, bend left, "q_{m+j+2}"] & \cdots \arrow[r] & H^0(C, \mathcal{Q}_j \otimes \mathcal{L}_k^h) \arrow[r, "\overline{q}_{j+1}"] \arrow[d] & H^0(C, \mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[d] \\
\mathbb{C}^n \otimes Sym^h W^* \arrow[r] \arrow[rrr, bend right, "q'_{m+j+2}"'] & \cdots \arrow[r] & H^0(C, \mathcal{Q}'_j \otimes \mathcal{L}_k^h) \arrow[r, "\overline{q}'_{j+1}"] & H^0(C', \mathcal{Q}'_{j+1} \otimes \mathcal{L}_k^h).
\end{tikzcd}$$

Taking the kernels of q_{m+j+2} and $\overline{q}'_{m+j+2}$, we obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(C, \mathcal{G}_{j+1}(h)) \arrow[r] \arrow[d] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r] \arrow[d, equal] & H^0(C, \mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(C', \mathcal{G}'_{j+1}(h)) \arrow[r] & \mathbb{C}^n \otimes Sym^h W^* \arrow[r] & H^0(C', \mathcal{Q}'_{j+1} \otimes \mathcal{L}_k^h) \arrow[r] & 0.
\end{tikzcd}$$

The analogue of $(\star\star)_j$ for the case of T a point shows that if we consider the cokernels of the compositions

$$H^0(C, \mathcal{G}_{j+1}(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to H^0(C, \mathcal{Q}_j \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \overline{\phi}_{\alpha *}(\mathcal{Q}_j \otimes \mathcal{L}_k^h)$$
$$H^0(C', \mathcal{G}'_{j+1}(h)) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathbb{C}^n \otimes Sym^h W^* \otimes \mathcal{O}_{\mathbb{P}(W)} \to H^0(C', \mathcal{Q}_j \otimes \mathcal{L}_k^h) \otimes \mathcal{O}_{\mathbb{P}(W)} \to \overline{\psi}_{\beta *}(\mathcal{Q}'_j \otimes \mathcal{L}_k^h)$$

(the middle and second maps coming from the inductive hypothesis) then we recover

$$\overline{\phi}_{\alpha *}(\mathcal{Q}_j \otimes \mathcal{L}_k^h) \to \overline{\phi}_{\alpha *}(\mathcal{Q}_{j+1} \otimes \mathcal{L}_k^h) \to 0$$
$$\overline{\psi}_{\beta *}(\mathcal{Q}'_j \otimes \mathcal{L}_k^h) \to \overline{\psi}_{\beta *}(\mathcal{Q}'_{j+1} \otimes \mathcal{L}_k^h) \to 0.$$

We get a commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[dd] & & 0 \arrow[dd] \\
H^0(C,\mathcal{K}_j\otimes\mathcal{L}_k^h)\otimes\mathcal{O}_{\mathbb{P}(W)} \arrow[rr] \arrow[dd] \arrow[dr] & & H^0(C',\mathcal{K}'_j\otimes\mathcal{L}_k^h) \arrow[dd] \arrow[dr] & \\
& \overline{\phi}_{\alpha *}(\mathcal{K}_j\otimes\mathcal{L}_k^h) \arrow[dd] & & \overline{\psi}_{\beta *}(\mathcal{K}'_j\otimes\mathcal{L}_k^h) \arrow[dd] \\
H^0(C,\mathcal{Q}_g\otimes\mathcal{L}_k^h) \arrow[rr] \arrow[dd] \arrow[dr] & & H^0(C',\mathcal{Q}'_j\otimes\mathcal{L}_k^h) \arrow[dd] \arrow[dr] & \\
& \overline{\phi}_{\alpha *}(\mathcal{Q}_j\otimes\mathcal{L}_k^h) \arrow[rr] \arrow[dd] & & \overline{\psi}_{\beta *}(\mathcal{Q}'_j\otimes\mathcal{L}_k^h) \arrow[dd] \\
H^0(C,\mathcal{Q}_{j+1}\otimes\mathcal{L}_k^h) \arrow[rr] \arrow[dd] \arrow[dr] & & H^0(C',\mathcal{Q}'_{j+1}\otimes\mathcal{L}_k^h) \arrow[dd] \arrow[dr] & \\
& \overline{\phi}_{\alpha *}(\mathcal{Q}_{j+1}^{\vee}\otimes\mathcal{L}_k^h) \arrow[dd] & & \overline{\psi}_{\beta *}(\mathcal{Q}'_{j+1}\otimes\mathcal{L}_k^h) \arrow[dd] \\
0 & & 0 & \\
& 0 & & 0
\end{tikzcd}$$

The same diagram chase we have already performed allows us to produce arrows

$$\overline{\phi}_{\alpha *}(\mathcal{Q}_{j+1}\otimes\mathcal{L}_k^h) \to \overline{\psi}_{\beta *}(\mathcal{Q}'_{j+1}\otimes\mathcal{L}_k^h)$$
$$\overline{\psi}_{\beta *}(\mathcal{Q}'_{j+1}\otimes\mathcal{L}_k^h \to \overline{\phi}_{\alpha *}(\mathcal{Q}_{j+1}\otimes\mathcal{L}_k^h).$$

The fact that the composition of these is the identity follows from the commutativity in either direction of

$$\begin{tikzcd}
\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] \arrow[d, equal] & \overline{\phi}_{\alpha *}(\mathcal{Q}_{j+1}\otimes\mathcal{L}_k^h) \arrow[d] \\
\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}(W)}(h) \arrow[r] & \overline{\psi}_{\beta *}(\mathcal{Q}'_{j+1}\otimes\mathcal{L}_k^h). \arrow[u, bend left]
\end{tikzcd}$$

Taking kernels and twisting by $\mathcal{O}_{\mathbb{P}(W)}(-h)$, we find the commutative diagram, all of whose rows are exact and whose columns are isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \overline{\phi}_{\alpha *}\mathcal{K}_j \arrow[r] \arrow[d] & \overline{\phi}_{\alpha *}\mathcal{Q}_j \arrow[r] \arrow[d] & \overline{\phi}_{\alpha *}\mathcal{Q}_{j+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\psi}_{\beta *}\mathcal{K}'_j \arrow[r] & \overline{\psi}_{\beta *}\mathcal{Q}'_j \arrow[r] & \overline{\psi}_{\beta *}\mathcal{Q}'_{j+1} \arrow[r] & 0.
\end{tikzcd}$$

As before, let $\tilde{\phi}_\alpha : C \to C''$, $\tilde{\psi}_\beta : C' \to C''$ be the isomorphisms with the embedded curve $C'' \subset \mathbb{P}(W)$. Pulling back to C'', we have a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[dd] & & 0 \arrow[dd] \\
i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{K}_j \arrow[rr] \arrow[dr] \arrow[dd] & & i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{K}'_j \arrow[dr] \arrow[dd] & \\
 & \tilde{\phi}_{\alpha*} \mathcal{K}_j \arrow[dd] & & \tilde{\psi}_{\beta*} \mathcal{K}'_j \arrow[dd] \\
i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{Q}_j \arrow[rr, equal] \arrow[dr] \arrow[dd] & & i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{Q}'_j \arrow[dr] \arrow[dd] & \\
 & \tilde{\phi}_{\alpha*} \mathcal{Q}_j \arrow[rr] \arrow[dd] & & \tilde{\psi}_{\beta*} \mathcal{Q}'_j \arrow[dd] \\
i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{Q}_{j+1} \arrow[rr] \arrow[dr] \arrow[dd] & & i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{Q}'_{j+1} \arrow[dr] \arrow[dd] & \\
 & \tilde{\phi}_{\alpha*} \mathcal{Q}_{j+1} \arrow[dd] & & \tilde{\psi}_{\beta*} \mathcal{Q}'_{j+1} \arrow[dd] \\
0 & & 0 & \\
 & 0 & & 0
\end{tikzcd}
$$

Here we recovered the arrows

$$
\begin{gathered}
\tilde{\phi}_{\alpha*} \mathcal{Q}_j \to \tilde{\phi}_{\alpha*} \mathcal{Q}_{j+1} \\
\tilde{\psi}_{\beta*} \mathcal{Q}'_j \to \tilde{\phi}_{\beta*} \mathcal{Q}'_{j+1}
\end{gathered}
$$

as the cokernels of

$$
\begin{gathered}
i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{K}_j \to i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{Q}_j \to \tilde{\phi}_{\alpha*} \mathcal{Q}_j \\
i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{K}'_j \to i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{Q}'_j \to \tilde{\psi}_{\beta*} \mathcal{Q}'_j
\end{gathered}
$$

from which we are able to recover the kernels.

Replacing $i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{K}_j$, $i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{K}'_j$ by their images in $i^*_{C''} i_{C''*} \tilde{\phi}_{\alpha*} \mathcal{Q}_j$, $i^*_{C''} i_{C''*} \tilde{\psi}_{\beta*} \mathcal{Q}'_j$, respectivley, we obtain a commutative diagram all of whose columns are exact, whose right arrows are isomorphisms, and whose downward-right arrows are surjections.

The same diagram chase as before produces an isomorphism $\tilde{\phi}_{\alpha*} \mathcal{Q}_{j+1} \cong \tilde{\psi}_{\beta*} \mathcal{Q}'_{j+1}$ fitting into a commutative diagram

$$\begin{tikzcd} \tilde{\phi}_{\alpha *}\mathcal{Q}_j \arrow[r] \arrow[d] & \tilde{\psi}_{\beta *}\mathcal{Q}'_j \arrow[d] \\ \tilde{\phi}_{\alpha *}\mathcal{Q}_{j+1} \arrow[r] & \tilde{\psi}_{\beta *}\mathcal{Q}'_{j+1}. \end{tikzcd}$$

Observing that $\tilde{\phi}_{\alpha *}$ and $\tilde{\phi}_{\alpha}^{-1\,*}$ are naturally isomorphic, and $\tilde{\psi}_{\beta *}$ and $\tilde{\psi}_{\beta}^{-1\,*}$ are naturally isomorphic, we can concatenate this square with the preexisting commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] \arrow[d, equal] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \tilde{\phi}_{\alpha}^{-1\,*}\mathcal{Q}_j \arrow[d] \\ \mathbb{C}^n \otimes \mathcal{O}_{C''} \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_1 \arrow[r] & \cdots \arrow[r] & \tilde{\psi}_{\beta}^{-1\,*}\mathcal{Q}'_j \end{tikzcd}$$

completes the induction. This proves that the two generalized stable quotients represent the same point in the moduli space. □

Therefore there is a set theoretic injective map from the moduli space to the $GL(W)$ orbits of the product of Grassmannians. It is clear from what we have shown above that the classifying map factors through this map. □

2.2.3 Ampleness

In the previous section, we produced a classifying map which was given by the data of $m+1+\ell$ vector bundle quotients

- $Sym^h \pi_* \mathcal{L}_k \to \mathcal{W}_j \to 0$, for $1 \le j \le m+1$
- $\mathbb{C}^n \otimes Sym^h \pi_* \mathcal{L}_k \to \mathcal{W}_{m+1+i} \to 0$, for $1 \le i \le \ell$.

By *Corollary 3.4i* of [Kol90], since $\pi_* \mathcal{L}_k$ is semipositive, all of the above vector bundle quotients are also semipositive.

Given any family of generalized stable quotients over a base scheme, we can consider the line bundle $\bigotimes_{v=1}^{m+1+\ell} det(\mathcal{W}_v)$ on the base. Since the line bundle is functorial, it is naturally a line bundle on the moduli stack.

By [KM97], the coarse moduli space exists as an algebraic space. Since generalized stable quotients have finite automorphism groups, some tensor power of this line bundle descends to a line bundle on the coarse moduli space. We will show in this section that a large enough tensor power of the line bundle $\left(\bigotimes_{v=1}^{m+1+\ell} det(\mathcal{W}_v)\right)^N$ is ample on the coarse moduli space, for $N >> 0$, using the techniques of [Kol90].

By *Theorem 16.6* of [LMB00], there exists a scheme T together with a finite surjective generically étale morphism to $\overline{\mathcal{Q}}_{g,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d})$, which yields a finite surjective map

$$T \to \overline{Q}_{g,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d})^c$$

where T has a universal family of stable quasimaps over it (by pulling back the universal family under the map to the moduli stack) and its moduli map is finite and surjective. The line bundle $\left(\bigotimes_{v=1}^{m+1+\ell} det(\mathcal{W}_v)\right)^N$ pulls back to the corresponding line bundle over T.

Since the moduli map is finite, it suffices to prove that $\left(\bigotimes_{v=1}^{m+\ell+1} det(\mathcal{W}_v)\right)^N$ is ample on T.

Here, the classifying map is finite (following [Kol90], [Has03]) if it has finite fibers and each point in a $GL(W)$ orbit in the image of the classifying map has the property that its stabilizer is finite. Since generalized stable quotients have finite automorphism groups, and the map to moduli is finite, it is clear we are in the above situation.

Using the reduction steps from [KP], *Lemma 4.6*, we can reduce to the case of the *Ampleness Lemma 3.9* in [Kol90], which allows us to work with just one vector bundle and one quotient vector bundle.

We will describe this reduction.

To start, we can embed

$$\prod_{j=1}^{m+1} Gr(w_j, Sym^h W^*) \times \prod_{i=1}^{\ell} Gr(w_{m+1+i}, \mathbb{C}^n \otimes Sym^h W^*)$$

in the Grassmannian

$$Gr\left(\sum_{v=1}^{m+\ell+1} w_v\,,\ \bigoplus_{j=1}^{m+1} Sym^h W^* \oplus \bigoplus_{i=1}^{\ell} \mathbb{C}^n \otimes Sym^h W^*\right).$$

We let $GL(W)$ act on

$$\bigoplus_{j=1}^{m+1} Sym^h W^* \oplus \bigoplus_{i=1}^{\ell} \mathbb{C}^n \otimes Sym^h W^*$$

via its natural action on $Sym^h W^*$. Notice that the embedding of Grassmannians is a $GL(W)$ invariant embedding, as in [KP]. Therefore we get an embedding

$$\begin{array}{c} \left(\prod_{j=1}^{m+1} Gr(w_j, Sym^h W^*) \times \prod_{i=1}^{\ell} Gr(w_{m+1+i}, \mathbb{C}^n \otimes Sym^h W^*)\right) / GL(W) \\ \big\downarrow \\ Gr\left(\sum_{v=1}^{m+\ell+1} w_v\,,\ \bigoplus_{j=1}^{m+1} Sym^h W^* \oplus \bigoplus_{i=1}^{\ell} \mathbb{C}^n \otimes Sym^h W^*\right) / GL(W). \end{array}$$

In this case, the classifying map is given by taking the direct sum of the quotients, and the determinant of this bundle is still $\bigotimes_{v=1}^{m+\ell+1} det(\mathcal{W}_v)$. The classifying map will still be finite.

By [Kol90] *Ampleness Lemma 3.9*, the line bundle $\left(\bigotimes_{v=1}^{m+\ell+1} det(\mathcal{W}_v)\right)^N$ is ample on T. Therefore $\left(\bigotimes_{v=1}^{m+\ell+1} det(\mathcal{W}_v)\right)^N$ is ample on $\overline{Q}_{g,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$.

This proves Theorems 0.4.1, 0.5.1.

3 Picard Rank Calculations I

In this chapter we will calculate the Picard rank of the moduli space $\overline{\mathcal{Q}}_{0,m}(Gr(r,n), d)$ using the natural 1-dimensional torus action.

3.1 Betti numbers via torus actions

In this subsection we recall the work of [Opr06a] necessary to perform the calculation of the dimension of the second cohomology group of the moduli stack of genus zero stable quotients to the flag variety.

Let $\mathbb{T} \cong \mathbb{C}^*$ act on $\mathbb{C}^n$ with weights $-\lambda_1, \ldots, -\lambda_n$. This induces an action on $Fl(\overline{r}, \mathbb{C}^n)$ and $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ by translation.

Lemma 3.1.1. *The Betti numbers of $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ are given by the following sum over the fixed loci $\mathcal{F}_i$*

$$h^m(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) = \sum_i h^{m-2n_i^-}(\mathcal{F}_i),$$

Here n_i^- is the number of negative weights on the tangent space at a point in the fixed locus $\mathcal{F}_i$.

The rational Chow rings and the rational cohomology are isomorphic:

$$A^*(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q} \cong H^{2*}(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q}.$$

Proof. By *Theorem 2.5* of [AHR], for any closed point q of $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$, there exists a reparameterization $\phi : \mathbb{T} \to \mathbb{T}$ and an étale neighborhood

$$(Spec(R), v) \to (\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}), q)$$

which is equivariant with respect to ϕ. This yields an affine smooth étale atlas which is $\mathbb{T}$-equivariant. This allows us to apply Theorems 0.4.1, 0.5.1, together with *Proposition 5* and *Lemma 6* of [Opr06a], from which the first result follows. We will see later that the rational Chow groups and the rational cohomology groups of the fixed loci are isomorphic, from which the second statement follows by [Opr06a]. □

Remark 3.1.1. *Given the above lemma, to calculate the Picard rank of the moduli space, it suffices to determine the second or zeroth Betti numbers of the fixed loci with* 0 *or* 1 *negative weight on their normal bundles, respectively.*

3.2 Tangent space calculations I

In this section we will count the number of fixed loci with either zero or 1 negative weight on their tangent bundles.

3.2.1 The weights on the tangent space I

Here we collect the weights on the different $\mathbb{C}^*$ representations that arise in the tangent space calculations.

We begin by recalling the facts needed for the calculation of the relevant fixed loci, following [MOP11]:

- $\mathbb{C}^*$ acts on $\mathbb{C}^n$ as $t\cdot(z_1,\dots,z_n) := (\,t^{-\lambda_1},\dots,t^{-\lambda_n}\,)$, $\lambda_j < \lambda_i$ for $i<j$, with fixed points $\langle e_{i_1},\dots,e_{i_r}\rangle := span(e_{i_1},\dots,e_{i_r})$, where $e_k := (0,\dots,1,\dots 0)$ with the 1 in the k^{th} position.

- The fixed rational curves in the Grassmannian are of the form
$$[s:t]\mapsto\langle e_{i_1},\dots,e_{i_{r-1}}, s\cdot e_{i_r}\ +t\cdot e_{i_{r+1}}\rangle$$
(see [Wit95]).

- Let $\mathcal{S}$, $\mathcal{Q}$ be the universal subbundle and quotient, respectively, over the Grassmannian. Then $TGr(r,n)\cong\mathcal{S}^*\otimes\mathcal{Q}$, and $TGr(r,n)|_{\langle e_{i_1},\dots,e_{i_r}\rangle}$ has weights $\lambda_{i_j}-\lambda_k$, where $j=1,\dots,r$ and $k\in\{1,\dots,n\}\backslash\{i_j\}_{j=1}^r$.

- Given a degree d map
$$f:\mathbb{P}^1\to\mathbb{P}^1 := [s:t]\mapsto\langle e_{i_1},\dots,e_{i_{r-1}}, s^d\cdot e_{i_r}+t^d\cdot e_{i_{r+1}}\rangle\subset Gr(r,n)$$
from a curve to a fixed curve in the Grassmannian, the weights on the tangent space to the preimages of $[1:0]$ and $[0:1]$ are $\frac{\lambda_{i_r}-\lambda_{i_{r+1}}}{d}$ and $\frac{\lambda_{i_{r+1}}-\lambda_{i_r}}{d}$, respectively; see *Exercise 27.2.2*, [HKK+03] page 538.

Associated to a $\mathbb{C}^*$-fixed genus zero stable quotient $(C, p_1, \ldots, p_m, q : \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q})$, we have an exact sequence defining the tangent space to the moduli stack $(*)$

$$0 \to Aut(C, p_1, \ldots, p_m) \to Def(q) \to Def(C, p_1, \ldots, p_m, q) \to Def(C, p_1, \ldots, p_m) \to 0.$$

Here, the first term is the group of infinitesimal automorphisms of the marked curve, the second is the group of first order deformations of the quotient, the third is the tangent space, and the last is the group of first order deformations of the marked curve.

- There are no obstructions since the genus is zero.
- The only infinitesimal automorphisms come from components with exactly two special points on them, and they can be contracted or noncontracted. The case that the component is noncontracted is covered in [HKK+03] page 543, and all weights on the space of infinitesimal automorphisms are zero. In the second case, since the component is contracted, the induced $\mathbb{C}^*$ action on the space of infinitesimal automorphisms of the marked component is trivial.
- Next we consider deformations of the quotient. By [FG05] pages 152 – 155, $Def(q) \cong \mathrm{Hom}(\mathcal{S}, \mathcal{Q})$. We have an exact sequence:
$$0 \to \mathcal{S}^* \otimes \mathcal{Q} \to \bigoplus_a (\mathcal{S}^* \otimes \mathcal{Q})|_{C_a} \to \bigoplus_j (\mathcal{S}^* \otimes \mathcal{Q})|_{n_j} \to 0.$$
If the component C_a is noncontracted, then
$$H^0(\mathcal{S}^* \otimes \mathcal{Q}|_{C_a}) \cong H^0(f^* TGr(r, n)).$$
If the component is contracted, $\mathcal{S}$ splits as a direct sum of $\mathcal{O}(-d_i)$'s and each of these injects into a distinct copy of $\mathcal{O}$:
$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}(-d_i) \to \mathbb{C}^n \otimes \mathcal{O}.$$
- The last term in the sequence $(*)$ is the group of first order deformations of the marked curve. As it is shown in [ACG11] *Theorem 3.17* page 186,
$$Def(C, p_1, \ldots p_m) \cong \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C(-\sum_{i=1}^m p_i)),$$
and it fits into a short exact sequence
$$0 \to H^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C(-\sum_{i=1}^m p_i))) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C(-\sum_{i=1}^m p_i)) \to H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}(-\sum_{i=1}^m p_i))) \to 0.$$
The only nonzero weights come from the last term,
$$H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}(-\sum_{i=1}^m p_i))) \cong \bigoplus_{n_\alpha} T_{n_{\alpha,1}} C_a \otimes T_{n_{\alpha,2}} C_{a'},$$
which is the subspace of deformations smoothing the nodes, where $n_{\alpha,1}$, $n_{\alpha,2}$ are the preimages of the node n_α under the normalization map ([ACG11]).

Now, we will determine the weights on the spaces $\mathrm{Hom}(\mathcal{S}|_{C_a}, \mathcal{Q}|_{C_a})$, $\mathrm{Hom}(\mathcal{S}|_{n_j}, \mathcal{Q}|_{n_j})$, $\mathrm{Hom}(\mathcal{S}, \mathcal{Q})$, and $T_{n_{\alpha,1}} C_a \otimes T_{n_{\alpha,2}} C_{a'}$, where C_a, $C_{a'}$ are irreducible components of C.

We first cover the case of a map from a component of C onto a $\mathbb{C}^*$ fixed curve $\mathbb{P}^1 \subset Gr(r,n)$. We assume the map takes the form

$$f : \mathbb{P}^1 \to \mathbb{P}^1 := [s:t] \mapsto \langle e_{i_1}, \ldots, e_{i_{r-1}}, s^d \cdot e_{i_r} + t^d \cdot e_{i_{r+1}} \rangle \subset Gr(r,n),$$

where here we drop the assumption $i_j < i_{j'}$ if $j < j'$ (for ease of notation). Notice that if we restrict $\mathcal{S}$ (the universal subbundle) to the curve $\mathbb{P}^1 \subset Gr(r,n)$, then

$$\mathcal{S}|_{\mathbb{P}^1} \cong V_{\langle e_{i_1}, \ldots, e_{i_{r-1}} \rangle} \otimes \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1),$$

where $V_{\langle e_{i_1}, \ldots, e_{i_{r-1}} \rangle}$ is the subspace of $\mathbb{C}^n$ spanned by $e_{i_1}, \ldots, e_{i_{r-1}}$. We can restrict the universal sequence over the Grassmannian to $\mathbb{P}^1$, then tensor with $\mathcal{S}^*|_{\mathbb{P}^1}$, pull back via f and take cohomology to yield the following exact sequence

$$0 \to H^0(f^*(\mathcal{S}^* \otimes \mathcal{S})) \to \mathbb{C}^n \otimes H^0(f^*\mathcal{S}^*) \to H^0(f^* TGr(r,n)) \to H^1(f^*(\mathcal{S}^* \otimes \mathcal{S})) \to 0.$$

Using this exact sequence, we determine that the weights on $H^0(TGr(r,n)|_{C_a})$ are

- $(\star\star)_1$ $\lambda_{i_k} - \lambda_\ell$ for $k \in \{1, \ldots, r-1\}$, $\ell \in \{1, \ldots, n\} \backslash \{i_k\}_{k=1}^{r-1}$
- $(\star\star)_2$ $\frac{(d-j)\lambda_{i_{r+1}} + j\lambda_{i_r}}{d} - \lambda_\ell$ for ℓ as above, $j \in \{0, \ldots, d\}$
- $(\star\star)_3$ $\lambda_{i_k} - \frac{(d-q)\lambda_{i_r} + q\lambda_{i_{r+1}}}{d}$ for k as above, $q \in \{1, \ldots, d-1\}$

Now, we determine the weights on $\mathrm{Hom}(\mathcal{S}|_{C_{a'}}, \mathcal{Q}|_{C_{a'}})$ where $C_{a'}$ is a contracted component. In this case, the quotient sequence takes the form

$$0 \to \bigoplus_{j=1}^{r} \mathcal{O}(-d_{i_j}) \to \mathbb{C}^n \otimes \mathcal{O} \to \bigoplus_{j=1}^{r} \mathcal{O}_{\Sigma_{i_j}} \oplus \bigoplus_{k \notin \{i_j\}_{j=1}^r} \mathcal{O} \to 0.$$

We find that the weights on $H^0(\mathcal{S}^*|_{C_{a'}} \otimes \mathcal{Q}|_{C_{a'}})$ are

- $(\star\star\star)_1$ $\lambda_{i_j} - \lambda_{i_h}$ (d_{i_h} times) for $j, h \in \{1, \ldots r\}$
- $(\star\star\star)_2$ $\lambda_{i_j} - \lambda_k$ ($d_{i_j} + 1$ times) for j as above, $k \notin \{i_j\}_{j=1}^r$

When we restrict $\mathcal{S}^* \otimes \mathcal{Q}$ to a node, notice that since the node must be mapped to a $\mathbb{C}^*$ fixed point, the weights on $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ are entirely dependent on which fixed point the node gets mapped to: if the node gets mapped to $\langle e_{i_1}, \ldots, e_{i_r} \rangle$, then the weights on $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ are

$$\lambda_{i_j} - \lambda_k, \text{ for } j \in \{1, \ldots, r\} \text{ and } k \notin \{i_j\}_{j=1}^r.$$

If a node is on a contracted component, the weight on the tangent space at the preimage of the node on the corresponding component in the normalization is zero.

Recall that we are counting the number of fixed loci such that the number n^- of negative weights on the tangent space to a general point in the fixed locus is ≤ 1.

We count the dual graphs of the fixed loci with the desired properties. Given a stable quotient, the dual graph associates ([MOP11]):

- to each contracted curve (does not have to be irreducible) a vertex labelled by the corresponding $\mathbb{C}^*$ fixed point it maps to and a tuple $(s_1, \ldots, s_r)$ (coming from the quotient sequence) together with an inclusion $\{1, \ldots, r\} \subset \{1, \ldots, n\}$
- to each noncontracted component an edge labelled by the degree of the map to the $\mathbb{C}^*$ fixed curve and the corresponding fixed endpoints of the fixed curve
- to each marking $p_1, \ldots, p_m$, the vertex corresponding to the curve containing the marking

A vertex in such a graph does not always correspond to a curve; in some cases it corresponds to a node where two noncontracted components come together or a marking on the curve; in both cases the valence is 2. The dual graph for genus zero stable quotients is necessarily acyclic.

3.2.2 The fixed loci of $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1, d)$

We let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $-\lambda_1, -\lambda_2$ where $\lambda_2 < \lambda_1$. Let the fixed points be p_1 and p_2. Notice that by stability, the curve must be a chain of curves with the markings 1 and 2 on the terminal components.

We begin with a lemma.

Lemma 3.2.1. *In order for $n^- < 2$, there cannot be a node mapped to p_2.*

Proof. We observe that if N consecutive nodes get mapped to p_2, then there are $N + 1$ components that are incident to those N nodes. This follows from the fact that the genus is zero, and thus the dual graph of the curve does not have any cycles.

For each collection of N consecutive nodes that get mapped to p_2, the negative weight $\lambda_2 - \lambda_1$ appears exactly N times in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$.

Notice that for each noncontracted curve C_a, the weight $\lambda_2 - \lambda_1$ appears in $H^0(f_a^* T\mathbb{P}^1)$ from $(\star\star)_2$. For every component $C_{a'}$ contracted to p_2, the weight $\lambda_2 - \lambda_1$ appears on $H^0(\mathcal{S}^*|_{C_{a'}} \otimes \mathcal{Q}|_{C_{a'}})$ at least once, as in $(\star\star\star)_2$.

Therefore, we see that the negative weight $\lambda_2 - \lambda_1$ appears at least $N+1$ times in $\bigoplus_{components} H^0(\mathcal{S}^*|_{C_a} \otimes \mathcal{Q}|_{C_a})$. This shows that the negative weight $\lambda_2 - \lambda_1$ appears at least once for each cluster of consecutive nodes mapped to p_2.

This argument shows that the curve cannot have two nodes both mapped to p_2 that do not lie on a common irreducible component.

Now we are reduced to the case where all nodes that map to p_2 are consecutive; that is to say, for each node mapped to p_2, there is another node mapped to p_2 on the same irreducible component (unless the node attaches to a terminal irreducible component). But this forces these components to be contracted. Since we only have to consider a general point of the fixed locus, we can assume that only one irreducible component of the curve is contracted to p_2, or the curve contains a single node which is mapped to p_2 but it does not contain any components contracted to p_2.

In the case that there is a component C_a which is contracted to p_2, we find another negative weight as follows:

- $\frac{\lambda_2-\lambda_1}{d_{a'}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$ if there is a noncontracted component $C_{a'}$ incident to C_a, where $d_{a'}$ is the degree of the map on the noncontracted component.

- $\lambda_2 - \lambda_1$ appears $d+1 > 1$ times on $H^0(\mathcal{S}^* \otimes \mathcal{Q})$ if the entire curve is a single irreducible component contracted to p_2, as in $(\star\star\star)_2$. Since there are no nodes, these weights also appear on the tangent space.

In the case that there is a single node mapped to p_2 and there are no components contracted to p_2, then we see that the negative weight $\frac{\lambda_2-\lambda_1}{d_a} + \frac{\lambda_2-\lambda_1}{d_{a'}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$, where d_a, $d_{a'}$ are the degrees of the maps on the noncontracted components C_a, $C_{a'}$, respectively.

In any case, if a node maps to p_2, then there are at least two negative weights on the tangent space. $\square$

Lemma 3.2.2. *In order for $n^- < 2$, the dual graph to the fixed locus cannot have an edge with degree ≥ 2.*

Proof. We handle the case where the dual graph has an edge of degree $d_a \geq 3$ first. In this case, $H^0(f^*T\mathbb{P}^1)$ has weights

$$\tfrac{(d_a-j)\lambda_2-(d_a-j)\lambda_1}{d_a} \text{ for } j \in \{0,\dots,d_a\}$$

as in $(\star\star)_2$. For $1 \leq j \leq d-1 \geq 2$, these weights do not appear as weights on $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ (where n is any node). Therefore we have at least 2 negative weights on the tangent space in this case.

Next, we consider the case where there is a degree 2 edge in the dual graph. As above, we obtain one negative weight on the tangent space, $\frac{\lambda_2-\lambda_1}{2}$, from $(\star\star)_2$. If the other negative weight, $\lambda_2-\lambda_1$, does not appear in $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$, then we are done.

If $\lambda_2-\lambda_1$ does appear in $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$, then we have a node mapped to p_2. This is excluded by the lemma above, so we are done. □

Thus, the dual graph can have at most two edges, with their common vertex being labelled by p_1 by what we showed above.

However, two-edged graphs are also ruled out since each edge will contribute the weight $\lambda_2-\lambda_1$ to $H^0(f_i^*T\mathbb{P}^1)$, as in $(\star\star)_2$, and this weight does not appear in $\mathcal{S}^*|_n \otimes \mathcal{Q}|_n$, leaving us with two negative weights in $H^0(\mathcal{S}^* \otimes \mathcal{Q})$.

If $d \geq 2$, the dual graph must consist of 1 edge and 2 vertices, one corresponding to a marking and the other to a contracted component, or it must be a single vertex, corresponding to a curve contracted to p_1.

- In the first case, one of the vertices is labelled by p_2 and it must correspond to a marking mapped to p_2, and the other vertex is a genus zero curve which is contracted to p_1. Since the noncontracted component yields the negative weights $\frac{(d_1-j)\lambda_2-(d_1-j)\lambda_1}{d_1}$ for $j = 0,\dots,d_1$, where d_1 is the degree of the quotient on the noncontracted component, then $d_1 = 1$.

- In the second case, the vertex has degree d and it is contracted to p_1. There are no negative weights on the tangent space to the stable quotient in this case.

We consider the case $d > 1$; the analysis for $d = 1$ is similar. We have the following relevant fixed loci:

- The generic point of the first type of fixed locus corresponds to an irreducible curve with 2 markings which is contracted to p_1. The quotient has degree d. For this fixed locus, $n^- = 0$.
- The generic point of the second type of fixed loci corresponds to a reducible curve with 2 irreducible components such that one component is contracted to p_1, and the other component is mapped $1:1$ to the fixed curve joining p_1 to p_2. The degree of the quotient on the contracted component is $d-1$. Each component has a marking. For these 2 fixed loci, $n^- = 1$.

These fixed loci are isomorphic to $\overline{M}_{0,2|d}/\mathcal{S}_d$ and $\overline{M}_{0,2|d-1}/\mathcal{S}_{d-1}$, respectively.

By Lemma 3.1.1, we see that, for $d \geq 2$,

$$h^2(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1, d)) = h^2(\overline{M}_{0,2|d}/\mathcal{S}_d) + 2.$$

It has already been shown in [MOP11] *Lemma 2* that $h^2(\overline{M}_{0,2|d}/\mathcal{S}_d) = d-1$. Thus, $h^2(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1, d)) = d+1$.

3.2.3 The fixed loci of $\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^1, d)$

Notice that Lemma 3.2.1 and Lemma 3.2.2 both still hold in the case that $m > 2$.

Therefore the relevant fixed loci cannot have any points corresponding to stable quotients with either a node mapped to p_2 or a noncontracted component with a map of degree ≥ 2.

Notice that the relevant fixed locus does not contain points corresponding to curves with more than one noncontracted component: in order for $n^- < 2$ the two components have to be incident to either a curve contracted to p_1 or a node mapped to p_1; either way, the weight $\lambda_2 - \lambda_1$ appears on $H^0(\mathcal{S}^* \otimes \mathcal{Q})$ using the same argument from before (no node is mapped to p_2).

Thus the relevant fixed loci cannot have more than one noncontracted component, and we see that the only contracted component can be mapped to p_1. Since we are assuming $m > 2$, the entire curve cannot be a single noncontracted component.

- The generic point of the first type of fixed locus corresponds to an irreducible curve with m markings which is contracted to p_1. The quotient has degree d. For this fixed locus, $n^- = 0$.

- The generic point of the second type of fixed loci corresponds to a reducible curve with two irreducible components such that one component is contracted to p_1, and the other component is mapped $1:1$ to the fixed curve joining p_1 to p_2. The degree of the quotient on the contracted component is $d-1$. The contracted component has $m-1$ markings, and the noncontracted component has 1 marking. For each of these m fixed loci, $n^- = 1$.

These fixed loci are isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$ and $\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}$, respectively. By Lemma 3.1.1, for $m > 2$, $d \geq 1$,

$$h^2(\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^1, d)) = h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) + m.$$

3.2.4 The fixed loci of $\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1}, d)$, for $m \geq 2$, $n \geq 3$

Let the $\mathbb{C}^*$ fixed points on $\mathbb{P}^{n-1}$ be denoted $p_1, \ldots, p_n$.

We will start by proving an analogous lemma to the case of $\mathbb{P}^1$.

Lemma 3.2.3. *In order for $n^- < 2$, there cannot be any nodes mapped to p_j for $j > 1$.*

Proof. The same start to the proof of Lemma 3.2.1 shows that if the curve contains a cluster of nodes mapped to p_j for $j > 1$, then the negative weight $\lambda_j - \lambda_1$ appears at least once. Thus, there cannot be more than one such cluster of nodes.

