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by

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Abstract

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The dominant image of mathematics as an abstract, universal, disinterested, and pure body of knowledge both misrepresents disciplinary practice and alienates many students. Undergraduate proof-based courses such as real analysis, which are supposed to introduce students to contemporary academic mathematics, often contribute to such idealized notions as well. To counter this idealized image, social scientists from various disciplines have characterized academic mathematics as an embodied, socially and materially distributed, socio-historically situated, and ideologically laden practice. Mathematics educators have been drawing on such insights to develop pedagogical approaches aimed at providing students with more authentic disciplinary experiences. Yet, not enough attention has been devoted to examining how (and why) proof-based mathematics is idealized to begin with. Specifically, how is it that real analysis lectures, taught by practicing research mathematicians who have first-hand experience with the discipline, give rise to idealizations that both distort and exclude?

In this dissertation I address this question through the close examination of language. That is, I set out to understand how mundane features of lecture discourse construct potentially problematic common sense ideas about mathematics. The reported study is a video-based micro-ethnography, conducted in three iterations of data collection in a prestigious mathematics department in a public research university in the United States. The primary data are video-recordings of lectures and lecture notes of real analysis courses taught by five different instructors, one in Spring 2015 and four in Fall 2020, all delivered on Zoom due to Covid-19.

Informed by socio-cultural theories and drawing on a variety of discourse analytic techniques, the dissertation examines three features of lecture discourse: the *stories* instructors told about the purpose of the real analysis course and academic mathematics as a whole, the *value-laden attributes* they deployed to characterize and appraise mathematics, and the way lecture discourse enacted *human subjectivity* in the context of mathematical activity. To situate the stories told to students in real analysis lectures, I also examined the *stories* about the purpose of mathematics that prominent mathematicians articulated in famous meta-reflective essays about disciplinary practice.

In chapter 3, I identify four different *stories* articulated by mathematicians in meta-reflective essays, distinguished by the kind of object they construct for mathematics as an activity system. I argue that the stories are consequential in that they correspond to different grammars of evaluations and can, in particular, construe radically different educational goals for courses such as real analysis. Furthermore, the existence of distinct high-profile perspectives on practice suggests that any presentation of mathematics as a practice with a single, universally agreed upon telos can function as an unproductive idealization of the discipline.

Chapter 5 examines *stories* about mathematics told in introductory real analysis lectures, where instructors face the rhetorical challenge of ‘motivating’ the subject and its new way of doing math. I identify five distinct meta-stories mobilized by instructors, characterizing each story by the assumptions it presupposes (what one needs to buy into to find the stories compelling). Assessing each story and its underlying assumptions for faithfulness to the discipline and relatability to students’ (likely) past experiences, I found that all five meta-stories were somewhat idealizing. One notable mechanism of idealization was the stories’ reliance on metaphors that render certain goals and values self-evident (e.g. “math is a structure, so it needs solid foundations.”)

Chapter 6 examines a more mundane and pervasive feature of lecture discourse: appraisals using value-laden attributes (e.g. “this is a *precise* definition”). I flagged all instances of instructors assigning a characteristic to a mathematical object or processes in four focal lectures, coding both the exact adjective or adverb used and the stance with which it was deployed. I found that instructors tended to comment much more on precision, validity, and difficulty, than on characteristics such as utility, interest and aesthetics, and that they repeatedly framed ambiguity with a negative stance. I argue that such patterns construe disciplinary orientations at odds both with what mathematicians say is important and with what many students might value as well.

In chapter 7, I tackle yet another mechanism of idealization in lectures: the discursive obfuscation of human agency. This phenomenon has already received significant attention in the literature. Here I focused on positive possibilities: I identify and characterize the discursive means by which instructors *humanized* the language of mathematics, which I operationalized as enacting aspects of human experience not typically reflected in the textual register. I argue that Bakhtin's (1981, 2010) construct of literary *chronotope* (discursive space-time configuration) is a useful conceptual tool for distinguishing different ways of enacting human agency in mathematical discourse, and propose a framework of three chronotopes for humanizing the language of mathematics: the here-and-now experience of doing mathematics, the socio-historical context of mathematical activity, and discursive hybridity. The framework also sheds light on how dominant epistemologies of mathematics are enacted in discourse: by omitting non-cognitive experiences, references to socio-historical context, and any allusions to other spheres of human activity, mathematical discourse constructs itself as transcendental, universal and pure.

Collectively, the analyses and findings in this dissertation show how mainstream idealizations of mathematics are discursively constructed in real analysis lectures. The identification of concrete mechanisms – such as stories, attributions, representation of human experience – is important. When mechanisms are no longer invisible, they can be used as levers for change.

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1 Introduction

The way professional mathematics is represented in folk narratives, books, and public displays (e.g. talks, lectures), does more to mystify disciplinary practice and sense-making than explain it (Davis & Hersh, 1981; Harris, 2017; Hersh, 1991). The mathematician Reuben Hersh (1991), for example, argued that mathematics, like other spheres of human activity (Goffman, 1978), has a "front" and a "back"; that there is a difference between discourse and practices that are displayed to the public and those that remain hidden from view. While front/back distinctions may not be intentional or malicious, they are not without consequence. They facilitate the preservations of enduring myths[1] about the discipline, such as there is only one true mathematics that is objective, universal and indubitable (Hersh, 1991). Buying into such myths broadens the gulf between those on the inside and those that are out.

Idealization of mathematics and the pedagogical activities that uphold them are problematic. First, they serve as major access barriers to students' effective and affective engagement with the discipline. When instruction "misrepresents mathematics, presenting it as a dead and deadly discipline" (Schoenfeld, 1994, p. 54), students have fewer opportunities to engage with important disciplinary practices such as problem-solving (Schoenfeld, 1988, 2016) and conceptual understanding (Dreyfus, 1991; Thurston, 1994). Besides restricting access to essential disciplinary ideas and techniques, idealized depictions of mathematics also provide less room for students to see themselves as reflected in and belonging to the discipline (Schoenfeld, 1994), and make it harder for teachers to notice and build-on students' mathematical strengths (Adiredja, 2019; Louie, 2018). Furthermore, such idealized narratives are commonly mobilized to argue against justice oriented reform (e.g. as part of the 90's math wars (Schoenfeld, 2004), or their current reincarnations (Schoenfeld & Daro, 2024)). On the whole, these idealized depictions stand in the way of mathematics, and in particular – advanced mathematics – living up to its potential to facilitate human flourishing (Su, 2020).

Given all of this, an overarching motivation of my work is to demystify mathematics in order to make educational experience both more authentic and more meaningful and affirming to all students. In this dissertation I focus on three types of beliefs along which mathematics is often idealized: (1) the purpose(s) of the discipline, (2) its values, and (3) the role of the human agent.

The overall aim of the dissertation research is two-fold: (1) to challenge the accuracy of common sense ideas about mathematics by examining mathematical practice through naturalistic observations of mathematical activity in lectures. The aim is to identify diverse resources used in math semiosis and explain how participants use them to accomplish disciplinary objectives, interactional common ground, sense-making, and other forms of human connection in and through advanced mathematics. This descriptive-analytic aspect of the work will contribute to the development of a theoretically and empirically grounded language with which to speak about how communication in abstract mathematics works. Having such language is important because

[1] Here I use the words myths, beliefs and idealizations synonymously, in a non-technical sense. Later, I will elaborate on a socio-cultural theorization of beliefs as embodied in the stories people tell, and in their regularities in behavior. In particular, I am not using myths in Barthes's sense of second-order semiosis.

it will make it easier to recognize, celebrate and deliberately use productive resources in teaching and learning interactions. Suggestions to make math communication in lectures more authentic, or any other pedagogical recommendations, will be most effective if they have ecological validity; that is, they need to be rooted in the reality of practice, and come with detailed and well conceptualized descriptions of what makes a particular approach work (Erickson, 1992).

The second objective of the proposed work is (2) to identify "counterproductive" processes by which important aspects of mathematical practice are obfuscated and devalued in lecture activity. Specifically, I want to examine how ideologies about mathematics are constituted in and shape teaching and learning interactions in the context of Real Analysis courses. Close observation of how front/back distinctions get accomplished in and through routine practices of undergraduate lectures is important because ideological influences are often invisible to participants, and hence difficult to change. Idealizations of mathematics can be explicit (e.g. through metapragmatic articulations such as "don't use visuals") or implicit (e.g. by systematically writing down only formal math (Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016)), and both types of processes need to be well understood if lecture interactions are to be reconfigured away from systematically devaluing certain aspects of mathematical semiosis and mystifying practice.

What is unique about the research reported here is the coordination of these two objectives in the same study and interactional context. That is, I naturalistically examine (1) displays of mathematical practice and (2) mystification *as these processes occur side-by-side*. Such investigation is needed because it is only when these two processes co-occur that one can see the tacit ways in which different aspects of mathematical activity are differentially valued; how some parts of doing math become the "front" and how some parts become the "back".

In mathematics education literature and other social sciences, there is a long standing tradition of studying mathematical practice in detail in ways that counter mainstream beliefs about it, i.e. by showing that mathematical activity is embodied, socially and materially distributed, socio-historically situated, and ideologically laden (e.g. (Alibali & Nathan, 2012; Csiszar, 2003; Greiffenhagen, 2014; Radford, 2003; Rotman, 1988)). Insights from this line of work are valuable, and I hope focus (1) of my own research will contribute to such conversations as well. However, in my experience, just showing that these the common sense beliefs about math are inaccurate hasn't been sufficient to engage mathematicians in productive discussions. Mathematicians either disagree with the characterizations of math that emerge from this work (Sfard, 1998), or think that lectures already adequately reflect them. Instructors repeatedly claim that their main pedagogical objective for advanced math courses is to *model* how an expert thinks through mathematics, and that the best way to do this – while also "covering" the required curriculum – is by lecturing (Johnson, Keller, & Fukawa-Connelly, 2018). Thus, what can be helpful is not only research that shows mathematicians the empirical reality of mathematical activity (i.e. focus (1)), but also one that shows how lecture discourse intentionally or unintentionally distorts certain aspects of it (i.e. focus (2)).

Enacting or perpetuating myths about mathematics is not simply a matter of knowing or not knowing that they are false. We must also acknowledge that these beliefs operate as ideologies; that is, they are intimately connected to issues of power. Narratives about what is or is not math are mobilized precisely in those situations in which power is most at stake; when decisions need to be made about who counts as legitimately mathematical (Csiszar, 2003; Louie, 2018). Thus, when an instructor enacts or articulates myths, it may be counterproductive to interpret these statements as indicators of their fixed belief about the discipline. Another feature of the proposed research is a theoretical and methodological approach that looks at such instances from an action-in-interaction perspective (Schegloff, 1996). Instead of asking what beliefs instructors hold, I ask – what does mobilizing idealized narratives accomplish in activity? What kind of mathematical work, and whose mathematical work, is justified as a result?

1.1.1 The current study

In the previous section I outlined the motivating problems, organizing principles and intended foci of this research project. I named three focal beliefs about contemporary academic mathematics and introduced two research objectives relevant to them: (1) understanding how mathematical semiosis differs from how the myths portray it, and (2) identifying and explicating mystification processes; how the three myths shape and get constituted in interactions. I argued that these two research objectives should be integrated in a single study design, because the underlying processes of semiosis and mystification are best studied when they occur side-by-side. I further suggested that using an interactional perspective that attends to issues of power is necessary to properly understand how these beliefs operate.

To address these questions for research, I conducted an observational study of instructional discourse and practices in upper division undergraduate math lectures. This is a productive context to address these questions because, being a point of transition into the discourse of contemporary academic mathematics in the educational trajectories of math majors, implicit differences in discursive norms and values are explicitly stated and tensions and contradiction inevitably rise to the surface. The specific course I choose to focus on is called Real Analysis (RA). Rationale for the selection of RA as a case within the undergraduate mathematics curriculum is provided section 4.1.2 in the methods chapter.

By placing pervasive meaning-making practices in lectures as a focus for empirical investigation, rather than as self-evident reality, I join other math education scholars (e.g. Weber, 2004), in arguing that folk descriptions about what advanced undergraduate math courses are like, while ring true, are too vague and polemic, and thus insufficient, to be effective ground for pedagogical change. Gloss descriptions of the problem are not enough. Put simply, telling mathematics instructors to stop teaching math in a formal, dehumanizing way without pointing to what that looks like, concretely, hasn't been effective.

My work is rooted in scholarly traditions that draw on sociocultural theory (Vygotsky, 2012; Wertsch, 2012) to conceptualize semiosis in mathematics (Lave, 1996; Saxe & Esmonde, 2012; Sfard, 2008), science (Lemke, 1990), and literacy (Sterponi, 2011). Central to my own use of socio-cultural theory is taking mediated action and discourse-in-interaction as basic units of analysis (Schegloff, 1996; Schiffrin, 1994; Wertsch, 2012). That is, I attend to how activity is distributed across and accomplished through different types of *semiotic resources* (C. Goodwin, 2000) as well as to the *functions* those deployments serve. I take discourse to be multi-functional (Halliday, 1993), in that a single utterance can accomplish multiple actions at the same time, and that resource choice is consequential as different types of resources have different affordances for action. Importantly, mathematical discourse does not only serve instrumental ends (e.g.

computation), but also indexes subject-positions, stances, values, and various social “voices” in and out of math (Agha, 2005; Bakhtin, 1981; Eckert, 2008; Silverstein, 2003). When mathematical activity is observed “in the wild”, it can be even clearer how mathematical tools are used to negotiate power and authority, and not just to meet practical ends (Wertsch & Rupert, 1993).

This dissertation follows the methodological tradition of video-based micro-ethnographic analysis of teaching and learning interactions (Derry et al., 2010; Erickson, 1992, 2006). A video-based micro-ethnography affords a close and repeated examination of instructional practice, which helps uncover details of routine conduct that may be otherwise invisible to participants and observers (i.e. not available to meta-pragmatic articulation (Silverstein, 1981)). Such detail is needed to address both of the questions for research. In terms of understanding how mathematical practice *really* works (1), a micro-analysis of interaction (Jordan & Henderson, 1995) that attends to how activity is distributed across different semiotic fields (C. Goodwin, 2000) is necessary because it is only through a coordination of different types of semiotic resources that mathematical reference is constructed (Greiffenhagen, 2008; Lemke, 1998; Radford, 2003). In terms of understanding processes of mystification (2), repeated observation is important because, as mentioned above, ideological mechanisms are often invisible. In particular, instructional discourse can perpetuate myths in tacit, nuanced, and un-intentional ways.

2 Background

2.1 Introduction: what gets taught and learned in a Real Analysis course?

The purpose of this section is to outline what gets taught and learned in a Real Analysis (henceforth, RA) course, in terms of both its explicit and hidden curriculum. I propose thinking about the explicit and implicit learning outcomes of RA courses on four levels: a (1) *content level* of concepts and techniques to be "covered," a (2) *practices level* of meta-mathematical (i.e. cross-domain) norms and ideas needed for effective participation in the Definition-Theorem-Proof (DTP) epistemic game, an (3) *axiological*[2] *level* of appropriating the values and objectives that guide the selection of what counts as good and appropriate mathematical activity, and finally, a (4) *stereotype level*, in which associations are made between practice and types of people. In short, the *what*, *how*, *why* and *who* of contemporary academic mathematics.

The levels range from most explicit and canonized (*content* level), to less explicit and broadly agreed upon (*axiology* level), and finally, to the largely unspoken (*stereotype* level). The first two levels are well known. The curricular level is the most explicit and canonized; its challenges and disciplinary significance are well familiar to RA instructors. A big accomplishment of mathematics education research of the past 50 years was to highlight the importance of the *practice* level, conceptualize it, document student difficulties, make it more explicit, and design instruction to address it (Rasmussen, Zandieh, King, & Teppo, 2005; Schoenfeld, 2014a). The third *axiological* level has been discussed less, and needs unpacking. As I will show in chapter 3, part of the challenge in addressing the third level is that there is much less consensus about purposes and values in the academic mathematics community, let alone among educators. Whatever is reproduced in instruction takes an active stand on these debates.

As I will argue in more detail in the sections below, all four levels shape students' learning and experience in classrooms. All four ultimately influence *who* gets to participate in academic mathematics. From a purely practical perspective of access to the discipline – of learning to "play the game" (R. Gutiérrez, 2002) – proficiency with the *content* of RA and the *practices* of the DTP epistemic game are both important for effective participation in academic mathematics and related fields. But, acculturation into a community of practice entails more than mastery of its norms and techniques. It also entails appropriating (or resisting) the values and purposes that guide the enterprise as a whole (Bakhtin, 1981; Dawkins & Weber, 2017; Polman, 2006; Solomon, 2007). And, while it may be unintentional by educators, it also entails 'learning' to connect mathematics with consequential identity markers (e.g. being 'smart') and discourses such as race and gender (Lave, 1996; Leyva, Quea, Weber, Battey, & López, 2021; Shah & Leonardo, 2017). The levels are, of course, interrelated. Purposes dictate what practices and content are valued, and conversely, changes in content and practices may shift values and purposes. Values and practices often index ethnic and gender identities (Ernest, 1995; Nasir,

[2] The first three levels map onto Laudan's (1984) three levels of commitment in a scientific community: *theories*, *methodologies*, and *axiology*. Here I chose to call the first two levels *content* and *practice*, rather than theories and methodologies, to align with terminology familiar to practitioners and used in mathematics education research.

Snyder, Shah, & Ross, 2012). In this dissertation, I focus primarily on the third, axiological level, while considering its relationships to the other levels of the *what*, *how* and *who* of mathematics.

The third level, of purposes and values, is important for several reasons. First, axiology is what renders practice sensible. Without purposes and values, norms associated with the DTP epistemic game seem arbitrary (Dawkins & Weber, 2017). Second, axiology influences students motivation and desire to belong in academic mathematics (Ernest, 1995). Do the values presented in the classroom align with what students value? For example, if math is about competition and looking smart, some students would find that appealing while others not. Finally, axiology also influences what student behaviors (and hence *which* students) are positioned as competent in mathematics classrooms (Adiredja, 2019; Gresalfi, Martin, Hand, & Greeno, 2009; Louie, 2018). If proof is the main business (the ultimate goal), and explanation, understanding etc. are epiphenomenal, student strengths related to proofs will be valued more than student strengths related to explanation and understanding.

In the following sub-sections I describe in more detail what is learned on each level, and the significance it has for students.

2.1.1 Content level: concepts and techniques.

In academic mathematics, the label "analysis" refers to a disciplinary sub-field (i.e. a body of knowledge and associated techniques). The concepts and techniques of analysis are central to both theoretical and applied contemporary research, even in seemingly far removed fields such as number theory and combinatorics. Boundaries between different areas of research are never clear cut, and hence there is no precise definition for what constitutes Analysis (as opposed to other related sub-fields). Emmett, for example, provided the following 'incomplete' definition in his lecture:

> "an informal and incomplete definition of analysis is that **analysis is that part of mathematics in which limits are used to solve problems**. So the key concept in analysis is that of a **limit**. I would say maybe the second concept is that of an **inequality**."
> Emmett, lecture 1.

In Wikipedia, Analysis is more comprehensively defined as:

> "the branch of mathematics dealing with continuous functions, limits and related theories, such as differentiation, integration, measure, infinite sequences, series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis."

Because the concepts and toolkit that constitute Analysis are so central to contemporary academic mathematical practice, courses such as RA (Analysis with Real numbers), Complex Analysis (Analysis with Complex numbers), and Functional Analysis (Analysis with abstract topological and metric spaces), are part of the core curriculum in most (if not all) undergraduate and graduate mathematics degrees worldwide. Proficiency with the concepts and techniques of

RA, the first course in the aforementioned sequence of Analysis courses, is thus an *institutional curricular goal* for all math majors. It is a prerequisite to many graduate level courses and core areas of contemporary research. Demonstration of proficiency is expected, for example, in applications to many graduate programs. Thus, one important objective of a RA course is to "cover" certain "content" not addressed in other undergraduate courses, both in terms of particular concepts (e.g. the epsilon-delta definition of limit, properties of Real numbers, uniform convergence), and in terms of important techniques specific to Analysis (e.g. epsilon-delta proofs, arguments with inequalities).

2.1.2 Practice level: the DTP epistemic game.

Besides mastery of topics and techniques specified in RA textbooks and curricula, a RA course serves other functions in the learning trajectories of undergraduate mathematics majors. Importantly, RA is one of the first "proof-based" courses many students take during their undergraduate studies[3]. A "proof-based" university math course, in contrast to courses such as Calculus, is centered on the Definition-Theorem-Proof (DTP) *epistemic game* (A. Collins & Ferguson, 1993) that is at the heart of contemporary academic mathematical practice.

The term epistemic game draws attention to the following constitutive characteristics of scientific practice: it is oriented toward the generation of knowledge (hence, the use of the term *epistemic*), it is form-mediated (scientists use tables, lists, diagrams, and above all - language), and it consists of normative routines for using the *epistemic forms* in processes of inquiry:

> "The forms and games we describe are epistemic in that they involve the construction of new knowledge. They are played to make sense of phenomena in the world. … We call them epistemic games, both because of the allusion to Wittgenstein's (1953) language games and because of the parallel to games such as tic-tac-toe. They are not simply inquiry strategies or methods; rather, they involve a complex of rules, strategies, and moves associated with particular representations (i.e. epistemic forms). As with any complex game, understanding all the subtleties of an epistemic game requires a long period of learning." (A. Collins & Ferguson, 1993, p. 26):

Contemporary "proof-based" mathematical practice can be usefully characterized as a *DTP epistemic game*, in the following way: mathematical activity is *epistemic* in that it is broadly oriented toward the generation of knowledge about mathematical objects and systems (e.g. truth-statements about numbers, functions, limits and so on). While mediation by language is a constitutive feature of all epistemic activity (Vygotsky, 2012), a special feature of contemporary academic mathematics is the use of *textual units* such as axioms, definitions, theorems and proofs. Thus, generic textual units such as definitions, theorems, proofs can be thought of *epistemic forms* that mediate the *epistemic games* that constitute mathematical practice. Finally, there will be no *practice* without normative routines (and associated understandings) for using those forms.

[3] In the U.S. context, Real Analysis is an upper division course, typically taken in students' third or fourth year of study, after they have completed computation-focused lower-division courses such as Calculus, Linear-Algebra, Differential Equations. In other countries, there is no separation to lower and upper division courses. Typically, students take a combined Calculus - Real Analysis course in their first year of study.

Effective participation in this game requires understanding and mastery of norms (e.g. how to write proofs), which is in turn supported by an understandings of various meta-mathematical ideas. Some of the practices and meta-mathematical ideas underpinning the RA epistemic game are not completely new to students. For example, the practice of deductive proof writing is taught in proof-based geometry courses in secondary school. However, the contemporary DTP epistemic game (of which the RA epistemic game is one instantiation) is substantially different from the classical DTP epistemic game, modeled on Euclidean geometry (Kleiner, 1991) and thus many practices and ideas will be new and, unsurprisingly, challenging for newcomers.

For example, in a classical DTP epistemic game, definitions *describe* (the essence of) already familiar mathematical objects, such as lines and triangles, whereas in a contemporary DTP game definitions play also a stipulating role. That is, by manipulating inscriptions and assumptions, we can stipulate – create – new mathematical objects (Dawkins, 2018). Another example of a "modern" mathematical practice is that of actively searching for "monster" counterexamples to disqualify theorems (Lakatos, 1976). Sørensen (2005) argued, for example, that prior to the 19th century, a "monster" counterexample would not invalidate a theorem, and that it was ok for theorems to work only in most, normative cases.

Yet another norm in the contemporary DTP epistemic game is that proofs should be deductive and inferences within proofs should rely *only* on properties stipulated within the previously assumed or validated textual units. For this reason, definitions are often expressed in technical, highly regimented language, reproduced verbatim within proofs, and manipulated by explicitly specified rules (e.g. one is allowed to exchange x+y with y+x only if the rule of commutativity x+y=y+x has been stipulated). Within the DTP epistemic game, such forms of "mechanical derivation," while not always explicitly realized, are an important tool for establishing consensus (Wagner, 2022) about the truth-value of proposed knowledge claims.

Section 2.2 below further elaborates on features of the language and generic units that mediate contemporary academic mathematics – i.e. the forms deployed in the DTP epistemic game.

2.1.3 Axiological level: purposes, values, and virtues.

RA courses are a context of enculturation to the academic mathematics community, which entails not just *mastery* of its mathematical and meta-mathematical ideas and practices, but also *appropriation* of (or resistance to) the beliefs, values, purposes and worldviews it espouses (Ernest, 1995; Polman, 2006). Whether explicitly intended by instructors or not, a RA course also "teaches" students about the purpose of mathematics as a discipline – what is the point of it as a human activity – and what counts as valuable mathematics. Such beliefs about what is important, can be grouped under the umbrella term axiology, which refers to the "shared values, goals, and principles that the discipline is trying to achieve" (Dawkins & Weber, 2017, p. 125). The category encompasses different constructs, which I clarify below.

First, I wish to make a distinction between the immediate *goals* of an activity and the *purposes* a social practice has as a whole. The same local goal – e.g. proving a theorem – can be contextualized in terms of radically different 'ultimate' purposes, which imbue the activity with different kinds meanings. The parable of the bricklayers (Baker, 2019) is a good illustration of this. In this story, three men are working at a construction site of a church. When asked what they are doing, one answered: “I’m a bricklayer. I’m working hard laying bricks to feed my family.” The second said: “I’m a builder. I’m building a wall.” And the third: “I’m a cathedral builder. I’m building a great cathedral to The Almighty.” All were performing the same local action – laying bricks – but viewed those activities as contributing to different kinds of collective “projects” (providing for a family, building structures, crafting a place of worship).

Similar distinctions can be made in mathematics. For example, a mathematician could be proving a theorem as part of a collective effort to increase the volume of valid knowledge (lists of true statements), as part of an effort to increase the community’s understanding of mathematical phenomena, or as part of a professional need to publish papers in mathematics journals. While one can argue that all of these purposes are always present to warrying degrees, I contend that mathematicians, as any community of practice, negotiate a shared telos – a shared *object* of activity (Cole & Engeström, 1993; Engeström, 1999; Roth & Lee, 2007) – that serves both to guide their work and render local activity, such as proving, meaningful. A goal is something a single person can have, and it can be complete. The purpose or object of an activity, in contrast, is collective and idealized. It cannot be accomplished by any one person, and cannot be ‘done’ once and for all. Mathematicians are never ‘done’ creating mathematical knowledge; it is a shared object they are continually crafting and passing along to future generations.

The second axiological construct I wish to unpack is values. Values have been defined and used in the literature in different ways. In the context of discussing mathematicians’ proving practice, Dawkins and Weber (2017) defined values as:

> … a community’s shared orientations and goals that underlie shared activity. We concur with Herbst, Nachlieli, and Chazan’s (2011) notion that values are what “members of a practice use to justify or otherwise discard possible actions” (p. 219) in a given situation. While values may justify actions, values themselves are generally assumed without justification. (p. 125)

In the context of epistemic activity more generally, Chinn, Buckland, and Samarapungavan (2011) define values as:

> Epistemic value refers to the worth of particular epistemic achievements. For example, a person who believes that scientific knowledge is worth attaining because it supports economic growth has a belief that scientific knowledge is valuable for practical reasons. (p. 142)

The above quotes define values as shared beliefs about worth that are used to justify action. This is a good general characterization, however, I concur with Agha (2003) that the construct of shared beliefs is empirically (and theoretically) problematic, and instead define values as “certain *regularities of evaluative behavior* [that] can be observed and documented as data.” (p. 242).

Different kinds of evaluative behaviors can be documented and interpreted as indicators of values. On the more explicit end, people can be observed *articulating* core principles and explaining how these principles justify actions. For example, a mathematician may explicitly say “precision is very important in mathematics, so this is not a good proof.” Alternatively, we could observe regularity in the kinds of proofs mathematicians discard, and infer from that what they value, even if they do not articulate a principle explicitly. In former example, of an explicit articulation, the mathematician uses a cultural artifact – the statement "precisions is very important in mathematics" – as a resource for making an evaluative decision (discarding a proof). In the latter example, the resource may mediate their decision internally, but we can only observe the regularity in evaluative behavior. Since values judgements about what counts as a legitimate proofs are related to distribution of resources (who gets to publish in journals, who gets high grades in exams), a statement such as "precision is very important in mathematics" is part of an arsenal of representational resources that operate as an ideology (S. Hall, 1986).

Most evaluative behavior is much more mundane and implicit than the two hypothetical scenarios considered above. Evaluation can manifest through very subtle stance taking in interaction (Ochs, 1996), including the use of extra linguistic resources such as tone of voice and facial expressions (M. H. Goodwin, Cekaite, & Goodwin, 2012). In classrooms, teachers' values can be gleaned through regularities in how they respond to student ideas, e.g. what they re-voice, and what they repair (Razfar, 2005; Yackel & Cobb, 1996). What teachers and researchers notice about students in the context of mathematical activity is a type of regularity in evaluative behavior, and hence an enactment of values, as well (Adiredja, 2019; Louie, 2018). In mathematics, what people choose to write down, as opposed to just say or gesture ephemerally, functions (in the eyes of students) as an evaluation of 'what's important' as well (Lew et al., 2016)

One feature of evaluative behavior that is expressed in language is the invocation of virtues and vices. A virtue is a characteristic that is deemed desirable. The notions of virtues and vices goes back to ancient Greek moral philosophy, notably the writing of Aristotle about virtues as a pathway to human flourishing (Su, 2017). In recent work, virtues have been defined in the context of practices – desirable characteristics for achieving particular aims.

Chinn, Buckland, and Samarapungavan (2011), for example, define epistemic virtues as:

> "… praiseworthy dispositions of character that aid the attainment of epistemic aims, such as intellectual courage and open-mindedness. In contrast, epistemic vices are those dispositions that hinder the achievement of epistemic aims." (p. 142)

Similarly, the moral philosopher Alasdair MacIntyre (1984) defined virtues as:

> "an acquired human quality the possession and exercise of which tends to enable us to achieve those goods which are internal to practices and the lack of which effectively prevents us from achieving any such goods." (p. 191) cited in (Aberdein, Rittberg, & Tanswell, 2021)

Note that in both of the above definitions, virtues are framed as characteristics of people. However, in the context of an epistemic practice such as mathematics, both objects of knowledge (theorems, proofs) and agents are routinely attributed with virtues and vices. For example, both the proof, the process of proving, and the prover can be described as *rigorous*. To highlight the possibility of appraising both objects and people, Aberdein, Rittberg, and Tanswell (2021) made the distinction between *theoretical* and *character virtues*.

To summarize the discussion thus far. Axiology encompasses purposes, values, and virtues. By purposes, I mean idealized collective objects that a community of practice is crafting, and which are negotiated as part of activity (Engeström, 1999). Values refer to regularities in evaluative behaviors, which can manifest in both explicit and subtle ways. One resource that is often used for evaluation is virtues – characteristics of people and objects that are deemed desirable for accomplishing the purposes of a particular practice. Clearly, all these constructs stand in dialectic relationship to one another. What one values dictates purposes, and purposes shape what one values. One way in which purposes influence values is in shaping what characteristics of people and objects of knowledge are deemed desirable, that is, in shaping what is considered virtues.

While currently there seems to be a general sense of agreement in the mathematics community about the mathematical and meta-mathematical ideas underpinning RA[4], once we move to this third axiological level practice we may notice less general or explicit agreement. Of course, if nothing else is said and done in RA courses besides "covering" canonical curriculum, one may easily get an unproblematic impression that "the purpose of mathematics is to produce theorems and proofs." But, however readily available this narrative is, it is far from generally agreed upon within the professional community and certainly isn't unproblematic. Chapter 3 is devoted to examining different positions about the purpose of mathematics evident in famous writings of mathematicians.

2.1.4 Stereotype level

The messages conveyed about mathematics and what it means for a person to be mathematical go beyond participants' explicit awareness, and consequentially, will not only be reflected in meta-pragmatic narratives about the discipline (Silverstein, 1981). In any activity, indexical valances are at play, which means participants may develop associations between different dimensions of situations even if those are not explicitly intended (Ochs, 1996). For example, if a mathematical activity is always displayed with particular performance of masculinity, one may learn to become more mathematical by simply looking, talking or acting more "like a man" (using certain rhetorical styles, tone of voice etc.). Furthermore, if sanctioned mathematical activity is always performed using a certain register (certain types of words and grammar), then the use of the register in and of itself becomes an index of being mathematical, regardless of its meaning and functionality in the context.

2.2 Language in contemporary academic mathematics

2.2.1 Background: speech genres and registers as functional-ideological resources

When groups of people come in recurrent contact with one another, e.g. in the context of joint activity (professional practice, courtyard play, neighborhood gossip), they develop patterned

[4] Though, there have been debates about the content and practices levels too. A famous example is the foundational debates between constructivists and formalist (Ferreirós, 2008), about whether it is sensible to study 'mathematical objects' that cannot be constructed directly. Or, in the 18th century, about whether non-referential manipulation of symbols are legitimate (Archibald, 2008; Kitcher, 1984).

forms of communication – characteristic ways of talking, writing and moving – that, over time, become associated with that context. The Russian literary critic and semiotician Mikhail Bakhtin (1895 – 1975) referred to such patterned ways of talking as *social languages*, *social voices* (Bakhtin, 1981), and elsewhere, *speech genres* (Bakhtin, 2010). From analyzing the unique characteristics of the novel as a literary genre, Bakhtin recognized that a seemingly single national language such as English, in fact consists of "a diversity of speech types" (Bakhtin, 1981, p. 263), and coined the term *heteroglossia* (*raznorečie*) to describe this phenomenon.

Besides their standardized denotational "content" (what words refer to), language forms also express, or "index" (Ochs, 1996; Silverstein, 2003), intentions, worldviews, and speaker identities. Thus, speech genres – the typical words, sentences, intonations, and compositional structures people use in certain contexts – are not just observable repetitions in the use of certain language forms (though such patterns can clearly be documented). They are entire "verbal-ideological and social belief systems" (Bakhtin, 1981, p. 288), which are "specific points of view on the world, forms for conceptualizing the world in words, specific world views, each characterized by its own objects, meaning and values (pp. 291-292). For example, when people use scientific terms (e.g. when they say words like "acceleration") they are not just invoking the referential function of language (directing attention to observables in the physical world such as motion). They are also acting out intentions in the social-ideological world; they are "being scientific", they speak in the voice of scientific authority, and impose a Western-scientific (e.g. static, decomposable, decontextualized) interpretation of experience that could also be interpreted in other ways (e.g. as fleeting, interconnected, historicized, spiritual).

These are not just abstract musings about the nature of language. The ideological associations of language forms are resources for action. That is, associated intentionality and value systems can be strategically mobilized in interaction. For example, a child can invoke scientific terminology they learned in school to renegotiate social authority during interactions with parents (Wertsch & Rupert, 1993).

The languages of science and math are often referred to as *registers* (Halliday & Martin, 2003). Register is a term that comes from the field of linguistics, specifically the scholarly tradition of Systemic Functional Linguistics (SFL) and social semiotics. It was coined by the linguist Michael Halliday (1925-2018) and refers to "the functional varieties of language, characteristic of particular activities in which language is used, defined by systematic differences in the probabilities of various grammatical and semantic features in the texts of each register" (Lemke, 2005, p. 26).

While similar to Bakhtin's notion of speech genre, the register construct used by the SFL tradition differs in its focal scale and degree of systematicity. In SFL there is an emphasis on detailed systematic analysis of language forms, their associated functions, and semantic meanings on the scale of vocabulary and grammar (as the name *systemic functional* linguistics suggests). SFL registers are explicitly defined through their characteristic lexico-grammar and corresponding "meaning-potential", i.e. what that words and grammar can represent and do (Halliday & Martin, 2003; Lemke, 2005). Bakhtin, on the other hand, is less concerned with language forms on the scale of words and sentences. He looks at entire utterance's compositional style and focuses instead on the ideological and identity undertones of different "voices" therein.

Here, I follow the SFL tradition and adopt the term "register" to refer to forms of language typically used in mathematical activities. Specifically, I refer to the language characteristic of written contemporary academic mathematics texts as the "textual" or "formal" register. I do, however, think of this register in the context of Bakhtin's more general theory of social voices. SFL is useful for the technical aspect, for attending to features of language. The technical toolkit developed in SFL has been used in other traditions, such as Critical Discourse Analysis (Fairclough, 1992). Also, much of the research on features of mathematical language stem from the SFL tradition. But Bakhtin is useful for understanding language-use more holistically. And for attending to heteroglossia, to small intrusions of "other voices".

With this theoretical terminology as the backdrop, the purpose of the following section is to describe some characteristic features of the language of contemporary academic mathematical texts and review literature on "the language of mathematics."

2.2.2 Language(s) of contemporary academic mathematics.

First, let me start with a qualification. There is no such thing as *the* language of mathematics. Rather, there are many mathematical languages; there are the characteristic ways of talking and writing in math classrooms, there are math languages used in monetary exchanges, there are the math talk of engineers, architects, candy-sellers, there is the mathematical language in newspapers and media etc. While there may be some commonality in mathematical languages across contexts (e.g. use of symbolism, discursively objectified processes), these languages differ significantly to the point that it does not make much sense to speak of a single mathematical language. Of course, even within "the same" context, a single uniform language is always an illusion (Bakhtin, 1981). Some features of language repeat and stabilize, while others emerge and transform with each new enactment of activity[5].

In this dissertation, I am concerned with language in *contemporary academic mathematics*. I use the term contemporary academic mathematics as a label for the *activity system* (Cole & Engeström, 1993; Roth & Lee, 2007) of research mathematics, i.e. the work done by professional mathematicians working in academia today (Davis & Hersh, 1981). This system includes activities like writing research articles, reviewing manuscripts, giving lectures, teaching, and mentoring. Thus, the "language of contemporary academic mathematics" is an umbrella term, referring to the somewhat different language varieties used in research articles, in personal correspondences, in lectures and professional presentations, in collaborative research conversations, in informal "hallway" discussions, and so on (Solomon & O'Neill, 1998). As explained above, each aforementioned "language type" is itself stratified. For example, there is always some variability in terms of how research articles are written (e.g. across different subfields, see also (Burton & Morgan, 2000)). Nevertheless, there are common, relatively stabilized characteristics of language in different academic math contexts and those are the features I refer to when using a definite article in expressions such as "*the* language of lectures" or "*the* register of the text."

[5] Bakhtin referred to these two tendencies of language – stabilization and diversification – as centripetal and centrifugal forces.

2.2.3 The importance of written text

Of particular interest is the language of written academic mathematical texts (e.g. journal articles, books, notes). It is of central importance because all other activities in academic mathematics – lecturing, proving, thinking – are organized around it. Text is the product and focus of all mathematical action. Davis & Hersh (1981), for example, describe math lectures as devoted primarily to writing: "the main thing was what you wrote down. As to spoken words, either from the class or from the teacher, they were important insofar as they helped to communicate the import of what was written." (p. 3). Proving, a focus of mathematicians' research activity (Rav, 1999), is also an inherently textual practice. Text is what mathematicians *do*. In activity theoretic terms, it is mediational means and the object of the activity system (Cole & Engeström, 1993; Wertsch, 2012). All other forms of practice in academic mathematics are oriented toward the production, interpretation and distribution of text.

What about mathematical thinking? Certainly, there are imaginative, tactile and embodied experiences that do not seem to involve text or even language (Hadamard, 1954). Yet, such experiences are situated, i.e. contextualized, within textual practice. A mathematician will have a flash of insight after reaching an impasse in constructing a proof, a textual artifact. She will imagine a tactile model as an interpretation of definition, a piece of text. As Davis & Hersh (1981) say "to do mathematics one must have, at the very least, instruments of writing or recording and of duplication." (p. 13). There could be no "abstract math" without text, as academic mathematics today deals with objects that are, to begin with, textually constructed. Taking a socio-cultural and distributed view of cognition, I posit that "advanced mathematical thinking" is inherently textually mediated. Mathematicians are scribblers (Rotman, 1988).

2.2.4 Features of academic mathematical texts: deductive structure & register.

Contemporary mathematical texts are written in a relatively uniform style. This style stabilized in the mid 20th century as the culmination of developments in the 19th and early 20th centuries (Solomon & O'Neill, 1998). It has distinctive features on two scales: (1) overall text organization and genre units therein, and (2) the formal register (i.e. the words and grammar).

2.2.4.1 Organization: deductive text structure and the Definition-Theorem-Proof genre units.

Nearly all contemporary academic mathematical texts can be parsed into two interweaving types of text, each with distinct rhetorical and epistemic functions. At the center is a "mathematical" part, comprising of recognizable and clearly demarcated generic units such as definitions, theorems and proofs. Around these units is "meta-mathematical" commentary, which serves as connective and interpretive tissue between the mathematical units.

The "mathematical" part of contemporary mathematical texts is organized in a structure modeled after Euclid's elements (Davis & Hersh, 1981, p. 7). In a Euclidean text structure, knowledge is organized deductively; claims are linearly "built up" through logical inferences from explicitly stated assumptions. An idealized deductive text starts with axioms, which are assumed-true statements that function as basic building blocks for deriving the rest of the mathematical theory at hand. The text then develops through a succession of definitions, theorems and proofs. Definitions name and stipulate objects within the system, and similarly to axioms, are taken as labeling assumptions that do not require justification. Theorems (and closely related generic units such as propositions, lemmas, and corollaries) are statements *about* mathematical objects that are not *assumed* true within the constructed system and thus require proof. Proofs are textual units that function as justifications for theorems. To be considered a true mathematical statement, each theorem must have (at least) one proof. The inferences within proofs rely only on stated axioms, definitions and previously proven theorems. This idealized logical structure is illustrated schematically in Figure 1 below.

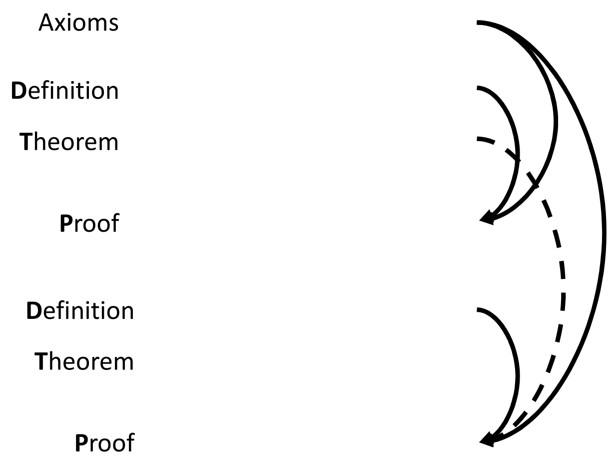
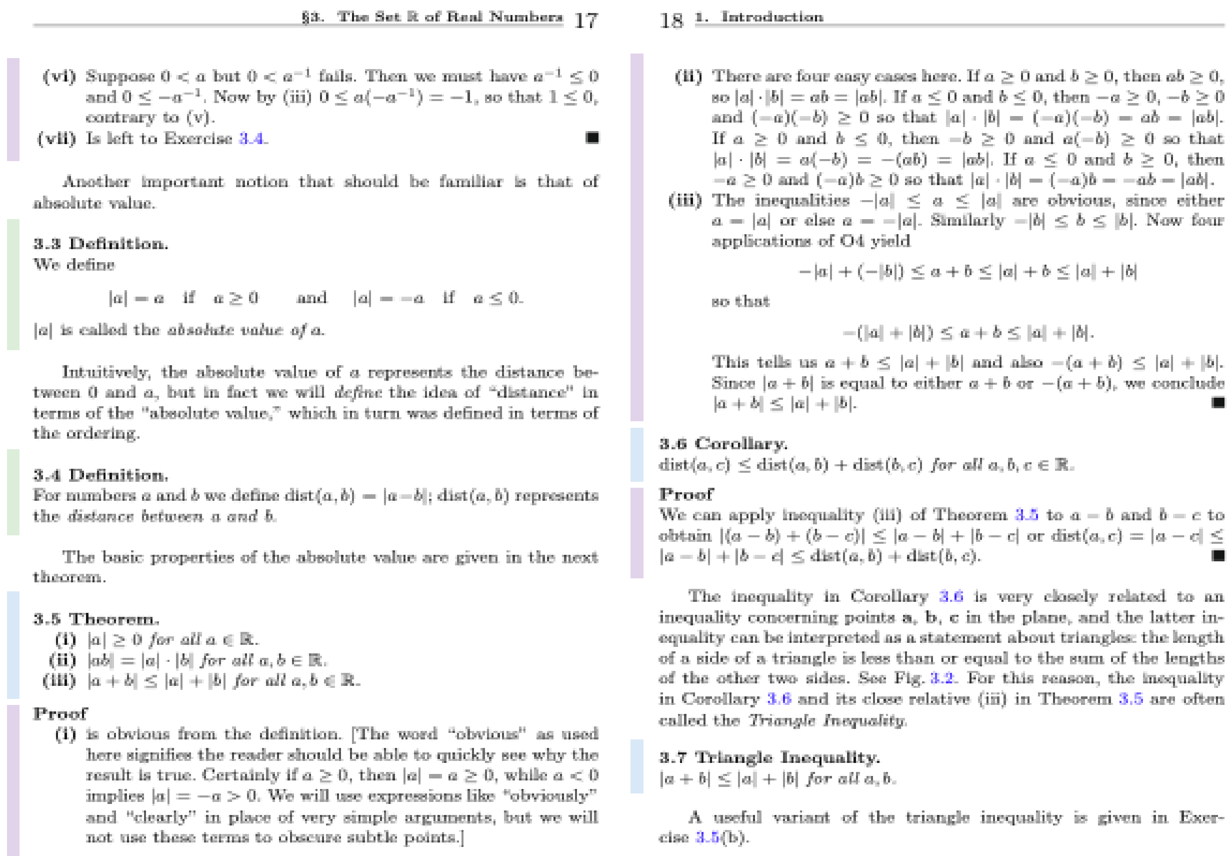


Figure 1: Schematic illustration of the deductive organization of mathematical text.

Actual mathematical texts are never as “clean” as this idealized model suggests. They include other generic units such as examples, graphs and diagrams, and in the case of textbooks, also units such as “exercises.” Furthermore, actual texts do not always embody the deductive linearity of its underlying arguments. Sometimes proofs are omitted, postponed, broken up to parts, or written non-linearly. Nevertheless, this idealized model of an axiomatic system is the underling structure that all contemporary academic texts take as a backdrop. The generic units of definitions, theorems, and proofs are nearly always present, dominant and organized in linear order (e.g. a proof will follow a theorem and not vice versa). Axioms are typically mentioned only in elementary texts that introduce a certain mathematical theory (e.g. textbooks). Most research articles and lectures tacitly assume underlying axioms and do not bother repeating them explicitly. Thus, the most salient feature of most contemporary mathematical texts is a succession of definitions, theorems and proofs, which is why the style is sometimes abbreviated as DTP (Thurston, 1994).

To illustrate this textual organization, consider pages 17-18 from (Ross, 1982) reproduced with highlights in Figure 2. These pages (starting from the fifth line on page 17) are devoted to the concept of a number’s “absolute value”. The mathematical units in the text are clearly demarcated through title formatting and indexing. Here, they are also color coded to highlight the deductive DTP structure. At the start of page 17 is the end of a proof from the previous page (purple). The section on absolute value starts with two definitions (green), then a theorem (blue) followed by its proof (purple), a corollary (blue) followed by its proof (purple), and another proposition called “Triangle Inequality” (blue).



§3. The Set ℝ of Real Numbers 17

(vi) Suppose $0 < a$ but $0 < a^{-1}$ fails. Then we must have $a^{-1} \leq 0$ and $0 \leq -a^{-1}$. Now by (iii) $0 \leq a(-a^{-1}) = -1$, so that $1 \leq 0$, contrary to (v).
(vii) Is left to Exercise 3.4. ■

Another important notion that should be familiar is that of absolute value.

3.3 Definition.
We define

$$|a| = a \quad \text{if} \quad a \geq 0 \qquad \text{and} \qquad |a| = -a \quad \text{if} \quad a \leq 0.$$

$|a|$ is called the *absolute value of* a.

Intuitively, the absolute value of a represents the distance between 0 and a, but in fact we will *define* the idea of "distance" in terms of the "absolute value," which in turn was defined in terms of the ordering.

3.4 Definition.
For numbers a and b we define $\text{dist}(a, b) = |a - b|$; $\text{dist}(a, b)$ represents the *distance between* a *and* b.

The basic properties of the absolute value are given in the next theorem.

3.5 Theorem.
(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$.
(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$.
(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$.

Proof
(i) is obvious from the definition. [The word "obvious" as used here signifies the reader should be able to quickly see why the result is true. Certainly if $a \geq 0$, then $|a| = a \geq 0$, while $a < 0$ implies $|a| = -a > 0$. We will use expressions like "obviously" and "clearly" in place of very simple arguments, but we will not use these terms to obscure subtle points.]

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(ii) There are four easy cases here. If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$, so $|a| \cdot |b| = ab = |ab|$. If $a \leq 0$ and $b \leq 0$, then $-a \geq 0$, $-b \geq 0$ and $(-a)(-b) \geq 0$ so that $|a| \cdot |b| = (-a)(-b) = ab = |ab|$. If $a \geq 0$ and $b \leq 0$, then $-b \geq 0$ and $a(-b) \geq 0$ so that $|a| \cdot |b| = a(-b) = -(ab) = |ab|$. If $a \leq 0$ and $b \geq 0$, then $-a \geq 0$ and $(-a)b \geq 0$ so that $|a| \cdot |b| = (-a)b = -ab = |ab|$.
(iii) The inequalities $-|a| \leq a \leq |a|$ are obvious, since either $a = |a|$ or else $a = -|a|$. Similarly $-|b| \leq b \leq |b|$. Now four applications of O4 yield

$$-|a| + (-|b|) \leq a + b \leq |a| + b \leq |a| + |b|$$

so that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

This tells us $a + b \leq |a| + |b|$ and also $-(a + b) \leq |a| + |b|$. Since $|a + b|$ is equal to either $a + b$ or $-(a + b)$, we conclude $|a + b| \leq |a| + |b|$. ■

3.6 Corollary.
$\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$ *for all* $a, b, c \in \mathbb{R}$.

Proof
We can apply inequality (iii) of Theorem 3.5 to $a - b$ and $b - c$ to obtain $|(a - b) + (b - c)| \leq |a - b| + |b - c|$ or $\text{dist}(a, c) = |a - c| \leq |a - b| + |b - c| \leq \text{dist}(a, b) + \text{dist}(b, c)$. ■

The inequality in Corollary 3.6 is very closely related to an inequality concerning points **a**, **b**, **c** in the plane, and the latter inequality can be interpreted as a statement about triangles: the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides. See Fig. 3.2. For this reason, the inequality in Corollary 3.6 and its close relative (iii) in Theorem 3.5 are often called the *Triangle Inequality*.

