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The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients

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Abstract. We prove an identity for Littlewood–Richardson coefficients conjectured by Pelletier and Ressayre. The proof relies on a novel birational involution defined over any semifield.

Keywords. Symmetric functions, Littlewood–Richardson coefficients, partitions, Schur functions, Schur polynomials, birational combinatorics, detropicalization, partitions

Mathematics Subject Classifications. 05E05

One of the central concepts in the theory of symmetric functions are the *Littlewood–Richardson coefficients* $c^{\lambda}_{\mu,\nu}$: the coefficients when a product $s_\mu s_\nu$ of two Schur functions is expanded back in the Schur basis $(s_\lambda)_{\lambda \in \mathrm{Par}}$. Equivalently, these coefficients are tensor product multiplicities of irreducible representations of GL_n (noting that each partition λ having length $\leqslant n$ has a certain irreducible representation $V(\lambda)$ of GL_n corresponding to it, the so-called *Weyl module* for shape λ). Various properties of these coefficients have been found, among them combinatorial interpretations, vanishing results, bounds and symmetries (i.e., equalities between $c^{\lambda}_{\mu,\nu}$ for different λ, μ, ν). A recent overview of the latter can be found in [2].

In [18], Pelletier and Ressayre conjectured a further symmetry of Littlewood–Richardson coefficients. Unless the classical ones, it is a partial symmetry (i.e., it does not cover every Littlewood–Richardson coefficient); it is furthermore much less simple to state, to the extent that Pelletier and Ressayre have conjectured its existence while leaving open the question which exact coefficients it matches up. In this paper, we answer this question and prove the conjecture thus concretized.

The conjecture, in its original form, can be stated as follows: Let $n \geqslant 2$, and consider the set $\mathrm{Par}[n]$ of all partitions having length $\leqslant n$. Let a and b be two nonnegative integers, and define the two partitions $\alpha = (a+b, a^{n-2})$ and $\beta = (a+b, b^{n-2})$ (where c^{n-2} means $\underbrace{c, c, \ldots, c}_{n-2 \text{ times}}$, as usual in partition combinatorics). Fix another partition $\mu \in \mathrm{Par}[n]$. Then, the families

$\left(c^{\omega}_{\alpha,\mu}\right)_{\omega\in\operatorname{Par}[n]}$ and $\left(c^{\omega}_{\beta,\mu}\right)_{\omega\in\operatorname{Par}[n]}$ of Littlewood–Richardson coefficients seem to be identical up to permutation. We can restate this in terms of Schur polynomials in the n variables $x_1, x_2, \ldots, x_n$; this then becomes the claim that the products $s_\alpha(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$ and $s_\beta(x_1, x_2, \ldots, x_n) \cdot s_\mu(x_1, x_2, \ldots, x_n)$, when expanded in the basis of Schur polynomials, have the same multiset of coefficients. Equivalently, this can be restated in terms of representations of GL_n; then, it becomes the claim that the tensor products $V(\alpha) \otimes V(\mu)$ and $V(\beta) \otimes V(\mu)$ decompose into irreps (i.e., irreducible representations) with the same multiplicities (in the sense that there is a multiplicity-preserving bijection between the irreps in $V(\alpha) \otimes V(\mu)$ and the irreps in $V(\beta) \otimes V(\mu)$).[1]

Pelletier and Ressayre have proved this conjecture for $n = 3$ (see [18, Corollary 2]) and in some further cases. We shall prove it in full generality, and construct what is essentially a bijection $\varphi : \operatorname{Par}[n] \to \operatorname{Par}[n]$ that makes it explicit (i.e., that satisfies $c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu}$ for each $\omega \in \operatorname{Par}[n]$). To be fully precise, φ will not be a bijection $\operatorname{Par}[n] \to \operatorname{Par}[n]$, but rather a bijection from $\mathbb{Z}^n$ to $\mathbb{Z}^n$, and it will satisfy $c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu}$ with the understanding that $c^{\omega}_{\alpha,\mu} = c^{\omega}_{\beta,\mu} = 0$ when $\omega \notin \operatorname{Par}[n]$. (Here, $\operatorname{Par}[n]$ is understood to be a subset of $\mathbb{Z}^n$ by identifying each partition $\lambda \in \operatorname{Par}[n]$ with the n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.)

We will define this bijection φ by explicit (if somewhat intricate) formulas that "mingle" the entries of the partition it is being applied to with those of μ (as well as a and b) using the $\min$ and $+$ operators. These formulas are best understood in the *birational picture*, in which these $\min$ and $+$ operators are generalized to the addition and the multiplication of an arbitrary semifield. (Our proof does not require this generality, but the birational picture has the advantage of greater familiarity and better notational support. It also reveals a connection with a known birational map known as a "birational R-matrix" (see Subsection 5.2), which could throw some light on the otherwise rather mysterious bijection.)

Another ingredient of our proof is an explicit formula for $s_\alpha(x_1, x_2, \ldots, x_n)$ for the above-mentioned partition α.

Remark on alternative versions

A number of proofs in this paper rely on long computations, inductions and laborious, if fairly straightforward, combinatorial arguments. In the present version of this paper, we only outline

[1] We note that the partition β is the complement of α with respect to the rectangle of height n and width $a+b$; thus, the representation $V(\beta)$ is isomorphic to the tensor product $(V(\alpha))^* \otimes \det^{a+b}$, where $(V(\alpha))^*$ denotes the dual representation to $V(\alpha)$, and where $\det$ is the 1-dimensional determinant representation of GL_n. Consequently, Pelletier's and Ressayre's conjecture can be reworded even further as saying that the tensor products $V(\alpha) \otimes V(\mu)$ and $(V(\alpha))^* \otimes V(\mu)$ have the same multiplicities of irreducible representations. Slightly more generally, we can replace α by an arbitrary *almost-rectangular* partition – that is, a partition $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \operatorname{Par}[n]$ satisfying $\gamma_2 = \gamma_3 = \cdots = \gamma_{n-1}$; however, this generalization follows easily from the original version, since any such partition satisfies

$$V((\gamma_1, \gamma_2, \ldots, \gamma_n)) \cong V\left(\underbrace{(\gamma_1 - \gamma_n, \gamma_2 - \gamma_n, \ldots, \gamma_{n-1} - \gamma_n, 0)}_{=\alpha \text{ for } a=\gamma_2-\gamma_n \text{ and } b=\gamma_1-\gamma_2}\right) \otimes \det{}^{\gamma_n}.$$

We cannot, however, replace α by an arbitrary partition in $\operatorname{Par}[n]$.

these proofs. The reader can consult the arXiv version [9] for the details. The even longer detailed version [8] also contains a few additional proofs that have been omitted even from [9], such as the proofs of the propositions in Subsection 5.1. Both arXiv versions also contain a "historical" subsection (§5.3) on the discovery of the maps φ and $\mathbf{f}_u$ appearing in this work.

