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# The influence of circulation on the stability of vortices to mode-one disturbances

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The initial value problem for the two-dimensional inviscid vorticity equation, linearized about an azimuthal basic velocity field with monotonic angular velocity, is solved exactly for mode-one disturbances. The solution behaviour is investigated for large time using asymptotic methods. The circulation of the basic state is found to govern the ultimate fate of the disturbance: for basic state vorticity distributions with non-zero circulation, the perturbation tends to the steady solution first mentioned in Michalke & Timme (1967), while for zero circulation, the perturbation grows without bound. The latter case has potentially important implications for the stability of isolated eddies in geophysics.

## 1. Introduction

The classical theory of flow instability has been of central importance in fluid dynamics for over a century. The problem of the instability of two-dimensional inviscid flow in simple geometries has naturally received much attention. The linearized vorticity equation governing the evolution of perturbations to a shear flow is Rayleigh's equation, and important conditions for stability, linked with the names of Rayleigh and others, have been derived from it. However, it was realised during the second half of this century that the classical approach of normal mode instability fails to completely describe the possible linear evolution of perturbations in unbounded inviscid flows. The continuous spectrum must also be taken into account. Maslowe (1985) gives a good overview of the ideas involved in the case of shear flows.

Rayleigh also derived a condition for the linear instability of a purely azimuthal flow, namely that the basic state vorticity should have an extremum somewhere in the flow. More recently, Michalke & Timme (1967) examined the stability of circular vortices produced by boundary layer breakdown. There has also been a large body of related work in dynamical meteorology and oceanography, since observed structures in the atmosphere and oceans, such as tropical cyclones and warm-core rings, can be modelled by quasi two-dimensional vortices (see e.g., Flierl 1987, Hopfinger & van Heijst 1993). Their properties are naturally of great interest.

Rayleigh's instability criterion for an azimuthal basic state is a condition for the existence of a growing normal mode perturbation. The related continuous spectrum seems to have attracted very little attention. For example, a well-known work on hydrodynamic instability, Drazin & Reid (1981), derives Rayleigh's condition for swirling flows, and mentions that a proper initial value treatment, taking the continuous spectrum into account, could be developed, but stops there. This paper completely solves the initial value problem of the linear evolution of a mode-one perturbation. Previous results have been restricted to the normal modes of the system (see e.g., Gent & McWilliams 1986). The solution is found in terms of a single integral whose large-time behaviour is studied; the latter depends crucially on the circulation of the basic state.

## 2. The initial value problem linearized about an azimuthal basic state

The two-dimensional motion of an incompressible fluid is governed by the principle of conservation of vorticity of the flow. Small disturbances to the basic flow may be investigated by expanding the equation about the basic state. The linearized vorticity equation for a perturbation $\psi(r,\theta)$ around a basic state corresponding to purely swirling motion may be written in the form

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta}\right)\nabla^2\psi-\frac{Q'}{r}\frac{\partial\psi}{\partial\theta}=0, \tag{2.1}$$

where $Q$ is the vorticity of the basic flow, and $\Omega$ is the angular velocity of the basic flow. These are related to the basic flow streamfunction $\Psi(r)$ by $Q=(r\Psi')'/r$ and $\Omega=\Psi'/r$ (using the geophysical convention that vorticity is the positive Laplacian of the streamfunction). This is Rayleigh's equation in radial geometry. The boundary condition on the streamfunction $\psi$ is that $|\nabla\psi|$ vanish at infinity. There is also a regularity condition that $|\nabla\psi|$ be non-singular at the origin.

Equation (2.1) is linear, and the coefficients depend only on $r$. The perturbation streamfunction may hence be decomposed into radial modes $\psi_n$ proportional to $e^{in\theta}$. The equation for mode $n$ is

$$\left(\frac{\partial}{\partial t}+\mathrm{i}n\Omega\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_n}{\partial r}\right)-\frac{n^2}{r^2}\psi_n\right]-\mathrm{i}n\frac{Q'}{r}\psi_n=0. \tag{2.2}$$

The solution for mode zero is merely $\psi_0=\psi_0^i$, the initial mode zero streamfunction perturbation. Any radial disturbance is just advected around by the basic flow, and could be trivially incorporated into the basic flow by altering $\Psi$. Hence mode zero is dynamically insignificant.