We are reduced to the case where the only nodes that map to p_j are consecutive; that is to say, for each node that maps to p_j, there is another node mapping to p_j on the same irreducible component (unless the node attaches to a terminal irreducible component).

Notice that if the entire curve is contracted to p_j, then since $d > 0$, one of the contracted components must have a quotient which has degree > 0, in which case our argument above shows that we get at least 2 negative weights on the tangent space.

We can assume that there is a noncontracted component C_a containing a node mapped to p_j. There are two cases to consider here.

- C_a maps to the line joining p_j to p_i, where $i > j$. In this case, since $i > j > 1$, the point mapped to p_i cannot be a node, so $\lambda_i - \lambda_j$ appears in $H^0(f^* T\mathbb{P}^{n-1})$, as in $(\star\star)_2$, but not in $\bigoplus_{nodes} \mathcal{S}^* \otimes \mathcal{Q}$.
- C_a maps to the line joining p_i to p_j, where $j > i$. If the other irreducible component $C_{a'}$ containing the node mapped to p_j is contracted, then $\frac{\lambda_j - \lambda_i}{d_a}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$, where d_a is the degree of the morphism from C_a onto the fixed curve.

  If the other irreducible component $C_{a'}$ containing the node mapped to p_j is noncontracted, it maps to the line joining p_j to p_k, for some $k \neq j$. If $k > j$, then we are in the case above. If $j > k$ then $\frac{\lambda_j - \lambda_i}{d_a} + \frac{\lambda_j - \lambda_k}{d_{a'}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$, where here d_a, $d_{a'}$ are the degrees of the maps on the noncontracted components C_a, $C_{a'}$, respectively.

Thus, either way we end up with at least 2 negative weights on the tangent space. □

Now, we would like to show that the relevant fixed loci cannot correspond to a stable quotient with a component mapped onto a fixed curve via a degree ≥ 2 morphism.

Lemma 3.2.4. *In order for $n^- < 2$, there cannot be any noncontracted components with degree ≥ 2 maps.*

The proof is analogous to the proof of Lemma 3.2.2 so we omit it.

Now I claim that there cannot be more than one noncontracted component.

Lemma 3.2.5. *In order for $n^- < 2$, there cannot be more than one noncontracted component.*

Proof. Suppose there was more than one noncontracted component. Then the only possibility is that each noncontracted component is incident to the same connected component, which is contracted to p_1.

The noncontracted components C_a must map to the curves joining p_1 to p_k where $k > 1$, for various values of $k > 1$. Then $\{\lambda_k - \lambda_1\}$, appear in $\bigoplus_{C_a} H^0(f_i^* T\mathbb{P}^{n-1})$, as in $(\star\star)_2$, and they do not appear in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$. Since we are assuming there is more than one such noncontracted component, this yields as least 2 negative weights on the tangent space. □

Thus the dual graph of the fixed locus can contain at most one edge.

Observe that if the noncontracted component is mapped to any curve other than the one joining p_1 to p_2, the weights

$$\lambda_j - \lambda_i$$

where $j > 2$, for all $i < j$, would appear in $H^0(f^* T\mathbb{P}^{n-1})$ as in $(\star\star)_2$, and not in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$. The edge must have vertices labelled by p_1 and p_2.

We omit the analysis of the case $d = 1$ as before.

We list the relevant fixed loci below:

- The generic point of the first type of fixed locus corresponds to an irreducible curve with m markings which is contracted to p_1. The quotient has degree d. For this fixed locus, $n^- = 0$.

- The generic point of the second type of fixed loci corresponds to a reducible curve with two irreducible components such that one component is contracted to p_1, and the other component is mapped $1:1$ to the fixed curve joining p_1 to p_2. The degree of the quotient on the contracted component is $d-1$. The contracted component has $m-1$ markings, and the noncontracted component has a single marking. For these m fixed loci, $n^- = 1$.

These fixed loci are isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$ and $\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}$, respectively.

By Lemma 3.1.1,

- for $n \geq 3$, $m = 2$, $d \geq 1$, $h^2(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) = h^2(\overline{M}_{0,2|d}/\mathcal{S}_d) + 2 = d + 1$ using *Lemma 2* in [MOP11]

- for $n \geq 3$, $m \geq 3$, $d > 0$, $h^2(\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1}, d)) = h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) + m$.

3.2.5 The fixed loci of $\overline{\mathcal{Q}}_{0,m}(Gr(r,n), d)$, $r \geq 2$, $n - r \geq 2$

We start by proving a lemma similar to the one we proved in the case of stable quotients to $\mathbb{P}^n$.

Lemma 3.2.6. *In order for $n^- < 2$, there cannot be any nodes mapped to $\langle e_{i_1}, \ldots, e_{i_r} \rangle$, where $i_j < i_{j+1}$ and $i_r > r$.*

Proof. If there is a node which is mapped to $\langle e_{i_1}, \ldots, e_{i_r}\rangle$ where $i_r > r$, then there exists an $h \notin \{i_j\}_{j=1}^r$ such that $h < i_r$.

We observe that (similar to the $\mathbb{P}^n$ case) for each collection of N consecutive nodes mapped to fixed points of the form:
$(**)$ $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ such that there exists an index ℓ_k with $\ell_k = i_r$ and $h \notin \{\ell_b\}_{j=1}^r$, there are $N+1$ irreducible components incident to these nodes. For each of these nodes, $\lambda_{i_r} - \lambda_h$ appears once in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$.

For every contracted component mapped to a fixed point $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ of the form $(**)$, $\lambda_{i_r} - \lambda_h$ appears at least once in $\bigoplus\limits_{components} \mathcal{S}^*|_{C_a} \otimes \mathcal{Q}|_{C_a}$, as in $(\star\star\star)_2$.

For every noncontracted component C_a that maps to a line joining such a fixed point $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ of the form $(**)$ to $\langle e_{\ell_1}, \ldots \hat{e}_{\ell_t}, \ldots e_{\ell_{r+1}}\rangle$, (here $\hat{e}_{\ell_t}$ means we omit this index, and ℓ_{r+1} is not necessarily greater than ℓ_r), there are several possibilities:

- $\ell_t \neq \ell_k = i_r$, and $h \neq \ell_{r+1}$, in which case $\lambda_{i_r} - \lambda_h$ appears in $H^0(f_a^* TGr(r,n))$, as in $(\star\star)_1$.
- $\ell_t = \ell_k = i_r$ and $h \neq \ell_{r+1}$, in which case $\lambda_{i_r} - \lambda_h$ appears in $H^0(f_a^* TGr(r,n))$, as in $(\star\star)_2$.
- $\ell_t \neq \ell_k = i_r$ and $h = \ell_{r+1}$, in which case $\lambda_{i_r} - \lambda_h$ appears in $H^0(f_a^* TGr(r,n))$, as in $(\star\star)_1$.
- $\ell_t = \ell_k = i_r$ and $h = \ell_{r+1}$, in which case $\lambda_{i_r} - \lambda_h$ appears in $H^0(f_a^* TGr(r,n))$, as in $(\star\star)_2$.

Therefore, at least one negative weight appears on the tangent space for each cluster of consecutive nodes mapped to fixed points of the form $(**)$, so there can be at most one such cluster.

Suppose there exists such a cluster of consecutive nodes mapped to fixed points of the form $(**)$.

Notice that if the entire curve is contracted, $\lambda_{i_r} - \lambda_h$ appears at least twice, as in $(\star\star\star)_2$. To see this, observe that the quotient must have strictly positive degree on at least one of the irreducible components C_a, in which case $\lambda_{i_r} - \lambda_h$ appears at least twice on $H^0(\mathcal{S}^*|_{C_a} \otimes \mathcal{Q}|_{C_a})$, as in $(\star\star\star)_2$. The fact that the curve is acyclic (using the same argument from before) shows that this weight must appear at least twice in the tangent space.

Therefore there must be at least one noncontracted component in the cluster. We will consider the irreducible noncontracted components incident to, or contained in, the cluster, i.e. those noncontracted components C_a mapping to the line joining such a fixed point $\langle e_{\ell_1}, \dots, e_{\ell_r}\rangle$ of the form $(**)$ to $\langle e_{\ell_1}, \dots \hat{e}_{\ell_t}, \dots e_{\ell_{r+1}}\rangle$, ($\ell_{r+1}$ is not necessarily greater than ℓ_r) where either : $\ell_t = \ell_k = i_r$ and $h \neq \ell_{r+1}$; $\ell_t \neq \ell_k = i_r$ and $h = \ell_{r+1}$; $\ell_t \neq \ell_k = i_r$ and $h \neq \ell_{r+1}$; or $\ell_t = \ell_k = i_r$ and $h = \ell_{r+1}$.

We handle each of these cases separately.

For $\ell_t = \ell_k = i_r$, $h \neq \ell_{r+1}$ there are two subcases to consider:

- $\ell_{r+1} > \ell_t$, in which $\lambda_{\ell_{r+1}} - \lambda_{\ell_t}$ appears in $H^0(f_a^* TGr(r,n))$ (as in $(\star\star)_2$) and it does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ since this would imply that there is another cluster of nodes mapped to fixed points of the form $(**)$.

- $\ell_{r+1} < \ell_t$, in which case we have to consider the quotient sequence on the other component $C_{a'}$ containing the node mapped to $\langle e_{\ell_1}, \dots, e_{\ell_r}\rangle$.

  If $C_{a'}$ is contracted, then $\frac{\lambda_{\ell_t} - \lambda_{\ell_{r+1}}}{d_{w_1}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$.

  If $C_{a'}$ is noncontracted, then it is mapped to the line joining $\langle e_{\ell_1}, \dots, e_{\ell_r}\rangle$ to $\langle e_{\ell_1}, \dots, \hat{e}_{\ell_{t'}}, \dots, e_{\ell_r}, e_{\ell_{r+2}}\rangle$. Notice that $h \neq \ell_{t'}$ since $h \notin \{\ell_1 \dots \ell_r\}$. Again, there are two cases here: either $\ell_{r+2} > \ell_{t'}$, or $\ell_{r+2} < \ell_{t'}$.

  - In the first case, $\lambda_{\ell_{r+2}} - \lambda_{\ell_{t'}}$ appears in $H^0(f_{a'}^* TGr(r,n))$ and it does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ for the same reason we outlined above (if it did appear it would yield the existence of another cluster of nodes mapped to fixed points of the form $(**)$).
  - In the second case, $\ell_{r+2} < \ell_{t'}$, and $\frac{\lambda_{\ell_t} - \lambda_{\ell_{r+1}}}{d_{w_1}} + \frac{\lambda_{\ell_{t'}} - \lambda_{\ell_{r+2}}}{d_{w_2}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$.

For the next case, $\ell_t \neq \ell_k = i_r$, $h = \ell_{r+1}$. In this case, either

- $h > \ell_t$, in which case $\lambda_h - \lambda_t$ appears in $H^0(f_a^* TGr(r,n))$ (as in $(\star\star)_2$), and does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ for this would imply the existence of another cluster of nodes mapped to fixed points with the property $(**)$, or

- $h < \ell_t$, which splits into two cases:

If the other component $C_{a'}$ containing the node mapped to $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ is contracted, then $\frac{\lambda_{\ell_t} - \lambda_h}{d_w}$ appears in $T_{n_1}C_{w_1} \otimes T_{n_2}C_{w_2}$.

If $C_{a'}$ is not contracted, then it is mapped to the line joining $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ to $\langle e_{\ell_1}, \ldots, \hat{e}_{\ell_{t'}}, \ldots, e_{\ell_{r+2}}\rangle$. Notice that $h \neq \ell_{t'}$ since $h \notin \{\ell_1 \ldots \ell_r\}$. Again, there are two cases here : either $\ell_{r+2} > \ell_{t'}$, or $\ell_{r+2} < \ell_{t'}$.

- In the first case, $\lambda_{\ell_{r+2}} - \lambda_{\ell_{t'}}$ appears in $H^0(f_{a'}^* TGr(r,n))$ (as in $(\star\star)_2$) and does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ for the same reason we have explained in the cases above.
- If $\ell_{r+2} < \ell_{t'}$, then $\frac{\lambda_{\ell_t} - \lambda_h}{d_a} + \frac{\lambda_{\ell_{t'}} - \lambda_{\ell_{r+2}}}{d_{a'}}$ appears in $T_{n_1}C_a \otimes T_{n_2}C_{a'}$.

In the third case, $\ell_t \neq \ell_k = i_r$, $h \neq \ell_{r+1}$. We split this into two cases:

- $\ell_{r+1} > \ell_t$, in which case $\lambda_{\ell_{r+1}} - \lambda_{\ell_t}$ appears in $H^0(f_a^* TGr(r,n))$ (as in $(\star\star)_2$) and it does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ (by the same argument above).

- $\ell_{r+1} < \ell_t$, in which case we consider the quotient sequence on the other component $C_{a'}$ containing the node mapped to $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$:

  If $C_{a'}$ is contracted, then $\frac{\lambda_{\ell_t} - \lambda_{\ell_{r+1}}}{d_a}$ appears in $T_{n_1}C_a \otimes T_{n_2}C_{a'}$.

  If $C_{a'}$ is not contracted, then it is mapped to the line joining $\langle e_{\ell_1}, \ldots, e_{\ell_r}\rangle$ to $\langle e_{\ell_1}, \ldots, \hat{e}_{\ell_{t'}}, \ldots, e_{\ell_r}, e_{\ell_{r+2}}\rangle$.

  - If $\ell_{t'} > \ell_{r+2}$, then $\frac{\lambda_{\ell_t} - \lambda_{\ell_{r+1}}}{d_a} + \frac{\lambda_{\ell_{t'}} - \lambda_{\ell_{r+2}}}{d_{a'}}$ appears in $T_{n_1}C_a \otimes T_{n_2}C_{a'}$.
  - If $\ell_{t'} < \ell_{r+2}$, we see that the point mapped to $\langle e_{\ell_1}, \ldots, \hat{e}_{\ell_{t'}}, \ldots, e_{\ell_r}, e_{\ell_{r+2}}\rangle$ cannot be a node otherwise we would have another (since $h \neq \ell_{t'}$) node mapped to a point of the form $(**)$. This node is still in the cluster of nodes, but applying the same argument as we made above to the collection of consecutive nodes mapped to fixed points of the form $\langle e_{\alpha_c}\rangle_{\alpha_c}$ such that there exists an index c' with $\alpha_{c'} = \ell_{r+2}$ and $\ell_{t'} \neq \alpha_c$ for any c, we see that we have another negative weight which is distinct from the negative weight we already found.
  - Thus, $\lambda_{\ell_{r+2}} - \lambda_{\ell_{t'}}$ appears in $H^0(f_{a'}^* TGr(r,n))$, and it does not appear in $\bigoplus\limits_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$.

In the final case, $\ell_t = \ell_k = i_r$ and $\ell_{r+1} = h$. We have two cases:

- If the other component $C_{a'}$ containing the node mapped to $\langle e_{\ell_1}, \dots, e_{\ell_r} \rangle$ is contracted, then $\frac{\lambda_{\ell_k} - \lambda_h}{d_a}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$.
- If $C_{a'}$ is not contracted, then we apply the same arguments from above to show that we obtain another negative weight, either
  - $\frac{\lambda_{\ell_k} - \lambda_h}{d_a} + \frac{\lambda_{\ell_{t'}} - \lambda_{\ell_{r+2}}}{d_{a'}}$ in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$, or
  - $\lambda_{\ell_{r+2}} - \lambda_{\ell_{t'}}$ in $H^0(\mathcal{S}^* \otimes \mathcal{Q})$

  using the notation we used in the cases above.

Thus, if a node is mapped to $\langle e_{i_1}, \dots, e_{i_r} \rangle$ where $i_j < i_{j+1}$ and $i_r > r$, then there are at least two negative weights on the tangent space. $\square$

We prove the analogue of Lemma 3.2.2 for the Grassmannian case.

Lemma 3.2.7. *In order for $n^- < 2$, there cannot be any noncontracted components with a degree ≥ 2 map.*

Proof. Suppose there was a noncontracted component C_a mapped to the line joining $\langle e_{i_1}, \dots, e_{i_r} \rangle$ to $\langle e_{i_1}, \dots, \hat{e}_{i_k}, \dots, e_{i_r}, e_{i_{r+1}} \rangle$ where $i_j < i_{j+1}$. Without loss of generality, we may assume $i_{r+1} > i_k$. Then $\frac{(d_a - e)(\lambda_{i_{r+1}} - \lambda_{i_k})}{d_w}$ appear in $H^0(f_a^* TGr(r,n))$, for $e = 0, \dots, d_a - 1 \geq 1$, and do not appear in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ since this would yield the existence of a node mapped to a fixed point of the form $(**)$. $\square$

I claim that there cannot be more than one noncontracted component.

Lemma 3.2.8. *In order for $n^- < 2$, there cannot be more than one noncontracted component.*

Proof. Suppose there were at least two noncontracted components, C_a, $C_{a'}$. The lemmas above show that the only possibility is that C_a and $C_{a'}$ meet the same connected component which is contracted to $\langle e_1, \dots, e_r \rangle$. Let C_a, $C_{a'}$ be mapped to the lines joining $\langle e_1, \dots, e_r \rangle$ to $\langle e_1, \dots, \hat{e}_k, \dots, e_r, e_\alpha \rangle$, $\langle e_1, \dots, \hat{e}_j, \dots, e_r, e_\beta \rangle$, respectively. Then, since $\alpha, \beta \notin \{1, \dots, r\}$, automatically $\alpha > k$, $\beta > j$. Thus $\lambda_\alpha - \lambda_k$ appears in $H^0(f_a^* TGr(r,n))$ and $\lambda_\beta - \lambda_j$ appears in $H^0(f_{a'}^* TGr(r,n))$. These do not appear in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ by Lemma 3.2.6. $\square$

Thus, the dual graph of the fixed locus can contain at most one edge.

One of the vertices must be labelled by $\langle e_1, \ldots, e_r\rangle$. If there is another vertex, it is labelled by $\langle e_1, \ldots, \hat{e}_k, \ldots, e_r, e_\alpha\rangle$.

Notice that if $\alpha > r+1$, then $\lambda_\alpha - \lambda_{r+1}$, $\lambda_\alpha - \lambda_k$ appear in $H^0(\mathcal{S}^* \otimes \mathcal{Q})$, as in $(\star\star)_2$. Therefore $\alpha = r+1$ (it must be greater than r and the argument above shows it must be $\leq r+1$).

Also, if $k \neq r$, then $\lambda_r - \lambda_k$ appears in $H^0(\mathcal{S}^* \otimes \mathcal{Q})$, as in $(\star\star)_1$. Thus, $k = r$.

We must consider the weights in $H^0(\mathcal{S}^*|_{C_c} \otimes \mathcal{Q}|_{C_c})$, where C_c is the contracted component. Studying the weights in $(\star\star\star)_1$ and $(\star\star\star)_2$, we can see that, for fixed $\ell \in \{1, \ldots, r\}$, if $d_\ell \neq 0$, the negative weights $\lambda_j - \lambda_\ell$ appear in the tangent space for each $\ell < j$.

In the case that we have a noncontracted component, there is already 1 negative weight, so we must have that $d_j = 0$ for all $j < r$, and $d_r = d - 1$.

If the entire curve is contracted, then either

- we must have $d_j = 0$ for all $j \leq r-2$, with $d_{r-1} = 1$ and $d_r = d-1$, which yields one negative weight, or
- $d_j = 0$ for all $j < r$, with $d_r = d$, which yields no negative weights.

We consider the cases $(m = 2, d > 1)$ and $(m \geq 3, d \geq 1)$; the analysis for $(m = 2, d = 1)$ is similar.

In these cases, the relevant fixed loci must take the following forms:

- The generic point of the first type of fixed locus corresponds to an irreducible curve with m markings which is contracted to $\langle e_1, \ldots, e_r\rangle$. The inclusion of subsheaves takes the form
$$\mathbb{C}^{r-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
where the inclusion respects the splittings. For this fixed locus, $n^- = 0$.
- The second type of fixed locus is almost the same as the first type, except now the inclusion of subsheaves takes the form
$$\mathbb{C}^{r-2} \otimes \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-d+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
where the inclusion respects the splittings. For this fixed locus, $n^- = 1$.
- The generic point of the third type of fixed loci corresponds to a reducible curve with two irreducible components such that one component is contracted

to $\langle e_1, \dots, e_r\rangle$, and the other component is mapped $1:1$ to the the fixed curve joining $\langle e_1, \dots, e_r\rangle$ to $\langle e_1, \dots, e_{r-1}, e_{r+1}\rangle$. The inclusion of the subsheaf on the contracted component has the form

$$\mathbb{C}^{r-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$

where the inclusion respects the splittings. The contracted component has $m-1$ markings, and the noncontracted component has a single marking. For these m fixed loci, $n^- = 1$.

These fixed loci are isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$, $\overline{M}_{0,m|d}/\mathcal{S}_{d-1}$, and $\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}$, respectively.

Using Lemma 3.1.1 together with *Lemma 2* in [MOP11], for $m \geq 3$, $r \geq 2$, $n-r \geq 2$, $d > 0$

$$\begin{aligned} h^2(\overline{\mathcal{Q}}_{0,2}(Gr(r,n),d) &= h^2(\overline{M}_{0,2|d}/\mathcal{S}_d) + h^0(\overline{M}_{0,2|d}/\mathcal{S}_{d-1}) + 2\cdot h^0(\overline{M}_{0,2|d-1}/\mathcal{S}_{d-1}) \\ &= d+2; \end{aligned}$$

and for $r \geq 2$, $n-r \geq 2$, $d > 0$,

$$\begin{aligned} h^2(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),d) &= h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) + h^0(\overline{M}_{0,m|d}/\mathcal{S}_{d-1}) + m\cdot h^0(\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}) \\ &= h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) + 1 + m. \end{aligned}$$

3.2.6 The fixed loci of $\overline{\mathcal{Q}}_{0,m}(Gr(n-1,n),d)$, for $n-1 \geq 2$

Lemma 3.2.9. *In order for $n^- < 2$, there cannot be any nodes mapped to $\langle e_1, \dots, \hat{e}_k, \dots, e_n\rangle$, where $k \neq n$.*

Lemma 3.2.10. *In order for $n^- < 2$, there cannot be any noncontracted components with maps of degree ≥ 2.*

Lemma 3.2.11. *In order for $n^- < 2$, there cannot be more than one noncontracted component.*

The proofs of the lemmas above are very similar to the proofs of Lemmas 3.2.6, 3.2.7, and 3.2.8 so we omit them.

We analyze the dual graphs. One of the vertices must be labelled by $\langle e_1, \dots, e_{n-1}\rangle$. If there is another vertex, it is labelled by $\langle e_1, \dots, \hat{e}_k, \dots, e_n\rangle$, for $k < n$. We see that $\lambda_n - \lambda_k$ appears in $H^0(f^*TGr(n-1,n))$, as in $(\star\star)_2$ and it does not appear in $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ by the first lemma above.

Notice that if $k < n-1$, then $\lambda_{n-1} - \lambda_k$ appears in $H^0(f^* TGr(r,n))$, as in $(\star\star)_1$, and it does not appear as a weight on $\bigoplus_{nodes} \mathcal{S}^*|_n \otimes \mathcal{Q}|_n$ by the first lemma above. Therefore, we see that $k = n-1$.

We must consider the form of the quotient sequence on the contracted component, C_c.

Considering the weights in $(\star\star\star)_1$, $(\star\star\star)_2$, we see that for fixed $\ell \in \{1, \dots, n-1\}$, if $d_\ell \neq 0$, $\lambda_j - \lambda_\ell$ appear, for each $\ell < j$. If $\ell < n-2$, then more than 1 negative weight appears.

When the fixed locus has a noncontracted component, there is already 1 negative weight, $\lambda_n - \lambda_{n-1}$, which forces $d_j = 0$ for all $j < n-1$, and $d_{n-1} = d-1$.

If the entire curve is contracted, then either

- $d_j = 0$ for all $j < n-2$, with $d_{n-2} = 1$ and $d_{n-1} = d-1$, in which case there is one negative weight on the tangent space, or
- $d_j = 0$ for all $j < n-1$, with $d_{n-1} = d$, in which case there are no negative weights on the tangent space.

We describe the relevant fixed loci for $(m = 2, d > 1)$ and $(m \geq 3, d > 0)$; the case $(m = 2, d = 1)$ is similar.

- The generic point of the first type of fixed locus corresponds to an irreducible curve with m markings which is contracted to $\langle e_1, \dots, e_{n-1}\rangle$. The inclusion of subsheaves takes the form
$$\mathbb{C}^{n-2} \otimes \mathcal{O} \oplus \mathcal{O}(-d) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
where the inclusion respects the splittings. For this fixed locus, $n^- = 0$.
- The second type of fixed locus is almost the same as the first type, except now the inclusion of subsheaves takes the form
$$\mathbb{C}^{n-3} \otimes \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-d+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
where the inclusion respects the splittings. For this fixed locus, $n^- = 1$.
- The generic point of the third type of fixed loci corresponds to a reducible curve with two irreducible components such that one component is contracted to $\langle e_1, \dots, e_{n-1}\rangle$ and the other component is mapped $1:1$ to the fixed curve joining $\langle e_1, \dots, e_{n-1}\rangle$ to $\langle e_1, \dots, e_{n-2}, e_n\rangle$. The inclusion of the subsheaf on

the contracted component has the form

$$\mathbb{C}^{n-2} \otimes \mathcal{O} \oplus \mathcal{O}(-d+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$

where the inclusion respects the splittings. The contracted component has $m-1$ markings, and the noncontracted component has a single marking. For these m fixed loci, $n^- = 1$.

These fixed loci are isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$, $\overline{M}_{0,m|d}/\mathcal{S}_{d-1}$, and $\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}$, respectively.

Using Lemma 3.1.1 and *Lemma 2* of [MOP11], we find: for $n-1 \geq 2$, $d > 0$

$$\begin{aligned} h^2(\overline{\mathcal{Q}}_{0,2}(Gr(n-1,n),\, d)) &= h^2(\overline{M}_{0,2|d}/\mathcal{S}_d) \;+\; h^0(\overline{M}_{0,2|d}/\mathcal{S}_{d-1}) \;+\; 2\cdot h^0(\overline{M}_{0,2|d-1}/\mathcal{S}_{d-1}) \\ &= \; d+2; \end{aligned}$$

and for $m \geq 3$, $n-1 \geq 2$, $d > 0$

$$\begin{aligned} h^2(\overline{\mathcal{Q}}_{0,m}(Gr(n-1,n),\, d)) &= h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) \;+\; h^0(\overline{M}_{0,m|d}/\mathcal{S}_{d-1}) + \\ &\quad\; m\cdot h^0(\overline{M}_{0,m|d-1}/\mathcal{S}_{d-1}) \\ &= h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) \;+\; 1 \;+\; m. \end{aligned}$$

3.2.7 The fixed loci for $\overline{\mathcal{Q}}_{0,m}(Gr(n,n),d)$, for $d > 0$, $n \geq 1$

In this case, all curves are contracted, so we only need to consider the weights on $H^0(\mathcal{S}^* \otimes \mathcal{Q})$ where the component is contracted.

If $n = 1$, then automatically we see that the quotient sequence is

$$0 \to \mathcal{O}(-d) \to \mathcal{O} \to \mathcal{O}_\Sigma \to 0.$$

There are no negative (or positive) weights on $H^0(\mathcal{S}^* \otimes \mathcal{Q})$, thus, $n^- = 0$. The only fixed locus corresponds to an irreducible curve with m markings and whose quotient sequence is as above. This fixed locus is isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$.

In fact, using *Proposition 3* in [MOP11], we see that we have an isomorphism of the following coarse moduli spaces

$$\overline{M}_{0,m|d}/\mathcal{S}_d \cong \overline{Q}_{0,m}(Gr(1,1),d).$$

Thus, $h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) \;=\; h^2(\overline{\mathcal{Q}}_{0,m}(Gr(1,1),d))$.

Now, suppose $n > 1$. Then, if the quotient sequence takes the form

$$0 \to \bigoplus_{j=1}^{n} \mathcal{O}(-d_j) \to \bigoplus_{j=1}^{n} \mathcal{O} \to \bigoplus_{j=1}^{n} \mathcal{O}_{\Sigma_j} \to 0$$

we have the following weights on $H^0(\mathcal{S}^* \otimes \mathcal{Q})$:

$$\lambda_j - \lambda_\ell \text{ appears } d_\ell \text{ times, where } j,\ell \in \{1,\ldots,n\}.$$

As we have already seen, for fixed ℓ, if $d_\ell \neq 0$, the negative weights $\lambda_j - \lambda_\ell$ appear d_ℓ times, for all $j > \ell$. If $\ell < n-1$ and $d_\ell \neq 0$, then there are at least two negative weights, so we must have $d_\ell = 0$ for all $\ell < n-1$.

For 1 negative weight to appear, we let $d_{n-1} = 1$, and $d_n = d - 1$.

For no negative weights to appear, we let $d_{n-1} = 0$ and $d_n = d$. To describe the fixed loci, we only have to specify the inclusion of sheaves:

- The first type of fixed loci corresponds to the inclusion of sheaves
  $$\mathbb{C}^{n-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
  where the inclusion respects the direct summands. This fixed locus has $n^- = 0$.

- The second type of fixed loci corresponds to the inclusion of sheaves
  $$\mathbb{C}^{n-2} \otimes \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-d+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$
  where the inclusion respects the direct summands. This fixed locus has $n^- = 1$.

These are isomorphic to $\overline{M}_{0,m|d}/\mathcal{S}_d$ and $\overline{M}_{0,m|d}/\mathcal{S}_{d-1}$, respectively.

Using Lemma 3.1.1, for $m \geq 2$, $d > 0$,
$$h^2(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),d)) = h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) + 1.$$

3.3 Calculating $h^2(\overline{M}_{0,m|d}/\mathcal{S}_d)$ for $m > 2$, $d > 0$

In this section we prove the following lemma:

Lemma 3.3.1. $h^2(\overline{M}_{0,m|d}/\mathcal{S}_d) = 2^{m-1} \cdot (d+1) - \frac{m^2+m+2d}{2}$ *, for* $m \geq 2$.

We stratify the moduli space $\overline{M}_{0,m|d}/\mathcal{S}_d$ based on the number of components the curve has together with its marking and degree distribution on the components.

Since the spaces $M_{0,m|d}$ are smooth, then after taking the quotient by $\mathcal{S}_d$ we can use Poincaré duality (when we take cohomology with coefficients in $\mathbb{Q}$) to calculate $h^{2m+2d-8} = h^2$. We will use the fact that the virtual Poincaré polynomial is additive on the strata, allowing us to calculate the coefficient on $t^{2m+2d-8}$ in the virtual Poincaré polynomial (and thus $h^{2m+2d-8}$) after removing pieces of codimension ≥ 2.

The relevant strata will be the interior of the moduli space, $M_{0,m|d}/\mathcal{S}_d$, and the locus of curves with exactly two irreducible components.

We have a forgetful map $M_{0,m|d}/\mathcal{S}_d \to M_{0,m}$. It is clear that the fibers of this map are Hilbert schemes of zero dimensional subschemes of length d of $\mathbb{P}^1\backslash m$ points. This is just $S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\})$.

We will show that

$$p^{vir}(M_{0,m|d}/\mathcal{S}_d) = p^{vir}(S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}) \cdot p^{vir}(M_{0,m})$$

where p^{vir} is the virtual Poincaré polynomial. Using *Lemma 2.1* in [Coo15], it is sufficient to show that the fundamental group of $M_{0,m}$ acts trivially on the cohomology groups of the fibers.

Notice that it suffices to prove the result in the case that $d = 1$, since in that case the fibers are $\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}$. The (compactly supported) cohomology of $S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\})$ is just the $\mathcal{S}_d$ invariant piece of the (compactly supported) cohomology of the d-fold product of $\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}$.

We will use the proof of *Proposition 4.1* in [Cha16].

Let $\Sigma_{0,m} = \mathbb{P}^1\backslash\{p_1,\ldots p_m\}$ where all p_i are distinct. As [Cha16] explains, the fundamental group of $M_{0,m}$ is isomorphic to the pure mapping class group, $\Gamma_{0,m}$.

By *Theorem 2.2.1* in [Sch03], $\Gamma_{0,m} \cong Out^*(\pi_1(\Gamma_{0,m}))$, where the $*$ denotes the subgroup of automorphisms which fix the conjugacy classes of loops around the punctures. The classes of these loops generate $H_1(M_{0,m})$. Thus the action of $\Gamma_{0,m}$ on $H_1(\Sigma_{0,m})$ is trivial. Since the corresponding cohomology dual elements to the classes of the loops surrounding the punctures generate the compactly supported cohomology, we see that the action of $\pi_1(M_{0,m})$ on the compactly supported cohomology of $\Sigma_{0,m}$ is trivial.

In order to calculate $p^{vir}(M_{0,m|d}/\mathcal{S}_d)$, it suffices to calculate $p^{vir}(M_{0,m})$ and $p^{vir}(S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}))$. We will begin by calculating $p^{vir}(S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}))$. We use a similar technique to the one in the proof of *Lemma 2* in [MOP11].

Lemma 3.3.2. $p^{vir}(S^\ell(\mathbb{C}^*\backslash\{p_1\})) = t^{2\ell} - 2t^{2\ell-2} + t^{2\ell-4}$

Proof. We will prove the result by induction on ℓ. The case $\ell = 2$ follows from the decomposition

$$S^2(\mathbb{C}^*) = S^2(\mathbb{C}^*\backslash\{p_1\}) \cup (\mathbb{C}^*\backslash\{p_1\}) \cup \{(p_1,p_1)\} :$$

$p^{vir}(S^2(\mathbb{C}^*)) = t^4 - t^2$ (as in [MOP11]), $p^{vir}(\mathbb{C}^*\backslash\{p_1\}) = t^2 - 2$, so

$$p^{vir}(S^2(\mathbb{C}^*\backslash\{p_1\})) = t^4 - 2t^2 + 1.$$

Suppose for some ℓ, $\forall k \leq \ell$, $p^{vir}(S^k(\mathbb{C}^*\backslash\{p_1\})) = t^{2k} - 2t^{2k-2} + t^{2k-4}$.

Again, we use the result in [MOP11]

$$p^{vir}(S^{k+1}(\mathbb{C}^*)) = t^{2k+2} - t^{2k}.$$

We can decompose $S^{\ell+1}(\mathbb{C}^*) = \bigcup_{k=0}^{\ell+1} S^k(\mathbb{C}^*\backslash\{p_1\})$. Therefore,

$$p^{vir}(S^{\ell+1}(\mathbb{C}^*)) = \sum_{k=0}^{\ell+1} p^{vir}(S^k(\mathbb{C}^*\backslash\{p_1\})),$$

and the result follows by induction. □

We generalize this to $S^d(\mathbb{C}^*\backslash\{p_1,\dots,p_{m-3}\})$.

Lemma 3.3.3. $p^{vir}(S^d(\mathbb{C}^*\backslash\{p_1,\dots,p_h\})) = \sum_{j=0}^{d}(-1)^j\binom{h+1}{j}t^{2d-2j}$.

Proof. We prove the result by induction on $q = h + d$.

Suppose the result is true for q, we will prove that it is true for $q+1$. Notice that we have the decomposition

$$S^d(\mathbb{C}^*\backslash\{p_1,\dots,p_{h-1}\}) = \bigcup_{j=0}^{d} S^j(\mathbb{C}^*\backslash\{p_1,\dots,p_h\}).$$

Using the induction hypothesis,

$$\sum_{k=0}^{d}(-1)^k\binom{h}{k}t^{2d-2k} = p^{vir}(S^d(\mathbb{C}^*\backslash\{p_1,\dots,p_h\})) + \sum_{j=0}^{d-1}\sum_{k=0}^{j}(-1)^k\binom{h+1}{k}t^{2j-2k}.$$

Look at the coefficient on $t^{2\ell}$ on both sides, for $\ell \in \{0,\dots,d\}$:

- On the left side, the coefficient is $(-1)^{d-\ell}\binom{h}{d-\ell}$
- On the right side, the coefficient is $\sum_{k=0}^{d-1-\ell}(-1)^k\binom{h+1}{k}$

Comparing coefficients, the coefficient on $t^{2\ell}$ in $p^{vir}(S^d(\mathbb{C}^*\backslash\{p_1,\dots,p_h\}))$ is

$$(-1)^{d-\ell}\binom{h}{d-\ell} + \sum_{k=0}^{d-1-\ell}(-1)^{k+1}\binom{h+1}{k}.$$

I claim that

$$(-1)^{d-\ell}\binom{h}{d-\ell} + \sum_{k=0}^{d-1-\ell}(-1)^{k+1}\binom{h+1}{k} = (-1)^{d-\ell}\binom{h+1}{d-\ell},$$

or, equivalently,

$$\sum_{k=0}^{d-\ell}(-1)^k\binom{h+1}{k} = (-1)^{d-\ell}\binom{h}{d-\ell}.$$

We see that it suffices to prove the following lemma.