3.7 Triangle Inequality.
$|a + b| \leq |a| + |b|$ *for all* a, b.

A useful variant of the triangle inequality is given in Exercise 3.5(b).

Figure 2: An example of the DTP structure (Ross, 1982, pp 17-18)*: Definitions (green), Theorems (blue) and Proofs (purple).*

This textual structure allows deductive organization of knowledge. Each theorem is followed by a supporting proof, and within each proof, inferences rely only on previously stated assumptions (axioms, definitions) and already proven theorems. The indexing facilitates this deductive organization by allowing explicit short hand referencing of previous claims. For example, in the above text, the proof of Corollary 3.6 explicitly references a previous theorem as a source of justification ("We can apply inequality (iii) of Theorem 3.5"). It also implicitly relies on the labeling conventions formulated in Definitions 3.3 and 3.4 (e.g. in equating the expressions "$dist(a, c)$" and "$|a - c|$").

The remaining non-highlighted paragraphs in the above text excerpt are the "meta-mathematical" commentary. Meta-mathematical statements serve various rhetorical functions in the text. Often, they make connections between different mathematical units, e.g. by announcing what comes next, how it relates to what came before, and its relative significance within the overall theory (e.g. the meta-commentary "the basic properties of the absolute value are given in the next theorem" preceding Theorem 3.5). At times, meta-mathematical commentary consists of interpretations of statements made within mathematical units. For example, in the above text, the commentary that follows Definition 3.3. provides an "intuitive" interpretation of the definition: "Intuitively, the absolute value of a represents the distance between 0 and a." Similarly, the longer paragraph at the end of page 18 discusses a geometric interpretation of Corollary 3.6, referencing an illustrating figure given on the next page of book (see Figure YY). This interpretation is then used to justify the subsequent labeling of 3.7 as “Triangle inequality.”

Exercises 19

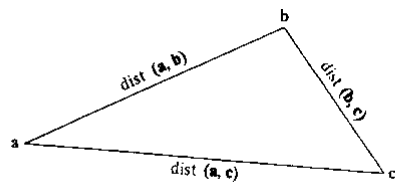

FIGURE 3.2

2.2.4.2 The textual register: symbolic notation, density, and the grammar of alienation. Mathematical texts also exhibit regularities on the smaller scale of words and grammar, i.e. on the scale of its register (Halliday & Martin, 2003). What features of language do we see if we zoom in to read within each genre unit?

Perhaps the most noticeable feature is the amount of specialized symbolic notation permeating the text. Some sentences are almost entirely made up of symbols (see e.g. Theorem 3.5). The multimodality of mathematical texts (O’Halloran, 2008, 2015; Schleppegrell, 2007) creates a significant access barrier for the uninitiated. Because of the symbols, most people attempting to read – including undergraduate students new to the practice (Shepherd & van de Sande, 2014) – would not be able to even “decode” (i.e. verbalize) the text, let alone read it with comprehension. They would be stuck at the very beginning.

However, inaccessible verbalization is far from the only challenge mathematical texts pose. A common complaint about academic mathematical texts is that they are *dense* (Schleppegrell, 2007). Symbols, as well as grammatical devices such as nominalization (verbs turned into nouns) and lexical density, compact potentially elaborate processes and logical relationships into very concise form. For example, the second statement in Theorem 3.5

$$|ab| \leq |a||b| \;\; for\ all\ a, b \in \mathbb{R}$$

can be restated, or “unpacked”, through much longer descriptions:

> *If we take any two real numbers a and b, multiply them ab and then compute the absolute value $|ab|$, we will always get the same result to if we first compute the absolute value of each number separately $|a|$, $|b|$, and then multiply the results $|a||b|$.*

Indeed, the subsequent proof of this statement requires unpacking it in yet another way, by considering four different possible situations, or “cases”: either both numbers are positive, either a is positive and b is negative, either a is negative and b is positive, or both are negative.

2.2.4.3 Alienating features of the textual register

Of particular significance in the context of this dissertation are features of the textual register that contribute to obfuscation of human agency and experience in mathematical activity, phenomena I refer to as *discursive alienation*. These include linguistic devices that obscure action altogether (e.g. nominalization, timelessness), omit the agent (e.g. passive voice, non-finite verb forms), or construct abstract or ambiguous agents (the use of imperatives, the mathematical 'we') whose actions fall almost exclusively within the cognitive-epistemic domain. And, perhaps even more importantly, many dimensions of human experience are systematically left out: emotions, the human body, social identity, goals, values, errors, conflict, and geographical context rarely appear in mathematical texts.

So, what is meant by discursive alienation? In a section dedicated to the cultural caricature of an 'ideal mathematician,' Davis and Hersh (1981) described the mathematician's writing style in the following way:

> "His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being. It gives the impression that, from the stated definitions, the desired results follow infallibly by a purely mechanical procedure." (p. 36)

Another perspective on discursive alienation was articulated by Jay Lemke (1990) in the context of the language of science:

> *"*The language of classroom science sets up *a pervasive and false opposition* between a world of objective, authoritative, impersonal, humorless scientific fact and the ordinary, personal world of human uncertainties, judgements, values, and interests." (pp. 129-130)

Sfard (2008), defined discursive alienation as:

> "discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (p. 295)

Different kinds of discursive forms, especially when coordinated together, contribute to the experience of alienation in mathematical discourse.

First, there is what has been referred to as *reification* or *objectification*, the practice of *"*... representing actions and events, and also qualities, as if they were objects." (Halliday & Martin, 2003). Linguistically, objectification can be accomplished through *grammatical metaphors* such as *nominalization*, which refers to the process by which verbs, adjectives, and adverbs are turned into nouns. For example, the verb 'to count' can be nominalized to the gerund 'counting' or simply treated as a noun: 'a count.' Symbolic notation, as well as specialized terminology, are another means for objectification in mathematical discourse. For example, the symbolic expression 7 and the term 'seven' reify the process of counting (Sfard, 2008).

Objectification, whether through nominalization, the use of symbolic notation, or coinage of new terminology, is a central discursive mechanism for doing mathematics and sciences, as it allows turning fleeting processes and experiences into objects for reflection. Through objectification, scientific and mathematical discourses construe the very reality they represent:

> "We have to abandon the naïve 'correspondence' notion of language, and adopt a more constructivist approach to it. The language of science demonstrates rather convincingly how language does not simply correspond to, reflect or describe human experience; rather, it interprets or, as we prefer to say, 'construes' it. A scientific theory is a linguistic construal of experience". (Halliday & Martin, 2003, p. 8)

While scientific theories can be seen as *about* the physical world, the fact that abstract discourses construe the very objects they refer to is particularly pronounced in mathematics, where the objects have no clear existence outside of the language games in which they are deployed (Sfard, 2008; Wittgenstein, 1953). Thus, objectification is in a way unavoidable in mathematics. If we want to talk about 'objects' such as numbers, functions, and derivatives, we have to encapsulate dynamic processes such as counting, co-varying, and approximating into nouns. Or, as Morgan (2016) clarifies: "it is not an arbitrary feature but is functional in that in enables complex phenomena to be assigned properties and to be put into relationship with one another." (p. 129)

What does this have to do with alienation? Well, as Sfard (2008) argued, one consequence of objectification is the disappearance of an agent, as now the processes the themselves (originally performed by a human) can grammatically become the performers of action. For example, a sentence such as "I counted and showed that…" could be nominalized to "the count showed…" In the original sentence, an 'I' was doing the counting and concluding. In the second sentence, turning the process of counting into a noun enabled positioning the process ('the count') as the subject performing the act of showing, "without the participation of human beings" (Sfard, 2008, p. 295).

Similar affects are achieved through the use of passive voice and non-finite verb forms (Morgan, 2016). For example, one can remove personal engagement in counting and showing by saying: "as shown by counting." While neither showing or counting have been nominalized – i.e., they still appear as verbs in the sentence – the agent doing those actions is absent.

Such discursive eliminations of the human agent from mathematical sentences also renders statements more objectively true. Instead of a fallible personal experience of counting and concluding, it was the count itself that showed the result. When human agents are grammatically absent, there is less room for personal judgment and error. Indeed, such absences of human agency can help texts seem more 'mathematical,' for example, in the eyes of teachers assessing students' written solutions to problem (Morgan, 1996, 2004), or in researchers' analyses of mathematicians' personal correspondences (Solomon & O'Neill, 1998).

Long texts written in nominalized, passive form can be difficult to read. In contemporary academic mathematics, a commonly used and recommended tactic for avoiding passive voice is using the pronoun 'we' as the agent performing mathematical actions (Knuth, Larrabee, & Roberts, 1989). Constructions such as "we show …" and "we count …" are pervasive in mathematical research articles (Tanswell & Inglis, 2023). Authors of mathematical writing guides explain that 'we' is preferable to using 'I,' the first person singular, which "most people consider it pompous and inappropriate." (Krantz, 1997, p. 33). Halmos (1970), for example, wrote that the use of the first person singular "I" sometimes has "a repellent effect, as arrogance or ex-cathedra preaching" (p. 141). The plural 'we' is also seen as preferable to the third person singular 'one', which "often leaves the writer struggling with awkward sentence structure." (Krantz, 1997, pp. 33-34).

Who are the 'we' in mathematical writing?

Halmos (1970), called it the "editorial 'we'" and advised writers of mathematical text to use it to "mean "the author and the reader" (or "the lecturer and the audience")." (p. 141). Similarly, Knuth et al., (1989) suggested that: ""we" should be used in contexts where it means "you and me together", not a formal equivalent of "I". Think of a dialog between author and reader." (p. 2). Krantz (1997) was not as explicit about the intended referent of 'we,' and instead commented that 'we' "stresses the participatory nature of the enterprise, and encourages the reader to push on." (p. 33). At the end, however, for Krantz (1997) the main reason for using 'we' is convention:

> "The custom in modern mathematics is to use the first person plural, or "we." … Moreover, since "we" is what people are accustomed to hearing, it is less likely to jar their ears, or to distract them, than one of the other choices." (p. 33)

Similarly, in the context of a post-structural semiotic analysis of the enunciative positions construed in one on of Gödel's papers, Wagner (2010b) wrote the following about the mathematical 'we':

> "… we needn't read too much into this we. We is simply part of the code. Its use is as imposed upon Gödel as is using I when I speak to a friend."

Burton & Morgan (2000) empirical study analyzing the construction of subjectivity in published mathematics research papers corroborated some of these claims. For example, they found that in co-authored papers, while 'we' was sometimes used to refer to the actual authors of the paper (what Burton and Morgan called an *agentic* use), in most cases 'we' was used in a manner they called *conventional*, that is, to avoid the passive voice through 'we' + present tense constructions.

Yet, whatever the convention is, there is also the experience of the reader to reckon with. Some mathematics educators have argued that the mathematical 'we' can be experienced as coercive, forcing the reader into actions and perspectives they have not had a chance to agree to (Pimm, 1987). At the very least, given all the above quotes, the mathematical 'we' is ambiguous.

Another method for grammatically enacting agency in mathematical text is the use of imperatives, that is, instructions such as " Show…" and "Count…" that acts as directives for action. The prevalence of such verb-forms was noted already by Pimm (1987), and like the use of 'we,' he interpreted them as coercive and alienating. A more substantive discussion of imperatives is provided by Rotman's (1988) semiotic analysis. Going beyond the general observation that imperatives are prevalent in mathematical texts, Rotman suggested that different kinds of imperatives correspond to different types of mathematical subjectivities. Drawing on ideas from Piercean semiotics, he identified three semiotic actors which the text construes as *doers* of mathematics. Two are abstract agencies discursively constructed through impeartives for the purpose of performing epistemic mathematical actions (e.g. “Conider…” and “Add…”). Only one of the three agencies corresponds to human subjectivity as it is experienced outside of mathematics, i.e. one that is situated in socio-historical time and “operates with the signs of natural language and can answer to the agency named by the “I” or ordinary nonmathematical discourse” (p. 15). This distinction is similar to the one Burton and Morgan (2000) made between agentic uses of ‘we’ (referring to the authors) and conventional ones, which are deployed only for the purpose of avoiding a passive tone and have an ambiguous referent.

Rotman's (1988) conceptual analysis highlights that in order to understand the nature of subjectivities a mathematical text constructs, whether through the use of personal pronouns or imperatives, it is important to also attend to the *kinds of verbs* used to depict agents’ mathematical actions. A recent empirical study by Tanswell and Inglis (2023) sheds more light on this. In this study, the authors leveraged corpus linguistic methods to analyze verb usage in a large[6] data set of mathematics research articles, looking specifically at the distribution of verbs across both the imperative and the ‘We + [verb]’ forms. Common verbs found in the imperative form included: let, suppose, note, consider, assume, recall, and define. Common verbs found in the ‘we +[verb]’ construction included: have, claim, consider, denote, prove, and show. While this was not the focus of the authors’ analysis, here I wish to point out that all the verbs their analysis flagged as common, especially those *most* common, refer to epistemic and cognitive action. Thus, even when an agency is grammatically present (e.g. there is a ‘we’ that is ‘claiming’), the constructed subject is not a regular, multi-dimensional human being. It is an abstract epistemic agent discursively construed to be the *doer* of mathematics.

One way by which mathematical discourse obfuscates human experience in mathematics is by what it omits. It does not reference the human body, or affective experience. The mathematical ‘we’ may ‘consider’ and ‘suppose,’ but it does not, for example, feel fear. It rarely references specific personal identities, their socio-historical context, and geographic locations. Chapter 7 will examine how lecture discourse at times breaks with these conventions of the textual register, and manages to construct dimensions of human experience that not traditionally enacted in mathematics.

[6] The primary data were all mathematics research articles uploaded to arXiv (an online repository of pre-prints used by many mathematicians) in the first four months of 2009. After appropriate scraping of natural language, the corpus contained 30,892,695 words. (Tanswell & Inglis, 2023)

3 Stories about the purpose of mathematics, values, and teaching.

3.1 Meta-stories in and about epistemic games.

Communities that engage in epistemic games typically have folk theories[7] *about* these games, which I hereafter refer to as meta-stories[8]. Mathematics is a community of practice centered on an epistemic game, and as such, it too has meta-stories such as 'math is objective and universal' and 'the purpose of math is to produce theorems and proofs.' Some of these meta-stories are dominant and mainstream, others less so.

Meta-stories serve different functions in activity. Practitioners may retell meta-stories to negotiate shared goals ('the purpose of math is to prove theorems'), demarcate disciplinary boundaries ('since there are no proofs, this is not math'), or affirm community membership ('her paper has proofs, so she is a mathematician').

One context in which meta-stories are often articulated is apprenticeship of newcomers. When insiders introduce newcomers to their community's epistemic games, like when the DTP game is taught in RA courses, they often feel the need to tell meta-stories about this game – what it is like, what it is good for, and how one does it – so that learners get a sense of what this game is all about and be motivated to participate in it.

A practice such as contemporary academic mathematics does not have a single, objectively true set of goals and values. In particular, there is no universal consensus on a single shared ultimate purpose for mathematical practice. The goals and values of a practice may appear fixed and stationary, but they are always in flux. In mathematics, as in any community, goals and values are historically contingent, and at any given point in time, there are disagreements between community members. So, the goals and values in math are always motion, even if that motion is slow, and they exhibit internal tensions due to forces pulling in different directions. Any sense we may have of a practice having clear and agreed-upon set of goals and values is *an accomplishment*. Goals and values are constantly negotiated and reproduced by a community of practice, in part, through the stories practitioners tell.

In contemporary academic mathematics, different meta-stories about the DTP epistemic game exist. Different stories provide different measures by which to evaluate whether particular instances of practice are legitimate, relevant, useful or important. In so doing, different stories also provide different opportunities for students to belong and see themselves as part of the game.

> "mathematicians are not a monolithic category; they, too, are multidimensional in the narratives they promote. The stories they tell will vary from their perspective, just as

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[7] Folk theories are stories *about* practice told by insiders to the practice. The label 'folk' is used to contrast them to theories offered by outsiders such as anthropologists who study the practice.

[8] I use the word meta following (Sfard, 2008), to contrast stories about mathematical objects which are constitutive of mathematical discourse (e.g. "two plus three equals five"), with meta-mathematical discourse which is discourse *about* the discourse of mathematics (e.g. "the statement 2+3=5 is objective truth"). Of course, one can discern many levels of 'meta'. Here, I gloss over this nuance, and use the term meta to refer to any discourse *about* mathematical practice.

> other actors (e.g., parents, students, and teacher leaders) have multidimensional perspectives." (Gutiérrez, Myers, & Kokka, 2023, p.13)

3.2 Meta-stories about the purpose of math

3.2.1 Four meta-stories about the purpose of mathematics

Meta-stories about the purpose of contemporary academic mathematics are a case in point. One dominant meta-story is that the purpose of math is to *produce knowledge* in the form of *lists of theorems and proofs*. For example, in a paper about the norms and values of proof, Dawkins & Weber (2017) articulated the following meta-story

> "At a broad level, the mathematical community aims to increase its *list of statements* that it believes to be true while minimizing the likelihood that this list contains a statement that is false." (Dawkins & Weber, 2017, p. 128)

In their paper, Dawkins and Weber used this "broad level" goal for mathematical practice as a whole as a starting point for discussing a list of values that they suggest are consensually shared among mathematicians and seen as supporting this top-level objective: a-priori and a-contextual arguments, understanding, and consistency of standards.

However, some mathematicians explicitly disagree with the above stated goal of *producing knowledge* by increasing the '*list of true statements*.' For example, in an oft cited meta-mathematical essay, the mathematician Thurston (1994) contrasted the goal of "proving theorems" (p. 161), which he said is tacitly assumed within the "the popular model" of mathematics (p. 162), with a goal he articulated as "advancing human understanding of mathematics" (p. 162). Thurston (1994) clearly favored the latter:

> "We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics." (p. 163)

In making a distinction between these two why-stories, I am not suggesting that they are necessarily in conflict with one another. In fact, they can overlap. Thurston (1994) discussed rigor and proof as one of many ways by which mathematical understanding is communicated (p. 168). Dawkins and Weber (2017) listed "understanding" as one of the values meant to support the ultimate purpose of producing *lists of true statements*. Though, citing Thurston (1994), they write that only *some* mathematicians desire understanding in addition to truth, and that these two goals may be at odds with one another:

> " Some mathematicians argue that what they desire is not only a list of true mathematical theorems but also (and perhaps primarily) an understanding of the mathematical theory that they are studying (e.g., Thurston, 1994; Rav, 1999) …
>
> Unfortunately, the quest of a priori truth and mathematical rigor is sometimes at odds with gaining understanding." (p. 130)

One way to characterize the difference between these two why-stories from Thurston (1994) and Dawkins and Weber (2017) is to say that they share the same goal of ‘producing knowledge,’ yet differ in where they interpret this knowledge to reside; in ‘lists of true statements’ or in ‘human understanding.’

But, even if one takes as given the overarching goal of increasing mathematical knowledge (whether operationalized as ‘lists of true statements’ or ‘human understanding”), there can still be disagreements about what counts as better or more significant contribution to this goal. Namely, what counts as good (or better) knowledge in mathematics? While it may seem that there is consensus about what counts as ‘good’ math, and there indeed may be mainstream or dominant perspectives, disagreements do exist, even among high-profile members.

This is reflected, for example, in the following quote by Timothy Gowers, taken from the preface to the Princeton Companion to Mathematics (Gowers, Barrow-Green, & Leader, 2008):

> “… in an era where so much mathematics is being published that no individual can understand more than a tiny fraction of it, it is useful to know not just which arrangements of symbols form grammatically correct mathematical statements, but also which of these statements *deserve* our attention.
>
> Of course, one cannot hope to give a fully objective answer to such a question, and *different mathematicians can legitimately disagree about what they find interesting*. For that reason, this book … has many authors with many *different points of view*.” (Gowers, Barrow-Green, & Leader, 2008, p. ix)

In this excerpt, Gowers highlighted the importance of value judgements in mathematics (“which of these statements *deserve* our attention”), and pointed out that these judgements cannot be “fully objective,” and that indeed, mathematicians “legitimately disagree” and have “different points of view.”

One example of such ‘legitimate disagreement’ was discussed by Gowers himself in an earlier meta-mathematical essay (Gowers, 2000) in which he wrote about ‘two cultures of mathematics’:

> “The “two cultures” I wish to discuss will be familiar to all professional mathematicians. Loosely speaking, I mean the distinction between mathematicians who regard their *central aim* as being to solve problems, and those who are more concerned with building and understanding theories.” (p. 1)

Gowers went on to explain that the focus on problems and theories is not mutually exclusive, as most mathematicians engage in both problem-solving and theory building. However, mathematicians do differ in which of these two purposes they prioritize. Gowers exemplified these two complementary priorities by articulating two statements, each framing one of the goals as subservient to the other (Gowers, 2000, p.1):

> “(i) The point of solving problems is to understand mathematics better.
> (ii) The point of understanding mathematics is to become better able to solve problems.”

Gowers took the position that both goals are an important part of mathematics: "It is obvious that mathematics needs both sorts of mathematicians" (p. 2), but then proceeded to lament that the two goals do not have equal status in the community:

> "… the subjects that appeal to theory-builders are, at the moment, much more fashionable than the ones that appeal to problem-solvers. Moreover, mathematicians in the theory-building areas often regard what they are doing as the central core (Atiyah uses this exact phrase) of mathematics, with subjects such as combinatorics thought of as peripheral and not particularly relevant to the main aims of mathematics."

He further repeated Atiyah's rationales for why theory-building is closer to the main aim of mathematics:

> "… [Atiyah] makes the point … that so much mathematics is produced that it is not possible for all of it to be remembered. The processes of abstraction and generalization are therefore very important as a means of making sense of the huge mass of raw data (that is, proofs of individual theorems) and enabling at least some of it to be passed on. The results that will last are the ones that can be organized coherently and explained economically to future generations of mathematicians. Of course, some results will be remembered because they solve very famous problems, but even these, if they do not fit into an organizing framework, are unlikely to be studied in detail by more than a handful of mathematicians.
>
> Thus, it is useful to think not so much about the intrinsic interest of a mathematical result as about how effectively that result can be communicated to other mathematicians, both present and future." (p. 4)

Here we can see echoes of Thurston's *human understanding* goal. According to Gowers, Atiyah suggested that what increases human understanding, what can be passed on to future generations, is not 'lists of true-statements' ("raw data") but theories that help make-sense of these lists. So, the ultimate goal is to produce knowledge in the form of coherent theories, so that humans across generations can understand. We might say that the goal here is *producing transmittable knowledge*.

In the rest of the essay Gowers went on to defend combinatorics, his discipline, against the critiques it gets from theory-builders like Atiyah. Namely, that combinatorics is 'just' a list of miscellaneous problems not connected by a coherent theory, lacks a sense of direction, and hence does not contribute to the main goal of *producing transmittable knowledge*. Gowers does so by explaining that combinatorics does in fact have a coherent, organizing "structure," albeit less explicit to outsiders: combinatorics, Gowers argued, is connected through general principles of problem-solving approaches. Importantly, Gowers did not abandon the top level why-story of *producing transmittable knowledge*, put forth by Atiyah. He simply suggested expanding what counts as transmittable knowledge to also include disciplines which cohere around problem-solving strategies, rather than only those that cohere around unifying theory for its 'content.'

Atiyah (1984), however, discussed other criteria of significance, which Gowers' argument did not address. Atiyah stressed coherence not only between different mathematical results, but also connections to past interests, both in mathematics and other subjects.

> "I strongly disagree with the view that mathematics is simply a collection of separate subjects, that you can invent a new branch of mathematics by writing down axioms 1, 2, 3 and going away and working on your own. Mathematics is much more of an organic development. It has a long history of connections with the past and connections with other subjects." (Atiyah, 1984, p. 11)

Indeed, for Atiyah (1984), mathematics is justifiable and valuable only to the extent that it is connected to its historical core, and thus can be seen as contributing to the scientific enterprise as a whole:

> "Why do we do mathematics? We mainly do mathematics because we enjoy doing mathematics. But in a deeper sense, why should we be paid to do mathematics? If one asks for the justification for that, then I think one has to take the view that mathematics is part of the general scientific culture. …
>
> If mathematics is to be thought of as fragmented specializations, all going off independently and justifying themselves, then it is very hard to argue why people should be paid to do this. We are not entertainers, like tennis players. The only justification is that it is a real contribution to human thought. Even if I'm not directly working in applied mathematics, I feel that I'm contributing to the sort of mathematics that can and will be useful for people who are interested in applying mathematics to other things." (Atiyah, 1984, p. 12)

According to Atiyah (1984), though it might not be a personal motivation, the ultimate justification for mathematics is in its utility as part of the general scientific endeavor. As long as one does work that contributes to the "sort of mathematics" that has historically been useful, one may be justified in producing that knowledge, because there is good reason to believe it will be useful for someone. Indeed, he later said "if you do interesting, basic mathematical work in the mainstream, then you shouldn't be surprised when others find it a useful tool." (Atiyah, 1984, p. 13). Atiyah (1984) contrasted work in the mathematical mainstream, which he deemed valuable, with:

> "the tendency today for people to develop whole areas of mathematics on their own, in a rather abstract fashion. They just go on beavering away. If you ask what is it all for, what is its significance, what does it connect with, you find that they don't know." (Atiyah, 1984, p. 11)

Thus, for Atiyah, a statement (or even an entire area of mathematics) being true is not enough to justify its production as part of mathematical activity. Mathematical knowledge needs to be motivated and connected to what is already deemed significant, interesting and useful.

A similar argument was put forth by Tao (2007) in a meta-essay about what constitutes good mathematics. In this essay, Tao argued that while it is possible to list individual qualities that constitute good mathematics (e.g. depth, rigor, elegance), and different mathematicians and subfields may find some more important than others, what is ultimately most important is whether the produced knowledge is part of *a great mathematical story*.

> "… I believe that good mathematics is more than simply the process of solving problems, building theories, and making arguments shorter, stronger, clearer, more elegant, or more rigorous, though these are of course all admirable goals. While achieving all of these tasks (and debating which ones should have higher priority within any given field), we should also be aware of any possible larger context that one's results could be placed in, as this may well lead to the greatest long-term benefit for the result, for the field, and for mathematics as a whole." (Tao, 2007, p. 633)

The majority of Tao's essay was devoted to a "case study" of one such larger context or story; the developments that led to the articulation and proof of Szemerédi's theorem. The story demonstrated how one problem can sprout interest in and discovery of a range of new mathematical phenomena and questions, which in turn lead to developments of techniques that weave together diverse mathematical topics, such as combinatorics, ergodic theory, and Fourier analysis. Thus, Tao's story can be seen as an instantiation of connections and cross-fertilization between different mathematical fields, highly prized by Atiyah.

A more recent example of a goal articulation for mathematics that provides a stark alternative to the dominant meta-stories about "producing knowledge" is Francis Su's (2020) "mathematics for human flourishing." First in an AMS-MAA retiring presidential address, and later elaborated in a book by the same name, Su put forth a vision and mission for mathematics that focuses on human experience. Like Atiyah, he asked (and answered) the question "Why do mathematics?" but his emphasis was elsewhere:

> "So if you asked me: why do mathematics? I would say: mathematics helps people flourish. Mathematics is for human flourishing." (Su, 2020, p. 10)

Su (2020) drew on the Greek notion of human flourishing, *eudaimonia*, to argue that

> "the proper practice of mathematics cultivates virtues that help people flourish. ... And the movement towards virtue is aroused by basic human desires – the universal longings that we all have – which fundamentally motivate everything we do" (Su, 2020, p. 10)

The rest of the book is then devoted to an overview of how different desires can be exercised in mathematics (e.g. play, beauty, truth), and the virtues they can cultivate (e.g. hopefulness, joy, humility).

Let's take stock. What meta-stories have we found so far, and how are they all connected or different from one another?

1. Lists of true-statements. (Dawkins & Weber, 2017)
2. Transmittable & useful knowledge in the form of theories (Atiyah, 1984), problem solving strategies (Gowers, 2000), or mathematical stories (Tao, 2007).

3. Human understanding of mathematics (Thurston, 1994)
4. Human flourishing (Su, 2020)

3.2.2 Meta-stories not contradictory, just different priorities

These four purposes are not mutually exclusive, in the sense that goals highlighted in one meta-story can be legitimately incorporated as part of another goal. For example, *advancing human understanding of mathematics*, positioned as *the* purpose in Thurston's story, is articulated as a value added to the purpose of *producing lists of true statement* in Dawkins' and Weber's meta-story. In contrast, rigorous proofs, positioned as the purpose in Dawkins' and Weber's story, is one mean to advance human understanding in Thurston's meta-story, and one mean for making knowledge *transmittable* in Atiyah's story:

> "… and the point of having a rigorous mathematical statement is so that something which in the first place is subjective and depends very much on personal intuition, becomes objective and capable of transmission. … in order for this to be conveyed to other people it must eventually be presented in such a way that it is unambiguous and capable of being understood by someone who does not necessarily have the same kind of insight as the originator." (Atiyah, 1974, p. 214)

Similarly, *producing transmittable & useful knowledge* may be seen as an ultimate purpose, or as one of many ways to advance human understanding of mathematics. Finally, we can think of *advancing human understanding* from the perspective of the human flourishing framework offered by Su, and frame "understanding" as one among many virtue cultivating desires realized by mathematical activity. Thus, similarly to the point Gowers made when contrasting problem-solvers and theory-builders, I see the main difference between the aforementioned meta-stories as a difference in priorities; which goal is the main purpose, and which are subservient to it.

3.2.3 Meta-stories differ in their focus on human beings

An important thing to note about this list of meta-stories is an increased focus on human beings, rather than on knowledge as something that resides outside and independently of people. As we go down the list of meta-stories, we see an increased importance given to human experience. In meta-story (1), the main criteria of truth does not reference human beings. Meta-story (2) adds the criteria that knowledge needs to be transmittable to future generations and useful for humanity's scientific enterprise as a whole. Indeed, in his many meta-essays Atiyah emphasized that mathematics is a human activity and its goals and values should be judged accordingly.

Meta-story (3) seems to share with (2) the concern with human understanding of mathematics. However, there is an important distinction in emphases. In the various articulations of the *transmittable knowledge* meta-story, the concern seems to be with the transmission of mathematical knowledge by *humanity* as a whole, and this can easily be interpreted as a concern for access by a *select few*. If knowledge is effectively passed down to an elite group of practitioners, that can be seen as sufficient to preserve the accumulation of mathematical knowledge by humanity. Thurston, in contrast, explicitly positioned the dissemination of mathematical understanding to the masses as an equally, if not more, important goal to producing new mathematical knowledge at the frontier[9].

Finally, Su expanded not only the scope of *who* matters, but also what dimensions of human experience matter. His call to action was "to draw more people into mathematics because they will see how mathematics connects to their deepest human desires." (Su, 2017, p. 485). Su also explicitly talked about justice as a virtue and desire that can and should be realized in mathematics.

3.3 Meta-stories, value judgements, and power.

Meta-stories about purpose clearly matter to mathematicians. While it's true that most mathematicians do not regularly questions the purpose of their practice, or think about these issues while they go about their daily business of "doing mathematics," the above quotes illustrate that high-profile members[10] of the mathematics community routinely take time to articulate and debate what is or should be the ultimate goal for doing mathematics.

One reason why purpose-stories matter so much to practitioners is that they are related to value-judgements. Mathematicians are regularly faced with making decisions about the significance and importance of particular instances of mathematical knowledge and activity. Common examples of professional situations in which such value judgements occur include reviewing journal articles and grants, deciding on research problems to tackle, and evaluating people's 'merit' for purposes of admission, hiring, and promotions. As Weber and Dawkins (2017) point out, mathematicians' awareness of their community's normative criteria for judgement shape not only how they evaluate others' work, but also the decisions they make in anticipation of others' judgements. That is, what they perceive to be normative community values influence how they choose to spend their time, the problems they choose to work on, and how they communicate their results. Thus, normative values in the community have immense influence on what gets done in mathematics.

To see how value judgements are related to the different purposes identified in the previous section let us consider how particular mathematical contributions and actions would be evaluated in relation to the four meta-stories for mathematics: (1) lists of true-statements, (2) coherent & useful knowledge, (3) human understanding, and (4) human-flourishing.

Within meta-story (1), any proved theorem can be seen as a desirable contribution as long as the proof is *valid* under the established criteria of rigor. In contrast, a proof that is not considered valid or rigorous would be seen as a less desirable, if not entirely inappropriate, mathematical contribution. Such value judgements are manifest, for example, in Jaffe & Quine's (1993) meta-essay complaining about Thurston's work receiving credit for not fully rigorous work. The authors suggest that results such as Thurston's should be considered speculative (or 'theoretical') mathematics, rather than mathematics proper. Relatedly, Wagner (2023) recently argued that

[9] That shift in emphasis from knowledge at the frontier to understanding at more basic levels is one thing that explains the popularity of Thurston's essay among mathematics educators.

[10] Thurston, Gowers, Atiyah and Tao are all recipients of the Fields Medal, the highest award for scholarly contributions in mathematics. Su was president of the Mathematical Association of America (MAA) and vice president of the American Mathematical Society (AMS), the two biggest professional math organizations in the US.

everything that cannot be formalized was pushed out of mathematics proper in the beginning of the 20th century.

Meta-story (2) introduces additional considerations: theorems and proofs need to not only be valid, but also *coherently connected* to the main progression of mathematical knowledge, which is motivated by problems outside of mathematics. New mathematics needs to be seen as contributing to the elaboration of existing frameworks, phenomena, and research directions of interest. In Tao's terms, they need to be part of some great mathematical story. In why-story (1), truth and validity, in the sense of justifications being a-priori and a-contextual, are the main criteria for judging whether a contribution is legitimate. In why-story (2), coherence, interest, utility, centrality and significance are also, and potentially even more, important. In light of goal (2), synthesizing knowledge into transmittable form, even if does not produce new lists of true statements, can be seen as an important contribution in and of itself. In contrast, producing knew "raw material" that is not connected to or motivated by existing concerns, can be seen as an undesirable contribution.

In a presidential address to the London Mathematical Society, Atiyah (1978), explicitly positioned the axiomatic method (hallmark of (1)) as *a tool* for coherent theory, rather than a goal in and of itself:

> "I feel that this needs to be emphasized because the axiomatic era has tended to divide mathematics into specialist branches, each restricted to developing the consequences of a given set of axioms. Now I am not entirely against the axiomatic approach so long as it is regarded as a convenient temporary device to concentrate the mind, but it *should not be given too high a status*."

So, the goals of axiomatic validity and coherent theory are not necessarily in conflict, but one needs to keep in mind which is the main goal (coherence), and which is subservient, and accordingly, not give "too high a status" to the axiomatic approach. In meta-story (3), the objective is human understanding, rather than an external body of knowledge. In light of this ultimate purpose, proofs that are illuminating but perhaps not sufficiently rigorous can be very valuable. In contrast, proofs that are correct but not illuminating (Thurston gave the example of the computer assisted proof of the 4-color theorem), are not as desirable. Also, actions other than proof, such as effective teaching and mentoring, or expository efforts are just as important for contributing to (3). Such activities may be seen as legitimate and perhaps even beneficial in why-stories (1) and (2), but, they are not *as important*. In contrast to stories (1) and (2) that place high value on developments at the frontier, story (3) justified focusing on effective communication at more well fundamental levels.

> "We need to focus far more energy on understanding and explaining the basic mental infrastructure of mathematics—with consequently less energy on the most recent results." (Thurston, 1994).

Why-story (4) expands the scope of what human experience mathematics can and should address, from a sole focus on understanding in why-story (3), to considering other virtues such as solidarity, joy, and humility. In (4), mathematical proofs that spark joy and facilitate human connection are desirable mathematical goals in and of themselves, without needing further justification in terms of how they advanced human understanding of mathematics. In contrast, mathematical joy for its own sake is not positioned as sufficiently important by Atiyah (1984):

> "We are not entertainers, like tennis players. The only justification is that it is a real contribution to human thought." (p. 12)

3.3.1 Values are regularities of evaluative behavior. Meta-stories are a discursive tool.

It is important to stress that meta-stories should not be analytically and theoretically equated with value-judgements. One can, and often does, make value judgements without ever thinking about an ultimate purpose to justify them. By claiming that meta-stories are connected to evaluations, I am also not suggesting we can infer a "belief" in a meta-story from certain evaluative behaviors. For example, if we find that a mathematician routinely valorized rigor over coherence or utility, we should not infer that this mathematicians *believes* the purpose of math is produce lists of theorems and proofs. Similarly, if we find that a mathematician values justice in mathematics, we should not infer they subscribe to the mathematics for human flourishing narrative. Cultural studies of ideology show that both evaluative behavior and narrative articulations are too transitory, episodic and context dependent to justify assigning the status of a stable belief or value to observed behavior. Thus, to avoid metaphysical constructs such as "shared beliefs," it is useful to make a distinction between defining cultural values as *regularities in evaluative behavior* (Agha, 2003), and *stories* about these value judgements.

So, what *is* the connection between evaluative behavior and meta-stories? As demonstrated above, a meta-story can be used to *justify and rationalize* a paradigm for evaluation; a metric to consider some actions more desirable than others. It is best to think of meta-stories as discursive tools deployed by participants in activity in order to influence evaluative behavior of self and others. Indeed, the above cited meta-stories are examples of this phenomenon of using narrative to influence evaluative behavior. In nearly all the above cited meta-essays, meta-stories were rhetorically mobilized to influence other mathematicians' value-judgements and priorities. Gowers' (2000) advocated for problem-solvers to get more credit for their work. Atiyah (1984) wanted to discourage the assignment of high status to the production of mathematical results that are true but not useful or sufficiently relevant to the historical core of the discipline. Jaffe and Quinn (1993) aimed to convince the mathematical community to not giving full credit to non-rigorous proofs, whereas Thurston (1994) argued that:

> "I think that our strong communal emphasis on theorem-credits has a negative effect on mathematical progress. If what we are accomplishing is advancing human understanding of mathematics, then we would be much better off recognizing and valuing a far broader range of activity." (p. 172)

Thus, already in the original context of the production of these meta-stories, they were not abstract philosophical musings about purpose. What the mathematicians were debating is the principles by which the community's resources should be allocated. These include both social goods (such as attention, credit, and status) and material goods (hiring, admission, and funding). Thus, such debates about the purpose of and values in mathematics are directly connected to power.

3.3.2 Values and the distribution of resources

Mathematics is not just a practice or activity, it is an institution. Mathematicians are payed (often by taxpayers) to do their job, and meta-stories can be used to influence who gets a share of the benefits that come along with legitimized as a mathematician. In meta-story (1), producing valid theorems and proofs is a necessary and sufficient condition for contribution to the advancement of mathematics, and hence anyone who produces valid theorems and proofs could be legitimately considered a mathematician. Without other criteria of quality, the mere pace of production of "raw data" can justify hiring. In meta-story (2), only those who contribute to the *core* of mathematical knowledge, whether theory-builders or problem-solvers, contribute to the purpose of mathematics, and hence only those at the core could justifiably 'ride the wave' of prestige that mathematics has acquired in society and be paid to do research work (Atiyah, 1984, p. 12). In meta-story (3), a teacher or expositor of mathematics contributes just as much as, and possibly even more than, theorem-provers to the ultimate goal of *advancing human understanding* of mathematics. Thus, one could use this meta-story to argue that funds currently used to support theorem-proving should be diverted to support educational efforts instead. One concrete policy change could be giving higher weight to teaching efficacy than research productivity in mathematics' departments hiring decisions. In why story (4), mathematicians that foster joy, belonging and justice through their work are seen as contributing more than those whose actions alienate and exclude. Such a meta-story can be used, for example, to justify the inclusion and prioritization of diversity statements in considerations of hiring

Values are not just in people's heads. Priorities are institutionalized in terms of what gets put in books, syllabi, exams, and admissions criteria. They influence both macro- and micro-judgements about what counts as mathematics. Since resources are at stake, mathematicians at times vehemently argue about the nature and purpose of mathematics.

3.4 Values, purposes and goals for learning

One context where disciplinary goals and values are extremely consequential is teaching and learning. What is prioritized, what counts as a central versus a peripheral aim, has significant impact on what kinds of mathematics students get to learn and which students get to experience access to and belonging in discipline (Schoenfeld, 2004).

Some mathematicians are well aware of this issue, at least in part. For example, at the end of his essay, Gowers (2000) expressed concern about the detrimental affect the relative dominance of one of the two cultures (theory vs. problem-solving) may have on students:

> "potential research students who are naturally suited to one culture can find themselves under pressure to work in an area from the other, and end up wasting their talent." (Gowers, 2000, p. 15)

While Gowers presents a scenario in which talent is not 'optimized', the concern for many educators may be with a more basic issue: if students find themselves "under pressure" to do certain kinds of mathematics, to be particular kinds of math people, they may get disaffected and not see themselves as math people at all. The meta-stories and evaluative behaviors we perpetuate affect who gets to be and see themselves as part of math.

The connection between meta-stories and learning was also a central concern for Su (2017). After posing the question "why do mathematics?" he said:

> "This is a simple question, but worth considerable reflection. Because how you answer will strongly determine who you think should be doing mathematics, and how you will teach it." (Su, 2017, p. 484)

Su argued that what we consider to be the purpose of mathematics affects two important judgements: *who* we envision to be doing mathematics, and *how* math should be taught.

3.4.1 The four meta-stories and goals for advanced mathematics courses.

Let us consider these questions in relation to the above articulated meta-stories. In most stories, the *who* is left implicit. Meta-story (1) seems to be agnostic about *who* is doing mathematics, as long as the objective – production of true-statements – gets done. Anyone who masters the skill of generating valid mathematical theorems and proofs can legitimately contribute to this goal. In fact, if the measure of success is *increase* in the list of true-statements (i.e. increase in the volume of "raw data"), then the more people engage in this work, the better the goal is accomplished. Thus, following the logic of meta-story (1) we may conclude that a reasonable goal of the teaching of advanced mathematics is to facilitate broad access to the practice of proving by current standards of rigor. Alternatively, the goal would be to preselect those individuals that can produce proofs most effectively and efficiently. Such priorities would then be reflected in curricula and assessments that focus on learning to produce valid (Dawkins & Weber, 2017).

There is evidence to suggest that most Real Analysis courses, if not the entire upper division undergraduate curriculum, indeed reflect this priority of proofs and validity. First, an examination of textbook exercises, homework problems, exam questions etc. would show that students are mainly accountable to the production of proofs, and within that, the only criteria for assessment is whether the proof is valid (as opposed to, say, illuminating, creative, joyful, or explanatory). Second, much of the meta-discourse about teaching advanced math courses – by both mathematicians and math education researchers – explicitly positions proof and rigor as *the main* learning objective of mathematics education at that level.

For example, in a post on his popular blog, Tao (n.d.) distinguished between three stages of mathematics education, each with their own overarching learning goals: (1) a "pre-rigorous" stage, encompassing school and early university years, where the goal is to use intuition and informal reasoning to solve mainly computational problems, (2) a "rigorous" stage, occupying upper division undergraduate and graduate courses, where the goal is to learn rigor and proof, and (3) a "post rigorous" stage, which is reached in late graduate years, where one learns to connect rigor with intuition and "the big picture."

The features of the "rigorous stage" highlighted by Tao (n.d.) – a focus on precision, theory, abstraction and "proof-type questions" – make the goal of advanced mathematics courses entirely compatible with meta-story (1), as training students in rigorous proving best supports the ultimate purpose of increasing the lists of true statement.

If one shifts to meta-story (2), however, where the goal of mathematics is to produce coherent and useful mathematical knowledge, proving per se becomes insufficient as a goal for mathematics education. Such a shift in priorities is reflected, for example, in Tao's arguments for the significance of what he called the "post rigorous" stage, where rigor is already mastered and "the emphasis is now on applications, intuition, and the "big picture"." In his meta-story, Tao does not discard or devalue rigor, but emphasizes the importance of fitting rigor within a larger organizing framework. Thus, learning at the post-rigorous stage may be seen as most aligned with what meta-story (2) articulates as the purpose of mathematics more generally: a coherent and useful body of knowledge.

There are several things to note about Tao's account of the "post-rigorous" stage in relation to the questions of *who* should do mathematics and how it should be *taught*. First, the transition to the "post-rigorous" stage does not correspond to any official coursework or training opportunities. It is unclear where and how students are meant to learn to integrate rigor and intuition and think about "the big picture." The learning objective advanced math of *courses* – taken by students in late undergraduate and earlier graduate education – remains rigor and proof throughout Tao's hypothetical learning trajectory. Second, situating "post rigorous" learning as something that happens in "late graduate years and beyond" suggests that only students who go to and *stay* in graduate school will get to experience it. This has serious implication for the question of *who* should do mathematics. Tao's proposed model for mathematics education seems to accept as legitimate that only a select group of few gets to experience mathematical practice that integrates rigor with big picture, and correspondingly, do work that aligns with the ultimate purpose of mathematics, as articulated in meta-story (2).

Indeed, if we return to some of the above quoted articulations of meta-story (2), we may note that while there is an acknowledgement that mathematics is a human endeavor, the concern is the preservation of mathematical knowledge by humanity as a whole, not broad access to mathematics. Atiyah's concern with passing down knowledge to future generation, for example, while ostensibly includes concern for people, does not necessarily reflect a broad idea of *who* should be the people that participate in this passing down of knowledge. The size of the group that participates in this inter-generational relay race does not necessarily affect its efficacy, and so an elite group of initiates can be seen as sufficient.

The meta-discourse in (2) about humanity as a whole and future generations, contrasts starkly with Thurston's and Su's explicit concern for students experience today.

3.5 Discussion

In this chapter, I showed that prominent mathematicians espouse different views about the purpose of mathematics. That these views are not abstract musings. They are related to concrete decisions pertaining to distribution of resources (i.e. power). In particular, they have significant implications for educational goals. In short, that there is disagreement about the purpose of mathematics and it is non-trivial.

3.5.1.1 Implications for teaching authentic disciplinary practice

Many mathematicians and mathematics educators strive to structure classroom mathematical experiences so that they authentically represent disciplinary practice (Brown, Collins, & Duguid, 1989; Schoenfeld, 2016). In particular, in the context of advanced mathematics courses, the goal of introducing students to the practice of mathematicians is often stated explicitly (Hoffmann & Even, 2023).

To date, most discussions of ‘authentic disciplinary practice’ tacitly assume uniformity. According to the dominant narrative, there is one correct or ‘authentic’ mathematical practice, the one done by mathematicians, and then there are different inauthentic misrepresentations of it in instruction. The above analysis of different purpose-narratives shows that different mathematicians seem to conceptualize their practice differently, and that those differences are substantial. Which conceptualization one adopts matters for what it means to represent a practice more authentically in instruction.

Many scholars seem to interpret the authenticity charge as a mandate to represent the mainstream practices of mathematicians, the majority view. Thus, while they acknowledge the possibility of non-normative outliers, they treat the dominant norm as an obligation for instruction. This creates a passive reproduction role for instructors: mathematicians get to decide, by popular vote, what their practice is ‘like’ and educators are charged with adequately reflecting those norms in instruction.

What this “neutral” position seems to ignore is the role education plays in solidifying those very same practices as the norm. Far from being side participants in mathematicians’ decisions about what is considered normative, an uncritical reproduction of mainstream mathematical practice actively contributes to the reproduction of hegemony. Glossing over disagreements is political (Apple, 2004).

For example, where does the view that the goal of mathematics produce theorems and proofs come from? I have found no writing by mathematicians that explicitly endorse this. I suggest instead that this goal is, at least in part, a biproduct of focusing instruction and evaluation criteria on proof. Uniformity of practice is an illusion, in part accomplished through education. Indeed, systemization and methodological consolidations were prompted by teaching (Archibald, 2008; Kleiner, 1991). The idea that teaching mathematics somehow stands outside of negotiations of what counts as math is a-historical.

As educators, we must critically reflect on what we reproduce under the banner of mathematics, whether it is the ‘popular view,’ a marginal position, or pluralistic account. No ‘authenticity’ is neutral. Any articulation of mathematics is political (Wagner, 2023), but in education that is even more true. Math education has an active role in the construction of what counts as math for future generations. This should not be treated lightly.

4 Methods

At the broadest level, this study is informed by the methodological tradition of video-based micro-ethnography (Derry et al., 2010; Erickson, 1992, 2006), in particular as it is applied to study mundane yet pervasive features of classroom discourse (Mehan, 1979; Razfar, 2005). This methodology consists of an ethnographic approach to data collection (video-recordings of naturally occurring pedagogical activity), and a micro-orientation to analysis (the honing in on small details of discourse-in-use, difficult to detect and reflect on without re-examinable records such as video and transcripts).

In this chapter I elaborate on the specific methodological choices I made in this study. These include the selection of a site and context for data collection (section 4.1), the data collection procedures (section 4.2), and finally, the general approach to data analysis (section 4.3). The specific analytic procedures used to address the different research questions are explained within the analytic chapters for which they were relevant.