Laplace transforming (2.2) in time leads to

$$-(r\phi_n')'+\left[\frac{n^2}{r}+\frac{\mathrm{i}nQ'}{p+\mathrm{i}n\Omega}\right]\phi_n=-\frac{rq_n^i}{p+\mathrm{i}n\Omega}, \tag{2.3}$$

where $q_n^i$ is the initial value of $(r\psi_n^{i}{}')'/r-n^2\psi_n^i/r^2$, i.e. the initial perturbation vorticity of mode $n$. The Laplace transform of the streamfunction mode $\psi_n$ is denoted by $\phi_n$. The solution for mode $n$ may formally be derived by Green's functions, as in Case (1960); this leads to a consistent solution to the initial value problem, incorporating the continuous spectrum. In the parallel-flow case, with lateral boundaries, considered by Case, the underlying Sturm–Liouville problem is regular, and a Green's function can always be constructed formally. In the geometry considered here however, the interval is infinite, and the underlying Sturm–Liouville problem is no longer regular.

For definiteness, the basic state vorticity will be taken to be bounded in magnitude, and localised in space (i.e. of compact support, or decaying faster than any power of $r$); the physical domain under consideration will be taken to be unbounded. The basic state angular velocity will be taken to be monotonic decreasing and positive. This corresponds to an initial circular vortex monopole. Consequently, the far-field angular velocity will be of the form

$$\Omega = \frac{\Gamma}{2\pi r^2} + O(r^{-\infty}), \tag{2.4}$$

where $\Gamma$ is the circulation, which will be taken positive (a derivation of this result is presented in Appendix A). The order infinity notation denotes a contribution that decays faster than any power of $r$ ($O$ could be replaced by $o$ in this notation). The basic state vorticity will also be taken to be piecewise continuous or smoother, which permits distributions such as the Rankine vortex. The initial perturbation vorticity will be taken to be localised and continuous.

## 3. The mode-one solution

Equation (2.2) has actually been solved for mode one in the context of plasma physics by Smith & Rosenbluth (1990)†. However, in their work, the equation is considered on a finite circular domain. As a result, the solution must either decay for large time, or have algebraic growth of order $t^{1/2}$ when the angular velocity has extrema. The first possibility is no longer true on an infinite domain, while the second is not relevant to monopolar vorticity distributions.

The mode-one solution to a forced version of (2.2) has been calculated in Reznik & Dewar (1994); the equation was solved with zero initial condition, and with forcing due to the beta-effect. The Green's function is the same for that problem and for the stability problem considered here, but the physical situation and properties of the solutions are quite different. It was also shown by Reznik and Dewar that there were no normal modes for mode one; this depends only on the linear differential operator, and holds for both cases. A derivation of the reduction to quadratures following Smith & Rosenbluth is given here, and the solution to the initial value problem is calculated to understand the stability properties of the system.

For mode one, Michalke & Timme (1967) showed that the function $r\Omega$ was a steady solution and mentioned that it did not appear to be the vanishing growth rate limit of any other normal mode solution. This steady solution was later shown in Gent & McWilliams (1986) to be the only non-singular solution of the eigenvalue problem for mode one. This form of this solution suggests the substitution $\phi_1 = r(p+\mathrm{i}\Omega)f$ in the governing equation. This leads to the equation

$$\frac{1}{r(p+\mathrm{i}\Omega)}\left[r^3(p+\mathrm{i}\Omega)^2 f'\right]' = \frac{r q_1^i}{p+\mathrm{i}\Omega}, \tag{3.1}$$

† The work in this paper was carried out before the author became aware of the results of Smith & Rosenbluth.

whose solution is

$$f = -\int_r^\infty \frac{\mathrm{d}v}{v^3(p+\mathrm{i}\Omega(v))^2}\int_0^v u^2 q_1^i(u)\,\mathrm{d}u + A(p) + B(p)\int_r^\infty \frac{\mathrm{d}v}{v^3(p+\mathrm{i}\Omega(v))^2}. \quad (3.2)$$