Lemma 3.3.0.1. *For all* $m, n \in \mathbb{N}^{\geq 0}$, $\sum_{k=0}^{m}(-1)^k\binom{n+1}{k} = (-1)^m\binom{n}{m}$

Proof. We have the identity

$$(1-x)^n - \sum_{k=0}^{m}(1-x)^{n+1}x^{m-k} = x^{m+1}(1-x)^n.$$

Comparing the coefficients on x^m on both sides yields the desired identity. □

The result now follows. □

We must calculate the virtual Poincaré polynomial of $M_{0,m}$.

Lemma 3.3.4. $p^{vir}(M_{0,m}) = \prod_{i=2}^{m-2}(t^2 - i)$ *for* $m \geq 4$.

Proof. We will prove this by induction. $M_{0,3}$ is a point, so $p^{vir}(M_{0,3}) = 1$. Next, $M_{0,4} \cong \mathbb{P}^1\backslash\{0,1,\infty\}$. We have already seen that $p^{vir}(\mathbb{P}^1\backslash\{0,1,\infty\}) = t^2 - 2$.

Suppose for some m, $p^{vir}(M_{0,m}) = \prod_{i=2}^{m-2}(t^2 - i)$ Now, using our earlier result, that the monodromy action of $\pi_1(M_{0,m})$ on the compactly supported cohomology of the fibers of $M_{0,m+1} \to M_{0,m}$ is trivial, we see that

$$p^{vir}(M_{0,m+1}) = p^{vir}(M_{0,m}) \cdot p^{vir}(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}).$$

We have already calculated that $p^{vir}(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\}) = t^2 - (m-1)$. By the inductive hypothesis together with the observation above, we see that

$$p^{vir}(M_{0,m+1}) = (t^2 - (m-1)) \cdot \prod_{i=2}^{m-2}(t^2 - i).$$

The result now follows. □

Thus, the virtual Poincaré polynomial of the open locus $M_{0,m|d}/\mathcal{S}_d$ is

$$p^{vir}(M_{0,m}) \cdot p^{vir}(S^d(\mathbb{P}^1\backslash\{p_1,\ldots,p_m\})) = \prod_{i=2}^{m-2}(t^2 - i) \cdot \sum_{j=0}^{d}(-1)^j\binom{m-1}{j}t^{2d-2j}.$$

We want to find the coefficient on $t^{2m+2d-8}$ in this product. Let $0 \leq h \leq m-3$. Then we must have $h + d - j = m + d - 4$, so $h = m - 4 + j$. But since $0 \leq h \leq m-3$, either $j = 0$ or $j = 1$. Therefore the coefficient on $t^{2m+2d-8}$ is

$$-(\textstyle\sum_{i=2}^{m-2} i) - (m-1) = \frac{-m\cdot(m-3)-2m+2}{2} = \frac{-m^2+m+2}{2}.$$

The case that $m = 3$ follows by direct calculation.

We consider the locus of reducible curves with exactly two components. In the case that $m = 3$, there are $3d$ components in the moduli space parameterizing reducible curves with exactly two components. Assume $m \geq 4$.

- Component A has 1 marking, degree $k > 0$, and component B has $m-1$ markings and degree $d - k \geq 0$
- Component A has $m - 2 \geq j \geq 2$ markings, degree $k \geq 0$, and component B has $m - j$ markings and degree $d - k$

There are md curves of the first type, and there are $\frac{d+1}{2} \cdot \sum_{j=2}^{m-2}\binom{m}{j}$ curves of the second type. Therefore, there are $2^{m-1}\cdot(d+1) - \frac{2\cdot(d+1)+2m\cdot(d+1)}{2} + md = 2^{m-1}\cdot(d+1) - \frac{2d+2+2m}{2}$ components corresponding to reducible curves with exactly two components.

Thus, the coefficient on $t^{2m+2d-8}$ in the virtual Poincaré polynomial of $\overline{M}_{0,m|d}/\mathcal{S}_d$ is $2^{m-1}\cdot(d+1)-\frac{m^2+m+2d}{2}$ for $m\geq 3$. This completes the proof of Lemma 3.3.1.

3.4 The rank of the Picard group

By *Corollary 2* of [Cey09], $A^*(\overline{M}_{0,m|d})\otimes\mathbb{Q}\cong H^{2*}(\overline{M}_{0,m|d})\otimes\mathbb{Q}$, so the same is true for $\overline{M}_{0,m|d}/\mathcal{S}_d$. This was needed in the proof of Lemma 3.1.1 in order to conclude that

$$A^*(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),\, d))\otimes\mathbb{Q}\cong H^{2*}(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),\, d))\otimes\mathbb{Q}.$$

Putting together the results of the previous subsection, we obtain:

Proposition 3.4.1. *For $m\geq 2$, $n\geq 2$, $d\geq 1$,*

$$rank(Pic(\,\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1},\, d))\otimes\mathbb{Q}\,)\;=\;2^{m-1}\cdot(d+1)-\frac{m^2-m+2d}{2}.$$

For $m\geq 2$, $r\geq 2$, $n-r\geq 1$, $d\geq 1$,

$$rank(Pic(\,\overline{\mathcal{Q}}_{0,m}(Gr(r,n),d))\otimes\mathbb{Q}\,)\;=\;2^{m-1}\cdot(d+1)-\frac{m^2-m+2d}{2}+1.$$

For $m\geq 2$, $r=n=1$, $d\geq 1$,

$$rank(Pic(\,\overline{\mathcal{Q}}_{0,m}(Gr(1,1),d))\otimes\mathbb{Q}\,)\;=\;2^{m-1}\cdot(d+1)-\frac{m^2+m+2d}{2}.$$

For $m\geq 2$, $r=n\geq 2$, $d\geq 1$,

$$rank(Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),d))\otimes\mathbb{Q}\,)\;=\;2^{m-1}\cdot(d+1)-\frac{m^2+m+2d}{2}+1.$$

Notice that since the rational Chow groups are isomorphic to the rational cohomology groups, then we see that numerical equivalence coincides with rational equivalence

$$Num(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),d))\otimes\mathbb{Q}\cong Pic(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),d))\otimes\mathbb{Q}.$$

We will use this when we intersect with curves to determine generators and relations for the Picard group when $m=2$.

4 Picard Group of Stable Quotients

In this chapter we find generators and relations for the rational Picard group of the moduli stack of stable quotients to the Grassmannian. When $m \geq 3$, we use an excision sequence to calculate generators for the Picard group. We show that the interior of the moduli stack is isomorphic to an open subscheme of a relative Quot scheme. Its Picard group is known [Str87]. By a dimension count and accounting for the relations pulled back from $\overline{M}_{0,m}$, we will have a complete set of generators and relations when $r \neq n$. When $r = n$, we will intersect with curves to find an additional relation.

When $m = 2$, the interior of the moduli space does not have as simple a description as before. Instead of the equivariant cohomology approach of [Opr06b], we produce a collection of generators and relations by intersecting with curves.

4.1 The Picard groups when $m \geq 3$

4.1.1 The analysis on the interior

We describe the Picard group over the interior of the moduli space of stable quotients. We begin by proving the following:

Lemma 4.1.1. *The interior $\mathcal{Q}_{0,m}(Gr(r,n),\ d)$ of the moduli of stable quotients is isomorphic to an open subscheme of the relative Quot scheme $Quot_{\mathbb{P}^1 \times M_{0,m}/M_{0,m}}(Gr(r,n),\ d)$ over $M_{0,m}$.*

In the proof we will use the analysis of the boundary in $Quot_{\mathbb{P}^1}(Gr(r,n),\ d)$ in [Ber94] and [Mar07].

Proof. As in [Ber94], [Mar07], the locus $Quot_{\mathbb{P}^1}(Gr(r,n),\ d)^{\tau=e}$ parameterizing those quotients whose quotient has a torsion subsheaf of degree exactly e has a morphism

$$Quot_{\mathbb{P}^1}(Gr(r,n),\ d)^{\tau=e} \to S^e(\mathbb{P}^1) \times Mor_{d-e}(\mathbb{P}^1, Gr(r,n)),$$

where $S^e(\mathbb{P}^1)$ is the e^{th} symmetric product of $\mathbb{P}^1$ and $Mor_{d-e}(\mathbb{P}^1, Gr(r,n))$ is the morphism space parameterizing degree $d-e$ morphisms from $\mathbb{P}^1$ to the Grassmannian. This map is obtained by mapping

$$[\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{Q} \to 0] \mapsto [Supp(\tau), \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{Q}/\tau \to 0]$$

where τ is the torsion subsheaf of $\mathcal{Q}$.

We will use the fact that $M_{0,m} \cong (\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta$, where here Δ is the union of all the diagonals. We can realize $S^e(\mathbb{P}^1)$ as $\mathbb{P}(H^0(\mathcal{O}(e))$ by taking the section which vanishes at the tuple of e points in $\mathbb{P}^1$, up to $\mathbb{C}^*$ multiplication.

Let y_i be coordinates on $(\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta$, and let

$$(x_0, \ldots, x_e) \in H^0(\mathbb{P}(H^0(\mathcal{O}(e)))\,, \mathcal{O}(1))$$

be a basis. We consider the closed subschemes of $((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1)$ given by the unions of the vanishing of the following equations:

- for each i, $\sum_{j=0}^{e} x_j \cdot y_i^{e-j}$, where here we think of y_i as an element of $\mathbb{P}^1$ via $[y_i : 1]$
- the preimages under the projection $((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1) \to S^e(\mathbb{P}^1)$ of the vanishing of x_0, x_e, or $\sum_{i=0}^{e} x_i$ in $S^e(\mathbb{P}^1)$, whose vanishing corresponds to the locus of points in $S^e(\mathbb{P}^1)$ where at least one of the elements in the (unordered) tuple in $S^e(\mathbb{P}^1)$ is $[1:0] = \infty$, $[0:1] = 0$, or $[1:1] = 1$, respectively.

The vanishing of the first collection of equations corresponds to the locus of tuples in $((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1)$ where there is an entry in the first factor which coincides with at least one entry in the second factor. The individual vanishing of each of the second equations corresponds to the locus of (unordered) tuples in $((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1)$ where at least one of the entries in the second factor is ∞, 0, or 1, respectively. Call the union of all of the above closed subschemes Δ^e. We can consider the preimage of Δ^e under the projection map

$$((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1) \times Mor_{d-e}(\mathbb{P}^1, Gr(r,n)) \to ((\mathbb{C}^*\backslash\{1\})^{m-3}\backslash\Delta) \times S^e(\mathbb{P}^1)$$

which we also call Δ^e by abuse of notation.

We consider the preimage of Δ^e under the morphism

$$M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)^{\tau=e} \to M_{0,m} \times S^e(\mathbb{P}^1) \times Mor_{d-e}(\mathbb{P}^1, Gr(r,n))$$

Call this $\Delta^{e'}$. Repeat this process for each $0 < e \le d$. Call the union

$$\bigcup_{e=1}^{d} \Delta^{e'} := \tau$$

a closed subscheme of $M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)$.

Points in $M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)\backslash\tau$ parameterize coherent quotient sheaves of $\mathbb{C}^n \otimes \mathcal{O}$ of rank $n-r$, degree d on an m pointed $\mathbb{P}^1$, where the first three markings are at $0, \infty, 1$, such that the quotient does not have torsion supported on the markings (if it has torsion at all).

Since the number of markings is at least 3, stable quotients whose underlying curve is smooth are automorphism-free. Since the coarse moduli space is a scheme, the substack $\mathcal{Q}_{0,m}(Gr(r,n),\ d)$ is representable by a scheme which we denote by $\mathcal{Q}_{0,m}(Gr(r,n),\ d)$ as well.

We claim that the scheme is isomorphic to $M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)$. We will produce a natural isomorphism between the two functors.

- If we consider a morphism
$$T \to M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)\backslash\tau$$
then we get a family $\mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q} \to 0$ of coherent quotient sheaves of $\mathbb{C}^n \otimes \mathcal{O}$ of rank $n-r$, degree d on $\mathbb{P}^1 \times T$ with m distinct sections (the first three of which correspond to the constant sections $0, \infty, 1$) and the quotient $\mathcal{Q}$ does not have torsion supported on any closed subscheme of the marked sections. This yields a map from T to $\mathcal{Q}_{0,m}(Gr(r,n),\ d)$.

- Since any family of stable quotients over T to $Gr(r,n)$ whose map to moduli has image contained in $\mathcal{Q}_{0,m}(Gr(r,n),\ d)$ has a map to $M_{0,m}$ by forgetting the quotient sequence, then the underlying family of curves is isomorphic to a trivial family with the first three marked sections being the constant sections $0, \infty, 1$. We can pull back the quotient sequence under the inverse of this isomorphism and this yields a map $T \to M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)\backslash\tau$ since the torsion of the quotient is away from the marked sections.

By construction, the two natural transformations we defined above are inverse to each other on objects and morphisms, so the result follows. $\square$

Let

$$\begin{tikzcd} \mathcal{C}^o \arrow[d] \\ \mathcal{Q}_{0,m}(Gr(r,n),\ d) \arrow[u, bend left, "\sigma_i"] \end{tikzcd} \quad , 0 \to \mathcal{S}_o \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_o \to 0$$

be the universal curve with its m sections and the restriction of the universal sequence over the universal curve.

Let $q = dim_{\mathbb{C}}(\mathcal{Q}_{0,m}(Gr(r,n),\ d)) = nd + r(n-r) + m - 3$.

We show

Lemma 4.1.2. $\pi^o_* c_1^2(\mathcal{Q}_0)$ *and* $\pi^o_* c_2(\mathcal{Q}_o)$ *generate* $A_{q-1}(\mathcal{Q}_{0,m}(Gr(r,n),\ d)) \otimes \mathbb{Q}$.

Proof. By excision,
$A_{q-1}(M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),d)) \otimes \mathbb{Q} \to A_{q-1}(\mathcal{Q}_{0,m}(Gr(r,n),\ d)) \otimes \mathbb{Q} \to 0$
which reduces the problem to the following lemma.

Lemma 4.1.3. $\rho_* c_1^2(\mathcal{F}')$ *and* $\rho_* c_2(\mathcal{F}')$ *generate* $A_{q-1}(M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),d))$ *over* $\mathbb{Q}$ *where*

$$\begin{tikzcd} \mathbb{P}^1 \times M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \arrow[d, "\rho"] \\ M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \end{tikzcd}$$

is the universal curve over the relative Quot scheme, and $\mathcal{F}'$ *is the universal quotient of* $\mathbb{C}^n \otimes \mathcal{O}$ *on the universal curve.*

We use the methods in [Opr06b].

Proof. We have the fiber diagram

$$\begin{tikzcd} \mathbb{P}^1 \times M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \arrow[r, "f'"] \arrow[d, "\rho'"] & \mathbb{P}^1 \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \arrow[d, "\rho"] \\ M_{0,m} \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \arrow[r, "f"] & Quot_{\mathbb{P}^1}(Gr(r,n),\ d) \end{tikzcd}$$

In [Str87], it is shown in *Theorem 2.2*, that $A^1(Quot_{\mathbb{P}^1}(Gr(r,n),\ d)) \otimes \mathbb{Q}$ is generated by $c_1(\rho_*\mathcal{F})$ and $c_1(\rho_*(\mathcal{F} \otimes p_1^*\mathcal{O}(-1)))$, where $\mathcal{F}$ is the universal quotient sheaf over $\mathbb{P}^1 \times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)$. [Str87] shows that there exist exact sequences $(***)$

$$0 \to \rho_*(\mathcal{F} \otimes p_1^*\mathcal{O}(k)) \to \bigoplus_{i=1}^{2} \rho_*(\mathcal{F} \otimes p_1^*\mathcal{O}(k+1)) \to \rho_*(\mathcal{F} \otimes p_1^*\mathcal{O}(k+2)) \to 0$$

We claim that $c_1(\rho_*\mathcal{F})$, $c_1(\rho_*(\mathcal{F} \otimes p_1^*\mathcal{O}(d))$ generate $A^1(Quot_{\mathbb{P}^1}(Gr(r,n),d)) \otimes \mathbb{Q}$
It suffices to prove the sublemma:

Lemma 4.1.0.1. *$c_1(\rho_*(\mathcal{F}\otimes p_1^*\mathcal{O}(h))) = (h+1)\cdot c_1(\rho_*\mathcal{F}) - h\cdot c_1(\rho_*(\mathcal{F}\otimes p_1^*\mathcal{O}(-1)))$ for all $h\geq 1$.*

Proof. The proof follows by induction together with $(***)$. $\square$

By Lemma 4.1.0.1, it follows that $c_1(\rho_*\mathcal{F})$, $c_1(\rho_*(\mathcal{F}\otimes p_1^*\mathcal{O}(d))$ generate $A^1(Quot_{\mathbb{P}^1}(Gr(r,n),d))\otimes\mathbb{Q}$.

By [Opr06b], we see that

$$\begin{aligned} A^1(M_{0,m}\times Quot_{\mathbb{P}^1}(Gr(r,n),d))\otimes\mathbb{Q} &\cong A_{q-1}(M_{0,m}\times Quot_{\mathbb{P}^1}(Gr(r,n),\ d))\otimes\mathbb{Q} \\ &\cong A_{q-1}(Quot_{\mathbb{P}^1}(Gr(r,n),d))\otimes\mathbb{Q} \end{aligned}$$

is generated by $f^*c_1(\rho_*\mathcal{F})$ and $f^*c_1(\rho_*(\mathcal{F}\otimes p_1^*\mathcal{O}(d))$. By base change, these generators can be rewritten as $c_1(\rho'_*\mathcal{F}')$ and $c_1(\rho'_*(\mathcal{F}'\otimes {p'_1}^*\mathcal{O}(d)))$.

Notice that on the fibers $\mathbb{P}^1_t$ of ρ', $det(\mathcal{F}')|_{\mathbb{P}^1_t}\cong\mathcal{O}(d)|_{\mathbb{P}^1_t}$, so by *Seesaw Theorem* ([Mum70]), $\exists$ a line bundle $\mathcal{N}$ on $M_{0,m}\times Quot_{\mathbb{P}^1}(Gr(r,n),\ d)$ such that

$$det(\mathcal{F}')\cong{\rho'}^*\mathcal{N}\otimes {p'}_1^*\mathcal{O}(d).$$

We will show that $c_1(\rho'_*\mathcal{F}')$ and $c_1(\rho'_*(\mathcal{F}'\otimes {p'}_1^*\mathcal{O}(d)))$ can be written as unique $\mathbb{Q}$ linear combinations of $\rho'_*c_1^2(\mathcal{F}')$ and $\rho'_*c_2(\mathcal{F}')$. This follows via Grothendieck-Riemann-Roch. Indeed, it is not difficult to compute:

- $c_1(\rho'_*\mathcal{F}') = c_1(\mathcal{N}) + \frac{1}{2}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}')$
- $c_1(\rho'_*(\mathcal{F}'\otimes {p'}_1^*\mathcal{O}(d))) = (d+1)c_1(\mathcal{N}) + \frac{1}{2}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}')$
- $c_1(\mathcal{N}) = \frac{1}{2d}\rho'_*c_1^2(\mathcal{F}')$.

Putting all of this together, we find:

$$\begin{aligned} c_1(\rho'_*\mathcal{F}') &= c_1(\mathcal{N}) + \frac{1}{2}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}') \\ &= \frac{d+1}{2d}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}'), \end{aligned}$$

$$\begin{aligned} c_1(\rho'_*(\mathcal{F}'\otimes {p'}_1^*\mathcal{O}(d))) &= (d+1)c_1(\mathcal{N}) + \frac{1}{2}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}') \\ &= \frac{2d+1}{2d}\rho'_*c_1^2(\mathcal{F}') - \rho'_*c_2(\mathcal{F}'). \end{aligned}$$

Since the matrix $\begin{pmatrix} \frac{d+1}{2d} & \frac{2d+1}{2d} \\ -1 & -1 \end{pmatrix}$ is invertible, we see that $\rho'_*c_1^2(\mathcal{F}')$ and $\rho'_*c_2(\mathcal{F}')$ generate $A_{q-1}(Quot_{\mathbb{P}^1}(Gr(r,n),\ d)\times M_{0,m})\otimes\mathbb{Q}$. $\square$

This concludes the proof of Lemma 4.1.2.

□

Corollary 4.1.1. *$\pi_* c_1^2(\mathcal{Q})$, $\pi_* c_2(\mathcal{Q})$, and the irreducible components of the boundary generate the Picard group of $\overline{\mathcal{Q}}_{0,m}(Gr(r,n),\, d)$ when $m \geq 3$.*

Proof. The proof follows from the excision sequence for stacks as in [Kre99]
$A_{q-1}(\Delta)\otimes\mathbb{Q} \to A_{q-1}(\overline{\mathcal{Q}}_{0,m}(Gr(r,n),\ d))\otimes\mathbb{Q} \to A_{q-1}(\mathcal{Q}_{0,m}(Gr(r,n),\ d))\otimes\mathbb{Q} \to 0$
where here Δ is the union of the boundary divisors. □

4.1.2 Picard group when $r \neq n$

In this subsection we will prove Theorem 0.4.2.

Proof. By a dimension count, we see that:

- for $m = 3$, $n \geq 2$, $Pic(\overline{\mathcal{Q}}_{0,3}(\mathbb{P}^{n-1}, d))\otimes\mathbb{Q}$ is generated by freely $\pi_* c_1^2(\mathcal{Q})$ and the irreducible components of Δ;
- for $m = 3$, $Pic(\overline{\mathcal{Q}}_{0,3}(Gr(r,n), d))\otimes\mathbb{Q}$ is generated freely by $\pi_* c_1^2(\mathcal{Q})$, $\pi_* c_2(\mathcal{Q})$, and the irreducible components of Δ for $r \neq n$, $r \geq 2$.

We consider the case $m > 3$.

We must count the boundary divisors. There are md boundary divisors corresponding to distinct reducible curves with one marking on one of the components and the other $m-1$ markings on the other component. There are $\frac{d+1}{2}\cdot\sum_{j=2}^{m-2}\binom{m}{j}$ boundary divisors corresponding to reducible domains with one component carrying $j \geq 2$ markings and the other component carrying $m-j \geq 2$ markings. Thus, there are $2^{m-1}(d+1) - m - d - 1$ such boundary divisors.

There is a map $\overline{\mathcal{Q}}_{0,m}(Gr(r,n),\ d) \to \overline{M}_{0,m}$ given by stabilizing the underlying curve. We can pull back relations among the boundary divisors using this map. [Kee92] calculated that there are $\binom{m-1}{2} - 1$ independent relations amongst the boundary divisors of $\overline{M}_{0,m}$. These relations will stay independent when we pull them back to $\overline{\mathcal{Q}}_{0,m}(Gr(r,n), d)$.

When $r = 1$, the collection of boundary divisors and the classes $\pi_* c_1^2(\mathcal{Q}) = \pi_* c_2(\mathcal{Q})$, with the relations pulled back from $\overline{M}_{0,m}$, yield the dimension count

$$(2^{m-1}(d+1) - m - d - 1) + 1 - (\tfrac{m^2-3m+2}{2} - 1) = 2^{m-1}(d+1) - \tfrac{m^2-m+2d}{2},$$

which agrees with our calculation of the Picard rank (Proposition 3.4.1). Thus, for $m > 3$, $n \geq 2$, $Pic(\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$ is generated by $\pi_* c_1^2(\mathcal{Q})$ and Δ, such that all relations are pulled back from $\overline{M}_{0,m}$.

Similarly, when $m > 3$, $r \geq 2$, $r \neq n$, $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(r,n), d)) \otimes \mathbb{Q}$ is generated by $\pi_* c_1^2(\mathcal{Q})$, $\pi_* c_2(\mathcal{Q})$, and the irreducible components of Δ, such that all relations are pulled back from $\overline{M}_{0,m}$.

This proves Theorem 0.4.2.

□

4.1.3 Test curves I

In this section we construct the curves used to calculate the generators and relations of the Picard group.

The curves $\mathsf{A}_{j,e}$

The first collection of curves are for the $r = n = 1$ case. Consider the curves $\mathsf{A}_{j,e}$ constructed as follows:

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.
- Pick the 2 trivial sections of p_2 at $[0:1]$, $[1:0]$, call them s_1, s_m.
- Pick $j-1$ smooth irreducible sections of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ whose only pairwise common vanishing points are $([1:0],[1:0])$ and $([0:1],[0:1])$, and at these points they have distinct tangent directions both from each other as well as from the 2 trivial sections above; call these s_i for $i \in \{2,\dots,j\}$ (we also allow $j = 1$, in which case $e > 0$ below).
- Pick $m-j-1$ trivial distinct sections of p_2, distinct from s_1, s_m above; call these s_k , $k \in \{j+1,\dots,m-1\}$ (we allow $j = 1$, which means that in this case $k \in \{2,\dots,m-1\}$).
- Pick $0 \leq e \leq d$ smooth irreducible sections of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ whose only pairwise common vanishing points are $([1:0],[1:0])$ and $([0:1],[0:1])$, such that at these points they have distinct tangent directions from each other as well as from s_i above, and they have distinct tangent directions from the marked

sections s_k at their points of intersection; call these δ_ℓ , $\ell = 1, \ldots, e$, ($e = 0$ is still allowed, but then $j > 1$ above).

- Pick $d - e$ trivial distinct sections of p_2, distinct from the sections above; call these σ_h, $h = 1, \ldots, d - e$ ($d - e = 0$ is still allowed).
- We also impose the conditions that $e + j \geq 2$, $(d - e) + (m - j) \geq 2$, for stability.

The conditions $j = 1 \implies e > 0$ and $e = 0 \implies j > 1$ ensure that the resulting family we obtain is nontrivial in moduli.

Now, we blow up $\mathbb{P}^1 \times \mathbb{P}^1$ in:

- $([1:0],[1:0])$ and $([0:1],[0:1])$ (when $m = 2$, these are the only points blown up; for arbitrary m we blow up further)
- the $(j-1)\cdot(m-j-1)$ intersection points of $\bigcup_i s_i \cap \bigcup_k s_k$
- the $e(m-j-1)$ intersection points of $\bigcup_k s_k \cap \bigcup_\ell \delta_\ell$
- the $(d-e)(j-1)$ intersection points of $\bigcup_i s_i \cap \bigcup_h \sigma_h$.

We have a family of stable quotients over $\mathbb{P}^1$ to $Gr(1,1)$ given by:

- $\tilde{p_2} : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^1$
- the strict transforms $\overline{s}_1 = Bl^* s_1 - E_1$, $\overline{s}_m = Bl^* s_m - E_m$ (if $j > 1$)
- the strict transforms $\overline{s}_i = Bl^* s_i - E_1 - E_m - \sum_k E_{i,k} - \sum_h E_{h,i}$, for each $i \in \{2, \ldots, j\}$ (if $j > 1$; if $j = 1$ then we do not have any of these sections)
- the strict transforms $\overline{s}_k = Bl^* s_k - \sum_i E_{i,k} - \sum_\ell E_{k,\ell}$ (if $e \neq 0$)
- the strict transforms $\overline{\delta}_\ell = Bl^* \delta_\ell - E_1 - E_m - \sum_k E_{k,\ell}$ if $e \neq 0$
- the strict transforms $\overline{\sigma}_h = Bl^* \sigma_h - \sum_i E_{h,i}$ (if $j > 1$)
- an inclusion of the subsheaf
$$\mathcal{O}(-\sum_\ell \overline{\delta}_\ell - \sum_h \overline{\sigma}_h) \hookrightarrow \mathcal{O}.$$

We can push forward these curves under the morphism

$$\overline{\mathcal{Q}}_{0,m}(Gr(1,1),\ d) \to \overline{\mathcal{Q}}_{0,m}(Gr(n,n),\ d)$$

given by

$$[\mathcal{S} \hookrightarrow \mathcal{O}] \mapsto [\mathcal{S} \oplus \mathbb{C}^{n-1} \otimes \mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathbb{C}^{n-1} \otimes \mathcal{O}]$$

where $n \geq 2$. We call the resulting curves $\mathsf{A}_{j,e}$ as well.

The curves $\mathsf{A}^2_{j,e}$

The next collection of curves are unique to the $n \geq 2$ case. Call these curves $\mathsf{A}^2_{j,e}$. The family of curves and sections is the same as $\mathsf{A}_{j,e}$, except we change the inclusion of the subsheaf to be

$$\mathcal{O}(-\sum_{\ell} \overline{\delta}_\ell) \oplus \mathcal{O}(-\sum_{h} \overline{\sigma}_h) \oplus \mathbb{C}^{n-2} \otimes \mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathcal{O} \oplus \mathbb{C}^{n-2} \otimes \mathcal{O}.$$

4.1.4 Intersections of curves with Δ

In order to find relations by intersecting with curves, we will need to be able to calculate the intersection numbers of various curves with the irreducible components of the boundary.

To do so, observe that given a map from a curve T to the moduli stack, the pullback of the boundary is the discriminant scheme of the corresponding family of curves $\pi : C \to T$ over T. This discriminant scheme is defined (as in [Eis00]) as the vanishing of the 0^{th} Fitting ideal of the pushforward of the structure sheaf of the singular scheme in the fibers of π, which in turn is defined as the subscheme given by the vanishing of the 1^{st} Fitting ideal of the relative cotangent sheaf of π.

Since all of our families are blowups of $\mathbb{P}^1$ bundles, and the singular scheme is defined locally, we can compute the discriminant scheme explicitly in all of our cases. In our families, the stable quotients whose underlying curve is reducible will be without automorphisms. Thus, to calculate the degree of Δ on T, all we must do is determine the length of the discriminant scheme.

Since the singular scheme is defined locally, on the $\mathbb{P}^1$ bundle we can restrict to an open subscheme of the base so that the $\mathbb{P}^1$ bundle has the form $\mathbb{P}^1 \times \mathbb{C} \to \mathbb{C}$. Since the blowup is local, we can consider the blowup of $\mathbb{C}^2 \to \mathbb{C}$ in a point. Picking coordinates on the blowup, we see that our family locally takes the form

$$Spec(\mathbb{C}[x, y, t]/(xy - t) \to Spec(\mathbb{C}[t]).$$

We have the following resolution of Ω_π :

$$\mathbb{C}[x,y,t]/(xy-t) \to \mathbb{C}[x,y,t](xy-t)dx \oplus \mathbb{C}[x,y,t]/(xy-t)dy \to \Omega_\pi \to 0,$$

where the first map is given by the matrix $\begin{pmatrix} ydx \\ xdy \end{pmatrix}$. Then, the 1^{st} Fitting ideal is given by $(x,y) + (xy-t) = (x,y,t)$. Pushing the stucture sheaf of this point forward to *Spec*$(\mathbb{C}[t])$, we obtain the structure sheaf of 0 in *Spec*$(\mathbb{C}[t])$, which has 0^{th} Fitting ideal given by (t). Thus, the discriminant subscheme has length 1. If our family of curves has more than one nodal fiber, we can repeat the same calculation for each nodal fiber one at a time in the same way we just did for a single nodal fiber since the discriminant subscheme commutes with base change.

Thus, for all the curves we consider, the intersection of Δ with our curve is the number of nodal fibers in the family over the curve.

4.1.5 The Picard group of $\overline{\mathcal{Q}}_{0,m}(Gr(n,n), d)$

Now we separately handle the case $r = n$.

The case $r = n = 1$

In the first case, we have the isomorphism of coarse moduli spaces (as in *Proposition 3* of [MOP11])

$$\overline{Q}_{0,m}(Gr(1,1), d) \cong \overline{M}_{0,m|d}/\mathcal{S}_d.$$

In [Cey09], the Picard group of $\overline{M}_{0,m|d}$ was calculated, where it is shown that the Picard group is generated by the irreducible components of Δ, as well as the classes $\mathcal{D}_{ij}$, parameterizing curves where the i^{th} and j^{th} weight ϵ markings coincide. All relations come from pushing forward the relations on $\overline{M}_{0,m+d}$. [Cey09] shows that

$$\mathcal{D}_{i,j} = \sum_{i\in B,\, j\notin B} \Delta_{A,B}$$

where $A \subseteq \{1,\dots,m\}$ and $B \subseteq \{1,\dots,d\}$. Then,

$$\sum_{i\leq j} \mathcal{D}_{i,j} = \frac{1}{2} \sum_{B\subset\{1,\dots,d\}} |B|(d-|B|)\Delta_{A,B}$$

is $\mathcal{S}_d$ invariant, so it gives us a relation in $Pic(\overline{M}_{0,m|d}/\mathcal{S}_d) \otimes \mathbb{Q}$. Thus, $Pic(\overline{M}_{0,m|d}/\mathcal{S}_d) \otimes \mathbb{Q}$ is generated by the $\mathcal{S}_d$ invariant sums of irreducible divisors in Δ.

However, we have an additional generator, $\pi_* c_1^2(\mathcal{Q}) = \pi_* c_2(\mathcal{Q})$, so we will write the generator $\pi_* c_1^2(\mathcal{Q})$ in terms of the irreducible components of Δ. To do so, we use the test families of curves. Notice that since $\mathcal{S}_m$ acts on the moduli space, assuming the relation has nonzero coefficients on all of the boundary terms, it will be $\mathcal{S}_m$ invariant, so the coefficient on $\Delta_{A,k}$ only depends on the cardinality of A, where here $\Delta_{A,k}$ refers to a reducible curve with A markings and degree k on one component.

If our desired relation has the form R_1 :

$$\sum_{j=1}^{m}\sum_{k=0}^{[\frac{d}{2}]} c_{j,k} \sum_{|A|=j} \Delta_{A,k} = \pi_* c_1^2(\mathcal{Q})$$

in the rational Picard group, then we have the following relation among the coefficients in the relation by intersecting with $\mathsf{A}_{j,e}$:

$$2c_{j,e} + (j-1)(m-j-1)c_{2,0} + (\,e(m-j-1) + (d-e)(j-1)\,)c_{1,1} =$$
$$2de - 2e^2 - e(m-j-1) - (d-e)(j-1)\,.$$

We must solve for the coefficients. Let $j=1$, $e=1$ to start:

$$mc_{1,1} = (2d-m) \implies c_{1,1} = \tfrac{2d}{m} - 1.$$

Next we let $j=2$, $e=0$:

$$(m-1)c_{2,0} + dc_{1,1} = -d \implies (m-1)c_{2,0} + \tfrac{2d^2}{m} - d = -d \implies c_{2,0} = \tfrac{-2d^2}{m(m-1)}.$$

Substituting these back into the original expression $\implies$

$$c_{j,e} = \tfrac{-jd^2(j-1)}{m(m-1)} - \tfrac{(m-2j)ed}{m} + e(d-e).$$

Thus, we see that we have the relation

$$\sum_{j=1}^{m}\sum_{k=0}^{[\frac{d}{2}]} \left(\tfrac{-jd^2(j-1)}{m(m-1)} - \tfrac{(m-2j)dk}{m} + k(d-k)\right) \sum_{|A|=j} \Delta_{A,k} = \pi_* c_1^2(\mathcal{Q})$$

in $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(1,1),\, d)\,)\otimes\mathbb{Q}$, for all $m \geq 3$. This proves Lemma 0.4.1 when $m \geq 3$.

The case $r = n \geq 2$

Now we consider the case of $\overline{\mathcal{Q}}_{0,m}(Gr(n,n),\, d)$, where $n \geq 2$. Notice that if $d = 1$, then we have an isomorphism

$$\overline{\mathcal{Q}}_{0,m}(Gr(n,n),1) \to \overline{M}_{0,m+1} \times \mathbb{P}^{n-1}$$

which goes as follows.

Suppose we are given a family of stable quotients

$$C \underset{p_i}{\overset{\pi}{\rightleftarrows}} T \;\;,\; 0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O}_C \to \tau \to 0$$

where the support of τ restricted to each fiber is a single point. We can push the quotient sequence forward to the base, yielding an exact sequence $\mathbb{C}^n \otimes \mathcal{O}_C \to \pi_*\tau \to 0$, noticing that the quotient is locally free by cohomology and base change

[Har77]. This defines a section of $\mathbb{P}^{n-1} \times T \to T$. We can take the determinant of the inclusion $\mathcal{S} \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_C$, yielding $\mathcal{L} \subset \mathcal{O}_C$. Taking the dual yields a section of $\mathcal{L}^*$, and we can consider the vanishing of this section in C, which is a section of π. By definition, this section does not vanish on the marked sections, so considering the family π with the extra section yields a map from T to $\overline{M}_{0,m+1}$. This yields a morphism $T \to \overline{M}_{0,m+1} \times \mathbb{P}^{n-1}$. Since cohomology and base change commute for $\pi_* \tau$, this defines a natural transformation.