4.1 Data collection context

The pedagogical context I chose for studying how ideas about the "image of mathematics" were conveyed by mathematicians was undergraduate Real Analysis courses taught in a research-focused mathematics department in the US. In the following sub-section I provide information about the data collection site, the RA course, and rationale for selecting this context.

4.1.1 Site: a prestigious, research-focused mathematics department

I collected data at a large[11], research-focused, and prestigious[12] mathematics department, at a large public university. Because of its size and high-status within the academic mathematics community, the department plays an outsized role not just in the production of cutting edge "knowledge," but also in shaping and propagating the culture of contemporary academic mathematics. The mathematical and meta-mathematical discourse in the department exerts influence outside its institutional bounds. People in the community pay attention to what high-profile mathematicians say, and to how they say it. Furthermore, the department exerts cultural influence through its many graduates, who, more so than in less prestigious departments, go on to have academic careers as faculty in other institutions.

In short, selecting this department as a research site for my study amounts to selecting a case (Yin, 2006, pp. 114-115). It is a good representation of the practices that perpetuate and shape the culture of contemporary academic mathematics not because it is a *typical* department – far from it. It is rather an *extreme* but *revelatory* case, a department whose cultural practices have particular significance with respect to the community of academic mathematics more broadly.

[11] 50-60 faculty members, 25-30 postdoctoral scholars, 150-200 graduate students.
[12] It is routinely ranked among the top 10 mathematics departments in the world.

4.1.2 The Real Analysis course: rationale for selection

"Real Analysis" [13] (RA) is an upper division undergraduate course, typically taught in a traditional lecture format. In the department in which this study takes place, RA is one of four required upper division courses for an undergraduate major in mathematics, the other three being Linear Algebra, Abstract Algebra, and Complex Analysis (for which RA is a prerequisite). Thus, RA is considered to be part of the core undergraduate curriculum in mathematics, and it is typically taken by undergraduate math majors in their third or fourth year of studies. Undergraduate and graduate students from other fields (e.g. physics, computer science, economics) also take this course; either as an elective, as a choice to strengthen their "mathematical portfolio," or as part of their program's requirement.

RA is important. Besides being a required course for an undergraduate degree (and a pre-requisite for one of the other required courses), it is also a stepping stone for future study of academic mathematics. Successful completion of RA is expected, whether explicitly or implicitly, for acceptance into graduate school in math and related disciplines. The course is also notoriously challenging, even compared to other upper-division math courses. Thus, it is a significant gatekeeper for participation in academic mathematics.

RA courses is where students are first introduced to the formal-axiomatic approach to the study of functions and real numbers dominant in academic mathematics today. RA curricula are centered on what practitioners in the discipline call the "epsilon-delta" language. The term refers to an articulation of analysis first put forth by the mathematician Karl Weierstrass in the late 19th century and that has since become the standard language of analysis in academic mathematics (Archibald, 2008). The Weierstrassian school deliberately replaced the until then prevalent dynamic-geometric conceptualization of functions and limits (e.g. in the works of Newton and Cauchy), with a static language that centers on logical relationships between inequalities[14]. The shorthand "epsilon-delta" comes from a visually recognizable feature of this approach: definitions of limits and continuity that use the Greek letters ε "epsilon" and δ "delta" to quantify rates of convergence.

The RA course, as a curricular milestone, is a particularly fruitful context for investigating the focal phenomena for several reasons. First, being a point of transition to the epistemic game of contemporary academic mathematics within the educational trajectory of undergraduate students, RA involves more explicit manifestations of ideologies than we might expect in other contexts in which contemporary academic mathematical discourse is practiced (e.g. Graduate-level courses and professional seminars). As explained above, it is a "high stakes" (gatekeeping) course in which boundaries between registers, practices (and disciplines!) – what does or does not count as *real* math – require explicit demarcation. In such a pedagogical context we can expect instructors to convey ideologies both explicitly (e.g. through the meta-stories they tell) and implicitly, through mundane and subtle discursive moves such as stance-taking and attributions of value.

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[13] The adjective *real* specifies the main objects of study; functions that map *real* numbers to *real* numbers. This naming convention contrasts with another mathematical subfield and corresponding courses – "complex analysis" – in which the objects of study are functions over *complex* numbers.

[14] The transition from a dynamic-geometric definition of limits, to a static inequality centered one, is part of a methodological transformation that has been referred to as the *arithmetization* of analysis (Archibald, 2008).

Second, unlike other upper-division proof-based math courses that feature a formal-axiomatic approach (e.g. courses such as “Linear Algebra” or “Abstract Algebra”) the referential universe in RA (functions, real numbers) is already well familiar to students from years of experience in Calculus courses. In Calculus, mathematical “objects” can be (and often are) embedded in space-time, and manipulated geometrically, dynamically etc. In RA, such semiotic practices are also utilized in the flow of activity, however, in the RA context they are now considered “informal” – out of the official *register of the text*. The shift between a spaciotemporal (spatial-dynamic) to the static epsilon-delta language entails a radical shift in the semiotic affordances of the utilized forms and referents (Núñez & Edwards, 1999). For students, it stands in stark contrast to how functions were used in their previous educational experiences. Thus, unlike other upper-division proof-based courses, RA courses require students to not only learn a new epistemic game, but to “forget” or intentionally suppress old ways of thinking about familiar objects. The heightened focus on the new epistemic game and register, even relative to other proof-based courses, is another reason why RA, in particular, is a good context for investigating ideologies in contemporary academic mathematics.

Finally, RA, as a subject matter or subfield of professional mathematics, has historical significance in relation to common idealizations of mathematics. The tension between the geometric-dynamic register of Calculus and the static register of Analysis, as it played out in the 19^{th} century, lead to famous meta-mathematical articulations by mathematicians (e.g. (Russell, 1918)), which then played a major role in the development of contemporary academic mathematics and its textual practices. In fact, I suspect that the juxtaposition between Calculus and Analysis in community narratives – performed and articulated in RA courses – functions as an *iconized distinction* between formal and informal math in general, *recursively* perpetuated in other subfields (Irvine & Gal, 2000). Thus, it is a distinction that affects a much larger context within contemporary academic mathematics than “just” one course and one subject matter in the undergraduate curriculum.

For all these reasons, I consider the RA course not just a *representative* of the transition into contemporary academic mathematics, but a site of socialization into this culture par excellence. It is thus not a *typical* case selection for a study of the enactment of idealized mathematics, but rather a particularly *revelatory* one (Yin, 2006, pp. 114-115).

4.2 Data collection procedures & corpora

My study of language practices and ideologies in RA lecture was conducted through several iterations of ethnographic data collection and analysis. In this section, I will explain the context for each data-collection iteration, its procedures, and the resulting data corpus.

4.2.1 Data set A – Spring 2015 (Henry & Greg)

In the Spring of 2015, I volunteered as a research assistant in Alon Pinto’s postdoctoral project for which Alan Schoenfeld was a PI. The project’s goal was to examine how instructional decisions in advanced mathematics are influenced by instructors’ Resources (knowledge, technical expertise), Orientations (beliefs), and Goals for their teaching (i.e. the ROG framework (Schoenfeld, 2010)). I assisted with data collection and corpus management, which consisted of video recording of lectures, interviews with instructors, and collection of auxiliary documents (e.g. lecture notes) in two RA courses taught by two math department faculty – Henry and Greg (all participant names are pseudonyms). I engaged in independent pilot analyses of the data-set from Henry's RA course. I chose Henry's course because I was more directly involved in data collection (I was not present in Greg's course, due to a schedule conflict). These pilot analyses led me to conduct an additional interview with Henry about his pedagogical practice in general, as well as his reaction to video-replays of a particular classroom episode I selected for closer investigation. I presented aspects of this analysis the Annual Conference for Research in Undergraduate Mathematics Education (Zarkh, 2017, 2022).

Henry's Real Analysis course consisted of three lectures per week, each lasting 50 minutes. For each lecture, Henry prepared handwritten lecture notes and distributed photocopies of the notes to attending students. The notes served as an interactive guide to the lectures; they consisted of (1) text paralleling, though not identical, to that Henry produced on the blackboard, (2) prompts for short pair activities for students, and, at the bottom of the notes, (3) homework questions relevant to that day's lecture. Such a pedagogical strategy for facilitating math lectures has been elsewhere referred to as "gappy", "partial" or "guided" notes (Alcock, 2018; Iannone & Miller, 2019).

As part of data collection, we photocopied these notes and video recorded a few lectures toward the end of the semester. The resulting data corpus is illustrated in the table in Figure 3.

Lecture	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
Lecture Notes																																					
Lecture Video																																					
Tutorial Video																																					

Figure 3: Henry, Spring 2015 data corpus

The top row represents each of the lectures in the semester (Spring 2015), 37 in total. Shaded boxes in the second row (grey) represent lectures for which there are lecture notes in the data corpus. Shaded boxes in the third row (green) represent lectures for which there are video recordings of lectures in the data corpus. The third row represents tutorial sessions, which occurred only a few times toward the end of the semester following a regular lecture. Here too, the shaded boxes (orange) represent tutorials for which there is a video recording.

Besides these data, Alon conducted one interview with Henry during the semester, which I transcribed and is now part of the data set. I conducted another interview with Henry in Spring 2016 (a year later), which included two components: general questions about his teaching practice and a video-based stimulated recall. The selected video clip centered on an episode from one of Henry's tutorial sessions that I chose for in depth analysis because it featured overt conflict between the textual (formal) register of Analysis (the epsilon-delta language) and a graphic-dynamic, in which functions are graphs manipulatable in space-time. I presented this analysis in (Zarkh, 2022)

The phenomena that intrigued me in this data set motivated me to articulate the research program of which this dissertation is part, conduct subsequent analyses of the existing data corpus, as well as collect new data in other instructors' courses, as explained below.

4.2.2 Data set B – Fall 2019 (Dr. A, Dr. B, & Dr. C)

Data Set B is a small data set of detailed fieldnotes from a few observations of Real Analysis lectures I conducted with Sebastian Geisler in Fall 2019. We observed three instructors – Dr. A, Dr. B, and Dr. C – teaching three sections of this course during a period of 3 weeks in September. Sebastian observed a total of 8 lectures taught by Dr. A, 4 lectures taught by Dr. B, and 6 lectures taught by Dr. C. I joined for a smaller subset of these observations; a total of 3, 1, 2 lectures for each of the three instructors, respectively.

While the data set is small, and consists of fieldnotes rather than video-recording, I include it here for several reasons. I engaged with this data extensively as part of my collaboration with Sebastian (Zarkh & Geisler, 2021), and this helped me identify more concrete examples of phenomena relevant to the focal issues the dissertation project engages with. While the analysis of this data corpus are not an "official" part of the dissertation, I mention it here because this was an important step in the development of ideas that would come to play a central role in this work. Namely, the framework presented in chapter 7.

As cases of lecturing "style" (in terms of recurrent features of the discursive routines deployed), Dr. A's and Dr. B's lectures offer two particularly interesting cases for examination. Dr. B's lecture discourse, in particular, featured performative aspects that seem unique in the data set and allowed me to identify and analyze phenomena consequential to the enactment human subjectivity in mathematical discourse, such as multivocality and performed affect. Thus I include the data set here for reference, even though it significantly differs from data sets A and C in extent (a few lectures vs. semester-long) and type (observational fieldnotes vs. video-recordings) of data sources.

4.2.3 Data set C – Fall 2020 (Alex, Bowen, Cai, David and Emmett)

Remote instruction in 2020, due to the Covid-19 pandemic, proved to be a unique opportunity for more data collection. At the end of Summer 2020, I contacted instructors assigned to teach different sections of RA during the forthcoming Fall semester. All five instructors I contacted – Alex, Bowen, Cai, David and Emmett (pseudonyms) – agreed to participate in the study. Participation involved providing me with access to all course materials made available to students online. This included: video-recordings of lectures, lecture notes (when relevant), automated transcriptions of lecture recordings (through Zoom or Kaltura), all course announcements sent by the instructor through bCourses, access to course discussion-forums (on Piazza or Discord), homework assignments, exams, and other auxiliary documents such as syllabi, and handouts.

Table 1 below compares the different instructors and courses along the basic dimensions of: the primary platform used to organize the course online, the weekly *lecture schedule*[15], the *textbook* (and hence, curriculum) used in the course, the mode of *lecture delivery* (how instructors chose to conduct online lecturing), the *body and inscription configuration* that resulted from choices of technology for online teaching (tablet scribbling vs. video of whiteboard) whether *lecture notes* were made available to students as separate documents, and the instructors' professional *position* at the time of the study.

[15] Most courses in the math department follow either a 2 x 80 min per week or a 3 x 50 min per week schedules

Table 1: Participating Instructors - Data Corpus C - Fall 2020

	Primary Platform	Lecture schedule[16]	Text	Lecture Delivery	Body & inscription configuration	Notes	Position
Alex	bCourses	3 x 50 min	Ross	Synch	Full/upper body + whiteboard	No	Postdoc
Bowen	bCourses	3 x 50 min	???	A-synch	Full/upper body + whiteboard	No	Lecturer
Cai	bCourses	2 x 80 min	Ross	Synch	Face + tablet screen	Yes	Postdoc
David	External	3 x 50 min	Rudin	Synch	Face + tablet screen	Yes	Postdoc
Emmett	bCourses	3 x 50 min	Pugh	Synch	Document camera Face + tablet screen	Yes	Senior Faculty

The resulting data corpus is extensive. The process of "scraping" all relevant data sources from the web and cataloging them for efficient retrieval took a long time. Due to both logistic and methodological reasons, I made the selection/data reduction decision to exclude Bowen and from the data set.

4.2.4 Selected participants: Henry, Alex, Cai, David and Emmett

Figure 4 below summarizes the three iterations of data collection described in the previous section. In this section that follows I provide more information about each of the focal instructors in the study: Henry (2015), and Alex, Cai, David and Emmett (2020).

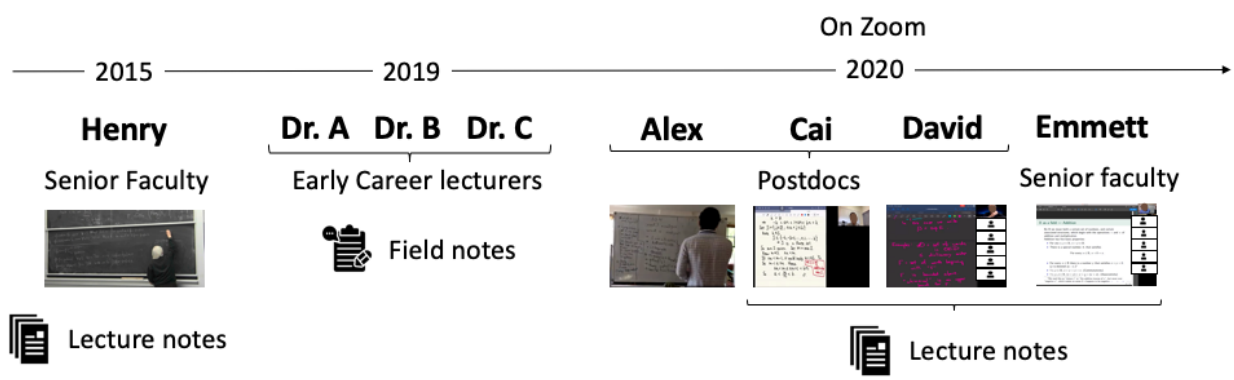


Figure 4: Timeline of the three phases of data collection and corresponding corporate.

The table below summarizes a few dimensions of variability between the five focal participants:

	Lecture schedule	Text	Lecture Delivery	Body & inscription configuration	Lecture notes	Position (at time of data collection)
Henry	3 x 50 min	Ross	In person	Full body + several blackboards	Guided	Senior Faculty
Alex	3 x 50 min	Ross	Remote	Full/upper body + whiteboard	None	Postdoc
Cai	2 x 80 min	Ross	Remote	Face + tablet screen	Standard	Postdoc
David	3 x 50 min	Rudin	Remote	Face + tablet screen	Standard	Postdoc
Emmett	3 x 50 min	Pugh	Remote	Face + slideshow	Slides	Senior Faculty

[16] I have not formally compared teaching workloads between lecturer and postdoc positions in Berkeley. I am using my general institutional knowledge to make these inferences.

4.2.4.1 Henry

At the time of data collection, Henry was a senior faculty member in the mathematics department. On top of a successful career as a research mathematician, Henry is also a very experienced and highly regarded teacher, as indicated, for example, by a university teaching award. In interviews, Henry expressed a strong commitment to serving all of his students and justified many of his pedagogical strategies through a conceptualization of student-learning as an active, social process. For example, in one of the interviews he stated that “students actually learn from each other by speaking to each other, and not necessarily from me. They would like me to speak all the time. But that's not how they learn.” At the same time, Henry also felt an obligation to deliver mathematical content, claiming that it is his “duty to go through the material” and “present the theory” in class.

Henry’s teaching style, as I observed it in the Real Analysis course, can be described as a partially flipped or “tilted” lecture (Alcock, 2018). In each class, Henry spent about half of class time “lecturing”, that is, engaging in “chalk-talk” at the board at the front of the room (Artemeva & Fox, 2011). However, unlike traditional lecturing, Henry deliberately stopped the flow of “chalk-talk” at regular intervals to have students work in pairs on strategically selected mathematical problems while he walked around the room to assist and observe (see Figure 5 for an illustrative diagrams of the classroom layout and participants’ positions during the two activity configurations).

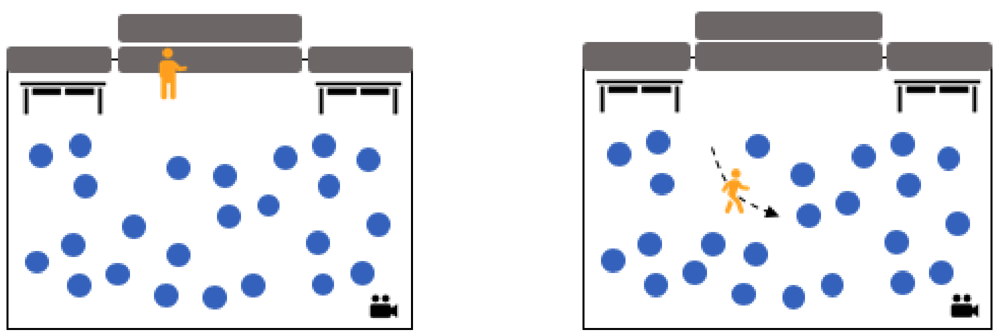


Figure 5: Two activity configuration in Henry’s tilted lecture: (a) “chalk-talk” at the blackboard (left), and (b) Pair work.

Henry used “gappy notes” (Alcock, 2018) to facilitate the “tilted lecture” classroom structure. At the beginning of each class, students received a copy of that day’s handwritten lecture notes. The notes contained the mathematical text Henry intended to write on the board, interspersed with text specifying short assignments for students. The notes were formatted to mimic the intended sequential unfolding of classroom activities. Each note started with a list of short recall activities for students to engage with at the beginning of class, similar to a “do now” activity used at the start of a lesson in many math classrooms in the U.S. Figure 6 below shows an example of these notes, with sections intended for student work highlighted in red, and sections intended for lecture in blue.

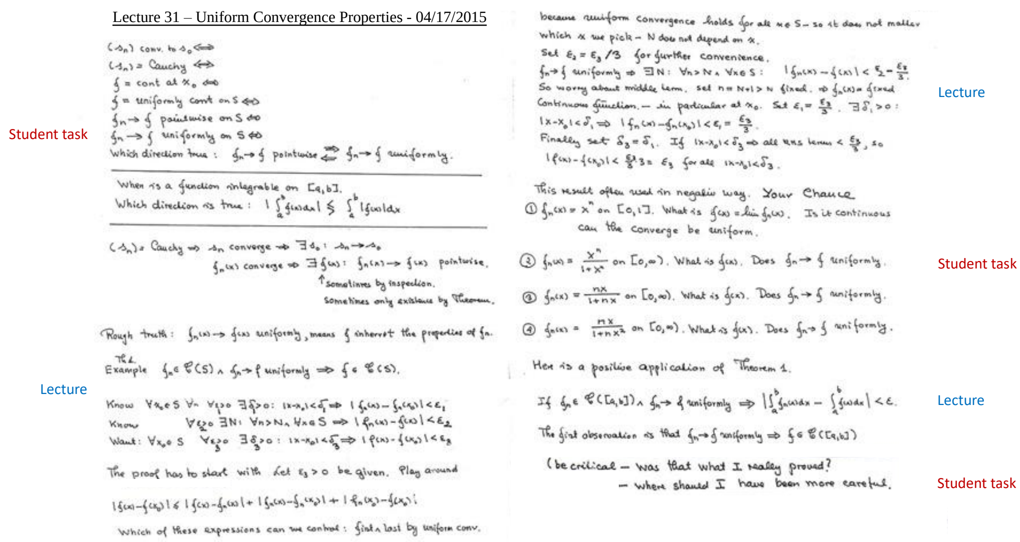


Figure 6: "Gappy" Lecture Notes. Highlights: (red) text specifying tasks for student (blue) text designated for "chalk-talk"

4.2.4.2 Alex

At the time of data collection, Alex held a research postdoc position which also included some teaching duties. Fall 2020 was the second time he taught RA in the department. The previous one was in Spring 2020, when instruction was moved online in the middle of the semester due to Covid-19.

Alex's synchronous online lectures had the following format: the video recordings showed him standing next to a small whiteboard, with which he delivered a lecture in a traditional chalk-talk style. Namely, he was able to (and did) use inscription, talk, gestures, and body positioning in synchrony, just as an instructor would in a traditional lecture room equipped with chalk or white boards (e.g. Henry). However, the whiteboard Alex used was significantly smaller than the ones in typical lecture rooms in the department, where chalk boards cover the entirety of one or more walls of the room. This constrained the amount inscription visible at any given time, and limited Alex's ability to use his entire body and large sections of the board to construe mathematical meanings (Greiffenhagen, 2014). Nevertheless, he was able to point to inscription with his hands and engage in other material coordination of semiotic resources not possible in lectures delivered via tablet, document camera or slides.

Another consequence of this format of online lecture delivery was that, unlike when using slides or a tablet, it did not automatically generate lecture notes that could be shared with students. If students wanted notes from the lectures, they would have had to copy Alex's whiteboard inscription down in the "old fashioned way." The small size of the whiteboard may have propelled Alex to use small and abbreviated writing, which meant that inscription was not always easily legible, in part also due to the rooms' lighting. For that reason, whenever I reference Alex's whiteboard inscriptions as data in this dissertation, I use my own reproductions of his inscriptions (attempting to stay as close as possible to the layout of the original), rather than screenshots from the video.

Alex's lectures were delivered synchronously, which meant that students joined live via Zoom. In the first lecture in the semester Alex told students how important it was, both for him and for students' learning experience, that there was a feeling of collective presence on Zoom. He encouraged students to turn on their cameras and participate in class by asking questions and writing comments on chat. Because Alex used a whiteboard for writing, he was not physically next to the device which ran the Zoom software. This meant that Alex was not able to see student chat comments as they came in. However, he did routinely come closer to the camera to check student comments. At times, he also explicitly invited student participation by asking them to provide answers to questions on chat. So, while the majority of time was spent in pure lecture format, there were instances in which Alex invited student participation explicitly and he routinely checked to see if there were any questions on Zoom.

4.2.4.3 Cai

At the time of data collection, Cai (like Alex) held the position of a research postdoc. Fall 2020 was his first semester in the department.

Cai delivered his synchronous online lectures first using a document camera (in lectures 1 and 2), and then using a tablet (lecture 3 onward). When using a document camera, Cai's hands were visible next to the inscriptions he produced, and so he was able to use pointing gestures. Once he transitioned to using a tablet, such pointing was no longer possible, but he used various tools (e.g. a tablet 'laser pointer') for similar functions. In both of these formats, a small window showing Cai's face and shoulders was visible at the upper right corner of the screen.

Both the document camera and the later adopted tablet format allowed Cai to share with students copies of the exact notes he produced during lecture. That meant that students had access to comprehensive notes without needing to copy inscription down during class. That also meant that I was able to use the original notes both for data analysis and as figures in the reporting of findings.

Cai invited student participation, both more frequently and in more substantial ways than other instructors in the 2020 dataset. For example, in the first lecture of the semester, he stopped the flow of lecture three times for students to work on a proof exercise themselves before he wrote a solution. This was planned and deliberate, and involved wait time of three minutes or more. In other lectures, while such explicit dedicated time for students to work on problems was not as frequent, he still routinely posed questions to students and waited for responses, either on the chat or verbally by students unmuting on Zoom. Corresponding, Cai's introductory lectures featured significant verbal participation from students.

4.2.4.4 David

At the time of data collection, David (like Alex and Cai) held the position of a research postdoc.

David delivered his synchronous online lectures using a tablet at the outset. As standard in this format, the video recordings showed the tablet inscriptions as the main backdrop, and David's face frame in a small window at the upper right corner of the screen. The video also often showed small windows corresponding to student participants on the Zoom call, some with video on and some off.

Like for Cai, the tablet format allowed David to share the exact notes he produced during lectures. And similarly, my analysis and reporting of findings use excerpts from the original notes rather than reproductions.

David also invited student participation, but the same substantial and systematic manner that Cai did. He routinely asked students if they had any questions, and paused to address them if they did. In the introductory lectures I observed, the atmosphere seemed comfortable and a few students participated verbally by asking questions. However, students were not explicitly given time to work on tasks during class.

4.2.4.5 Emmett

At the time of data collection, Emmett was a senior faculty member in the department. This makes him an outlier in the 2020 data set, as the three other participants were postdocs. Also, the RA course Emmett taught was the only designated *honors course*. An honors RA course satisfies the same course requirements as a regular RA course, but is intended to be more challenging. There is a minimum GPA requirement for enrollment in such courses.

In the first three lectures, Emmett used slides. The video showed the slides as the main backdrop and his face frame in a small window in the upper right corner. Later, he transitioned to using a document camera. The slides were made available to students after class. Thus similarly to Cai and David's lectures, my analysis and reporting of findings use excerpts from these slides.

In the three observed lectures, Emmett made minimal attempts to explicitly invite students participation. He did, however, verbally respond to some questions students posed on the Zoom chat. Because I do not have access to the chat data, it is not possible for me to determine whether all student questions and comments were responded to, nor ascertain the timing. My sense was that Emmett attended to the Zoom chat only at the end of major sections of lecture, and answered questions when he saw them at those points in time.

4.2.5 Selection of focal lectures

The labor intensity of micro-ethnographic data analysis necessitated a selection of focal lectures. Originally, my intention was to conduct comparative analyses between lectures within the curricular unit of uniform convergence in the RA courses of Alex, Cai, and Henry. The rationale for that selection was that in the 2015 data set (Henry), lectures on uniform convergence were the only ones for which video recordings were available, and of the 2020 dataset, Alex and Cai were the best comparison as they used the same text book as a curricular base. As part of that initial focus, I transcribed and created activity logs for several lectures on uniform convergence in Alex's and Cai's courses.

In Fall 2022, I started working with an undergraduate research assistant (Kaya Poff), and as an entry point into the data, we decided to watch the lectures of Alex and Cai from the beginning (i.e. lecture 1). Those introductory lectures featured discursive phenomena that I (and Kaya) found very interesting – namely, the meta-stories analyzed in chapter 5 – and so I decided to shift the focus from lecture on uniform convergence, to introductory lectures. Because introductory lectures were available for both David and Emmett, I decided to include them in the analysis. Subsequently, with Kaya's help, I watched, transcribed, and created activity logs for the first three lectures of each of the four instructors in the 2020 dataset: Alex, Cai, David, and Emmett. At the second, more detailed stage of analysis of meta-stories (chapter 5) and values (chapter 6) I focused exclusively on the very first lectures of each instructor.

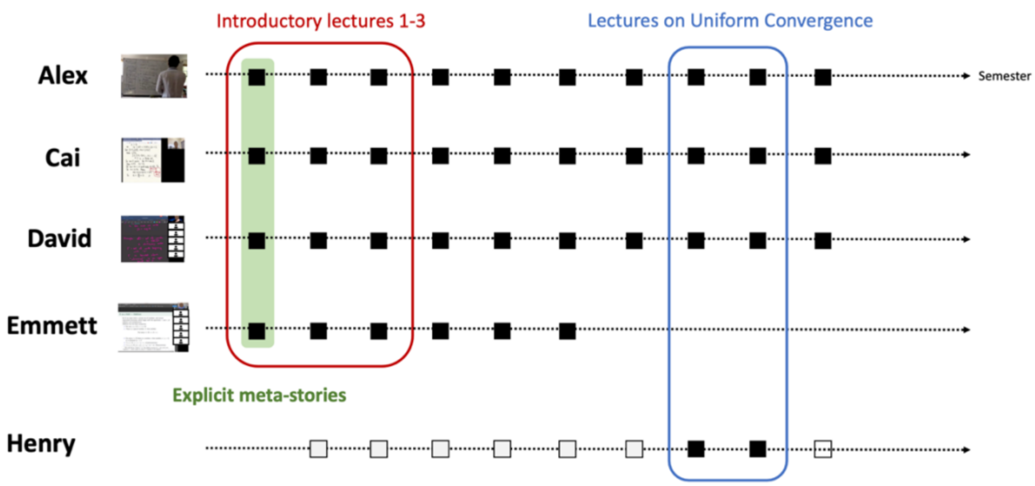


Figure 7: Schematic illustration of the lecture video data corpora and selection of focal lectures for analysis.

Figure 7 summarizes the selection of lectures. The blue rectangle highlights the original focus of comparing lectures on uniform convergence. The red rectangle represented the twelves lectures which I observed, transcribed and conducted initial analysis on. The green rectangle represents the lectures selected for the detailed analysis reported on in chapters 5 and 6.

The development of the humanizing framework presented in chapter 7 was informed primarily by the 2015 and 2019 data collection cycles, and accordingly most of the examples in the chapter are from Henry's course (either from the video recordings or the lecture notes). The other examples are drawn from the 2020 dataset, both from the introductory lectures, and form the first transcribed lectures on uniform convergences.

4.3 Data analysis

In line with the micro-ethnographic tradition (Erickson, 1992), I conducted data analysis in two main phases (see Figure 8).

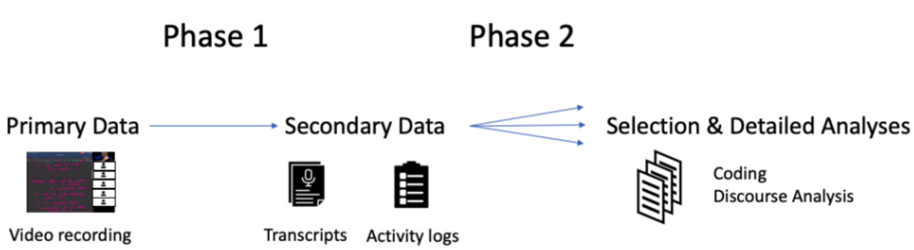


Figure 8: Two phases of micro-ethnographic data analysis (Erickson, 1992)

4.3.1 Segmented transcripts

The first phase involved the generation of secondary data artifacts (transcripts, activity logs) from primary data (video recordings). Figure 9 serves as an example and illustration of this process.

Each of the introductory lecture videos was transcribed for talk and salient gestures. Transcripts were segmented into short episodes, of 1 min length or less. Segmentation was based on "natural" transitions. These included linguistic markers (e.g. "So" and "Okay"), and shifts in topic or inscriptional focus. Each 1 min episode was given a title that served as its descriptive summary, using a mixture of content and in-vivo coding (Saldaña, 2021). These are highlighted in blue in Figure 9. I then grouped the short 1 min episodes into larger coherent wholes, according to the overall type of classroom activity (e.g. if episodes 3-8 were all part of a single proof, they were grouped together under the heading "proof of …"). The titles for these episode groups are highlighted in yellow in Figure 9. This multi-step process resulted in a hierarchically structured outline of each lecture's transcript (Erickson, 1992), shown to the right in Figure 9. The structured transcripts were the data sources used for subsequent analysis.

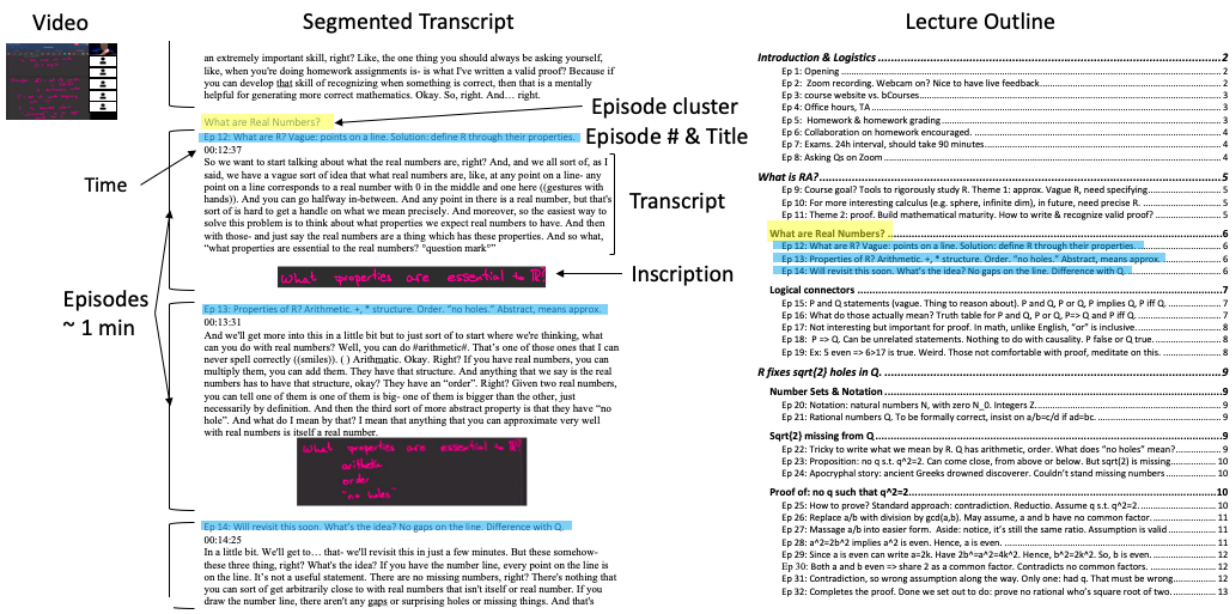


Figure 9: An example of secondary data artifacts (transcripts, activity logs)created on the basis of lecture video recordings.

The second phase of detailed analysis was different for each of the studies comprising chapters 5, 6, and 7, and will be explained in detail in the context of each analytic chapter.

5 Motivating Real Analysis: lecture meta-stories about math.

5.1 Introduction

The stories we tell about the nature and purpose of mathematical practice matter. They frame mathematical activity in and out of the classroom, render it sensible (or not), and help delineate what actions are deemed appropriate, valuable or even feasible in a given context (Dawkins & Weber, 2017; Schoenfeld, 1989). They influences *who* is seen by teachers and researchers as a valuable contributor in mathematics (Adiredja, 2019; Louie, 2018), as well as *who* gets to consider mathematics desirable (Ernest, 1995; Solomon, 2007).

Stories about math can be problematic. Many mainstream stories, such as "there is only one math" (Hersh, 1991) or "math is a young man's game" (Barany, 2021), both misrepresent disciplinary practice and alienate students by constructing a world that few can see themselves in. Furthermore, stories such as "mathematics is sequential" or "mathematics is a-political" have been used to justify curricular decisions with significant impact on students (R. Gutiérrez et al., 2023; Parks, 2010; Schoenfeld, 2004).

Because stories about mathematics exert a strong (and often negative) influence on educational experiences and practice, unpacking dominant narratives – recognizing them as stories, identifying their weaknesses and affordances, and understanding their underlying assumptions – is important. Gutiérrez, Myers and Kokka (2023), for example, argued that such an unpacking of meta-stories should be considered an essential component of what teachers should do as part of *political* knowledge in teaching mathematics (p. 3). In this chapter, I take on this task of unpacking stories about mathematics by looking closely at the narratives mathematicians tell about the nature of purpose of Real Analysis (RA) in the context of the course's introductory lectures.

5.1.1 RA as an important context for meta-mathematical story-telling

Proof-based university courses such as RA are an important site of meta-mathematical story-telling. In such courses, students are first introduced to the epistemic game of contemporary academic mathematics, and thus instructors may feel the need to *explicitly* explain and justify the course and its new way of doing math. Among proof-based university courses, RA is a particularly interesting case. Because of the significant content overlap with calculus, there is a heightened focus on the novelty of the epistemic game and associated discursive routines. For that reason, we can expect RA lectures to be a context rife with meta-mathematical story telling.

The significance and likely impact of meta-stories told in proof-based courses extends beyond the immediate curricular context of the course itself. Because of the courses' position in the educational trajectory of undergraduate math majors, the meta-stories told in these contexts can serve as a proxy for meta-stories about advanced mathematics in general. In addition, while lower division courses are often outsources to lecturers and graduate students, upper division courses are typically taught by research mathematicians, which wield unique authority to determine what counts as "real mathematics."

5.1.2 Meta-stories and the "giving motivation" genre

This chapter examines specific types of meta-stories told by mathematicians in RA courses: those articulated as answers to the questions: "what is Real Analysis, and why is it worth studying?" These questions were rhetorically posed in the lectures by the instructors themselves.

In the culture of academic mathematics, this kind of talk is commonly referred to as "giving motivation" for a topic. The "giving motivation" genre is not unique to undergraduate lectures or even to teaching. It is common, for example, to provide "motivation" before presenting mathematical results in a paper or a professional talk. Mathematicians use the "motivation" genre to convince their audience that the results they are about to share are interesting and important.

The "motivation" talk I examine here is an extreme example of this genre in that it was deployed to justify not only specific mathematical results or topics, but an entire sub-field of mathematics (Real Analysis) and, in some cases, explicitly framed as a justification for the epistemic game of contemporary academic mathematics more generally.

Because the primary rhetorical function of "giving motivation" is to convince an audience that something is worth doing or attending to, meta-stories deployed in that context are rife with discursive displays that signal values and goals (Dawkins & Weber, 2017). For example, a story such as "in mathematics, we want to make things simple and elegant" presupposes simplicity, elegance, purity, and ease as positive values. Similarly, the story "mathematics is a structure that needs solid foundations" suggests validity and reliability as valuable attributes.

In so doing, meta-stories told with a "motivational" purpose also construct subject positions with certain desires, values and epistemic experiences. These constitute an image of an idealized mathematical subjectivity, a mathematical point of view on the world. Depending on the story, being "mathematical" can be ostensibly defined as being "precise", being "useful," or being "playful." Because of this, meta-stories are an important resources for identity making. Depending on the dispositions the meta-stories enact and valorize, student can find the mathematical practice described relatable and desirable, or off-putting.

Thus, it is important to examine what meta-stories students are exposed to, and critically evaluate the kind of mathematics they construct.

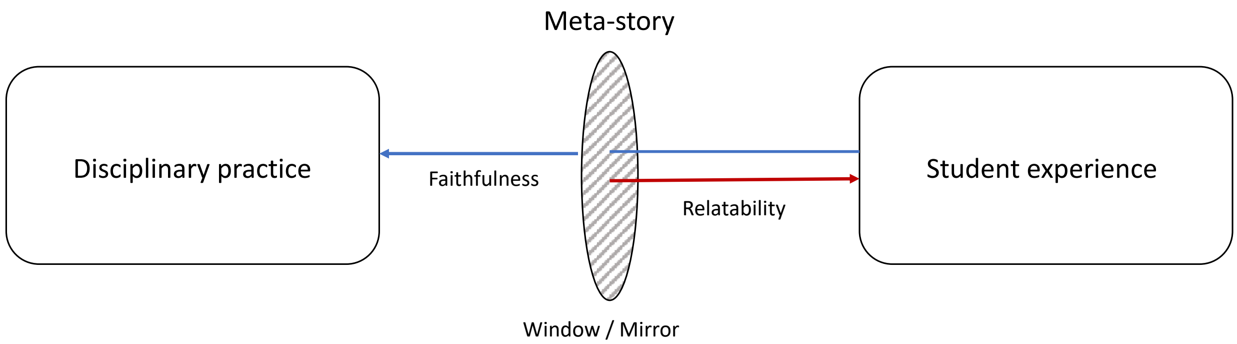
5.1.3 What makes a meta-story "healthy"? The criteria of faithfulness and relatability

The criteria I call faithfulness and relatability are connected to two goals for mathematics instruction long lauded by education researchers: on the one hand, to enact mathematical experiences that are authentic to the discipline (Brown et al., 1989; Lampert, 1990; Schoenfeld, 1994, 2016), and on the other, to make sure that classroom activity is sensible and relatable to all students (Schoenfeld, 2014b). A useful metaphor for thinking about these two complementary goal was proposed by R. Gutiérrez (2018): classroom experiences ought to provide students with *windows* onto mathematics, and *mirrors* to see themselves reflected in it. In this chapter, I adopt this metaphor to assess affordances of different meta-stories by instructors. In particular, I posit that a good meta-story is one that accomplishes both[17]: it is both faithful to disciplinary practice *and* relatable to students.

Accomplishing both goals within a single story or pedagogical practice is not without issue. While the goals of windows and mirrors are not contradictory, researchers have noted that there can be irreducible tensions between them (Ball, 1993), in particular in the context of proof (Weber & Melhuish, 2022). For example, imposing norms of writing proof in a specific register, while to some extent authentic to the conventions of contemporary mathematical practice, may create classroom experiences that are inaccessible and marginalizing for students (Weber & Melhuish, 2022, p. 310). Similarly, enacting a competitive environment of proof production may be faithful to prevalent professional practices, but not relatable to students who do not value competitiveness (Weber & Melhuish, 2022, p. 308), and can be overall seen as harmful.

Such tensions between faithfulness and relatability can be one reason why mathematicians and mathematics educators seem to often disagree about the nature of mathematics. It is not uncommon for educators to craft responsive and relatable classroom activities and curricula, only for those to then be labeled as “not real math” (Sfard, 1998), and at times vehemently opposed (Schoenfeld, 2004). Nevertheless, an assumption shared by mathematics educators, certainly those devoted to equity, is that classroom experiences can and should navigate these tensions and provide both windows and mirrors. That is, classroom mathematics should be both faithful to the discipline and relatable to students.

Meta-stories – what we say about mathematics – can be assessed in terms of faithfulness and relatability too. Some stories can be seen as faithful to the discipline as practiced to date (e.g. “math is mostly done by white men”), but are far from relatable to most students. Others may be relatable to (at least some) students (e.g. “mathematics is not about competition”), but are not sufficiently reflective of current disciplinary practice to adequately equip students with resources to “play the game” later on.



Drawing on these considerations of the importance of meta-stories, as well as the criteria of faithfulness and relatability, in this chapter, I address the following research questions:

[17] Importantly, I am not claiming that these are *sufficient* criteria for meta-stories to be educationally “healthy”. One critical piece that is missing here is considerations of justice and the greater social good. As educators, we do not have to blindly reproduce disciplinary practice, nor do we need to uncritically respond to students’ desires. The values we hold for education (e.g. jointly crafting a more just and communal society) should shape the kinds of stories we choose to tell in classrooms as well.

1. What stories about the nature and purpose of RA do instructors tell in lectures?
2. What assumptions about mathematical practice do these stories presuppose? (what claims do students need to "buy-into" for the stories to make sense?)
3. How do these assumptions relate to what is known about (1) the nature of disciplinary practice, and (2) students past curricular experiences.

5.2 Data analysis

Data analysis proceeded in four steps. First, as described in the method chapter 4, I generated secondary data through transcription, segmentation and descriptive coding of episodes. Then, I coded episodes for meta-talk, consolidated into story themes, and iteratively constructed summary descriptions for each theme. Then, I identified underlying assumptions constitutive of each story. Finally, I assessed the faithfulness and relatability of each assumption.

5.2.1 Coding meta-talk & meta-story summaries.

Meta talk appeared in lectures in two ways. There were entire episodes devoted to meta talk (the "giving motivation" genre), and there were shorter instances of meta-talk (e.g. a single sentence) that occurred in the midst of an episode with a different focus. The coding process was inductive and proceeded as follows. First, I flagged episodes of meta-talk that addressed either what the course or the sub-field of RA are, or why they are worth studying or doing. To determine whether an episode or utterance counted as meta-talk, I relied on explicit linguistic indicators such as "our goal in this course" and "Real Analysis is." I then went through each episode that was flagged for meta talk, and created sub-codes to capture different story types. Shifts to different stories were often marked by the instructors themselves (e.g. "another goal of this course is…"). Through an iterative process of refinement, I delineated distinct categories of meta-stories about RA that together account for all meta-talk episodes and instances. With these meta-story categories, I went through the structured transcripts again, looking for confirming and disconfirming evidence and more instances of meta-talk. This process led to further refinement of the codes and consolidation in the form of story summaries.

5.2.2 Constitutive assumptions

I determined underlying assumptions by interrogating the meta-story summaries with the question: "what does one need to buy into in order to find this story compelling?" For example, for the meta-story I call "fix-calculus" I identified three underlying assumptions: (1) calculus is a desirable activity, (2) calculus sometimes breaks, and (3) analysis helps fix or avoid calculus breakdowns. An identification of assumption enabled me to decompose each story down into constitutive elements, and in so doing, highlight the structural similarities between different instantiations of a meta-story my coding identified as the same. The breakdown into constitutive assumptions also allowed a more structured discussion of the variation between different articulations of the same meta story.

5.2.3 Faithfulness and relatability

Once I produced a list of underlying assumption, I assessed each assumption separately in terms of its faithfulness and relatability. To assess faithfulness, I examined the assumption in relation to what is known about that aspect of disciplinary practice from the historical, sociological and psychological accounts of mathematical practice, as well as famous reflections by mathematicians. I also assessed faithfulness in relation to what is known about the institutional context in which these courses were taught. To assess how relatable these stories might be to students, I relied on what can be reasonably assumed about students' past curricular experiences based on the institutional context (e.g. the course pre-requisites), as well as research on student experiences, in particular in proof-based university courses.

5.3 Results

5.3.1 Overview

5.3.1.1 Episodes of meta-talk

All four instructors devoted class time to meta-talk about the nature and purpose of RA during their introductory lectures. Figure 10 below shows a coded outline of the first lecture for each of the instructors. Each segment corresponds to approximately 5 minutes of video. The segments are color-coded, with each color indicating the lecture activity featured most prominently during the corresponding 5 minutes. The key to the color coding is at the bottom of the figure. Episodes coded as "meta-talk" appear in orange. Other episode types are: logistics (light grey), framing/transition commentary (yellow), definitions or notation (green), proposition statements (blue), proof (purple) and examples (dark grey).

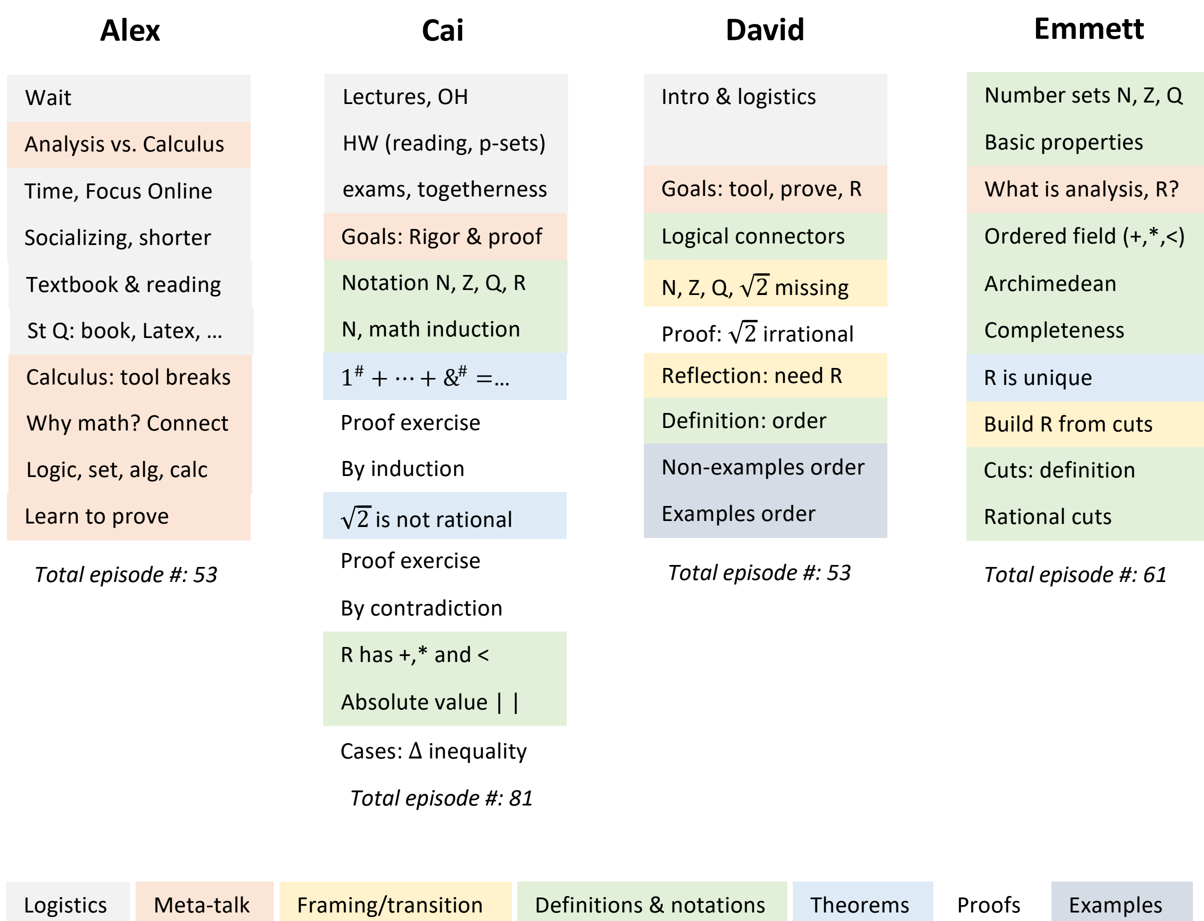


Figure 10: Segmented, color-coded outlines of the first lectures of Alex, Cai, David & Emmett.