The boundary condition at infinity may be rewritten as $r(p+\mathrm{i}\Omega)f \to 0$; this is why the upper limit for the outer integral has been taken to be infinity. This boundary condition must be satisfied for all $p$, so $A$ must be zero. Near the origin, the inner integral behaves like $v^3$, since the initial perturbation vorticity is bounded near the origin (and everywhere in fact). Hence $B$ must be zero, otherwise $f$ would behave like $r^{-2}$, which would be unacceptable. This leads to the solution

$$\phi_1 = -r(p+\mathrm{i}\Omega(r))\int_r^\infty \frac{m(v)}{(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v, \quad (3.3)$$

where

$$m(v) = \frac{1}{v^3}\int_0^v u^2 q_1^i(u)\,\mathrm{d}u = \frac{\partial}{\partial v}\left(\frac{\psi_1^i(v)}{v}\right). \quad (3.4)$$

The inverse Laplace transform of (3.3) leads to

$$\psi_1 = -r\int_r^\infty [1 + t(-\mathrm{i}\Omega(v) + \mathrm{i}\Omega(r))]\, m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v \quad (3.5)$$

$$= -r\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right) t\int_r^\infty m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v \quad (3.6)$$

$$= \psi_1^i(r)e^{-\mathrm{i}\Omega(r)t} - \mathrm{i}r\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right) t^2\int_r^\infty \frac{\psi_1^i(v)}{v}\Omega'(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v \quad (3.7)$$

The solution clearly satisfies the initial condition. The special case $m = \Omega'$ leads to the solution $\psi_1 = r\Omega$. This is natural, since that choice of $m$ corresponds to taking $r\Omega$ as the initial condition, and this is known to be a steady solution.

The corresponding expression for the vorticity may be found from (2.3), using the fact that $\exp(-\mathrm{i}\Omega t)$ is annihilated by the differential operator $\partial_t + \mathrm{i}\Omega$. Hence

$$q_1 = -\mathrm{i}Q't\int_r^\infty m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v + q_1^i(r)\mathrm{e}^{-\mathrm{i}\Omega(r)t}. \quad (3.8)$$

The vorticity is always localised in space, and the far-field behaviour of $\psi_1$ is still as in Appendix A.

## 4. Large time asymptotic behaviour

The full solution may be examined in the limit of large $t$. It is useful to rewrite the integral in (3.6) as

$$I = \int_0^{1/r} m\left(\frac{1}{s}\right)\mathrm{e}^{-\mathrm{i}\Omega(1/s)t}\,\frac{\mathrm{d}s}{s^2}. \quad (4.1)$$

The limit of large time corresponds to very rapid oscillation in the exponential function, while $m$ varies comparatively slowly. The method of steepest descents may be used to obtain an asymptotic approximation to the integral, but care needs to be taken, since $\Omega$ may not have simple analytic behaviour at critical points. The ultimate behaviour of $I$ will depend on the circulation of the basic flow, since that quantity determines the form of the flow at infinity, as the analysis will show.

### (*a*) *Non-zero circulation*

For basic states whose vorticity vanishes beyond a certain radius $r_M$, the angular velocity beyond that distance is $\Gamma/2\pi r^2$. The angular velocity and $m$ must both be continuous at this radius, although the vorticity may have a jump there (as in the case of the Rankine vortex). Then $I$ may be rewritten as

$$I = \int_0^{1/r_P} m\left(\frac{1}{s}\right) \mathrm{e}^{-\mathrm{i}\Gamma s^2 t/2\pi} \frac{\mathrm{d}s}{s^2} + \int_{1/r_P}^{1/r} m\left(\frac{1}{s}\right) \mathrm{e}^{-\mathrm{i}\Omega(1/s)t} \frac{\mathrm{d}s}{s^2}, \tag{4.2}$$

when $r_P$ is the maximum of $r$ and $r_M$. If $r_M < r$, the second integral will be zero.