An older version of this paper (missing the connection with the birational R-matrix) appeared as Oberwolfach Preprint OWP-2020-18.

1. Notations

We will use the following notations (most of which are also used in [10, §2.1]):

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- We fix a commutative ring $\mathbf{k}$; we will use this $\mathbf{k}$ as the base ring in what follows.
- A *partition* means an infinite sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ such that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots$ and such that all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $\alpha_i = 0$.
- For any partition α and any positive integer i, we let α_i denote the i-th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots))$. More generally, we use this notation whenever α is an infinite sequence of any kind of objects.
- We let Par denote the set of all partitions.
- We will often omit trailing zeroes from partitions: i.e., a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ will be identified with the k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whenever $k \in \mathbb{N}$ satisfies $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. For example, we have $(3, 2, 1, 0, 0, 0, \ldots) = (3, 2, 1) = (3, 2, 1, 0)$.

  As a consequence of this, an n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ (for any given $n \in \mathbb{N}$) is a partition if and only if it satisfies $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$.
- The *length* of a partition λ means the smallest $k \in \mathbb{N}$ such that $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. Equivalently, the length of a partition λ is the number of nonzero entries of λ (counted with multiplicity). This length is denoted by $\ell(\lambda)$. For example, $\ell((4, 2, 0, 0)) = \ell((4, 2)) = 2$ and $\ell((5, 1, 1)) = 3$.
- We will use the notation m^k for "$\underbrace{m, m, \ldots, m}_{k \text{ times}}$" in partitions and tuples (whenever $m \in \mathbb{N}$ and $k \in \mathbb{N}$). (For example, $(2, 1^4) = (2, 1, 1, 1, 1)$.)
- We let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. This is a subring of the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series. To be more specific, Λ consists of all power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that are symmetric (i.e., invariant under permutations of the variables) and of bounded degree (see [10, §2.1] for the precise meaning of this).

We shall use the symmetric functions h_n and s_λ in Λ as defined in [10, Sections 2.1 and 2.2]. Let us briefly recall how they are defined:

- For each $n \in \mathbb{Z}$, we define the *complete homogeneous symmetric function* $h_n \in \Lambda$ by
$$h_n = \sum_{i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n} x_{i_1} x_{i_2} \ldots x_{i_n}.$$
Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.

- For each partition λ, we define the *Schur function* $s_\lambda \in \Lambda$ by
$$s_\lambda = \sum \mathbf{x}_T,$$
where the sum ranges over all semistandard tableaux T of shape λ, and where $\mathbf{x}_T$ denotes the monomial obtained by multiplying the x_i for all entries i of T. We refer the reader to [10, Definition 2.2.1] or to [21, §7.10] for the details of this definition and further descriptions of the Schur functions.

  The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ, and is known as the *Schur basis*. It is easy to see that each $n \in \mathbb{N}$ satisfies $s_{(n)} = h_n$.

- We shall use the *Littlewood–Richardson coefficients* $c^\lambda_{\mu,\nu}$ (for $\lambda, \mu, \nu \in \mathrm{Par}$), as defined in [10, Definition 2.5.8], in [21, §7.15] or in [4, Chapter 10]. One of their defining properties is the following fact (see, e.g., [10, (2.5.6)] or [21, (7.64)] or [4, (10.1)]): Any two partitions $\mu, \nu \in \mathrm{Par}$ satisfy
$$s_\mu s_\nu = \sum_{\lambda \in \mathrm{Par}} c^\lambda_{\mu,\nu} s_\lambda. \tag{1.1}$$

2. The theorem

Convention 2.1.

(a) For the rest of this paper, we fix a positive integer n.

(b) Let $\mathrm{Par}[n]$ be the set of all partitions having length $\leqslant n$. In other words,
$$\begin{aligned}\mathrm{Par}[n] &= \{\lambda \in \mathrm{Par} \mid \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\} = \mathrm{Par} \cap \mathbb{N}^n \\ &= \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0\}\end{aligned}$$
(where we are using our convention that trailing zeroes can be omitted from partitions, so that a partition of length $\leqslant n$ can always be identified with an n-tuple).

(c) A family $(u_i)_{i \in \mathbb{Z}}$ of objects (e.g., of numbers) is said to be *n-periodic* if each $j \in \mathbb{Z}$ satisfies $u_j = u_{j+n}$. Equivalently, a family $(u_i)_{i \in \mathbb{Z}}$ of objects is n-periodic if and only if it has the property that
$$(u_j = u_{j'} \text{ whenever } j \text{ and } j' \text{ are two integers satisfying } j \equiv j' \bmod n).$$

Thus, an n-periodic family $(u_i)_{i\in\mathbb{Z}}$ is uniquely determined by the n consecutive entries $u_1, u_2, \ldots, u_n$ (because for any integer j, we have $u_j = u_{j'}$, where j' is the unique element of $\{1, 2, \ldots, n\}$ that is congruent to j modulo n).

Example 2.2. If $n = 3$, then both partitions $(3, 2)$ and $(3, 2, 2)$ belong to $\operatorname{Par}[n]$, while the partition $(3, 2, 2, 2)$ does not. The n-tuples $(4, 2, 1)$ and $(3, 3, 0)$ are partitions, while the n-tuples $(1, 0, -1)$ and $(2, 0, 1)$ are not. If ζ is an n-th root of unity, then the family $(\zeta^i)_{i\in\mathbb{Z}}$ of complex numbers is n-periodic.