As can be seen in Figure 10, instructors varied in the amount of time they devoted to meta-talk during their first RA lecture. Table 2 shows the length of meta-talk episodes for each instructor.

Table 2: Total time of meta-talk for each instructor in hours : minutes : seconds

Alex	Cai	David	Emmett
0:20:29	0:05:39	0:05:16	0:03:21

Cai, David, and Emmett all devote a relatively short amount of time to introductory meta-talk, around 5 minutes each. Alex was an outlier; he devoted the entire first lecture (50 minutes) to introductory discussions, of which 20 minutes were devoted to meta-talk about the nature and purpose of the RA course. Accordingly, the stories about RA in Alex's lecture were the most elaborated of the four instructors. However, as I will show in the rest of the finding section, similar stories were present, albeit in shorter form, in the other instructors' lectures as well.

5.3.1.2 The five meta-stories

Across the data set, I identified the following five meta-stories about RA:

1. To do calculus, we need to specify R (Real numbers). Our current understanding of R is vague. In RA, we define R precisely and build calculus from it.
2. RA covers the same topics as calculus, but unlike calculus, it is proof-based.
3. Calculus is a tool that sometimes breaks. RA is a theory of how the tool work. It is good to learn the theory for future tool use and new tool development.
4. RA is a theory of connections between calculus and more fundamental topics. It is good because it makes calculus simpler and more elegant.
5. RA is good for learning to prove.

Table 3 below shows the presence or absence of the five meta-stories in each of the instructors' lectures.

Table 3: Meta-stories in the lectures of the four instructors

	Alex	Cai	David	Emmett
S1: precise R			precise R	precise R
S2: calc. with proof	calc. with proof	calc. with proof		
S3: fix calculus	fix calculus	fix calculus		
S4: connections	connections			
S5: learn-to-prove	learn-to-prove	learn-to-prove	learn-to-prove	

In this chapter, I interpret meta-stories as shared cultural artifacts made available to students, rather than individual instructors' beliefs (Louie, 2018; Philip, 2011). Relatedly, individual differences between instructors are not part of the analysis. The reason I am sharing the breakdown of the differences in deployment of narratives across the four instructors is to provide support to the claim, underpinning the current analysis, that these stories are distinct. For, the distribution shows, it is possible to tell one meta-story without telling the other.

5.3.2 Meta-story 1: To do calculus, we need precise R.

In this section I will describe how Emmett and David each constructed a version of the "precise R" story in their first lecture, and then compare their versions across key underlying assumptions: (1) calculus is a desirable activity, (2) students' current understanding of R is lacking, and (3) to do calculus, there is a need to specify R.

5.3.2.1 Emmett – construct R to develop calculus

Emmett discussed the two constitutive terms of the course title, 'real' and 'analysis,' separately. Starting with 'analysis,' he offered the following definition:

> … an informal and incomplete definition of analysis is that analysis is that part of mathematics in which limits are used to solve problems. So, the key concept in analysis is that of a limit. I would say maybe the second concept is that of an inequality. But maybe the most fundamental concept is the limit. And that's really what this course is about. Limits and closely related concepts. (episode 10)

In the above quote, Emmett defined analysis as a subfield of math, characterized by its focal concepts (limits, inequality) that are framed as tools for solving problems. Next, he shifted attention to the other part of the title, namely, the adjective 'real.' He posed the question "so, what about real numbers?" and shared a slide with the following famous quote from the German mathematician Leopold Kronecker: "God made the whole numbers; humans made the rest." Emmett interpreted the distinction Kronecker's quote makes between god-made and man-made numbers as implying that the real numbers are "not as fundamental as the whole numbers," and declared that the course will take that "point of view."

On the next slide, he elaborated on what makes real numbers less fundamental. He asked "what is a real number?" rhetorically and provided three "informal descriptions" as answers: real numbers are (1) points on the number line ("when you teach a calculus class, you draw a line on the board and you say here are the real numbers"), (2) expressible using decimals ("Any number you can write like that, positive or negative, is a real number"), or (3) possible values of a variable x in a calculus course. Emmett then explicitly framed these descriptions as inadequate:

> So that's obviously not a very rigorous notion, but that captures what we're talking about. I hope that you'll all agree with me that none of these answers are anywhere near as clear and unambiguous as the positive whole numbers. We all are confident that we can start with one, add one to it, add one to it, add one to it, and keep going. And we more or less understand what sort of objects we're getting that way. The real numbers are a bit more mysterious. (episode 12)

In the perspective offered above, familiar descriptions of real numbers (standard in calculus courses) are declared problematic because they are "informal" and not "very rigorous." The ambiguity and mystery of real numbers is justified through a comparison to the presumed clarity with which whole numbers are comprehended. According to Emmett, whole numbers as mathematical objects are clear and unambiguous; by imagining repeated addition, "we can more or less understand what sort of objects we're getting." Emmett uses this appeal to a presumably shared experience of different degrees of ontological clarity to justify assigning real and whole

numbers a different epistemic status: “And in this course, we will not take basic properties of the real numbers for granted. Unlike these more basic kinds of numbers.”
In Emmett’s story, filling this ontological and epistemological void is a key problem the course aims to solve. Accordingly, constructing the real numbers in a clear an unambiguous way played a central role in “the program” Emmett laid out next:

> Instead, we're going to spend some time and we're going to use the rationals, which we've agreed, we do understand that we can assume things about. We're going to use the rationals to construct some objects. And this collection of objects will have all the properties that the real numbers ought to have. And then we'll just call these objects the real numbers. And as the course goes on, we'll use them. We'll use them, we will develop the concept of a limit. And we'll use them with, together with limits to develop the theorems that underpin calculus and to solve various other kinds subproblems. Ok, so that's the program. (episode 13)

The “program” Emmett articulated describes RA as sequential coverage of mathematical content (R, limits, calculus), where each step is ‘built’ on top of the previous one: rational numbers are used to construct real numbers, real numbers are used to develop limits, limit are used to develop calculus theorems. Analysis is situated later than real numbers in the sequential development of content. But, how are these two connected? Why are real numbers needed for doing analysis, assuming the latter is an agreed upon goal? Later in the lecture, Emmett made additional meta-comments, that can be seen as filling this gap in the story:

> … the rationals are not complete. And that, in a nutshell, is why this course is real analysis instead of rational analysis, the rationals are defective from our point of view. We want to be able to take limits, completeness, the rationals are not complete and there will be instances when we can't take limits when we want to. So the rationals don't work for us and we need the reals. (episode 34)

Here, the course’s focus on real numbers is justified by their necessity for doing limits. In this argument, we don’t care about real numbers *just* because they are unclear and complex. We care about them because we want “to take limits,” and rational numbers do not always allow it. The problem is that rational numbers, unlike real numbers, “are not complete” which makes them “defective from our point of view.”

In a later meta-comments, Emmett repeated the course’s program description, adding detail:

> … we're going to construct a set of objects that has all the properties that the real numbers ought to have. We'll just call that set the real numbers, and then we'll use it together with limits to solve problems and to justify the foundations of calculus. OK. And the objects that we construct are called cuts, or they are called Dedekind cuts after, after a mathematician named Dedekind. (episode 43)

In this last comment, Emmett provided additional information. Namely, that the to-be constructed real numbers are “objects” called Dedekind cuts. He immediately followed this programmatic outline with a comment about the resulting ontological status of real numbers:

> So notice that this approach sidesteps the basic question of what the real numbers actually are. And, you know, that's some sort of metaphysical question that doesn't really have a mathematical meaning. So, we're simply not going to go there. We're simply going to construct something and we'll call it the real numbers. Because of the theorem I told you at the end of the last set of slides, that's a reasonable approach. Any fully mathematical question you can ask will have the same answer for our real numbers, as for anybody else's real numbers. (episode 43)

In this last quote, Emmett clarified that the course's "program" will not provide an answer to the ontological question of "what the real numbers actually are." Despite the fact that he used the (presumed) ontological ambiguity of real numbers to motivate the program at the beginning of the lecture, in this later comment, he framed the ontological question as "metaphysical" and devoid of "mathematical meaning." To justify this dismissal of the original question, he referenced the theorem that any two complete, Archimedean, ordered fields are isomorphic.

Based on this meta-talk, I summarized Emmett's narrative as follows:

We are interested in doing analysis, which is using limits to solve problems. To do limits, we need real numbers. But, we don't know what real numbers are in a clear and unambiguous way. We do, however, clearly understand natural, whole and rational numbers. So, the course starts with constructing [something we call] real numbers from rational numbers, and develops calculus from that.

5.3.2.2 David – specify R to do advanced calculus

David began his meta-story by rhetorically posing the question: "what's the goal of this course?", and immediately providing the following answer:

> This is introduction to analysis or real analysis. So what we're <u>really</u> trying to do is come up with a set of tools to rigorously study what happens on… what the real numbers are and how the real numbers behave. (episode 9)

The above description features some of the themes we saw in Emmett's presentation. David claimed that the course's objective ("what we're really trying to do") is to develop "tools." The tool metaphor is similar to Emmett's framing of limits as something that can be used to solve problems. However, unlike in Emmett's story where the nature of the problems remains underdetermined, in David's description the objective is characterized as the exploration of a mathematical reality in which real numbers are the landscape and focal characters: "what happens on [the number line]" "what real numbers are," "how they behave." At this point in David's story, understanding phenomena pertaining to real numbers is the goal, whereas in Emmett's story, it was a necessary step in a 'program' whose ultimate objective was calculus, i.e. 'using limits to solve problems'. But, this was just the start of David's story. He continued:

> … the core ideas in this course, and the core themes we are going to explore are. Well, I guess one core theme is essentially approximation. And the real question we have, right?

> You know you have a vague notion, probably sort of what a real number is, but if you actually want to specify what a real number is, it can be a little tricky. (episode 9)

David mentioned approximation as a theme but did not connect it to the declared focus on real numbers. It seemed as though he was about to list several core ideas, not just approximation, but he quickly shifted to what he called "the real question," which is that of specifying real numbers. Similarly to Emmett, David problematized real numbers by invoking a binary opposition between two epistemic stances: having a "vague notion" versus "actually" specifying. This binary allows seeing the specification of "what a real number is" as a problem to be solved. David proceeded to justify why specifying real numbers is useful. Recall that in Emmett's story the only justification is the purported experience of ambiguity (for David, "vagueness") with familiar descriptions of real numbers. David's narrative features more rationales. Specifically, he claimed that specifying real numbers is necessary for more advanced calculus:

> … if we want to start doing complicated things like calculus or analysis of things in higher dimensions or working on- or doing calculus on a surface that isn't actually the real line. If you want to do calculus on the surface of a sphere, for example, it gets a little bit tricky and it becomes important to know precisely what sort of thing you're talking about. And so the goal of this class is to sort of lay that groundwork that dealing with the real numbers are, so that we can then in the future expand it to talk about more interesting, more interesting topics. (episode 10)

Similarly to Emmett, David invoked a building-foundation metaphor ("lay the groundwork") to interpret the significance of 'specifying real numbers' as a base step in the development of a hierarchically organized system of mathematical knowledge. However, unlike in Emmett's story where the end point was 'regular calculus,' here the motivating horizon is more advanced versions of calculus – calculus in higher dimensions or on surfaces such as a sphere – that are "interesting," yet "tricky" and "complicated." Thus far in David's story it is the complexity of these more advanced topics, rather than any difficulty with regular calculus, that necessitates an ontological specification of R ("it *becomes* important to know precisely what sort of thing you're talking about"). He developed this idea further in relation to infinite dimensional calculus:

> You might be curious. Can you do something that looks like calculus in infinitely many dimension? I mean, you've probably taken- I'm sure one of the courses you've taken as a prerequisite to this is a multivariable calculus class that probably did two or three, maybe n-dimensional calculus. But can you have meaningful kinds of calculus in infinitely many dimensions? And the answer is yes. But developing that, there are many, many subtleties to get through to develop that. So we're going to start somewhere to have something to build on. And so we really want to make these ideas of what makes the real numbers, what they are, precise. (episode 10)

We see here a repetition of the same theme. Doing more advanced calculus (in this case, infinite dimensional) is subtle and complex. Mathematical knowledge is a hierarchical structure. To be able to (in the future) cope with this complexity of more advanced levels, it is necessary to do preparatory work at the base ("start somewhere to have something to build on"). In particular, the prep work involves specifying real numbers. Unlike Emmett, David did not provide an explanation for why real numbers, specifically, is the "somewhere" where one "starts."

I summarized David's narrative as follows:

We are interested in doing advanced calculus, such as calculus on surfaces, and calculus in multiple and infinite dimensions . To do advanced calculus, we need precise real numbers. Our current understanding of real numbers is vague. So, the course focuses on specifying real numbers, to have something to build on.

5.3.2.3 Assumptions in the "precise R" meta-story

The first assumption in the "precise R" meta-story is (1) students are interested in doing calculus. Emmett treated interest in calculus as a taken-for-granted goal, both in the definition he offered for "analysis" and in positioning calculus theorems and limits as the end goals of content development. In David's meta-story, the objective is not regular calculus, but rather more advanced versions, such as calculus on surfaces and in infinite dimensions. But, like for Emmett, the interest in such topics is assumed and positioned as an ultimate goal. A RA instructor may reasonably assume that a motivation for calculus – What kinds of problems do limits help solve? Why should we care? – was addressed in previous courses. However, David's framing of interest in infinite dimensional calculus as something one is naturally curious about after doing finite dimensional calculus can be problematic. It is historically inaccurate and potentially alienating, as few students are likely to see themselves reflected in this positionality of wondering about artificial extensions of theory.

The second assumption is that (2) students' current understandings of real numbers are lacking. Emmett conveyed this by listing descriptions of real numbers students are familiar with and framing them as "not very rigorous," unclear, and ambiguous. David, similarly, claimed that students only "have a vague notion … of what a real number is." In both stories, the negative epistemic stance is not fully justified. Emmett's use of phrases such as "obviously" and "I hope that you'll all agree," actually highlights the absence of an explanation. Indeed, a negative epistemic and ontological stance toward real numbers is not self-evident. Students (and various professionals, including mathematicians) have been successfully using 'informal descriptions' of real numbers for a long time (e.g. in calculus) without feeling the need to specify them in the way Emmett and David suggested. To date, in many (if not most) contexts of doing calculus, people unproblematically rely on representations of real numbers which Emmett and David labeled as ambiguous and vague. Thus, the offered stance can be experienced as an imposition.

The final assumption is that (3) to do calculus, R needs to be specified. Emmett did not provide explicit justification for this claim, but the programmatic description he offered relies heavily on building-foundation metaphors for mathematical knowledge, and within such a metaphorical conceptualization of math it may seem as self-evidently true that concepts need to be 'built on solid ground' to be viable. David's story incorporated building-foundation metaphors too (e.g. "lay the ground work"). However, David offered an additional rationale by positioning the specification of R, not as something done for its own sake, or as something needed for doing regular calculus (that may directly contradict students' experiences of doing calculus without specifying R), but rather as something necessary for doing more advanced calculus, one that students have not yet seen. This story defers a compelling motivation for specifying R to a future context (e.g. infinite dimensional calculus). The fact that students are tasked with 'taking David's word for it' further highlights the absence of an accessible rationale.

5.3.3 Meta-story 2: Analysis is calculus with proof

5.3.3.1 Alex – Analysis is rigorous calculus

Alex opened his lecture by framing calculus and analysis as covering overlapping content:

> So what is Real Analysis? I'm going to start writing a few things that we are going to do in analysis. We are going to study and define certain nice classes of functions. With these functions we are going to learn to compute derivatives, learn when that is possible, how to do this, and we're also, close to the end of this class, are going to carefully learn how to integrate. So, the class that I'm describing so far, you could call it analysis, but it also goes by another name, a class that all of you have taken before. So this is calculus.

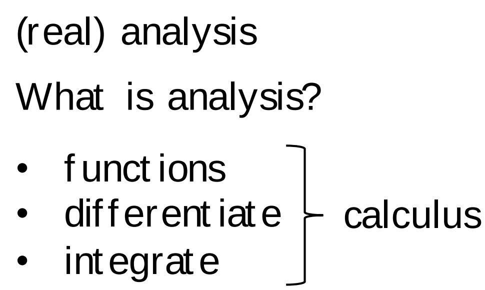


Figure 11: Alex whiteboard inscription: "What is analysis?"

In the above excerpt, Alex listed some mathematical ideas and processes that the RA course will address (defining functions, computing derivatives, integration). The items on this list are part of what is traditionally referred to as the course's *content*, i.e. what the course is about. Upon making this brief list of content goals for the course, Alex then remarked that the proposed content-focused description of the course is applicable to both Analysis and Calculus.

Alex used the assertion that Calculus and Analysis seem to cover 'the same' content, to raise the question of how the two courses may in fact be different, with the framing suggesting that the difference is in aspects *other than content*:

> So how is analysis different than calculus? Are we just going to be doing a different kind of calculus all semester? So let me open up this question to you. So, clearly analysis includes doing things we would like to do in calculus, but how is analysis bigger? How is analysis different from calculus? And, and people who don't maybe know much about calculus or have not seen a calculus result proven, I want you to guess.

In the above excerpt, Alex posed "how is analysis different than calculus?" as a question for students to answer ("let me open this question to you"). Even before addressing any student response, Alex offered several hints at an answer with a succession of elaborating questions and comments. In two subsequent questions Alex suggested that analysis is not *just* a different kind of calculus, but is in some way bigger. Furthermore, in suggesting that "people who … have not seen a calculus result proven" should guess, he hinted that proofs are relevant for that difference.

These questions were not merely rhetorical. Alex solicited student responses, asking them to participate by unmuting themselves or writing an answer on the Zoom chat. He also wrote the prompts "Guess" and "What is difference b/w calculus and analysis?" on the white board.

Students then shared answers on the Zoom chat, which Alex proceeded to revoice, record on the board, and briefly comment on:

> "Analysis *is* more rigorous." ((writes "+rigorous")) So, that's part of it for sure. What else? "It's proof based." Exactly. So, it's rigorous, and specifically the rigor is by having proofs. ((writes "+proofs!")) What else? Great, great answer. "Application versus theory." So calculus, mainly is concerned with applications. It's concerned with calculating things. In analysis we understand how these things fit together into a coherent theory. Theory emphasis over applications.((writes "+theory emphasis")). Great.

Alex agreed with student suggestions that analysis is more rigorous and proof based, and further elaborated a connection between these two observations: that rigor is achieved by having proofs. He then positively evaluated both responses as "Great." Next, he re-voiced a student suggestion that the difference between calculus and analysis is in applications versus theory. He elaborated that "calculus mainly is concerned with applications … with calculating things," whereas "in analysis we understand how these things fit together into a coherent theory." In Alex's elaboration, theory is about understanding and coherence. Application is about calculations.

Alex proceeded to summarize the discussion:

> So all of these are great answers. And these, uhm, these are the main practical ways in which our class is going to be different from analysis. However, later I'm going to- later in this class, I'm going to give you- uhm- tell you a certain story about what I think the importance of these differences are, how to contextualize these differences, and why one would want to have a class that's a lot like calculus but just has these additional properties of theory emphasis and rigor, why this is a good idea, and what I expect you to learn by the end of the term.

In the above quote, Alex summarized student answers as "the main practical ways" in which the two courses will be different. He then promised to tell "a certain story" about what he thinks is the "importance" and context of "these differences." That is, he promised to address not only the question of *how* these two courses are different (additional properties of theory emphasis and rigor), but *why* having this difference is helpful ("why this is a good idea") He also promised to connect this story to his learning goals for the course ("what I expect you to learn by the end of the term.")

Alex signaled desire to proceed to the next phase of the lecture (discussing logistics), but then noticed more student responses to the original question ("how is analysis different than calculus?") and decided to attend to them too:

Analysis deals with edge cases or, so, behaviors of functions we didn't know exist in calculus. So weird functions. ((writes "+weird functions")). So these are edge cases and extremities. Functions that blow up or go to infinity or just have really crazy behavior. More formal definitions. Right. That goes with the rigor. Plus formal definitions. Calculus is applied. Right, that goes with the theory emphasis. So analysis emphasizes theory versus calculus emphasizes applications. Right.
(Alex lecture 1, episode 8)

<u>Guess</u>
+ rigorous + formal definitions
+ proofs
+ theory emphasis vs applications
+ weird functions

Figure 12: Alex whiteboard inscription: How are analysis and calculus different? Student guesses

Here too, Alex both affirmed and expanded on student responses. On the student suggestion that "Analysis deals with edge cases," he elaborated that those edge cases refer to "behaviors of functions we didn't know exist in calculus," which are "weird functions," "extremities," or "functions that blow up or go to infinity or just have really crazy behavior." Alex also affirmed the suggestion that analysis involves more formal definitions, but added that this aspect "goes with the rigor," and thus merely reinforces what was already discussed. Similarly, to the suggestion that "calculus is applied," he responded by saying that this was already addressed in the "theory emphasis" point.

Board inscription	Student answer on Chat	Alex's Comment
+ rigorous	Analysis is more rigorous	So, that's part of it for sure.
+ proofs!	It's proof based	Exactly. So, it's rigorous, and specifically the rigor is by having proofs.
+ theory emphasis vs. applications	Application vs theory Calculus is applied	So calculus, mainly is concerned with applications. It's concerned with calculating things. In analysis we understand how these things fit together into a coherent theory. Theory emphasis over applications Right, that goes with the theory emphasis. So analysis emphasizes theory versus calculus emphasizes applications.
+ weird functions	Analysis deals with edge cases	So, behaviors of functions we didn't know exist in calculus. So weird functions. So these are edge cases and extremities. Functions that blow up or go to infinity or just have really crazy behavior.
+ formal def's	More formal definitions	Right. That goes with the rigor. Plus formal definitions.

I summarize Alex's "analysis is calculus with proof" meta-story in the follow way:

Analysis covers similar content to calculus, but with the additional features of rigor (formal definitions, proofs) and theory emphasis. One might legitimately ask: Why is it "a good idea" to repeat the same topics with these additional properties of rigor and theory?

5.3.3.2 Cai – Analysis is Calculus with Proof

Cai explicitly signaled the beginning of a meta-discussion by stating that he wants to talk about the course's goals, and writing a corresponding header in the lecture notes (see Figure 13below)

Figure 13: Cai notes inscription: "Goals of this course"

He then proceeded to give a short, somewhat tautological answer:

> "And of course as the title suggests we are working on real analysis and so of course the first goal is, of course, learn a solid foundation of real analysis."

This prompted him to immediately articulate the question of *what is* real analysis:

> Then of course you may have to ask what is real analysis? Does anyone have some idea what we mean by real analysis?

Similarly to Alex, Cai posed this as a question to students. However, unlike in Alex's case where there was a clear expectation that students actually provide answers to this question (indicated by extended wait time and repeated prompting to type or say their answers), Cai's question seemed to be rhetorical, as he immediately provided the following answer:

> Maybe one way to say this very simply, it's more like calculus with proof. Maybe that's one way to say.

In the above quote, Cai acknowledged that the description "Analysis is Calculus with Proof" is a gloss ("very simply"). He also hedged the suggested answer, using the words "maybe" and "like", and framing it as one of possible answers. He then elaborated:

> So we mainly focus on how to prove theorems, which you learn in calculus, like how to guarantee some series converges, how to compute derivatives, how to do integration like integration by parts and so on.

Like Alex, Cai highlighted content overlap with calculus by listing topics students already encountered ("how to guarantee series convergence", "how to compute derivatives", "how to do integration") and stating that in RA, the main focus is on "how to prove" those mathematical procedures, which he framed as "theorems." While he did not write the list of topics on the board as Alex did, Cai kept a written record of this meta-discussion in the lecture notes:

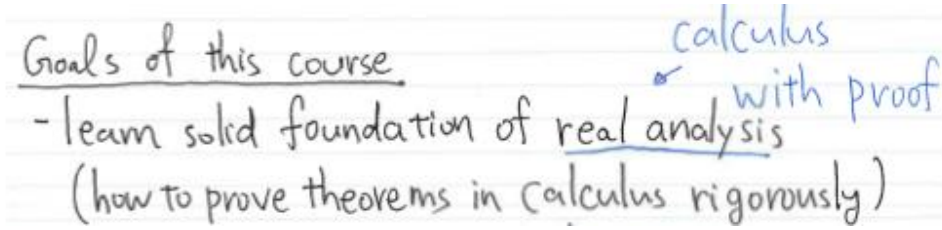


Figure 14: Cai notes inscription: "learn solid foundation of real analysis"

This discussion prompted Cai to ask:

> And of course, you may have to ask, why (do) we have to learn this again?

Thus, similarly to Alex, Cai used the story that analysis is calculus with proof to justify asking a “why” question. Because, if we assume the courses indeed share the same content, it naturally raises the question “why do we have to learn this again?”

I summarize Cai’s “analysis is calculus with proof” meta-story in the follow way:

RA is calculus with proof. The focus is on proving calculus theorems rigorously. This begs the question, why do we have to do this again?

5.3.3.3 Assumptions in the “analysis is calculus with proof” meta-story

The main assumption underlying this narrative is that the two courses, Analysis and Calculus, indeed have the same “content.” Alex substantiated this claim by writing down a list of topics to be covered the RA course – functions, differentiate, integrate – and pointed out that these same topics were already addresses in calculus courses. Cai, similarly, listed mathematical procedures students already engaged with in calculus – determining series convergence, differentiating, integrating – as the focus of RA too. For both Alex and Cai, highlighting the overlap in topics served as a rhetorical tool to motivate the question they wanted students to engage with. In Alex’s case, the question was: if not in *content*, how are analysis and calculus different? That is, highlighting the overlap helped direct attention away from content to other potential dimensions of difference between the two courses. In the case of Cai, the difference in approach (a focus on proof) was stated explicitly, rather than posed as a question to students, but the suggested overlap in content was similarly used to justify asking the question: “why do we have to learn this again?” Indeed, in saying “you may *have to* ask”, Cai suggested that analysis and calculus having the same content renders the “why question” natural.

It is possible to outline content goals for the RA course in a way that does not emphasize the overlap with calculus. For example, the mathematical topics listed in the department’s course descriptions of the RA and Calculus courses diverge more than they overlap (see Table 4 below). The RA course description mentions topics such as the real number system, metric space, and uniform convergence, not mentioned in the calculus description, whereas the description of calculus courses includes ordinary differential equations, not covered in RA. Some topics that are covered in both courses (e.g. differential calculus, limits), are in fact not listed in either of the top level descriptions.

Table 4: Comparison between departmental descriptions of RA and calculus courses

Department description of RA	**Department descriptions of Calculus**
The real number system. Sequences, limits, and continuous functions in R and R. The concept of a metric space. Uniform convergence, interchange of limit operations. Infinite series. Mean value theorem and applications. The Riemann integral.	**Calc 1**. An introduction to differential and integral calculus of functions of one variable, with applications and an introduction to transcendental functions. **Calc 2.** Continuation of Calc 1. Techniques of integration; applications of integration. Infinite sequences and series. First-order ordinary differential equations. Second-order ordinary differential equations; oscillation and damping; series solutions of ordinary differential equations.

Only calculus	**Both calculus & analysis**	**Only RA**
The real number system	Sequences	Differential calculus
Limits	Functions	Transcendental functions
Metric space	Series	First-order ODEs
Uniform convergence	Integrals	Second-order ODEs
Interchange of limits		Oscillation & damping
Mean value theorem		

The point of comparing these descriptions of topics with the ones offered by Alex and Cai is not to suggest that their descriptions are *wrong*. Any short description of course syllabi necessarily omits details, and hence highlights some aspects of the course at the expense of others. What I wish to draw attention to is that the emphasize on the content-overlap between the two courses, rather than their differences, does not give a full and accurate picture and serves a rhetorical function in subsequent meta-stories.

5.3.4 Meta-story 3: Fix calculus

5.3.4.1 Alex – RA is the "under the hood" of calculus. Good for fixing and building cars.

Alex transitioned from talking about course logistics to an extended episode of meta talk about RA by reiterating the questions he posed earlier, about why analysis – " a class that's a lot like calculus but just has these additional properties of theory emphasis and rigor" – is worth studying. He elaborated:

> So why study analysis? … So why not just study calculus and then say that, well yeah, we can make everything rigorous. Why bother with actually getting into the nitty gritty of proofs and rigor, in this particular context? And so there's two reasons.

The first of the two reasons Alex articulated was what I refer to as the "fix calculus" meta-story:

> … calculus is great when it works. And so there's- there's an analogy that I like which is that a body of knowledge that's sort of more application based is like a car. Calculus is

> like a car. And so you can drive the car and you can learn what all the dials and the levers do. And you can become very good at driving a car. But, sometimes your car will break. And when your car breaks it's really helpful, especially if you don't have a mechanic around, to learn what's under the hood of the car. So, and that is what analysis is. Analysis is when we take this very well developed, shiny theory of calculus and look under the hood. "Under the hood."

Central to Alex's fix calculus meta-story is the "calculus is like a car" metaphor (which he referred to as "an analogy"). The "calculus is like a car" metaphor construes calculus as a useful, "well developed" and "shiny" tool, which, like a car, can be used effectively without knowing the underlying mechanisms that make it work. But sometimes tools like cars and calculus break. In the case of a car, one way to deal with the tool breaking is call a mechanic. However, if there is no mechanic around, it is helpful for a user to know the underling mechanisms that make the car work ("what's under the hood of the car"), presumably to be able to fix the car by themselves. And so by analogy, to be handle calculus breaking, it is useful to know the *underling mechanisms* that make calculus work. Analysis is a study of these underling mechanisms, a look "under the hood" of calculus.

Alex continued by providing two reasons for why looking "under the hood" of calculus is useful, the first being a reiteration of the "fix calculus when it breaks" motivation:

> So there's two reasons why you'd want to look under the hood. So it's useful either if your car breaks. So this is what someone mentioned, that some functions are weird, they diverge or they are not continuous or are not differentiable. So sometimes, even when you are doing things that are essentially calculus you will end up with functions that don't behave like you expect. And then it's really nice to understand how the car actually works to see if you can still get them to do what you want.

In the above excerpt, Alex elaborated on what a "calculus breaks" scenario might look like. It is possible, Alex claimed, to do things "that are essentially calculus" but still end up with functions that exhibit strange, unexpected behaviors like divergence or discontinuity. Understanding the "under the hood" of calculus (i.e. its underling mechanisms) then allows one to better control these "weird functions" ("see if you can still get them to do what you want").

Next, Alex provided a second reason for looking "under the hood" of calculus:

> And it's also useful if you want to build a new car. So if you want to study more interesting- study and especially do research in more interesting versions of calculus, where maybe things are multi-dimensional or complex, or even infinite dimensional, or don't always converge, it's really useful to understand how a car really works in order to be able to then build a fancier one, or a different one.

Continuing with the "calculus is a car" metaphor, Alex's second motivation likened the study of advanced calculus ("multi-dimensional or complex, or even infinite dimensional") to building new, "fancier" cars. In the case of cars, understanding how a car "really works" is helpful (if not necessary) for building new cars. And so by analogy, understanding how calculus "really works" is helpful for studying more advanced versions of calculus.

Across the different episodes, Alex's "fix calculus" meta-story relied heavily on the "calculus is a car" metaphor. The table below shows the different elements of this metaphor Alex used to construct his meta-story.

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

I summarize Alex's "fix calculus" meta-story in the following way:

Calculus, like a car, can be used effectively without understanding the underlying mechanisms that make it work. However, when calculus breaks (e.g. when you get weird functions), analysis – an understanding of underlying mechanisms – is useful. Furthermore, understanding the underlying mechanisms of calculus allows one to develop more advanced versions of calculus.

5.3.4.2 Cai – Calculus is a powerful tool that sometimes does not work.

Similarly to Alex, Cai's "fix-calculus" narrative relies on a conceptualization of calculus as a *tool* or set of tools:

> "… there are very powerful helpful tools from calculus. First, some of you might have heard the name l'Hospitals like theorem with which sometimes you can compute particular limits very easily."

Cai began his *fix calculus* meta-story by stating that "there are very powerful helpful tools from calculus." He mentioned l'Hospital theorem as an example of one such tool, and added that this tool is used to "compute particular limits very easily." In this story, calculus tools are framed in a positive way – they are powerful, helpful, and easy to use. But, as Cai explained next, despite being very useful, these tools sometimes to do not work:

> “Now, as we will see in the class. Sometimes you realize you can’t apply this theory. And to judge whether a particular question you can apply l'Hospital theorem or not is very subtle. And in fact, many mathematicians have made mistakes. And then they realized they really need a rigorous foundation for Real Analysis. So that the theorem is actually true and can be proved.”

In the above excerpt, Cai claimed that in certain situations the tools of calculus cannot be applied (“Sometimes you realize you can’t apply this theory”). Or, rather, sometimes using these tools can lead to mistakes, which “many mathematicians” have done in the past. He promised that later in the class, students will get to see these problematic situations themselves (“as we will see in this class”).

To avoid making these mistakes, it’s necessary to determine in which situations one can or cannot apply a calculus tool such as l’Hospital’s rule. Making such a determination is a “very subtle” problem. According to Cai, mathematicians realized that to address this subtlety, to avoid making mistakes using calculus tools, “they really need a rigorous foundation for Real Analysis.” A rigorous foundation solves these problems because it allows theorems to be “actually true” and “proved.” It is implied that theorems that are “actually true” and “proved” resolve the issue of determining conditions for correct tool applicability. Meaning, with a rigorous foundation, calculus tools can be applied without error.

Finally, Cai connected this story to an explicit learning goal for the RA course:

> “And we want to just follow their efforts to learn and understand the super correct foundation of real analysis.”

I summarize Cai’s fix calculus meta-story in the following way:

Calculus has powerful, easy-to-use tools that sometimes cannot be applied. Determining when a tool can be used without error is difficult, and in the past, many mathematicians made mistakes applying the tools. They found that to avoid mistakes, theorems need to be proved true, which a rigorous foundation makes possible. The analysis course follows this historical process of using rigorous foundation to actually prove calculus theorems so they can be applied without error.

5.3.4.3 Assumptions in the “fix calculus” meta-story

Similarly to the “precise R” meta-story, the “fix calculus” meta story assumes that calculus is a desirable activity. Indeed, both Alex and Cai explicitly framed calculus in a positive light. Alex referred to calculus as a “well developed, shiny theory,” akin to a well-functioning car. Cai described the tools of calculus as “very powerful,” “helpful” and easy to use. But, the problems these tools help solve, what makes using them desirable, is left unexplained. Thus, similar to David and Emmett “precise R” meta-story, Alex and Cai’s stories take interest in calculus for granted. Here too such an assumption seems reasonable, as motivation for calculus is likely addressed in calculus courses. Similarly to David, Alex takes interest in calculus a step further and posits developing new versions of calculus (“building a new car”) as another desirable objective.

The second constitutive assumption in both Alex's and Cai's "fix calculus" meta-stories is that using the tools of calculus can lead to breakdowns. Alex described such instances metaphorically as "the car" breaking. The metaphoric "car breaks" scenario refers to "doing things that are essentially calculus" (driving the car) and ending up with functions that are "weird," "divergent," "not continuous," "not differentiable," or "don't behave like you expect." Thus, in Alex's metaphoric conceptualization, getting "weird" functions is likened to breakdown. In Cai's story, breakdowns feature as well but they are characterized as errors of judgement. Cai said that "many mathematicians made mistakes" using calculus theorems by wrongly applying them to situations in which they are not applicable. Cai used "l'Hospital theorem," likely familiar to students from calculus courses, as an example of a theorem for which judgement of application is "very subtle" and hence can lead to errors of inappropriate application.

The third assumption I wish to highlight in Alex's and Cai's fix-calculus story is that analysis – what they both earlier defined as proof based calculus – is what's needed to resolve or avoid calculus breakdowns. Alex's fix calculus story used the metaphors of car and under the hood to position analysis (looking "what's under the hood of the car") as *a resource* for being able to independently (without a "mechanic around") resolve problems with calculus breaking. More specifically, Alex said, analysis ("understanding how the car actually works") allows control over "weird" and unexpected outcomes ("get them to do what you want"). Cai's story similarly positioned "a rigorous foundation for Real Analysis" as something mathematicians realized *they needed* to avoid doing mistakes with calculus theorems.

How does this story relate to what we know about disciplinary practice? While Alex's and Cai's stories do not identify the specific mathematical contexts in which, as they claim, calculus breaks, it is possible to speculate given the descriptions they offer.

For example, Alex's story of situations in which one ends up with "weird functions" from doing things that are "essentially calculus" is compatible with the discovery of discontinuous limits of trigonometric series in the early 1800s. When solving differential equations by trigonometric approximations (which is, "doing things that are essentially calculus"), the French mathematician Joseph Fourier found examples of infinite sums of trigonometric expression whose limits are (from our contemporary perspective) discontinuous.

The very first example of using "trigonometric series in the theory of heat" in his book (Fourier, 1878, section II), he found the following solution function to heat equation

$$y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots$$

and claimed that the equation

> "belongs to a line which … is composed of separated straight lines, each of which is parallel to the axis …. These parallels are situated alternately above and below the axis… and joined by *perpendiculars which themselves make part of the line*." (Fourier, 1878, p. 144; my emphasis)

Figure 15 below illustrates (in blue) the graph of the limit function that the above quote from Fourier described verbally.

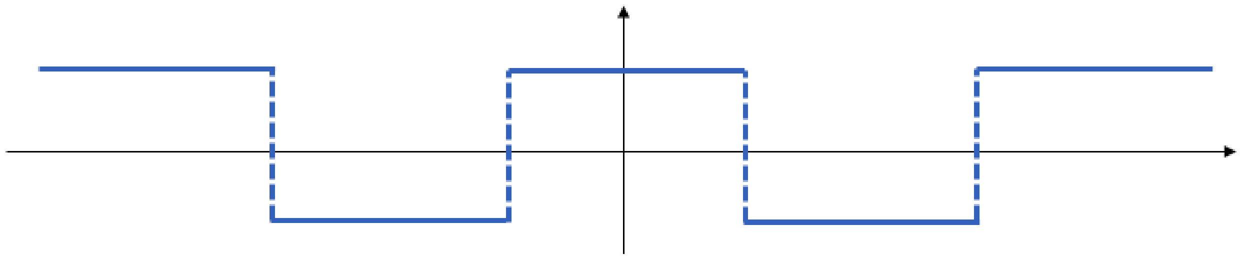


Figure 15: A graph of the limit function of the trigonometric series given by $y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots$

Note that in Fourier's description, the perpendiculars (dashed lines in the figure) are considered to be *part of* the graph representing the function $y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots$ ("perpendiculars which themselves make part of the line"). Per our current conventions, perpendicular segments like these should not be considered as part of a graph of *a function*[18]. Instead, we interpret a graph of this type as having *discontinuities* at the x values in which these perpendiculars appear. To highlight this understanding that the perpendiculars are not technically part of the graph, it is conventional to draw them with dashes (as I did in Figure 15). In short, assuming the now consensually accepted definitions of functions and continuity, the graph depicted above is that of a discontinuous function. Thus, this example matches the descriptions put forth by Alex: Fourier obtained a "weird" ("discontinuous") function by "doing things that are essentially calculus."

This example of obtaining a discontinuous limit function also exemplifies situations in which outcomes do not "behave like you expect." Underling the then widespread method of finding solutions with approximations by simpler functions as building blocks (in this case, trigonometric) is an expectation that the limit function, the one obtained after adding infinitely many expressions, shares the properties of its constitutive elements, such as continuity. This expectation that the limit function is continuous was widely shared by mathematicians of the day (Gray, 2015). In fact, it was famously canonized as *a theorem* in Augustine Cauchy's textbook Course d'Analyse: "if a sequence of continuous functions converges to a limit, then the limit function is continuous." It was soon pointed out (by the Norwegian mathematicians Niels Abel) that Fourier's example features a discontinuous limit function and is thus an *exception* to this rule (Sørensen, 2005). This, as well as Fourier's own insistence on the function's continuity, illustrate that a discontinuous limit was an unexpected outcome for mathematicians.

Cai's story of "many mathematicians made mistakes" in applying calculus theorems is also compatible with this historical example. Both Fourier's and Cauchy's claims quoted above are considered mistakes from our contemporary vantage point[19]. The theorem stated in Cauchy's book, when applied to the example given by Fourier, implies that the limit function depicted in Figure 15 is supposed to be continuous, which, as explained above, we consider to be wrong by

[18] In contemporary calculus education discourse, the graph Fourier describes fails "the vertical test."

[19] Importantly, these "mistakes" were extremely productive in that they spurred many important mathematical and meta-mathematical developments (Kitcher, 1984; Lakatos, 1976; Trninic et al., 2018).

the now standard definitions of continuity. This mismatch between the theorem statement and Fourier's example was quickly recognizes as an issue also by 19th century mathematicians (Gray, 2015; Kitcher, 1984), and led them to focus on the "very subtle" problem (to use Cai's language) of figuring out exactly under what conditions the theorem does hold true. Indeed, as Lakatos (1976) points out, this problem was so difficult for the mathematicians of the day, that it took more than 20 years to resolve[20].

And, it is this very problem of "fixing" Cauchy's theorem and proof that compelled mathematicians to introduce the distinction between pointwise and uniform convergence (what Lakatos (1976) calls a proof-generated concept). And, it is this very distinction between pointwise and uniform convergence that most clearly showcased the advantage of Weierstrass's epsilon-delta language of limits over the geometric-dynamic definitions put forth earlier by Cauchy. Because, unlike the geometric-dynamic language, the epsilon-delta definition allows one to *symbolically encode different rates of convergence*. In this sense, Cai's claim that mathematicians "realized they really need a rigorous foundation for Real Analysis" in response to making "mistakes" can be seen as compatible with this historical narrative. To solve the "very subtle" problem of discerning under what conditions Cauchy's theorem holds mathematicians had to introduce the distinction between point-wise and uniform convergence, which was rendered easily encodable by the epsilon-delta language. In this historical story, the reformulation of all basic calculus definitions (limits, continuity, series convergence) in the epsilon-delta language – which, in Cai's terms, may be thought of as "a rigorous foundation for Real Analysis"– came forth as *a tool* to solve a pertinent problems of practice (Kitcher, 1984).

There are other historical narratives to match Cai's story of mathematicians making "mistakes" using calculus, i.e. applying theorems in ways that we today consider weird, inappropriate or plain wrong. For example, leading 18th century mathematicians (among them Leibniz and Euler), used and justified claims such $1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$, which we today, with Cauchy' canonized definition of series convergence as a backdrop, consider absurd. The story of such computations also exemplifies how mathematicians' desire to rigorize calculus emerged as *a need* to resolve what was increasingly seen as error-prone practice, and not as un-motivated interest in foundations for their own sake (Kitcher, 1984). In this sense, computations with infinite sums of numbers (chronologically an earlier issue than that surfaced by Fourier's trigonometric methods), is yet another historical that matches Cai's story of how "mistakes" compelled mathematicians to realize that "they need a super correct foundation of Real Analysis."

The point of this extended discussion of how different episodes from the history of analysis match the two main assumptions in Alex's and Cai's "fix-calculus" story – calculus breaks and rigor is needed to fix it – is to demonstrate that this story can be seen as a reasonably accurate reflection of (past) professional practice. Thus, the story, while rife with metaphorical glosses[21], captures important aspects of authentic disciplinary practice . The question that remains is: is

[20] In fact, Lakatos (1976) attributes the discovery of the method of "proofs and refutations" – a major *meta*-mathematical transformation of 19th century mathematics – to this exact problem.

[21] For example, the car metaphor may imply that once problems emerge (the car "breaks"), one can no longer use the tool (calculus). This blatantly contradicts practice, as one can (and many did and do) effectively use calculus without resolving these issues.

there good reason to believe that this story and its constitutive assumptions are compelling to students?

The biggest issue I see is with the claim that calculus breaks. By the time students take the RA course, they already had extensive experience with calculus working. Using Alex's car metaphor, they spent many years driving the car without witnessing or being told of any potential problems with the machinery. To buy into the fix calculus story, students need to believe not only that it is possible for the car to break, in theory, but also that it happens often or in a context that matters. If problems are extremely rare, and only happen in use-cases that outside the scope of my practice, why should I as a user bother to go through the labor-intensive process of "looking under the hood" and "understanding how the car actually works"?

Neither Alex's nor Cai's story provided students with enough information about the problems to allow independent judgement of whether the issues are important or interesting enough to justify all the fuss of analysis. To buy into the story, students have to take Alex's and Cai's word for it. And in fact, the stories use a variety of rhetorical moves to compensate for the absence of explicit justification. In Alex's case, the car metaphor itself renders the need to "understand how the car really works" as self-evident. *Everyone knows* that cars can break and that it is helpful to know something about what's under the hood. In Cai's story, there is no overarching metaphor to do the rhetorical work. Instead, Cai explicitly acknowledged the absence of supporting detail by promising that these will be available later ("as we will see in the class"). His story also recruited the authority of "many mathematicians." This renders the analysis effort important and worthwhile by the sheer fact that mathematicians considered it to be. Students' motivation to learn analysis is then framed as a desire to "follow *their* efforts."

5.3.5 Meta-story 4: Connections

The meta-story I call "connections" was the most lengthy and elaborated meta-story in the data set. I attributed this meat-story only to Alex, who devoted a total of 10 minutes to it in his lecture (parsed into 11 episodes in the transcript). For context, recall that the total length of meta-talk in Alex's lecture was about 20 minutes, whereas for the three other instructors, it was on the order of five minutes.

5.3.5.1 Alex – Simplify through connections

The "connections" meta-story was the second motivation Alex provided for studying RA. He articulated it immediately following the "fix calculus" meta-story (which I analyzed in the previous section) and marked the transition explicitly:

> But there is another reason, that's actually dearer to me because my main research is not analysis. It's not building cars that are similar to calculus. It's a little bit different. And so the second reason - and I think that the second reason is the main reason why analysis is on your curriculum. And this gets down to what I think is the point of math in general. So what is the point of math. Why do math?

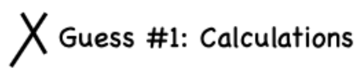
In the above quote, Alex explicitly framed the forthcoming meta-story as more significant than the "fix calculus" one: it is more *personally* meaningful to him ("dearer to me because my main research is not in analysis"), it is the "main reason" for the *institutional* decision of including RA in the undergraduate curriculum, and it is connected to the *bigger purpose question* of why do math at all (what "is the point of math in general").

He quickly clarified that the posed question was only in relation to contemporary academic (proof-based) mathematics, not mathematical practices more generally:

> Well, by math I don't mean high school math, I don't mean mathematical modeling. I mean, maybe more abstract math. Like analysis, algebra, etc.

Before providing his endorsed answer to the "why do math" question, Alex suggested a couple of "wrong" responses, immediately following them with rebuttals:

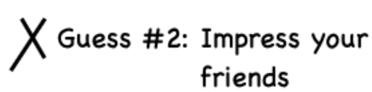
> …two possible answers to this question that are wrong. So, guess number one. If you ask someone on the street why do math, well, they will think that what mathematicians do is calculations. But that's not the point of math. Even if we do end up doing calculations, even if we are applied mathematicians and want to get a calculation (at the end), the point is not the calculations. It's something deeper.



In the above excerpt, Alex proposed "calculations" as a lay person's ("someone on the street") response to the "why do math" question. He immediately framed this answer as wrong ("But that's not the point of math"). Alex went on to explain that on a surface level this observation is correct: calculations can be something mathematicians do, and, for "applied mathematicians", it can even be the immediate goal of activity (what they "want to get"). But, since it is not the ultimate *purpose* of this activity, claiming that the point of math is calculations is akin to confusing a surface level feature (what mathematicians do) with "something deeper" (why they do it).

He continued with a second 'wrong answer':

> And maybe a bigger problem for undergrads, and maybe a common reason to do math is that math is known to be complicated. And so what you can do is you can essentially design some very complicated game with symbols and complex relationships between them and then play it and get really good at playing it to impress your friends. But that is also not the point of math. It's very much not. The point – we don't want to make things complicated and show off how good we are at playing arbitrary games. We actually want to… to make things simpler and more elegant.



Alex framed the second "wrong answer" as the one most relevant to undergraduate students ("biggest problem for undergrads"). He suggested that because "math is known to be complicated," a common reason why undergraduate students choose to do it (e.g. take RA courses) is to gain status within their peer group ("to impress your friends"). This perspective, as voiced by Alex, characterizes math as a "game with symbols" that is "arbitrary" and intentionally designed to "complicated" so that by getting "really good at playing it" would be an impressive feat, and allows on to "show off."

Like the previous mock answer of "calculations," the doing math to "impress your friends" rationale is empathetically rebutted by Alex ("this is also not the point of math. It's very much not.") This motivation was positioned as antithetical to mathematical practice, because mathematicians desire simplicity ("we actually want … to make things simpler and more elegant.")

Finally, Alex was ready to articulate the endorsed answer, which is the "connections" meta-story:

> And so what I think the point of math is, or at least something that gets very close to the essence of what math is, is math is about making connections. It's making connections between things like calculations and things like geometry. Things like, different, maybe slightly more complicated algebraic structures which, nevertheless, are there in order to more easily make connections between different scientific and sort of numerically and geometrically aligned disciplines. So, so the math is really in the connections. And in order to really learn how to do abstract math, you need to learn to make connections. And so what I think, what my main answer is to why study calculus, is that it is in itself a connection.