The first integral may be evaluated as an asymptotic series by the method of steepest descents, by treating $s = x + \mathrm{i}y$ as a complex variable. The path is transformed onto the steepest descent contour starting from the origin, going to $e^{-\mathrm{i}\pi/4}\infty$, and returning along the hyperbola $x^2 - y^2 = 1/r_P^2$. The behaviour of $m$ is only required near the origin and the point $1/r_P$; the former is given in Appendix A. Parametrising the contour from the origin by

$$s = (2\pi/\Gamma)^{1/2} e^{-\mathrm{i}\pi/4} u^{1/2} + O(u^{3/2}) \tag{4.3}$$

(where $u$ is real, since it describes a constant phase contour) leads to a contribution

$$I_1^0 = -\frac{2\mathrm{i}\pi M}{\Gamma} \int_0^\infty \mathrm{e}^{-ut} \left[u^{3/2} + O(u^\infty)\right] \frac{u^{-1/2}\mathrm{d}u/2 + O(u^{1/2})\mathrm{d}u}{u + O(u^2)} \tag{4.4}$$

$$= -\frac{\mathrm{i}\pi M}{\Gamma t} + O\left(\frac{1}{t^2}\right). \tag{4.5}$$

The notation $\Omega'(r_P^+)$ will be used to highlight the discontinuity in the slope of $\Omega$ at $r_P$. The second contour may be locally parametrised near the point $s = 1/r_P$ by

$$s = \frac{1}{r_P} + \frac{\mathrm{i}u}{r_P^2 \Omega'(r_P^+)} + O(u^2). \tag{4.6}$$

The contribution from this point to the first integral is then

$$I_1^1 = \mathrm{e}^{-\mathrm{i}\Omega(r_P)t} \int_\infty^0 \mathrm{e}^{-ut} \left[r_P^2 m(r_P) + O(u)\right] \left[\frac{\mathrm{i}}{r_P^2 \Omega'(r_P^+)} + O(u)\right] \mathrm{d}u \tag{4.7}$$

$$= -\frac{\mathrm{i}m(r_P)}{\Omega'(r_P^+)t} \mathrm{e}^{-\mathrm{i}\Omega(r_P)t} + O\left(\frac{1}{t^2}\right). \tag{4.8}$$

The second integral has no stationary points, and can be evaluated asymptotically by repeated integration by parts. This is more easily done in the original variable. Then

$$I_2 = \int_r^{r_P} m(v) \mathrm{e}^{-\mathrm{i}\Omega(v)t} \, \mathrm{d}v \tag{4.9}$$

$$= \frac{\mathrm{i}}{t}\left[\frac{m(r_P)\mathrm{e}^{-\mathrm{i}\Omega(r_P)t}}{\Omega'(r_P^-)} - \frac{m(r)\mathrm{e}^{-\mathrm{i}\Omega(r)t}}{\Omega'(r)}\right] - \frac{\mathrm{i}}{t}\int_r^{r_P}\left(\frac{m(v)}{\Omega'(v)}\right)' \mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v. \quad (4.10)$$

This integration by parts can be repeated to give an asymptotic series in $1/t$. There are now two cases, depending on whether $\Omega$ is smooth at $r_M$. If $\Omega$ is smooth there, i.e. if $\Omega'(r_M) = -\Gamma/\pi r_M^3$, then all the terms in $r_P$ will cancel with the equivalent terms in $I_1$, and the leading order behaviour of $I$ is given by

$$I = -\frac{\mathrm{i}\pi M}{\Gamma t} - \frac{\mathrm{i}m(r)}{\Omega'(r)t}\mathrm{e}^{-\mathrm{i}\Omega(r)t} + O\left(\frac{1}{t^2}\right). \quad (4.11)$$

This is also the solution when the second integral is not present, and $r_P = r$. The second term is annihilated by the $\partial_t + \mathrm{i}\Omega$ operator in front of the integral, so the large time behaviour of the solution is given by

$$\psi_1 = -\frac{\pi M}{\Gamma} r\Omega(r) + O\left(\frac{1}{t}\right). \quad (4.12)$$

Thus any smooth initial distribution with compact vorticity support, and non-zero circulation, tends to the steady solution found by Michalke and Timme.