We can now state our main theorem, which is a concretization of [18, Conjecture 1]:

Theorem 2.3. *Assume that $n \geqslant 2$. Let $a, b \in \mathbb{N}$. Define the two partitions $\alpha = (a + b, a^{n-2})$ and $\beta = (a + b, b^{n-2})$. Fix any partition $\mu \in \operatorname{Par}[n]$. Define a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows: Let $\omega \in \mathbb{Z}^n$. Define an n-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$ by*

$$\nu_i = \omega_i - a \qquad \text{for each } i \in \{1, 2, \ldots, n\},$$

where ω_i means the i-th entry of ω. For each $i \in \mathbb{Z}$, we let $i\#$ denote the unique element of $\{1, 2, \ldots, n\}$ congruent to i modulo n. For each $j \in \mathbb{Z}$, set

$$\begin{aligned}\tau_j = \min \{ &\left(\nu_{(j+1)\#} + \nu_{(j+2)\#} + \cdots + \nu_{(j+k)\#}\right) \\ &+ \left(\mu_{(j+k+1)\#} + \mu_{(j+k+2)\#} + \cdots + \mu_{(j+n-1)\#}\right) \\ &\mid\ k \in \{0, 1, \ldots, n-1\} \}.\end{aligned}$$

Define an n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_{i\#} + \left(\mu_{(i-1)\#} + \tau_{(i-1)\#}\right) - \left(\nu_{(i+1)\#} + \tau_{(i+1)\#}\right) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Let $\varphi(\omega)$ be the n-tuple $(\eta_1 + b, \eta_2 + b, \ldots, \eta_n + b) \in \mathbb{Z}^n$. Thus, we have defined a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$. Then:

(a) *The map φ is a bijection.*

(b) *We have*

$$c^{\omega}_{\alpha,\mu} = c^{\varphi(\omega)}_{\beta,\mu} \qquad \text{for each } \omega \in \mathbb{Z}^n.$$

Here, we are using the convention that every n-tuple $\omega \in \mathbb{Z}^n$ that is not a partition satisfies $c^{\omega}_{\alpha,\mu} = 0$ and $c^{\omega}_{\beta,\mu} = 0$.

This theorem will be proved at the end of this paper, after we have shown several (often seemingly unrelated, yet eventually useful) results.

Example 2.4. Let $n = 4$ and $a = 1$ and $b = 4$. The partitions α and β defined in Theorem 2.3 then take the forms $\alpha = (1 + 4, 1^2) = (5, 1, 1)$ and $\beta = (1 + 4, 4^2) = (5, 4, 4)$. Let $\mu \in \operatorname{Par}[n]$ be the partition $(2, 1) = (2, 1, 0, 0)$. Let $\omega \in \operatorname{Par}[n]$ be the partition $(5, 3, 2) = (5, 3, 2, 0)$. We shall compute the n-tuple $\varphi(\omega)$ defined in Theorem 2.3. Indeed, the n-tuple ν from Theorem 2.3 is

$$\begin{aligned}\nu &= (\omega_1 - a, \omega_2 - a, \omega_3 - a, \omega_4 - a) \\ &= (5-1, 3-1, 2-1, 0-1) = (4, 2, 1, -1).\end{aligned}$$

The integers $i\#$ from Theorem 2.3 form an n-periodic family

$$\begin{aligned}(i\#)_{i\in\mathbb{Z}} &= (\ldots, 0\#, 1\#, 2\#, 3\#, 4\#, 5\#, 6\#, 7\#, \ldots) \\ &= (\ldots, 4, 1, 2, 3, 4, 1, 2, 3, \ldots).\end{aligned}$$

The integers τ_j (for $j \in \mathbb{Z}$) from Theorem 2.3 are given by

$$\begin{aligned}\tau_1 &= \min\left\{\left(\nu_{2\#} + \nu_{3\#} + \cdots + \nu_{(k+1)\#}\right) + \left(\mu_{(k+2)\#} + \mu_{(k+3)\#} + \cdots + \mu_{4\#}\right) \right.\\ &\qquad\qquad \left.\mid\ k \in \{0,1,2,3\}\right\} \\ &= \min\{\mu_{2\#} + \mu_{3\#} + \mu_{4\#}, \quad \nu_{2\#} + \mu_{3\#} + \mu_{4\#}, \\ &\qquad\qquad \nu_{2\#} + \nu_{3\#} + \mu_{4\#}, \quad \nu_{2\#} + \nu_{3\#} + \nu_{4\#}\} \\ &= \min\{\mu_2 + \mu_3 + \mu_4, \quad \nu_2 + \mu_3 + \mu_4, \quad \nu_2 + \nu_3 + \mu_4, \quad \nu_2 + \nu_3 + \nu_4\} \\ &= \min\{1+0+0, \quad 2+0+0, \quad 2+1+0, \quad 2+1+(-1)\} \\ &= \min\{1,2,3,2\} = 1\end{aligned}$$

and

$$\begin{aligned}\tau_2 &= \min\left\{\left(\nu_{3\#} + \nu_{4\#} + \cdots + \nu_{(k+2)\#}\right) + \left(\mu_{(k+3)\#} + \mu_{(k+4)\#} + \cdots + \mu_{5\#}\right) \right.\\ &\qquad\qquad \left.\mid\ k \in \{0,1,2,3\}\right\} \\ &= \min\{\mu_{3\#} + \mu_{4\#} + \mu_{5\#}, \quad \nu_{3\#} + \mu_{4\#} + \mu_{5\#}, \\ &\qquad\qquad \nu_{3\#} + \nu_{4\#} + \mu_{5\#}, \quad \nu_{3\#} + \nu_{4\#} + \nu_{5\#}\} \\ &= \min\{\mu_3 + \mu_4 + \mu_1, \quad \nu_3 + \mu_4 + \mu_1, \quad \nu_3 + \nu_4 + \mu_1, \quad \nu_3 + \nu_4 + \nu_1\} \\ &= \min\{0+0+2, \quad 1+0+2, \quad 1+(-1)+2, \quad 1+(-1)+4\} \\ &= \min\{2,3,2,4\} = 2\end{aligned}$$

and

$$\begin{aligned}\tau_3 &= \min\left\{\left(\nu_{4\#} + \nu_{5\#} + \cdots + \nu_{(k+3)\#}\right) + \left(\mu_{(k+4)\#} + \mu_{(k+5)\#} + \cdots + \mu_{6\#}\right) \right.\\ &\qquad\qquad \left.\mid\ k \in \{0,1,2,3\}\right\} \\ &= \min\{\mu_{4\#} + \mu_{5\#} + \mu_{6\#}, \quad \nu_{4\#} + \mu_{5\#} + \mu_{6\#}, \\ &\qquad\qquad \nu_{4\#} + \nu_{5\#} + \mu_{6\#}, \quad \nu_{4\#} + \nu_{5\#} + \nu_{6\#}\} \\ &= \min\{\mu_4 + \mu_1 + \mu_2, \quad \nu_4 + \mu_1 + \mu_2, \quad \nu_4 + \nu_1 + \mu_2, \quad \nu_4 + \nu_1 + \nu_2\} \\ &= \min\{0+2+1, \quad (-1)+2+1, \quad (-1)+4+1, \quad (-1)+4+2\} \\ &= \min\{3,2,4,5\} = 2\end{aligned}$$