In this story, "connections" is interchangeably positioned as the purpose ("the point of") and the nature of mathematics ("the essence of what math is," "math is really in the connections," "calculus … is in itself a connection"). That "connections" are the *purpose* of math is evident in Alex's claim that any complications ("slightly more complicated algebraic structures") are introduced in the service of making connections ("are there in order to more easily make connections"), and not, as the "impress your friends" story suggests, to intentionally add difficultly. Since connections are also the *nature* of math, to "really learn how to do abstract math," one needs "to learn to make connections."

What is being connected by these connections? In the above excerpt, when talking about "abstract math" in general, Alex mentioned calculations, geometry, algebraic structures, different scientific disciplines, and numerical methods. In the next six episode (5 minutes of lecture time, in total), he proceeded to describe the specific connections that are relevant to the RA course.

First, he talked about logic, which he framed as a "very fundamental subject in math that is so fundamental it is also a subject in philosophy." He explained that "people realized in the early 20th century ... that all of math can be reduced to … very basic logical language," but because it "would just be too much work," it is not something that they (Alex and students) are "going to do in this class."

Then, he moved on to talk about set theory, which, unlike basic logic, *is* "something that we will use " in the RA class. He characterized set theory as "related to" many different kinds of math. First, to Boolean logic, which he described as an interesting connection, albeit one studied in a course on logic, not RA. He also mentioned connections between set theory and basic algebra, namely, arithmetic operations on integers and fractions. Numbers too can be reduced to "basic logical statements," Alex said, which is yet another connection that is "not the main point of our class," but students "would at least get a feel for" if they "took a logic class."

The final topic he mentioned in this overview of to-be-connected mathematical subjects is calculus, which he described as something students learned "in high school or college." Alex pointed out that in calculus courses "they don't teach you … how to reduce things like taking a derivative… to basic logic and algebra." In those contexts, he continued, "you know roughly that a derivative is something like a tangent line, but you don't know how to reduce it to these more fundamental branches of math."

It is this missing link that Alex subsequently frames as the purpose of the RA course:

> And so, this class of analysis is going to be- its main purpose is going to be establishing a connection between calculus and generalizations of calculus, and more powerful versions of calculus. And connecting them down, really expressing them rigorously in terms of more fundamental mathematics. So, let me write it down like this. So, analysis is really a theory of connections. And the connections that it encompasses are, or at least the ones we will see the most, are connections first of all between calculus and basic algebra. So integers, operations on integers and their formalism, and also to set theory.

In Alex's "connections" meta-story, RA is "a theory of connections". The main purpose of the RA course is to connect calculus and its generalizations to "more fundamental mathematics," in particular, to "basic algebra" and "set theory." In Alex's story, this connecting has a downward trajectory; from a top level of a structure to its base ("connecting them down"). Making such connections is accomplished by "rigorously" expressing one level in terms of more basic ones.

Throughout his discussion of different topics and connections between them, Alex made corresponding whiteboard inscriptions. The resulting inscription is reproduce in Figure 16. Boxed inscriptions represent the different topics mentioned in the narrative: logic, set theory, basic algebra, and calculus. Arrows in black represent connections that, according to Alex, are addressed in a logic rather than a RA course. He even added the code of the department's logic course (XXX in the figure, for anonymity) in parentheses next to the relevant arrows (see Figure 16). The connections that comprise analysis are represented by red[22] arrows. Note that unlike the black arrows which are bi-directional, the red arrows only point in the direction from calculus to "more fundamental mathematics," again reinforcing the downward trajectory of RA connections in Alex's meta-story.

[22] The color coding in Figure 16 reflects the color-coding in Alex's original whiteboard inscriptions.

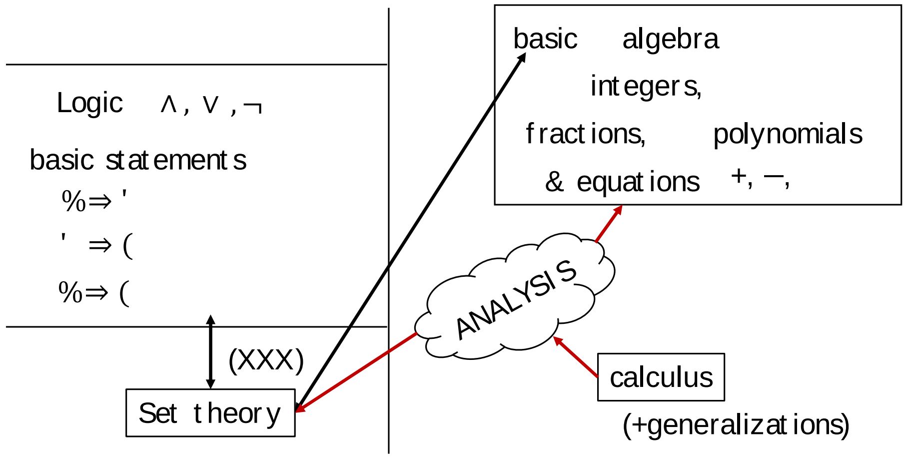


Figure 16: Alex whiteboard inscription: different topics connected in real analysis

I summarize Alex's connections meta-story in the following way:

The point of doing abstract math is not computations or impressing people by playing complicated games. Instead, the point of doing abstract math is to make things simpler and more elegant through connections. In RA, those connections are between calculus and its generalizations down to the more fundamental topics of basic algebra and set theory.

5.3.5.2 Assumptions in the "connections" meta-story

The first assumption in Alex's connections meta-story is that making things simpler and more elegant is desirable. In particular, in the context of calculus. Alex emphasized this point by juxtaposing the desire for simplicity with a seemingly opposing perspective: namely, that of making calculus complicated *on purpose* in order to "impress your friends." Alex attributed the latter motivation – to look smart in the eyes of peers – to undergraduate students ("a bigger problem for undergrads").

In Alex's narrative the vocalizers of "wrong answers" are undergrads and people on the street. The "correct answer," however, was not explicitly attributed to particular types of people. Instead, Alex voiced the desire for simplicity through an unspecified "we" (e.g. "*we* don't want to make things complicated…" and "*we* actually want to… to make things simpler and more elegant").

Who is the "we" in Alex's story? It can be interpreted as an opinion he shares with a few others, or, that he speaks with authority for the entire community of mathematicians, or that the "we" incorporates the very students he is addressing, thus rendering the statement prescriptive rather than descriptive. That is, Alex could be describing a current state of affairs (many if not all mathematicians, himself included, desire simplicity and "very much not" complication to look smart) or outlining a perspective he urges students to adopt about mathematics. Or, of course, as discourse is multifunctional and speakers are laminated (Goffman, 1979), the use of "we" in this context could be accomplishing both. Indeed, other scholars have noted the ambiguous meaning of personal pronouns in pedagogical mathematical discourse (Rowland, 1999).

Alex's stated desire for simplicity and elegance, and framing that desire as the essence of mathematics, echoes articulations by many mathematicians (Inglis & Aberdein, 2015). For example, the Atiyah (1978) stated in an interview that:

> "Both unity and simplicity are essential, since the aim of mathematics is to explain as much as possible in simple basic terms." (Atiyah, 1978, p. 76)

Thus, in some sense, the claim that (at least some) mathematicians "want to… to make things simpler and more elegant" can be considered an accurate portrayal of professional practice, if we consider how professional practice has been represented in *mathematicians' reflections*. However, there is a big difference between what mathematician say they find desirable in mathematics when reflecting about the purpose or significance of their work post hoc, and what compelled them to take particular actions in the moment[23]. In particular, what compelled them to develop, study and use RA.

Hardy's famous essay "A Mathematicians Apology" (Hardy, 1940) is a case in point. In the culture of contemporary academic mathematics, this essay is most commonly remembered as an exemplar of the "mathematics is pursued for its beauty" narrative, exemplified by quotes such as:

> "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way." (p. 14)

However, within this same essay, Hardy also clearly states that the actual "motives" people have for doing research are far less "noble" (to use his terminology). Instead, he wrote that "intellectual curiosity, professional pride, and ambitions are the dominant incentives to research" (p. 11).

In the very last, autobiographical section of his Apology, Hardy admits that his own trajectory into mathematics is an example of these more down to earth desires:

> "I do not remember having felt, as a boy, any passion for mathematics, and such notions as I may have had of the career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships: *I wanted to beat other boys*, and this seemed to be the way in which I could do so most decisively." (Hardy, 1940, p. 46, my emphasis)

Hardy's description of his initial motivation to study mathematics ("to beat other boys") can be interpreted as a counterexample to some of the claims in Alex's connections meta-story. In Hardy's recollections, we see a novice mathematician that is driven not by desire for simplicity but by a wish to impress his peers. Hardy thus is not one of Alex's "we" that do not want to "show off how good we are." While I am not aware of empirical research systematically examining mathematicians' motives for doing and learning mathematics, I find it unlikely that Hardy's account is out of the norm. I believe that similar pathways into mathematics were treaded by many successful mathematicians. Furthermore, to maintain success in the competitive

[23] Similar points were made by Schoenfeld (2014) and Inglis & Aberdein (2015), about how mathematicians' self-reflections are not a reliable source of knowledge about the nature of their practice.

field of academic mathematics one is often compelled to complicated things on purpose to gain professional status. Thus, Alex's story implying that mathematicians are not primarily motivated by a desire to "impress their friends" is, at best, an idealization. A state of affairs he may wish to be true, but not an accurate description of current reality.

Another aspect of mathematical practice to consider in weighing the faithfulness of the "we desire simplicity" claim, is whether the desire to simplify and make more elegant is what motivated mathematicians to develop RA to begin with. As was discussed in the context of the "fix calculus" meta-story, many of the specific mathematical and meta-mathematical changes that transformed calculus into what we today recognize as RA were developed as solutions to specific mathematical problems, such as the need to reconcile Fourier's examples of a discontinuous limit function with the normatively held assumption (and Cauchy's explicitly stated theorem) that limit functions 'inherit' the properties of the functions that approximate them (Kitcher, 1984).

However, other kinds of considerations and historical conditions shaped these developments as well. At the end of the 18th and early 19th centuries, the rise of new models of academic institutions (the grand Echoles in France, the Humboldt research university in Germany) initiated a monumental shift in the material and institutional conditions for 'doing' research mathematics (R. Collins, 2009). In particular, the need to train large cohorts of students in the techniques of calculus, which until then were practiced in a somewhat ad hoc manner by 'virtuoso' mathematicians such as Euler, created a demand to consolidate the techniques into a systematic, coherent and accessible body of knowledge. That is, the need to teach obliged mathematicians to simplify, unify and render *reliably reproducible* the available methods of calculus. Formalization and rigor accomplish just that. The biggest landmarks of the transformation from calculus to analysis in the 19th century – Cauchy's geometrico-axiomatic consolidation of calculus around the concept of limit in his course d'Analyse, Dedekind's arithmetic definition of real numbers, Weierstrass's epsilon-delta (non-dynamic) reformulation of limits and continuity – were all developed in the context of teaching[24] (R. Collins, 2009; Kleiner, 1991). In that sense, desires for ease and elegance *can* be seen as central to the rigorization of analysis, but they emerged from pedagogical needs, whereas Alex's story positions them as main driving force of mathematical work.

No doubt, the desire for simplicity and elegance is experienced by many mathematicians. They certainly repeatedly say so! (Atiyah, 1984; Inglis & Aberdein, 2015) However, there is a difference between claiming that *some* mathematicians are *sometimes* motivated by the desire to make things simpler and more elegant and Alex's meta-story, which is much more grandiose: simplicity and elegance are framed the single most important (and correct!) reason for doing abstract mathematics.

In fact, the idea that math has a single main "point," "essence" or motive is yet another assumption built into the connections meta-story. Even before providing "connections" as the

[24] In fact, R. Collins (2009) points out that many contributions to rigorization in mathematics in the 19th century, not only in analysis, came from practicing school teachers. Weierstrass is one such example (Kleiner, 1991). This further suggests resonance between the practice of teaching and the epistemic aims of systematizing knowledge through rigor.

correct answer, Alex repeatedly framed “abstract math” as having a single main “point” or “essence”:

> … **the main reason** why analysis is on your curriculum. And this gets down to what I think is **the point** of math in general. So what is **the point** of math. (episode
>
> But that’s not **the point** of math. Even if we do end up doing calculations, even if we are applied mathematicians and want to get a calculation (at the end), **the point** is not the calculations.
>
> But that is also not **the point** of math. It’s very much not. **The point** –
>
> what I think **the point** of math is, or at least something that gets very close to **the essence** of what math is, is math is about making connections

However, as the above discussion of different motivations demonstrates, positioning any single desire as *the* “essence” of mathematics inevitably creates some distortions.

The front-back metaphor (Goffman, 1978; Hersh, 1991) may be useful for thinking about this kind of distortion. In the ‘back’, actual human mathematicians are motivated by down to earth desires such as competition and status. At the ‘back,’ goals such as simplifying emerge, at least in part, from institutionally mandated teaching requirements. But, that’s a messy sociological reality. When we present the meal at the ‘front’ of the restaurant, to others and maybe even to ourselves in the process of reflection, we need a clean story. Mathematics having a single main purpose or essence is reasonably seen as a more palatable story to serve and consume. The question is, what effects does this particular plating choice have (to push the metaphor even further)? Are students benefitting from it?

To what extent can we expect students to see themselves reflected in the idealized “we” that doesn’t want to “show off” and instead strives for simplicity and elegance? On the one hand, it is plausible that students who dislike competitiveness in mathematics will find the vision of mathematics depicted in Alex’s narrative inviting. If this is the case, it has important equity implications. Many dominant narratives in mathematics education center and foster competitiveness, and in so doing, systematically privilege those students who (like Hardy) value and desire individual success and comparison with other. Promoting values such as competitiveness and individual success through both the practice of education (testing, gradings) and corresponding meta-storytelling, can also be systematically privileging men (Ernest, 1995). In this way, Alex’s story about valuing simplicity over showing off, can be experienced as a relatable counter narrative to the dominant one inscribed by educational practice, and a more compelling one for students that are marginalized by the values embedded in mainstream practice.

On the other hand, it is important to acknowledge that the RA course is embedded in an institutional context seeped in neoliberal ideology. In particular, many students choose to major in mathematics because of the perceived market value of this degree (Solomon, 2006). That is, for many students, the motive for taking a RA course may not be to “impress their friends” per se, but rather, to impress future employers or graduate school admissions committees. Thus, while the sentiment Alex expressed may resonate with at least some students, some may also perceive it as out of touch or even alienating. A student may get the impression that if their intention is not pure and idealistic, “noble” to use Hardy’s term, it must mean they do not have what it takes to do mathematics. Mathematicians, Alex’s story implies, should be motivated by intrinsic passion for elegance and simplicity.

Furthermore, students do not develop a utilitarian view of RA as a stamp of approval for ‘smartness’ in a vacuum. And not just because their friends consider the game to be complicated. Institutionally, RA courses often play precisely this smartness-labeling role. For example, RA is often added as a requirement by non-mathematics programs that want to brand themselves as “more rigorous” and “hard,” regardless of the practical utility of the course’s content for students’ subsequent academic work. Some math graduate programs single out success in RA courses as an unofficial litmus test for ‘math grad school readiness’ in their application reviews, more so than any other upper division undergraduate course-work. Within that context of grad school admissions, the more one can show that their RA course was complicated (indexed by the type of textbook used in the course), the better the impression made on the admission committee. So, many features of the higher education system set up the RA course to function in exactly the way Alex says it does (or should) not: an arbitrary complicated game with symbols used to impress and show off.

To sum up, the statement “in math, we desire simplicity and elegance” is faithful to some mathematicians’ reflections about the nature of their practice, but positioning it as *the* one single purpose of mathematics renders invisible other salient motivations. The statement “we do not want to show off” may be compelling to some students, particularly those marginalized by the dominant competitiveness promoting practices. At the same time, the framing of the desire to make an impression as foreign to ‘real’ mathematics may alienate students who, legitimately, due to the institutional configuration of RA courses, are motivated by the stamp of approval succeeding in this complicated course provides. It may create the wrong impression that to be a mathematician, one should be motivated to do math by the “noble” desire for simplicity and elegance.

The final assumption I wish to highlight in Alex’s connections meta-story is that rigorous connections to fundamental topics indeed make calculus more simple and elegant. In a way, this is a statement about epistemic affect (Jaber & Hammer, 2016). The claim is that going through process of connecting calculus to fundamental topics, or perhaps experiencing the outcome of such a process, will lead to feelings of greater ease and aesthetic appreciation.

There are a lot of introspective accounts by mathematicians suggesting that they often experience ease and aesthetic appreciation when doing mathematics (Hardy, 1940; Inglis & Aberdein, 2015). But, is there evidence suggesting that the activity of connecting calculus to fundamental topics specifically leads to such experiences? Above, I discussed how positioning the desire for simplicity and elegance as the main driving forces for learning or developing analysis is misleading. However, even if simplicity and elegance were not the primary goals for the rigorization of calculus, or the choice to study analysis, it is still possible that the process did lead to widespread experiences of aesthetic appreciation among mathematicians.

Is it reasonable to assume that students would experience such "downward connections" as an increase in simplicity? Research about student difficulties with RA (Alcock & Weber, 2005), as well as Alex's own framing of RA as "known to be complicated", suggests that that's not likely. Such a claim can thus be seen as out of touch with students' lived experiences. It may create a perceived divide between mathematicians, who do experience ease and aesthetic appreciation from reducing calculus to fundamental topics, and students, for whom the same experience is difficult and messy. In so doing, the narrative may enact a mathematical subjectivity that few students can see themselves in.

The word simple has other meanings, which if evoked, alter how the statement "we want to make things simpler' can be interpreted and broaden our analysis of the assumptions' faithfulness to disciplinary practice. We could, for example, consider the word simple to mean "plain," "basic" or "noncompound." In that sense, connecting calculus to fundamental topics is seen as accomplishing simplicity not because it is easy to do, but because it reduces the number of constitutive elements in the theory.

The use of a small number of constitutive 'building blocks' to develop bodies of knowledge is a central feature of the Euclidean method. When a body of knowledge is organized in the axiomatic textual style, a few ideas are chosen as primitives and the rest are developed from them through chains of deductive arguments. For example, Cauchy's famous analysis textbook *course d'anylse* is considered to be a rigorization of calculus because it derives a plethora of ideas and methods from a few basic definitions of limits and continuity (Barany, 2015). Cauchy's book was successful in part because it replaced a seemingly disarrayed collection of techniques, with different mathematicians using different arguments and assumptions to justify them (18^{th} century analysis), with an epistemically coherent and parsimonious rendering that allowed building consensus around a relatively small set of assumptions (Archibald, 2008; Kitcher, 1984; Wagner, 2022).

This idea of deriving knowledge from a small set of building blocks was pushed to an extreme by the foundational movements of the late 19^{th} and early 20^{th} centuries (Ferreirós, 2008). Reducing the amount of assumed primitives became the new ideal to aspire to in all branches of mathematics. Ultimately, this led to the claim that *all* mathematical knowledge can be built up from a single primitive language, such as logic. The idea that all of math can be reduced to logic, referred to as logicism in the philosophy of mathematics, are most famously articulated in the writing of the mathematician Bertrand Russell (1918):

> "All pure mathematics – Arithmetic, Analysis, and Geometry – is built up by combinations of the primitive ideas of logic … this is no longer a dream or an aspiration. … Philosophers have disputed for ages whether such deduction was possible; mathematicians have sat down and made the deduction." (p. 39)

> "It must be admitted that what a mathematician has to know to begin with is not much. There are at most a dozen notions out of which all the notions in all pure mathematics (including Geometry) are compounded." (p. 41)

The above quotes from Russell (1918) illustrate that the reduction of mathematics, and in particular analysis, to a few constitutive elements – making it *simpler* – was highly desirable ("a dream or and aspiration"), for at least some mathematicians. Note, however, that unlike in Alex's meta-story, the aspirations articulated by Russel are clearly couched in the "noncompound" interpretation of the word simple. Indeed, in the same famous essay, Russell explains that reduction to logical symbolism is useful precisely because it makes things more difficult:

> "The fact is that symbolism is useful because it makes things difficult. … in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy to correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious." (p. 40)

In contrast, Alex's meta-story seems to confound the two meanings of "simple." On the one hand, the juxtaposition of the desire to make things simpler with the desire to make things complicated to impress, readily suggests the "ease" interpretation of simple. Such an interpretation, as discussed above, renders the "rigorization makes calculus simpler" statement both less faithful and likely unrelatable to students. On the other hand, Alex's subsequently stated objective of making connections to more fundamental topics aligns better with the "noncompound" meaning of simple. A simple as "noncompound" interpretation seems more apt for the second half of his meta-story because it renders the statement "rigorization makes calculus simpler" faithful to the discipline and, not as obviously dissonant with student experiences.

How likely are students to see value in "reducing" calculus to few constitutive elements, given that such a reduction renders a familiar context more difficult? Can they appreciate the ideal exemplified in Russel's quote? The desire for epistemic coherence and parsimony, a goal in and of itself in the type of simplification championed by Alex and Hardy, is positioned here as something students must except if they are to join mathematics.

5.3.6 Meta story 5: Learn to prove

Three out of the four instructors (Alex, Cai and David) explicitly articulated a 'learn to prove' goal for the RA course. In David's story, 'learning to prove' was referred to as "building mathematical maturity," and encompassed both processes of writing and validating proofs. Cai only referred to "proof writing skills," but elaborated on how some of his pedagogical decisions (including proof problems in homework) are intended to help achieved this learning goal. Finally, Alex referred to learning to prove as learning to make 'rigorous connections' and positioned this as one of the main learning goals for the course.

5.3.6.1 David – develop mathematical maturity

David articulated the 'learn to prove' meta-story immediately following his articulation of the 'precise R' meta-story, described in the previous section. He marked the transition to a different meta-story explicitly by framing it as "another related theme" of the course:

> "And another related theme about the course, especially because I- looking at least at the prerequisites and what people have said, maybe you haven't taken too many proof-based

> courses before. And another thing that's going to be important here is we're just going to work on building mathematical maturity. "

David started this story by establishing that RA is one of the first experience students have with proof-based courses. He did so both by referencing the course's positioning in the undergraduate curriculum ("looking . at the prerequisites"), and students' self-reporting of their past experience with proof (" what people have said, maybe you haven't taken too many proof-based courses before."). His story then linked this assertion about students' relative inexperience with proof, to the learning goal of "building mathematical maturity."

"Mathematical maturity" isn't idiosyncratic phrasing; it is a commonly used term in the professional meta-discourse of contemporary mathematicians. In general, it refers to a person's ability to effectively engage with the language and practices of contemporary academic mathematics. The Wikipedia entry on 'mathematical maturity,' for example, defines it as "the quality of having a general understanding and mastery of the way mathematicians operate and communicate." (Wikipedia) It can be found in the introductions of many advanced textbooks, where authors state explicit assumptions about the intended readers of the text. However, as a folk-theory term, it does not have a precise, broadly agreed upon definition. Certainly, there is no reason to assume that students taking the RA course have encountered the term before. And accordingly, David elaborated on what he means by "mathematical maturity":

> "The idea is to make you aware of how to write proofs and how to think about proofs and how to recognize when something is a valid proof. And that is actually an extremely important skill, right? Like, the one thing you should always be asking yourself, like, when you're doing homework assignments is- is what I've written a valid proof? Because if you can develop <u>that</u> skill of recognizing when something is correct, then that is a mentally helpful for generating more correct mathematics."

In the above quote, David defined mathematical maturity as the ability to effectively engage in different aspects of proving, in particular, the two skills of writing and validating proofs. He further explained that the ability to do the latter ("recognizing when something is correct") is helpful for doing the former ("generating more correct mathematics"), and accordingly advised students to engage in proof validation when doing homework assignments.

David's use of the term 'mathematical maturity' is notably narrower than the more encompassing way it is often used. It restricts the whole of mathematical practice and language to proving, and within proving, zooms in on the practices of writing and validating (not including, for example, proof comprehension).

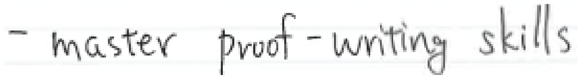
I summarized David's **learn-to-prove** meta-story in the following way:

For many students in the class, proof-based mathematics is relatively new. So, learning the skills of writing and validating proofs is a goal for the course.

5.3.6.2 Cai – master proof writing skills

Similarly to David, Cai discussed proving as an additional goal, after already discussing other goals, and he signaled the transition to this new meta-story explicitly by framing it as "another aspect" and writing it up in the lecture notes as a separate item in a list of course goals:

> But, I also want to emphasize there is another aspect about this course, which is like "master proof writing skill."



Also, like David, Cai proceeded to explain what he meant by this goal:

> What (do) I mean by this? So, this Real Analysis, Math ###. It is one of the mandatory subjects in addition to XXX, YYY and ZZZ. YYY is Algebra, ZZZ is Complex Analysis. I guess those two courses are very new to you in some sense, you learn very new concepts and have to understand how to, what are the typical examples of these new concepts and what kinds of theorems are there. … And then of course, developing new like theorems or theories require proof.

In the above excerpt, Cai explicitly situated RA as one of several upper division math courses offered by the department, which are "mandatory" for the undergraduate major. He referred to them by their course numbers (blinded here for anonymity) and subject matter. Then, he pointed out that that the latter two courses ("Algebra" and "Complex Analysis") are "very new to [students] in some sense" because they entail learning "new concepts," their "typical examples," new "kinds of theorems," and correspondingly, "proofs." In other words, that these courses cover *content* that is likely to be new for students.

> And then, so in those courses your attention, it's more like understanding new concepts and then get familiar with it. So it doesn't provide you enough opportunity to kind of master or improve your proof writing skills. Whereas Real Analysis, many concepts are relatively familiar to you. And then we just try to understand how prove rigorously, or how to give rigorous solutions to some questions. And then along the way. So I want you to master proof writing skills.

In the above excerpt Cai suggested that the novelty of the content in other upper division courses such Algebra and Complex Analysis monopolizes students' attention, and so provides less opportunities to learn the skill of proving per se. He then contrasted that situation with RA, where content is "relatively familiar" and one can instead "*just* try to understand how (to) prove rigorously." Thus, in Cai's story, the relative familiarity of content in RA affords a special opportunity to "master proof writing skills." In other proof-based courses, learning content competes with proving for students' attention. In RA, one can focus on proof.

Cai further explained that the learning-to-prove goal is reflected some pedagogical choices he made for the course:

> For that reason, in homework, I also include at least two or three proof problems. And I asked my grader to just read your proofs very carefully. So hopefully, you also pay attention to how to write proofs in a clear and rigorous way throughout this course.

In the above quote, Cai alerted students to a deliberate inclusion of proof problems in homework, in support of the stated learning goal of mastering proof writing skills. This focus on proof is further reinforced by the instructions Cai gave to the grader of reading the submitted proofs "very carefully." These two decisions – the inclusion of proof problems in homework, and careful assessment of solutions – are meant to encourage students to "pay attention to how to write proofs in a clear and rigorous way." So, unlike David who emphasized both writing and validation as part of learning-to-prove, in Cai's story, the explicit focus is on proof *writing* skills. Of course, the requirement to write proofs that are *clear* and *rigorous*, can be interpreted as implicitly implying the ability to determine when a proof is clear and rigorous, e.g. through validation.

I summarized Cai's **learn-to-prove** meta-story in the following way:

RA is one among several proof-based required courses. Unlike other courses, however, the content in RA is familiar, and so students can more easily focus on learning the skill of writing clear and rigorous proofs.

5.3.6.3 Alex – learning how to make rigorous connections called proofs

At the beginning of his lecture, Alex (with the help of student contributions) articulated and substantiated the assertion that "analysis is calculus with proof." He used this framing to legitimate a "why is this a good idea" question, and subsequently provided two answers: the "fix calculus" meta-story, and the "connections" meta-story. It is in relation to this second purpose, of making connections to more fundamental topics in order to simplify calculus, that Alex articulated a learn-to-prove goal for students:

> All of these branches of math I put down are very fundamental and connections between them and learning to make these connections rigorously and work with them consistently is extremely important and its very useful that the first time many of you do this, namely in this class, you do it in a context where everything is known. So calculus is something you know. All of these – well, maybe you don't know logic, but we are not going to use any very fundamental logic. So you certainly know basic algebra and fractions. You certainly know calculus and you at least have some idea of what set theory is.

Similarly to Cai, Alex emphasized that RA is a particularly "useful" context for "learning to make these connections rigorously" because in RA the content is familiar ("everything is known"). To substantiate the claim that "everything is known", he listed the different topics and ideas he previously outlined as being connected in RA (Figure 16) – calculus, logic, basic algebra, fractions and set theory – and shared assessments about how familiar they are to students. Calculus, basic algebra and fractions are topics students "certainly know." Set theory, they "at least have some idea of." Logic was framed as the least familiar topic ("maybe you don't know logic”), but that was compensated with the statement that it will not be used as much (“we are not going use any very fundamental logic”).

With the claim that “everything is known” substantiated, Alex moved on to further unpacking what he meant by “learning to make these connections rigorous”:

> And so, in this class, the connections that we’ll make, and this idea that many people have mentioned of making these rigorous connections called proofs. So what is a proof but just a rigorous connection between two statements? These will be the bread and butter of this class. And so this is something to keep in mind, that the main thing I want you to get out of this, the main thing that we are learning, is how to make connections. How to write proofs, how to be rigorous, how to make these connections that comprise later, that you will see comprise all of mathematics. And that you will then use in the rest of your mathematical career.

In the above excerpt, Alex articulated a conceptualization of proof and proving within the conceptual architecture of the connections meta-story: a proof is “a rigorous connection between two statements” and correspondingly, the process of proving is the making of such connections. Alex subsequently positioned learning “how to make connections” – that is, learning to prove – as the “main thing” to get out of the RA course. Learning to make connections entails both learning a skill (“how to write proof”), and a disposition (“how to be rigorous”).

Alex also commented on the broader significance of learning to prove. Continuing the thread of the connections meta-story, he claimed that rigorous connections “comprise all of mathematics,” suggesting that students will get to experience it for themselves (“you will see”). Consequently, accordingly to Alex, making connections is something students will use in the rest of their “mathematical career.”

I summarize Alex’s learn-to-prove meta-story in the following way:

Because all of mathematics is comprised of connections, proving – making rigorous connections – is very important. When learning to prove for the first time, familiar content is helpful. In RA, most of the content – calculus, basic algebra, some set theory – is known. So, RA is a great context for learning to prove. Thus, the main thing to learn in a RA course is how to prove and be rigorous, which are essential for doing math in the future.

5.3.6.4 Assumptions in the “learn-to-prove” meta-story.

The central assumption in the learn-to-prove meta-story is that proving and its associated dispositions (e.g. being “rigorous,” being “mathematical mature”) are desirable. This is certainly a position endorsed by many mathematicians and math educators (Rav, 1999; Stylianides, Stylianides, & Weber, 2017).

An exclusive focus on proof – framing proving as the single *main* feature of contemporary academic mathematics, and the one students need to *master* if they wish to join – can lead to problematic images of the discipline. No doubt, proving is an activity most research mathematicians engage in. Indeed, proving is a methodological requirement for publishing in mathematics research journals. However, it is by far not the only activity constitutive of the DTP epistemic game, and certainly not of professional practice as a whole. Even within the scope of the DTP model, an exclusive focus on proof omits, for example, disciplinary practices such as defining and conjecturing. As Thurston (1994) put it, proving alone “doesn't explain the source of the questions” (p. 163).

Singling out proving as the main skill one needs to have to participate in contemporary academic mathematics also privileges the production of new “raw data” as the locus of mathematical progress. However, historically, many significant disciplinary developments in mathematics (among them rigorization!) came from activity geared toward systematization, communication, and teaching of disciplinary knowledge (R. Collins, 2009; Kleiner, 1991). Thus the idea that doing mathematics equals doing proofs is a distortion of the practice, at least historically and in the eyes of some prominent mathematicians (Atiyah, 1984; Thurston, 1994).

Another assumption in the learn-to-prove meta-story is that proving is a content-independent skill and hence transferrable. Namely, that if you learn to prove in the context of a RA course, you do not have to learn the same skill anew in other proof-based courses such as abstract algebra or complex analysis. This assumption is only tacitly present in David’s “learn-to-prove” meta-story, but is a central feature of both Alex’s and Cai’s narratives.

A great deal of research on learning suggests that any naïve assumptions about transfer should be regarded with skepticism. It has been repeatedly documented that successful demonstration of skills in one context does not necessarily translate to successful performance in another, even when the two tested skills can be viewed as identical form an expert’s perspective. One famous example of such findings in the context of mathematical activity is Saxe (1988) study of youth candy-sellers in Brazil. Saxe found that the candy-sellers performed complex arithmetic operations in the context of trade, using currency to mediate their activity, but ‘failed’ to do similar computations when those were presented in the standard numerical orthography used in schools. The realization that cognitive tasks are embedded in situations that are multi-dimensions, and that many situational dimensions do not remain the same when contexts vary even when an underlying cognitive or epistemic process seems the same from an experts’ blind-spot perspective, is a central pillar of situated theories of learning (Lave, 1996; Sfard, 2008).

Similar doubts about transferability have been recently raised in the context of research on proof as well, in particular regarding the presumed content-independence of proving. Dawkins and Karunakaran (2016), for example, urged mathematics education researchers to attend to the content-specific aspects of proving, and illustrated how a content-independent interpretive lens misses important aspects of reasoning in students’ proving activity. Savic (2017) examined student generated proofs and found structural differences between RA and abstract algebra proofs, and suggested that such differences may hinder the transfer of learning to prove from one subject areas to another. Interestingly, the mathematicians interviewed in Savic’s study mentioned several differences between proving in RA and abstract algebra (e.g. the greater prevalence of logical quantifiers in the former). Yet, at the same time, like the instructors in the current study, they thought that a content course such as RA can be effectively used to teach the general practice of proof.

In sum, the view that proving is essentially the same in RA and other upper division courses such as Algebra seems to be unproblematically assumed by both mathematicians and most math education researchers (if judging the by published literature to date). Empirical studies examining the transfer hypothesis in this context are limited, but the few existing studies suggest there is at least some reason to be skeptical pending more evidence. The broader literature about (lack of) transfer recommends such skepticism too.

The final assumption I wish to highlight in the learn-to-prove meta-story is the argument (articulated by Alex and Cai, but not David) that: because content in RA is relatively familiar to students, RA is a particularly useful context for learning to prove. The question of whether or not there is indeed a content overlap between courses was addressed in section 5.3.3.3. Here, I will comment on the claim that this familiarity necessary implies easier engagement with and more effective learning of proving.

There is reason to be doubtful. First of all, the fact that students have encountered topics in a calculus class does not mean they are comfortable with them. The topics of infinite sequences and series, for example, are notoriously challenging for students, even after they have successfully completed calculus courses (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012)[25]. Indeed, introduction to proof courses – i.e. courses designed specifically to teacher the practice of proving – are not centered on RA content (David & Zazkis, 2020).

Furthermore, when topics *are* familiar, the to-be-proved claims often seem self-evident. When a statement is already well understood and believed to be true, proving may seem like a redundant pedantic exercise rather than a solution to a genuine intellectual need.

5.4 Summary and discussion.

Table 5 below summarizes the five meta-stories about RA I found in the introductory lectures of Alex, Cai, David & Emmett.

Table 5: The five meta stories: meta-story summaries and their constitutive assumptions

	Meta-stories	Meta-story Summaries via Constitutive Assumptions
1	precise R	We want to do calculus$_1$. To do calculus, we need specified R (real numbers)$_2$. Our current understanding of R is vague$_3$. In RA, we define R precisely, and build calculus from it.
2	calculus + proof	Calculus and RA share the same content$_1$. They differ in that RA, unlike calculus, is proof-based.
3	fix calculus	Calculus has powerful tools$_1$. These tools sometimes break$_2$. To fix and avoid these breakdowns, we need to understand the theory of how the tool works, which is RA$_3$.
4	connections	We want mathematics to be simple and elegant$_1$. Calculus can be simplified through reduction to more fundamental topics$_2$.

[25] The opening sentence of this paper is: “As every college professor knows, infinite series cause great difficulty in students.”

5	learn-to-prove	Proving is a desirable skill[1]. Since proving is transferable[2], RA is an opportunity to learn the general skill of proving. Since RA is calculus with proof, the content, unlike in other proof-based courses, is known[3]. Hence, RA is a particularly good context for learning to prove.

What makes a story about mathematics healthy? In this chapter, I proposed evaluating stories on two criteria: faithfulness to professional practice and relatability to students. The analyses in this chapter suggest that these stories exhibit problematic features in relation to both of the faithfulness and relatability criteria.

5.4.1 Faithfulness

Each of the stories contained some unfaithful idealizations of disciplinary practice. In the 'precise R' story, it was the claim that a precise definition of R is *needed* in order to do calculus. This contrasts with the empirically verifiable fact that most calculus activity, both today and throughout the field's history, has been done without use of or reference to what the stories frame as a *precise* definition of real numbers. In the 'fix calculus' story, a key assumption is that calculus breakdowns are frequent and important enough to merit the fuss of RA. Again, in most contexts of activity, calculus works just fine without the fixes that RA offers. Even if calculus breaks, in the sense that, say, convergence fails, these problems can (and historically have been) solved using ad-hoc tactics rather than recourse to a rigorous overhaul of the entire theory. In the 'connections' meta-story, a crucial component was the claim that mathematicians (Alex's "we") are motivated by a desire to make things less complicated. While this may be true in some contexts and for some mathematicians, it is certainly not accurate as a blanket statement about mathematicians' motivations to do real analysis. In particular, some prominent mathematicians explicitly stated that when first learning mathematics, they were motivated by desires to look smart in comparison to others. Finally, constitutive of the 'learn-to-prove' meta-story is an assumption that proving is a content independent skill. While there is no conclusive evidence to prove or disprove this claim, the existing research on proving and transfer in general suggests that context independence should not be easily assumed.

The observation that these narratives are not entirely faithful to disciplinary practice and instead function as distortions or idealizations, raises the question: why do mathematicians tell *these* stories?

This question can be taken up empirically. Future research could, for example, probe mathematicians' rationales for using such stories in instruction through various interview techniques. Of course, only a part of the cultural dynamics at play can be reliably discerned through conscious reflections. A complementary approach could look into the circulation of these narratives in various context of mathematical meta-talk, including, for example: other lectures (in mathematics and related fields), research talks, mathematicians' reflective writings, textbooks, depictions of mathematics in popular culture, and discussions on social media.

Here, however, I cannot address this question with an additional empirical investigation, and will instead suggest a few hypotheses based on the current analyses, available data, and my familiarity with the context.

So, why *these* narratives? First of all, faithfulness may simply not be an important concern for instructors. The main function of these meta-stories in lectures is rhetorical: they are deployed to convince students that RA is worth studying, to influence students' evaluative behavior. An idealized depiction of practice may be seen as more effective to accomplish that goal. Such a perspective is well captured in Hersh's (1991) application of Goffman's (1978) front/back metaphor to mathematics. Hersh likened the idealization of mathematics (e.g. in lectures) to the way restaurants do not show costumers the cooking process, and the way theater productions do not show viewers their actors without make up. Such idealizations of practice, Hersh (1991) suggested, "adds to the costumer's enjoyment of the performance; it may even be essential" (p. 129). And so here too, instructors may find it is more important for the meta-stories to be enjoyable, compelling, and convincing, rather than technically accurate.

In addition, instead of faithfulness, instructors may prioritize promoting a purpose that they themselves hope to see realized, even if it is not an accurate description of the majority of practice to date. For example, Alex is likely well aware that mathematicians are often motivated by competition, but nevertheless chose to advocate for an idealistic, simplicity-oriented mathematical practice. Su's (2017) "mathematics for human flourishing" can be interpreted in a similar vein. It is not a description of current reality. Su (2017) would likely agree that much of mathematics as practiced today does not contribute to human flourishing. Rather, it is a vision articulation for desirable forms of practice.

Second, it is possible that of all the stories that can satisfy the rhetorical function of 'selling' the RA curriculum, the stories instructors tell are the ones most easily accessible to them. One potential channel for making these stories widespread and accessible is textbooks. While I have not yet conducted systematic analysis, I think it is quite likely that there is a strong relationship between the meta-stories instructors tell to students in lectures, and the meta-stories that appear in the textbooks they use for the class. For example, the preface in (Ross, 1982) – the textbook Alex and Cai used in their RA course – contains statements reminiscent of the 'learn-to-prove' meta-story, e.g. in its opening paragraph: "An ability to read and write proofs will be stressed." (p. v). The preface in (Pugh, 2015), the textbook Emmett used, contains descriptions echoing the 'precise R' meta-story: "Chapter 1 gets you off the ground. The whole of analysis is built on the system of real numbers R …" (p. vii). Since many instructors read the textbooks in preparation for lectures (Mesa & Grittiths, 2012), the influence of such narratives on instructors' discourse is certainly plausible. Of course, people do not just mindlessly repeat what they read. Rather, my (Bakhtinian) claim is that the narratives made available in textbooks function as raw cultural material for instructors' subsequent crafting of their own meta-stories in lectures (Bakhtin, 1981; Dyson, 1993; Holland, D., Lachicotte, W., Skinner, D., & Cain, 1998). Moreover, since textbooks are artifacts also heavily used by students, the meta-stories therein exert a direct influence too. This all suggests that analyzing the meta-stories promoted in textbooks, in conjunction with those instructors tell in lectures, may be a fruitful direction for future research on the meta-mathematical storytelling ecology students are immersed in.

Finally, instructors may be unaware of the stories' idealizing aspects. As R. Gutiérrez et al., (2023) pointed out, when people talk about things that appear to be *common sense*, they often do not realize that what they are telling are *stories*, and not objective facts about states of affair in the world:

> “… many people are not even conscious that they are (re)telling stories about mathematics (e.g., “Some people are good at math and others are not”) because it feels like they are simply responding to what is the “truth” or reality in which we are all engaged, that our reality could not be otherwise” (R. Gutiérrez et al., 2023, p. 2)

R. Gutiérrez et al., (2023) argued that interrogating such meta-mathematical stories by assessing the extent to which they reflect the discipline and its history is part of *political mathematical knowledge for teaching* that can be developed. The analysis presented in this paper can be seen as contributing to increasing the field’s collective awareness of the idealizing effects familiar meta-stories about RA have. A potentially fruitful direction for future research is design work akin to that reported on in R. Gutiérrez et al., (2023), in which the authors worked with teachers to critically reflect on different meta-stories and how they circulate. One can imagine similar kinds of interventions with current and future RA instructors.

One reason why these meta-stories may seem so common sensical, and hence difficult to recognize as idealizing, is their reliance on metaphors. In three of the four meta-stories, metaphors play a central role in rendering the story’s underlying assumptions self-evidently true. In the ‘precise R’ meta-story, it is the “math is a building” metaphor. If math is a building, the need for “solid foundations” may seem self-explanatory. Similarly, in the ‘fix calculus’ meta-story, the “calculus is a tool” metaphor sets the stage for the need to fix calculus when it breaks to seem natural. In the ‘connections’ meta-story, though not as explicitly articulated, a fitting metaphor is “calculus is an impure/confound substance.” If calculus is impure or confound, “reducing” it to basic elements is clearly desirable.

In each case, the metaphor functions as a gloss for features of practice, a black-boxing of sorts (Latour, 1987). Each metaphor also naturalizes certain values and goals. If math is a building, of course we want it to be solid. If math is a tool, of course we want it to be useful and reliable. If math is a substance, of course we want it to be pure. Even more than the stories, it may be *the metaphors* that are ubiquitous in mathematical discourse, and are thus a core mechanism for the construction of meta-mathematical commonsense.

A concrete takeaway for instruction, and indeed any context of cultural broadcasting of meta-mathematical ideas (e.g. textbooks, mathematics education research articles), is to be weary of metaphors. Meaning, if one is telling a meta-story that heavily relies on a certain metaphor to conceptualize practice and define its telos, then that could be a possible indication that something remains not fully justified or unpacked. To be clear, my suggestion is not to avoid metaphors. Metaphors are powerful discursive means that allow us to collectively make sense of our experiences in the world (Lakoff & Johnson, 2008). Arguably, avoiding metaphors in abstract domains such as mathematics is not even feasible (Lakoff & Núñez, 2000). Instead, based on findings that the use of metaphors for communication of mathematical axiology can lead to idealizing affects, I suggest that metaphorically constituted stories, perhaps more so than other stories, should be treated as an object of critical and careful reflection.

5.4.2 Relatability

The meta-stories relied on assumptions incompatible with (likely) student experiences. A key claim in the 'precise R' story is that decimal and number-line representations of real numbers are problematic. However, students (and again, most professionals) have been using these representations in the context of calculus without any problem. Similarly, in the 'fix calculus' story, the claim that calculus sometimes breaks is not reflective of students' past experiences. In calculus courses, erroneous applications of theorems, if happen, are solved without the need to invoke anything remotely close to the epsilon-delta rigorization proposed in analysis. In the 'connections' meta-story, the claim that the reduction of calculus to fundamental topics such as logic and set theory *simplifies* it is not likely to ring true, at least not in the usual sense of the word 'simple.' Students have seen the epsilon-delta definition of limits in their calculus courses, and for many (if not most) students that experience was far from simple. The claim that such definitions simplify calculus may thus be experienced as a big disconnect from students' own experience grappling with these notions. Finally, in the 'learn-to-prove' meta story, the value of proving as a skill is not self-evident and may not be shared by students. An unqualified reliance on the assumption that learning-to-prove is desirable may be experienced as a disconnect. In short, the meta-stories presuppose epistemic values and goals that many students might not share or have had the opportunity to experience.

A potential way out of this problem, also advocated by (Dawkins & Weber, 2017)[26], is to first provide students with access to epistemic experiences that embody these values and goals, and only then discuss such experiences in meta-reflection. That is, rather than *telling* students that there are problems in calculus which real analysis solves, that their ideas of real numbers are problematic, or that analysis simplifies calculus, *show* – or better yet, have students directly engage with – mathematical situations that give rise to such epistemic judgements. In short: show, don't tell.

The history of mathematics is one source for findings mathematical situations that can serve the pedagogical function of providing *intellectual need* for new concepts. For example, to substantiate the 'fix calculus' meta-story, one could invite students to grapple with mathematical problems that arose in the context of calculus and compelled mathematicians to develop the toolkit we now recognize as real analysis. Two of the mathematical examples mentioned earlier in this chapter can be used for this purpose.

One can, for instance, ask students to discuss what the value of the expression $1-1+1-1+1-1+1-1+\cdots$ should be. Rather than making the point that there is a wrong and a right answer from the vantage point of our current consensus, one could instead explore – together with students – the different opinions prominent mathematicians of the day held. Only *after* weighing the relative affordances of different arguments, one could introduce the resolution proposed by Cauchy: agree to always[27] define the value of an infinite sum to be the limit of the

[26] "We do not advocate explaining to students why they should adopt mathematicians' values … we anticipate that students are more likely to accept an epistemic aim if they experience that epistemic aim being achieved." (Dawkins & Weber, 2017, p. 138)

[27] I use the word always to stress the difference between Cauchy's treatment of infinite sums and that of, for example, Euler. Euler also used the interpretation of infinite sums as limits of partial sums, but his treatment,

partials sums. This illustrate not only the power of the specific definition to resolve this specific debate, but also the utility of the *practice* of systematically relying on consensually agreed upon definitions to resolve enduring debates, which amounts to teaching and learning that integrate the content, practice and axiological level.

A recourse to actual historical debates, rather than problems devised especially for pedagogical purposes, can serve additional functions. First, it can illustrate (beyond the recreation of debate among students) that this was a *genuine* problem in the mathematical community, not just one artificially constructed for students. Second, it can humanize mathematics, in the sense that will be discussed more in chapter 7. It illustrates that mathematical definitions are not pre-determined, but rather chosen and endorsed by people, to achieve certain ends. Finally, sharing disagreements between mathematicians, some of which used arguments that we would today consider as 'wrong,' promotes a fallibilistic, and hence more reliable and possibly more inviting, image of mathematics (Ernest, 1995; Trninic, Wagner, & Kapur, 2018).

Another potentially useful example to consider for these pedagogical purposes is the Fourier problem. The Fourier problem – the surprising phenomena that an infinite sum of trigonometric functions, each individually continues, yields a discontinuous limit – is much more technically involved than the sum of integers considered above. Yet, of all the calculus conundrums vexing 19^{th} century mathematicians which ultimately led to the development and widespread adoption of the epsilon-delta formulation of analysis (Gray, 2015), questions about Fourier series (when do they exist? when do they converge to continuous limits?) are likely the most accessible to students taking real analysis courses. Thus, while it might take more time and effort to engage students with questions about the Fourier problem, the reward of providing a genuine intellectual need for the epsilon-delta language, rather than imposing it as 'better' than calculus in a top down matter, may be worth it.

5.5 Conclusion

Mainstream stories about math are rife with idealizations of the discipline that both alienate many learners and do not tell the full story of what the practice is like and what, and for whom, it is good for (Hersh, 1991; Wagner, 2022). This chapters aims to contribute to our understanding of how such idealizations are constructed in the gatekeeping educational context of RA lectures. Critically examining the assumptions instructors' meta-stories rely on and how these assumptions relate to past and current professional practice and students' past curricular experiences, can help us craft meta-stories that are both realistic and more compelling.

5.5.1 Limitations & future work

A limitation of this work is the lack of explicit dialogue with teachers and students. This is particularly important for making claims about the stories' resonance with students' experiences. In that sense, this work raises more questions than answers. How do students interpret and experience being told statements such as "you only have a vague idea of what real numbers are"? Do students see themselves included in the "we" that do not want to "show off" and instead want to "make things simpler and more elegant"?

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unlike that of Cauchy, did not insist on using that one interpretation systematically across all contexts (Archibald, 2008; Kitcher, 1984).