If $\Omega$ is not smooth, then the exponential terms in $r_P$ will not cancel, and the leading behaviour will be given by

$$\psi_1 = -\frac{\pi M}{\Gamma} r\Omega(r) - r(\Omega(r) - \Omega(r_P))\mathrm{e}^{-\mathrm{i}\Omega(r_P)t} m(r_P)\left[\frac{1}{\Omega'(r_P^+)} - \frac{1}{\Omega'(r_P^-)}\right] + O\left(\frac{1}{t}\right). \quad (4.13)$$

Any localised, discontinuous basic state vorticity distribution will therefore tend to the steady solution in the region with zero vorticity. However, there will be an oscillatory term in the region inside the discontinuity. It is clear that this argument can be extended to any basic state profile with jumps in $\Omega'$ by decomposing the integral into suitable portions. Each point of discontinuity will give an oscillating contribution exactly as above. There will be no oscillatory contribution beyond the point of discontinuity with the largest radius.

The preceding argument may be generalised to cover the case of any basic state with non-zero circulation. Decomposing the integral (4.1) into sub-ranges, and using appropriate order relations leads to exactly the same result for continuous vorticity distributions, albeit with a slightly weaker order condition:

$$\psi_1 = -\frac{\pi M}{\Gamma} r\Omega(r) + o(1). \quad (4.14)$$

Discontinuities in the basic state vorticity will force oscillatory terms as before.

### (*b*) *Zero circulation*

For basic states with zero circulation, the Taylor series of the term $\mathrm{i}\Omega(1/s)$ in the exponential vanishes at the origin (i.e. its righthand derivatives all vanish at that point). This corresponds to an essential singularity in the complex $s$-plane. The method of steepest descents is no longer valid, and a different procedure must be followed to obtain the large time behaviour. The Riemann–Lebesgue lemma shows that $I(t)$ must vanish for large $t$. The failure of steepest descents shows that the decay must be slower than any inverse power of $t$, hence $tI(t)$ must be larger than $O(1)$ in the large $t$ limit. This means that the solution grows without bound, albeit slower than $t$, since $I(t)$ decays in time.

As an example, consider the Gaussian streamfunction $\Psi = -\exp(-r^2)/2$, which has zero circulation. The associated basic state angular velocity is $\Omega = \exp(-r^2)$, so the integral to be evaluated is

$$I = \int_0^{1/r} \exp\left(-\mathrm{i}e^{-1/s^2}t\right) \frac{m(1/s)}{s^2}\,\mathrm{d}s. \tag{4.15}$$

This is very similar to an integral treated by Bender & Orszag (1978, Sec. 6.6 Example 3). The change of variable $v = \mathrm{i}\exp(-1/s^2)$ leads to the integral

$$I = \int_0^{\mathrm{i}\exp(-r^2)} \mathrm{e}^{-vt} \frac{m(1/s)}{s^2} \frac{ds}{dv}\,\mathrm{d}v, \tag{4.16}$$

where $s$ is a function of $v$. The contours of stationary phase in the $v$-plane are the real axis, and the line $v = x + \mathrm{i}\exp(-r^2)(0 \leqslant x < \infty)$. The contribution from the point $\mathrm{i}\exp(-r^2)$ is exponentially small compared with that from the origin, so the leading order behaviour is obtained by considering

$$I \sim \int_0^{\infty} \mathrm{e}^{-vt} \frac{m(1/s)}{s^2} \frac{ds}{dv}\,\mathrm{d}v. \tag{4.17}$$

Integrating once by parts removes the apparent singularity at the origin and leads to

$$I \sim -t \int_0^{\infty} \mathrm{e}^{-vt} \psi_1^i(1/s) s\,\mathrm{d}v. \tag{4.18}$$

Following the steps in Bender & Orszag (1978), and using $\psi_1^i = -Ms/2 + O(s^\infty)$ (see Appendix A), now leads to the result

$$I = -\frac{M}{\ln t}\left\{1 + \frac{\mathrm{i}\pi/2 - \gamma}{\ln t} + O\left(\frac{1}{(\ln t)^2}\right)\right\}, \tag{4.19}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. An equality sign may be used due to the presence of an explicit order term. Hence the long-time behaviour of $\psi_1$ is given by

$$\psi_1 = \frac{\mathrm{i}Mt}{\ln t} r\Omega(r) + O\left(\frac{t}{(\ln t)^2}\right). \tag{4.20}$$

There is a phase shift of $\pi/2$ in the solution, since it is the imaginary part which dominates in the long time limit. It is clear that angular velocities of the form $\exp(-r^a)$ will lead to growth rates of the form $(\ln t)^{-2/a}$.