and

$$\begin{aligned}\tau_4 &= \min\left\{\left(\nu_{5\#}+\nu_{6\#}+\cdots+\nu_{(k+4)\#}\right)+\left(\mu_{(k+5)\#}+\mu_{(k+6)\#}+\cdots+\mu_{7\#}\right)\right.\\ &\qquad\qquad \left.\mid\ k\in\{0,1,2,3\}\right\}\\ &= \min\left\{\mu_{5\#}+\mu_{6\#}+\mu_{7\#},\quad \nu_{5\#}+\mu_{6\#}+\mu_{7\#},\right.\\ &\qquad\qquad \left.\nu_{5\#}+\nu_{6\#}+\mu_{7\#},\quad \nu_{5\#}+\nu_{6\#}+\nu_{7\#}\right\}\\ &= \min\left\{\mu_1+\mu_2+\mu_3,\quad \nu_1+\mu_2+\mu_3,\quad \nu_1+\nu_2+\mu_3,\quad \nu_1+\nu_2+\nu_3\right\}\\ &= \min\left\{2+1+0,\quad 4+1+0,\quad 4+2+0,\quad 4+2+1\right\}\\ &= \min\left\{3,5,6,7\right\}=3\end{aligned}$$

and

$$\tau_j=\tau_{j'} \qquad \text{whenever } j\equiv j' \bmod 4$$

(the latter equality follows from the n-periodicity of the family $(i\#)_{i\in\mathbb{Z}}$). Thus, the n-tuple $\eta=(\eta_1,\eta_2,\ldots,\eta_n)$ from Theorem 2.3 is given by

$$\eta_1=\underbrace{\mu_{1\#}}_{=\mu_1=2}+\left(\underbrace{\mu_{0\#}}_{=\mu_4=0}+\underbrace{\tau_{0\#}}_{=\tau_4=3}\right)-\left(\underbrace{\nu_{2\#}}_{=\nu_2=2}+\underbrace{\tau_{2\#}}_{=\tau_2=2}\right)=1$$

and

$$\eta_2=\underbrace{\mu_{2\#}}_{=\mu_2=1}+\left(\underbrace{\mu_{1\#}}_{=\mu_1=2}+\underbrace{\tau_{1\#}}_{=\tau_1=1}\right)-\left(\underbrace{\nu_{3\#}}_{=\nu_3=1}+\underbrace{\tau_{3\#}}_{=\tau_3=2}\right)=1$$

and

$$\eta_3=\underbrace{\mu_{3\#}}_{=\mu_3=0}+\left(\underbrace{\mu_{2\#}}_{=\mu_2=1}+\underbrace{\tau_{2\#}}_{=\tau_2=2}\right)-\left(\underbrace{\nu_{4\#}}_{=\nu_4=-1}+\underbrace{\tau_{4\#}}_{=\tau_4=3}\right)=1$$

and

$$\eta_4=\underbrace{\mu_{4\#}}_{=\mu_4=0}+\left(\underbrace{\mu_{3\#}}_{=\mu_3=0}+\underbrace{\tau_{3\#}}_{=\tau_3=2}\right)-\left(\underbrace{\nu_{5\#}}_{=\nu_1=4}+\underbrace{\tau_{5\#}}_{=\tau_1=1}\right)=-3,$$

so $\eta=(1,1,1,-3)$. Hence, $\varphi(\omega)=(1+b,1+b,1+b,-3+b)=(5,5,5,1)$ (since $b=4$). This is a partition. Theorem 2.3 **(b)** now yields $c^{\omega}_{\alpha,\mu}=c^{\varphi(\omega)}_{\beta,\mu}$, that is, $c^{(5,3,2)}_{(5,1,1),(2,1)}=c^{(5,5,5,1)}_{(5,4,4),(2,1)}$. And indeed, this equality holds (both of its sides being equal to 1).

Question 2.5. Can the bijection φ in Theorem 2.3 be defined in a more "intuitive" way, similar to (e.g.) jeu-de-taquin or the RSK correspondence? (There is no tableau being transformed here, just a partition.)

3. A birational involution

The leading role in our proof of Theorem 2.3 will be played by a certain piecewise-linear involution (which is similar to the bijection φ in Theorem 2.3, but without the shifting by $-a$ and b). For the sake of convenience, we prefer to study this involution in a more general setting, in which the operations $\min$, $+$ and $-$ are replaced by the structure operations $+$, $\cdot$ and $/$ of a semifield. This kind of generalization is called *detropicalization* (or *birational lifting*, or *tropicalization* in the older combinatorial literature); see, e.g., [12], [17], [5, Sections 5 and 9] or [19, §4.2] for examples of this procedure (although our use of it will be conceptually simpler).

3.1. Semifields

We recall the definition of a semifield (more precisely, the one we will be using, as there are many competing ones):[2]

Definition 3.1. A *semifield* means a set $\mathbb{K}$ endowed with

- two binary operations called "addition" and "multiplication", and denoted by $+$ and $\cdot$, respectively, and both written infix (i.e., we write $a+b$ and $a \cdot b$ instead of $+(a,b)$ and $\cdot(a,b)$), and
- an element called "unity" and denoted by 1

such that $(\mathbb{K},+)$ is an abelian semigroup and $(\mathbb{K},\cdot,1)$ is an abelian group, and such that the distributivity laws

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c) \qquad \text{and} \qquad (a+b) \cdot c = (a \cdot c) + (b \cdot c)$$

hold for all $a \in \mathbb{K}$, $b \in \mathbb{K}$ and $c \in \mathbb{K}$.

Thus, a semifield is similar to a field, except that it has no additive inverses and no zero element, but, on the other hand, has multiplicative inverses for all its elements (not just the nonzero ones).

Example 3.2. Let $\mathbb{Q}_+$ be the set of all positive rational numbers. Then, $\mathbb{Q}_+$ (endowed with its standard addition and multiplication and the number 1) is a semifield.

Example 3.3. Let $(\mathbb{A},*,e)$ be any totally ordered abelian group (whose operation is $*$ and whose neutral element is e). Then, $\mathbb{A}$ becomes a semifield if we endow it with the "addition" $\min$ (that is, we set $a+b := \min\{a,b\}$ for all $a,b \in \mathbb{A}$), the "multiplication" $*$ (that is, we set $a \cdot b := a * b$ for all $a,b \in \mathbb{A}$), and the "unity" e. This semifield $(\mathbb{A},\min,*,e)$ is called the *min tropical semifield* of $(\mathbb{A},*,e)$.