Suppose we are given a family of $m+1$ pointed stable curves together with a section of projective $n-1$ space over T

$$\begin{tikzcd} C' \arrow[r, "\pi"] & T \arrow[l, bend right, "p_{m+1}"'] \arrow[l, bend left, "p_i"] \end{tikzcd} \ , \ \begin{tikzcd} \mathbb{P}^{n-1} \times T \arrow[r] & T \arrow[l, bend left, "\sigma"] \end{tikzcd} .$$

Pulling back the universal quotient sequence over $\mathbb{P}^{n-1} \times T$ along σ, we get an exact sequence $\mathbb{C}^n \otimes \mathcal{O}_T \to \sigma^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) \to 0$. We can pull back the quotient sequence along π, and restrict the quotient to the $m+1^{st}$ section, $\mathbb{C}^n \otimes \mathcal{O}_C \to \pi^* \sigma^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)|_{p_{m+1}}$. By construction, the quotient is supported on the $m+1^{st}$ marking, and if we forget the $m+1^{st}$ marking, this defines a map from T to $\overline{\mathcal{Q}}_{0,m}(Gr(n,n), 1)$. This yields a natural transformation.

By construction, these natural transformations are inverses.

Given the universal quotient sequence, $0 \to \mathcal{S} \to \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q} \to 0$, by cohomology and base change, since $\mathcal{Q}$ is flat over $\overline{\mathcal{Q}}_{0,m}(Gr(n,n),1)$ and it meets every fiber in a single point, $\pi_* \mathcal{Q}$ is a line bundle.

Similarly, $\pi_* \mathcal{S}$ has no higher direct images. Thus, we get an exact sequence $\mathbb{C}^n \otimes \mathcal{O} \to \pi_* \mathcal{Q} \to 0$. This yields the map to $\mathbb{P}^{n-1}$. Under this map $\pi_* \mathcal{Q}$ is the pullback of $\mathcal{O}(1)$.

To recover $\mathcal{Q}$, we start with $\mathbb{C}^n \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$ on $\mathbb{P}^{n-1}$. We can pull this back to $\overline{M}_{0,m+1} \times \mathbb{P}^{n-1}$, and pull it back further to $\overline{M}_{0,m+2} \times \mathbb{P}^{n-1}$, the universal curve. Then, we consider the restriction

$$\mathbb{C}^n \otimes \mathcal{O} \to \pi^* p_2^* \mathcal{O}(1) \to \pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}} \to 0,$$

where Σ_{m+1} is the image of the universal section corresponding to the last marking. Then, $\mathcal{Q} \cong \pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}$.

Using the multiplicativity of the Chern polynomial applied to

$$0 \to \pi^* p_2^* \mathcal{O}(1) \otimes \mathcal{O}(-\Sigma_{m+1}) \to \pi^* p_2^* \mathcal{O}(1) \to \pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}} \to 0$$

we see that

- $c_1(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) + c_1(\pi^* p_2^* \mathcal{O}(1) \otimes \mathcal{O}(-\Sigma_{m+1})) = \pi^* p_2^* c_1(\mathcal{O}(1))$
$\implies c_1(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) = \Sigma_{m+1} \implies \pi_* c_1^2(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) = \pi_* \Sigma_{m+1}^2$

- $c_2(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) + c_1(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) \cdot c_1(\pi^* p_2^* \mathcal{O}(1) \otimes \mathcal{O}(-\Sigma_{m+1})) = 0 \implies$
$c_2(\pi^* p_2^* \mathcal{O}(1)|_{\Sigma_{m+1}}) = \Sigma_{m+1}^2 - \Sigma_{m+1} \cdot \pi^* p_2^* \xi.$

We will show that $\pi_* \Sigma_{m+1}^2$ is in the span of the pull backs of the boundary divisors under $p_1 : \overline{M}_{0,m+1} \times \mathbb{P}^{n-1} \to \overline{M}_{0,m+1}$. However, notice that

$$\pi_* c_2(\mathcal{Q}) = \pi_* \Sigma_{m+1}^2 - p_2^* \xi,$$

and the latter of the two terms is clearly not in the span of the pull backs of the boundary divisors under p_1. This will show that the Picard group is generated by the boundary classes and $\pi_* c_2(\mathcal{Q})$, with all relations among the boundary divisors being pulled back from $\overline{M}_{0,m+1}$.

Now, we express $\pi_* \Sigma_{m+1}^2$ in terms of the boundary divisors. Notice that Σ_{m+1} is the pullback of the universal $m+1^{st}$ section over $\overline{M}_{0,m+1}$, so it suffices to calculate the self intersection on $\overline{M}_{0,m+2}$, and then pull this back under the projection map p_1.

Recall that $\pi'_* \Sigma'^2_{m+1} = -\psi_{m+1}$, where $\pi' : \overline{M}_{0,m+2} \to \overline{M}_{0,m+1}$ is the universal curve over $\overline{M}_{0,m+1}$, and Σ'_{m+1} is the $m+1^{st}$ universal section.

Thus, $\pi_* \Sigma_{m+1}^2 = \pi'_* \Sigma'^2_{m+1} = -p_1^* \psi_{m+1}$.

Let $i, j \in \{1, \ldots, m\}$. There is a forgetful morphism $\overline{M}_{0,m+1} \to \overline{M}_{0,\{i,j,m+1\}}$. By [Koc01],

$$-\psi_{m+1} = -\sum_{m+1 \in A;\, i,j \notin A} \Delta_A.$$

If we sum over all pairs $(i,j) \in \{1, \ldots, m\}$, we see that

$$\binom{m}{2} \pi'_* \Sigma'^2_{m+1} = \sum_{m+1 \in A} -\binom{m+1-|A|}{2} \Delta_A \implies \pi'_* \Sigma'^2_{m+1} = \sum_{m+1 \in A} -\frac{\binom{m+1-|A|}{2}}{\binom{m}{2}} \Delta_A.$$

Now, we see that $\Delta_A \in Pic(\overline{M}_{0,m+1}) \otimes \mathbb{Q}$, where $m+1 \in A$, pulls back to the divisor $\Delta_{A \setminus \{m+1\},1}$, which is the divisor parameterizing reducible curves with one component containing $A \setminus \{m+1\}$ markings and degree 1. Thus,

$$\pi_* c_1^2(\mathcal{Q}) = \sum_{j=1}^{m-2} \left(\frac{2jm - j^2 - j}{m^2 - m} - 1\right) \sum_{|A|=j} \Delta_{A,1} \text{ in } Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),1)) \otimes \mathbb{Q}.$$

Therefore, $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),1)) \otimes \mathbb{Q}$ is generated by the boundary divisor classes, $\pi_* c_1^2(\mathcal{Q})$, and $\pi_* c_2(\mathcal{Q})$, with all relations among the boundary divisors pulled back from $\overline{M}_{0,m+1}$, and the relation

$$\pi_* c_1^2(\mathcal{Q}) = \sum_{j=1}^{m-2} \left(\frac{2jm - j^2 - j}{m^2 - m} - 1\right) \sum_{|A|=j} \Delta_{A,1}.$$

We will see below that this agrees with the expression we find for $\pi_* c_1^2(\mathcal{Q})$ in terms of the boundary divisors in $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),\, d)\,)\otimes\mathbb{Q}$ where $d>1$, $n\geq 2$.

Now we handle the case $d>1$.

If our desired relation has the form

$$\sum_{j=1}^{m}\sum_{k=0}^{[\frac{d}{2}]} c_{j,k}\sum_{|A|=j}\Delta_{A,k} + c_\alpha\pi_* c_1^2(\mathcal{Q}) + c_\beta\pi_* c_2(\mathcal{Q}) = 0$$

in the Picard group, then we see that we have the following relation among the coefficients after intersecting with $\mathsf{A}_{j,e}$

$2c_{j,e} + (j-1)(m-j-1)c_{2,0} + (\,e(m-j-1) + (d-e)(j-1)\,)c_{1,1} +$
$(2de-2e^2 - e(m-j-1) - (d-e)(j-1)\,)c_\alpha +$
$(2de-2e^2 - e(m-j-1) - (d-e)(j-1)\,)c_\beta = 0.$

We see that we have the two relations

- $mc_{1,1} + (2d-m)c_\alpha + (2d-m)c_\beta = 0$
- $mc_{1,1} + (2d-m)c_\alpha + (-m+d+1)c_\beta = 0$

among the coefficients after intersecting the relation with $\mathsf{A}_{1,1}$ and $\mathsf{A}_{1,1}^2$, respectively, which allows us to conclude that $c_\beta = 0 \implies c_{1,1} = (1-\frac{2d}{m})c_\alpha$. Now the argument is identical to the one we gave in the case of $n=1$, and so we see that we have the relation

$$\sum_{j=1}^{m}\sum_{k=0}^{[\frac{d}{2}]}\left(\frac{-jd^2(j-1)}{m(m-1)} - \frac{(m-2j)dk}{m} + k(d-k)\right)\sum_{|A|=j}\Delta_{A,k} = \pi_* c_1^2(\mathcal{Q})$$

in $Pic(\overline{\mathcal{Q}}_{0,m}(Gr(n,n),\, d)\,)\otimes\mathbb{Q}$, where $n\geq 2, d\geq 2$. By rank considerations (Proposition 3.4.1), there are no relations in the rational Picard group other than the one above and those among the boundary divisors, which are pulled back from $\overline{\mathcal{Q}}_{0,m}(G(1,1),\, d)$ under the map $det: \overline{\mathcal{Q}}_{0,m}(Gr(n,n),\, d)\to\overline{\mathcal{Q}}_{0,m}(G(1,1),\, d)$

Notice that this agrees with what we found when $d=1$. Putting together everything we have done above concludes the proof of Proposition 0.4.1 when $m\geq 3$.

4.2 The Picard groups for $m=2$

The goal of this section is to completely describe $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n),\, d)\,)\otimes\mathbb{Q}$, thus proving Lemma 0.4.1, Proposition 0.4.1, and Theorem 0.4.3.

4.2.1 Test curves II

We construct the curves needed in the calculation of the Picard group.

We have the same curves $\mathsf{A}_{1,e}$, $\mathsf{A}^2_{1,e}$ as in the $m \geq 3$ case. We can pushforward the curves $\mathsf{A}_{1,e}$, $\mathsf{A}^2_{1,e}$ under the morphism

$$\overline{\mathcal{Q}}_{0,2}(Gr(r,r),\ d) \to \overline{\mathcal{Q}}_{0,2}(Gr(r,n),\ d)$$

given by

$$[\mathcal{S} \hookrightarrow \mathbb{C}^r \otimes \mathcal{O}] \mapsto [\mathcal{S} \oplus \mathbb{C}^{n-r} \otimes \mathcal{O} \hookrightarrow \mathbb{C}^r \otimes \mathcal{O} \oplus \mathbb{C}^{n-r} \otimes \mathcal{O}]$$

where $n > r$. We call the resulting curves $\mathsf{A}_{1,e}$, $\mathsf{A}^2_{1,e}$.

The curves $\mathsf{B}_{1,e}$

The next collection of curves is defined for $r = n = 1$, but as before we can push them forward. Call these curves $\mathsf{B}_{1,e}$.

- The underlying family of curves is $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ together with the two trivial marked sections s_1, s_2.
- We keep the same $d - e$ trivial sections σ_h as in $\mathsf{A}_{1,e}$.
- We can pick our e smooth irreducible sections δ_ℓ of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ so that they simultaneously vanish along the first marked section s_1 at $([1:0], [1:0])$, but they do not vanish simultaneously or even in pairs along the second marked section s_2.
- We blow up the point on s_1 and the e points $\bigcup_\ell (\delta_\ell \bigcap s_2)$ along s_2.
- We consider the strict transforms of s_1, s_2, δ_ℓ, σ_h : $\overline{s}_1 = Bl^* s_1 - E_1$, $\overline{s}_2 = Bl^* s_2 - \sum_\ell E_{2,\ell}$, $\overline{\delta}_\ell = Bl^* \delta_\ell - E_1 - E_{2,\ell}$, $\overline{\sigma}_h = Bl^* \sigma_h$.
- We have the inclusion of the subsheaf
$$\mathcal{O}(-\sum_\ell \overline{\delta}_\ell - \sum_h \overline{\sigma}_h) \hookrightarrow \mathcal{O}.$$

The curve C

We need a curve which only meets the divisor $\Delta_{1,1}$. The resulting curve C is constructed as follows:

- Start with the Hirzebruch surface $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$.

- It has the two sections s_1, s_2 corresponding to the subbundles $\mathcal{O}$ and $\mathcal{O}(1)$ of $\mathcal{O}(1) \oplus \mathcal{O}$. These sections have numerical classes $\xi + f$, ξ, respectively, by [Har77] *Proposition V. 2.6.*

- Pick d smooth irreducible sections δ_i of $H^0(\mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{O}(1))$ which do not vanish pairwise simultaneously along s_1. By construction they do not vanish along s_2.

- We blow up each of the d points on the first marked section.

- We have the following strict transforms: $\overline{s}_1 = Bl^* s_1 - \sum_{i=1}^{d} E_i$, $\overline{s}_2 = Bl^* s_2$, $\overline{\delta}_i = Bl^* \delta_i - E_i$.

- We have the inclusion of the subsheaf
$$\mathcal{O}(-\sum_{i=1}^{d} \overline{\delta}_i) \hookrightarrow \mathcal{O}.$$

The next collection of curves are for $r = 1, n > r$.

The curve D

We consider the curve D in the moduli space obtained by the following construction:

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

- Pick n smooth irreducible sections $\{\delta_i\}_{i=1}^n$ of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ whose only pairwise common vanishing points are $([1:1], [1:1])$ and $([-1:1], [-1:1])$.

- Use the same two sections at $0, \infty$ from before as the two marked sections s_1, s_2.

- Pick $n(d-1)$ trivial distinct sections $\sigma_{i,h}$ of p_2 which are not s_1, s_2.

- Consider the inclusion
$$\mathcal{O}(-1) \boxtimes \mathcal{O}(-1) \otimes \bigotimes_{h=1}^{d-1} p_1^* \mathcal{O}(-1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$
given by the sections δ_i, $\sigma_{i,h}$ above.

The curve F

We construct the curve F as follows:

- Start with the vector bundle $\mathcal{O}(v) \oplus \mathcal{O}(w) \to \mathbb{P}^1$, where $v \neq w$, $v, w > 0$.
- Consider the projective bundle $\rho : \mathbb{P}(\mathcal{O}(v) \oplus \mathcal{O}(w)) \to \mathbb{P}^1$ with two sections s_1, s_2 given by the subbundles $\mathcal{O}(w)$, $\mathcal{O}(v)$ of $\mathcal{O}(v) \oplus \mathcal{O}(w)$. These sections have numerical classes $\xi + vf$, $\xi + wf$, respectively, by [Har77] *Proposition V. 2.6.*
- Fix $h > 0$ such that $-vd + h$, $-wd + h > 0$, and $H^0(\mathcal{O}_{\mathbb{P}}(d) \otimes \rho^*\mathcal{O}(h)) >> 0$. We pick sections $\{x_i\}_{i=1}^n$, $x_i \in H^0(\mathcal{O}_{\mathbb{P}}(d) \otimes \rho^*\mathcal{O}(h))$ such that the induced rational map to $\mathbb{P}^{n-1}$ has no basepoints along the marked sections: first pick a section x_1; this meets the two marked sections in $-wd + h$ and $-vd + h$ points (counted with multiplicity), respectively; it is an open condition that the second section does not vanish at these points; now the rest of the sections can be chosen arbitrarily.

This gives a family of stable quotients to $\mathbb{P}^{n-1}$ over $\mathbb{P}^1$ with 2 marked sections.

The next two curves are for the $d = 1$ case.

The curves $\mathsf{G}_1, \mathsf{G}_2$

The first curve G_1 is obtained as follows:

- Consider the Hirzebruch surface $\rho : \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^1$ with two sections s_1, s_2 given by the subbundles $\mathcal{O}$, $\mathcal{O}(1)$ of $\mathcal{O}(1) \oplus \mathcal{O}$. These sections have numerical classes $\xi + f$, ξ, respectively, by [Har77] *Proposition V. 2.6.*
- Pick n smooth irreducible sections $\{\delta_i\}_{i=1}^n$ of $\mathcal{O}_{\mathbb{P}}(1) \otimes \rho^*\mathcal{O}(1)$ which do not vanish pairwise simultaneously along the first marked section (they do not intersect the second marked section).
- Consider the inclusion of sheaves $\mathcal{O}_{\mathbb{P}}(-1) \otimes \rho^*\mathcal{O}(-1) \to \mathbb{C}^n \otimes \mathcal{O}$ given by $\bigoplus_{i=1}^{n} \delta_i$.

G_2 is defined by reversing the roles of 1 and 2 in G_1.

The curve H

The next curve is for $d = 1$, $n - r \geq 1$, $r \geq 2$.

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the two trivial sections s_1, s_2 at 0 and ∞.
- Pick a trivial section σ of p_2 which is distinct from the two trivial sections above.
- Pick two disjoint sections of $p_2^* \mathcal{O}(1)$; call them f_1 and f_2.
- Consider the map of sheaves

  $$\mathbb{C}^{r-2} \otimes \mathcal{O} \oplus (p_1^* \mathcal{O}(-1) \oplus p_2^* \mathcal{O}(-1)) \hookrightarrow \mathbb{C}^{r-2} \otimes \mathcal{O} \oplus \left(\mathcal{O} \oplus \mathbb{C}^2 \otimes \mathcal{O}\right) \oplus \mathbb{C}^{n-r-1} \otimes \mathcal{O}$$

  where the map $p_1^*(\mathcal{O}(-1)) \subset \mathcal{O}$ is given by σ, and the map $p_2^* \mathcal{O}(-1) \subset \mathbb{C}^2 \otimes \mathcal{O}$ is given by $\bigoplus_{j=1}^{2} f_j$.

4.2.2 Independence of the boundary divisors

We begin with a lemma which holds in all cases.

Lemma 4.2.1. *The collection of boundary divisors $\Delta_{a,b}$ for $a = 1, 2$, $1 \leq b \leq d - 1$ in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r, n), d)) \otimes \mathbb{Q}$ are linearly independent, $\forall r, n$.*

Proof. First, we shall prove the result in the case that $r = n = 1$. In the case that $d = 1$, there are no boundary divisors, so the result is trivially true. In the case that $d = 2$, there is a single boundary divisor, so again the result is trivially true since the family $\mathsf{A}_{1,1}$ has nontrivial intersection with the single boundary divisor (when $d = 2$).

Suppose $d \geq 3$. We know that if there is a relation among the boundary divisors, then it must take the form $\sum_{k=1}^{d} c_{1,k} \Delta_{1,k} = 0$, where $\Delta_{a,k}$ parameterizes the locus of reducible curves with one component containing the marking p_a and whose quotient has degree k when restricted to this component.

It is clear that $\sum_{k=1}^{d} c_{1,k} \Delta_{1,k} \cdot \mathsf{A}_{1,e} = c_{1,e} + c_{1,d-e}$, so we see that $c_{1,e} = -c_{1,d-e}$.

Notice that $\sum_{k=1}^{d} c_{1,k} \Delta_{1,k} \cdot \mathsf{B}_{1,e} = c_{1,e} + e\, c_{1,d-1} = 0$. Using these relations, we just need to show that $c_{1,d-1} = 0$ (or $c_{1,1} = 0$) in order to show that all of the coefficients are zero.

It is clear that $\sum_{i=1}^{d-1} c_{1,i}\Delta_{1,i} \cdot \mathsf{C} = d\, c_{1,1} = 0$.

This shows that these divisors are linearly independent for $r = n = 1$.

Now, having established the result for $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\, d)) \otimes \mathbb{Q}$, notice that there are morphisms

$$\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\, d) \to \overline{\mathcal{Q}}_{0,2}(Gr(r,n),\, d)$$

given by

$$[\mathcal{S} \hookrightarrow \mathcal{O}] \mapsto [\mathcal{S} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \oplus \mathbb{C}^{n-r} \otimes \mathcal{O}].$$

Notice that the boundary divisors in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n),\, d)) \otimes \mathbb{Q}$ pull back to the same boundary divisors in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\, d)) \otimes \mathbb{Q}$ under this morphism. If we have a relation $\sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k} = 0$ in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n),\, d)) \otimes \mathbb{Q}$, we can pull it back to $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\, d)) \otimes \mathbb{Q}$, where we see that all $c_{1,k} = 0$ from our argument above. Thus, the boundary divisors are linearly independent $\forall r, n$ when $m = 2$. □

To determine the Picard group completely, we handle the individual cases $(r = n = 1)$; $(r = n \geq 2)$; $(r = 1,\, n \geq 2)$; $(r \geq 2,\, n - r = 1)$; $(r \geq 2,\, n - r \geq 2)$ separately.

4.2.3 $r = n = 1$

In the case that $r = n = 1$, we have already calculated that the rank of the rational Picard group is $d-1$, and there are $d-1$ such linearly independent divisors we found above. These divisors form a basis.

We also have the class $\pi_* c_1^2(\mathcal{Q})$ which can be expressed in terms of the boundary divisors

$$\sum_{e=1}^{d} c_{1,e}\Delta_{1,e} = \pi_* c_1^2(\mathcal{Q}).$$

Intersecting with $\mathsf{A}_{1,e}$ gives the relation on coefficients

$$c_{1,e} + c_{1,d-e} = 2e(d-e).$$

Intersecting with $\mathsf{B}_{1,e}$ gives the relation on coefficients

$$c_{1,e} + e\, c_{1,d-1} = e(d-e) + e(d-1).$$

Finally, intersecting with C gives the relation

$$d\, c_{1,1} = d^2 - d,$$

which implies $c_{1,1} = d-1$. We see that $c_{1,d-1} = d-1$ as well. Plugging this back in to the second relations we found, we see that $c_{1,e} = e(d-e)$ for all $1 \leq e \leq d$.

Thus, $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\, d)\,) \otimes \mathbb{Q}$ is generated by $\Delta_{1,k}$, $k = 1, \ldots, d-1$ with the relation

$$\sum_{k=1}^{d-1} k(d-k)\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q}).$$

This proves Lemma 0.4.1 when $m = 2$.

4.2.4 $r = n \geq 2$

In the case that $r = n \geq 2$, we have already seen that the rank of the rational Picard group is d.

First we consider the case $d > 1$.

We have the $d-1$ linearly independent boundary divisors $\Delta_{1,k}$ as well as the two classes $\pi_* c_1^2(\mathcal{Q})$ and $\pi_* c_2(\mathcal{Q})$.

We claim that we have the same relation as in the $r = n = 1$ case,

$$\sum_{k=1}^{d-1} k(d-k)\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q}),$$

and $\pi_* c_2(\mathcal{Q})$ is not in the span of the boundary divisors.

Suppose we had a relation

$$\sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k} + c_\alpha \pi_* c_2(\mathcal{Q}) = \pi_* c_1^2(\mathcal{Q}).$$

Notice that $\mathsf{A}_{1,e}$ and $\mathsf{A}^2_{1,e}$ intersect the boundary and $\pi_* c_1^2(\mathcal{Q})$ the same, but they intersect $\pi_* c_2(\mathcal{Q})$ differently since the subsheaf in the quotient sequence has a nonzero second Chern class in the family of curves $\mathsf{A}^2_{1,e}$. Thus, we see that $\pi_* c_2(\mathcal{Q})$ does not appear in the relation. The rest of the calculation is identical to the $r = n = 1$ case.

For $d > 1$, $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(n,n),\, d)\,) \otimes \mathbb{Q}$ $(n \geq 2)$ is generated by $\Delta_{1,k}$, for $1 \leq k \leq d-1$, as well as the classes $\pi_* c_1^2(\mathcal{Q})$ and $\pi_* c_2(\mathcal{Q})$, with the only relation being

$$\sum_{k=1}^{d-1} k(d-k)\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q}).$$

When $d = 1$, we invoke the isomorphism of the previous section

$$\overline{\mathcal{Q}}_{0,2}(Gr(n,n),\, 1) \cong \overline{M}_{0,3} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}.$$

Under this isomorphism, $\pi_* c_1^2(\mathcal{Q}) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Thus,

$$Pic(\overline{\mathcal{Q}}_{0,2}(Gr(n,n),\, 1)) \otimes \mathbb{Q} \cong \mathbb{Q} \cdot \pi_* c_1^2(\mathcal{Q}), \text{ for } n \geq 2.$$

This proves Proposition 0.4.1 when $m = 2$.

4.2.5 $r = 1, n \geq 2$

First we consider the case $d > 1$. When $r = 1$, $n \geq 2$, we know that $\pi_* c_1^2(\mathcal{Q}) = \pi_* c_2(\mathcal{Q})$. The following two lemmas are identical to *Lemma 1.2.1 i), iii)* in [Pan99] from the stable maps case. We claim that

Lemma 4.2.0.1. *$\pi_* c_1^2 \notin span(\Delta)$.*

Proof. Suppose we had a relation

$$\sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k} = \pi_* c_1^2(\mathcal{Q})$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$.

It is clear that $\Delta_{1,k} \cdot \mathsf{D} = 0$, $\forall k = 1, \ldots, d-1$. However,

$$\pi_* c_1^2(\mathcal{Q}) \cdot \mathsf{D} = (-d\xi - f)^2 = 2d \neq 0,$$

so we have reached a contradiction. $\square$

We claim

Lemma 4.2.0.2. *The classes $ev_1^* c_1(\mathcal{O}(1))$, $ev_2^* c_1(\mathcal{O}(1))$ are linearly independent modulo $span(\Delta)$.*

Proof. Notice that $\Delta_{a,b} \cdot \mathsf{F} = 0$ for any $a = 1, 2$, $1 \leq b \leq d-1$. However,

$$ev_1^* c_1(\mathcal{O}(1)) \cdot \mathsf{F} = -wd + h, \text{ and } ev_2^* c_1(\mathcal{O}(1)) \cdot \mathsf{F} = -vd + h.$$

If there was a relation

$$c_1 ev_1^* c_1(\mathcal{O}(1)) + c_2 ev_2^* c_1(\mathcal{O}(1)) = \sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k}$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$, then I claim that the $\mathbb{Q}$ linear span of the relation must be $\mathcal{S}_2$ invariant.

Given the claim, the coefficients c_1 and c_2 would have to be equal in absolute value (assuming the coefficients c_1, c_2 are nonzero, there exists $q \in \mathbb{Q}$ such that $q \cdot c_1 = c_2$ and $q \cdot c_2 = c_1$, forcing $c_1^2 = c_2^2$). Since $-wd - vd + 2h \neq 0$ and $(-w+v)d \neq 0$, we see that the coefficients $c_1 = c_2$ are zero.

If the $\mathbb{Q}$ linear span is not $\mathcal{S}_2$ invariant, then there is another independent relation with c_1, c_2 interchanged and $c_{1,e}$, $c_{1,d-e}$ interchanged. Using this, we can solve for c_1 or c_2 in terms of the coefficients on the boundary divisors. The intersection with F above shows that either c_1 or c_2 is zero, from which it follows that the other is also zero. $\square$

By rank considerations, $\{ev_1^* c_1(\mathcal{O}(1)), ev_2^* c_1(\mathcal{O}(1)), \Delta_{1,1}, \ldots, \Delta_{1,d-1}\}$ form a basis for $Pic(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$. However, we have the additional class $\pi_* c_1^2(\mathcal{Q})$. We will write this class in terms of the basis. Therefore we expect a relation

$$\pi_* c_1^2(\mathcal{Q}) = c_1 ev_1^* c_1(\mathcal{O}(1)) + c_2 ev_2^* c_1(\mathcal{O}(1)) + \sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k}$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$.

Intersecting with $\mathsf{A}_{1,e}$ yields the relation on coefficients

$$2e(d-e) = c_{1,e} + c_{1,d-e}.$$

The relation above must be $\mathcal{S}_2$ invariant since the space of relations has rank 1. Hence $c_{1,e} = c_{1,d-e} = e(d-e)$.

After intersecting with F, we see that

$$(-v-w)d^2 + 2hd = c_1(-vd - wd + 2h).$$

Therefore, $c_1 = c_2 = d$.

This yields the relation

$$\pi_* c_1^2(\mathcal{Q}) = d(ev_1^* c_1(\mathcal{O}(1)) + ev_2^* c_1(\mathcal{O}(1))) + \sum_{k=1}^{d-1} k(d-k)\Delta_{1,k}$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q}$.

We consider the case $d = 1$ separately. This can be done by intersecting with G_1 and G_2.

This proves the first part of Theorem 0.4.3.

4.2.6 $r \geq 2$, $n - r \geq 1$

In this case, we do not have $\pi_* c_1^2(\mathcal{Q}) = \pi_* c_2(\mathcal{Q})$. As before, first we consider the case $d > 1$.

We will prove the following lemma

Lemma 4.2.0.3. *The collection*

$$\{\pi_* c_2(\mathcal{Q}), ev_1^* c_1(\mathcal{O}_G(1)), ev_2^* c_1(\mathcal{O}_G(1)), \Delta_{1,1}, \ldots, \Delta_{1,d-1}\}$$

is linearly independent in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n), d)) \otimes \mathbb{Q}$, for $r \geq 2$, $n - r \geq 1$..

Proof. Suppose there was relation :

$$c_\alpha \pi_* c_2(\mathcal{Q}) + c_{\beta,1} ev_1^* c_1(\mathcal{O}_G(1)) + c_{\beta,2} ev_2^* c_1(\mathcal{O}_G(1)) + \sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k} = 0$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n), d)) \otimes \mathbb{Q}$.

There is a morphism $\overline{\mathcal{Q}}_{0,2}(Gr(1,1),\ d) \to \overline{\mathcal{Q}}_{0,2}(Gr(r,n),\ d)$ given by

$$[\mathcal{S} \hookrightarrow \mathcal{O}] \mapsto [\mathcal{S} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \oplus \mathbb{C}^{n-r} \otimes \mathcal{O}].$$

We can push forward all of our curves along this morphism.

Intersecting with $\mathsf{A}_{1,e}$ yields the relation among the coefficients

$$2e(d-e)c_\alpha + c_{1,e} + c_{1,d-e} = 0.$$

Notice that there is a morphism $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1,\ d) \to \overline{\mathcal{Q}}_{0,2}(Gr(n-1,n),\ d)$ given by

$$[\mathcal{S} \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O}] \mapsto [\mathcal{S} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O} \oplus \mathbb{C}^{r-1} \otimes \mathcal{O} \oplus \mathbb{C}^{n-r-1} \otimes \mathcal{O}].$$

Using this, we can push forward the curve F from the $\mathbb{P}^1$ case. All the intersection numbers can be computed on $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1,\ d)$, therefore we have

- $ev_1^* c_1(\mathcal{O}_G(1)) \cdot \mathsf{F} = -wd + h$
- $ev_2^* c_1(\mathcal{O}_G(1)) \cdot \mathsf{F} = -vd + h$
- $\pi_* c_2(\mathcal{Q}) \cdot \mathsf{F} = (d\xi + hf)^2 = (-v-w)d^2 + 2hd$
- $\Delta_{1,k} \cdot \mathsf{F} = 0$ for all $1 \le k \le d-1$.

The same argument as in the $\mathbb{P}^1$ case shows that we have $c_{\beta,1} = c_{\beta,2}$, and we have the relation on the coefficients

$$c_\alpha + dc_\beta = 0.$$

Intersecting with $\mathsf{A}^2_{1,e}$ yields the relation on the coefficients:

$$e(d-e)c_\alpha + c_{1,e} + c_{1,d-e} = 0.$$

Combining this relation with the first collection of relations we found, we see that $c_\alpha = 0$, but then this forces $c_\beta = 0$. Since the boundary divisors were already shown to be linearly independent, the result follows. □

By rank considerations,

$$\{\pi_* c_2(\mathcal{Q}),\ ev_1^* c_1(\mathcal{O}_G(1)),\ ev_2^* c_1(\mathcal{O}_G(1)),\ \Delta_{1,1}, \ldots, \Delta_{1,d-1}\}$$

form a basis for $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n),\ d)) \otimes \mathbb{Q}$ for $r \ge 2$, $n - r \ge 1$.

However, we still have the class $\pi_* c_1^2(\mathcal{Q})$. We will write this class in terms of the basis above by intersecting with curves. We expect a relation

$$\pi_* c_1^2(\mathcal{Q}) = c_\alpha \pi_* c_2(\mathcal{Q}) + c_{\beta,1} ev_1^* c_1(\mathcal{O}_G(1)) + c_{\beta,2} ev_2^* c_1(\mathcal{O}_G(1)) + \sum_{k=1}^{d-1} c_{1,k}\Delta_{1,k}.$$

Intersecting with $\mathsf{A}_{1,e}$, we find the relation on the coefficients

$$2e(d-e) = 2e(d-e)c_\alpha + c_{1,e} + c_{1,d-e}.$$

Intersecting with F, we find the relation on the coefficients

$$(-v-w)d^2 + 2hd = ((-v-w)d^2 + 2hd)c_\alpha + (-vd + h)c_{\beta,1} + (-wd + h)c_{\beta,2}.$$

Intersecting with $\mathsf{A}^2_{1,e}$, we find the relation on coefficients

$$2e(d-e) = e(d-e)c_\alpha + c_{1,e} + c_{1,d-e}.$$

Combining the first and last relations on the coefficients, we find that $c_\alpha = 0$. Plugging this into the second relation on the coefficients, combined with the same argument from the $\mathbb{P}^{n-1}$ case, shows that $c_{\beta,1} = c_{\beta,2} = d$.

The rest of the proof of the relation is identical to the $r = n = 1$ case (intersect with the curves $\mathsf{B}_{1,e}$, C). Putting this all together, we see that we have the relation

$$\pi_* c_1^2(\mathcal{Q}) = d(ev_1^* c_1(\mathcal{O}_G(1)) + ev_2^* c_1(\mathcal{O}_G(1))) + \sum_{k=1}^{d-1} k(d-k)\Delta_{1,k}$$

in $Pic(\overline{\mathcal{Q}}_{0,2}(Gr(r,n), d)) \otimes \mathbb{Q}$ for $r \geq 2$, $n - r \geq 1$.

We separately handle the case $d = 1$.

In this case, the rank of the Picard group is 3. We will see that it is generated by the classes $\pi_* c_1^2(\mathcal{Q})$, $\pi_* c_2(\mathcal{Q})$, $ev_1^* c_1(\mathcal{O}_G(1))$, and $ev_2^* c_1(\mathcal{O}_G(1))$, with a single relation.

We show that the divisors $\pi_* c_2(\mathcal{Q})$, $ev_1^* c_1(\mathcal{O}_G(1))$, and $ev_2^* c_1(\mathcal{O}_G(1))$ are linearly independent.

Suppose we had a relation

$$c_\alpha \pi_* c_2(\mathcal{Q}) + c_{\beta,1} ev_1^* c_1(\mathcal{O}_G(1)) + c_{\beta,2} ev_2^* c_1(\mathcal{O}_G(1)) = 0.$$

We can pushforward the curves G_1 and G_2 from the $\mathbb{P}^1$ case and consider the intersection of our relation with the resulting curves. Intersecting with $\mathsf{G}_1, \mathsf{G}_2, \mathsf{H}$ yields: $c_\alpha + c_{\beta,1} = 0$, $c_\alpha + c_{\beta,2} = 0$, and $c_\alpha + c_{\beta,1} + c_{\beta,2} = 0$. Thus all coefficients are zero.

We would like to write $\pi_* c_1^2(\mathcal{Q})$ in terms of this basis, so we expect a relation

$$\pi_* c_1^2(\mathcal{Q}) = c_\alpha \pi_* c_2(\mathcal{Q}) + c_{\beta,1} ev_1^* c_1(\mathcal{O}_G(1)) + c_{\beta,2} ev_2^* c_1(\mathcal{O}_G(1)).$$

Intersecting our relation with the curves G_1, G_2, and H we find the relations on the coefficients: $1 = c_\alpha + c_{\beta,1}$, $1 = c_\alpha + c_{\beta,2}$ and $2 = c_\alpha + c_{\beta,1} + c_{\beta,2}$. We see that $c_{\beta,1} = c_{\beta,2} = 1$ and $c_\alpha = 0$.