6 What is ‘good’ mathematics? Valued attributes in RA lectures

6.1 Introduction

When mathematicians talk about mathematics, they often deploy attribute-words that evaluate the mathematical object or process under consideration. For example, they might call a proof *beautiful*, *intricate*, *deep*, or *precise*. As Inglis and Aberdein (2015) noted, mathematicians refer to such characteristics when justifying award decisions and choosing research directions. However, value-laden attributes also abound in much more mundane contexts, such as undergraduate lectures. In this chapter, I examine the value-laden attributes instructors used to characterize mathematical ideas and practices in the context of introductory RA lectures.

Recurrent displays of appraisal is one way by which *cultural values* – community regularities in evaluative behavior (Agha, 2003) – are perpetuated in the classroom[28]. For example, when an instructor positively appraises a proof for being *precise* in front of students in lectures, it “sends the message” that precision matters in mathematics. This is a dual message, for it not only conveys that precision is better than ambiguity, but also that precision, as a category, is important. That is, the message is not only which pole is positive or negative within a certain attribute spectrum, but also what dimensions of a mathematical situation are to be noticed at all. For example, even if an instructor positively evaluates, say, beauty, but does so much less frequently than attending to precision, the overall “message” is that in mathematical activity, precision is a more relevant feature than beauty. Some attributes are, of course, not mentioned at all in the context of mathematical activity and are thus rendered invisible. In that way, appraisals enact a certain mathematical gaze (C. Goodwin, 1994); one that attends to, for example, precision (or lack thereof) more so than other possible characteristics of a situation.

What attributes are presented as desirable, or even mentionable, in mathematics classrooms can affect student experience in several ways. If an instructor repeatedly appraises proofs for precision, for example, students can come to expect that if they create *precise* proofs, those proofs will be positively evaluated by the instructor, and perhaps the broader mathematical community they represents. While students may not personally relate to the displayed values, their decisions to act one way or another are nevertheless informed by what they expect the evaluative behavior of others, in this case mathematicians, to be (Dawkins & Weber, 2017). This includes decisions about how to act in the context of the course – e.g. how to talk in the classrooms, how to write in exams – as well as bigger scale decisions about disciplinary pursuits – e.g. what courses to take next, whether to continue studying mathematics. Instructor appraisals are part of how lecture discourse enacts of what Ernest (1995) called the *image of mathematics.* Students who do not relate to this image and its constitutive values may find mathematics unappealing and disengage (Solomon, 2007).

[28] Sociologists have made a finer distinction between “valuation practices (giving worth or value) and evaluative practices (assessing – how an entity attains a certain type of worth)” (Lamont, 2012, p. 205), noting that in practice the processes are often interrelated. The appraisals considered here are in the former category, of valuation practices, but I do not make this distinction explicit and use the more colloquially familiar word evaluation, rather than valuation, throughout the chapter.

6.1.1 Values and classroom practice.

There is a long tradition in mathematics education research of studying how classroom practice perpetuates beliefs about mathematics, in ways that could be either productive or exclusionary. Originally, the focus has been on activity; how the kinds of tasks students are asked to engage in (bite-size computations versus genuine problems) influence their beliefs about the nature of the discipline and their ability to participate in it (Schoenfeld, 1988, 2016). Then, attention shifted to include participation structures and divisions of labor in the classroom. Researchers were asking not only *what* kinds of tasks are enacted, but *who* gets to do the heavy lifting and intellectual work (Schoenfeld, 2014b). Taken together, task types, participation structures, and regularities of action constitute *classroom mathematical practices* (Cobb, Stephan, McClain, & Gravemeijer, 2001), which construct ideas about what counts as mathematically competent (Gresalfi et al., 2009), and who gets to develop positive mathematical identities as a result (Cobb, Gresalfi, & Hodge, 2009).

Another line of research looked at how beliefs about mathematics are constructed through seemingly mundane yet pervasive features of classroom discourse. Early work in the discursive tradition attended to the role that the frequently used personal pronouns ‘we’ and ‘you’ have in establishing the discipline’s and teacher’s authority in the classroom (Pimm, 1987; Rowland, 1999). More recently, Herbel-Eisenmann and Wagner (2010) conducted a corpus linguistic study of lexical-bundles (combinations of 3 words or more) in secondary mathematics classroom discourse. They found that the most common type of word combinations was the *stance* bundle (e.g. phrases such as “*I want you to*”), that the high frequency was unique to math even in comparison to other pedagogical corpora (Herbel-Eisenmann, Wagner, & Cortes, 2010), and, similarly to earlier work on pronouns, argued that such bundles index hegemonic authority relationships between the discipline, teachers and students.

Across the different research traditions that examined how classroom practices shapes beliefs about mathematics, a major focus has been the construction of authority relationships between students, teachers and the discipline (which teachers are seen to represent). That is, how classroom activity gives rise to student beliefs about “*who* can do mathematics,” or “*who* can determine what is correct.” Such beliefs are seen as crucial for shaping student opportunities to learn disciplinary practice and develop positive disciplinary identities.

Less attention has been given to how classroom discourse shapes beliefs about *what attributes* constitute good mathematics. This is present but implicit in earlier studies. For example, Yackel and Cobb (1996), in their famous paper that introduced the idea of socio-mathematical norms as the cultural, collective counterpart of individual beliefs (Cobb et al., 2001), showed how teacher contributions in classroom discussions (e.g. their responses to a student ideas) can shape students’ beliefs about what counts as *efficient* or *sophisticated* in mathematics. However, *that* efficiency and sophistication were constructed as desirable attributes is not examined or questioned. In fact, the labels *efficiency* and *sophistication* were introduced by the researchers themselves to identify aspects of student responses the teacher was praising. Similarly, values such as *accuracy* and *sensibility* were included in Gresalfi et al. (2009) characterization of systems of competence. However, the valued attributes themselves were not a major focus of their analysis, which instead centered on the kinds of tasks, *who* students were accountable to, what kinds of *agency* they got to exercise in activity.

Thus, as Dawkins and Weber (2017) noted, mathematics education research on values is limited. In the same paper, Dawkins and Weber claimed that this lack of attention to values is problematic because, especially in the context of proving, teaching disciplinary norms without explicating and negotiating the values these norms are meant to uphold may amount to an arbitrary imposition on students:

> "we hypothesize that when students do not recognize how these norms support *their* mathematical values, the norms will appear arbitrary." (Dawkins & Weber, 2017, p. 136; emphasis in original)

Dawkins and Weber (2017) further argued that adopting mathematicians' values with respect to proof is an essential component of enculturation to the disciplinary practice. Accordingly, they characterized proof-based math courses as an activity in which "students and teachers … are engaging in cross-cultural interactions according to distinct sets of values" and concluded that "proof instruction must also seek to expose the underlying values that guide that practice" (p. 138). The authors then hypothesized that values will be most effectively communicated to students by allowing students to experience the realization of the values first hand. That is, if you want students to value, say, *precision*, provide students with opportunities to experience precision as a positive attribute in a mathematical situation. In other words: show, don't tell.

In this chapter, rather than testing the efficacy of different approaches to the teaching values, I aim to empirically examine the baseline of current practice – what values do mathematicians already convey in proof-based lectures?

Since the publication of Dawkins and Weber's (2017) influential paper, some mathematics education research focusing on values with respect to proofs has emerged. For example, in a recent study adopting the values and norms framework, Rupnow and Randazzo (2023) interviewed algebraists about the characteristics they value in definitions. They found that mathematicians in their study valued *clarity* for the purpose of communication, and *freedom* of choice in the creation and use of definition. They also found interesting differences in what mathematicians prioritized in the contexts of research and teaching. Six of the nine interviewed algebraists claimed that they emphasized *precision* more in teaching than they would in the context of research and discussion with fellow mathematicians. This finding suggests that what mathematicians say they value in mathematics may not necessarily be the same values they convey to students when teaching.

Similar conclusions were reached by Sommerhoff and Ufer (2019). In a large-scale survey study comparing proof acceptance criteria used by secondary school students, university students and mathematicians in response to proof validation tasks, they found that mathematicians, similarly to both groups of students, responded to the proof validation tasks using primarily structure-oriented criteria. Sommerhoff and Ufer noted that these findings contrast with previous research on mathematicians' validation practices (Weber, 2008; Weber, Inglis, & Mejia-Ramos, 2014), which found that in the context of research, mathematicians often rely on meaning-oriented criteria, empirical evidence, and the perceived authority of the proof's author to accept a proof as correct. The authors interpret the discrepancy as indicative of differences in context. That is, proofs in teaching and proofs in research are judged using different criteria, with the former focusing almost exclusively on structural, formal features, and the latter including also empirical evidence, understanding, and social factors.

In a recent commentary, Weber and Melhuish (2022) problematized the relationship between values in teaching and values in mathematics research from a critical perspective on equity. They highlighted an irreducible tensions between mathematicians' values, students' values, and the values educators may want to promote, giving the example of competition. Competition and competitiveness can be seen as an authentic feature of mathematicians' practice, but may not be desirable to reproduce in instruction, both because it is in tension with some broader societal values educators may wish to promote (Ball, 1993), and because it has been shown that competitive environments systematically disadvantages women and students of color (Herzig, 2004a; Leyva et al., 2021). The potential for such tensions suggests that identifying which values mathematicians convey in instruction, and how, is important to inform the design of mathematics instruction that is both more equitable and more authentic to practice.

6.1.2 Values in the history and philosophy of mathematics.

While values have not been a major focus of mathematics education research to date, they have been receiving increased attention in historical and philosophical studies of disciplinary practice. For example, a recent paper that synthesized research in the *philosophy of mathematical practice* tradition (Hamami & Morris, 2020), listed 'value judgements' as one of five focal topics in that area. Another indicator of heightened attention to values is a recent topical collection in the journal *Synthese* devoted to virtue theory of mathematical practice (Aberdein et al., 2021). In both of these cases, the focus has been on value judgements in the form of appraising mathematics with virtues such as purity, beauty, and precision – what I call value-laden attributes in this chapter.

In their review of virtue-theoretic research on mathematical practice, Aberdein et al. (2021) made a distinction between theory virtues, which are desirable characteristics of the product or object of epistemic activity (e.g. a *rigorous* proof), and character virtues, which are attributes of epistemic agents (e.g. a *meticulous* person). Most of the research they reviewed entails deep dives into the history of specific virtues in mathematical practice. For example, the rise and shifting meanings of *purity* as a value in the German speaking academic world in the 19^{th} and early 20^{th} century (Ferreirós, 2016).

Particularly relevant to the analysis in this chapter, however, is the work of Inglis and Aberdein (2015). The authors noted that mathematicians routinely use many different words to appraise mathematics. As an example, they mentioned Tao's (2007) long list of attributes that constitute 'good' mathematics (e.g. *rigorous, beautiful*, *elegant*, *creative*, *deep*, *strong*) and subsequent claim that "the concept of mathematical quality is a high-dimensional one" (p. 624). Inglis and Aberdein questioned whether words such as elegant and beautiful necessarily refer to different aspects of mathematics, and thus, instead of accepting Tao's claim that mathematical quality is a high-dimensional construct, they reformulated it as an empirical question – on how many different dimensions do appraisals of mathematics vary?

To address this question, Inglis and Aberdein (2015) conducted a survey study in which they asked mathematicians to recall a recent proof they read as part of their research work and (reflectively) assess it on a list of 80 attributes generated by the authors. Then, they performed factor analysis, which is a statistical method that aims to "model the covariation among a set of observed variables in terms of functions of a small number of latent constructs, or factors, which are themselves unobservable." (p. 92). Through factor analysis, they identified four distinct groups of attributes that co-varied in mathematicians' proof appraisals: *aesthetics, intricacy, utility* and *precision*. These four theory-virtues are then seen as the only four dimensions on which proofs are really appraised.

This study is relevant to the current analysis both in terms of its operationalization of values and its methods. First, unlike previous studies that inferred values from patterns in classroom interactions (Yackel & Cobb, 1996), interviews (Rupnow & Randazzo, 2023), and literature syntheses (Dawkins & Weber, 2017), Inglis and Aberdein's (2015) focused on the empirically documented phenomenon of mathematicians using attribute words to evaluate mathematical units such as proof. Here, even though my data is naturalistic talk and not surveys, I also operationalize values – defined as regularities in evaluative behavior (Agha, 2003) – through instructors' explicit use of attribute words (virtues) to characterize mathematics. In addition, I agree with Inglis and Aberdein's (2015) basic premise that different attribute words are often used to refer to the same underlying characteristic (e.g. "precise" and "rigorous" are often used synonymously), and, as I explain in the methods of analysis section, also take this into account in my coding.

In sum, while we have many reflective accounts by mathematicians about what they value in mathematics (Atiyah, 1984; Rav, 1999; Tao, 2007, the discussion in chapter 3), and some insights from recent interview and survey studies (Inglis & Aberdein, 2015; Rupnow & Randazzo, 2023; Sommerhoff & Ufer, 2019), research on what mathematicians *display to students* as valuable attributes during lectures is missing. This chapter addresses this gap. In particular, I ask the following research questions:

1. What value-laden attributes did instructors use to characterize mathematics in lectures?
2. How frequently were the different attributes evoked by the instructors?
3. Which attributes were positively and negatively evaluated?

6.2 Data analysis

This chapter examines the value-laden attributes that the four instructors – Alex, Cai, David and Emmett – conveyed in their introductory (first) lectures. The primary data used for this analysis are video recordings of four lectures, three of which were approximately 50 minutes long (Alex, David and Emmett's) and one of which was approximately 80 minutes long.

Data analysis proceeded in three steps. First, I generated secondary data through transcription, segmentation and descriptive coding of episodes. Transcripts were segmented into short episodes, of 1 min length or less, based on natural transitions. Each 1 min episode was given a title that served as its descriptive summary, using a mixture of content and in-vivo coding (Saldaña, 2021). I then grouped the short 1 min episodes into larger coherent wholes, according to the overall type of classroom activity. This multi-step process resulted in a hierarchically structured outline of each lecture's transcript (Erickson, 1992).

6.2.1 In-vivo value coding

In the second step, I conducted in-vivo value coding (Saldaña, 2021) of the structured transcripts, flagging every instance of an instructor assigning a value-laden attribute to a mathematical idea. Grammatically, value-laden attributes were realized as adjectives describing mathematical objects ("this is a *precise* definition" or "this is a *beautiful* number"), and adverbs describing mathematical processes ("let's define this *precisely*"). To reduce the amount of distinct in-vivo codes, I coded adverb forms of attributes as their adjective equivalent (e.g. “precisely” was coded as “precise”).

The value codes were applied on a level of a single sentence in which they were deployed. Meaning, if an attribute was repeated twice in the same sentence in the transcript, only one code was applied. But, if the same attribute was deployed in two different sentences, two codes would be applied, even if the sentence were in same episode or even adjacent.

Not all adjective and adverbs that appeared in the discourse were coded as value laden attributes. First, to be coded, the attribute had to be about mathematics. A statement such “this lecture is short” would qualify the attribute “short” for a code. In contrast, if the instructor said “this proof is short,” short would be applied as a code. Second, to be coded, the attribute (in the form of adjective or adverb) had to constitute a value judgement that went beyond the standardized mathematical register. For example, the word “even” in the statement “this number is *even*” is part of the technical vocabulary of mathematics designating a type of number, and was thus not coded as a value-laden attribute for the purpose of this analysis. In contrast, the word “beautiful” in the statement “this is a *beautiful* number” is not standardized register, and would have been flagged with an attribute code. The guiding question for making these coding decisions was: *does the use of the attribute in the sentence convey ideas about what counts as good, bad, or relevant characteristics in mathematics?*

To take note of whether attributes were desirable, unfavorable, or merely mentionable, I labeled attribute-codes with an additional + or -, depending on whether the speaker conveyed a positive or negative evaluative stance (Ochs, 1996). If the evaluation was neutral, or difficult to determine, I coded the attribute without adding a + or -. Determination of stance relied on extra linguistic data such as speakers’ intonation, facial expressions, as well as the immediate discursive context (i.e. what was done and said prior and after the episode).

As an illustration of this procedure, consider the following example of episode 37 from David’s lecture:

> The first and sort of easiest property to talk about when we're talking about numbers is this idea of ordering. But, so we should be a little bit precise about what we mean by, by an ordering. And it turns out it's actually quite useful to put orderings on sets other than just numbers. There are times when you might want to order other things because the idea of an ordering is a useful property to talk about in general. And so we have the following definition.

Here is how this excerpt was parsed and coded:

	Sentences & Indicators (highlighted)	Codes
1	The first and sort of easiest property to talk about when we're talking about numbers is this idea of ordering.	easy
2	But, so we should be a little bit precise about what we mean by, by an ordering.	precise+
3	And it turns out it's actually quite useful to put orderings on sets other than just numbers.	useful+
4	There are times when you might want to order other things because the idea of an ordering is a useful property to talk about in general.	useful+ general
5	And so we have the following definition.	

A few things to note about this example. First, the decision to apply a stance marker or not was based on extra-linguistic indicators in addition to the transcript. It was also applied conservatively. Meaning, when uncertain about whether the stance was positive, neutral or negative, I opted for a neutral mark. Second, not all adjectives used to describe mathematics were coded as valued-laden attributes. For example, the words *first* in line (1) and *following* in line (5) were not coded because they function as cohesive devices in discourse (Morgan, 2004), rather than conveying an attribute to be valued (or not).

6.2.2 Value clusters

This in-vivo coding process resulted in a large number of distinct codes. To identify patterns in the results, in the third step of analysis, I grouped different attribute codes according to a shared theme. As Inglis & Aberdein (2015) found through factor analysis, mathematicians use different words to refer to a similar underlying characteristic. For example, they may use words such as "precise," "exact," "specified," or "rigorous" to convey roughly the same idea. Thus, it makes sense to collapse those different attributions under the same attribute code.

I clustered together both positive and negative evaluations of the same theme. For example, the cluster *precision* grouped together positively evaluated words such as *rigorous*+ and *exact*+, with negatively evaluated oppositions such as *vague*- and *ambiguous*-. Similarly, the cluster *difficulty* grouped together words such as *hard*, *easy*, *simple*, and *complicated*, which were sometimes positively and sometimes negatively evaluated.

Here, unlike in Inglis & Aberdein's (2015) study that used factor-analysis, I relied on my own interpretation of the contextual meaning of deployed attribute words to determine when different words were used to refer to the same underlying characteristic. This procedure introduces a degree of subjective judgement into the analysis. However, it is important to note that the clustering decisions I made were informed by the contexts in which words were used, not just by assumed dictionary meaning of the words in isolation. That is, to determine whether a certain attribute word belonged to a cluster, I went back to the episode in which it was deployed, and judged the sense in which the word was used in that particular context. While this does not eliminate validity and reliability threats, the procedure helped mitigate them.

My choice to use an inductive approach to coding had the following rationale. In the absence of similar analyses in the literature, I thought that a deductive, top-down approach to coding would not be appropriate. The categories of *precision*, *utility*, *aesthetics* and *intricacy* identified by Inglis

& Aberdein's (2015), the most relevant result to the present study, is one set of potentially relevant a-priori codes. However the context of their study – mathematicians evaluating a single proof, read in the context of their research work, and reflected on from memory – is significantly different than the context of the current study: mathematicians commenting on attributes of a variety of mathematical units and processes (not just proof) and in the pedagogical context of lecture. This difference in function, context and scope renders an a-priori restriction to the previously identified value categories ill advised. It is reasonable to assume that other attributes, or a finer grained distinction between attributes, becomes relevant when evaluating aspects of mathematics other than proof and in a pedagogical, rather than research, context. Thus, I opted for an inductive approach for forming the value-clusters, and related the findings to categories previously identified in the literature only at the end of the analytic process.

Another analytic choice I made was to start with in-vivo codes, and only then collapse the codes into thematic clusters. An alternative approach would have been do develop a coding scheme with value-clusters as codes, and then use those to code the transcripts directly. There are two reasons why I chose the procedure of coding in-vivo first, and clustering later. First, an in-vivo approach to coding afforded a high degree of systematicity. It allowed formulating low inference rules for flagging attributes, and at clearly designated units (at the level of the sentence in which the attribute word appeared). These features of the in-vivo approach helped increase the reliability[29] of the coding.

A second reason for choosing the in-vivo approach is that the results are then multilayered. With both in-vivo codes and subsequent clustering, I am able to show not only that mathematicians commented on a certain attribute, and how frequently, but also in what manner. As a result, the analysis provides a richer picture of the value-signaling aspect of lecture discourse than coding with cluster categories alone would.

6.3 Results

I coded a total of 411 instances of instructors deploying a value-laden attributes in their introductory lectures, using a total of 151 distinct[30] in-vivo codes .

Figure 17 below shows the breakdown of counts across the different instructors. I found 100 instances in Alex's lecture, 85 in Cai's, 92 in David's and 134 in Emmett's. Note that Cai's lecture was longer than the lectures of the other instructors. Taking that into consideration, I found that Cai deployed value laden attributes at a much lower rate than the other instructors (about 1 value per episode, compared to ~ 2 value per episode for the other instructors). One thing that explains this is the topic of Cai's lecture. Whereas David, Emmett and especially Alex devoted a significant amount of time to introductory meta-talk in their first lecture, Cai opted to 'jump right in' and do several proof-writing exercise in the first lecture (see Figure 10 in chapter 5 for an outline of topics covered in the lecture).

[29]Two different coders (Kaya Poff & Vasiliki Laina) assisted me with in-vivo coding of one of the lectures (David), and unofficially, I can report that the coding procedure yielded a surprisingly high degree of reliability. A concrete next step for this work is to provide official measures of reliability for this coding, a work I plan to pursue in collaboration with Vasilki Laina.

[30] Value attributes with a different stance indicator, as well as negations are counted as a distinct code (e.g. *easy*, *easy-*, *easy+* and *not easy* are counted separately).

	Alex	Cai	David	Emmett	Total
# of applied codes	**100**	**85**	**92**	**134**	411
# of episodes	53	81	53	61	
Fequency	1.887	1.049	1.7358	2.19672	1.71719
					Average

Figure 17: Total number of applied in-vivo value codes for each instructor and per episode frequency.

Many of the codes were applied only once, suggesting these words were not part of a standardized way of characterizing mathematics in the disciplinary meta-discourse. For example, the attributes *shiny* and *defective* appeared only once in the data set. Some attribute words appeared many times in the data. The five most commonly occurring attribute words[31] were *basic* (35 instances), *true* (25 instance), *rigorous* (18 instances), *easy* (17 instances) and *important* (16 instances). Note that these numbers represent the total number of applied codes for all four lectures together. Figure 18 shows the distribution of the use of these words across the four instructors. Looking at the distribution, we can see that one or two instructors were responsible for the high frequency counts of each of these words. For example, Emmett is responsible for 24 out of the 35 (~69%) uses of the words *basic* in the data, whereas David did not use the word a single time. Similarly, David is responsible for 15 out of 25 (60%) of the uses of the word *true* in the data, whereas Cai used the word *true* only twice. Nevertheless, for each of the words, at least two of the instructors used the word at least 5 times in their lectures. This suggests that these words are not idiosyncratic, and are rather part of a standard evaluative meta-discourse in RA lectures.

	Total	Alex	Cai	David	Emmett
Basic	35	10	1	0	24
True	25	3	2	15	5
Rigorous	18	9	7	1	1
Easy	17	1	6	6	4
Important	16	2	7	2	5

Figure 18: The five most frequently used attribute words, and their total number of occurrences by each instructor.

Note also that the fact that an instructor did not use a particular attribute word often does not mean they did not comment on the same feature in their lecture. David, for example, said the word *rigorous* only once, but used the word *precise* 7 times in his lecture, in roughly the same sense as the word *rigorous* by other instructors in the data set. Thus, the low counts in the above table should not be interpreted as instructors not commenting on those attributes in their lecture. Rather, it only means that they did not use that specific word. To assess the extent to which instructors commented on certain attributes, it is instructive to cluster words with similar meaning in this context, such as rigorous and precise. This will be the focus of the next section.

[31] For these counts of the frequently occurring words, I counted different stance codes together (e.g. true+ and true were counted together), and negative form of an attribute word (e.g. non-rigorous was counted together with rigorous).

6.3.1 Value clusters

From clustering together words used with similar meaning, I found the following value-laden attribute themes: *precision*, *validity*, *generality*, *difficulty*, *level*, *density*, *pace*, *significance*, *normativity*, *utility*, *interest*, and *aesthetic*. A few of the flagged attribute words did not cohere into a single attribute-theme, and were thus grouped together in a category named *other*. I will provide more details on the exact words used in each of the categories, as well as the words that did not fit in any, in the subsequent sections.

Figure 19 below presents the counts for each of the identified attribute clusters, broken down across the four instructors (leftmost column). To enhance clarity, I color-coded related themes. *Precision*, *validity* and *generality* (green) are traits traditionally associated with rigor in contemporary mathematics. *Difficulty*, *level*, *density* and *pace* (blue) pertain to the perceived complexity of concept, akin to the intricacy category in (Inglis & Aberdein, 2015). *Significance* and *normativity* (yellow) assess the idea's relative position within the mathematical body of knowledge. Finally, *utility*, *interest*, *aesthetic*, and *other* are considered separately, each with its own color (purple, orange, red, and grey).

	Precision	Validity	Generality	Difficulty	Level	Density	Pace	Significance	Normativity	Utility	Interest	Aesthetic	Other	Total
Alex	15	9	6	9	16	3	0	9	7	6	7	9	4	100
Cai	15	6	0	16	1	1	9	14	14	3	3	1	2	85
David	19	28	1	13	1	0	0	5	7	5	7	4	2	92
Emmett	23	10	1	28	27	1	2	8	16	3	7	4	4	134
	72	53	8	66	45	5	11	36	44	17	24	18	12	411

Figure 19: Counts of attribute codes in Alex, Cai, David and Emmett's lectures, color-coded by thematic relatedness

Note that unlike Figure 18 from the previous subsection, which looked at the specific attribute words most frequently used, here we are considering thematic clusters, and hence we find a more uniform distribution between instructors in at least some of the categories. For example, all four instructors frequently commented on *precision*, even though not all of them used the exact same words to do it. Figure 20 and Figure 21 on the next page display these results in a bar-chart format.

At the aggregate level, ignoring difference between instructors, there seem to be three tiers of how frequently attributes were commented on:

Top	precision (72), validity (53), difficulty (66), level (45), significance (36), normativity (44)
Mid	utility (16), interest (23), aesthetic (17)
Low	generality (8), density (5), pace (11),

At the lowest level, not represented in the above three-tier summary, were the attributes that did not cluster into a distinct category and compiled under the code *other*.

In subsequent sections I provide more detail about the top and mid frequency categories.

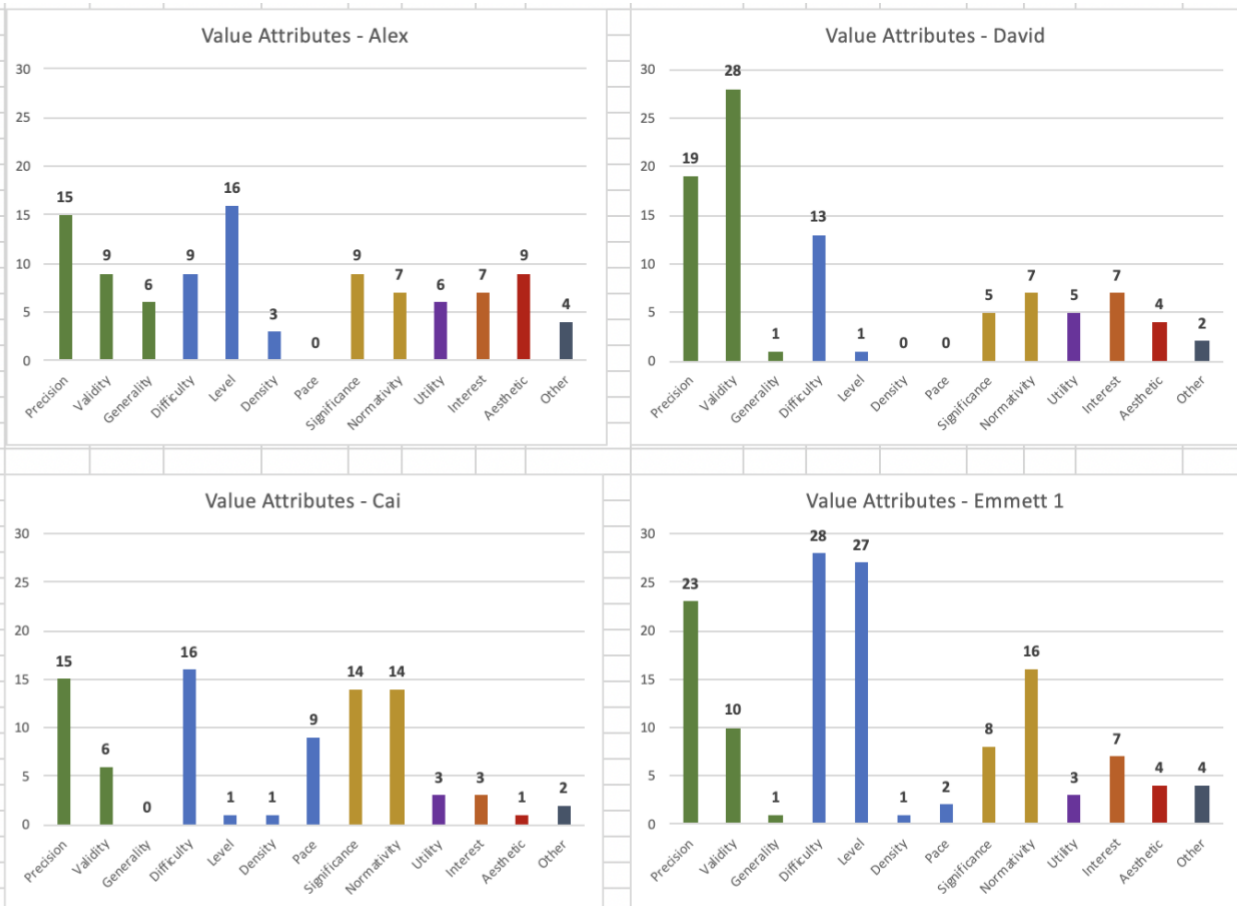


Figure 20: Frequency of value clusters in each of the four instructors' lectures

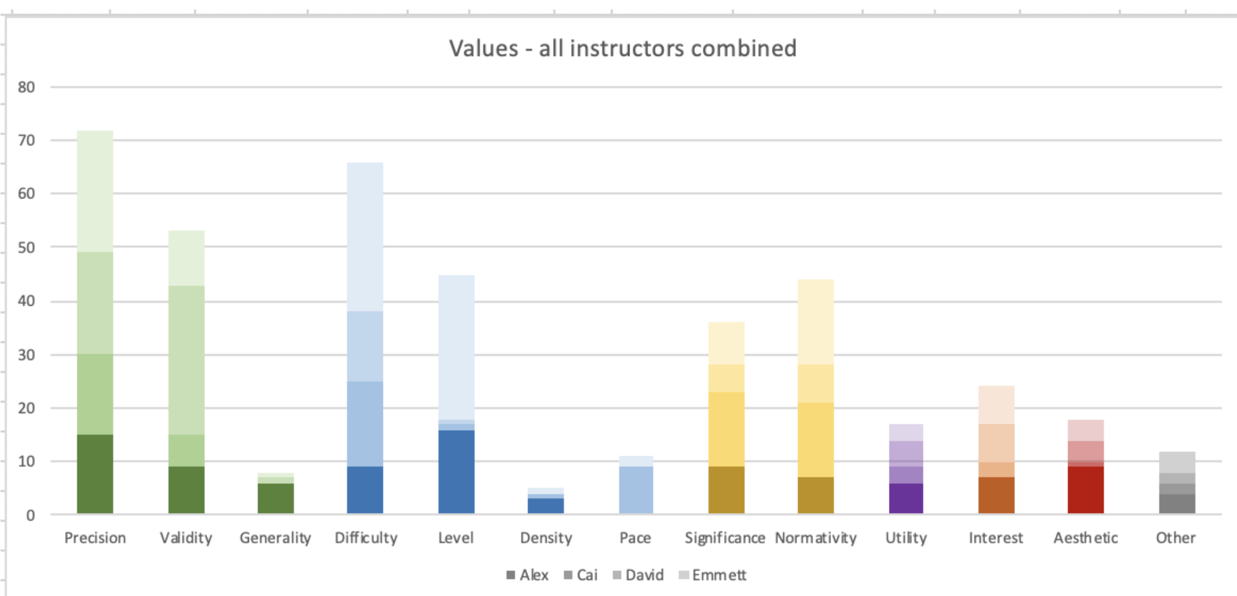


Figure 21: Frequency of value clusters across the four lectures: Alex, Cai, David and Emmett, shaded in decreased order of opacity

6.3.2 Precision & validity

6.3.2.1 Precision

The category of precision groups together adjectives and adverbs that refer to exactness (or lack thereof) of a mathematical unit or process. On the positively evaluated pole, it included words such as: *rigorous*, *precise*, *careful*, *exact*, and *clear*. On the negative one, it includes words such as: *vague*, *informal*, and *heuristic*. Figure 22 below shows the breakdowns of in-vivo codes (exact words and evaluative stances used) applied to each instructor's lecture.

Alex	Precision						
	Rigorous +	9			Vague -	1	15
	Close +	2			Rough -	1	
	Careful +	1					
	Formal +	1					
		13		0		2	

David	Precision						
	Precise +	7	Intuitive	2	Vague -	3	19
	Formal +	2	Abstract	1	Astract -	1	
	Specified +	1					
	Rigorous +	1					
	Exact +	1					
		12		3		4	

Cai	Precision						
	Rigorous +	7					15
	Careful +	3					
	Precise +	2					
	Exact +	1					
	Clear +	2					
		15		0		0	

Emmett	Precision						
	Clear +	4	Implicit	1	Informal -	2	14
	Unambiguous +	1	Geometric	1	Heuristic -	2	
					Not rigorous -	1	
					Rough -	1	
					Incomplete -	1	
		5		2		7	

*Figure 22: Positive, neutral and negative **precision** appraisals by each of the four instructors.*

Within each of the four tables, the leftmost column contains words deployed with a positive evaluative stance, the middle column with neutral, and the rightmost with a negative evaluation. A consistent pattern across all four instructors is that *precision* and its counterparts (e.g. *rigor*, *clarity*, *exactness*) was positively evaluated. Whereas Cai only made positive appraisals, Alex, David, and especially Emmett also negatively evaluated imprecision, indicated by using words such as *vague*, *rough*, *heuristic* and *informal* with a negative evaluative stance.

To see in more detail how words such as precision and rigor were used, consider the following excerpts from two consecutive episodes in Cai's lecture:

> And of course, you can just use a sentence to describe a set. And in this course, I don't pay precise attention how to define those numbers precisely, but we will just discuss some properties, especially for reals. (Cai, episode 30)

> So, first of all, set of natural numbers. So, it's easy to describe numbers, just 1, 2, 3, 4. Just count all numbers one by one. And then, of course, in mathematics, sometimes it's better to give a rigorous definition. So, so maybe let me just write this out. So first of all this can be defined very rigorously using some axioms called Peano's axioms. (Cai, episode 31)

In the above excerpts, precision was evoked to characterize a type of awareness ("*precise* attention"), a way of defining ("define those numbers *precisely* " and "can be defined very *rigorously* "), and a possible feature of the definition itself ("*rigorous* definition"). The temporal and thematic proximity of the two excerpts suggest that the words rigor and precision were used by Cai in a similar manner: as a contrast with descriptive ways of defining sets of numbers. While Cai did not explicitly use negative attribute words such as ambiguous, imprecise or vague, it is clear from the juxtaposition he made that the opposite of precision is “just use a sentence to describe a set,” and the opposite of rigor is “describe numbers, just 1, 2, 3, 4. Just count all numbers one by one.” That is, he contrasted “just” describing, which is easy, with defining precisely, which is “sometimes” better in mathematics.

Like Cai, David drew attention to *precision* in the context discussing definitions. This occurred several times throughout the lecture:

> we all sort of, as I said, we have a vague sort of idea that what real numbers are, like, at any point on a line- any point on a line corresponds to a real number with 0 in the middle and one here ((gestures)). And you can go halfway in-between. And any point in there is a real number, but that's sort of is hard to get a handle on what we mean precisely. (David, episode 22)
>
> R. Is the set of real numbers. And the first chunk of the course is going to be getting to a point where we can be more precise about what we actually mean by that. (David, episode 22)
>
> we should be a little bit precise about what we mean by an ordering. (David, episode 37)
>
> of course, if you wanted to be <u>really</u> precise, you'd need to sit down and formally define what less than or equal to means on Q, and then verify that it satisfies all these properties. (David, episode 51)

As can be seen in the above excerpts, David repeatedly assigned the attribute of *precise* to the process of expressing meaning:

> “hard to get a handle on **what we mean** precisely,”
> “be more precise about **what we actually mean** by that,”
> “we should be a little bit precise about **what we mean**”.

In the above excerpts, being *precise* about meaning is positively marked as a desirable (“we *should*,” “if you *want* to”) yet difficult (“hard to get a handle on”) objective. The word “actually” in the phrase “what we actually mean” implies that precision is *necessary* for fully realizing meaning. The absence of precision is characterized as unstable and undesirable: without precision, one only “sort of” has a “*vague* idea”.

The attribute of precision was also used in the context of discussing proofs and proving. For example, Alex referred to rigor as a defining characteristic of what a proof is, and to proving as a constitutive feature of rigor:

> So, [analysis] is rigorous, and specifically the rigor is by having proofs (Alex, episode 5)

… making these rigorous connections called proofs. So what is a proof but just a rigorous connection between two statements? (Alex, episode 52)

Positive precision appraisals appeared in lecture discourse in more mundane (not meta) ways as well. For example, during a step in a proof by induction, Cai said:

> Now, I have to be careful. I have to work on n plus one. So in that- What that means. So this means I have to replace all of n by n plus one. Or more precisely, in this situation, I just need to add the last term, which should be n plus one squared. (Cai, episode 43)

6.3.2.2 *Validity*

The category of *validity* groups together adjectives and adverbs that refer to the truth-value of a mathematical statement (e.g. words such as *true*, *correct*, *valid*, *false*, *wrong*), as well as those that refer to the certainty or skepticism of the epistemic agent (e.g. words such as *certain*, *definite*, *trusting*). Figure 23 below shows the breakdowns of the corresponding in-vivo codes across the four lectures.

	Validity						
Alex	True+	3	Proof-based	4	Trusting -	1	9
	Questioning +	1					
		3		4		1	

	Validity						
David	True +	7	True	8	Absurd -	3	28
	Valid +	3	False	5			
	Correct +	2					
		12		13		3	

	Validity						
Cai	True +	2			Wrong -	2	4
	Solid +	1					
	Correct +	1					
		4		0		2	

	Validity						
Emmett	True +	3	True	2	Nervous -	1	8
	Certain +	2					
	Confident +	1					
	Definite +	1					
		5		2		1	

Figure 23: Positive, neutral and negative ***validity*** *appraisals by each of the four instructors*

Unsurprisingly, the general pattern across the four lectures was a positive stance toward truth and certainty. Falsehood (a statement being *false*, *wrong* or *absurd*) was assessed neutrally or negatively. In terms of the attitude of the epistemic agent, Emmett displayed a positive stance toward being *confident*, *certain*, and *definitive*, and a negative one toward being *nervous* about a statement's validity. Alex positively evaluated a skeptical (in his words, *questioning*) attitude toward mathematical texts, and a *trusting* one negatively. Overall, the image that emerges is of a mathematical subject that prizes validity and certainly, and pursues those epistemic aims through virtues such as skepticism.

Another observation that can be made by examining at the above tables is that David used validity attributes much more frequently than the other three instructors, Alex, Cai, and Emmett. In fact, David's total count of 28 instances of validity appraisals is greater than the total counts for the three other instructors combined (9+4+8=21). This is partly explained by the fact that David, unlike the other instructors, devoted a few minutes of lecture time to discussing the idea of truth tables. Those episodes alone (episodes 15-19) account for 15 out of the 28 applied validity codes. For illustration, the two sentences below from episode 19 were flagged with in-vivo validity codes four time, for the words *true* (x2), *absurd*, and *false*.

> And remember that an implication being true doesn't have anything to do with… whether the first term is true or the last term is true they can both be absurdities. A false statement implies anything. (David, episode 19)

However, even if we discount the validity codes applied to those episodes, we still retain a total of 13 instances of reference to validity in David's lecture, more than any of the other instructors.

Alex, Cai and David all emphasized truth and validity when talking about practices of reading and writing proofs. For example, Cai emphasized that as part of homework he expected students to thoroughly read and not just skim the text, a process he described in the following way:

> But if you really want to understand mathematics, you have to think why this computation is true, why this logic is true. (Cai, episode 7)

Like Cai, Alex also announced an intention to give reading assignments as homework and spent time explaining what he expected students to do:

> … you want to question what the book says. Pretend like you don't really believe it. And then try to understand really why it's true. So the idea of reading and doing mathematics is Socratic. You really want to pretend like you're discovering it for the first time. Check, you know, if somebody tells you something that doesn't mean that it is true. It means … that this is an opportunity for … for a mistake or a misconnection, and so you should really figure out every single connection. Make sure that it's- that you believe it. So, the idea of reading math, it is a conversation where you are trying to really believe the statements in a way that … that is questioning rather than trusting.

In the above excerpt, Alex characterized the process of reading mathematical texts as one oriented exclusively toward the determination of statements' truth-value. To engage in this practice, Alex explained, one needs to adopt a highly skeptical stance ("*questioning* rather than *trusting*") and treat every statement as "an opportunity … for a mistake or a misconnection."

David, unlike Alex and Cai, did not assign dedicated reading homework. In David's lecture, ideas about what it means to read and write mathematical proofs came up in the context articulating a 'learn-to-prove' goal for the course (discussed in chapter 5):

> The idea is to make you aware of how to write proofs and how to think about proofs and how to recognize when something is a valid proof. And that is actually an extremely important skill, right? Like, the one thing you should always be asking yourself, like, when you're doing homework assignments is- is what I've written a valid proof? Because if you can develop <u>that</u> skill of recognizing when something is correct, then that is a mentally helpful for generating more correct mathematics.

What I wish to draw attention to in this chapter is that in David's entire presentation of the "learn-to-prove" meta-story, truth and validity are positioned as the only important attributes of proofs and proving. Similarly, when writing proofs, the only objective is being *correct*.

In mathematical practice, proofs are often read and written for purposes other than validation (determining correctness and truth-value). For example, proofs can be read for comprehension, for learning a proving-technique, or even for aesthetic appreciation (Sinclair, 2009). In the above excerpts, however, students were not asked to assess whether the proof is illuminating, useful, or creative. The accounts of proving provided by Alex, David, and Cai focus exclusively on the validation function of reading, and in so doing, position validity as only value the proving is meant to uphold (Dawkins & Weber, 2017).

6.3.3 Difficulty and level

6.3.3.1 Difficulty

The *difficulty* category groups together adjectives and adverbs used to assess whether a mathematical idea or process is easy or challenging. On one end, it includes words such as: *hard*, *difficult*, *complicated*, *complex*, and on the other, words such as: *easy*, *obvious*, *familiar*, and *simple*. Figure 24 below shows the breakdowns of the in-vivo codes across the four lectures.

	Difficulty						
Alex	Complex +	1	Complicated	3	Complicated -	1	9
	Simple +	1	Compex	1	Obvious -	1	
	Easy +	1					
		3		4		2	

	Difficulty						
David	Easy +	5	Subtle	1	Tricky -	3	13
	Complicated +	1	Easy	1	Hard -	2	
		6		2		5	

	Difficulty						
Cai	Easy +	6	Simple	2	Complicated -	1	16
	Familiar +	2	Subtle	1	Challenging -	1	
			New	2	Simple -	1	
		8		5		3	

	Difficulty						
Emmett	Easy +	3	Obvious	3	Obvious -	8	28
	Complicated +	2	Simple	3	Simple -	3	
	Sneaky +	2	Easy	1	Difficult -	1	
	Subtle +	1					
	Simple +	1					
		5		6		12	

*Figure 24: Positive, neutral and negative **difficulty** appraisals by each of the four instructors*

Unlike in the previous two attribute clusters (*precision* and *validity*), in the difficulty cluster there is no clear positive or negative pole. At times, *complexity* is evaluated positively, at times negatively. Similarly, words such as easy and simple were sometimes deployed with a positive stance, and sometimes with a negative one, even by the same instructor.

For example, Emmett used simplicity with a positive stance to motivate a notational abbreviation:

> It's just simpler than having to constantly talk about A and B when the information that B encodes is completely redundant. (Emmett, episode 61)

But, in other episodes, he used simple in a negative way:

> This is really simple, but- let me do it again. (Emmett, episode 4)

> So I guess that's the first theorem we proved this semester (smile). It's a pretty simple one. But that illustrates how many, many basic properties that you know are true can be deduced from the short list that I already gave. (Emmett, episode 25)

In the two excerpts above Emmett positioned simplicity as something to apologize for (indicated by the use of the word "but"). Even though the idea is simple, he will repeat it. Even though the theorem is simple, it is good for illustration. These two uses suggest that simple is not a desirable attribute.

In a similar vein, Emmett also used the word *obvious* with a negative stance to justify not spending time on something:

> In the obvious ways, I won't go through those, and they have corresponding properties. (episode 23)

Such uses of the words *simple* and *obvious* in lectures have been called out for potentially being experienced as a mathematical micro-slights by students (Su, 2015).

Complexity was also used in both positive and negative ways by the instructors. When deployed in a positive light, it was typically used as an indicator of an idea's richness to justify interest.

> The reals are actually very complicated. Which is why we have a whole course about real analysis and a couple of graduate courses after this too. (Emmett, episode 30)

> And then if we want to start doing complicated things like calculus or analysis of things in higher dimensions or working on- or doing calculus on a surface that isn't actually the real line. (David, episode 10)

Complexity was also used in a negative way to motivate interest. For example, David repeatedly used the words *tricky* and *hard* to problematize real numbers, so that the effort to define them precisely would seem warranted:

> if you <u>actually</u> want to specify what a real number is, it can be a little tricky. (David, episode 9)

> it gets a little bit tricky and it becomes important to know precisely what sort of thing you're talking about. (David, episode 10)

> is hard to get a handle on what we mean precisely. (David, episode 12)

> it's a little tricky to write down exactly what we mean by real numbers. (David, episode 22)

Cai used difficulty and complexity with a negative stance to make a mathematical experience relatable to students. For example, at the end of a proof by contradiction (of the statement that square root of two is irrational), Cai said:

So it's complicated, but this means square root of two is not rational. So this is proof by contradiction. (episode 64)

6.3.3.2 *level*

The *level* category groups together adjectives and adverbs assessing a concept's position within the hierarchical structure of mathematical knowledge, using words such as *basic* and *fundamental*. Figure 25 below shows the breakdown of in-vivo codes.

	Level						
Alex	Fundamental +	5	Basic	10			16
	Deep +	1					
		6		10		0	

	Level						
David	Fundamental +	1					1
		1		0		0	

	Level						
Cai			Basic	1			1
		0		1		0	

	Level						
Emmett	Fundamental +	2	Basic	24			27
			Non-basic	1			
		2		25		0	

*Figure 25: Positive, neutral and negative **level** appraisals by each of the four instructors*

The words *basic* and *fundamental* can be interpreted (and are often used) to signal *ease*, a meaning which would merit including them in the difficulty category. However, I found it instructive to create a separate category for intricacy characterizations that focus on the perceived level of a concept. First, it is possible to describe an idea as fundamental or basic, but not easy. For example, mathematical work in fields such as logic or category theory can be considered foundational yet complex. Second, the fact that these two meaning (difficulty and level) get conflated in meta-mathematical (in this case pedagogical) discourse is a phenomena that deserves attention. Let us consider some examples.

As was already discussed in section 5.3.5, Alex's *connections* meta-story framed RA as a process of reducing the complex (in the sense of compound) subject of calculus into *simpler* (in the sense of non-compound) constitutive units. Those constitutive units were then repeatedly described as *basic*:

> Well logic tells you that you can develop really a lot of very complex arguments starting with some very basic logical rules. (Alex, episode 44)

> so something that people realized in the early 20th century is that all of math can be reduced to this very, very basic logical language. (Alex, episode 45)

> And that's not what we're going to do in this class (4s) because that would just be too much work, to reduce all statements down to basic logic. (Alex, episode 45)

> So [Algebra] is a branch of math that you think of- you learn as just its own branch, but in fact, of course now we know that it can be reduced down to these very basic logical statements. (Alex, episode 48)

> something that they don't teach you when you first take a class in calculus is how to reduce things like taking a derivative to basic- to basic logic and algebra. (Alex, episode 49)

On the one hand, the word *basic* evokes a building-foundation metaphor of mathematical knowledge, and in so doing, conveys that logic is at the ground level of a vast structure built on top. On the other hand, Alex also repeatedly used the word reduce, which evokes a metaphor of a compound substance simplified to more pure, non-compound form. Neither of these metaphoric conceptualizations render logic, or the complementary processes of building from logic and reduction to logic, straightforward. Indeed, in episode 45, Alex explicitly framed reducing to statements to logic as "too much work." Rather than ease, these statements signal the virtues of parsimony and purity: "*all* of math" and "a*ll* statements" can be reduced to a single origin.

However, the line between *basic* and *easy* is undoubtedly blurry. Consider, for example, episode 44:

> many of you know that there is this very fundamental subject in math that is so fundamental that it is also a subject in philosophy, which is logic. And so what is logic? Well logic tells you that you can develop really a lot of very complex arguments starting with some very basic logical rules. So you start with some symbols. You know, things like Booelan operations "and", "or", "not." And you start with some basic statements that you know are true. These are sometimes called axioms. So a basic- an example of a basic statement is that if we have a statement A that implies statement B and statement B implies statement C then together these two facts imply that statement A implies statement C. That is a fundamental logical fact, which even though it's quite obvious to most people, you need to put into this machinery of weird symbols ((points to "and", "or", "not" symbols)) in order to get out an interesting theory.