## 5. Conclusions

The exact solution to the mode-one linearized vorticity equation about an azimuthal basic state has been derived. For basic states with non-zero circulation and continuous vorticity, the perturbation tends to the steady solution first found by Michalke and Timme. This explains why the steady solution is not the limit of any other amplified or decaying normal mode of the system (there are none). Discontinuities in the basic state vorticity lead in addition to oscillatory behaviour, with frequency given by the angular velocity of the basic flow at the point of discontinuity. These oscillations may be interpreted as waves propagating on the basic state vorticity discontinuity (i.e. Rossby waves), and whose influence is felt over the disc bounded by the furthest radius of discontinuity of the vorticity.

Distributions with zero circulation eventually grow without bound as $t$ increases, at a rate slower than $t$. The disturbances exhibit a $\pi/2$ phase shift in general for large time. Vortices with zero circulation have attracted a lot of attention in geophysical applications, since they have a weak signature in the far-field. However, the preceding result shows that these isolated eddies are potentially unstable to any perturbation with a mode-one component.

The large time limiting behaviour is not a uniform limit. The higher order terms in (4.10) will not decay for large $r$, and neither will the higher order terms in (4.20). However, these equations can only be expected to hold for $\Omega(r)t \gg 1$. In the non-zero circulation case, this corresponds to $\Gamma t \gg r^2$ for large $r$. For zero circulation, the corresponding condition does not take such a simple form.

The evolution of the initial perturbation is entirely due to the continuous spectrum. Nonlinear effects are not considered. However, when the continuous spectrum can grow in time, it must be of importance in the appearance of nonlinear effects, even though the basic state is stable to normal mode-one disturbances. Of course, the nonlinear evolution of the system will involve mode coupling, but the possibility of nonlinear growth being triggered by the algebraic growth of the continuous spectrum component of the mode-one deformation is very intriguing, and shows the vital importance of the circulation of the basic state.

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## Appendix A. The far-field behaviour of the streamfunction

The streamfunction $\psi(\mathbf{r})$ associated with a localised vorticity distribution $q(\mathbf{r})$ may be calculated for each mode by a Green's function method. Poisson's equation for mode $n$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_n}{\partial r}\right) - \frac{n^2}{r^2}\psi_n = q_n. \tag{A 1}$$

The regularity and decay conditions on $\psi$ lead to the following Green's functions:

$$G(r,\xi) = \xi \ln r_> \tag{A 2}$$

for mode 0 and

$$G(r,\xi) = -\frac{\xi}{2n}\left(\frac{r_<}{r_>}\right)^n \tag{A 3}$$

for mode $n$, where $r_<$ and $r_>$ are the minimum and maximum respectively of $(r,\xi)$.

For a radial vorticity distribution $q_0(r)$, the solution for $\psi_0$ is

$$\psi_0 = \ln r \int_0^r \xi q_0(\xi)\,\mathrm{d}\xi + \int_r^\infty \xi \ln \xi q(\xi)\,\mathrm{d}\xi. \tag{A 4}$$

For large $r$, the asymptotic behaviour of $\psi_0$ is given by

$$\psi_0 = \frac{\Gamma}{2\pi} \ln r + O(r^{-\infty}), \tag{A 5}$$

as can be shown by a formal expansion about the point $r = \infty$. This is the result for the basic state. As a consequence,

$$\Omega = \frac{\Gamma}{2\pi r^2} + O(r^{-\infty}). \tag{A 6}$$

For a mode-one vorticity distribution, the solution is

$$\psi_1 = -\frac{1}{2r} \int_0^r \xi^2 q_1(\xi)\, \mathrm{d}\xi - \frac{r}{2} \int_r^\infty q_1(\xi)\, \mathrm{d}\xi, \tag{A 7}$$

which leads to

$$\psi_1 = -\frac{M}{2r} + O(r^{-\infty}), \tag{A 8}$$

for large $r$, where

$$M = \int_0^\infty \xi^2 q_1(\xi)\, \mathrm{d}\xi \tag{A 9}$$

is related to the first moment in the multipole expansion of $\psi$. Since $m = (\psi_1/r)'$, its far-field behaviour is given by

$$m = \frac{M}{r^3} + O(r^{-\infty}). \tag{A 10}$$

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