This concludes the proof of Theorem 0.4.3.

5 Picard Rank Calculations II

As in *Chapter 3*, here we compute the rank of the rational Picard group of $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ using the torus action on the moduli stack and Lemma 3.1.1.

5.1 Tangent space calculations II

In this section we will count the fixed loci in $\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ with either one or zero negative weights on their tangent bundles. As before, we must determine the weights on the tangent space to a $\mathbb{C}^*$ fixed stable quotient.

5.1.1 The weights on the tangent space II

First, we describe the fixed loci in the flag variety.

- The fixed points of the $\mathbb{C}^*$ action on the (partial) flag variety are the flags whose steps are the subspaces that are spanned by the coordinate vectors. More precisely, to each fixed point we can associate a flag of subsets

  $$I^\bullet := (I_1 \subset \ldots \subset I_\ell \subset [n])$$

  so that the fixed point is

  $$e_{I^\bullet} := (\langle e_i \rangle_{i \in I_1} \subset \ldots \subset \langle e_i \rangle_{i \in I_\ell} \subset \mathbb{C}^n)\,.$$

- Two fixed points corresponding to flags of indices $I^\bullet$, $J^\bullet$ can be joined by a rational curve if there exist indices $1 \leq p < q \leq \ell$ and $1 \leq \alpha < \beta \leq n$ such that

  $$I_1 = J_1 := K_1, \ldots, I_p = J_p := K_p,$$

  $$I_{p+1} \backslash \{\alpha\} = J_{p+1} \backslash \{\beta\} := K_{p+1}, \ldots I_q \backslash \{\alpha\} = J_q \backslash \{\beta\} := K_q,$$

$$I_{q+1} = J_{q+1} := K_{q+1}, \dots, I_\ell = J_\ell := K_\ell$$

where K_h denotes the h^{th} mutual indices. We write the rational curve as

$$\langle e_k \rangle_{k \in K_1} \subset \dots \subset \langle e_k \rangle_{k \in K_p} \subset \langle e_k \rangle_{k \in K_{p+1}} \oplus (se_\alpha + te_\beta) \subset \dots$$
$$\subset \langle e_k \rangle_{k \in K_q} \oplus (se_\alpha + te_\beta) \subset \langle e_k \rangle_{k \in K_{q+1}} \subset \dots \subset \langle e_k \rangle_{k \in K_\ell} \subset \mathbb{C}^n.$$

Given this data, we can describe the fixed loci in the generalized stable quotients moduli space. We do this by looking at the irreducible components of the curve corresponding to the fixed stable quotient.

- If the component has ≥ 3 markings or nodes on it, then the component must be contracted, and the flag of subsheaves takes the form
$$\bigoplus_{i \in I_1} \mathcal{O}(-d_{1,i}) \hookrightarrow \dots \hookrightarrow \bigoplus_{i \in I_\ell} \mathcal{O}(-d_{\ell,i}) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$
where each line bundle in each direct sum injects into a copy of $\mathcal{O}$ and $\forall\, 1 \leq k \leq \ell$, $\sum_{i \in I_k} d_{k,i} = d_k$, where $|I_k| = r_k$.

- If the subsheaf has exactly two markings or nodes on it, then the flag of subsheaves can either take the form above, where at least one $d_{k,i} \neq 0$, for some k and some i, or it can come from a map to the flag variety whose image is a $\mathbb{C}^*$ fixed rational curve.

- Notice that this map must be a genuine morphism to the flag variety; it cannot be a morphism only after projecting to a factor of one of the Grassmannians. To see this, it suffices to consider several cases.

- The first case is that of a line bundle sitting in a copy of $\mathcal{O}$, $\mathcal{O}(-d_1) \hookrightarrow \mathcal{O}$.
  - I claim that this cannot factor through $f^*\mathcal{O}(-1) \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O}$, where this second inclusion is given by pulling back the $\mathcal{O}(-1)$ on a rational curve in one of the Grassmannians, and one of the copies of $\mathcal{O}$ is the same as the above copy of $\mathcal{O}$.
  - The second inclusion is given by the matrix
$$\begin{pmatrix} x_0^{d_2} & 0 \\ 0 & x_1^{d_2} \end{pmatrix}$$
whereas the first is given by inclusion into the first copy of $\mathcal{O}$, and the claim follows.

- Thus, the only way the first sheaf can be in a flag with the second is if the flag has the form
$$\mathcal{O}(-d_1) \hookrightarrow \mathcal{O} \oplus f^*\mathcal{O}(-1) \hookrightarrow \mathcal{O} \oplus \mathbb{C}^2 \otimes \mathcal{O}$$
with $f^*\mathcal{O}(-1) \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O}$.
- But in order for this to be fixed, we use the fact that the curve has a $\mathbb{C}^*$ worth of automorphisms, which we can use to absorb the torus action on the second subsheaf.
- However, the first subsheaf will not be fixed under this action unless the support of the quotient of $\mathcal{O}(-d_1) \hookrightarrow \mathcal{O}$ is one of the torus fixed points, which are the nodes or markings, and this is not allowed.

- The second case is $f^*\mathcal{O}(-1) \hookrightarrow \mathbb{C}^2 \otimes \mathcal{O}$ factoring through the direct sum of two line bundles each sitting in their own copy of $\mathcal{O}$. The same reason as above shows that this cannot happen: in order for the first inclusion to be invariant, we use the $\mathbb{C}^*$ automorphisms of the marked curve, but then this changes the support of the quotients of $\mathbb{C}^2 \otimes \mathcal{O}$ by the two line bundles necessarily.

- Notice that the morphism from the component to the rational curve must have degree $(0, \dots, 0, d_i, \dots, d_i, 0, \dots, 0)$ if it is to be fixed by the $\mathbb{C}^*$ action.

By [Lau88], [CF99], *Proposition 2.5*, *Theorem 1.2* the tangent space to the HyperQuot scheme $HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ at a point
$$0 \hookrightarrow \mathcal{S}_1 \hookrightarrow \dots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to \mathcal{Q}_1 \to \dots \to \mathcal{Q}_\ell \to 0$$
has the following expression
$$[\bigoplus_{i=1}^{\ell} H^0(\mathcal{S}_i^* \otimes \mathcal{Q}_i)] - [\bigoplus_{j=1}^{\ell-1} H^0(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})]$$
in K-theory. Since the flag variety can be viewed as a particular case of the HyperQuot scheme, the same result holds for the tangent space to the flag variety.

Using this, we can determine the weights on the tangent space to a $\mathbb{C}^*$-fixed generalized stable quotient. The calculations are simillar to the Grassmannian case.

First, we shall determine the weights on the tangent space to the HyperQuot scheme $HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ at a $\mathbb{C}^*$ fixed point q. Let
$$0 \hookrightarrow \bigoplus_{i_1 \in I_1} \mathcal{O}(-d_{1,i_1}) \hookrightarrow \dots \hookrightarrow \bigoplus_{i_\ell \in I_\ell} \mathcal{O}(-d_{\ell,i_\ell}) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$
be a fixed point of the $\mathbb{C}^*$ action on the HyperQuot scheme. We have the following

weights in

$$[THQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})_q]$$

- $(\dagger)_1$ $\lambda_h - \lambda_\gamma$ appears $d_{j,h} + 1$ times, where j is the largest index such that $h \in I_j$ but $\gamma \notin I_j$.
- $(\dagger)_2$ $\lambda_\mu - \lambda_\tau$ appears
  - $d_{k,\tau}$ times if $\mu \notin I_{k-1}$
  - and 0 times if $\mu \in I_{k-1}$, where k is the smallest index such that both $\mu, \tau \in I_k$.

Next, we determine the weights on $H^0(f^* TFl(\overline{r}, \mathbb{C}^n))$ given a map of degree $(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0)$ from an irreducible component to a $\mathbb{C}^*$-fixed rational curve in the flag variety.

In this case, the flag of subsheaves has the form

$$V_{I_1} \otimes \mathcal{O} \hookrightarrow \ldots \hookrightarrow V_{I_q} \otimes \mathcal{O} \hookrightarrow V_{I_{q+1}} \otimes \mathcal{O} \oplus f^* \mathcal{O}_{\mathbb{P}^1_{\alpha,\beta}}(-1) \hookrightarrow \ldots \hookrightarrow V_{I_u} \otimes \mathcal{O} \oplus f^* \mathcal{O}_{\mathbb{P}^1_{\alpha,\beta}}(-1)$$
$$\hookrightarrow V_{I_{u+1}} \otimes \mathcal{O} \hookrightarrow \ldots \hookrightarrow V_{e_\ell} \otimes \mathcal{O} \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}.$$

Using the description of the tangent space to the flag variety above, together with what we have already found in the Grassmannian case (by considering each of the ℓ short exact sequences associated to the flag sequence), we determine that the weights on $H^0(f^* TFl(\overline{r}, \mathbb{C}^n))$ are the following:

- $(\dagger\dagger)_1$ $\lambda_\delta - \lambda_\gamma$ appears as a weight once if $\exists k$ such that $\delta \in I_k$ but $\gamma \notin I_k$.
- $(\dagger\dagger)_2$ For $0 \le j \le d$, $\frac{(d-j)\lambda_\alpha + j\lambda_\beta}{d} - \lambda_\theta$ each appear once if $\exists k$ such that $q+1 \le k \le u$ where $\theta \notin I_k$ but $\theta \in I_{u+1}$.
- $(\dagger\dagger)_3$ For $1 \le g \le d-1$, assuming $d > 1$, $\lambda_\epsilon - \frac{(d-g)\lambda_\alpha + g\lambda_\beta}{d}$ each appear once if $\exists k$ such that $q+1 \le k \le u$ where $\epsilon \in I_k$ but $\epsilon \notin I_q$.

We want to calculate the weights on $[\bigoplus_{i=1}^{\ell} (\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1} (\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n]$, where n is a node of the domain curve mapping to $V_{I_1} \subset \ldots \subset V_{I_\ell}$. Using the same methods as our calculations above, we see that the weights are:

$\lambda_\delta - \lambda_\gamma$ appears as a weight once if $\exists k$ such that $\delta \in I_k$ but $\gamma \notin I_k$.

The last weights we want to calculate are the weights on the tangent space to the domain curve at the preimages of the fixed points when we have a map from a component of the domain curve to a $\mathbb{C}^*$-fixed curve in the flag variety.

These are the same as in the Grassmannian case: if the component C maps $d:1$ to the rational curve joining two flags whose flags of indices differ from each other by α, β, respectively, with preimages $[1:0]$, $[0:1]$, then the weights are:

- $T_{[1:0]}C$ has weight $\frac{\lambda_\alpha - \lambda_\beta}{d}$.
- $T_{[0:1]}C$ has weight $\frac{\lambda_\beta - \lambda_\alpha}{d}$.

The tangent space to the moduli stack sits in analogous exact sequence to the one from the Grassmannian case, the only difference is that we replace deformations of the quotient with the global sections of the kernel of

$$\bigoplus_{i=1}^{\ell} \mathcal{H}om(\mathcal{S}_i,\, \mathcal{Q}_i) \to \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\mathcal{S}_j,\, \mathcal{Q}_{j+1}) \to 0$$

(the tangent space to the HyperQuot scheme from the first chapter).

5.1.2 The fixed loci with < 2 negative weights

We first prove a lemma which holds independent of the number of markings, the rank type of the flag variety, and the degree type.

By the standard flag, we mean the flag whose i^{th} subspace is the subspace spanned by the first r_i standard basis vectors.

As before, since we are looking at the generic point of the fixed locus, we can assume that there are no clusters of contracted components.

Lemma 5.1.1. *In order for $n^- < 2$, the fixed locus cannot correspond to a generalized stable quotient with a node that gets mapped to any flag other than the standard flag.*

Proof. Suppose we have a curve with a node that is mapped to a flag $V_{I_1} \subset \ldots \subset V_{I_\ell}$ which is not the standard flag.

Then $\exists h, k$ such that $h \in I_k$ and $h > r_k$. This implies that $\exists \gamma \leq r_k$ such that $\gamma \notin I_k$. The weight $\lambda_h - \lambda_\gamma < 0$ appears in $[\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n]$.

The key observation is the same observation we made in the Grassmannian case: if there are N consecutive nodes with the property that $\lambda_h - \lambda_\gamma < 0$ appears in $[\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n]$ for each of the N nodes, then there are $N+1$ irreducible components incident to these nodes.

If $\lambda_h - \lambda_\gamma$ appears in $[\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n]$, then this means that the node maps to a flag with the property

$$(***)\ \exists \text{ an index } \mu \text{ with } h \in I_\mu \text{ and } \gamma \notin I_\mu.$$

The claim is that for each such component C_a incident to a node mapping to a flag with the property $(***)$, the weight $\lambda_h - \lambda_\gamma$ appears at least once in

$$[\bigoplus_{i=1}^{\ell} H^0((\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_{C_a})] - [\bigoplus_{j=1}^{\ell-1} H^0((\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_{C_a})].$$

We will consider the different cases.

In the first case, where C_a is contracted, $\lambda_h - \lambda_\gamma$ appears at least once on $[\,THQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})_q]$, as in $(\dagger)_1$.

The next cases are when the component is not contracted. We handle them individually.

- First, suppose the component is mapped to the rational curve joining two $\mathbb{C}^*$-fixed flags whose flags of indices differ from each other by indices other than h, γ. Then, $\lambda_h - \lambda_\gamma < 0$ appears in $H^0(f^*\,TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_1$.

- Next, suppose the component is mapped to the rational curve joining two $\mathbb{C}^*$-fixed flags whose flags of indices differ from each other by h, γ, respectively. Then $\lambda_h - \lambda_\gamma < 0$ appears in $H^0(f^*\,TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$.

- Suppose the component is mapped to the rational curve joining two $\mathbb{C}^*$-fixed flags whose flags of indices differ from each other by h, ψ, respectively, where $\psi \neq \gamma$. The map has degree type $(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0)$, where there are $q \geq 0$ zeroes initially, $u - q > 0$ d's, followed by $\ell - u \geq 0$ zeroes at the end in the degree type. Notice that e_h cannot be in one of the first q mutual subspaces. Let k be the largest index $\geq q+1$ such that $\gamma \notin I_k$.
  - If $k \leq u$, then $\lambda_h - \lambda_\gamma$ appears in $H^0(f^*\,TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$.
  - If $k \geq u+1$, then $\lambda_h - \lambda_\gamma$ appears in $H^0(f^*\,TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_1$.

- Finally, suppose the component is mapped to the rational curve joining two torus fixed flags whose flags of indices differ from each other by γ, ϕ, respectively, where $\phi \neq h$. The map has degree type $(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0)$, where there are $q \geq 0$ zeroes initially, $u - q > 0$ d's, followed by $\ell - u \geq 0$ zeroes at the end in the degree type.

- If e_h appears in any of the first q mutual subspaces, then $\lambda_h - \lambda_\gamma$ appears in $H^0(f^* TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_1$.
- If e_h does not appear in the first q mutual subspaces, then the smallest index b such that $h \in I_b$ must be $\leq u$ (by assumption there exists an index c such that $h \in I_c$ but $\gamma \notin I_c$, and for $\nu \geq u+1$, $\gamma \in I_\nu$), $\lambda_h - \lambda_\gamma$ appears in $H^0(Fl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_1$.

Therefore, there is at least one negative weight on the tangent space to the fixed locus if there is a node mapping to any flag other than the standard flag.

The argument above shows that if there is more than one cluster of nodes which map to fixed flags of the form $(***)$, then there is more than one negative weight on the tangent space.

Also, if a cluster of nodes map to a collection of $\mathbb{C}^*$-fixed flags such that there are at least two indices γ_1, γ_2, both $< h$, such that for each flag, there exists an index k with $\lambda_h \in I_k$ but $\gamma_1, \gamma_2 \notin I_k$, then the above argument repeated for each index shows that there is more than one negative weight on the tangent space.

Therefore, if a node maps to a flag, based on what we proved above, it can only differ from the standard flag as follows:

- V_{I_k} is spanned by the first $r_k - 1$ basis vectors and e_{r_k+1}
- V_{I_ν} is spanned by the first r_ν basis vectors for $\nu \in \{1, \ldots, \ell\} \backslash \{k\}$.

We will show that this still yields too many negative weights.

To prove this, consider the extremal components of the cluster of components whose nodes map to a flag as above.

If both components C_a, $C_{a'}$ incident to the node are mapped to the rational curve joining the above flag to the standard flag, then $\frac{\lambda_{r_k+1} - \lambda_{r_k}}{d_a} + \frac{\lambda_{r_k+1} - \lambda_{r_k}}{d_{a'}}$ appears in $T_{n_1} C_a \otimes T_{n_2} C_{a'}$.

The next three cases all involve the scenario where one component C_a is mapped to the rational curve joining the flag above to another nonstandard fixed flag. They all rely on the following fact, which we will prove by considering the various cases:

(Δ) If $J^\bullet$ denotes the flag of indices for the second nonstandard fixed flag, $\exists \mu, \beta, \tau$ such that $\beta \in J_\mu$ and $\beta > \tau$ for some $\tau \notin J_\mu$. I claim that $\lambda_\beta - \lambda_\tau$ (distinct from $\lambda_{r_k+1} - \lambda_{r_k}$) appears in $H^0(f_a^* \, TFl(\overline{r}, \mathbb{C}^n))$.

- First, suppose that $I_k \backslash \{\alpha\} = J_k \backslash \{\beta\}$, where $\beta, \alpha \neq r_k, r_k + 1$. Then $\alpha \in I_k$, and necessarily $\beta > r_k + 1 > \alpha$, so $\lambda_\beta - \lambda_\alpha < 0$ appears in $H^0(f^* \, TFl(\overline{r}, \dots, r_\ell; \mathbb{C}^n))$ as in $(\dagger\dagger)_2$, and it does not appear in
$$\sum_{nodes} ([\bigoplus_{i=1}^{\ell} (\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1} (\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$
since the point which is mapped to the nonstandard fixed flag with the flag of indices $J^\bullet$ cannot be a node from what we saw above.

- Next, suppose that $I_k \backslash \{\alpha\} = J_k \backslash \{r_k\}$ for $\alpha \neq r_k + 1$. Then $\alpha \in I_k$ and $\alpha < r_k$, so $\lambda_{r_k} - \lambda_\alpha < 0$ appears in $H^0(f^* \, TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$, and it does not appear in
$$\sum_{nodes} ([\bigoplus_{i=1}^{\ell} (\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1} (\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$
for the same reason as above

- Suppose that $I_k \backslash \{r_k + 1\} = J_k \backslash \{\beta\}$ for $\beta \neq r_k$. Since $\beta \notin I_k$, and $\beta \neq r_k$, then $\beta > r_k + 1$, so $\lambda_\beta - \lambda_{r_k+1} < 0$ appears in $H^0(f^* \, TFl(\overline{r}, \mathbb{C}^n))$ as in $(\dagger\dagger)_2$, and it does not appear in
$$\sum_{nodes} ([\bigoplus_{i=1}^{\ell} (\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1} (\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$
for the same reason as in the first case above.

- If $I_{k'} \backslash \{\alpha\} = J_{k'} \backslash \{\beta\}$ for $k' \neq k$, then $\beta > r_{k'} \geq \alpha$ since $I_{k'} = [r_{k'}]$ for $k' \neq k$. We claim that $(\beta, \alpha) \neq (r_k + 1, r_k)$. If $\beta = r_k + 1$, then $k' < k$, in which case $r_k \notin I_{k'}$, so $\alpha \neq r_k$. If $\alpha = r_k$, then $k' > k$, in which case $r_k + 1 \in I_{k'}$, so $\beta \neq r_k + 1$. Thus, $\beta \in J_{k'}$, $\beta > \alpha$, and $\alpha \notin J_{k'}$ for $(\beta, \alpha) \neq (r_k + 1, r_k)$. Then $\lambda_\beta - \lambda_\alpha$ appears in $H^0(f^* \, TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$, and it does not appear in
$$\sum_{nodes} ([\bigoplus_{i=1}^{\ell} (\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1} (\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$
for the same reason as in the first case above.

Therefore, in the remaining cases where:

- both components incident to the node each map to rational curves joining the first nonstandard fixed flag to another nonstandard fixed flag

- one component maps to the rational curve joining the first nonstandard fixed flag to the standard flag, and the other maps to a rational curve joining the first nonstandard fixed flag to another nonstandard fixed flag
- one component is contracted, and the other maps to a rational curve joining the first nonstandard fixed flag to another nonstandard fixed flag

we see that we have produced another negative weight in $H^0(f^*TFl(\overline{r}, \mathbb{C}^n))$ which does not appear in $\sum_{nodes}([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$.

The last case to consider is when one component $C_{a'}$ is contracted, and the other C_a is mapped to the rational curve joining this nonstandard fixed flag to the standard flag. In this case, the weight $\frac{\lambda_{r_k+1}-\lambda_{r_k}}{d_a} < 0$ appears on $T_{n_1}C_{a'} \otimes T_{n_2}C_a$.

In any case, there are at least 2 negative weights on tangent space. □

The same arguments above prove the following lemma:

Lemma 5.1.2. *In order for $n^- < 2$, there cannot be more than one noncontracted component.*

Proof. If there was more than one noncontracted component, then the node(s) would have to be mapped to the standard flag. Then, the incident noncontracted components would have to be mapped to the rational curves joining the standard flag to nonstandard fixed flags. As we saw in the proof above, each of these yield a negative weight on $H^0(f_a^*TFl(\overline{r}, \mathbb{C}^n))$, for each noncontracted component C_a. These weights do not appear in

$$\sum_{nodes}([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$

since the nodes only map to the standard flag. □

Let us now determine what degree type the noncontracted component must take in order for $n^- < 2$, assuming the curve has two components.

Lemma 5.1.3. *The degree type of the noncontracted component must be $(0, \ldots, 0, 1, 0, \ldots, 0)$ in order for $n^- < 2$.*

Proof.
- If the map has degree type $(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0)$, and the node maps to the standard flag, then we see that

$$[r_p]\backslash\{\alpha\} = I_p\backslash\{\beta\}, \ldots, [r_q]\backslash\{\alpha\} = I_q\backslash\{\beta\}$$

and elsewhere the two flags of indices are equal. Since $\alpha \leq r_p$ and $\beta \notin [r_p]\backslash\{\alpha\}$, $\beta \neq \alpha$, then $\beta > r_p \geq \alpha$. Thus, $\lambda_\beta - \lambda_\alpha < 0$ appears as a weight on $H^0(f^* TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$, and it does not appear as a weight on

$$\sum_{nodes}\left([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n]\right)$$

since the node maps to the standard flag.

- If we assume $q - p > 1$, then $\exists \gamma \in I_{p+1}\backslash I_p$. We must have $\gamma > \alpha$. To see this, notice first that $\gamma \in I_{p+1}\backslash\{\beta\} = [r_{p+1}]\backslash\{\alpha\}$, and $\gamma \neq \alpha$ since $\alpha \notin I_{p+1}$. Thus $\gamma \notin [r_{p+1}] \implies \gamma > r_{p+1} > \alpha$. The negative weight $\lambda_\gamma - \lambda_\alpha$ appears as a weight on $H^0(f_\nu^* TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_1$, and it does not appear as a weight on

$$\sum_{nodes}([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$

since the node maps to the standard flag.

Thus, if the curve has two components, then the noncontracted component must have degree type $(0, \ldots, 0, d, 0, \ldots, 0)$. It is immediate from our calculation of the weights on $H^0(f_\nu^* TFl(\overline{r}, \mathbb{C}^n))$ that the degree d must equal 1 since otherwise $\frac{j\lambda_\beta - j\lambda_\alpha}{d}$ appear $(j = 1, \ldots, d \geq 2)$. □

We would like to determine the image of the single marked point on the noncontracted component C_a.

Suppose the flag of indices is $I^\bullet$. Then, $I_k\backslash\{\beta\} = [r_k]\backslash\{\alpha\}$, and $I_j = [r_j]$ for $j \neq k$.

If $\alpha < r_k$, then $\lambda_{r_k} - \lambda_\alpha < 0$ appears in $H^0(f_\nu^* TFl(\overline{r}\mathbb{C}^n))$, as in $(\dagger\dagger)_1$, since $r_k \in I_k$ but $\alpha \notin I_k$. We also have the weight $\lambda_\beta - \lambda_\alpha < 0$ on $H^0(f_\nu^* TFl(\overline{r}, \mathbb{C}^n))$. Notice that neither of these weights appear in

$$\sum_{nodes}([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$

since the node maps to the standard flag. Therefore, we must have $\alpha = r_k$.

If $\beta > r_k + 1$, then $r_k + 1 \notin I_k$ but $\beta \in I_k$. Thus, $\lambda_\beta - \lambda_{r_k+1} < 0$ appears in $H^0(f_\nu^* TFl(\overline{r}, \mathbb{C}^n))$, as in $(\dagger\dagger)_2$, and it does not appear in

$$\sum_{nodes}([\bigoplus_{i=1}^{\ell}(\mathcal{S}_i^* \otimes \mathcal{Q}_i)|_n] - [\bigoplus_{j=1}^{\ell-1}(\mathcal{S}_j^* \otimes \mathcal{Q}_{j+1})|_n])$$

since the node maps to the standard flag. Together with the negative weight $\lambda_\beta - \lambda_\alpha < 0$, there are at least two negative weights on the tangent space, so β must equal $r_k + 1$.

Thus, $I_k\backslash\{r_k + 1\} = [r_k - 1]$, and $I_j = [r_j]$ for $j \neq k$. This shows us that, if the curve has two components, then

- the contracted component maps to the standard flag
- the noncontracted component maps $1 : 1$ to the rational curve joining the standard flag to the flag whose flag of indices satisfies $I_k \backslash \{r_k + 1\} = [r_k - 1]$, and $I_j = [r_j]$ for $j \neq k$.

We must determine the distribution of the degrees on the subsheaves of the flag sequence on the contracted component.

In order for there to be no negative weights coming from the contracted component (there is already one negative weight coming from the noncontracted component), we must have that, for $j \neq k$, $d_{j,\tau} = 0$ for $\tau \neq r_j$ and $d_{j,r_j} = d_j$; for $j = k$, $d_{k,\tau} = 0$ for $\tau \neq r_k$ and $d_{k,r_k} = d_k - 1$.

There are $m \cdot |\{ 1 \leq i \leq \ell | d_i > 0 \}|$ fixed loci with 1 negative weight on their tangent space as above.

Before we begin describing the fixed loci, we introduce some notation :

- c, nc, b stand for contracted, noncontracted, or both; these letters will be superscripts
- sf, nf stand for standard flag and nonstandard flag; these will be superscripts
- i_s, j_m will stand for i^{th} step, j^{th} markings; these will be subscripts
- $'$ will be used to distinguish further.

Call the following collection $\mathcal{F}^b_{i_s,j_m}$, where $1 \leq j \leq m$. We can describe them as follows:

The generic point of the fixed locus corresponds to a reducible curve with 2 irreducible components such that one component is contracted to the standard flag and the other component is mapped $1 : 1$ to the fixed curve joining the standard flag to a flag whose flag of indices satisfies $J_i = [r_i - 1] \bigcup \{r_i + 1\}$, and $J_h = [r_h]$ for $h \neq i$. The contracted component carries $m - 1$ markings, and the noncontracted component carries a single marking, the j^{th}. The flag sequence of sheaves on the contracted component takes the form

$$0 \hookrightarrow \mathbb{C}^{r_1-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_1) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_i+1) \hookrightarrow \ldots$$
$$\hookrightarrow \mathbb{C}^{r_\ell-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_\ell) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$

where each inclusion of sheaves respects the splitting. For this fixed locus, $n^-(\mathcal{F}^b_{i_s,j_m}) = 1$.

We consider the case where the curve has a single component which is contracted.

By considering the weights above, we see that the component cannot map to a flag such that one of the indices k in the associated flag of indices has the property that $\exists \delta, \gamma_1, \ldots, \gamma_v$ such that $\delta \in I_k$ but $\gamma_i \notin I_k$ for $1 \leq i \leq v \geq 2$. Thus, the contracted component either maps to the standard flag, or a flag whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for $j \neq k$.

If the component is mapped to the standard flag, then we see that either

- $d_{\gamma,\tau} = 0$ for $\tau < r_\gamma$ and $d_{\gamma,r_\gamma} = d_\gamma$ for all $1 \leq \gamma \leq \ell$. The corresponding fixed locus has no negative weights on its tangent space; or

- $\exists k$ such that, $\forall \gamma \neq k$, $d_{\gamma,\tau} = 0$ for $\tau < r_\gamma$, $d_{\gamma,r_\gamma} = d_\gamma$; $d_{k,\tau} = 0$ for all $\tau < r_k - 1$, $d_{k,r_k-1} = 1$, $d_{k,r_k} = d_k - 1$. The corresponding fixed locus has 1 negative weight on its tangent space. This can occur only if either $r_k - r_{k-1} > 1$ or $d_{k-1} > 0$.

Now, we consider what the weights on the tangent space are if we contract the curve to a flag other than the standard flag.

- Assume $d_k > 0$. If the component is mapped to a a flag whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for $j \neq k$, then $\lambda_{r_k+1} - \lambda_{r_k}$ appears at least once on the tangent space, as in $(\dagger)_1$. Therefore, we must have $d_{k,r_k+1} = 0$, but this forces $d_{k,\tau} > 0$ for some $\tau \leq r_k - 1$. Then $\lambda_{r_k+1} - \lambda_\tau < 0$ appears at least once in $[THQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})_q]$, as in $(\dagger)_2$, and there are too many negative weights.

- Assume $d_k = 0$. Following the argument above, we do not run into the same problem if $d_k = 0$. However, in order for the degrees to actually yield a valid flag sequence, we must have $d_{j,h} \geq d_{j+1,h}$, where j refers to the step in the flag sequence and h refers to the basis element of $\mathbb{C}^n$.

  If $r_{k+1} - r_k > 1$, then we let $d_{j,\tau} = 0$ for $\tau < r_j$ and $d_{j,r_j} = d_j$, for $j \neq k$. This fixed locus has 1 negative weight on its tangent space.

If $r_{k+1} - r_k = 1$, then this does not work unless $d_{k+1} = 0$ since $d_{k,r_k+1} = 0$ yet $d_{k+1,r_k+1} = d_{k+1}$. In the case that $r_{k+1} - r_k = 1$ and $d_k = d_{k+1} = 0$, we can define the fixed locus just as we have done for $r_{k+1} - r_k > 1$.

In the case that $r_{k+1} - r_k = 1$ and $d_{k+1} > 0$, we must study the weights in $(\dagger)_1$ and $(\dagger)_2$ more carefully. We cannot have $d_{k+1,r_k+1} > 0$ from what we saw above. Therefore, we must have $d_{k+1,\tau} = d_{k+1}$ for some $\tau \leq r_k$. If $\tau < r_k$, then this does not define a valid flag sequence (the degree type does not satisfy $d_{k,\tau} \geq d_{k+1,\tau}$). Thus, $\tau = r_k$, and in this case we have no further weights on the tangent space, as in $(\dagger)_2$.

Thus, if the curve is contracted and all $d_k > 0$, it must map to the standard flag.

The first type of contracted fixed loci (where the curve is contracted to the standard flag) will be denoted $\mathcal{F}^{c,sf}$; it can be described as follows (there is only one such fixed locus):
The generic point of this fixed locus corresponds to an irreducible curve with m markings which is contracted to the standard flag. The flag sequence takes the form

$$0 \hookrightarrow \mathbb{C}^{r_1-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_1) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_\ell-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_\ell) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}.$$

This fixed locus has $n^-(\mathcal{F}^{c,sf}) = 0$.

The second type of contracted fixed loci (where the curve is contracted to the standard flag and $d_i > 0$) will be denoted $\mathcal{F}^{c,sf}_{i_s}$; and it can be described as follows:
The generic point of this type of fixed loci corresponds to an irreducible curve with m markings which is contracted to the standard flag. The flag sequence of sheaves takes the form

$$0 \hookrightarrow \mathbb{C}^{r_1-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_1) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_i-2} \otimes \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-d_i+1) \hookrightarrow \ldots$$
$$\hookrightarrow \mathbb{C}^{r_\ell-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_\ell+1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O},$$

where each inclusion respects the splitting. For these fixed loci, $n^-(\mathcal{F}^{c,sf}_{i_s}) = 1$.

Now we come to the fixed loci parameterizing curves contracted to flags other than the standard flag.

If $d_k = 0$ and $r_{k+1} - r_k > 1$, then we have the following fixed locus, which we denote by $\mathcal{F}^{c,nf}_{k_s}$:
The generic point of this fixed locus corresponds to an irreducible curve with m markings whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for

$j \neq k$. The flag sequence takes the form

$$0 \hookrightarrow \mathbb{C}^{r_1-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_1) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_k-1}\mathcal{O} \oplus \mathcal{O}^{(r_k+1)} \hookrightarrow \ldots$$
$$\hookrightarrow \mathbb{C}^{r_\ell-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_\ell) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$

where all inclusions respect the splitting, but the inclusion of the k^{th} subsheaf maps the last copy of $\mathcal{O}$ into the $r_k + 1^{st}$ summand, $\mathcal{O}$, in the $k + 1^{st}$ subsheaf (here is where we are using $r_{k+1} - r_k > 1$). This fixed locus has $n^-(\mathcal{F}^{c,nf}_{k_s}) = 1$.

If $d_k = 0$ and $r_{k+1} - r_k = 1$, then we have the following fixed locus, which we denote by $\mathcal{F}^{c,nf}_{k_s}{}'$:

The generic point of this fixed locus corresponds to an irreducible curve with m markings which is contracted to the flag whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for $j \neq k$. The flag sequence takes the form

$$0 \hookrightarrow \mathbb{C}^{r_1-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_1) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_k-1} \otimes \mathcal{O} \oplus \mathcal{O}^{(r_{k+1})} \hookrightarrow \mathbb{C}^{r_k-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_{k+1}) \oplus \mathcal{O}$$
$$\hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_\ell-1} \otimes \mathcal{O} \oplus \mathcal{O}(-d_\ell) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$

where all inclusions respect the splitting, but the inclusion of the k^{th} subsheaf maps the last copy of $\mathcal{O}$ into the r^{th}_{k+1} summand, $\mathcal{O}$, in the $k + 1^{st}$ subsheaf. This fixed locus has $n^-(\mathcal{F}^{c,nf}_{k_s}{}') = 1$.

The only case left to consider is when the number of markings is 2, and the entire curve is a single noncontracted component. The degree type must be uniform in order for this to occur, as we have already seen.

If the rational curve is $se_\alpha + te_\beta$, then without loss of generality we may assume $\beta > \alpha$. Then, if $d > 1$, where the degree type of the map is $(0, \ldots, 0, d, \ldots, d, 0, \ldots, 0)$, we have the negative weights $\frac{j\lambda_\beta - j\lambda_\alpha}{d}$ for $1 \leq j \leq d$. Thus, the map must have degree type $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ in order for there to be less than two negative weights on the tangent space to the fixed locus.

Automatically we have one negative weight on the tangent space, $\lambda_\beta - \lambda_\alpha < 0$.

If we denote by I_k the k^{th} mutual index set of the two $\mathbb{C}^*$ fixed flags, referring back to the weights on the tangent space $((\dagger \ldots \dagger)_i)$, it is clear that we must have that the indices in I_k are less than every index in I_k^c (^{c} refers to the complement), $\forall\ 1 \leq k \leq \ell$.

This forces

- $I_j = \{1, \ldots, r_j\}$ for $1 \leq j \leq q$ or $u + 1 \leq j \leq \ell$ (all the places in the degree type where there is a zero)

- $I_k = \{1, \ldots, r_k - 1\}$ for $q + 1 \leq k \leq u$ (all the places in the degree type where there is a 1)

where we used the same notation from our weight calculations.

By considering the second collection of weights, we see that we must have β, α less than or equal to every index in I_k^c, $\forall\ q+1 \leq k \leq u$. This translates into $\alpha \leq r_{q+1}$, which forces $\alpha = r_{q+1}$. The same argument we used before when $m > 2$ shows that $q+1 = u$. Therefore the curve is mapped to the rational curve joining the standard flag to a $\mathbb{C}^*$ fixed flag whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for $j \neq k$.