Convention 3.4. All conventions that are typically used for fields will be used for semifields as well, to the extent they apply. Specifically:

[2] We recall that semigroups are associative but not necessarily have a neutral element.

- If $\mathbb{K}$ is a semifield, and if $a, b \in \mathbb{K}$, then $a \cdot b$ shall be abbreviated by ab.
- We shall use the standard "PEMDAS" convention that multiplication-like operations have higher precedence than addition-like operations; thus, e.g., the expression "$ab + ac$" must be understood as "$(ab) + (ac)$".
- If $\mathbb{K}$ is a semifield, then the inverse of any element $b \in \mathbb{K}$ in the abelian group $(\mathbb{K}, \cdot, 1)$ will be denoted by b^{-1}. Note that this inverse is always defined (unlike when $\mathbb{K}$ is a field).
- If $\mathbb{K}$ is a semifield, and if $a, b \in \mathbb{K}$, then the product ab^{-1} will be denoted by a/b and by $\dfrac{a}{b}$. Note that this is always defined (unlike when $\mathbb{K}$ is a field).
- Finite products $\prod_{i \in I} a_i$ of elements of a semifield are defined in the same way as in commutative rings. The same applies to finite sums $\sum_{i \in I} a_i$ as long as they are nonempty (i.e., as long as $I \neq \varnothing$). The empty sum is not defined in a semifield, since there is no zero element.

3.2. The birational involution

For the rest of Section 3, we agree to the following two conventions:

Convention 3.5. We fix a positive integer n and a semifield $\mathbb{K}$. We also fix an n-tuple $u \in \mathbb{K}^n$.

Convention 3.6. If $a \in \mathbb{K}^n$ is an n-tuple, and if $i \in \mathbb{Z}$, then a_i shall denote the $i\#$-th entry of a, where $i\#$ is the unique element of $\{1, 2, \ldots, n\}$ satisfying $i\# \equiv i \bmod n$. Thus, each n-tuple $a \in \mathbb{K}^n$ satisfies $a = (a_1, a_2, \ldots, a_n)$ and $a_i = a_{i+n}$ for each $i \in \mathbb{Z}$. Therefore, if $a \in \mathbb{K}^n$ is any n-tuple, then the family $(a_i)_{i \in \mathbb{Z}}$ is n-periodic.

We shall soon use the letter x for an n-tuple in $\mathbb{K}^n$; thus, $x_1, x_2, \ldots, x_n$ will be the entries of this n-tuple. This has nothing to do with the indeterminates $x_1, x_2, x_3, \ldots$ from Section 1 (that unfortunately use the same letters); we actually **forget all conventions from Section 1** (apart from $\mathbb{N} = \{0, 1, 2, \ldots\}$) for the entire Section 3.

The following is obvious:

Lemma 3.7. *If $a \in \mathbb{K}^n$ is any n-tuple, then $a_{k+1}a_{k+2}\cdots a_{k+n} = a_1 a_2 \cdots a_n$ for each $k \in \mathbb{Z}$.*

Definition 3.8. We define a map $\mathbf{f}_u : \mathbb{K}^n \to \mathbb{K}^n$ as follows: Let $x \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, define an element $t_{r,j} \in \mathbb{K}$ by

$$t_{r,j} = \sum_{k=0}^{r} \underbrace{x_{j+1}x_{j+2}\cdots x_{j+k}}_{=\prod_{i=1}^{k} x_{j+i}} \cdot \underbrace{u_{j+k+1}u_{j+k+2}\cdots u_{j+r}}_{=\prod_{i=k+1}^{r} u_{j+i}}.$$

Define $y \in \mathbb{K}^n$ by setting

$$y_i = u_i \cdot \frac{u_{i-1}t_{n-1,i-1}}{x_{i+1}t_{n-1,i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Set $\mathbf{f}_u(x) = y$.

Example 3.9. Set $n = 4$ for this example. Let $x \in \mathbb{K}^n$ be an n-tuple; thus, $x = (x_1, x_2, x_3, x_4)$. Let us see what the definition of $\mathbf{f}_u(x)$ in Definition 3.8 boils down to in this case. Let us first compute the elements $t_{n-1,j} = t_{3,j}$ from Definition 3.8. The definition of $t_{3,0}$ yields

$$\begin{aligned} t_{3,0} &= \sum_{k=0}^{3} x_1 x_2 \cdots x_k \cdot u_{k+1} u_{k+2} \cdots u_3 \\ &= u_1 u_2 u_3 + x_1 u_2 u_3 + x_1 x_2 u_3 + x_1 x_2 x_3. \end{aligned}$$

Similarly,

$$\begin{aligned} t_{3,1} &= u_2 u_3 u_4 + x_2 u_3 u_4 + x_2 x_3 u_4 + x_2 x_3 x_4; \\ t_{3,2} &= u_3 u_4 u_5 + x_3 u_4 u_5 + x_3 x_4 u_5 + x_3 x_4 x_5 \\ &= u_3 u_4 u_1 + x_3 u_4 u_1 + x_3 x_4 u_1 + x_3 x_4 x_1 \\ &\qquad (\text{since } u_5 = u_1 \text{ and } x_5 = x_1); \\ t_{3,3} &= u_4 u_5 u_6 + x_4 u_5 u_6 + x_4 x_5 u_6 + x_4 x_5 x_6 \\ &= u_4 u_1 u_2 + x_4 u_1 u_2 + x_4 x_1 u_2 + x_4 x_1 x_2 \\ &\qquad (\text{since } u_5 = u_1 \text{ and } x_5 = x_1 \text{ and } u_6 = u_2 \text{ and } x_6 = x_2). \end{aligned}$$

We don't need to compute any further $t_{3,j}$'s, since we can easily see that

$$t_{3,j} = t_{3,j'} \text{ for any integers } j \text{ and } j' \text{ satisfying } j \equiv j' \bmod 4. \tag{3.1}$$

Thus, in particular, $t_{3,4} = t_{3,0}$ and $t_{3,5} = t_{3,1}$. Now, let us compute the 4-tuple $y \in \mathbb{K}^n = \mathbb{K}^4$ from Definition 3.8. By its definition, we have

$$\begin{aligned} y_1 &= u_1 \cdot \frac{u_{1-1} t_{3,1-1}}{x_{1+1} t_{3,1+1}} = u_1 \cdot \frac{u_0 t_{3,0}}{x_2 t_{3,2}} = u_1 \cdot \frac{u_4 t_{3,0}}{x_2 t_{3,2}} \qquad (\text{since } u_0 = u_4) \\ &= u_1 \cdot \frac{u_4 (u_1 u_2 u_3 + x_1 u_2 u_3 + x_1 x_2 u_3 + x_1 x_2 x_3)}{x_2 (u_3 u_4 u_1 + x_3 u_4 u_1 + x_3 x_4 u_1 + x_3 x_4 x_1)} \end{aligned}$$