In the above excerpt, different characteristics of mathematical statements are referenced: ease (a statement being "quite *obvious* to most people"), position at the base of structure ("starting from some very *basic* logical rules") and an agent's epistemic stance toward it ("statements that you know are *true*"). However, the way that these characteristics are juxtaposed, the distinctions are difficult to tease out. In the axiomatic game, it is the positionality of a statement at the base (its declared status as an axiom) that should grant it its truth value. Here, however, this direct is not that clear. The sentence "and you start with some basic statements that you know are true" seems to suggest the opposite order: you choose statements as basic because you know they are true, and the example (and subsequent comment) suggests that one know the statement is true because it is obvious. To justify choosing something as a building block, we often use experienced ease. The polysemy of the word *basic* renders this subtle distinction difficult to discern.

This was a recurrent feature in Emmett's lecture, who frequently used both the words easy and basic. At times, basic seemed to just reinforce statements about ease:

> And in this first lecture, which is supposed to be an easy lecture, we'll just review some basic things about the Real numbers. (Emmett, episode 2)

But, Emmett soon ran into trouble, because he wanted to use the perceived ease of a property to justify using it as a foundation in the deductive organization of knowledge:

Okay, so, obvious basic things we'll assume and hopefully there won't be much doubt about what really counts as basic and what doesn't, okay? (Emmett, episode 9)

Here we see the polysemy of *basic* in action.

6.3.4 Position

6.3.4.1 Significance

The *significance* category groups together adjectives and adverbs assessing a concept's importance and centrality within the mathematical body knowledge. On the positive end, it includes words such as: *important*, *core*, *key*, *big*, *essential* and *crucial*. On the negative, there was one statement about an idea being *not important*. There were also a few neutral attributions of ideas being less central, using terms such as *small* and *extra-curricular*. Figure 26 below shows the breakdown of in-vivo codes.

	Significance						
Alex	Important +	2	Main	5			9
	Big +	1	Extra-curricular	1			
		3		6		0	

	Significance						
David	Important +	2	Essential	1			5
	Core +	2					
		4		1		0	

	Significance						
Cai	Important +	6	Target	1	Not important -	1	14
	Key +	2	Small	1			
	Famous +	1	Slight	1			
			Main	1			
		9		4		1	

	Significance						
Emmett	Important +	5					8
	Key +	1					
	Famous +	1					
	Crucial +	1					
		8		0		0	

Figure 26: Positive, neutral and negative ***significance*** *appraisals by each of the four instructors*

Instructors commented on the significance of a mathematical idea or topic within the course, or mathematical body of knowledge more generally, to direct students' attention. For example, when discussing the definition of natural numbers using Peano axioms, Cai first said:

this is an interesting topic, but it's not super important in our course. (Cai, episode 31)

But then immediately followed with:

But what's most important consequence of these Peano axioms is the principle of mathematical induction, or just induction. (Cai, episode 32)

So, at first, he signaled to students that they should not pay much attention to Peano axioms for the purpose of this course. But then highlight one property (mathematical induction) that does require their attention.

Similarly, Cai used significance attributions to highlight steps students should attend to in proofs:

And then the key part is to show assuming Pn holds. Then we have to prove Pn plus one ho- is also true. So that's the most important part. (Cai, episode 42)

Significance attributions were also used to convey the centrality of ideas or topics in the course as a whole:

the main thing I want you to get out of this, the main thing that we are learning, is how to make connections. (Alex, episode 52)

> [inequalities] is a key point, important in analysis (Cai, episode 67)

> Well, I guess the one core theme is essentially approximation. (David, episode 9)

Other attributions of significance highlighted the conceptual role played by mathematical ideas:

> So [the distributive property] is important of course, because that's the one property that links the two operations.

Many of the above quoted instances of significance attributions could be used in similar ways in non-mathematical contexts. Given their function as directors of students' attention, it is plausible that frequent references to significance are a general feature of pedagogical discourse.

6.3.4.2 Normativity

The category of *normativity* groups together adjectives and adverbs that comment on whether something is within or outside the norm, as well as whether ideas are similar or connected to others. Normativity is indicated by word such as *common*, *typical*, *standard*, *natural*, whereas deviation by words such as *weird*, *special*, *crazy*, *arbitrary*, *surprising*, *funny*, *occasional*. Connectedness (or lack thereof) by words such as *similar, same, different, unrelated.* Figure 27 shows the breakdowns of the corresponding in-vivo codes across the four lectures.

	Normatibity						
Alex	Consistent +	1	Weird	2	Weird -	1	7
			Different	1	Arbitrary -	1	
			Crazy	1			
		1		4		2	

	Normatibity						
David			Common	2	Weird -	2	6
			Standard	1	Surprising -	1	
					Unrelated -	1	
		0		3		4	

	Normatibity						
Cai	Natural +	2	Typical	6			14
			Standard	2			
			Similar	1			
			Same	1			
			Extra	1			
			Frequent	1			
		2		12		0	

	Normatibity						
Emmett	Special +	2	Related	2	Crazy -	1	16
	Reasonable	1	Normal	2	Wrong -	1	
			Different	1			
			Funny	2			
			Close (proxin	1			
			Occasional	1			
			Typical	1			
			Similar	1			
		3		11		2	

*Figure 27: Positive, neutral and negative **normativity** appraisals by each of the four instructors*

Most references to normativity and cohesion were deployed with a neutral, or not easy to determine, stance. The few instances that were deployed with a positive or negative stances, seem to mark deviance with negative affect:

> some functions are weird, they diverge or they are not continuous or are not differentiable. (Alex, episode 38)

> a lot of the proofs that we're going to do, kind of, in some sense depend on this weird meaning of what implies actually is. (David, episode 19)

> And there are fields like this, F sub two that are really different from the reals (Emmett, episode 19)

In the above excerpts, deviance attributes such as *weird* and *different* were used to direct students' attention to mathematical situations that they may or should experience as unexpected or uncomfortable. The negative affect seems to also serve the function of relatability, as if saying: yes, this is weird and somewhat uncomfortable for me too, but we should attend to this nevertheless.

Normativity attributions were used to convey notational conventions:

> So that's typically called blackboard font. (Cai, episode 25)

> So I also want to introduce some typical, like, standard abbreviation in mathematics (Cai, episode 58)

As well as the typicality (and hence importance) of certain techniques:

> So typically it's good to just unwind what this means. (Cai, episode 42)

> The typical strategy is you just compute left-hand side and then adding lots of new equalities you finally reach right hand side, we can just say, those quantities are the same. (Cai, episode 43)

> I want to give you another exercise which typically require another type of proof method (Cai, episode 51)

> the approach is, one of the standard, proof by contradiction, a reductio ad absurdum. (David, episode 25)

Thus, similar to the significance category, the main function of normativity attributions was to organize knowledge and steer students' attention accordingly. Ideas and techniques marked as central or typical are important, and hence should be attended to. Deviance form the norm, a mathematical object being *different* or *weird*, deserves attention because it makes things stand out.

6.3.5 Utility, interest & aesthetics

6.3.5.1 Utility

The *utility* category groups together adjectives and adverbs assessing how useful a mathematical idea or process is. It includes words such as *useful, helpful, powerful*, and *practical*. Figure 28 shows the breakdown of in-vivo codes.

	Utility						
Alex	Useful +	4					6
	Powerful +	1					
	Helpful +	1					
		6		0		0	

	Utility						
David	Useful +	4					5
	Helpful+	1					
		5		0		0	

	Utility						
Cai	Helpful +	2					3
	Powerful +	1					
		3		0		0	

	Utility						
Emmett	Efficient +	1	Practical	1	Defective -	1	3
		1		1		1	

*Figure 28: Positive, neutral and negative **utility** appraisals by each of the four instructors*

In the majority of cases, utility was commented on in a positive way. That is, it was evoked when an idea or technique was deemed useful or helpful for something. For example, David referred to *utility* to justify an abstract definition of order:

> And it turns out it's actually quite useful to put orderings on sets other than just numbers. There are times when you might want to order other things because the idea of an ordering is a useful property to talk about in general. (David, episode 37)

Cai described calculus theorems as very *powerful* and *helpful*, because they allow easy computation of limits:

> … there are very powerful helpful tools from calculus. First, some of you might have heard the name l'Hospitals like theorem with which sometimes you can compute particular limits very easily. (Cai, episode 21)

A complementary perspective was offered by Emmett, where he highlighted the non-utility of rational numbers for taking limits:

> And that in a nutshell, is why this course is real analysis instead of rational analysis, the rationals are defective from our point of view. We want to be able to take limits, completeness, the rationals are not complete and there will be instances when we can't take limits when we want to. So the rationals don't work for us and we need the reals. (Emmett, episode 34)

In the above excerpt, Emmett characterized rational numbers as "*defective* from our point of view," and subsequently explained that the problem is with how rationals can or cannot be used: the rationals cannot be reliably used to take limits. Hence, the defect is a *utility* problem.

In a few cases, instructors commented on certain activities as being useful for learning:

> And if you're taking, say, an abstract course, a course on abstract algebra, where you build Q from the ground up then it's a useful exercise to do. (David, episode 51)

… learning to make these connections rigorously and work with them consistently is extremely important and its very useful that the first time many of you do this, namely in this class, you do it in a context where everything is known. (Alex, episode 51)

6.3.5.2 Interest

The *interest* category groups together adjectives and adverbs assessing how interesting a mathematical idea or process is. It includes words such as *interesting*, *curious*, and *fun*. Figure 29 shows the breakdown of in-vivo codes.

	Interest						
Alex	Interesting +	4					7
	Curious +	3					
		7		0		0	

	Interest						
David	Fun +	2			Not interesting -	1	7
	Interesting +	2					
	Curious +	1					
	Meaningful +	1					
		6		0		1	

	Interest						
Cai	Intersting +	2					3
	Desired +	1					
		3		0		0	

	Interest						
Emmett	Interesting +	5			Technical -	1	7
	Mysterious +	1					
		6		0		1	

*Figure 29: Positive, neutral and negative **interest** appraisals by each of the four instructors*

As with the *utility* category, most *interest* attributions were on the positive pole: instructors tended to mention when something was interesting, rather than when something was not interesting.

Emmett, for example, used *interest* to characterize Dedekind cuts:

But there are other cuts. And this is what makes cuts interesting. And here's an example of an interesting cut. (Emmett, episode 55)

In the above excerpt, Emmett did not only refer to certain examples of cuts as *interesting*, but frames the existence of such interesting examples as something that makes the very idea of cuts interesting. Prior to this episode, Emmett talked about rational cuts, which are cuts that correspond to rational numbers. In this episode, he presented examples of non-rational cuts, which are cuts that do not correspond to rational numbers. The specific example he shared immediately following the above quote was a cut corresponding to the irrational number $\sqrt{2}$.

Like Emmett, David also deployed the word interesting to characterize an example:

… that can sometimes be a somewhat interesting example to think about as well. (David, episode 53)

The example described as *interesting* in the above quote was "dictionary order" as an example of a set (collection of words) with an order (which word appears before the other in a dictionary). This was the very last example David gave in his lecture.

What function does the word *interesting* have here? Similar to significance, normativity, and utility, interest motivates or justifies attention. However, the words interesting, useful, typical and significant are not interchangeable. Something being *interesting* suggests a different kind of attention. Not just remembering or keeping something in mind for future use, but digging deeper into that one thing. Furthermore, the attention to be given to the two examples David and Emmett highlighted as *interesting* is motivated not by their position in the body of knowledge and recurrence in application. Rather, by a certain epistemic satisfaction from engaging with that particular idea, for its own sake. The dictionary order example is a good illustration of this. It is not typical, central, or useful in RA. Nevertheless, it is interesting.

Indeed, something can be interesting but not very important. This is how, for example, Cai framed Peano axioms (in the context of a RA course):

> this is an interesting topic, but it's not super important in our course. (Cai, episode 31)

Interest motivating attention due to richness is evident in the following quote by Alex:

> That is a fundamental logical fact, which even though it's quite obvious to most people, you need to put into this machinery of weird symbols in order to get out an interesting theory. (Alex, episode 44)

Interest was deployed as a theory virtue, that is, to describe the mathematical object, idea or theory. But, for a topic to be interesting, someone needs to be interested in it. The counterpart character virtue was curiosity:

> if we want to start doing complicated things like calculus or analysis of things in higher dimensions … And so the goal of this class is to sort of lay that groundwork that dealing with the real numbers are, so that we can then in future expand it to talk about more interesting, more interesting topics. You might be curious. Can you do something that looks like calculus in infinitely many dimension? (Davide, episode 10)

In the above quote, David positioned multi-dimensional versions of calculus as more complicated and hence interesting. And, in saying "you might be curious" he suggested that student may assume the mathematical virtue of *being interested* in those topics.

Like David, Alex also used the word *curious* as a mathematical character virtue:

> I am going to ask you to take a piece of text from the book, a result or a theorem, and sort of read it as if you are curious. (Alex, episode 29)
>
> check everything and try to be curious. (Alex, episode 33)

Note that both David and Alex impose the virtue of curiosity on students. David's framing suggests that being curious about infinite dimensional calculus is something a student might naturally do. In Alex's case the imposition is very explicit: he asked students to "pretend" and "try" to be curious. In either case, the implication is that to as part of mathematical activity one needs to exercise the virtue of being *curious*.

6.3.5.3 *Aesthetics*

The category of *aesthetics* refers to appraisals of beauty. The instructors in this study did not often use words that refer to beauty and appearances explicitly (the word beautiful, for example, did not appear at all). However, they did use words that conveyed affective reactions to pieces of mathematics, that, due to their general character, I interpreted as instances of aesthetic appreciation. These included words such as *nice*, *wonderful*, *spectacular*, *fresh*, *shiny*, and *elegant*. Figure 30 shows the breakdown of in-vivo codes.

	Aesthetics						
Alex	Shiny +	1					9
	Elegant +	1					
	Refreshing +	1					
	Fancy +	1					
	Nice +	1					
	Coherent +	1					
	Fresh +	1					
	Well written +	1					
	Well developed +	1					
		9		0		0	

	Aesthetics						
Cai	Nice +	2					2
		2		0		0	

	Aesthetics						
David	Wonderful +	1			Not rich -	1	4
	Nice +	1			Not nice -	1	
		2		0		2	

	Aesthetics						
Emmett	Spectacular +	1			Awkward -	1	4
	Favorite +	2					
		3		0		1	

Figure 30: Positive, neutral and negative ***aesthetics*** *appraisals by each of the four instructors*

This category more so than others is not dominated by recurrent uses of the same exact words. *Utility*, for example, was signaled by recurrent uses use of the words *useful* and *helpful*. But here, aesthetic appreciation is conveyed using a variety of words (especially by Alex).

Here are a few examples:

> And in fact, in our text, and I encourage you to read this on pages 20 and 21 of our text, our author gives a really spectacular example of an ordered field. (Emmett, episode 29)

In two separate instances, Cai used the word *nice* as a desirable attribute of algebraic expressions:

> replace them by these other two numbers which have a nicer property (Cai, episode 29)

> I can just have this more nicely. (Cai, episode 46)

6.4 Summary & Discussion

In this chapter I examined the value-laden attributes instructors used to characterize mathematics.

6.4.1 What value-laden attributes were used?

Through in-vivo coding of virtue attributions in lecture talk and subsequent aggregation into clusters, I found twelve attributes used by instructors: *precision*, *validity*, *generality*, *difficulty*, *level*, *density*, *pace*, *significance*, *normativity*, *utility*, *interest*, and *aesthetic*.

This list can be seen as replicating and extending the four factors found by (Inglis & Aberdein, 2015), which were precision, intricacy, interest, and utility (see Figure 31). Their precision factor maps nicely unto the precision category found here, as it encompasses some of the same attribute words I identified through in-vivo coding (e.g. ‘precise,’ ‘careful,’ ‘rigorous,’ ‘clear,’ ‘unambiguous’). My category of *difficulty*, when considered together with *density*, can be mapped onto the *intricacy* factor identified in (Inglis & Aberdein, 2015); there is overlap in the specific words ‘difficult,’ ‘simple,’ and ‘dense’ appearing both in the intricacy factor and in my in-vivo codes in these categories. My category of aesthetic can be matched with the similarly named factor in (Inglis & Aberdein, 2015), even though only one of my in-vivo codes ‘elegant’ appears as a word in that cluster. This lack of overlap can be explained by the somewhat idiosyncratic use of words to refer to aesthetics in my data; Alex, for example, referred to aesthetics 9 types, using 9 different words. Finally, the factor *utility* matches the here similarly named attribute-category, indicated by overlap in the words ‘efficient’ and ‘useful’ clustered under that label in both studies.

This study	Precision	Validity	Generality	Difficulty	Desnity	Level	Pace	Significance	Normativity	Utility	Interest	Aesthetics
IA(2015)	Precision			Intricacy						Utility		Aesthetics
Word overlap	precise		general	difficult	dense					efficient		elegant
	careful			simple						useful		
	rigorous											
	clear											
	unambiguous											

Figure 31: Comparison between values identified in this study, and those found by factor analysis in (Inglis & Aberdein, 2015)

Seven of the categories identified here, namely, *validity*, *generality*, *level*, *pace*, *significance*, *normativity*, and *interest* do not appear in (Inglis & Aberdein, 2015). The discrepancy can be explained by the differences in methodology and context. In terms of methodology, in (Inglis & Aberdein, 2015), the researcher-generated list of attributes constrained what clusters could possibility emerge from participants’ evaluations. With an exception of one word (‘generality’), none of the in-vivo codes grouped under these seven clusters appeared on the list of 80 adjectives presented to participants in (Inglis & Aberdein, 2015). This illustrates the importance of complementing studies that rely on researcher-generated instruments with ethnographic research where attributions are not constrained in a predetermined manner.

The significant difference in disciplinary function can also explain the emergence of new categories. The survey instrument used by (Inglis & Aberdein, 2015) asked participants to assess a single proof. In the current naturalistic study, instructors used attributes to characterize a wide variety of mathematical units, objects, and processes. It is quite possible that different attributes become salient when evaluating different types of units and objects. For example, in the lectures, the attribute *interest* was applied to mathematical objects (“this is what makes cuts interesting”, Emmett), examples (“a somewhat interesting example to think about,” David), and topics (“this is an interesting topic” Cai), but never to proofs. Is there systematic correlation between the kinds of attributes used and the focus of what is being evaluated? Addressing this questions, whether through lab or naturalistic methods, is a fruitful direction for future research.

Evaluating a proof as a mathematical unit considered in isolation could explain the absence of the categories *significance* and *normativity* in (Inglis & Aberdein, 2015), as both of these categories refer to positionality within a larger body of knowledge.

Another notable difference between the two studies is that of disciplinary context: evaluating a proof read as part of research work versus valuation of mathematics in front of students in lectures. The proofs mathematicians recalled reading as part of the survey prompt in (Inglis & Aberdein, 2015) were likely either already published or, perhaps, under review[32]. In a published research context, where lack of validity is disqualifying, the truth value of proofs may seem as an uninteresting attribute to comment on. In a pedagogical context, however, the validity of proofs (e.g. student generated proofs) is less presumed and hence can be seen as worth highlighting.

The absence of what I called *level* in (Inglis & Aberdein, 2015) can be again explained by the difference in methodology and study context: an introductory real analysis lecture renders attention to level (whether something is 'basic' and 'fundamental') more relevant than in the context of recently published research, which is likely neither basic or fundamental.

Finally, that *pace* was commented on by instructors but not included in the study of (Inglis & Aberdein, 2015), can again be easily explained by the pedagogical context and the fact that the coding scheme used here flagged adverbs describing processes (e.g. 'let me go over this *quickly'*, Cai) in addition to adjectives.

It is entirely possible that the twelve categories identified here are not truly distinct, in the sense examined by (Inglis & Aberdein, 2015). Because the methodology employed here is not conducive for resolving this issue with factor analysis, the question of whether there is systematic conflation between different attributes (e.g. do *difficulty* and *level* always refer to the same underlying characteristic?), as well as whether the specific words were grouped appropriately (e.g. does 'fundamental' refer to *level* or to *significance*?) remains open.

A possible direction for future work is thus to conduct more studies using factor analysis, with naturalistically identified attributes and influencing contextual factors in mind. That is, expand the list of adjectives to include the those found here through in-vivo codes, and, as already mentioned above, systematically vary the survey prompts, to include evaluation of different units (proofs, definitions, topics), and in different contexts of disciplinary practice (research, teaching). The latter is of interest also in relation to the contrast between the findings of (Sommerhoff & Ufer, 2019) and (Weber, 2008), that mathematicians seem to use different acceptance criteria when evaluating proofs in the context of teaching and research.

[32] Most of the participants in (Inglis & Aberdein, 2015) were graduate students, and hence less likely to read proofs in the context of reviewing proofs in a journal article.

6.4.2 What value-laden attributes were used more and less often?

The methodology of flagging all instances of naturally occurring attributions during lectures, allows to not only identify which attributes were evoked, but also examine their relative frequencies. Namely, it allows comparing which attributes were invoked more and less often.

Considering all instructors in aggregate (see Figure 32), the most mentionable of the twelve attributes were *precision* (72), *validity* (53), *difficulty* (66), *level (*45), *significance* (36) and *normativity* (44).

	Precision	Validity	Generality	Difficulty	Level	Density	Pace	Significance	Normativity	Utility	Interest	Aesthetic	Other	Total
Alex	15	9	6	9	16	3	0	9	7	6	7	9	4	100
Cai	15	6	0	16	1	1	9	14	14	3	3	1	2	85
David	19	28	1	13	1	0	0	5	7	5	7	4	2	92
Emmett	23	10	1	28	27	1	2	8	16	3	7	4	4	134
	72	53	8	66	45	5	11	36	44	17	24	18	12	411

Figure 32: Counts of attribute codes in Alex, Cai, David and Emmett's lectures, color-coded by thematic relatedness

That *precision* and *validity* were highly mentionable by the instructors in this study is not very surprising. These are attributes traditionally associated with proof-based mathematics. Indeed, together with generality (e.g. whether something is abstract or concrete), they may be considered as defining characteristics of contemporary academic mathematics (Dawkins & Weber, 2017); dimensions along which it is distinguished from other disciplines. Because real analysis lectures are contexts of transition to the DTP epistemic game, it makes sense that instructors would highlight those attributes along which this game stands out. It would be interesting to see if more advanced mathematics lectures (e.g. graduate courses, research talks), where the transition to proof is no longer as salient, or even later lectures within the same real analysis course, would exhibit similar heightened attention to precision and validity.

None of the identified attributes are unique to mathematics. Mathematical proofs can be deemed *beautiful*, *difficult*, and *significant*. But so can, for example, works of art. Precisions and validity, while clearly taking on specialized form in mathematics, are attributes of concern in other fields as well. What may be unique to mathematics, and even unique to real analysis, is the relative frequency with which different attributes are invoked. It is possible to imagine that lectures in different fields would exhibit different patterns of reference to various categories of virtue. We can hypothesize, for example, that lectures in fields like art would invoke the aesthetics category more frequently, and would not invoke validity at all. Or, that lectures on mathematical topics with more applications, e.g. statistics, would features more references to utility. These musings point to a potentially productive direction of conducting comparative analyses with lectures in other disciplines and subfields of mathematics. A specific question I find interest for future comparative studies is whether frequent invocations of difficulty is unique to mathematics, or the so-called "hard" sciences.

The prominence of the *level* category in the lectures can be explained, at least in part, by the fact that these are introductory lectures. It is possible to imagine, for example, that references to attributes such as 'basic' or 'fundamental' would decrease as the course progresses. On the other hand, there is no reason to assume that *difficulty* would disappear as a salient category. Indeed, it may increase as topics and problems become more challenging. It would be interesting to compare the patterns identified here, in the context of the first lecture in the semester, to patterns of valuation in later lectures by the same instructors.

Instructors' frequent invocation of *significance* and *normativity* can be understood in light of the context being pedagogical. When teaching RA, instructors do not just teach individual facts. Likely, they wish to provide the lay of the land of the entire subject. This requires connecting facts to one another (e.g. saying: "this is similar," "this is different") and describing individual units' positionality within the larger body of knowledge (e.g. saying "this is central," "this is peripheral"). Attributions grouped under the *significance* and *normativity* categories can serve that function in activity. It would be interesting to see if those categories play a much less prominent role in non-pedagogical contexts, such as research talks.

References to *significance* and *normativity* also help moderate students' attention. Claims about a topic's or a technique's importance ("the key part is to show … Pn+1 is true," Cai) or typicality ("So typically it's good to just unwind what this means," Cai), as well as deviance from the norm ("weird meaning of … implies," David), can signal to students what of the massive amount of information in lectures to pay attention to. Of course, that instructors intend to convey that something is important does not necessarily mean that students interpret it that way. Effects of other valuation processes (e.g. what gets written down, what is 'just' said orally) may trump the effects of *significance* and *normativity* attributions.

On the next tier, in terms of how frequently categories were deployed, are: *utility* (17), *interest* (24) and *aesthetics* (18). These attributes are commonly associated with mathematics. Indeed, two of the categories – utility and aesthetics – were also identified as factors by (Inglis & Aberdein, 2015), and, as the authors noted in the introduction to their paper, considerations of utility and aesthetics are commonly deployed by mathematicians as justification for award decisions.

Additionally, utility, interest and aesthetics frequently feature in mathematicians' meta-reflections about what they value in their practice (Atiyah, 1984; Hardy, 1940; Tao, 2007). In fact, these attributes are often given high priority in mathematical practice. Hardy (1940), for example, famously claimed that beauty is necessary for mathematics to have any significance:

> "Beauty is the first test: there is no permanent place in the world for ugly mathematics." (Hardy, 1940, p. 14)

And, while Hardy (1940) explicitly devalued utility, Atiyah (1984) considered such a point of view dangerous and instead positioned utility as a primary justification for doing mathematics:

> "The only justification is that it is a real contribution to human thought. Even if I'm not directly working in applied mathematics, I feel that I'm contributing to the sort of mathematics that can and will be useful for people who are interested in applying mathematics to other things." (Hardy, 1940, p. 12)

In light of this, the relative infrequency[33] with which these attributes were referred to in lectures can be seen as a misrepresentation of disciplinary practice. Of course, it is quite possible that mathematicians *say* they value beauty, interest and utility, but do not engage in or are guided by such value judgements in their daily practice – this is an open empirical question. What we can say with some confidence is that utility, interest and aesthetics are prioritized in high profile meta-discourse (e.g. famous reflection essays, award justifications). One way to gauge whether the relative insignificance of utility, interest and aesthetics is a feature of contemporary academic mathematics discourse in general or of introductory lectures more specifically, is to compare patterns of attribution in lectures with those deployed in professional contexts such as research talks.

A limitation of the current study that should be taken into account when interpreting these results is that the data analyzed here was of the first lecture of the semester only. It is possible that as the semester progressed, different values became more salient, and in particular, that the relative significance of utility, interest and aesthetics increased. This is a question that can be taken up by analysis of other lectures in the same data set.

That said, a reasonable interpretation of these preliminary results is that introductory lecture discourse skews toward assigning more importance to rigor (precision and validity), complexity (difficulty and level), and position within the body of knowledge (significance and normativity), than to utility, interest and aesthetics, a distribution at odds with the values (purported) by professionals in the practice. In that sense, these findings provide further empirical support to the observation made by (Sommerhoff & Ufer, 2019) that proof-based undergraduate lectures such as real analysis enact a mathematical culture different than that of professional mathematicians.

This contrasts with the traditional view of both mathematicians and mathematics education researchers. Namely, that real analysis lectures introduce students to the culture of contemporary academic mathematics (Sfard, 2014).

6.4.3 What was positively and negatively evaluated?

The twelve attribute clusters identified in this study group together both positive and negative evaluations of what I interpreted to be the same underlying characteristic. For example, positive stances toward rigor and negative stances toward vagueness were interpreted as commenting on the opposite poles of the same attribute: *precision*. The discussions in the preceding two sections about which categories were present in lecture discourse and what was their relative frequency, did not take this division into positive and negative evaluations into account. The distinction between positive and negative evaluations is the focus of this section.

Most attribute categories exhibited a clear positive and negative pole. Within the *precision* category, precision was marked with positive stance, whereas the lack of precision – indicated by words such as ‘vague,’ ‘heuristic,’ ‘rough,’ and ‘informal’ – was evaluated negatively. Within the *validity* category, ‘true,’ ‘valid,’ and ‘correct’ were used positively, whereas ‘wrong’ and ‘absurd’ negatively. *Significance* also featured a clear positive pole: words such as ‘important,’

[33] Certainly in comparison to precision, which was invoked more than 3 times as often as each of these categories: *precision* (72) $\geq$ 3* *utility* (17), *interest* (24), *aesthetics* (18).

‘core,’ and ‘key,’ were deployed with a positive stance. Unlike *precision* and *validity*, however, negative or even neutral significance attributions – i.e. something being insignificant – were rare. *Utility*, *interest*, and *aesthetics*, also had a clear positive pole: utility, interest, and beauty were framed as desirable. As with significance, most of the attributions in these categories were positive, with only a few instance of instructors exhibiting a negative stance with phrases such as ‘defective,’ ‘not interesting,’ and ‘awkward.’

Two of the categories, *level* and *normativity*, did not seem to lean positively or negatively in any clear way; most instances were deployed with a neutral (or underdetermined) stance. Within *level*, there was some skewness toward positively evaluating something for being ‘fundamental,’ and within *normativity*, there was some skewness toward negatively evaluating deviance from the norm (e.g. words such as ‘weird’ and ‘surprising’ deployed with a negative stance.

The most interesting category in terms patterns of positive and negative stances was *difficulty*. While there were both positive and negative attributions, the usage was not very consistent, in the sense that the same attribute pole could at times be evaluated positively and at times negatively. For example, the word ‘complicated’ was used eight times across the four lectures, of which three were coded with a positive stance, three with a neutral stance, and two with a negative one. This is not just a between-instructor difference. Three of the four instructors – Alex, David and Emmett – evaluated difficulty both positively and negatively. The most consistent of the four instructors was Cai, who, with the exception of one negative stance toward ‘simple,’ evaluated ease positively and difficulty negatively.

Overall, attributions tended to be positive. Meaning, whenever instructors commented on an attribute, they typically highlighted positive aspects. For example, they would more often comment on something being ‘interesting’ rather than say that something is ‘not interesting’ (though there was one instance of that in the data). Figure 33 shows the distribution of positive, neutral, and negative attributions for each instructor, counted as a percentage of the total number of attributions each made (Alex, N=100, Cai, N=85, David, N=92, Emmett, N=134).

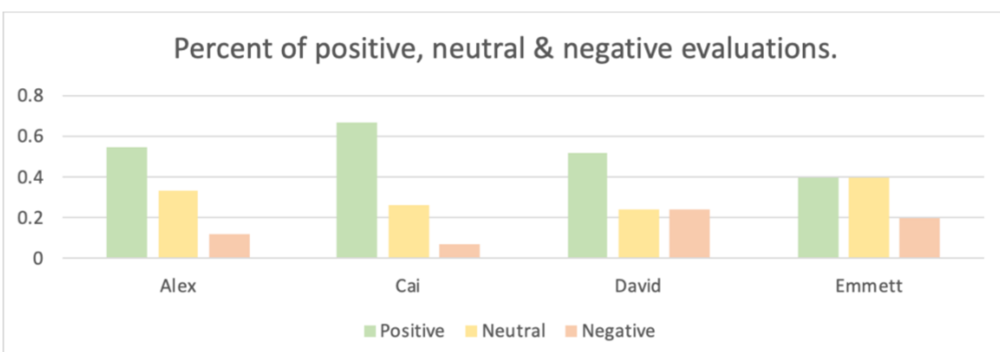


Figure 33: Percent of positive, neutral & negative evaluations. for Alex (N=100), Cai (N=85), David (N=92), & Emmett (N=134).

Even though negative evaluations were overall less frequent, I think it is worth paying close attention to them because negative framings of attributes can function as implicit messages about what (and who) does not belong in mathematics. Su (2015), for example, characterize such messaging as “mathematical micro-aggressions”:

> "Microaggressions may not seem like a big deal (they're micro, so we can just shrug them of, right?), but if you get them repeatedly, their cumulative effect reinforces the message that you don't belong. They have the strongest effect on those who already doubt themselves. … in what ways do we communicate to students "you don't belong in mathematics"? (Su, 2015, p. 36)

In the essay quoted above, Su focused on how certain language choices in lectures may unintentionally function as microaggressions, and because most instructors use them without awareness of their potentially negative effect, he opted to refer to them as micro-slights (a further diminutive of the term micro-aggression). Su's first example of a micro-slight is instructors making statements such as "it is obvious" and claimed that such statements may send the unintentional message "You're stupid if you can't see it right away." (Su, 2015, p. 37). Here such an utterance would be likely be coded as 'obvious' with a negative stance.

Su (2015) claimed that what mathematicians do intend to convey with such negative attributions of "obvious" is that the mathematical idea is "straightforward for someone who has already mastered the material." (pp. 36-37). He then suggested that instructors use a more level-headed phrase such as "It is straightforward, with some work, to show that . . ." (p. 37) instead of "it is obvious." In the language of the current study, that suggestion would translate to – I think – deploying straightforward with a neutral stance. In short, it is OK to comment on things being hard or easy, just be matter of fact about it and do not imbue it with judgement.

This is a good general heuristic, but here I wish to make an additional point. The distinction Su (2015) made about the meaning mathematicians presumably intend to convey with statements such as "this is obvious" and the meaning students may interpret, is in *who* the idea is obvious to: someone who already mastered the material versus students who are just learning it. The question I wish to raise is why instructors, to begin with, make references to how easy something is for those proficient with material when they are addressing students who are learning it for the first time? Furthermore, one may argue that all the material presented in undergraduate lectures is "straightforward with some work" for experts. Why comment on some parts and not others? And again, why make so many comments about ease and difficulty to begin with? As I speculated in the previous section, this may be characteristic of mathematics lecture discourse more than other fields.

One answer lies in considering not just the supposed referential meaning of the statement "this is obvious" but what function such a statement is meant to accomplish in activity. As I have argued earlier in this chapter, in lectures statements such as "this is obvious" often serve the rhetorical function of justifying what the instructor chooses to spend time on. It also serves the function of directing students' attention. Thus, I suggest a more accurate translation of "this is obvious" is: "this is more straightforward than other parts, and hence I chose to spend less time on it in class."

Another function of difficulty attributions is relatability with students. An instructor may say "this step is a bit tricky," not because it is experientially difficult for them, but because they expect students to experience it as difficult, and want to express epistemic empathy through shared epistemic affect (Jaber, 2021; Jaber & Hammer, 2016). The problem with statements such as "this is obvious" is that these two functions – justifying omission and epistemic empathy – are in tension with one another. What I think happens in lectures is that the desire to direct students attention to X and not Y may prevent an instructor from noticing the detrimental effect of difficulty attributions on students. Thus, rather than explaining that the described epistemic stance is that of an expert rather than a novice, as Su (2015) suggested, what I recommend is making the distinction between the two functions more explicit, and perhaps separating them altogether.

Another set of attributes that were repeatedly evaluated negatively are those grouped as the opposite pole to *precision*: e.g. words such as 'vague,' 'ambiguous,' 'informal,' and 'heuristic.' Vagueness and ambiguity were framed negatively by three of the instructors: Alex (2 times), David (4 times) and Emmett (8 times). While negative stances toward ambiguity did not appear on Su's (2015) list of examples, they too can be considered as micro-slights his terminology, as they may signal to students – if your ideas are vague, you don't belong in mathematics.

It is very unlikely that instructors wish to convey such a strong message. However, it is not far-fetched to imagine that after recurrent exposure to negative evaluations of vagueness and ambiguity by authority figures, at least some students would be reluctant to share imprecise ideas in class. That is, students who internalize such messages may not feel comfortable sharing ideas for fear of being perceived as vague, or, alternatively, impose a precise-sounding language, just to look smart. In either case, the result might constitute 'checking your identity at the door' (R. Gutiérrez, 2018). Negative attributions of vagueness and ambiguity, especially when applied to student ideas (e.g. statements such as "you may have a vague idea of what real numbers are") can index deficit narratives about students and their thinking (Adiredja, 2019).

An unqualified message that ambiguity is bad in mathematics is problematic not only because it can constrain access and feelings of belonging for students, but also because it is a distortion of disciplinary practice. Ambiguity is an essential aspect of mathematical semiosis (Wagner, 201AD), and historically been a driving force of development (Trninic et al., 2018).

The mismatch between what instructor intend to convey and the unintentional implications of what they say can again be explained by different functions in activity. In the lectures considered here, negative evaluations of vagueness and ambiguity served a rhetorical function of justifying precision as a goal and ideal: our goal is precision because ambiguity is bad (see section 5.3.2). At the same time, students may interpret them literally (e.g. through the statement's referential function) as a description and negative evaluation of students' epistemic state. Such an interpretation may be experienced as a micro-slight.

6.4.4 An image of 'good' mathematics

I opened this chapter by arguing that recurrent displays of appraisal through the use value-laden attributes are one way by which cultural values are perpetuated. Now it is time to take stock: what image of 'good' mathematics was conveyed in the observed lectures?

> Mathematics is good when it is precise and valid. When it is vague and ambiguous, it is bad. Mathematics is good when it is interesting, useful and beautiful, but those are *not as important* as precision and validity. How difficult things are is an important aspect of mathematical experience as well.

As I have discussed in more detail in the previous sections, such an image both misrepresents mathematics and potentially alienates many students. In the reflective writings of mathematicians, interest, utility and aesthetics are explicitly framed as more important than precision (Atiyah, 1984; Hardy, 1940). Historical and philosophical analyses of mathematical practice suggest that ambiguity is an essential and productive component of disciplinary activity (Wagner, 201AD).

For students, negative evaluations of ambiguity and a relative insignificance of interest, utility and aesthetics, may be off putting. Namely, students who value interest, utility or aesthetics more than precision may find the presented mathematics undesirable and alienating (Ernest, 1995; Herzig, 2004b; Solomon, 2007). Furthermore, students who internalize messages such as 'vagueness is bad' may not feel comfortable sharing ideas for fear of being perceived as vague, or, alternatively, impose a precise-sounding language, just to look smart. In either case, the result might constitute 'checking your identity at the door' (R. Gutiérrez, 2018).

Mathematics, as any discourse, cannot be value free. However, it is important to notice the tacit ways by which we as educators convey values in classroom talk, and critically reflect on the implications these messages have for who gets to see themselves as belonging in mathematics. This study offers one dimension of classroom discourse to attend to: it suggests that educators should be more conscious about the attributes of mathematics we valorize in instruction. Both in terms of what is mentioned more and less (e.g. precision vs. interest), and it terms of what is deployed with a positive and negative stance. Since negative attributions by instructors can often be experienced by students as micro-slights (Su, 2015), it may be particularly important be very careful when using them.

Stances are mobilized for different functions in activity. The analysis in this chapter suggests that attention to value-function relations may be a key to presenting a level-headed image of mathematics. Rather than saying "this is basic", we may want to explained: "I am saying basic to highlight that this should be taken as axiom, not to say it is easy to understand." Rather than saying "ambiguity is bad, and precision is good" we may want to make statements of the form "precision is good for X, ambiguity is good for Y." A functionality conscious description of mathematical practice, however, is hard. It requires deliberate reflection on practice, knowledge of its history, and critical analysis of its narratives: mathematical knowledge for teaching that many mathematicians, and we as a scholarly community, might simply not yet have.

This study has several implications for research. First, it illustrates the importance of looking at valuation processes in naturalistic contexts, and not just use what mathematicians say they value to inform laboratory studies. There may be a systematic disconnect between mathematicians' meta-talk and their evaluative behavior in the flux of activity. To ascertain that, more research is needed. Second, the current work hints at the importance of value-function relationship. Future research should address this issue more explicitly through studies that systematically examine what values are mobilized when and for what purpose. Future studies should both more comprehensively examine the lecture context and compare evaluations in lecture with other contexts of activity, both mathematical and in other disciplines. Finally, it is important to do more research on students' perspectives on these values.

7 Humanizing mathematical language: a chronotope framework.

7.1 Introduction

Math, and in particular, contemporary academic math (i.e. the discursive practice of research mathematicians), is often experienced as alienating and decontextualized from human issues and experiences (Davis & Hersh, 1981; Schoenfeld, 1994), especially among students who are already underrepresented and marginalized in the discipline (R. Gutiérrez, 2018; Herzig, 2004b). One aspect of practice that contributes to such feelings of alienation are the discipline's norms of talking and writing (Lemke, 1990). Language practices in contemporary academic mathematics are centered on the register of the formal text, which is known to obfuscate human agency, e.g. through linguistic devices such as passive verbs or nominalization (Burton & Morgan, 2000). These dehumanizing features are pervasive in part because they help constitute the community's epistemic goals of creating abstract, indubitable, objective and universal knowledge (Hersh, 1991). Language choices, however, are not set in stone. In communication contexts such as lectures, instructors can embellish the textual register with movement, speech and inscription that put "humanity in the machine" (Shaw, 2001, p. 27) and have the potential to cultivate a feeling of disciplinary belonging for students (Rodd, 2003).

The term humanizing has been gaining popularity in mathematics education in recent years, following R. Gutiérrez's (2018) call to shift away from mainstream equity discourses, which are obsessed with "fixing" students' (perceived) deficits and hence counterproductive (Adiredja, 2019; R. Gutiérrez, 2018), to a conversation that focuses instead on how classroom mathematics itself can be reconfigured to better reflect the human beings it lives through and serves. Yet, despite the increased popularity of the term, there is no clear agreement on what humanizing actually means or looks like in concrete mathematical interactions. In this paper, I aim to unpack what humanizing can mean for communication in contemporary academic mathematics. I propose a framework for thinking about this issue and illustrate how lecturers' small discursive moves in advanced math courses can contribute to a more humanizing experience and image of the discipline.

7.1.1 What about mathematical language and interactions can be humanized?

R. Gutiérrez (2018) listed eight (non-exhaustive) dimensions along which math classrooms can be rehumanized for "students and teachers who are Latinx, Black and Indigenous … They include (1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotion, and (8) ownerships." (p. 4)

This list casts a wide net. Other scholarship focuses more specifically on one or two of these issues, and does not necessarily frame them under a single umbrella concept such as humanization. Also, in contrast to Gutiérrez's work, the starting point for such research is not necessarily the concerns and experience of marginalized students. Nevertheless, similar aspects of discourse are taken up. For example, in the embodied cognition tradition, researchers attend to how mainstream math teaching discourse, seeped in Cartesian dualism, can devalue and make invisible important affective and embodied resources by focusing primarily on purely mental processes, abstraction and formalism (e.g. J. F. Gutiérrez, 2018; Núñez, Edwards, & Matos, 1999). In the social-semiotics tradition, researchers studied the linguistic features of written mathematics, such as academic research articles and textbooks, to highlight how pervasive practices such as nominalization (turning verbs into nouns; e.g. “I counted” to “counting” to “a count”) and the use of passive voice (e.g. “as shown by the count in theorem 2.7”) contribute to the obfuscation of human agency in the creation of mathematics (Burton & Morgan, 2000; O’Halloran, 2008). Rotman (1988), building on Pierce’s semiotic theory, observed that mathematical texts make extensive use of inclusive imperatives such as “formalize” “define” and “let”, and suggested that these are instructions for an abstract semiotic agent he called “Subject”. The Subject is an idealized reader and writer of the text, a decontextualized scribbler that does the epistemic work of mathematics. Importantly, this idealized writer-reader has no social identity, nor is it situated in any cultural-historical context. It does not have preferences, goals, nor does it get excited or scared. There is nothing besides text and cognition in the universe the formal text evokes.

7.2 Chronotopes: discursive infrastructure for mathematical subjectivity.

To provide room for human experience, mathematical texts need to construct a world in which human beings can be situated. Bakhtin's (1981, 2010) concept of chronotope is a useful tool for thinking about how this can be accomplished through language. In his study of the evolution of the novel as a literary genre, Bakhtin coined the term chronotope (in Greek, “time-space”) to refer to “the intrinsic connectedness of temporal and spatial relationships that are artistically expressed in literature.” (p. 84). This construct allowed Bakhtin to effectively contrast and characterize different novel types (e.g. adventure, metamorphosis, biographical novel), which in turn allowed him to track the historical evolution of the genre through concrete examples of texts. Important for my purpose here is that a text’s chronotope “determines to a significant degree the image of man … The image of man is always intrinsically chronotopic.” (Bakhtin, 1981, p. 85)

In this chapter, I use the concept on a much smaller scale of speech genres (Bakhtin, 2010) to characterize three broad types of humanizing mathematical utterances. I propose that human experience in advanced mathematics can be discursively situated in three time-space arenas: (1) the here-and-now experience doing math, (2) the social-historical context of math activity, and (3) cultural-discursive hybridity. Figure Figure 34 illustrates these three chronotopes within a single landscape of space-time. The first chronotope (in red) is about the in-the-moment mathematical experience, the micro-genesis of doing math. The second chronotope (in blue) expands both the space and time scales, attending to the synchronic and diachronic dimensions of activity; it invites us to look at how math is a collective cultural practice, spanned across historical time and across socio-geographic space. The third choronotope (in green) entails a further expansion of space-time, inviting us to consider other discourses and cultural-historical activity systems that are parallel, interwoven, and seep to and from the social history of what we typically label as “math”.

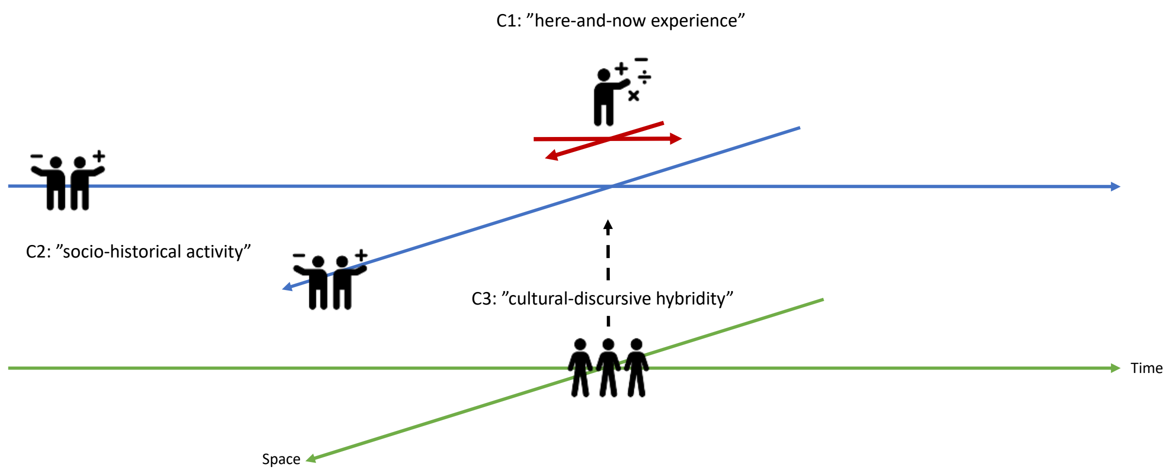


Figure 34: A chronotopic framework for situating the human in mathematical discourse

7.3 Data analysis

The framework emerged through inductive coding of transcripts and document data (Saldaña, 2021). A focal question orienting analysis was: What aspects of advanced math lecture discourse may contribute to a more humanizing experience and image of the discipline? In an initial round of coding, I flagged and labeled lecturers' discursive moves that I broadly interpreted as humanizing. This process generated a long lists of codes such as "performed affect", "personification" and "out-of-math metaphors". I then synthesized the codes into broad categories. The theory of chronotopes was then brought in to help characterize the categories and situate them within a single coherent theory. The resulting framework, of three chronotopic scales for humanizing mathematical discourse, is elaborated below.

7.4 A three-chronotope framework for humanizing the language of mathematics

7.4.1 Chronotope 1: Multidimensional here-and-now experience doing math.

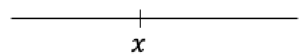
How do we talk about and enact the in-the-moment experience of doing math; how does mathematical activity *feel like* to the person doing it? If we look at the formal register, the vast majority of actions are attributed to an abstract agent (Rotman, 1988) and are cognitive and epistemic in nature. But when actual people engage with math, they do more than "consider", "assume" or "know". They also have affective experiences such as excitement and fear, they imagine pictures and motion, and they have bodily experiences moving in and around inscriptions and virtual mathematical landscapes (Ochs, Gonzales, & Jacoby, 1996). These experiences are rarely reported on in writing, in part because that would jeopardize the epistemic goal of the text by making the mathematical ideas bound to the concrete human body. An abstract free floating cognitive agency "assumes," but a person also feels, imagines, and moves.

7.4.1.1 Example 1.1 – Embodied interpretation

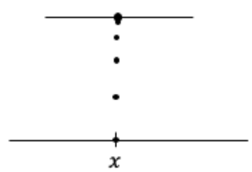
When math is done by humans, it is always done with bodies and material artifacts (Núñez et al., 1999). Mainstream discourse about math, seeped in traditions of mind-body dualism, tends to erase or at least decenter the material and bodily aspects of math activity (J. Gutiérrez, 2018). It either doesn't exist at all, or is inconsequential. One way to humanize math, to frame math as done by actual human beings, is to explicitly talk about and enact the body in mathematical activity. The example below illustrates how attending to the body can be done in lectures within the specific context of "interpreting formal definitions", a discursive practice central to doing academic math.

The episode begins after Henry finished writing the formal definitions for pointwise and uniform continuity. Actions are described in double brackets (( )). Participants' inscriptions are reproduced within the transcript as well.

1 The question is what image do I get in my mind. So, let's try to see this one here
2 ((points at 'pointwise' convergence condition, sketches a diagram)). My image of it is



3 It says. For every x in S. OK. So, let me stand on x. What does it then say? It says that
4 fn of x minus f of x should be less than epsilon. That means that the fn of x <u>converges</u>
6 to f of x. So, that simply means going up going up going up and converging to a point



7 over there. I have (been)- ((stomps feet)) I stand on a point ((pulls fists down)) and
8 then I look (at) ((raises hands up)) fn going up to the point that's sitting up there.
9 That's my feeling of it.