Thus, in the case that $m = 2$, we have the following fixed loci with 1 negative weight on their tangent space, call them $\mathcal{F}_{1_m}^{nc}$ and $\mathcal{F}_{2_m}^{nc}$:
The generic point of these fixed loci corresponds to an irreducible curve with 2 markings which is mapped 1 : 1 to the curve joining the standard flag to the nonstandard fixed flag whose flag of indices satisfies $I_k = [r_k - 1] \bigcup \{r_k + 1\}$, and $I_j = [r_j]$ for $j \neq k$. The two markings are mapped to the standard flag and the nonstandard flag. These fixed loci have $n^- = 1$.

5.1.3 The fixed loci of $\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r}, \mathbb{C}^n), (0, \ldots, 1, 0, \ldots, 0))$

Suppose the 1 in the degree type sits in the k^{th} position. If $k = 1$, then set $r_0 = 0$.

First we describe the contracted fixed locus.

The first type of contracted fixed locus is $\mathcal{F}^{c,sf}$, which has no negative weights on its normal bundle.

If $r_k - r_{k-1} = 1$, then the second type does not occur. Otherwise we have the fixed locus $\mathcal{F}_{k_s}^{c,sf}$. This fixed locus has 1 negative weight on its normal bundle.

The third and fourth type of contracted fixed loci are $\mathcal{F}_{i_s}^{c,nf}$ and $\mathcal{F}_{i_s}^{c,nf}{}'$, for each $i \neq k$, depending on whether $r_{i+1} - r_i > 1$ or $r_{i+1} - r_i = 1$, respectively. There are $\ell - 1$ such fixed loci, and each has 1 negative weight on its normal bundle.

The noncontracted fixed loci are $\mathcal{F}_{1_m}^{nc}$ and $\mathcal{F}_{2_m}^{nc}$. Both of these have 1 negative weight on their normal bundle.

The fixed locus with no negative weights on its normal bundle is isomorphic to $\overline{M}_{0,2|1}$, which is a point.

Using Lemma 3.1.1, we see that

- if $r_k - r_{k-1} > 1$, then

$$h^2(\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r},\mathbb{C}^n),(0,\ldots,0,1,0,\ldots,0))) = h^2(\overline{M}_{0,2|1}) + \ell + 2$$
$$= \ell + 2.$$

- if $r_k - r_{k-1} = 1$, then

$$h^2(\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r},\mathbb{C}^n);(1,0,\ldots,0))) = h^2(\overline{M}_{0,2|1}) + \ell - 1 + 2$$
$$= \ell + 1.$$

5.1.4 The fixed loci of $\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r},\mathbb{C}^n),\overline{d})$

Assume $\exists i$ such that $d_i > 1$ or more than one $d_i > 0$.

There are four types of fixed loci which are contracted.

The first type of fixed locus where the curve is contracted is $\mathcal{F}^{c,sf} \cong \overline{M}_{0,2|\sum_{i=1}^{\ell} d_i}/\prod_{i=1}^{\ell} \mathcal{S}_{d_i}$, which has no negative weights on its normal bundle.

The second type are $\mathcal{F}^{c,sf}_{k_s}$. There are $|\{1 \le \gamma \le \ell \,|\, r_\gamma - r_{\gamma-1} > 1,\ d_\gamma > 0\}| + |\{1 < \eta \le \ell \,|\, r_\eta - r_{\eta-1} = 1;\ d_\eta,\ d_{\eta-1} > 0\}|$ such fixed loci, and each has 1 negative weight on its normal bundle.

The third and fourth types of fixed loci are $\mathcal{F}^{c,nf}_{i_s}$ and $\mathcal{F}^{c,nf}_{k_s}{}'$, depending on whether $r_{i+1} - r_i > 1$ or $r_{i+1} - r_i = 1$, respectively. There are $\ell - |\{1 \le i \le \ell \,|\, d_i > 0\}|$ such fixed loci, and each has one negative weight on its normal bundle.

Lastly, we have the fixed loci with one contracted component and one noncontracted component: $\mathcal{F}^b_{i_s,j_m}$ for $j = 1,2$. There are $2|\{1 \le i \le \ell \,|\, d_i > 0\}|$ such fixed loci, and each has 1 negative weight on its normal bundle.

Using Lemma 3.1.1, we find that

$$\begin{aligned} h^2(\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r},\mathbb{C}^n),\overline{d})) = {} & h^2(\overline{M}_{0,2|\sum_{i=1}^{\ell} d_i}/\prod_{i=1}^{\ell}\mathcal{S}_{d_i}) + |\{1 \le i \le \ell \,|\, d_i > 0\}| \\ & + |\{1 < \eta \le \ell \,|\, r_\eta - r_{\eta-1} = 1;\ d_\eta,\ d_{\eta-1} > 0\}| + \ell \\ & + |\{1 \le \gamma \le \ell \,|\, r_\gamma - r_{\gamma-1} > 1,\ d_\gamma > 0\}| \end{aligned}$$

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5.1.5 The fixed loci of $\overline{\mathcal{Q}}_{0,m}(\,Fl(\overline{r},\mathbb{C}^n),\,\overline{d})$, for $m \geq 3$

Assume that at least one $d_i > 0$.

As before, we start with the contracted fixed loci.

The first type is $\mathcal{F}^{c,sf} \cong \overline{M}_{0,m|\sum_{i=1}^{\ell} d_i}/\prod_{i=1}^{\ell} \mathcal{S}_{d_i}$, which has no negative weights on its normal bundle.

The second type is $\mathcal{F}^{c,sf}_{k_s}$. There are

$|\{1 \leq \gamma \leq \ell \,|\, r_\gamma - r_{\gamma-1} > 1,\ d_\gamma > 0\}| + |\{\,1 < \eta \leq \ell \,|\, r_\eta - r_{\eta-1} = 1;\ d_\eta,\ d_{\eta-1} > 0\,\}|$ such fixed loci, and each has 1 negative weight on its normal bundle.

The last two types of contracted fixed loci correspond to curves contracted to a nonstandard flag: $\mathcal{F}^{c,nf}_{i_s}$ and $\mathcal{F}^{c,nf}_{i_s}{}'$, depending on whether $r_{i+1} - r_i > 1$ or $r_{1+1} - r_i = 1$, respectively. There are $\ell - |\{1 \leq i \leq \ell | d_i > 0\}|$ such fixed loci, and each has 1 negative weight on its normal bundle.

The last type of fixed loci parameterizes reducible curves with one component contracted and the other noncontracted. These are $\mathcal{F}^b_{i_s,j_m}$. There are $m|\{\,1 \leq i \leq \ell \,|\, d_i > 0\}|$ such fixed loci, and each has 1 negative weight on its normal bundle.

Using Lemma 3.1.1,

$$\begin{aligned} h^2(\,\overline{\mathcal{Q}}_{0,m}(\,Fl(\overline{r},\mathbb{C}^n),\overline{d})\,) &= h^2(\,\overline{M}_{0,m|\sum_{i=1}^{\ell} d_i}\,/\prod_{i=1}^{\ell}\mathcal{S}_{d_i}\,) + (m-1)|\{\,1 \leq i \leq \ell \,|\, d_i > 0\}| \\ &+ |\{\,1 < \eta \leq \ell \,|\, r_\eta - r_{\eta-1} = 1;\ d_\eta,\ d_{\eta-1} > 0\,\}| + \ell \\ &+ |\{1 \leq \gamma \leq \ell \,|\, r_\gamma - r_{\gamma-1} > 1,\ d_\gamma > 0\}| \end{aligned}$$

for $m \geq 3$.

5.2 Calculation of $h^2(\,\overline{M}_{0,m|\sum_{i=1}^{\ell} d_i}\,/\,\prod_{i=1}^{\ell}\mathcal{S}_{d_i}\,)$

As before, we will calculate the coefficient on $t^{2m+2\sum_{i=1}^{\ell} d_i - 8}$ in the virtual Poincaré polynomial of $\overline{M}_{0,m|\sum_{i=1}^{\ell} d_i}\,/\,\prod_{i=1}^{\ell}\mathcal{S}_{d_i}$.

5.2.1 $m = 2$

When $m = 2$ stratify the space by the number of components of the curve.

In the first case, the curve is irreducible. We can use the $Aut(\mathbb{P}^1)$ action to move the markings to 0 and ∞. Then there is a $\mathbb{C}^*$ action on the curve that fixes the 2 markings. Therefore, the coarse moduli space of the interior is isomorphic to $(\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^*))/\mathbb{C}^*$, where the action is the diagonal action.

We already computed that the virtual Poincaré polynomial of $S^{d_i}(\mathbb{C}^*)$ is $t^{2d_i} - t^{2d_i-2}$, where if $\alpha < 0$, then $t^\alpha = 0$.

Thus,

$$p^{vir}\Big(\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^*)\,/\,\mathbb{C}^*\Big) = \frac{1}{t^2-1}\prod_{i=1}^{\ell}(t^{2d_i} - t^{2d_i-2}),$$

using [GP06]. The coefficient on $t^{\sum_{i=1}^{\ell} d_i - 4}$ is $-|\{1 \le i \le \ell \,|\, d_i > 0\}| + 1$.

We can disregard the curves with three or more components, since these will not contribute to the coefficient on $t^{\sum_{i=1}^{\ell} d_i - 4}$ for dimension reasons. Therefore all we have to do is count boundary divisors with two irreducible components, as we did before. There are $\prod_{i=1}^{\ell}(d_i + 1) - 2$ such boundary divisors.

Putting this together,

$$h^2\Big(\overline{M}_{0,2|\sum_{i=1}^{\ell} d_i}\,/\,\prod_{i=1}^{\ell}\mathcal{S}_{d_i}\Big) = \prod_{i=1}^{\ell}(d_i + 1) - |\{1 \le i \le \ell \,|\, d_i > 0\}| - 1.$$

5.2.2 $m = 3$

Assume $m = 3$. We have a map $M_{0,3|\sum_{i=1}^{\ell} d_i}\,/\,\prod_{i=1}^{\ell}\mathcal{S}_{d_i} \to M_{0,3}$ which has fibers $\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^* \setminus \{1\})$. By the same argument from before,

$$p^{vir}\Big(M_{0,3|\sum_{i=1}^{\ell} d_i}\,/\,\prod_{i=1}^{\ell}\mathcal{S}_{d_i}\Big) = p^{vir}\Big(\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^* \setminus \{1\})\Big)\cdot p^{vir}(M_{0,3}).$$

Thus,

$$\begin{aligned} p^{vir}\Big(\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^* \setminus \{1\})\Big) &= \prod_{i=1}^{\ell} p^{vir}\big(S^{d_i}(\mathbb{C}^* \setminus \{1\})\big) \\ &= \prod_{i=1}^{\ell}(t^{2d_i} - 2t^{2d_i-2} + t^{2d_i-4}). \end{aligned}$$

We need to find the coefficient on $t^{6+\sum_{i=1}^{\ell} d_i - 8}$ in this expression. By counting, we see that this coefficient is $\sum_{i|d_i>0}(-2) = -2|\{1 \le i \le \ell \,|\, d_i > 0\}|$.

We must count the number of boundary divisors. There are $3(\prod_{i=1}^{\ell}(d_i + 1) - 1)$ such boundary divisors: one of the components has one marking, and at least one of the degrees must be positive on this component; repeating for each marking gives the result.

Thus, $h^2(\overline{M}_{0,3|\sum_{i=1}^{\ell} d_i} / \prod_{i=1}^{\ell} \mathcal{S}_{d_i}) = 3(\prod_{i=1}^{\ell}(d_i + 1) - 1) - 2|\{1 \le i \le \ell \,|\, d_i > 0\}|$.

5.2.3 $m > 3$

As above, we have a map $M_{0,m|\sum_{i=1}^{\ell} d_i} / \prod_{i=1}^{\ell} \mathcal{S}_{d_i} \to M_{0,m}$ which has fibers $\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^* \setminus \{p_3, \ldots, p_m\})$. By the same argument from before,

$$\begin{aligned} p^{vir}(M_{0,m|\sum_{i=1}^{\ell} d_i} / \prod_{i=1}^{\ell} \mathcal{S}_{d_i}) &= p^{vir}(\prod_{i=1}^{\ell} S^{d_i}(\mathbb{C}^* \setminus \{p_3, \ldots, p_m\})) \cdot p^{vir}(M_{0,m}) \\ &= \prod_{i=1}^{\ell} p^{vir}(S^{d_i}(\mathbb{C}^* \setminus \{p_3, \ldots, p_m\})) \cdot p^{vir}(M_{0,m}) \\ &= \prod_{i=1}^{\ell} \sum_{j_i=0}^{d_i} (-1)^{j_i} \binom{m-1}{j_i} t^{2d_i - 2j_i} \cdot \prod_{k=2}^{m-2} (t^2 - k). \end{aligned}$$

We want to find the coefficient on $t^{2m + \sum_{i=1}^{\ell} d_i - 8}$ in this expression. After counting, we see that this coefficient is

$$-\sum_{k=2}^{m-2}(k) - \sum_{i\,|\,d_i>0} \binom{m-1}{1} = -\frac{m\cdot(m-3)}{2} - (m-1)|\{1 \le i \le \ell \,|\, d_i > 0\}|.$$

Next we count the number of boundary divisors:

- There are $m\prod_{i=1}^{\ell}(d_i + 1) - m$ boundary divisors with 1 marking on one component and $m - 1$ markings on the other component.
- There are $\binom{m}{k}\prod_{i=1}^{\ell}(d_i + 1)$ boundary divisors with k markings on one component and $m - k$ markings on the other component, for $2 \le k \le m - 2$.
- Thus, there are

  $$m\prod_{i=1}^{\ell}(d_i + 1) - m + \frac{1}{2}\sum_{k=2}^{m-2}\binom{m}{k}\prod_{i=1}^{\ell}(d_i + 1) = \frac{1}{2}\prod_{i=1}^{\ell}(d_i + 1)(2^m - 2) - m$$

  boundary divisors in total.

Putting this all together, we see that

$$\begin{aligned} h^2(\overline{M}_{0,m|\sum_{i=1}^{\ell} d_i} / \prod_{i=1}^{\ell} \mathcal{S}_{d_i}) = &\frac{1}{2}\prod_{i=1}^{\ell}(d_i + 1)(2^m - 2) - m - \frac{m^2 - 3m}{2} \\ &- (m-1)|\{1 \le i \le \ell \,|\, d_i > 0\}|. \end{aligned}$$

5.3 The rank of the Picard group II

As in the Grassmannian case, we use *Corollary 2* of [Cey09] together with Lemma 3.1.1 to conclude that

$$h^2(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\,\mathbb{C}^n),\,\overline{d}))\otimes\mathbb{Q}\cong Pic((\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\,\mathbb{C}^n),\,\overline{d}))\otimes\mathbb{Q}.$$

We can put this together with what we found in the previous sections to conclude that

Proposition 5.3.1. *For $m\geq 2$,*

$$\begin{aligned} rank(Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))) = {} & \frac{1}{2}\prod_{i=1}^{\ell}(d_i+1)(2^m-2)-m-\frac{m^2-3m}{2}+\ell \\ & +\left|\{1<\eta\leq\ell\mid r_\eta-r_{\eta-1}=1;\,d_\eta,\,d_{\eta-1}>0\}\right| \\ & +\left|\{1\leq\gamma\leq\ell\mid r_\gamma-r_{\gamma-1}>1,\,d_\gamma>0\}\right|. \end{aligned}$$

The reader can check that the case $m=2$ agrees with the more general formula found above.

In particular, when all $d_i>0$, the Picard rank is

$$\prod_{i=1}^{\ell}(d_i+1)(2^m-1)-\tfrac{m^2-m}{2}+\ell+2\ell$$

if $r_1>1$, and

$$\prod_{i=1}^{\ell}(d_i+1)(2^m-1)-\tfrac{m^2-m}{2}+\ell+2\ell-1$$

if $r_1=1$.

The same argument as in the Grassmannian case shows that

$$Num(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}\cong Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}.$$

6 Picard Group of Generalized Stable Quotients

In this chapter, we use the calculations from the previous chapter to find generators and relations for the Picard group. We will split the chapter into 2 parts, the first part being when the number of markings is ≥ 3, and the second part being when the number of markings is equal to 2. We obtain the full result when $m \geq 3$, and a partial result ($r_i - r_{i-1} > 0$, $d_i > 0$ for all $1 \leq i \leq \ell + 1$) when $m = 2$. The first is obtained using similar methods to those we used in the Grassmannian case, and the second is obtained by intersecting with curves.

6.1 Calculation of the Picard group for $m \geq 3$

6.1.1 Analysis of the interior II

Assume $m \geq 3$. Let $\mathcal{Q}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ be the interior of the moduli space, corresponding to the locus where the curve is irreducible, but the quotients are allowed to have torsion. We prove that

Lemma 6.1.1. *$\mathcal{Q}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ is isomorphic to an open subscheme of a relative HyperQuot scheme over $M_{0,m}$:*

$$HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}) \times M_{0,m} \to M_{0,m}.$$

Let q be the dimension of the moduli space.

Then, by similar arguments to the ones we made in the Grassmannian case, we will see that from the exact sequence ([Kre99]) :

$$A_{q-1}(\Delta) \otimes \mathbb{Q} \to A_{q-1}(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q} \to A_{q-1}(\mathcal{Q}_{0,m}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q} \to 0$$

together with the fact from [Opr06b], that $A_*(T) \cong A_*(M_{0,m} \times T)$ for any scheme

T, the Picard group is generated by the generators of the Picard group of the HyperQuot scheme and the boundary divisors of the moduli space.

Proof. We have a natural morphism
$$\alpha : HQuot \times M_{0,m} \to \prod_{i=1}^{\ell} Quot_{\mathbb{P}^1}(Gr(r_i, n), d_i) \times M_{0,m}$$
We can consider the compositions
$$\alpha_i : HQuot \times M_{0,m} \to Quot_{\mathbb{P}^1}(Gr(r_i, n), d_i) \times M_{0,m}$$
With the same notation we used in the case of the Grassmannian, we consider the closed subschemes
$$\tau_i = \bigcup_{e_i=1}^{d_i} \Delta^{e_i'} \subset Quot_{\mathbb{P}^1}(Gr(r_i, n), d_i) \times M_{0,m}$$
parameterizing quotients with torsion supported on the markings.

We consider the preimages of these subschemes in the HyperQuot scheme under the morphisms α_i. It is clear that this closed subscheme is exactly the locus of flags such that at least one of the quotients (of $\mathbb{C}^n \otimes \mathcal{O}$) has torsion supported on a marking. Notice that if a quotient $\mathcal{Q}_i$ has torsion supported on a marking, then the inclusion of subsheaves
$$\mathcal{S}_i \hookrightarrow \ldots \hookrightarrow \mathcal{S}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$$
has to drop rank at the same marked point somewhere in the chain of inclusions, say at the j^{th} inclusion. We see that in particular, the preimage of τ_i contains the locus where the inclusion of the i^{th} subsheaf into the $i+1^{st}$ subsheaf drops rank at a marked point. For the same reason, the preimages of the τ_i will have nontrivial intersection.

It is clear that every point in the union of $\alpha_i^{-1}(\tau_i)$ corresponds to a flag of subsheaves where one of the inclusions of subsheaves drops rank at a marked point. Conversely, by the reasoning above, every point corresponding to a flag of subsheaves with the property that the inclusion of the i^{th} subsheaf into the $i+1^{st}$ subsheaf drops rank at a marked point is contained in $\bigcup_{i=1}^{\ell} \alpha_i^{-1}(\tau_i)$.

Thus, the locus of flags where one of the inclusions of subsheaves drops rank along a marked section is a closed subscheme of $HQuot \times M_{0,m}$.

By construction, the complement is isomorphic to the locus of stable quotients to the flag variety whose underlying curve is smooth and irreducible (using the same argument as in the Grassmannian case).

□

6.1.2 The Picard rank of $HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$

As in [Kim], the rational cohomology groups of the HyperQuot scheme coincide with the rational Chow groups of the HyperQuot scheme.

In order to determine how many relations there are among the generators of the HyperQuot scheme, we can calculate the rank of $h^2(\,HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})\,)$ using [Che01]. [Che01] gives an algorithm for computing the Betti numbers of the HyperQuot scheme in *Proposition 3*.

We will recall the notation of [Che01], except we reverse the roles of r_i and s_i.

Let $S \subset \mathcal{S}_n$ be the collection of permutations σ such that

$$\sigma(r_{i-1}+1) < \ldots < \sigma(r_i) \text{ for each } 1 \le i \le \ell \text{ where } r_0 = 0 \text{ and } r_{\ell+1} = n.$$

Let P be the collection of tuples $(\{a_{i,k}\}_{1\le i\le \ell,\, 1\le k\le r_i}\,,\ \{b_{i,k}\}_{1\le i\le \ell,\, 1\le k\le r_i}\,,\ \sigma)$ where

- $a_{i,k}, b_{i,k} \ge 0$, $\sigma \in S$
- $\sum_{k\le r_i} (a_{i,k} + b_{i,k}) = d_i$
- $a_{i,k} \ge a_{i+1,k}$, $b_{i,k} \ge b_{i+1,k}$.

Let $\epsilon^{\sigma}_{j,k} = 1$ if $\sigma(j) < \sigma(k)$ and 0 otherwise.

Then from *Proposition 3* of [Che01] we see that

$$h^2(\,HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})\,) = |\{h^{-1}(1)\}|$$

where

$$h(a,b,\sigma) = \sum_{i=1}^{\ell}\sum_{k\le r_i}(a_{i,k} + b_{i,k} + 1)\sum_{r_i<j\le r_{i+1}}\epsilon^{\sigma}_{k,j} + \sum_{i=1}^{\ell}\sum_{k\le r_i}(a_{i,k} + b_{i,k})\sum_{r_{i-1}<j\le r_i}\epsilon^{\sigma}_{j,k}$$
$$+ \sum_{i=1}^{\ell}\sum_{r_{i-1}<k\le r_i} b_{i,k}.$$

It is clear that, in order for $h(a,b,\sigma) = 1$, we must have that exactly one of the terms is 1 and the other two are 0.

- Suppose that the first term is 0. Then for each i, if $k \le r_i < j$, we must have $\sigma(k) > \sigma(j)$. This forces σ to be the following permutation :
$$\sigma = \begin{pmatrix} 1 & \ldots & r_1 & \ldots & r_\ell & \ldots & n \\ n+1-r_1 & \ldots & n & \ldots & 1 & \ldots & n+1-r_\ell \end{pmatrix}$$
Since σ has the property that, for each i, if $k \le r_i < j$, $\sigma(k) > \sigma(j)$, then we see that, for $1 \le i \le \ell$, if $k \le r_{i-1}$, $\sum_{r_{i-1}<j\le r_i}\epsilon^{\sigma}_{j,k} > 0$. If we require that the

second term in the expression $h(a, b, \sigma)$ is 0, then we see that we must have $a_{i,k} = b_{i,k} = 0\ \forall k \leq r_{i-1}$, for each $1 \leq i \leq \ell$.

Given this,

$$h(a, b, \sigma) = \sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} (a_{i,k} + b_{i,k}) \sum_{r_{i-1}<j\leq r_i} \epsilon_{j,k}^{\sigma} + \sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} b_{i,k}$$

Notice that, for each $r_{i-1} < k \leq r_i$ such that $\exists j$ such that $r_{i-1} < j < k \leq r_i$, $\sum_{r_{i-1}<j\leq r_i} \epsilon_{j,k}^{\sigma} \geq 1$. If we want this term to be zero, we must have that $a_{i,k} = b_{i,k} = 0$ for all $r_{i-1} + 1 < k \leq r_i$.

Thus, we must have $a_{i,r_{i-1}+1} + b_{i,r_{i-1}+1} = d_i$ for each $1 \leq i \leq \ell$. Notice that if $r_{i-1} + 1 = r_i$, then this term still does not appear in

$$\sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} (a_{i,k} + b_{i,k}) \sum_{r_{i-1}<j\leq r_i} \epsilon_{j,k}^{\sigma}$$

since $\epsilon_{r_i,r_i}^{\sigma} = 0$. Since we want $h(a, b, \sigma) = 1$, we must have that $\exists j$ such that $b_{j,r_{j-1}+1} = 1$, and for each $i \neq j$, $b_{i,r_{i-1}+1} = 0$.

This forces $a_{j,r_{j-1}+1} = d_j - 1$, and for $i \neq j$, $a_{i,r_{i-1}+1} = d_i$.

There are $|\{1 \leq i \leq \ell \,|\, d_i > 0\}|$ such points in $h^{-1}(1)$.

- Let σ be as above, so $\sum_{i=1}^{\ell} \sum_{k\leq r_i} (a_{i,k} + b_{i,k} + 1) \sum_{r_i<j\leq r_{i+1}} \epsilon_{k,j}^{\sigma} = 0$, and now suppose the last term $\sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} b_{i,k}$ is also zero. Then $b_{i,k} = 0$ for all $1 \leq i \leq \ell$, $1 \leq k \leq r_i$. Thus,

  $$h(a, b, \sigma) = \sum_{i=1}^{\ell} \sum_{k\leq r_i} (a_{i,k}) \sum_{r_{i-1}<j\leq r_i} \epsilon_{j,k}^{\sigma}.$$

  Suppose now that $\exists i$ such that $r_i - r_{i-1} = 1$.

  We need

  $$\sum_{k\leq r_i} a_{i,k}\epsilon_{r_i,k} = 1, \text{ and } \sum_{j\neq i} a_{j,k} \sum_{r_{j-1}<s\leq r_j} \epsilon_{s,k}^{\sigma} = 0.$$

  Notice that, for each $k \leq r_{j-1}$, $\sum_{r_{j-1}<s\leq r_j} \epsilon_{s,k}^{\sigma} \geq 1$, for $j \neq i$. In order for $\sum_{j\neq i} a_{j,k} \sum_{r_{j-1}<s\leq r_j} \epsilon_{s,k}^{\sigma} = 0$, we must have that $a_{j,k} = 0$, for $j \neq i$, $k \leq r_{j-1}$. Then, by the same argument we used above, we must have $a_{j,r_{j-1}+1} = d_j$, and all other $a_{j,k} = 0$. Notice that, in order for $\sum_{k\leq r_i} a_{i,k}\epsilon_{r_i,k} = 1$, we must have $a_{i,k} = 1$ and $a_{i,r_i} = d_i - 1$, for some $k < r_i$. The only k for which $a_{i-1,k} > 0$ is $k = r_{i-2} + 1$. Since $a_{i-1,k} > a_{i,k}$, we must have that $a_{i,r_{i-2}+1} = 1$. If $i = 2$, then we let $a_{2,1} = 1$ and $a_{2,r_2} = d_2 - 1$.

  There are $|\{1 < i \leq \ell \,|\, r_i - r_{i-1} = 1,\, d_i, d_{i-1} > 0\}|$ such points in $h^{-1}(1)$.

If $r_i - r_{i-1} > 1$, then we must have $a_{j,r_{j-1}+1} = d_j$, and all other $a_{j,k} = 0$ for $j \neq i$, $k \neq r_{j-1} + 1$. Then, a similar argument to the one above shows that we must have $a_{i,r_{i-1}+1} = d_i - 1$, and $a_{i,r_{i-1}+2} = 1$, with $a_{i,k} = 0$ otherwise. Unlike in the case above, since $r_i - r_{i-1} > 1$, if $a_{i,k} > 0$ for some $k < r_{i-1}$, then

$$a_{i,k} \sum_{r_{i-1}<s\leq r_i} \epsilon^{\tau}_{s,k} = a_{i,k} \cdot (r_i - r_{i-1}) > a_{i,k} \geq 1,$$

so this cannot happen.

There are $|\{1 \leq j \leq \ell \,|\, r_j - r_{j-1} > 1,\, d_j > 0\}|$ such points in $h^{-1}(1)$.

- In the next case the third term is 0, so all $b_{i,k} = 0$, and the second term is zero.

  We have
  $$h(a, 0, \tau) = \sum_{i=1}^{\ell} \sum_{k\leq r_i} (a_{i,k} + 1) \sum_{r_i<j\leq r_{i+1}} \epsilon^{\tau}_{k,j} + \sum_{i=1}^{\ell} \sum_{k\leq r_i} (a_{i,k}) \sum_{r_{i-1}<j\leq r_i} \epsilon^{\tau}_{j,k}.$$
  The second term breaks into
  $$\sum_{i=1}^{\ell} \sum_{k\leq r_{i-1}} (a_{i,k}) \sum_{r_{i-1}<j\leq r_i} \epsilon^{\tau}_{j,k} + \sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} (a_{i,k})(k - r_{i-1} - 1).$$
  Therefore, we must have $a_{i,k} = 0$ for each $r_{i-1} + 1 < k \leq r_i$ if $r_i - r_{i-1} > 1$. We want the first term $= 1$, so we need there to exist unique indices γ, μ, ν such that $\nu \leq r_\gamma$, $r_{\gamma+1} \geq \mu > r_\gamma$ for which $\tau(\nu) < \tau(\mu)$ and otherwise if $\alpha \leq r_j < \beta$ then $\tau(\alpha) > \tau(\beta)$. Then this forces $\mu = r_{\gamma+1}$ and $\nu = r_{\gamma-1} + 1$.

  Therefore, τ differs from σ by interchanging the locations of $\sigma(r_{\gamma-1} + 1)$ and $\sigma(r_{\gamma+1})$.

  We must have $a_{i,k} = 0$ for $k \leq r_{i-1}$ for all $1 \leq i \leq \ell$. This kills the second term. Thus, $h(a, 0, \tau) = \sum_{i=1}^{\ell} \sum_{r_{i-1}<k\leq r_i} (a_{i,k} + 1) \sum_{r_i<j\leq r_{i+1}} \epsilon^{\tau}_{k,j}$. For $i \neq \gamma$, we must have $a_{i,r_{i-1}+1} = d_i$ or $a_{i,r_i} = d_i$ if $r_i - r_{i-1} = 1$. If $r_\gamma - r_{\gamma-1} = 1$, we let $a_{\gamma,r_\gamma} = d_\gamma$, and if $r_\gamma - r_{\gamma-1} > 1$, then we let $a_{\gamma,r_{\gamma-1}+1} = d_\gamma$. We must have $d_\gamma = 0$ in order for these points to appear in $h^{-1}(1)$. This yields $|\{1 \leq \gamma \leq \ell \,|\, d_\gamma = 0\}|$ points in $h^{-1}(1)$.

Putting this together with the fact that the Chow groups of the HyperQuot scheme coincide with the cohomology groups, we see that

$$\begin{aligned} rank(Pic(HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d}))) = &|\{1 \leq \gamma \leq \ell \,|\, r_\gamma - r_{\gamma-1} > 1,\, d_\gamma > 0\}| + \\ &|r_\eta - r_{\eta-1} = 1,\, d_\eta,\, d_{\eta-1} > 0\}| + \ell. \end{aligned}$$

6.1.3 The Picard group of the HyperQuot Scheme

We will use the methods and results of [Opr06b].

Let $\mathcal{S}_i$, $\mathcal{Q}_i$ be the i^{th} universal subsheaf and quotient, respectively, over $\mathbb{P}^1 \times HQuot$.

[Opr06b] shows that $\exists$ an exact sequence of sheaves on $\mathbb{P}^1 \times HQuot \times HQuot$:

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{\ell} \mathcal{H}om(\pi_1^*\mathcal{S}_i, \pi_2^*\mathcal{Q}_i) \to \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\pi_1^*\mathcal{S}_j, \pi_2^*\mathcal{Q}_{j+1}) \to 0$$

where $\pi_i : \mathbb{P}^1 \times HQuot \times HQuot \to \mathbb{P}^1 \times HQuot$ is the projection onto the i^{th} factor of $HQuot$ and the identity on $\mathbb{P}^1$, and the last map pre-composes the i^{th} entry with the map $\pi_1^*\mathcal{S}_{i-1} \to \pi_1^*\mathcal{S}_i$ and composes the $i-1^{st}$ entry with the map $\pi_2^*\mathcal{Q}_{i-1} \to \pi_2^*\mathcal{Q}_i$, then takes the difference.

It is shown that $p_*\mathcal{K}$ is a vector bundle of rank equal to the dimension of the HyperQuot scheme, where $p : \mathbb{P}^1 \times HQuot \times HQuot \to HQuot \times HQuot$ is the natural projection.

We recall the argument of [CF99].

Fix two geometric points of $HQuot \times HQuot$,

$$\begin{array}{c} 0 \hookrightarrow S_1' \hookrightarrow \ldots \hookrightarrow S_\ell' \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to Q_1' \to \ldots \to Q_\ell' \to 0 \\ 0 \hookrightarrow S_1'' \hookrightarrow \ldots \hookrightarrow S_\ell'' \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to Q_1'' \to \ldots \to Q_\ell'' \to 0 \end{array} .$$

This yields a map from $\mathbb{P}^1$ to $\mathbb{P}^1 \times HQuot \times HQuot$. We can pull back $\mathcal{K}$ along this map. Notice that we have a natural map

$$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \otimes \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{\ell} \mathcal{H}om(S_i', Q''_i) \to \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(S_j', Q''_{j+1}) \to 0$$

which factors through the pullback of $\mathcal{K}$. This map is generically surjective as we already proved in the first chapter.

Since $\mathcal{K}$ admits a generically surjective morphism from a trivial bundle, it follows that $\mathcal{K}$ has no higher cohomology on the fibers of p.

Next, a section of $p_*\mathcal{K}$ is obtained, coming from the section of $\mathcal{K}$ from the collection of morphisms $\{\pi_1^*\mathcal{S}_i \to \mathbb{C}^n \otimes \mathcal{O} \to \pi_2^*\mathcal{Q}_i\}_{i=1}^{\ell}$. This section vanishes set theoretically along the diagonal. To see this, notice that if

$$\begin{array}{c} 0 \hookrightarrow S_1' \hookrightarrow \ldots \hookrightarrow S_\ell' \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to Q_1' \to \ldots \to Q_\ell' \to 0 \\ 0 \hookrightarrow S_1'' \hookrightarrow \ldots \hookrightarrow S_\ell'' \hookrightarrow \mathbb{C}^n \otimes \mathcal{O} \to Q_1'' \to \ldots \to Q_\ell'' \to 0 \end{array}$$

represents a geometric point in the vanishing locus of this section of $p_*\mathcal{K}$, then since the compositions

$$S_i' \to \mathbb{C}^n \otimes \mathcal{O} \to Q''_i$$

are zero, we have a factorization

$$S_i' \to S_i'' \to \mathbb{C}^n \otimes \mathcal{O}.$$

Since the first Chern classes of S_i', S_i'' are equal, the cokernel must have rank 0 and first Chern class 0, so the map $S_i' \to S_i''$ is an isomorphism for each $1 \leq i \leq \ell$. This shows that, set theoretically, the vanishing of this section of $p_*\mathcal{K}$ is the diagonal.

We would like to express the class of the diagonal in the Chow ring of $HQuot \times HQuot$ as a multiple of the top Chern class of $p_*\mathcal{K}$.

If the vanishing of this section has nonreduced structure along the diagonal, we can consider the preimage of the generic point of the zero section under this section, which will have a certain multiplicity, say α. Since we are only interested in the top dimensional piece of the rational equivalence class of this subscheme, the class of this subscheme is $\alpha[\Delta]$. Thus, the class of the diagonal in the Chow ring of $HQuot \times HQuot$ is given by a rational multiple of the top Chern class of $p_*\mathcal{K}$, by *Proposition 14.1* in [Ful98]. Using the exact sequence

$$0 \to p_*\mathcal{K} \to p_* \bigoplus_{i=1}^{\ell} \mathcal{H}om(\pi_1^*\mathcal{S}_i, \pi_2^*\mathcal{Q}_i) \to p_* \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\pi_1^*\mathcal{S}_j, \pi_2^*\mathcal{Q}_{j+1}) \to 0,$$

we see that the Chern character of $p_*\mathcal{K}$ can be written as

$$ch(p_* \bigoplus_{i=1}^{\ell} \mathcal{H}om(\pi_1^*\mathcal{S}_i, \pi_2^*\mathcal{Q}_i)) - ch(p_* \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\pi_1^*\mathcal{S}_j, \pi_2^*\mathcal{Q}_{j+1})).$$

Recursively, we can determine the top Chern class of $p_*\mathcal{K}$ from the Chern characters of $p_* \bigoplus_{i=1}^{\ell} \mathcal{H}om(\pi_1^*\mathcal{S}_i, \pi_2^*\mathcal{Q}_i)$ and $p_* \bigoplus_{j=1}^{\ell-1} \mathcal{H}om(\pi_1^*\mathcal{S}_j, \pi_2^*\mathcal{Q}_{j+1})$.