(by our formulas for $t_{3,0}$ and $t_{3,2}$). Similar computations lead to

$$\begin{aligned} y_2 &= u_2 \cdot \frac{u_1 (u_2 u_3 u_4 + x_2 u_3 u_4 + x_2 x_3 u_4 + x_2 x_3 x_4)}{x_3 (u_4 u_1 u_2 + x_4 u_1 u_2 + x_4 x_1 u_2 + x_4 x_1 x_2)}; \\ y_3 &= u_3 \cdot \frac{u_2 (u_3 u_4 u_1 + x_3 u_4 u_1 + x_3 x_4 u_1 + x_3 x_4 x_1)}{x_4 (u_1 u_2 u_3 + x_1 u_2 u_3 + x_1 x_2 u_3 + x_1 x_2 x_3)}; \\ y_4 &= u_4 \cdot \frac{u_3 (u_4 u_1 u_2 + x_4 u_1 u_2 + x_4 x_1 u_2 + x_4 x_1 x_2)}{x_1 (u_2 u_3 u_4 + x_2 u_3 u_4 + x_2 x_3 u_4 + x_2 x_3 x_4)}. \end{aligned}$$

Of course, knowing one of these four equalities is enough; the expression for y_{i+1} is obtained from the expression for y_i by shifting all indices (other than the "3"s that were originally "$n-1$"s) forward by 1.

Remark 3.10. Instead of assuming $\mathbb{K}$ to be a semifield, we could have assumed that $\mathbb{K}$ is an infinite field. In that case, the $\mathbf{f}_u$ in Definition 3.8 would be a birational map instead of a map in the usual sense of this word, since the denominators $x_{i+1}t_{n-1,i+1}$ in the definition of y can be zero. Everything we say below about $\mathbf{f}_u$ would nevertheless still hold on the level of birational maps.

The map $\mathbf{f}_u$ we just defined has the following properties:

Theorem 3.11.

(a) *The map $\mathbf{f}_u$ is an involution (i.e., we have $\mathbf{f}_u \circ \mathbf{f}_u = \mathrm{id}$).*

(b) *Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,*

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

(c) *Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,*

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right)$$

for each $i \in \mathbb{Z}$.

(d) *Let $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ be such that $y = \mathbf{f}_u(x)$. Then,*

$$\prod_{i=1}^{n} \frac{u_i + x_i}{x_i} = \prod_{i=1}^{n} \frac{u_i + y_i}{u_i}.$$

Theorem 3.11 will be crucial for us; but before we can prove it, we will need a few lemmas.

Lemma 3.12. *Let $x \in \mathbb{K}^n$ be an n-tuple. Let $t_{r,j}$ and y be as in Definition 3.8. Then:*

(a) *We have $t_{r,j} = t_{r,j'}$ for any $r \in \mathbb{N}$ and any two integers j and j' satisfying $j \equiv j' \bmod n$. In other words, for each $r \in \mathbb{N}$, the family $(t_{r,j})_{j \in \mathbb{Z}}$ is n-periodic.*

(b) *We have $t_{0,j} = 1$ for each $j \in \mathbb{Z}$.*

(c) *For each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have*

$$x_j t_{r,j} + u_j u_{j+1} \cdots u_{j+r} = t_{r+1,j-1}.$$

(d) *For each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have*

$$u_{j+r+1} t_{r,j} + x_{j+1} x_{j+2} \cdots x_{j+r+1} = t_{r+1,j}.$$

(e) *For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, we have*

$$x_a t_{n-1,a} + u_{b-1} t_{n-1,b-1} = x_b t_{n-1,b} + u_{a-1} t_{n-1,a-1}.$$

(f) *For each $i \in \mathbb{Z}$, we have*

$$x_{i+1}t_{n-1,i+1} + u_{i-1}t_{n-1,i-1} = (x_i + u_i)\, t_{n-1,i}.$$

(g) *For each $j \in \mathbb{Z}$ and each positive integer q, we have*

$$t_{n-1,j+q+1} \cdot x_{j+2}x_{j+3}\cdots x_{j+q+1} + u_j t_{n-1,j} t_{q-1,j+1} = t_{n-1,j+1} t_{q,j}.$$

(h) *For each $i \in \mathbb{Z}$, we have*

$$y_i = u_i \cdot \frac{u_{i-1}t_{n-1,i-1}}{x_{i+1}t_{n-1,i+1}}.$$

Now, for each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, let us define an element $t'_{r,j} \in \mathbb{K}$ by

$$t'_{r,j} = \sum_{k=0}^{r} \underbrace{y_{j+1}y_{j+2}\cdots y_{j+k}}_{=\prod_{i=1}^{k} y_{j+i}} \cdot \underbrace{u_{j+k+1}u_{j+k+2}\cdots u_{j+r}}_{=\prod_{i=k+1}^{r} u_{j+i}}.$$

(This is precisely how $t_{r,j}$ was defined, except that we are using y in place of x now.) Then:

(i) *For each $j \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have*

$$\frac{t'_{q,j}}{u_{j+1}u_{j+2}\cdots u_{j+q}} = \frac{t_{n-1,j+1}}{t_{n-1,j+q+1}} \cdot \frac{t_{q,j}}{x_{j+2}x_{j+3}\cdots x_{j+q+1}}.$$

(j) *For each $j \in \mathbb{Z}$, we have*

$$\frac{t'_{n-1,j}u_j}{u_1u_2\cdots u_n} = \frac{t_{n-1,j+1}x_{j+1}}{x_1x_2\cdots x_n}.$$

(k) *For each $i \in \mathbb{Z}$, we have*

$$x_i = u_i \cdot \frac{u_{i-1}t'_{n-1,i-1}}{y_{i+1}t'_{n-1,i+1}}.$$

Proof of Lemma 3.12. The proof of this lemma is long but unsophisticated: Each part follows by rather straightforward computations (and, in the cases of parts **(g)** and **(i)**, an induction on q) from the previously proven parts.[3] We therefore omit it. □

Lemma 3.13. *Let $x \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let*

$$q_j = \sum_{k=0}^{n-1} x_{j+1}x_{j+2}\cdots x_{j+k} \cdot u_{j+k+1}u_{j+k+2}\cdots u_{j+n-1}.$$

Let $z \in \mathbb{K}^n$ be such that

$$z_i = u_i \cdot \frac{u_{i-1}q_{i-1}}{x_{i+1}q_{i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Then, $\mathbf{f}_u(x) = z$.