The interpretation is personal. Henry speaks with the voice of an "I" when he says "what image do *I get in my* mind" (line 1), "*my* image of it is" (line 2) and "that's *my* feeling of it" (line 9). He also inserts himself into the graph: "let me stand on x" (line 3), "I stand on a point" (line 7), and "I look at" (line 8). Henry constructs a subjectivity that imagines (lines 1 & 2) and feels (line 9). This subject imagines moving in and around the world of inscriptions (Ochs et al., 1996); he "stands" on a point and looks at function values "going up going up going up" to a point above. But this is not just visual imagery. It is a full body experience. To "stand on x" Henry firmly stomps his feet on the ground and holds his hands tight to stay put (lines 8-9). Looking up (together with students) involves raising his hands and pointing up in the graph (line 9). The subjectivity Henry enacts is vastly different than the one in the formal math text (in fact, the definition text Henry wrote does not refer to a human subjectivity at all). Henry's "I" is one that imagines and feels, not just "considers" and "defines." And it does this imagining with a body.

In this episode, Henry enacted interpretations of the pointwise convergence definition using language and gestures that depict an embodied experience. These interpretations explicitly evoked and mobilized the whole body, and in so doing, conveyed the idea that math is felt and that we do it with our whole body.

7.4.1.2 Example 1.2 – Emotional reactions and stances

Discourse about math and math learning often describes "cold" cognition (Pintrich, Marx, & Boyle, 1993). However, whenever people do math, they do not cease to have feelings. Emotions and emotional stances toward the objects of activity are not only part of mathematical experience, but also central to it. It is often the very emotions bundled up with the math that propel our mathematical activity forward (Jaber & Hammer, 2016). Thus, one way to more accurately and expansively depict what math *feels like,* is to explicitly center emotional reactions and stances in our discourse in and about math.

In the observed lectures, there were many instances of instructors displaying affective reactions to the math they were talking about. At times, affective stances were labeled explicitly (e.g. "I'm scared" or "I'm excited). Most often, however, affect is conveyed through describing the object of focus and through other modalities such as prosody and gestures. The excerpt below illustrates this type of "performed affect." In this episode, Henry discussed errors in recalling the radius of convergence formula ($\mathcal{R} = \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$):

1 Some people were confused about $\mathcal{R} = \overline{\lim}\sqrt[n]{|a_n|}$. That's of course wrong.

2 [should be] one divided by it $\mathcal{R} = \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$. That's why it's a bad way of remembering it.

3 In some sense, the best way of remembering it is $\overline{\lim}\sqrt[n]{|a_n||x|^n} < 1$. Right?

4 That's a lim sup. That's maybe the most natural way of remembering it.

5 I know this is this horrible beta ( ) instead of 1 over R

The episode ends with a display of an affective stance (Ochs, 1996) toward a "horrible beta" (line 7). The word "beta" refers to terminology introduced in the textbook $\beta = \lim \sup |a_n|^{\frac{1}{n}}$. What I wish to call attention to is Henry's use of the the adjective "horrible", which enacts an affective stance toward the mathematical signifier.

What makes the beta horrible? And importantly, to whom? The beta is not "horrible" in a platonic mathematical universe. It is horrible to Henry (and possibly students) in the here and now of doing math, and perhaps also in past experiences of encountering the signifier (e.g. when reading the textbook). Thus, this affective stance marker centers the here-and-now experience of people doing math. The overt affective display constructs engagement with math not as an unemotional view from nowhere, but as a person that can get intimidated by the presence of difficult symbols.

7.4.2 Chronotope 2: Socio-historical context of math activity

The first chronotope entailed zooming in on how it *feels* to do math *in-the-moment,* attending to how mathematical experience is, like any other human experience, multidimensional. In particular, doing math is not purely a cognitive and epistemic experience; it involves affective, somatic, and aesthetic dimensions as well. In the second chronotope, we zoom out and look at the context of mathematical activity.

Doing math is not just a random experience that people come to have. Mathematical activity is a *social* practice (Kitcher, 1984; Schoenfeld, 2016); it has a history and it is shared by a community of other people who “do math”, past, present and future. That is, mathematical experience is part of a cultural-historical activity system (Cole, 1998). The socio-historical context of mathematical activity is not typically centered in disciplinary discourse, with the formal register often avoiding it altogether. The history of mathematics – the fact that mathematical texts such as definitions and theorems were and continue to be *written by people*, based on decisions, circumstances, purposes and preferences – is rarely discussed in official texts. Likewise, the fact that mathematical texts are not decontextualized archives of truths but rather artifacts that people use (today and in the past) for different kinds of purposes and in different kinds of ways is not emphasized in the formal register. If mathematical texts were to be historically contextualized, they run the risk of being seen as historically contingent, of being one of many possible alternatives, and thus lose their status of universality and objectivity.

7.4.2.1 Example 2.1 – Authorship, social persona and choice

One way to portray mathematical activity as situated in socio-historical context is to explicitly attribute authorship to mathematical text units (e.g. theorems, definitions and proofs). In the formal register, mathematical text units often come from “nowhere”. Definitions are “defined” by abstract semiotic agents (e.g. “we define a metric space…”) or there is no agent at all (e.g. “a metric space is defined…”). In contexts of practice, however, definitions are always written by people. These people have names, purposes, and preferences. Classroom discourse can highlight that. As an example, consider the following excerpt from Henry’s lecture notes (shown in figure 2 below).

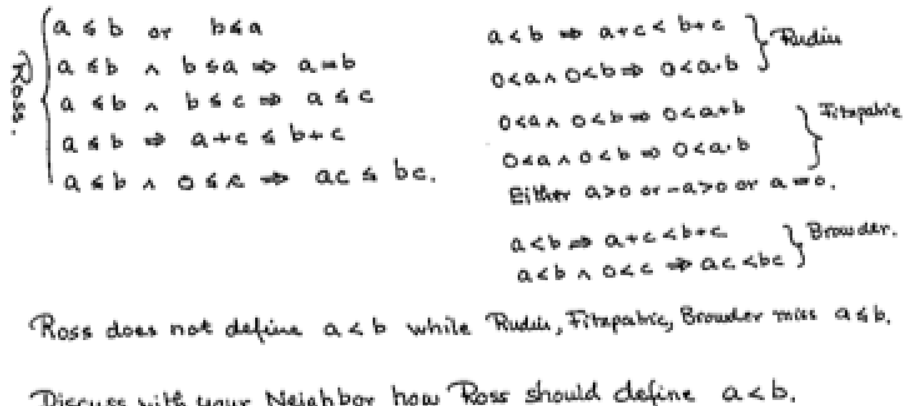


Figure 35: Excerpt from Henry’s lecture notes excerpt inviting students to compare definitions in four textbooks

In the above notes, Henry presents excerpts of definitions from four different textbooks, one for the sign $\leq$ (top left, labeled “Ross”) and three others for sign $<$ (top right, labeled “Rudin”, “Fitzpatric” and “Browder”). Presenting these definitions side-by-side invites students to engage in a mathematical practice that is ubiquitous in mathematicians’ day-to-day work, yet not often centered in pedagogy: comparing and contrasting alternative definition formulations for “the same” (or similar) mathematical object. Designing a classroom task that mimics this practice is an interesting pedagogical move in and of itself, but what I wish to highlight for the purpose of the discussion on humanization is that Henry does not “just” ask students to compare features of different definitions. By labeling each definition by its respective textbook author Henry associates them to people. This does not only raise the consideration of alternatives but further suggests that these definitions were chosen and written by specific people, identifiable by their last name.

Furthermore, the verbs in the next sentence in the notes ("Ross does not define $a < b$" and "Rudin, Fitzpatric, Browder miss $a \leq b$") explicitly attribute definition formulation to the mentioned authors. Ross, Rudin, Fitzpatric and Browder are presented here as agentic mathematical actors that choose between alternatives (even if not knowingly, as suggested by the verb "miss"). Such a formulation promotes an image of mathematics as a living practice (Gutiérrez, 2018), one that "… underscores mathematics as something in motion. When students can see mathematics as full of not just culture and history, but power dynamics, debates, divergent answers, and rule breaking, it highlights the human element and helps promote a vision that is a verb rather than a noun" (p. 5). While this excerpt from Henry's notes does not go as far as revealing "behind the scenes" power dynamics, it does suggest at least the possibility of debate.

Arguably, the invocation of social persona in this example is still rather abstract. Is "Ross" a person or a book? What kind of person is Ross? What is his full name? Where is Ross from and how does he look like? When did he write this definition, and what for? Why did he choose to define $\leq$ and not $<$? Still, what we see in this example from Henry's notes is more than just attaching a name to a mathematical text unit (a common practice in math texts, e.g. "Cauchy's theorem"). Here, the names are positioned as agents. They perform actions such as "define" and "miss" and students are invited engage in mathematical activity from these agents' perspectives.

7.4.2.2 Example 2.2 – Contemporary research as context of mathematical activity

Mathematics is a living practice. People today do mathematics all the time, for different purposes and in different contexts, including the academic one. Connecting classroom practice to contemporary mathematical activity is an important avenue for humanization, because "… seeing mathematics as living practice means individuals can recognize modern mathematics as relatively young and look to practice it in different ways, for their own purposes (not just for school or credentialing)." (Gutiérrez, 2018, p. 5). This is crucially important for imagined futures and belonging.

An episode from Alex's lecture on uniform convergence illustrates how referencing "contemporary research" can look like in Real Analysis lectures.

Prior to the focal episode, Alex presented a "problematic" example of function convergence. It was an example of a sequence of continuous functions that converges to a limit function that is discontinuous. The example is "problematic" because discontinuity is undesirable (in Alex's own words: "we like our functions to be continuous") and one may expect, as mathematicians before the 19th century did, that if a function is obtained as a limit of continuous functions, it should itself be continuous too. To summarize the "takeaway" from this example, Alex wrote the following "warning" on the whiteboard:

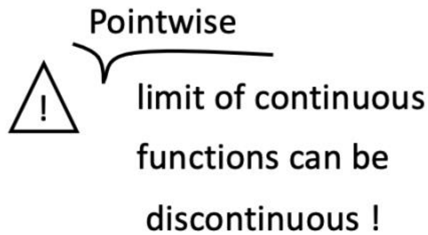


Figure 36: Alex's "warning" about pointwise convergence: "pointwise limit of continuous functions can be discontinuous"

Alex used this (and later another) example to "motivate" the uniform convergence definition. Meaning, he framed the concept of uniform convergence as a solution to this "problem". What followed is our episode of focus. It is a short "digression" about how other solutions are possible, and that looking for such solutions is at the cutting edge of research:

> But there's a- there's another kind of so, so- Let me point- I'll also say that uniform convergence is not the only solution to these problems. And then, so depending on how deep you want to go with all those there's different conditions of convergence on functions. This is actually a- sort of cutting edge. Modern day analysis is exactly about different- different conditions of convergence and there's like an incredible depth of theory to this where you can get a Fields Medal or, you know, win a million dollar problem if you understood that these kinds of conditions on convergence different cases. But what we are going to solve today is we're going to solve this- this issue of uniform convergence.

In math community meta-pedagogical lingo, this kind of move is often characterized as "giving motivation" for the uniform convergence definition. In terms of a pedagogical meta-function, "motivation" is indeed an apt top-level description. But let's unpack it a bit more. What is accomplished in this episode? Uniform convergence is framed here not only as a solution to a problem (a link already established through the previous "problematic" example), but as one of many possible solutions (lines 2-4). It is not only a useful tool, it is part of a *broader context of human activity*; a larger enterprise of solving "these problems" (line 2), that people engage with to this day (it's "cutting edge", line 5). Moreover, these problems are very significant in contemporary practice. Not only are people still interested in them, they are at the heart of "modern day analysis" (line 5), there is an "incredible depth of theory" (lines 6-7), and you can win the most prestigious mathematical award for solving them ("a Fields Medal … a million dollar problem" lines 7-8). The episodes ends with connecting this enterprise back to the topic of the day.

What is *motivating* about this commentary? It is not the promise of fame and money (as focusing on lines 7-8 may suggest). Alex paints a picture of a vibrant field of research, with deep ideas, awards, and problems of interest. And it is the possibility of *belonging to this community* that functions as motivation. Studying uniform convergence thus becomes a legitimate peripheral way of participating in a real, contemporary community of practice (Lave & Wenger, 1991).

7.4.2.3 Example 2.3 – Students' learning trajectory as context for mathematical activity

Contemporary academic mathematics, though can serve as "motivation" (opportunity for belonging), is probably not the most prominent context in students' own lived experience. The most dominant activity system students participate in when they sit in a math lecture is schooling. Students are taking courses, earning grades, and working toward degrees. Acknowledging that, situating classroom math experiences as part of that, is humanizing.

To see how this can be integrated with the mathematical activity of Real Analysis, let's look at another example from Henry's lecture notes. The excerpt below (Figure 37) is taken from the 12th lecture in the semester. By that point, Henry and students spent several weeks learning Real Analysis together, "covering" topics such as number systems and axioms, completeness, limits of sequences, subsequences and various properties. This excerpt marks a transition to a new topic, normed and metric spaces (Ross, 1982, section 2.13).

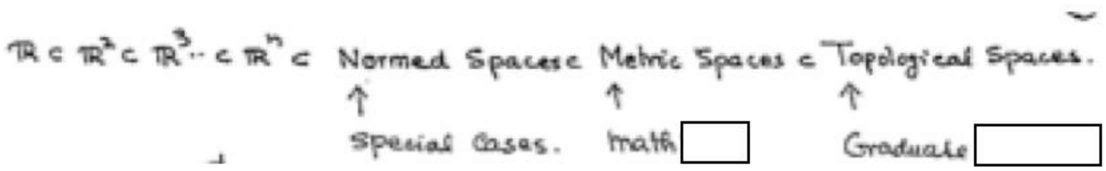


Figure 37: Inclusion sequence notation relating finite-Euclidean, Normed, Metric and Topological spaces. Each space is associated to a different course in students' undergraduate curriculum. Henry, lecture #12

In this excerpt, Henry lists different types of "spaces" featured in Analysis and uses set inclusion notation ("$\subset$ ") to specify a relationship between them. Reading left to right, the first part ("$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \ldots \subset \mathbb{R}^n$") lists an inclusion relationship between finite-dimensional Euclidean spaces; the one dimensional space $\mathbb{R}$ (the Real "number line") is a subset of the two dimensional space $\mathbb{R}^2$ (the Real plane), which is a subset of the three dimensional space $\mathbb{R}^3$ and so on.

The remaining part is what's supposed to be novel for students. It is a list of three space-types: "Normed Spaces", "Metric Spaces" and "Topological Spaces". Each category is no longer a specific space in the way "$\mathbb{R}^2$" and "$\mathbb{R}^3$" are. Rather, it is an entire family of spaces unified by their type. The inclusion relationship "Normed Spaces" $\subset$ "Metric Spaces" $\subset$ "Topological Spaces" reads (from left to right): The collection of all possible normed spaces is a subset of the collection of all possible metric spaces, which is itself a subset of all possible topological spaces. Another way of thinking about this is: any normed space is a metric space, and any metric space is a topological space. Or yet in other words, metric spaces are a generalization of normed spaces and topological spaces are a generalization of metric spaces[34].

[34] The connection $\mathbb{R}^n \subset$ "Normed Spaces" can be understood in this latter sense. Strictly speaking, $\mathbb{R}^n$ is not a subset of "Normed Spaces" (the set of all normed spaces). Rather, it is one type of normed space. A more appropriate notation would have been $\mathbb{R}^n \in$ "Normed Spaces". Yet, if we interpret the notation $\subset$ non-canonically, reading $X \subset Y$ as "Y is a generalization of X", this connection makes sense and reads: normed spaces are generalizations of finite-dimensional Euclidean spaces.

Our focus here is not, however, on how the relationship between the mathematical categories is instantiated per-se. It is the connection Henry draws to students' curricular trajectories that I would like to highlight. The arrows locate students *in relationship* to these categories. The middle arrow, linking the real analysis course catalog number (blinded in Figure 37) to the category "Metric Spaces", functions as a "you are here" indicators on this curricular map. To the right is mathematical content linked with students' potential future trajectories; Topological Spaces and graduate courses. To the left is present and past. Normed Spaces are "a special case" of interest in the real analysis course, whereas $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ are implied past as they should be familiar to students from previous courses like calculus.

What is humanizing about this? Windows and mirrors. It acknowledges students' experience and opens a window to the future. In terms of the chronotope; topics are not just "there". They are there as part of human activity. Students going through them as part of their learning trajectory.

7.4.3 Chronotope 3: Discursive hybridity

In the first two chronotopes we looked at the human experience in mathematical activity on different time scales; the micro (chronotope 1), and meso and macro (chronotope 2). Each dimension further involved an expansion of features and contexts under consideration. From a purely cognitive micro experience, to one that is also affective and multimodal. From decontextualized abstract mathematics, to one that incorporates various mathematical histories and futures. So far, we stayed within discourses that are traditionally considered mathematical, whether it is graphic-embodied interpretations, academic research, history, or students' educational trajectories. However, mathematical activity is just a small fraction of the human experience. For people to feel fully human in math classrooms, to be reflected in and affirmed by mathematical interactions, non-mathematical discourses and activities should be allowed to permeate too (R. Gutiérrez, 2018). Such a hybridization of mathematics does not only affirm students; it also expands the mathematical meaning potential to new contexts, thus enriching mathematics itself (Adiredja & Zandieh, 2020).

Looking at the textual register and many math classrooms, one may have the impression that other human activities and discourses do not exist. In written text, there will sometimes be a reference to "applications" in other STEM fields, but that is as far off from "pure" math that the text would stir from. But no activity is ever in isolation, and no discourse is ever entirely separated from other voices. Hybridity is an inherent feature of all languages (Bakhtin, 1981), and so too in math, other voices seep through. How can language used in math communication incorporate voices indexing other spheres of social activity, those typically considered outside of mathematics? To what extent are outside-of-math contexts and voices mobilized in mathematical activity? Are boundaries clearly demarcated (e.g. "now we are talking 'off topic'") or is the mathematical discourse permeable and truly diverse?

7.4.3.1 Example 3.1 – Out of math context as a metaphorical resource for math concepts

One way to hybridize the discourse in math classrooms is to use out-of-math contexts or situations to make sense of mathematical concepts. Everyday experiences are a powerful resource for making sense of mathematical ideas, and also used by experts (Sfard, 1994). Yet, it is rarely discussed out in the open. In advanced math courses such as Real Analysis, even when "informal" interpretations are allowed, the evoked context (e.g. physical-dynamical models such as moving graphs) still tends to be removed from the everyday world. The example below (Figure 38), taken from Henry's lecture notes, illustrates how the everyday activity of "looking up at the night sky" can be leveraged to make sense of the relationship between the sets of real and rational numbers.

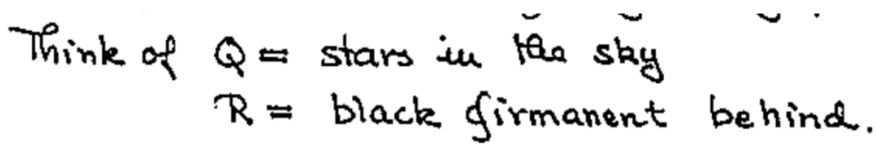


Figure 38: The night sky context as the sets of Real and Rational numbers, Henry's lecture notes

In this excerpt, Henry invites students to think about $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers) in terms of two parts of the night sky: the stars and black background "behind" them. The metaphor helps highlight important properties of these two sets: the set inclusion relationships $\mathbb{Q} \subset \mathbb{R}$, and the density of $\mathbb{Q}$ in $\mathbb{R}$. It illustrates the crucial and complex idea of different types of infinity. Of course, the metaphor has limitations. Because the stars and the black firmament are distinct entities in the metaphor, it may suggest that $\mathbb{Q}$ is not contained in $\mathbb{R}$ (which is wrong per normative usage of the terms). But metaphors are powerful not because they are absolutely correct. When mathematicians leverage metaphors as tactical tools of sense making in local contexts, they capitalize on their affordances all the while acknowledging their limitations.

It is important to exploit metaphors, not only because they buttress the mathematical sense-making of experts (Sfard, 1994), but because they can be humanizing. They provide important spaces for intersection between official mathematical texts and students' lived experience (Adiredja & Zandieh, 2020). Henry's invocation of the night sky context provides students with opportunities to connect everyday experience with mathematical ideas. A student can recall or imagine themselves looking up at the night sky, and use that rich and imaginative experience as a sensemaking resource in the math classroom. And perhaps more importantly, by explicitly referencing this metaphor in the notes, Henry does not only invite students to make sense of numbers in terms of the night sky. He implicitly sanctions the practice of leveraging everyday experiences for mathematical sense making as legitimate in math classrooms. It makes drawing on everyday experiences not only "allowed," but encouraged.

7.4.3.2 Example 3.2 – Using language resources not typically associated with math

In the previous example, we looked at how an everyday situation ("looking up at the night sky") can be evoked to make sense of mathematical ideas. Next, I would like to examine how math lectures can use various types of language resources from spheres of human activity not commonly associated with math (e.g. MEMEs, slang words, everyday forms of speech, etc.). Why is seeing social diversity in the language of mathematical activity important? This goes back to R. Gutiérrez's (2018) idea of Mirrors and to a general call for more diverse representation in mathematics. Bringing in out-of-math language forms can help youth see themselves reflected in activity. The more we diversify the language practice, the more diverse forms of self can be reflected in "official classroom discourse," the more inviting the space becomes. What I am talking about here is a very subtle effect. By using particular words, phrases, even intonations, a speaker can invoke out-of-math voices that make the activity feel a bit more humanizing. What's humanizing about it is the idea that as people doing math, we are also always embedded in other discourses, and bringing resources from those other discourses acknowledges that aspect of our humanity.

The example below (Figure 39) can serve as a small illustration of this. In the third lecture in the semester, Henry wrote a proof of the $a \geq 0$ case of the inequality $-|a| \leq a \leq |a|$ in the following way:

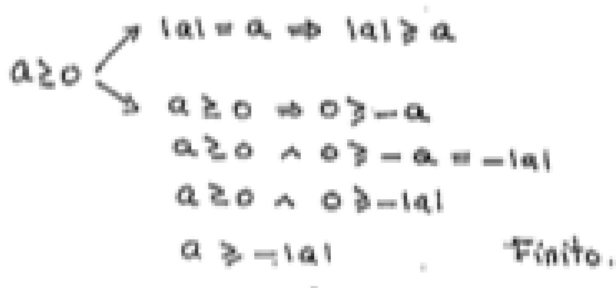


Figure 39: Lecture notes excerpt: Finito in place of Q.E.D.

What stands about this example in terms of discursive hybridity is the use of the word "finito" at the end of the proof. It is common to mark the end of mathematical proofs. Two commonly used signs to mark ends of proofs in math are: "Q.E.D.", which initializes quod erat demonstrandum (in Latin "which was to be demonstrated") and the tombstone symbol ■, imported into mathematical use by Paul Halmos. Finito is borrowed from Spanish and Italian, where it means "finished" or "finite". In English, it has a playful connotation. Why use the word "finito" instead of the more traditional ■ or Q.E.D? It lightens the mood. It signals feelings of relief. Putting such word-use in the notes can sanction this kind of blending for students too.

7.5 Discussion

In this chapter, I argued that Bakhtin's (1981, 2010) chronotopes are a useful construct for conceptualizing the discursive construction of human-math relations. I proposed three mathematical chronotopes – (1) here-and-now experience of doing math, (2) social-historical context of math activity, and (3) cultural-discursive hybridity – as concrete arenas for situating the human in mathematical discourse. I provided illustrations of small discursive moves in advanced math lectures that can be considered humanizing in terms of these three broad categories.

The proposed framework can be used to humanize advanced mathematical language in two ways. On one hand, it can help scholars and university math educators who routinely observe talk in advanced math courses notice (van Es, Hand, Agarwal, & Sandoval, 2022) humanizing discursive moves lecturers make. This includes researchers who use lecture discourse as data, practitioners who observe lectures for the purposes of peer-evaluation and feedback, and instructors themselves, who may reflect on how they themselves talk and routinely evaluate the talk and writing of students. I suggest that the moves I highlighted are helpful for humanizing, and thus critically important to note in the talk of both instructors and students. Second, I aim the framework to be helpful to instructors who want to humanize their lectures but are not sure where to start in tackling this broad and perhaps vague objective. What I offer is three dimensions to think with, and some examples of things one might try in classroom talk or writing. This is not a one size fits all recommendation of "pedagogical tricks." I provided examples of how some instructors humanize their lecture, and hope it can function as a springboard for new ideas about how advanced math discourse can be humanized.

There is no intention or presumption that this study gives a definitive answer to what "re-humanizing" math lecture discourse is or can be. Such an objective is neither feasible nor desirable (arguably, any researcher's attempt to give a definitive answer in a top-down manner is in itself dehumanizing, see e.g. Vossoughi & Gutierrez (2017)). Rather, I aimed to provide an initial conceptual framework and empirically grounded taxonomy of discursive phenomena that can be considered humanizing in advanced mathematics lectures. This objective – of descriptive detail and theoretical grounding – is motivated by the assumption that the resulting language can help practitioners talk about "teaching math in a humanizing way" in ways that will better support us to consciously and collaboratively work toward this goal. Simply agreeing to "teach math in a humanizing way", or tell others to do so, without agreeing on what that means in the micro-detail of classroom practice is not sufficiently specific as a description of practice to be actionable (Erickson, 1992).

8 Discussion

In this dissertation I set out to understand how the discourse of real analysis lectures constructs common sense ideas about mathematics that both misrepresent the discipline and alienate many students. Understanding the subtle ways by which mathematics is idealized in lectures is important, both because mundane yet pervasive features of classrooms discourse operate as hidden curriculum to exert significant influence on student experiences and opportunities to learn and belong, and because those subtle forms of idealization may be invisible to practitioners, most of whom genuinely strive to represent disciplinary practice authentically in their teaching.

I examined three features of lecture discourse: the *stories* instructors told about the purpose of the real analysis course and mathematics as a whole (chapter 5), the *value-laden attributes* they deployed to characterize and appraise mathematics (chapter 6), and the way lecture discourse enacted *human subjectivity* in the context of mathematical activity (chapter 7). To situate the stories instructors told to students in real analysis lectures, I also examined *stories* about the purpose of mathematics that prominent mathematicians articulated in famous meta-reflective essays about disciplinary practice (chapter 3).

In this final chapter, I take stock of the different findings and identify crosscutting themes. Specifically, I attempt to provide answers to the broad questions that motivated this work: how is mathematics idealized in real analysis lectures, and why? I address this question in relation to each of the analytic foci – meta-stories, valued attributed, and discursive construction of human experience – separately (section 8.1), and then synthesize by suggesting a focus on text as an underlying issue (section 8.2). I end with a short discussion of limitations (section 8.3), and brief conclusions (section 8.4)

8.1 How is mathematics idealized in lectures? Stories, values, and human agency.

8.1.1 Single purpose stories.

Arguably, no single story can ever provide a fully comprehensive and accurate picture of the messy sociological and materially distributed reality of an epistemic practice such as mathematics. Chapter 3 illustrates the existence of significant disagreements among mathematicians even about the purpose of their practice. Thus, the very fact that most real analysis instructors mobilized a single story to characterize mathematics already functions as an idealization of the discipline, in that the existence and importance of multitude of opinions and conflict is obscured.

The existence of different opinions about the nature and purpose of mathematics within the professional community is not a bug, but a feature of disciplinary practice. Historically, conflict has been a significant driver of development in mathematics and theoretical knowledge more broadly (R. Collins, 2009; Trninic et al., 2018)[35]. Thus, hiding conflict in mathematics teaching,

[35]The book "Sociology of philosophies" by Randall Collins (2009) is a massive analysis of network relationships within different philosophical (among them mathematical) traditions throughout history: ancient Greece, China, Japan, India, medieval Islamic, Jewish and Christian worlds, modern Europe up to the work of the Vienna circle.

curricula and other public displays of the discipline does not only distort *an* aspect of practice but erases its essential driving force (Apple, 2004)[36].

A way out of this kind of idealization is to center conflict, or at the very least, a multitude of perspectives in instruction. Teaching about the existence of conflicting positions does not mean teaching a relativistic cacophony of "anything goes," as the straw counter argument might say. Rather, what I suggest is the presentation of concentrated debates between a few rival position (as well as leaving space open at the seams for new debates to emerge) as the lifeblood of knowledge building practices such as mathematics.

There is reason to believe that a pluralistic position that emphasizes the existence of debate may be more appealing to students, particularly those currently alienated by mathematics. The presentation of any one story as the dogma is likely off-putting to students whose goals and values 'happen to' misalign with what's currently considered mainstream. Of course, what is presented as the dogma has everything to do with *who* has historically been in positions of power to dictate it. Paul Ernest (1995), for example, argued that the mainstream "image of mathematics" as cold, objective, and universal aligns with values traditionally associated with men. A broader mathematics, diverse in its axiological positions, can provide more pathways for identifying with the discipline. Thus, diversity in purpose-stories and values is also an issue of access.

A mathematics in which people debate goals and values also provides more room for student agency. It may be easier for students to desire to participate in mathematics if they see that the game is constantly being negotiated and that one can *join* the debate by aligning with different positions or even articulating new ones. In short, the game of mathematics may be more appealing to students who wish to exercise agency if it has room for both *playing* and *changing* the game.

How can conflict, debate and disagreement be centered in mathematics instruction? One option is deliberately tell pluralistic meta-stories[37], or at the very least, allude to the existence of debate. Meaning, rather than declare that *the* purpose of mathematics is some personally endorsed telos, tell students about different perspectives one can have on the purpose of mathematics, without assigning strong value judgements (e.g. even if one is a 'pure' mathematician, avoid positioning applied mathematics as somehow less prestigious).

Another way to center debate is through smaller scale framing moves, as illustrated by the examples of the "sociohistorical context" chronotope in chapter 7. Rather than presenting definitions as pre-determined, frame them as authorial decisions. Even better, contextualize decisions about definitions historically. For example, rather than framing one definition as rigorous and correct, and another as not, explain why mathematicians came to adopt one over the other, for what purposes, and also share contemptuous voices that disagreed. A good example is the story of Weierstrass, who, on the one hand, popularized the epsilon-delta definition of

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The main conclusion from that work is what Collins called 'the law of small numbers': a concentrated debate between a small number of rival positions is the key ingredient for significant intellectual progress.

[36] Michael Apple (2004) made a similar point about the absence of conflict in science curricula.

[37] This would be the pedagogical parallel of what Wagner (2017) called a "yes, please" philosophy of mathematics.

convergence, today framed as the gold standard of rigor in analysis, but, on the other hand, opposed the abstract definition of functions we now treat as baseline (Ferreirós, 2008). Examples like this from the history of mathematics illustrate that mathematical practice is extremely multi-faceted, and that any single dividing line (e.g. rigorous or not-rigorous) misses a diversity of positions.

8.1.2 Unfaithful meta-stories

The meta-stories identified in chapter 5 were idealizing not just in that they offered a singular and hence incomplete perspective. Even when considered independently, each of the stories unfaithful in that it constructed and presupposed *causal dependencies* (e.g. to do calculus, we *need* a precise definition of real numbers, or, to fix calculus, we *need* real analysis) with no clear parallels in current or historical professional practice, as well as *subject positions* with desires, needs, and experiences (e.g. we desire precision, we desire simplicity, calculus breaks) not obviously shared by practitioners. Furthermore, neither the casual dependencies, nor the subject positions constitutive of the identified meta-stories are accessible and relatable to students given their likely past experiences in the discipline (i.e. as students in calculus courses).

In the discussion of chapter 5, I speculated that *these* (and not other) stories were deployed by instructors because they are the ones most easily accessible (e.g. from textbooks, meta-mathematical discourse) and serve the rhetorical functions at hand. Furthermore, the problematic presuppositions constituting each of the stories may seem unproblematic because the *metaphors* underlying each of the stories render them self-evidently true (e.g. if mathematics is a building, of course it needs to be solid). Both of these suggest that to move away from one-dimensional idealizing narratives about practice, there is a need for instructors (and mathematics education research) to critically examine the broadly available narratives about real analysis and mathematics more generally. This process amounts to teacher-learning of *political knowledge for teaching mathematics* (R. Gutiérrez et al., 2023).

To render the stories more accessible and compelling to students, it may be productive to first provide students with opportunities to experience the needs and desires constitutive of the story themselves. That is, I concur with Dawkins and Weber (2017) that a "show, don't tell" approach is likely more effective in conveying disciplinary axiology. In the context of the stories identified here, this may mean that instead of telling students that calculus breaks, have them explore a mathematical situation in which it does. Or, instead of telling students that they should desire precision, have them experience intellectual need for it. Dawkins (2018) has been developing and testing a "show, don't tell" approach for teaching axiomatizing in geometry courses. Designing and studying similar pedagogical experiences in the real analysis context is a useful direction for future research.

A "show, don't tell" strategy may require a significant reconfiguration of the real analysis curriculum. For example, starting with the contexts in which motivating problems emerge (approximation techniques), then closely examining the problematic situations (trigonometric series convergence), and only then introducing different aspects of rigor (systematic reliance on definitions of convergence, the epsilon-delta language) as *means to an end.* While such a trajectory would time consuming, it may be worth it for conveying this meta-mathematical message. Because, when rigor is not positioned as means to an end, it becomes an end in and of itself. This may then give rise to the problematic mainstream narrative that the purpose of mathematics is to produce valid theorems and proofs (chapter 3).

8.1.3 Value-laden attributes

Another mechanism of idealization explored in this dissertation is value-laden attributions. Value laden attributes such as *precision*, *validity* and *simplicity* featured heavily in the meta-stories instructors told. As mentioned above, the desire to see such virtues such as precision and simplicity realized was a central characteristic of the subject positions constituting the stories. However, framing certain characteristics as desirable in mathematics (or not), was not limited to the meta stories alone. Instructors used different attribute words to characterize mathematical objects, processes and experiences throughout the lecture. Values conveyed in subtle mundane ways may have an even greater effect on construing an idealized image of the discipline, as they are so pervasive (Herbel-Eisenmann et al., 2010).

Looking at the value-laden attributions across the entirety of the instructors' first lectures, I identified several patterns that could be contributing to unproductive idealizations of the discipline. First, different attributes were mentioned more and less frequently. Across the four instructors, precision, validity, and difficulty were most mentioned, closely followed by the knowledge organizing characteristics of centrality and normativity. Mentioned, but much less frequently, were utility, interest, and aesthetic appreciation. These patterns construct an image of mathematics at odds with what many mathematicians say is important to them. And, such an image may also be unappealing to students who value utility, interest and aesthetics more than precision and challenge.

Another idealization of the discipline identified through a close look at value-attributions in the lectures was negative framings of vagueness and ambiguity. Mathematical practice clearly features instances in which ambiguity is perceived negatively. Historically, vagueness and ambiguity have led to what we today consider mistakes. However, consistently framing ambiguity only in negative light functions as an idealization of disciplinary practice, because it obscures the productive and generative role ambiguity has historically served in mathematical development (Trninic et al., 2018). It also constitutes an access barrier for students (e.g. by discouraging them from sharing vague ideas), which is particularly problematic considering that vagueness and ambiguity can be productive resources for learning as well (Barwell, 2005).

Taken together these two findings suggests that a conscious effort to increase the relevance of utility, interest, aesthetic, and ambiguity in real analysis instruction may be a productive means for increasing participation in mathematics. An important question, to be taken up in future design work, is how. That is, future research can address questions such as: what needs to be reconfigured in real analysis instruction so that utility, interest and aesthetics would emerge as more relevant dimensions to comments on? How can precision be celebrated without dismissing and devaluing ambiguity?

8.1.4 Obfuscation of human experience.

In chapter 7, I tackled yet another mechanism of idealization in lectures: the discursive obfuscation of mathematical subjectivity. Here, rather than documenting the linguistic mechanisms by which human agency is obscured in mathematical discourse – a phenomena that already received significant attention in the literature (see review in section 2.2.4.3) – I focused on positive possibilities. That is, I identified and characterized the discursive means by which instructors *humanized* the language of mathematics, namely, enacted aspects of human experience not typically reflected in the textual register. I argued that Bakhtin's (1981, 2010) construct of literary *chronotope* (discursive space-time configuration) is a useful analytic and conceptual tool for distinguishing different ways of enacting human subjectivity in mathematical discourse, and proposed a framework of three chronotopes for humanizing the language of mathematics: the here-and-now experience of doing mathematics, the socio-historical context of mathematical activity, and discursive hybridity.

While the focus of chapter 7 has been on how discursive obfuscation of human agency can be resisted, the chronotopic framework also sheds light on how and why human agency is obscured to begin with. The different chronotopes help highlight how what I called *humanizing moves* perturb dominant epistemologies of mathematics. For, if mathematics is enacted in ways that bound it to the human body and its affective experiences (chronotope 1), it become more difficult to perceive mathematics as transcendental and abstract. When discourse situates mathematics in socio-historical context (chronotope 2), it can no longer be seen as timeless, universal and disinterested. If mathematics is explicitly integrated with other discourses and contexts (chronotope 3), it is no longer pure or unique. In this way, we can understand the epistemological ideals typically associated with mathematics as *discursive accomplishments*. By omitting non-cognitive here-and-now experiences, references to socio-historical context, and any allusions to other spheres of human activity, mathematical discourse construct itself as transcendental, universal and pure.

The examples in chapter 7 showcase one way to push against traditional idealizations of mathematics: through small humanizing moves in lectures. While these amount to small-scale discursive shifts, I hypothesize that their cumulative effect on the presented ‘image of mathematics’ (Ernest, 1995) is non-negligible. To put it more explicitly, I hypothesize that lectures that feature more humanizing moves (both overall, and in diversity of kinds), are experienced as more welcoming and affirming by students.

This suggests many avenues for future empirical work. A first step may be to conduct interview studies with students, either stimulated recall of humanizing moves in lectures they experienced directly, or gauge their general reactions and interpretation of different kinds of *humanizing moves*. On a larger scale, the chronotope framework can be used for comparative analyses of lectures aimed at characterizing how and to what extent different lectures feature humanizing moves. The resulting ‘humanizing profiles’ could be then examined in relation to different probes and metrics of student experience, including interview reflections, survey responses, dropout rates etc. At a more mature stage of such a research program, one could even imagine systematically manipulating humanizing features in the discourse, and examining different effects.

As a design tool, the chronotope framework can guide deliberate use of humanizing discursive moves in instruction. For example, as an instructor, I can try to consciously infuse my classroom talk, writing and movement with references to here-and-now experiences, sociohistorical context, and extra-mathematical discourses. Or, the framework could be used to mediate design efforts at a larger scale. What kinds of classroom tasks, participation structures, and curricula center the here-and-now experience, socio-historical context, and discursive hybridity? Considered independently, such design questions have of course been thoroughly explored in various research traditions (e.g. problem-solving, embodied cognition, culturally sustaining pedagogy). What the humanizing framework offers is an integrated view that could facilitate the coordination of different approaches. For example, one can use tasks that leverage bodily and aesthetic experience for learning mathematical concepts (de Freitas & Sinclair, 2012; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012), *and* tasks that integrate the history of mathematics in curriculum (Chorlay, Clark, & Tzanakis, 2022), *and* explicitly connect traditional curricula with local communities' funds of knowledge (González, Andrade, Civil, & Moll, 2001).

8.2 Contemporary academic mathematics: toward a humanized textual practice

8.2.1 A focus on text

A focus on text permeates the discourse of contemporary academic mathematics and RA lectures. In activity theoretic terms (Cole & Engeström, 1993; Wertsch, 2012), text is both the mediational means of mathematical practice and, in contemporary academic mathematics, also its *object*: the idealized collective thing that mathematicians are crafting. This is reflected in the mainstream stories told about the discipline (chapter 3), which frame the production of knowledge in the form of texts as the main purpose of mathematics. A prioritization of text also permeates the stories instructors told about real analysis in their lectures (chapter 5), and the values they emphasized (chapter 6). The humanizing framework suggests ways of moving beyond the text, but also highlights the absence of important dimensions of human experience in the dominant textual register (chapter 7). As I have argued in the previous section, such a focus on text functions as an idealization of disciplinary practice.

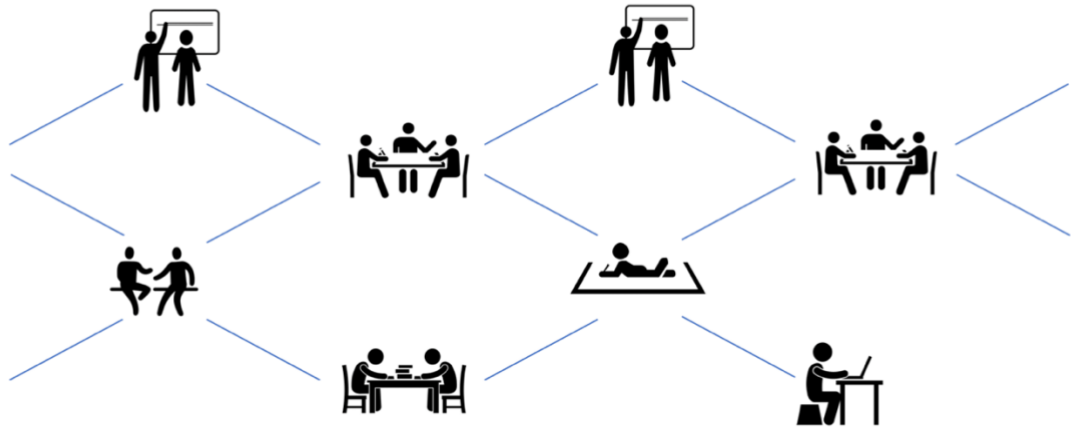
The centrality of text in real analysis lectures can be explained by the fact that in this context, instructors introduce students to the Definition-Theorem-Proof epistemic game (section 2.1.2). To explain the DTP epistemic game, one has to enact an epistemic perspective that makes the text work (Rotman, 1988). In deductive mathematical texts, this is an idealized epistemic agency that "doesn't know" what real numbers are before they are explicitly defined, and "doesn't believe" claims until they are proven. This agency is not the same as the actual human beings writing the text, who have knowledge of what real numbers are before they define them (otherwise, where do the definitions come from?). In mathematics, this is one difference between ready-made science and science in the making (Latour, 1987). Just like scientific articles construct an artificial never realized in practice chronology of research-questions, methods, findings, conclusions, mathematical text construct an artificial epistemic perspective, not literally experienced by anyone, but temporarily adopted by everyone in order to reach consensus (Wagner, 2022). This is the DTP language game.

The problem is that lecturers talk to actual students, and do so without making it explicitly clear that they are discursively constructing an imaginary perspective. When they make epistemic claims about knowledge states the referent is, at best, ambiguous. At worst, it imposes on students epistemic experiences they might not share ("you have a vague idea", "you don't know what real numbers are") or explicitly asks them to pretend ("pretend like you don't believe it"). Instructors' enactment of a textual epistemic agency vis-à-vis students' literal interpretations of talk as claims about *them* could explain why pervasive features of lecture discourse ("this is basic" "we all know this") function as unintentional micro-aggressions: instructors are talking about an idealized text, students are listening for clues about their possible not belonging (Leyva et al., 2021; Su, 2015).

Mathematicians' focus on text is not surprising given their literal perspective on practice. To see this, let's engage in a phenomenological exercise. If you are a mathematician doing mathematics, what is in your immediate perception? Most likely, a text. Either in the form of printed text, organized deductively, or various kinds of scribbles. As you read/write your text you may experience it as having referents, which you perceive as a reality external to the text – what the text is *about*. You may also be reflectively aware that as part of the process of reading/writing *you* are thinking and imagining things. As Rotman (1988) argued, these three dimensions of a mathematical situations map onto Pierce's tripartite model of semiotics – the text (signifier), the referent (signified) and interpretation (interpreter) – and, when *considered independently*, correspond to traditional philosophies of mathematics (formalism, platonism and intuitionism).

Now, imagine that you are a social scientist studying mathematical practice by, say, observing mathematicians do their work. You may even be pointing a video-camera at them. What you see and focus on is different from what mathematicians are seeing. You see *people*, sometimes alone sometimes in groups, using their entire bodies, and having a full range of affective experiences while reading/writing their texts. You see activity that is distributed across individuals, tools, media, and bodies, and imbued with affect, value judgements, and interpersonal positionings. Once you look from the outside, by literally shifting the position of your analytic 'camera' (R. Hall, 2000), it is easier to notice that mathematical activity involves not just deductive textual systems, free floating cognition and platonic referents, but material reality and full-body affective experience too. In short, you see a multi-dimensional here-and-now.

And, perhaps depending on your research tradition, you may zoom out to consider a larger space-time landscape. Any observable in-the-moment mathematical experience is situated in a larger context of practice: it is part of an interaction ritual chain (R. Collins, 2004), with a past, present and future. Considering these dimensions of mathematical practice, you may realize that mathematical activity, even when done "alone," is always inherently dialogic – it consists of communicational acts in response to and in anticipation to the reaction of others (Bakhtin, 1981; Sfard, 2008). Whether you caught your mathematician working alone or in collaboration, it no longer makes sense to explain the practice in terms that only consider individual subjectivity, even if that individual is defined expansively (e.g. as having a full body, emotions and tools).



Zooming-out to consider synchronic and diachronic dimensions of practice opens up a host of other considerations. We may note that contemporary mathematics has a history, that in other times and places different kinds of mathematics were practiced. We may realize that we see and experience today as mainstream has not always been so; that changes in mathematics were shaped by problems of practice, people's decisions, preferences, values, debates, power struggles, and various institutional pressures (Archibald, 2008; R. Collins, 2009; Kitcher, 1984; Kleiner, 1991; Lakatos, 1976). Furthermore, the fact that there is a dominant or mainstream practice at any given epoch does not mean that other practices do not co-exist, even within the same individual (Ferreirós, 2015).

And finally, we may zoom out even further, and see that the landscape of mathematical practice is never isolated. What mathematics is done and how has always been influenced by other discourses and activity systems. Work in the history of mathematics has, for example, traced the emergence of deduction in ancient Greece to aspects of its political culture (Novaes, 2020), and examined how styles of mathematical writing were influenced by contemporaneous literary aesthetics (Netz, 2009). People cross boundaries. As they enter mathematics they do not leave their experience of other contexts, their repertoires of practice (K. D. Gutiérrez & Rogoff, 2003), or their identities "at the door" (R. Gutiérrez, 2018).

These three zooming-out exercises correspond, of course, to the three chrontopes suggested in chapter 7. The dimensions also point toward potential ways of humanizing textual practice, without ignoring or eliminating it altogether.

8.2.2 Reclaiming rather than decentering text.

A focus on text, I have argued, both distorts and alienates. One way to overcome this is to get rid of text altogether. Such an attitude is evident in some reform rhetoric: to center student agency and active participation, get rid of the text. However, getting rid of the text in the context of academic mathematics amounts to getting rid of the main mediator and object of epistemic activity. No wonder that mathematicians look at reform efforts and think "this is not math" (Sfard, 1998).

Instead, what I would like to suggest as a potential way out, and an agenda for future research and development, is text-centered pedagogy that engages with the DTP textual practice and *humanizes* it by situating it in the space-time scales suggested in the chronotope framework. This could take the form, for example, of a mathematical critical literacy class, in which: (1) diverse here-and-now experienced of text interpretation are examined, (2) norms of mathematical writing are interrogated as culturally and historically contingent, and (3) influences of other discourses on mathematical writing are analyzed. A critical literacy approach could open new pathways into mathematics, not currently made available by the gatekeeping rigor-first approach to undergraduate mathematics education and real analysis courses.

8.3 Limitations

My use of the zooming out metaphor in section 8.2.1 may unintentionally suggest that there is a position from which one can finally see everything, an objective view from nowhere, a god's eye. I do not wish to imply that. I simply want to highlight that different perspectives offer new insights about the nature of practice that should be taken into account when enacting an *image of mathematics* in classrooms. Personally, I am content with a fragmented view of the 'reality' of mathematical practice; we increase our understanding by projecting it on different planes of analytic 'vision.'

One perspective that is glaringly absent in my three-chronotope account, for example, is considerations of place. The chronotopes I identified, while spanning different time scales and activities, are not situated in any specific physical reality. They could be happening anywhere. But mathematics always happens *somewhere*. What affect does place have? How has place contributed to shaping mathematical practice, and how has mathematics shaped place?

This omission may be partially explained by a blind-spot rooted in my own positionality and life experiences. My perspective, shaped by life-long immersion in Western tradition of thought and multi-generational experiences of immigration, exhibits deep detachment to place. There is much to be learnt, for example, from indigenous forms of knowledge, in which space and time are connected in ways not reflected in the space-ambivalent chronotopic model I proposed.

8.4 Conclusions

Because mathematics carries high social status, and serves as gatekeeper to professional and financial opportunities, any articulation of mathematics – any instance of identifying an activity as mathematical – carries an ethical charger (Wagner, 2023). What we enact and label as mathematics affects *what* would be seen by others as mathematical (and hence high status), and *who* can consider themselves and be considered by others as mathematical. This is especially true for mathematics enacted in educational contexts, and even more so when the instructors are mathematicians who hold unique authority to tell others what mathematics is.

In this dissertation I examined different discursive mechanisms by which mathematical practice is unintentionally idealized in real analysis, and speculated on the potential effects of such idealizations on student experience. I found that disciplinary practice is obfuscated through the stories told about mathematics, the characteristics repetitively valorized, and the way human subjectivity is enacted in discourse. These all reflect an emphasis on rigor and text, that can be transcended by considering mathematics as a textual, *human* practice.

9 References

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