Now, apply Grothendieck Riemann Roch to each term (note that $\pi_1^*\mathcal{S}_i^*$ and $\pi_2^*\mathcal{Q}_i$ have no higher cohomology on the fibers of p) :

$$ch(p_*(\pi_1^*\mathcal{S}_i^* \otimes \pi_2^*\mathcal{Q}_i) = p_*(\pi_1^*ch(\mathcal{S}_i^*) \cdot \pi_2^*ch(\mathcal{Q}_i) \cdot \phi^*td(T\mathbb{P}^1))$$

where $\phi : \mathbb{P}^1 \times HQuot \times HQuot \to \mathbb{P}^1$ is the projection to $\mathbb{P}^1$. Notice that $\phi^*td(T\mathbb{P}^1) = 1 + \phi^*\xi$, where ξ is the hyperplane class on $\mathbb{P}^1$.

Lemma 6.1.2. *The rational Chow ring of $HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})$ is generated by the classes*

- $\pi'_*(p(c_1(\mathcal{Q}_j), \ldots, c_{r_j}(\mathcal{Q}_j)))$
- $\pi'_*(q(c_1(\mathcal{Q}_k), \ldots, c_{r_k}(\mathcal{Q}_k)) \cdot \phi'^*\xi)$

where p and q are monomials in the Chern classes of $\mathcal{Q}_i$, for $1 \leq i \leq \ell$.

Proof. The method follows the proof of *Theorem 2.1* in [ES93].

The discussion above shows that the diagonal can be written as a sum of classes $p_*(\pi_1^* m_\alpha(\mathcal{S}_j^*)\cdot\pi_2^* m_\beta(\mathcal{Q}_k))$, $p_*(\pi_1^* m_\gamma(\mathcal{S}_h^*)\cdot\pi_2^* m_\mu(\mathcal{Q}_i)\cdot\phi^*\xi)$ in the Chow ring of $HQuot \times HQuot$, where m_α, m_β, m_γ, m_μ are monomials in the Chern classes of $\mathcal{S}_j^*$, $\mathcal{Q}_k$, $\mathcal{S}_h^*$, and $\mathcal{Q}_i$, respectively, for $1 \le h, i, j, k \le \ell$.

Take any class $z \in A^*(HQuot)\otimes\mathbb{Q}$. We have the formula $\rho_{2\,*}(\rho_1^* z\cdot\delta) = z$, where $\rho_i : HQuot \times HQuot \to HQuot$ is the projection to the i^{th} factor.

Looking at the individual terms of the intersection, we have

$$\rho_{2\,*}(\rho_1^* z \cdot p_*(\pi_1^* m_\alpha(\mathcal{S}_j^*)\cdot\pi_2^* m_\beta(\mathcal{Q}_k))\,) = \rho_{2\,*}(p_*(p^*\rho_1^* z\cdot\pi_1^* m_\alpha(\mathcal{S}_j^*)\cdot\pi_2^* m_\beta(\mathcal{Q}_k))).$$

We have the following fiber diagrams :

$$\begin{tikzcd}
\mathbb{P}^1 \times HQuot \times HQuot \arrow[r, "p"] \arrow[d, "\pi_i"'] & HQuot \times HQuot \arrow[d, "\rho_i"] \\
\mathbb{P}^1 \times HQuot \arrow[r, "\pi'"'] & HQuot
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}^1 \times HQuot \times HQuot \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & \mathbb{P}^1 \times HQuot \arrow[d, "\phi'"] \\
\mathbb{P}^1 \times HQuot \arrow[r, "\phi'"'] & \mathbb{P}^1.
\end{tikzcd}$$

The combination of the two diagrams and the projection formula allow us to write this as

$$\pi'_*\pi_{2\,*}(p^*\rho_1^* z\cdot\pi_1^* m_\alpha(\mathcal{S}_j^*)\cdot\pi_2^* m_\beta(\mathcal{Q}_k)) = \pi'_*(\pi_{2\,*}(\,\pi_1^*\pi'^* z\cdot\pi_1^* m_\alpha(\mathcal{S}_j^*)\,)\cdot m_\beta(\mathcal{Q}_k)\,).$$

Notice that, from the second diagram,

$$\pi_{2\,*}\pi_1^*(\pi'^* z\cdot m_\alpha(\mathcal{S}_j^*)\,) = \phi'^*\phi'_*(\pi'^* z\cdot m_\alpha(\mathcal{S}_j^*).\,)$$

$\phi'_*(\pi'^* z\cdot m_\alpha(\mathcal{S}_j^*)\,)$ is either 0, $x\,[pt]$, or $x\,[\mathbb{P}^1]$ for some nonzero $x\in\mathbb{Q}$.

In the first case, the intersection is zero.

In the second case, $\phi'^*\phi'_*(\pi'^* z\cdot c_\alpha(\mathcal{S}_j^*)\,) = x\,\phi'^*\xi$. Then,

$$\begin{aligned}
\pi'_*(\pi_{2\,*}(\,\pi_1^*\pi'^* z\cdot\pi_1^* m_\alpha(\mathcal{S}_j^*)\,)\cdot m_\beta(\mathcal{Q}_k)\,) &= \pi'_*(x\,\phi'^*\xi\cdot m_\beta(\mathcal{Q}_k)\,)\\
&= x\,\pi'_*(\phi'^*\xi\cdot m_\beta(\mathcal{Q}_k)\,).
\end{aligned}$$

In the third case,

$$\pi'_*(\pi_{2\,*}(\,\pi_1^*\pi'^* z\cdot\pi_1^* m_\alpha(\mathcal{S}_j^*)\,)\cdot m_\beta(\mathcal{Q}_k)\,) = x\,\pi'_*(m_\beta(\mathcal{Q}_k)).$$

The calculation for the terms of the second type is similar.

□

This proves:

Lemma 6.1.3. *Pic*($HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q}$ *is generated by*

- $\pi'_* c_1^2(\mathcal{Q}_i)$ *for* $1 \leq i \leq \ell$
- $\pi'_* c_2(\mathcal{Q}_j)$ *for* $1 \leq j \leq \ell$
- $\pi'_*(\, c_1(\mathcal{Q}_k) \cdot \phi'^* \xi\,)$ *for* $1 \leq k \leq \ell$.

We can improve on this lemma and will do so by calculating several intersections below.

- If $d_i > 0$, then we have that $det(\mathcal{Q}_i)|_{\mathbb{P}^1_h} \cong \mathcal{O}(d_i)$ for each $h \in HQuot$. As in the Grassmannian case, $\exists$ a line bundle $\mathcal{N}_i$ on $HQuot$ such that $det(\mathcal{Q}_i) \cong \pi'^* \mathcal{N}_i \otimes \phi'^* \mathcal{O}(d_i)$. Then, $c_1(\mathcal{Q}_i) = \pi'^* c_1(\mathcal{N}_i) + \phi'^* d_i \xi$. A direct calculation yields $\frac{\pi'_* c_1^2(\mathcal{Q}_i)}{2d_i} = c_1(\mathcal{N}_i)$.
- If $d_j, d_k > 0$, then

$$\begin{aligned} \pi'_*(c_1(\mathcal{Q}_j) \cdot c_1(\mathcal{Q}_k)) &= \pi'_*(\,(\pi'^* c_1(\mathcal{N}_j) + \phi'^* d_j \xi) \cdot (\pi'^* c_1(\mathcal{N}_k) + \phi'^* d_k \xi)) \\ &= d_j\, c_1(\mathcal{N}_k) + d_k\, c_1(\mathcal{N}_j) \\ &= \frac{d_j\, \pi'_* c_1^2(\mathcal{Q}_k)}{2d_k} + \frac{d_k\, \pi'_* c_1^2(\mathcal{Q}_j)}{2d_j} \end{aligned}$$

  using what we saw above.
- If $d_j = 0$, then $\mathcal{Q}_j$ is the pullback of the universal quotient over the Grassmannian $Quot_{\mathbb{P}^1}(\mathbb{C}^n \otimes \mathcal{O}, n - r_j, 0) = Gr(r_j, n) \times \mathbb{P}^1$; call the pullback of the quotient to the HyperQuot scheme $\mathcal{F}_j$. We see that

$$\begin{aligned} \pi'_* c_1^2(\mathcal{Q}_j) &= \pi'_*(\pi'^* c_1^2(\mathcal{F}_j)) \\ &= c_1^2(\mathcal{F}_j) \cdot \pi'_*[\mathbb{P}^1 \times HQuot] \\ &= 0 \end{aligned}$$

  since π' has positive dimensional fibers. Similarly, $\pi'_* c_2(\mathcal{Q}_j) = 0$.

  However,

$$\begin{aligned} \pi'_*(c_1(\mathcal{Q}_j) \cdot \phi'^* \xi) &= \pi'_*(\pi'^* c_1(\mathcal{F}_j) \cdot \phi'^* \xi) \\ &= c_1(\mathcal{F}_j) \neq 0. \end{aligned}$$

- Also, observe that if $d_j = 0$, then for k such that $d_k > 0$,

$$\begin{aligned}\pi'^*(c_1(\mathcal{Q}_j)\cdot c_1(\mathcal{Q}_k)) &= \pi'_*(\pi'^*c_1(\mathcal{F}_j)\cdot(\pi'^*c_1(\mathcal{N}_k)+\phi'^*d_k\xi))\\ &= c_1(\mathcal{F}_j)\cdot c_1(\mathcal{N}_k)\cdot\pi'_*[\mathbb{P}^1\times HQuot]+d_kc_1(\mathcal{F}_j)\\ &= d_k\,c_1(\mathcal{F}_j).\end{aligned}$$

- If $d_j = d_k = 0$, then

$$\begin{aligned}\pi'_*(c_1(\mathcal{Q}_j)\cdot c_1(\mathcal{Q}_k)) &= c_1(\mathcal{F}_j)\cdot c_1(\mathcal{F}_k)\cdot\pi'_*[\mathbb{P}^1\times HQuot]\\ &= 0.\end{aligned}$$

This shows that the following classes generate the rational Picard group of the HyperQuot scheme:

- $c_1(\mathcal{F}_j) = \pi'_*(c_1(\mathcal{Q}_j)\cdot\phi'^*\xi)$ for all j such that $d_j = 0$
- $\pi'_*c_1^2(\mathcal{Q}_k)$ for all k such that $d_k > 0$
- $\pi'_*c_2(\mathcal{Q}_h)$ for all h such that $d_h > 0$

We will further restrict to a subcollection of these classes by making some observations below.

Consider the situation where j is such that $d_j > 0$ but $d_{j-1} = 0$. Then we have the short exact sequence

$$0\to\mathcal{K}_{j-1}\to\mathcal{Q}_{j-1}\to\mathcal{Q}_j\to 0$$

on $\mathbb{P}^1\times HQuot$. Taking the total Chern class, we see that

$$c_1(\mathcal{K}_{j-1})\cdot c_1(\mathcal{Q}_j)+c_2(\mathcal{K}_{j-1})+c_2(\mathcal{Q}_j) = c_2(\mathcal{Q}_{j-1}).$$

Pushing forward and reducing, we have

$$\pi'_*(c_1(\mathcal{Q}_{j-1})\cdot c_1(\mathcal{Q}_j)) - \pi'_*c_1^2(\mathcal{Q}_j)+\pi'_*c_2(\mathcal{K}_{j-1})+\pi'_*c_2(\mathcal{Q}_j) = 0.$$

Notice that since $\mathcal{Q}_{j-1}$ is actually a vector bundle, we claim that $\mathcal{K}_{j-1}$ must also be a vector bundle. Given this for now, we see that if $r_j - r_{j-1} = 1$, then $\mathcal{K}_{j-1}$ is a line bundle, and so it has zero second Chern class. With our earlier calculations, we have that

$$d_j\,c_1(\mathcal{F}_{j-1}) - \pi'_*c_1^2(\mathcal{Q}_j)+\pi'_*c_2(\mathcal{Q}_j) = 0.$$

By our calculation of the rank earlier together with our work above, we can conclude

Proposition 6.1.1. *$Pic(\,HQuot_{\mathbb{P}^1}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q}$ is freely generated by the following classes:*

- *$c_1(\mathcal{F}_j)$ for all $1 \leq j \leq \ell$ such that $d_j = 0$*
- *$\pi'_* c_1^2(\mathcal{Q}_k)$ for all $1 \leq k \leq \ell$ such that $d_k > 0$*
- *$\pi'_* c_2(\mathcal{Q}_i)$ for all $1 \leq i \leq \ell$ such that $r_i - r_{i-1} > 1$ and $d_i > 0$*
- *$\pi'_* c_2(\mathcal{Q}_h)$ for all $1 < h \leq \ell$ such that $r_h - r_{h-1} = 1$ and $d_h, d_{h-1} > 0$.*

It suffices to prove the following lemma.

Lemma 6.1.0.1. *In the case that $d_{j-1} = 0$ above, $\mathcal{K}_{j-1}$ is locally free.*

Proof. Notice that when restricted to fibers over closed points, $\mathcal{K}_{j-1}|_{\mathbb{P}^1_h} \subset \mathcal{Q}_{j-1}|_{\mathbb{P}^1_h}$ is an inclusion of sheaves, and a subsheaf of a locally free sheaf on a smooth curve over a field is necessarily locally free. We use the Auslander Buchsbaum formula ([Eis95]), which tells us that if q is a closed point in the fiber $\mathbb{P}^1_h$, then

$$projdim_{\mathcal{O}_{\mathbb{P}^1_h},q}(\mathcal{K}_{j-1}|_{\mathbb{P}^1_h,q}) + depth(\mathcal{K}_{j-1}|_{\mathbb{P}^1_h,q}) = depth(\mathcal{O}_{\mathbb{P}^1_h,q}).$$

The first term is 0 since $\mathcal{K}_{j-1}$ is locally free when restricted to fibers over closed points. Since $HQuot$ is smooth ([Kim], [CF95]), passing from $\mathcal{O}_{\mathbb{P}^1 \times HQuot,q}$ to $\mathcal{O}_{\mathbb{P}^1_h,q}$ is given by taking the quotient by the extension of the maximal ideal of $\mathcal{O}_{HQuot,h}$ which is generated by a regular sequence (the sequence remains regular on $\mathcal{O}_{\mathbb{P}^1 \times HQuot,q}$ by flatness). Also, since $\mathcal{K}_{j-1,q}$ is flat over $\mathcal{O}_{HQuot,h}$, we see that the sequence is regular for $\mathcal{K}_{j-1,q}$. Thus,

- $depth(\mathcal{K}_{j-1}|_{\mathbb{P}^1_h,q}) = depth(\mathcal{K}_{j-1,q}) - dim(HQuot)$
- $depth(\mathcal{O}_{\mathbb{P}^1_h,q}) = depth(\mathcal{O}_{\mathbb{P}^1 \times HQuot,q}) - dim(HQuot)$.
- Using the Auslander Buchsbaum formula ([Eis95]) for $\mathcal{K}_{j-1,q}$, we see that $projdim(\mathcal{K}_{j-1,q}) = 0$.

Thus, $\mathcal{K}_{j-1}$ is locally free. $\square$

6.1.4 Generators and relations for $Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}$

As we saw above, $Pic(M_{0,m}\times HQuot)\otimes\mathbb{Q}\cong Pic(HQuot)\otimes\mathbb{Q}$. We have already seen that $\mathcal{Q}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d})$ is isomorphic to an open subscheme of $M_{0,m}\times HQuot$. From the exact sequence ([Ful98] *Proposition 1.8* page 21)

$$A_{q-1}(M_{0,m}\times HQuot)\otimes\mathbb{Q}\to A_{q-1}(\mathcal{Q}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}\to 0$$

together with the isomorphism above, we see that:

Lemma 6.1.4. *$Pic(\mathcal{Q}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}$ is generated by the following classes:*

- $\phi_j^{0\,*}c_1(\mathcal{F}_j)$ *for each* $1\le j\le\ell$ *such that* $d_j=0$
- $\pi_*^0 c_1^2(\mathcal{Q}_k^0)$ *for each* $1\le k\le\ell$ *such that* $d_k>0$
- $\pi_*^0 c_2(\mathcal{Q}_i^0)$ *for each* $1\le i\le\ell$ *such that* $r_i-r_{i-1}>1$ *and* $d_i>0$
- $\pi_*^0 c_2(\mathcal{Q}_h^0)$ *for each* $1<h\le\ell$ *such that* $r_h-r_{h-1}=1$ *and* $d_h,d_{h-1}>0$.

There are several items that need clarification.

Here, π^0 is the restriction of the universal curve to the interior of the moduli stack (the complement of the boundary divisors).

We will explain what is meant by ϕ_j^0 now. Since the degree $d_j=0$, the curve underlying each stable quasimap is contracted to a point in the j^{th} Grassmannian, $Gr(r_j,n)$. This yields a morphism ϕ_j from the moduli stack to $Gr(r_j,n)$. $\mathcal{F}_j$ is the universal quotient on this Grassmannian. Then, ϕ_j^0 is the restriction of this class to the interior of the moduli stack.

Now as in the Grassmannian case, we have the exact sequence ([Kre99])

$$A_{q-1}(\Delta)\otimes\mathbb{Q}\to Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}\to Pic(\mathcal{Q}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}\to 0$$

Since Δ is purely $q-1$ dimensional, the first term is the $\mathbb{Q}$ vector space whose basis elements are the classes of the irreducible components of Δ.

Thus, $Pic(\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))\otimes\mathbb{Q}$ is generated by:

- $\phi_j^*c_1(\mathcal{F}_j)$ for each $1\le j\le\ell$ such that $d_j=0$
- $\pi_*c_1^2(\mathcal{Q}_k)$ for each $1\le k\le\ell$ such that $d_k>0$
- $\pi_*c_2(\mathcal{Q}_i)$ for each $1\le i\le\ell$ such that $r_i-r_{i-1}>1$ and $d_i>0$
- $\pi_*c_2(\mathcal{Q}_h)$ for each $1<h\le\ell$ such that $r_h-r_{h-1}=1$ and $d_h,d_{h-1}>0$

- the irreducible components of the boundary.

We have already counted that there are $\frac{1}{2}\prod_{i=1}^{\ell}(d_i+1)(2^m-2)-m$ boundary divisors (there are exactly the same number of boundary divisors as there are in $\overline{M}_{0,m|\sum_{i=1}^{\ell} d_i}/\prod_{i=1}^{\ell}\mathcal{S}_{d_i}$). Just as in the Grassmannian case, there is a forgetful morphism :

$$\overline{\mathcal{Q}}_{0,m}(Fl(\overline{r},\mathbb{C}^n),\overline{d}) \to \overline{M}_{0,m}$$

along which we can pull back the $\binom{m-1}{2}-1$ relations among the boundary divisors [Kee92]. Counting dimensions, we see that the vector space spanned by the above classes with the relations pulled back from $Pic(\overline{M}_{0,m})\otimes\mathbb{Q}$ has dimension

$$\ell+|\{1\leq i\leq\ell\,|\,d_i>0,\, r_i-r_{i-1}>1\}|+|\{1<\gamma\leq\ell\,|\,d_\gamma,\,d_{\gamma-1},\,r_\gamma-r_{\gamma-1}=1\}|$$
$$+\frac{1}{2}\prod_{i=1}^{\ell}(d_i+1)(2^m-2)-m-\binom{m-1}{2}+1.$$

We already calculated that the Picard group has rank

$$\frac{1}{2}\prod_{i=1}^{\ell}(d_i+1)(2^m-2)-m-\frac{m^2-3m}{2}+|\{1\leq\gamma\leq\ell\,|\,r_\gamma-r_{\gamma-1}>1,\;d_\gamma>0\}|$$
$$+|\{\,1<\eta\leq\ell\,|\,r_\eta-r_{\eta-1}=1;\,d_\eta,\,d_{\eta-1}>0\,\}|+\ell.$$

Since $\binom{m-1}{2}-1=\frac{m^2-3m}{2}$, we see that this agrees with the dimension count we performed earlier using localization (Proposition 5.3.1), and so there are no other relations.

This concludes the proof of Theorem 0.5.2.

6.2 The Picard group when $m = 2$

Going forward we assume all $d_i>0$. We will use the same method from the Grassmannian case of intersecting with curves to calculate the Picard group when $m=2$.

6.2.1 Test curves III

In this section we construct the test curves to be used in the proof of Theorem 0.5.3.

The curves $\mathsf{A'}_{1,\overline{e}}$

Fix a tuple $\overline{e} := (e_1, \dots, e_\ell) \in \mathbb{N}^\ell_{\geq 0}$ such that $e_i \leq d_i$ for all $1 \leq i \leq \ell$, $\sum_{i=1}^{\ell} e_i > 0$ and $\sum_{i=1}^{\ell}(d_i - e_i) > 0$. $\mathsf{A'}_{1,\overline{e}}$ are constructed as follows:

- Consider the Hirzebruch surface $\rho : \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^1$ with two sections s_1, s_2 given by the subbundles $\mathcal{O}$, $\mathcal{O}(1)$ of $\mathcal{O}(1) \oplus \mathcal{O}$. These sections have numerical classes $\xi + f$, ξ, respectively, by [Har77] *Proposition V. 2.6.* Notice that $(\xi + f) \cdot \xi = 0$.

- Pick $\sum_{i=1}^{\ell} e_i$ smooth irreducible sections $\{\delta_{i,k_i}\}_{i=1,k_i=1}^{\ell,e_i}$ of $\mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{O}(1)$ such that $\bigcup_{i \mid e_i > 0} \{\delta_{i,k_i}\}_{k_i=1}^{e_i}$ vanish simultaneously at a single point on s_1, yet they have distinct tangent directions at this point.

- Next we pick $\sum_{q=1}^{\ell}(d_q - e_q)$ smooth irreducible sections $\{\sigma_{q,h_q}\}_{q=1,h_q=1}^{\ell,d_q-e_q}$ of $\mathcal{O}_{\mathbb{P}}(1) \otimes p^*\mathcal{O}(1)$ which do not have any pairwise common vanishing points on s_1, with each other or with the δ_{i,k_i}.

- Notice that none of the sections listed above vanish on s_2.

- We blow up the intersection points of the δ_{i,k_i}, σ_{q,h_q} with the first marked section.

- The strict transforms of the sections above are given by:
$$\overline{s}_1 = Bl^* s_1 - E_1 - \sum_{q \mid d_q - e_q > 0} \sum_{h_q=1}^{d_q-e_q} E_{q,h_q},\ \overline{s}_2 = Bl^* s_2,$$
$$\overline{\delta}_{i,k_i} = Bl^* \delta_{i,k_i} - E_1, \text{ and } \overline{\sigma}_{q,h_q} = Bl^* \sigma_{q,h_q} - E_{q,h_q}.$$

- The i^{th} inclusion in the flag sequence of sheaves is given by
$$\begin{array}{c} \mathbb{C}^{r_i - 1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{k_i=1}^{e_i} \overline{\delta}_{i,k_i} - \sum_{h_i=1}^{d_i-e_i} \overline{\sigma}_{i,h_i}) \\ \big\downarrow \\ \mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus \mathcal{O} \oplus \mathbb{C}^{r_{i+1}-r_i-1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{k_{i+1}=1}^{e_{i+1}} \overline{\delta}_{i+1,k_{i+1}} - \sum_{h_{i+1}=1}^{e_{i+1}} \overline{\sigma}_{i+1,h_{i+1}}) \end{array}$$
in which the nontrivial factor in the i^{th} subsheaf maps into the r_i^{th} factor of $\mathcal{O}$ in the $i+1^{st}$ subsheaf, and the map is the identity elsewhere.

The next collection of curves are a slight modification of $\mathsf{A'}_{1,\overline{e}}$.

The curves $\mathsf{A}'^{\gamma}_{1,\overline{e}}$

Fix an index $1 \leq \gamma \leq \ell$. We define the curves $\mathsf{A}'^{\gamma}_{1,\overline{e}}$ as follows:

- We retain the same sections from $\mathsf{A}'_{1,\overline{e}}$.
- The only change from $\mathsf{A}'_{1,\overline{e}}$, is in the γ^{th} subsheaf, which is given by
$$\mathbb{C}^{r_\gamma - 2} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{k_\gamma=1}^{e_\gamma} \overline{\delta}_{\gamma,k_\gamma}) \oplus \mathcal{O}(-\sum_{h_\gamma=1}^{d_\gamma - e_\gamma} \overline{\sigma}_{\gamma,h_\gamma})$$
where the inclusion into the $\gamma + 1^{st}$ subsheaf is given by the sections
$$\bigotimes_{k_\gamma=1}^{e_\gamma} \overline{\delta}_{\gamma,k_\gamma} \oplus \bigotimes_{t_\gamma=1}^{d_\gamma - e_\gamma} \overline{\sigma}_{\gamma,t_\gamma}$$
on the $r_\gamma - 1^{st}$ and the r_γ^{th} factor, and the identity elsewhere.

We produce two collections of curves for when $d_j = 1$.

The curves B_j

We next construct curves B_j for each j such that $d_j = 1$.

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the two trivial sections s_1, s_2 at 0 and ∞, respectively.
- For each $i \neq j$, pick d_i trivial sections of p_2, σ_{i,k_i}, which are disjoint both from each other and from s_1, s_2.
- Pick 2 smooth irreducible sections δ_1, δ_2 of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ without common vanishing points on s_1, s_2.
- The i^{th} subsheaf is given by
$$\mathbb{C}^{r_i - 1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{k_i=1}^{d_i} \sigma_{i,k_i})$$
such that the last factor maps into the r_i^{th} copy of $\mathcal{O}$ in the $i + 1^{st}$ subsheaf, and the map is the identity on the first $r_i - 1$ factors.
- The j^{th} subsheaf is given by
$$\mathbb{C}^{r_j - 1} \otimes \mathcal{O} \oplus \mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$$
where the inclusion into the $j + 1^{st}$ subsheaf is given by $\bigoplus_{u=1}^{2} \delta_u$ on the last factor, and the identity on the first $r_j - 1$ factors .

The curves B'_j

The next family will be very similar. The only difference lies in the j^{th} subsheaf.

- Pick one trivial section σ of p_2 (distinct from s_1, s_2), and choose two disjoint sections f_1, f_2 of $p_2^*\mathcal{O}(1)$.
- Everything stays the same as in B_j, except now the j^{th} subsheaf is given by
$$\mathbb{C}^{r_j-2}\otimes\mathcal{O}\oplus\mathcal{O}(-\sigma)\oplus p_2^*\mathcal{O}(-1)$$
where the inclusion into the $j+1^{st}$ subsheaf is given by σ on the r_j-1^{st} factor, $\bigoplus_{u=1}^{2} f_u$ on the r_j^{th} factor, and the identity elsewhere.

The curves C_j

Fix an index $1\le j\le \ell$ such that $d_j>1$. The curves C_j are constructed as follows:

- Start with $p_2:\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$.
- Fix the two marked sections s_1, s_2 to be at 0 and ∞, respectively.
- For each $i\ne j$, pick d_i trivial sections of p_2 which are distinct from both each other and s_1, s_2; call them σ_{i,h_i}.
- Pick $2(d_j-1)$ trivial distinct sections of p_2 (distinct from σ_{i,h_i} for $i\ne j$ as well as s_1, s_2); call them $\sigma_{j,h_j,1}$ and $\sigma_{j,h_j,2}$.
- Choose two smooth irreducible sections of $\mathcal{O}(1)\boxtimes\mathcal{O}(1)$ which do not vanish simultaneously on the marked sections; these will be denoted δ_1,δ_2.
- The i^{th} subsheaf is given by
$$\mathbb{C}^{r_i-1}\otimes\mathcal{O}\oplus\mathcal{O}(-\sum_{h_i=1}^{d_i}\sigma_{i,h_i})$$
where the inclusion into the $i+1^{st}$ subsheaf is given by the sections
$$\bigotimes_{h_i=1}^{d_i}\sigma_{i,h_i}$$
on the r_i^{th} factor, and the identity elsewhere.
- The j^{th} subsheaf is given by
$$\mathbb{C}^{r_j-1}\otimes\mathcal{O}\oplus\bigotimes_{h_j=1}^{d_j-1}p_1^*\mathcal{O}(-1)\otimes(\mathcal{O}(-1)\boxtimes\mathcal{O}(-1))$$

where the inclusion into the $j+1^{st}$ subsheaf is given by the sections
$$\bigoplus_{k=1}^{2}(\bigotimes_{h_j=1}^{d_j-1}\sigma_{j,h_j,k})\otimes\delta_k$$
on the r_j^{th} factor, and the identity elsewhere.

The curves D_j

Fix an index $1 \le j \le \ell$. Define a new curve D_j as follows:

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the two marked sections s_1, s_2 the trivial sections at 0 and ∞, respectively.
- Pick $2d_j$ smooth irreducible sections $\{\delta_{j,h_j,u}\}_{h_j=1,u=1}^{d_j,2}$ of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ without any pairwise common vanishing points on the two marked sections.
- Pick $\sum_{i\neq j} d_i$ trivial sections of p_2, $\bigcup_{i\neq j}\{\sigma_{i,k_i}\}_{k_i=1}^{d_i}$, which are distinct from each other and from s_1, s_2.
- The j^{th} subsheaf is given by
$$\mathbb{C}^{r_j-1}\otimes\mathcal{O}\oplus\bigotimes_{h_j=1}^{d_j}\mathcal{O}(-1)\boxtimes\mathcal{O}(-1)$$
where the inclusion into the $j+1^{st}$ subsheaf is given $\bigoplus_{u=1}^{2}\bigotimes_{h_j=1}^{d_j}\delta_{j,h_j,u}$ on the r_j^{th} factor and the identity elsewhere.
- For $i \neq j$, the i^{th} subsheaf is given by
$$\mathbb{C}^{r_i-1}\otimes\mathcal{O}\oplus\mathcal{O}(-\sum_{k_i=1}^{d_i}\sigma_{i,k_i})$$
where the inclusion into the $i+1^{st}$ subsheaf is given by $\bigotimes_{k_i=1}^{d_i}\sigma_{i,k_i}$ on the r_i^{th} factor and the identity elsewhere.

The curves F_j

Fix an index $1 \le j \le \ell$. Call the next curve F_j, which is constructed as follows:

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the two marked sections s_1, s_2 being the trivial sections at 0 and ∞, respectively.
- Pick $\sum_{i\neq j} d_i$ trivial sections of p_2, $\bigcup_{i\neq j}\{\sigma_{i,h_i}\}_{h_i=1}^{d_i}$, which are distinct from each other and from s_1, s_2.

- Pick $d_j - 1$ trivial sections of p_2, $\{\sigma_{j,h_j}\}_{h_j=1}^{d_j-1}$, which are distinct from s_1, s_2, and σ_{i,h_i} for each i and each $1 \leq h_i \leq d_i$.

- Choose a smooth irreducible section δ of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$.

- Blow up the intersection points $\delta \bigcap s_1$, $\delta \bigcap s_2$.

- Let the i^{th} $(i \neq j)$ subsheaf be given by
$$\mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{h_i=1}^{d_i} \overline{\sigma}_{i,h_i})$$
where the inclusion is given by the strict transforms $\bigotimes_{h_i=1}^{d_i} \overline{\sigma}_{i,h_i}$ on the r_i^{th} factor and the identity elsewhere.

- Let the j^{th} subsheaf be given by
$$\mathbb{C}^{r_j-1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{h_j=1}^{d_j-1} \overline{\sigma}_{j,h_j} - \overline{\delta}),$$
where the inclusion is given by the strict transforms $(\bigotimes_{h_j=1}^{d_j-1} \overline{\sigma}_{j_{h_j}}) \otimes \overline{\delta}$ on the r_j^{th} factor and the identity elsewhere.

The curves G_j

Fix an index $1 \leq j \leq \ell$. We define a new curve which is similar to both D_j and F_j. Call this curve G_j.

- Start with $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ with the two marked sections s_1, s_2 being the trivial sections at 0 and ∞.

- Pick $\sum_{i \neq j} 2d_i$ smooth irreducible sections $\bigcup_{i \neq j} \{\delta_{i,h_i,1}, \delta_{i,h_i,2}\}_{h_i=1}^{d_i}$ of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ without any pairwise common vanishing points on the two marked sections.

- Pick $d_j - 1$ trivial sections $\{\sigma_{j,h_j}\}_{h_j=1}^{d_j-1}$ of p_2, distinct from each other and from s_1, s_2.

- Pick a smooth irreducible section δ_j of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$.

- Blow up the two intersection points $\delta \bigcap s_1$, $\delta \bigcap s_2$.

- Let the i^{th} subsheaf (for $i \neq j$) be given by
$$\mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus \bigotimes_{h=1}^{d_i} Bl^*(\mathcal{O}(-1) \boxtimes \mathcal{O}(-1))$$

where the inclusion into the $i+1^{st}$ subsheaf is given by the strict transforms $\bigoplus_{u=1}^{2} \bigotimes_{h_i=1}^{d_i} \overline{\delta}_{i,h_i,u}$ on the r_i^{th} factor, and the identity elsewhere.

- Let the j^{th} subsheaf be given by
$$\mathbb{C}^{r_j-1} \otimes \mathcal{O} \oplus \mathcal{O}(-\sum_{h_j=1}^{d_j-1} \overline{\sigma}_{j,h_j} - \overline{\delta}_j),$$
where the inclusion into the $j+1^{st}$ subsheaf is given by the strict transforms $(\bigotimes_{h_j=1}^{d_j-1} \overline{\sigma}_{j,h_j}) \otimes \overline{\delta}_j$ on the r_j^{th} factor, and the identity elsewhere.

The curves H_j

Fix an index $1 \le j \le \ell$. Construct the curve H_j as follows:

- Start with $\rho : \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^1$ with the two marked sections s_1, s_2 given by the subbundles $\mathcal{O}$, $\mathcal{O}(1)$ of $\mathcal{O}(1) \oplus \mathcal{O}$. These have numerical classes $\xi + f$ and ξ, respectively by [Har77] *Proposition V. 2.6.*

- Pick $(\sum_{i \neq j} 2d_i) + 2d_j - 2$ smooth irreducible sections
$$\bigcup_{i \neq j} \{\delta_{i,h_i,1}, \delta_{i,h_i,2}\}_{h_i=1}^{d_i} \bigcup \{\delta_{j,h_j,1}, \delta_{j,h_j,2}\}_{h_j=1}^{d_j-1}$$
of $\mathcal{O}_{\mathbb{P}}(1) \otimes \rho^*\mathcal{O}(1)$ without common vanishing points on s_1 (they do not intersect s_2), and pick 2 smooth irreducible sections σ_1, σ_2 of $\mathcal{O}_{\mathbb{P}}(1) \otimes \rho^*\mathcal{O}(1)$ with a shared vanishing point on s_1 such that σ_1, σ_2 have distinct tangent directions at this point.

- Blow up the intersection point $\sigma_1 \bigcap \sigma_2$ on s_1.

- For $i \neq j$, the i^{th} subsheaf is given by
$$\mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus \bigotimes_{h_i=1}^{d_i} Bl^*(\mathcal{O}_{\mathbb{P}}(-1) \otimes \rho^*\mathcal{O}(-1))$$
where the inclusion into the $i+1^{st}$ subsheaf is given by the strict transforms $\bigoplus_{u=1}^{2} \bigotimes_{h_i=1}^{d_i} \overline{\delta}_{i,h_i,u}$ on the r_i^{th} factor and the identity elsewhere.

- The j^{th} subsheaf is given by
$$\mathbb{C}^{r_j-1} \otimes \mathcal{O} \oplus \bigotimes_{h_j=1}^{d_j-1} Bl^*(\mathcal{O}_{\mathbb{P}}(-1) \otimes \rho^*\mathcal{O}(-1)) \otimes Bl^*(\mathcal{O}_{\mathbb{P}}(-1) \otimes \rho^*\mathcal{O}(-1)) \otimes \mathcal{O}(E)$$
where the inclusion into the $j+1^{st}$ subsheaf is given by the sections
$$\bigoplus_{u=1}^{2} (\bigotimes_{h_j=1}^{d_j-1} \overline{\delta}_{j,h_j,u} \otimes \overline{\sigma}_u)$$
on the r_j^{th} factor and the identity elsewhere.