[3]Moreover, the hardest parts of the proof – namely, the proofs of parts **(g)**, **(i)**, **(j)** and **(k)** – can be sidestepped entirely, as these parts will only be used in the proof of Theorem 3.11 **(a)**, but we will give an alternative proof of Theorem 3.11 **(a)** later on (in Remark 3.16), which avoids using them.

Proof of Lemma 3.13. Let $t_{r,j}$ and y be as in Definition 3.8. Then, $t_{n-1,j} = q_j$ for each $j \in \mathbb{Z}$ (by comparing the definitions of $t_{n-1,j}$ and q_j). Hence, $z_i = y_i$ for each $i \in \{1, 2, \ldots, n\}$ (by comparing the definitions of z_i and y_i). Hence, $z = y = \mathbf{f}_u(x)$ (since $\mathbf{f}_u(x)$ was defined to be y). $\square$

For future convenience, let us restate Lemma 3.13 with different labels:

Lemma 3.14. *Let $y \in \mathbb{K}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let*

$$r_j = \sum_{k=0}^{n-1} y_{j+1} y_{j+2} \cdots y_{j+k} \cdot u_{j+k+1} u_{j+k+2} \cdots u_{j+n-1}.$$

Let $x \in \mathbb{K}^n$ be such that

$$x_i = u_i \cdot \frac{u_{i-1} r_{i-1}}{y_{i+1} r_{i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Then, $\mathbf{f}_u(y) = x$.

Proof of Lemma 3.14. Lemma 3.14 is just Lemma 3.13, with x, q_j and z renamed as y, r_j and x. $\square$

We are now ready for the proof of Theorem 3.11:

Proof of Theorem 3.11. **(a)** Let $x \in \mathbb{K}^n$. We shall prove that $(\mathbf{f}_u \circ \mathbf{f}_u)(x) = x$.

Let $t_{r,j}$ and y be as in Definition 3.8. Then, $\mathbf{f}_u(x) = y$ (by the definition of $\mathbf{f}_u$). Let $t'_{r,j}$ (for each $r \in \mathbb{N}$ and $j \in \mathbb{Z}$) be as in Lemma 3.12. The definition of $t'_{n-1,j}$ shows that

$$t'_{n-1,j} = \sum_{k=0}^{n-1} y_{j+1} y_{j+2} \cdots y_{j+k} \cdot u_{j+k+1} u_{j+k+2} \cdots u_{j+n-1}$$

for each $j \in \mathbb{Z}$. Lemma 3.12 **(k)** shows that

$$x_i = u_i \cdot \frac{u_{i-1} t'_{n-1,i-1}}{y_{i+1} t'_{n-1,i+1}} \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Thus, Lemma 3.14 (applied to $r_j = t'_{n-1,j}$) yields that $\mathbf{f}_u(y) = x$. In view of $\mathbf{f}_u(x) = y$, this rewrites as $\mathbf{f}_u(\mathbf{f}_u(x)) = x$. In other words, $(\mathbf{f}_u \circ \mathbf{f}_u)(x) = x$.

We have proved this for each $x \in \mathbb{K}^n$. In other words, $\mathbf{f}_u \circ \mathbf{f}_u = \mathrm{id}$. This proves Theorem 3.11 **(a)**.

(b) Let $t_{r,j}$ be as in Definition 3.8. Note that the y from Definition 3.8 is precisely the y in Theorem 3.11 **(b)** (because both y's satisfy $\mathbf{f}_u(x) = y$).

The definition of y in Definition 3.8 shows that

$$\begin{aligned} y_1 y_2 \cdots y_n &= \left(u_1 \cdot \frac{u_0 t_{n-1,0}}{x_2 t_{n-1,2}}\right)\left(u_2 \cdot \frac{u_1 t_{n-1,1}}{x_3 t_{n-1,3}}\right) \cdots \left(u_n \cdot \frac{u_{n-1} t_{n-1,n-1}}{x_{n+1} t_{n-1,n+1}}\right) \\ &= (u_1 u_2 \cdots u_n) \cdot \underbrace{(u_0 u_1 \cdots u_{n-1})}_{\substack{=u_1 u_2 \cdots u_n \\ \text{(by Lemma 3.7)}}} \cdot \underbrace{(t_{n-1,0} t_{n-1,1} \cdots t_{n-1,n-1})}_{\substack{=t_{n-1,1} t_{n-1,2} \cdots t_{n-1,n} \\ \text{(by Lemma 3.12 (a))}}} \\ &\qquad / \left(\underbrace{(x_2 x_3 \cdots x_{n+1})}_{\substack{=x_1 x_2 \cdots x_n \\ \text{(by Lemma 3.7)}}} \cdot \underbrace{(t_{n-1,2} t_{n-1,3} \cdots t_{n-1,n+1})}_{\substack{=t_{n-1,1} t_{n-1,2} \cdots t_{n-1,n} \\ \text{(by Lemma 3.12 (a))}}} \right) \\ &= (u_1 u_2 \cdots u_n) \cdot (u_1 u_2 \cdots u_n) \cdot (t_{n-1,1} t_{n-1,2} \cdots t_{n-1,n}) \\ &\qquad / ((x_1 x_2 \cdots x_n) \cdot (t_{n-1,1} t_{n-1,2} \cdots t_{n-1,n})) \\ &= \frac{(u_1 u_2 \cdots u_n)^2}{x_1 x_2 \cdots x_n}, \end{aligned}$$

so that

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

This proves Theorem 3.11 **(b)**.

(c) Let $t_{r,j}$ be as in Definition 3.8. Note that the y from Definition 3.8 is precisely the y in Theorem 3.11 **(c)** (because both y's satisfy $\mathbf{f}_u (x) = y$).