The curve I

The last curve I is constructed as follows:

- Start with the vector bundle $\mathcal{O}(v) \oplus \mathcal{O}(w) \to \mathbb{P}^1$, where $v \neq w$, $v, w > 0$.
- Consider the projective bundle $\rho : \mathbb{P}(\mathcal{O}(v) \oplus \mathcal{O}(w)) \to \mathbb{P}^1$ with two sections s_1, s_2, given by the subbundles $\mathcal{O}(w)$, $\mathcal{O}(v)$ of $\mathcal{O}(v) \oplus \mathcal{O}(w)$. These have numerical classes $\xi + vf$, $\xi + wf$, respectively, by [Har77] *Proposition V. 2.6.*
- Fix $h > 0$ such that $-vd_i + h$, $-wd_i + h > 0$ and $H^0(\mathcal{O}_{\mathbb{P}}(d_i) \otimes \rho^*\mathcal{O}(h)) >> 0$, for all $1 \le i \le \ell$.
- For each $1 \le i \le \ell$, we pick two sections $\gamma_{i,1}$, $\gamma_{i,2}$ of $\mathcal{O}_{\mathbb{P}}(d_i) \otimes \rho^*\mathcal{O}(h)$ which do not have any common vanishing points on the two marked sections.
- The i^{th} subsheaf is given by
$$\mathbb{C}^{r_i-1} \otimes \mathcal{O} \oplus (\mathcal{O}_{\mathbb{P}}(-d_i) \otimes \rho^*\mathcal{O}(-h))$$
where the inclusion into the $i{+}1^{st}$ subsheaf is given by $\bigoplus_{u=1}^{2} \gamma_{i,u}$ on the r_i^{th} factor, and the identity elsewhere.

One can check that all of the families above satisfy the stability condition.

6.2.2 The Picard group when $r_i - r_{i-1} > 1$ for all $1 \le i \le \ell + 1$

We begin by showing that the boundary divisors are linearly independent.

Lemma 6.2.1. *The irreducible components of the boundary divisors are linearly independent in* $Pic(\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r}, \mathbb{C}^n), \overline{d})) \otimes \mathbb{Q}$.

Proof. Suppose we had a relation $\sum c_{1,\overline{k}} \Delta_{1,\overline{k}} = 0$. Intersecting this relation with $\mathsf{A}'_{1,\overline{e}}$ we find the relation on the coefficients
$$c_{1,\overline{e}} + \sum_{i=1}^{\ell} (d_i - e_i) c_{1,(0,\dots,1,\dots,0)} = 0$$
where the 1 is in the i^{th} position in $(0, \dots, 1, \dots, 0)$ in the i^{th} term of the sum. Intersecting with H_i we see that $c_{1,(0,\dots,1,\dots,0)} = 0$ for each $1 \le i \le \ell$. Together with the relation above, we see that all $c_{1,\overline{e}} = 0$.

□

Recall that (Proposition 5.3.1)

$$rank(Pic(\overline{\mathcal{Q}}_{0,2}(Fl(\overline{r},\mathbb{C}^n),\overline{d}))) = \prod_{i=1}^{\ell}(d_i+1) + 2\ell - 1$$

when all $d_i > 0$.

There are $\prod_{i=1}^{\ell}(d_i+1) - 2$ boundary divisors. We claim that the 2ℓ classes are coming from $\pi_* c_1^2(\mathcal{Q}_i)$, $\pi_* c_2(\mathcal{Q}_i)$ for $1 \le i \le \ell$. We are off by a one dimensional subspace. Based on what happened for the Grassmannian, we can add the two classes $ev_1^* c_1(\mathcal{F}_k)$, $ev_2^* c_1(\mathcal{F}_k)$ for some k. We will expect a relation among these generators, which we will show excludes $\pi_* c_2(\mathcal{Q}_i)$ for every $1 \le i \le \ell$.

Lemma 6.2.2. *The collection*

$$\{\Delta_{1,\overline{e}}\} \bigcup \{\pi_* c_1^2(\mathcal{Q}_j)\}_{j=1}^{\ell} \bigcup \{\pi_* c_2(\mathcal{Q}_k)\}_{k=1}^{\ell}$$

is linearly independent.

Proof. Suppose we had a relation

$$\sum_{i=1}^{\ell} c_{\alpha,i}\pi_* c_1^2(\mathcal{Q}_i) + \sum_{j=1}^{\ell} c_{\beta,j}\pi_* c_2(\mathcal{Q}_j) + \sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} = 0.$$

Intersecting with $\mathsf{A'}_{1,\overline{e}}$ yields the relation on the coefficients

$$\sum_{i=1}^{\ell}(d_i^2 - e_i^2 - d_i + e_i)\cdot(c_{\alpha,i} + c_{\beta,i}) + c_{1,\overline{e}} + \sum_{q=1}^{\ell}(d_q - e_q)c_{1,(0,\ldots,0,1,0,\ldots,0)} = 0$$

where the 1 is in the q^{th} spot in $(0,\ldots,1,\ldots,0)$ in the q^{th} term of the sum.

Notice that the only difference between the intersection of the relation with $\mathsf{A'}^{\gamma}_{1,\overline{e}}$ and $\mathsf{A'}_{1,\overline{e}}$, is in the $\pi_* c_2(\mathcal{Q}_\gamma)$ term, and an explicit Chern class calculation yields

$$e_\gamma(d_\gamma - e_\gamma)c_{\beta,\gamma} = 0 \implies c_{\beta,\gamma} = 0$$

for all γ such that $d_\gamma > 1$.

When $d_j = 1$, we use the curves B_j. Intersecting with B_j, we find the relation on the coefficients

$$2c_{\alpha,j} + 2c_{\beta,j} = 0.$$

Intersecting with B'_j yields

$$2c_{\alpha,j} + c_{\beta,j} = 0.$$

Combining this with the result we found for B_j, we see that $c_{\beta,j} = c_{\alpha,j} = 0$ in this case as well.

Thus, the relation cannot involve $\pi_* c_2(\mathcal{Q}_\gamma)$ for each $1 \le \gamma \le \ell$.

Intersecting with C_j (for each $1 \le j \le \ell$) yields the relation

$$2d_j c_{\alpha,j} = 0.$$

Since linear independence of the boundary divisors has already been established, the result follows. □

Next we prove a result involving the evaluation classes.

Proposition 6.2.1. *Fix $j \in \{1, \dots, \ell\}$. Then the collection*

$$ev_1^* c_1(\mathcal{F}_j) \cup ev_2^* c_1(\mathcal{F}_j) \cup \{\Delta_{1,(e_1,\dots,e_\ell)}\} \cup \{\pi_* c_1^2(\mathcal{Q}_i)\}_{i \neq j} \cup \{\pi_* c_2(\mathcal{Q}_k)\}_{k=1}^{\ell}$$

is linearly independent.

Notice that the subspace spanned by the above classes has dimension equal to the rank of the Picard group from our previous calculations (Proposition 5.3.1), so for each j, the above classes form a basis.

Proof. Suppose we had a relation

$$\sum_{\overline{e}} c_{1,\overline{e}} \Delta_{1,\overline{e}} + c_1 ev_1^* c_1(\mathcal{F}_j) + c_2 ev_2^* c_1(\mathcal{F}_j) + \sum_{i \neq j} c_{\alpha,i} \pi_* c_1^2(\mathcal{Q}_i) + \sum_{k=1}^{\ell} c_{\beta,k} \pi_* c_2(\mathcal{Q}_k) = 0.$$

Since we are assuming $r_h - r_{h-1} > 1 \ \forall\, 1 \leq h \leq \ell$, we can use the same curves we used above ($\mathsf{A}'_{1,\overline{e}}$, $\mathsf{A}'^{\gamma}_{1,\overline{e}}$, B_j, and B'_j) to show that all $c_{\beta,k} = 0$.

Therefore, the relation has the form

$$\sum_{\overline{e}} c_{1,\overline{e}} \Delta_{1,\overline{e}} + c_1 ev_1^* c_1(\mathcal{F}_j) + c_2 ev_2^* c_1(\mathcal{F}_j) + \sum_{i \neq j} c_{\alpha,i} \pi_* c_1^2(\mathcal{Q}_i) = 0.$$

Notice that it suffices to show that $c_1 = c_2 = 0$, since our lemma above showed that the remaining classes are linearly independent.

We make use of the curve D_j. Intersecting with D_j, we find that

$$c_1 + c_2 = 0.$$

Intersecting with F_j, we find

$$c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = 0,$$

where the 1 is in the j^{th} position in $(0, \dots, 1, \dots, 0)$.

Intersecting with G_j, we find that

$$\sum_{i \neq j} 2d_i^2 c_{\alpha,i} + c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = 0,$$

where the 1 is in the j^{th} position in $(0, \dots, 1, \dots, 0)$. Putting this together with the relation we found on the coefficients above,

$$c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = 0,$$

we see that $\sum_{i \neq j} d_i^2 c_{\alpha,i} = 0$. By considering the intersection with I, we find the relation on the coefficients

$$(-wd_j + h)c_1 + (-vd_j + h)c_2 + \sum_{i \neq j} (d_i^2(-v - w) + 2hd_i) c_{\alpha,i} = 0.$$

Using the fact that $c_1 = -c_2$, and the relations on the coefficients we found above, we see that this relation on the coefficients becomes

$$(-wd_j + vd_j)c_1 + \sum_{i \neq j} 2hd_i c_{\alpha,i} = 0.$$

Regarding this as a polynomial in w, v, h, we see that $c_1 = 0$, which forces $c_2 = 0$. □

By the preceding proposition, we expect that we can express $\pi_* c_1^2(\mathcal{Q}_j)$ in terms of this basis.

Lemma 6.2.3. *For each $1 \leq j \leq \ell$, there is a relation*

$$\sum_{\overline{e}} e_j(d_j - e_j)\Delta_{1,\overline{e}} + d_j(ev_1^* c_1(\mathcal{F}_j) + ev_2^* c_1(\mathcal{F}_j)) = \pi_* c_1^2(\mathcal{Q}_j).$$

Proof. We expect a relation

$$\sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} + c_1 ev_1^* c_1(\mathcal{F}_j) + c_2 ev_2^* c_1(\mathcal{F}_j) + \sum_{i=1}^{\ell} c_{\alpha,i}\pi_* c_1^2(\mathcal{Q}_i) + \sum_{k=1}^{\ell} c_{\beta,k}\pi_* c_2(\mathcal{Q}_k) = 0$$

for each $1 \leq j \leq \ell$.

Just as before, $c_{\beta,k} = 0$ for each $1 \leq k \leq \ell$. Therefore, the relation takes the form

$$\sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} + c_1 ev_1^* c_1(\mathcal{F}_j) + c_2 ev_2^* c_1(\mathcal{F}_j) + \sum_{i=1}^{\ell} c_{\alpha,i}\pi_* c_1^2(\mathcal{Q}_i) = 0.$$

Intersecting with D_j, we find the relation on the coefficients

$$2d_j^2 c_{\alpha,j} + d_j(c_1 + c_2) = 0.$$

Intersecting with D_k, we see that $c_{\alpha,k} = 0$, for each $k \neq j$. Thus, our relation has the form

$$\sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} + c_1 ev_1^* c_1(\mathcal{F}_j) + c_2 ev_2^* c_1(\mathcal{F}_j) + c_{\alpha,j}\pi_* c_1^2(\mathcal{Q}_j) = 0.$$

Next, we intersect with F_j. This yields the relation on the coefficients

$$2(d_j - 1)c_{\alpha,j} + c_{1,(0,\ldots,1,\ldots,0)} + c_{1,(d_1,\ldots,d_j-1,\ldots,d_\ell)} = 0.$$

Intersecting with F_k $(k \neq j)$, we find (recall $c_{\alpha,k} = 0$ for $k \neq j$)

$$c_{1,(0,\ldots,1,\ldots,0)} + c_{1,(d_1,\ldots,d_k-1,\ldots,d_\ell)} = 0.$$

Intersecting with the curve I, we find the relation on the coefficients

$$(-wd_j + h)c_1 + (-vd_j + h)c_2 + (d_j^2(-v - w) + 2hd_j)c_{\alpha,j} = 0.$$

Regarding this as a polynomial in w, v, h, we see that

$$c_2 = c_1 = -d_j c_{\alpha,j}.$$

Thus, our relation has the form

$$\sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} - d_j c_{\alpha,j}(ev_1^* c_1(\mathcal{F}_j) + ev_2^* c_1(\mathcal{F}_j)) + c_{\alpha,j}\pi_* c_1^2(\mathcal{Q}_j) = 0.$$

Intersecting with H_j, we find

$$c_{1,(0,\ldots,1,\ldots,0)} = -(d_j - 1)c_{\alpha,j},$$

where the 1 is in the j^{th} position in $(0,\dots,1,\dots,0)$. Since

$$2(d_j - 1)c_{\alpha,j} + c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = 0,$$

as we saw from intersecting our relation with F_j, we see that

$$c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = -(d_j - 1)c_{\alpha,j}.$$

Intersecting with H_k, we have

$$c_{1,(0,\dots,1,\dots,0)} = 0$$

where the 1 is in the k^{th} position in $(0,\dots,1,\dots,0)$, for each $k \neq j$. Intersecting with the curves $\mathsf{A'}_{1,\overline{e}}$, we see that we have the relation on the coefficients

$$c_{1,\overline{e}} + (d_j - e_j)c_{1,(0,\dots,1,\dots,0)} + (d_j - e_j)\cdot(d_j + e_j - 1)c_{\alpha,j} = 0.$$

where the 1 is in the j^{th} position in $(0,\dots,1,\dots,0)$. Since

$$c_{1,(0,\dots,1,\dots,0)} = -(d_j - 1)c_{\alpha,j},$$

we see that

$$c_{1,\overline{e}} = -e_j(d_j - e_j)c_{\alpha,j}.$$

This completes the proof of the lemma.

□

By Proposition 6.2.1, we see that there must be some relations among the evaluation classes. We seek to prove

Lemma 6.2.4. *There is a relation*

$$-\tfrac{1}{d_j}ev_1^*c_1(\mathcal{F}_j) + \tfrac{1}{d_k}ev_1^*c_1(\mathcal{F}_k) + \tfrac{1}{d_j}ev_2^*c_1(\mathcal{F}_j) - \tfrac{1}{d_k}ev_2^*c_1(\mathcal{F}_k)$$
$$+\sum_{\overline{e}}\left(\tfrac{e_k}{d_k} - \tfrac{e_j}{d_j}\right)\Delta_{1,\overline{e}} = 0$$

for each $j \neq k$.

Proof. One way to do this is to express the two ψ classes in terms of the various bases we have found.

We expect a relation

$$c_\gamma\psi_1 + \sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} + c_1 ev_1^*c_1(\mathcal{F}_j) + c_2 ev_2^*c_1(\mathcal{F}_j) + \sum_{i\neq j} c_{\alpha,i}\pi_*c_1^2(\mathcal{Q}_i) + \sum_{k=1}^{\ell} c_{\beta,k}\pi_*c_2(\mathcal{Q}_k) = 0.$$

Again, the same method as before shows that all $c_{\beta,k} = 0$. Intersecting with the curve D_j, we see that we have the relation on the coefficients $c_1 + c_2 = 0$. Intersecting with the curves D_k, we see that $c_{\alpha,k} = 0$ for $k \neq j$. Therefore, the relation has the form

$$c_\gamma\psi_1 + \sum_{\overline{e}} c_{1,\overline{e}}\Delta_{1,\overline{e}} + c_1 ev_1^*c_1(\mathcal{F}_j) - c_1 ev_2^*c_1(\mathcal{F}_j) = 0.$$

Intersecting with the curve F_j, we see that we have the relation on the coefficients

$$-c_\gamma + c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_j-1,\dots,d_\ell)} = 0$$

where the 1 is in the j^{th} position in $(0,\dots,1,\dots,0)$. Similarly, intersecting our relation with the curves F_k, we find that

$$-c_\gamma + c_{1,(0,\dots,1,\dots,0)} + c_{1,(d_1,\dots,d_k-1,\dots,d_\ell)} = 0$$

where the 1 is in the k^{th} position in $(0,\dots,1,\dots,0)$, for each $k \neq j$. Next we intersect with I to find the relation on the coefficients

$$\begin{aligned}(-wd_j + h)c_1 + (-vd_j + h)c_2 + (\xi + vf)^2 c_\gamma = 0 &\implies\\ (-wd_j + h + vd_j - h)c_1 + (-v - w + 2v)c_\gamma = 0 &\implies\\ c_1 = \tfrac{-1}{d_j}c_\gamma.&\end{aligned}$$

Intersecting with H_j, we find the relation on the coefficients

$$\begin{aligned}c_{1,(0,\dots,1,\dots,0)} + (d_j - 1)c_1 + (Bl^*(\xi + f) - E)^2 c_\gamma = 0 &\implies\\ c_{1,(0,\dots,1,\dots,0)} + (d_j - 1)c_1 = 0 &\implies\\ c_{1,(0,\dots,1,\dots,0)} = \tfrac{d_j-1}{d_j}c_\gamma,&\end{aligned}$$

where the 1 is in the j^{th} position in $(0,\dots,1,\dots,0)$. Using what we found above, this implies $c_{1,(d_1,\dots,d_j-1.\dots,d_\ell)} = \frac{1}{d_j}c_\gamma$. Intersecting with H_k, for each $k \neq j$, we find the relation on the coefficients

$$\begin{aligned}d_j c_1 + c_{1,(0,\dots,1,\dots,0)} + (Bl^*(\xi + f) - E)^2 c_\gamma = 0 &\implies\\ d_j c_1 + c_{1,(0,\dots,1,\dots,0)} = 0 \implies c_{1,(0,\dots,1,\dots,0)} = c_\gamma,&\end{aligned}$$

where the 1 is in the k^{th} position in $(0,\dots,1,\dots,0)$, for any $k \neq j$. Intersecting with $\mathsf{A'}_{1,\overline{e}}$, we find

$$\begin{aligned}c_{1,\overline{e}} + \sum_{k\neq j}(d_k - e_k)c_\gamma + (d_j - e_j)\tfrac{d_j-1}{d_j}c_\gamma - \sum_{k=1}^{\ell}(d_k - e_k)c_\gamma = 0&\\ \implies c_{1,\overline{e}} = \tfrac{d_j-e_j}{d_j}c_\gamma.&\end{aligned}$$

Thus, our relation has the form

$$\psi_1 + \sum_{\overline{e}} \tfrac{d_j-e_j}{d_j}\Delta_{1,\overline{e}} - \tfrac{1}{d_j}ev_1^*c_1(\mathcal{F}_j) + \tfrac{1}{d_j}ev_2^*c_1(\mathcal{F}_j) = 0.$$

Comparing the relation above for two indices j, k and taking the difference yields the lemma. □

Only $\ell - 1$ of these equations are linearly independent; an example of a linearly independent collection is obtained by letting $j = 1$ and letting $2 \leq k \leq \ell$. By our rank calculations (Proposition 5.3.1), we have:

- $2\prod_{i=1}^{\ell}(d_i + 1) - 2$ boundary divisors
- ℓ of the $\pi_* c_1^2(\mathcal{Q}_i)$ classes

- ℓ of the $\pi_* c_2(\mathcal{Q}_i)$ classes
- 2ℓ of the $ev_k^* c_1(\mathcal{F}_i)$ classes $(k = 1, 2)$
- ℓ independent relations
$$\sum_{\bar{e}} e_j(d_j - e_j)\Delta_{1,\bar{e}} + d_j(ev_1^* c_1(\mathcal{F}_j) + ev_2^* c_1(\mathcal{F}_j)) = \pi_* c_1^2(\mathcal{Q}_j)$$
- and $\ell - 1$ independent relations
$$-\tfrac{1}{d_j} ev_1^* c_1(\mathcal{F}_j) + \tfrac{1}{d_k} ev_1^* c_1(\mathcal{F}_k) + \tfrac{1}{d_j} ev_2^* c_1(\mathcal{F}_j) - \tfrac{1}{d_k} ev_2^* c_1(\mathcal{F}_k) + \sum_{\bar{e}} (\tfrac{e_k}{d_k} - \tfrac{e_j}{d_j})\Delta_{1,\bar{e}} = 0.$$

Subtracting the number of independent relations from the cardinality of the collection of generators yields
$$2(\prod_{i=1}^{\ell}(d_i + 1)) + 2\ell - 1$$
which agrees with the rank of the Picard group (Proposition 5.3.1) we found earlier. This proves Theorem 0.5.3.

A Appendix

A.1 GIT construction of the flag variety

We carry out the GIT construction of $V/\!/G$ and show that the lineariztion yields the correct line bundle on the flag variety.

Recall our situation

$$V \cong \bigoplus_{i=1}^{\ell} \mathrm{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r_{i+1}})$$
$$G \cong \prod_{i=1}^{\ell} GL(r_i, \mathbb{C}).$$

G acts on V as

$$(g_1, \ldots, g_\ell) \cdot (A_1, \ldots, A_\ell) := (g_2 \circ A_1 \circ g_1^{-1}, \ldots, A_\ell \circ g_\ell^{-1}).$$

Notice that G embeds in $GL(V)$, and the image contains $\mathbb{C}^* \cdot I_V$. The preimage of $t \cdot I_V$ is

$$(t^{-\ell} \cdot I_{r_1}, \ldots, t^{-1} \cdot I_{r_\ell}) \in G.$$

We would like to take the GIT quotient $V/\!/G$ with the linearization coming from the trivial line bundle on V endowed with the nontrivial representation $\prod_{i=1}^{\ell} det_i$.

We will first explain why the linearization is correct.

Notice that we have G-equivariant bundles $\mathbb{C}^{r_i}_{GL(r_i,C)} \times V$ on V where the action on V comes from the embedding of G in $GL(V)$, and the action on $\mathbb{C}^{r_i}$ is given by considering the projection $G \to GL(r_i, \mathbb{C})$ and then using the right action of left multiplication by g^{-1}, where $g \in GL(r_i, \mathbb{C})$.

Since $\mathbb{C}^*$ acts freely on $V \backslash \{0\}$, we can start by taking the GIT quotient of V by $\mathbb{C}^*$ with the linearization given by $\mathbb{C}_{\prod_{i=1}^{\ell} det_i} \times V$. The bundle $\mathbb{C}^{r_i}_{GL(r_i,C)} \times V$ is also C^* equivariant, with $t \in \mathbb{C}^*$ acting as $t^{\ell+1-i} \cdot I_{r_i}$ on $\mathbb{C}^{r_i}$. Taking the determinant of this bundle yields $\mathbb{C}_{det_i^{-1}} \times V$, which is also $\mathbb{C}^*$ equivariant, and t acts on $\mathbb{C}$ as $t^{r_i(\ell+1-i)}$. The bundle $\mathbb{C}^{r_i}_{GL(r_i,C)} \times V$ yields a $SL(V) \bigcap G$-equivariant bundle on $\mathbb{P}(V)$:

$$\mathbb{C}^{r_i}_{GL(r_i,\mathbb{C})} \times_{\mathbb{C}^*} V \setminus \{0\} \cong \mathbb{C}^{r_i} \otimes \mathcal{O}(-\ell - 1 + i)$$

from our above description. Thus, we see that

$$\mathbb{C}_{\prod_{i=1}^{\ell} det_i} \times V \cong \bigotimes_{i=1}^{\ell} det((\mathbb{C}^{r_i}_{GL(r_i,\mathbb{C})} \times V)^*),$$

and so the induced line bundle on $\mathbb{P}(V)$ is $\bigotimes_{i=1}^{\ell} \mathcal{O}(r_i(\ell + 1 - i))$.

Once we see that the flag variety is the GIT quotient $\mathbb{P}(V)//SL(V) \bigcap G$ and that $\mathbb{C}^{r_i} \otimes \mathcal{O}(-\ell - 1 + i)$ induces $\mathcal{E}_i$ on the flag variety, it will be clear that the chosen linearization induces $\bigotimes_{i=1}^{\ell} det(\mathcal{E}_i^*)$ on the flag variety.

Notice that we have morphisms

$$\mathbb{C}^{r_i}_G \times V \to \mathbb{C}^{r_{i+1}}_G \times V$$

of G representations over V, where the action of G on $\mathbb{C}^{r_i}$ is given by projecting to the i^{th} factor of the product, for $i \neq \ell + 1$. For $i = \ell + 1 = n$, we endow $\mathbb{C}^n$ with the trivial G action. After taking the $\mathbb{C}^*$ quotient, we obtain a flag of vector bundles over $\mathbb{P}(V)$

$$\mathbb{C}^{r_1} \otimes \mathcal{O}(-\ell) \hookrightarrow \ldots \hookrightarrow \mathbb{C}^{r_\ell} \otimes \mathcal{O}(-1) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}.$$

If we pick coordinates on each $\mathbb{C}^r_i$, call them x^i_j, for $1 \leq j \leq r_i$, then we can write the map between $\mathbb{C}^{r_i} \otimes \mathcal{O}(-\ell - 1 + i) \hookrightarrow \mathbb{C}^{r_{i+1}} \otimes \mathcal{O}(-\ell + i)$ as

$$\begin{pmatrix} x^{i*}_1 \otimes x^{i+1}_1 & \cdots & x^{i*}_{r_i} \otimes x^{i+1}_1 \\ \vdots & \ddots & \vdots \\ x^{i*}_1 \otimes x^{i+1}_{r_i} & \cdots & x^{i*}_{r_i} \otimes x^{i+1}_{r_i} \\ \vdots & \ddots & \vdots \\ x^{i*}_1 \otimes x^{i+1}_{r_{i+1}} & \cdots & x^{i*}_{r_i} \otimes x^{i+1}_{r_{i+1}} \end{pmatrix}$$

where each $x^{i*}_j \otimes x^{i+1}_k \in H^0(\mathcal{O}(1))$. Over the stable locus, which we shall see consists entirely of the points corresponding to projectivized full rank matrices (after picking bases), these morphisms are injective as morphisms of vector bundles. Notice that these bundles are $SL(V) \bigcap G$-equivariant bundles, which induce bundles on $\mathbb{P}(V)//SL(V) \bigcap G$. Thus, we get a flag of subbundles on the flag variety

$$\tilde{\mathcal{E}}_1 \hookrightarrow \ldots \hookrightarrow \tilde{\mathcal{E}}_\ell \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}.$$

By the universal property, this induces a morphism from the flag variety to itself such that the flag sequence above is the pullback of the tautological flag sequence over the flag variety. Since we have the above morphisms of vector bundles, if we show that the fibers of each $\tilde{\mathcal{E}}_i$ can be canonically identified with the corresponding subspaces of $\mathbb{C}^n$, then we see that $\tilde{\mathcal{E}}_i \cong \mathcal{E}_i$. We can prove this at the level of sets using:

$$\tilde{\mathcal{E}}_i \cong \mathbb{C}^{r_i}_{\chi^{r_i}_{\ell+1-i}} \times_{\mathbb{C}^*} V^s / SL(V) \bigcap G \cong \mathbb{C}^{r_i} \times_G V^s.$$

Closed points of $\mathbb{C}^{r_i} \times_G V^s$ have the form $((z_1, \ldots, z_\ell), (A_1, \ldots, A_\ell))$ (having picked coordinates on each $\mathbb{C}^{r_i}$ as above). We can define a map from this fiber to $\mathbb{C}^n$ by $\phi(\,((z_1, \ldots, z_\ell), (A_1, \ldots, A_\ell))\,) := A_1 \cdots A_i (z_1, \ldots, z_{r_i})^t$. To see that this is well defined, suppose $((z_1, \ldots, z_\ell), (A_1, \ldots, A_\ell)) \sim (w_1, \ldots, w_{r_i}), (B_1, \ldots, B_\ell)$. Then, $\exists\,(g_1, \ldots, g_\ell) \in G$ such that

$$(g_i^{-1} \cdot (z_1, \ldots, z_{r_i}),\, (g_2^{-1} \cdot A_1 \cdot g_1, \ldots, g_{i+1}^{-1} \cdot A_i \cdot g_i, \ldots, A_\ell \cdot g_\ell)) = ((w_1, \ldots, w_{r_i}),\, (B_1, \ldots, B_\ell)).$$

Now it is clear that

$$B_\ell \cdots B_i (w_1, \ldots, w_{r_i})^t = A_\ell \cdots A_i (z_1, \ldots, z_{r_i})^t.$$

This map is onto the image of $A_1 \cdots A_\ell$ by construction, and it is injective since all A_i have full rank. This is actually the map $\tilde{\mathcal{E}}_i \hookrightarrow \ldots \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}$ restricted to fibers induced by the tautological sequence on V. Thus, we have $\tilde{\mathcal{E}}_i \cong \mathcal{E}_i$ for all $1 \leq i \leq \ell$.

We will now determine the stable points and unstable points of the action of $SL(V) \bigcap G$ on $\mathbb{P}(V)$, where $\mathbb{P}(V)$ has the natural $SL(V)$ linearization given by $\mathcal{O}(-1)$, as in [Tho06].

Lemma A.1.1. *For the above action of $G \bigcap SL(V)$ on $\mathbb{P}(V)$, we have that $\overline{A} = \overline{(A_1, \ldots, A_\ell)}$ is stable and semistable with respect to the action of G if and only if all A_i are injective. Otherwise it is unstable.*

We were unable to find a proof in the literature so we have included a short proof here, following the proof for the Grassmannian construction given in [Tho06]. Having proven this, we get that the flag variety $Fl(r_1, \ldots, r_\ell; \mathbb{C}^n)$ is isomorphic to the GIT quotient $\mathbb{P}(V)//G \bigcap SL(V)$.

Proof. The proof is by the Hilbert-Mumford numerical criterion, [Tho06], *Theorem 3.9.* Let $(A_1, \ldots, A_\ell) \in V$ be a lift to V of a point lying in $\mathbb{P}(V)$, $\overline{A}$.

First we shall prove that if one of the A_i's is not injective, then $\overline{A}$ is unstable.

To see this, we must exhibit a 1 parameter subgroup of $G \bigcap SL(V)$ such that $\mu(\overline{A}, \lambda) > 0$, where $\mu(\overline{A}, \lambda)$ is defined to be the weight of the action of λ on the fiber of $\mathcal{O}(-1)$ over the limiting fixed point $\lim_{t \to 0} \lambda(t) \cdot \overline{A}$.

Suppose $A = (A_1, \ldots, A_\ell) \in V$ such that A_k is not injective. Pick a basis for each $\mathbb{C}^{r_i}$. Using the $G \bigcap SL(V)$ action, we can assume that a matrix for A_k is given by an $r_{k+1} \times r_k$ matrix whose last column is 0.

We will now produce a one parameter subgroup with the required properties.

We pick the one parameter subgroup that acts on $\mathbb{C}^{r_i}$ as $\lambda_i(t) = t^{(n-r_i)(r_{k-1}-r_{k+1})} \cdot I_{r_i}$ for $i \neq k$, and it acts on $\mathbb{C}^{r_k}$ as

$$\lambda_k(t) = \begin{pmatrix} t^{(n-r_k)(r_{k-1}-r_{k+1})} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t^{(n-r_k)(r_{k-1}-r_{k+1})} & 0 \\ 0 & \cdots & 0 & t^{\xi} \end{pmatrix}$$

where

$$\xi = \sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(n - r_i)r_i + \sum_{j=k+1}^{\ell}(r_{j+1} - r_{j-1})(n - r_j)r_j \\ + (r_{k+1} - r_{k-1})(r_k - 1)(n - r_k) > 0.$$

We can regard $\lambda(t)$ as a block diagonal matrix. Now, consider the induced action of $\lambda(t)$ on $\mathbb{C}^{r_i\,*} \otimes \mathbb{C}^{r_{i+1}}$. It acts on $\mathbb{C}^{r_i\,*}$ as $\lambda_i^{-1}(t)$, and it acts on $\mathbb{C}^{r_{i+1}}$ as $\lambda_{i+1}(t)$. Thus, the determinant of this block is

$$t^{-r_i r_{i+1}(n-r_i)(r_{k-1}-r_{k+1}) + r_{i+1} r_i (n-r_{i+1})(r_{k-1}-r_{k+1})}.$$

We see that the exponent on t in the determinant of $\lambda(t)$ is

$$\sum_{i=1}^{k-1}(r_{i-1} - r_{i+1})(r_{k-1} - r_{k+1})(n - r_i)r_i + (r_{k-1} - r_{k+1})^2(r_k - 1)(n - r_k) \\ + (r_{k-1} - r_{k+1})\xi + \sum_{j=k+1}^{\ell}(r_{j-1} - r_{j+1})(r_{k-1} - r_{k+1})(n - r_j)r_j = 0$$

so this subgroup does lie in $G \bigcap SL(V)$. Notice that, in our basis for V, $\lambda(t)$ acts on $e^{r_i}_{h,j}$ as

$$t^{(r_{k-1}-r_{k+1})(n-r_{i+1}) - (r_{k-1}-r_{k+1})(n-r_i)} = t^{(r_{k+1}-r_k)(r_{i+1}-r_i)}$$

for all i, h, j except the cases $(i = k-1,\, h = r_k)$ and $(i = k,\, j = r_k)$. For $(i = k-1, h = r_k)$, $\lambda(t)$ acts on $e^{r_k}_{r_k,j}$ as

$$t^{\xi - (r_{k-1}-r_k)(n-r_{k-1})}$$

and in the second case the entries that would have a negative weight on them are zero in A_k. Thus all the nonzero entries of A are such that $\lambda(t)$ acts on them with a positive weight, so $\mu(\overline{A}, \lambda) > 0 \implies$ the point $\overline{A}$ is unstable.

We must show that if $\overline{A}$ is such that all A_i have full rank, where $A = (A_1, \ldots, A_\ell)$ is a lift of $\overline{A}$, then $\overline{A}$ is stable.

We can diagonalize a given one parameter subgroup λ so that it acts on $\mathbb{C}^{r_i}$ as

$$\begin{pmatrix} t^{\lambda_{i,1}} & \cdots 0 & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda_{i,r_i}} \end{pmatrix}.$$

In our basis $\{v^{r_i}_{h,j}\}$ of $\mathbb{C}^{r_i\,*} \otimes \mathbb{C}^{r_{i+1}}$, we have that $\lambda(t)$ acts on $v^{r_i}_{h,j}$ as $t^{\lambda_{i+1,h} - \lambda_{i,j}}$. It will follow that $\overline{A}$ is stable if we can show that $\exists\, v^{r_i}_{h,j}$ such that $\lambda_{i+1,h} - \lambda_{i,j} < 0$.

Suppose for a contradiction that for each $v^{r_i}_{h,j}$ such that the coefficient in A on $v^{r_i}_{h,j}$ is nonzero, we have $\lambda_{i+1,h} - \lambda_{i,j} > 0$.

I claim that this forces all $\lambda_{i,k} < 0$, which cannot happen if the subgroup is to lie in $SL(V)$ since this occurs if and only if the condition $\sum\limits_{i=1}^{\ell}(r_{i-1} - r_{i+1}) \sum\limits_{j=1}^{r_i} \lambda_{i,j} = 0$ is satisfied.

Since the matrix for A_ℓ has full rank we see that each column of A_ℓ must have at least one nonzero entry, say the coefficient on $v^{r_\ell}_{h_j,j}$, thus forcing $\lambda_{\ell,j} < 0$ for all $1 \le j \le r_\ell$. By descending induction, suppose that for some $h < \ell$, all $\lambda_{h,k} < 0$, $1 \le k \le r_h$. Now pick $1 \le w \le r_{h-1}$. We claim that $\lambda_{h-1,w} < 0$. Since A_{h-1} has full rank, the w^{th} column has a nonzero entry, say the coefficient on $v^{r_h}_{q,w}$. By assumption, we must have $\lambda_{h,q} - \lambda_{h-1,w} > 0$, which tells us that $\lambda_{h-1,w} < 0$. Repeating for all w completes the induction. Thus, $\overline{A}$ is stable with respect to every diagonal one parameter subgroup λ.

The property that every column of a matrix for A has a nonzero entry in it is independent of the choice of matrix representation of A since A has full rank. Thus this statement holds true for all $g \cdot \overline{A}$. Since $\mu(g \cdot \overline{A}, \lambda) = \mu(\overline{A}, g^{-1}\lambda g)$, we see that $\overline{A}$ is stable with respect to every conjugate of λ, for every diagonal one parameter subgroup λ. Thus, $\overline{A}$ is stable. □

We can take the GIT quotient of $\mathbb{P}(V)$ by $G \bigcap SL(V)$ and from the lemma above it is clear that the result is the flag variety.

We will now prove the following claim made in the introduction regarding the character group of G.

Lemma A.1.2. *$\chi(GL(n, \mathbb{C})) \cong \mathbb{Z}\, det$ for all $n \in \mathbb{Z}$.*

Proof. It suffices to prove the result for diagonalizable matrices since these are dense in $GL(n, \mathbb{C})$ and characters are continuous in the Zariski topology. Thus, if

$A = P^{-1} \cdot D \cdot P$, then for $D = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_n \end{pmatrix}$, $\chi(A) = \prod_{i=1}^{n} z_i^{\alpha_i}$. If we show that all α_i are equal, then it will follow that χ is a power of the determinant, and we will be done.

However, this is clear since this depends on the order of the eigenvalues in the diagonal matrix. $\square$

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