Let $i \in \mathbb{Z}$. Then, Lemma 3.12 **(h)** yields

$$\begin{aligned} u_i + y_i &= u_i + u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} = u_i \cdot \frac{x_{i+1} t_{n-1,i+1} + u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\ &= u_i \cdot \frac{(x_i + u_i)\, t_{n-1,i}}{x_{i+1} t_{n-1,i+1}} \end{aligned} \tag{3.2}$$

(by Lemma 3.12 **(f)**). Now,

$$\begin{aligned} \frac{1}{u_i} + \frac{1}{y_i} &= \underbrace{(u_i + y_i)}_{\substack{=u_i \cdot \frac{(x_i + u_i)\, t_{n-1,i}}{x_{i+1} t_{n-1,i+1}} \\ \text{(by (3.2))}}} / (u_i \cdot y_i) = \frac{(x_i + u_i)\, t_{n-1,i}}{x_{i+1} t_{n-1,i+1}} / \underbrace{y_i}_{\substack{=u_i \cdot \frac{u_{i-1} t_{n-1,i-1}}{x_{i+1} t_{n-1,i+1}} \\ \text{(by Lemma 3.12 (h))}}} \\ &= \frac{(x_i + u_i)\, t_{n-1,i}}{u_i u_{i-1} t_{n-1,i-1}}. \end{aligned}$$

The same argument (applied to $i + 1$ instead of i) yields

$$\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}} = \frac{(x_{i+1} + u_{i+1})\, t_{n-1,i+1}}{u_{i+1} u_i t_{n-1,i}}.$$

Multiplying (3.2) with this equality, we obtain

$$
\begin{aligned}
(u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) &= u_i \cdot \frac{(x_i + u_i)\, t_{n-1,i}}{x_{i+1} t_{n-1,i+1}} \cdot \frac{(x_{i+1} + u_{i+1})\, t_{n-1,i+1}}{u_{i+1} u_i t_{n-1,i}} \\
&= \underbrace{(x_i + u_i)}_{=u_i + x_i} \cdot \underbrace{\frac{x_{i+1} + u_{i+1}}{x_{i+1} u_{i+1}}}_{= \frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}} = (u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right).
\end{aligned}
$$

This proves Theorem 3.11 **(c)**.

(d) Let $t_{r,j}$ be as in Definition 3.8. Every $i \in \mathbb{Z}$ satisfies (3.2) (as we have shown in the proof of Theorem 3.11 **(c)** above). Hence, taking the product of the equalities (3.2) over all $i \in \{1, 2, \ldots, n\}$, we find

$$
\begin{aligned}
\prod_{i=1}^{n} (u_i + y_i) &= \prod_{i=1}^{n} \left(u_i \cdot \frac{(x_i + u_i)\, t_{n-1,i}}{x_{i+1} t_{n-1,i+1}} \right) \\
&= \left(\prod_{i=1}^{n} u_i\right) \cdot \frac{\left(\prod_{i=1}^{n} (x_i + u_i)\right)\left(\prod_{i=1}^{n} t_{n-1,i}\right)}{\left(\prod_{i=1}^{n} x_{i+1}\right)\left(\prod_{i=1}^{n} t_{n-1,i+1}\right)} \\
&= \left(\prod_{i=1}^{n} u_i\right) \cdot \frac{\left(\prod_{i=1}^{n} (x_i + u_i)\right)\left(\prod_{i=1}^{n} t_{n-1,i}\right)}{\left(\prod_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} t_{n-1,i+1}\right)} \\
&\qquad \left(\text{since Lemma 3.7 yields } \prod_{i=1}^{n} x_{i+1} = \prod_{i=1}^{n} x_i\right) \\
&= \left(\prod_{i=1}^{n} u_i\right)\left(\prod_{i=1}^{n} \frac{x_i + u_i}{x_i}\right) \underbrace{\prod_{i=1}^{n} \frac{t_{n-1,i}}{t_{n-1,i+1}}}_{\substack{= \frac{t_{n-1,1}}{t_{n-1,n+1}} \\ \text{(by the telescope principle)}}} \\
&= \left(\prod_{i=1}^{n} u_i\right)\left(\prod_{i=1}^{n} \frac{x_i + u_i}{x_i}\right) \underbrace{\frac{t_{n-1,1}}{t_{n-1,n+1}}}_{\substack{=1 \\ \text{(by Lemma 3.12 (a))}}} = \left(\prod_{i=1}^{n} u_i\right) \prod_{i=1}^{n} \frac{x_i + u_i}{x_i}.
\end{aligned}
$$

Dividing both sides of this by $\prod_{i=1}^{n} u_i$, we obtain

$$
\prod_{i=1}^{n} \frac{u_i + y_i}{u_i} = \prod_{i=1}^{n} \frac{x_i + u_i}{x_i} = \prod_{i=1}^{n} \frac{u_i + x_i}{x_i},
$$

which proves Theorem 3.11 **(d)**. □

Let us observe one more property of the involution $\mathbf{f}_u$ (even though we will never use it):

Proposition 3.15. *Let $x \in \mathbb{K}^n$ be such that $x_1 x_2 \cdots x_n = u_1 u_2 \cdots u_n$. Then, $\mathbf{f}_u(x) = x$.*

Proof of Proposition 3.15. The main step is to show that $u_{i-1} t_{n-1,i-1} = x_i t_{n-1,i}$ for each $i \in \mathbb{Z}$ (where $t_{r,j}$ are as in Definition 3.8). □

Remark 3.16. There is an alternative proof of Theorem 3.11 **(a)** that avoids the use of the (rather complicated) parts **(g)**, **(i)**, **(j)** and **(k)** of Lemma 3.12. Let us outline this proof: The claim of Theorem 3.11 **(a)** can be restated as the equality $\mathbf{f}_u(\mathbf{f}_u(x)) = x$ for each $x \in \mathbb{K}^n$ and each $u \in \mathbb{K}^n$ (we are not regarding u as fixed here). This equality boils down to a set of identities between rational functions in the variables $u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n$ (since each entry of $\mathbf{f}_u(x)$ is a rational function in these variables, and each entry of $\mathbf{f}_u(\mathbf{f}_u(x))$ is a rational function in the former entries as well as $u_1, u_2, \ldots, u_n$). These rational functions are subtraction-free (i.e., no subtraction signs appear in them), and thus are defined over any semifield. But there is a general principle saying that if we need to prove an identity between two subtraction-free rational functions, it is sufficient to prove that it holds over the semifield $\mathbb{Q}_+$ from Example 3.2. (Indeed, this principle follows from the fact that any subtraction-free rational function can be rewritten as a ratio of two polynomials with nonnegative integer coefficients, and thus an identity between two subtraction-free rational functions can be rewritten as an identity between two such polynomials; but the latter kind of identity will necessarily be true if it has been checked on all positive rational numbers.) As a consequence of this discussion, in order to prove Theorem 3.11 **(a)** in full generality, it suffices to prove Theorem 3.11 **(a)** in the case when $\mathbb{K} = \mathbb{Q}_+$. So let us restrict ourselves to this case. Let $x \in \mathbb{K}^n$. We must show that $\mathbf{f}_u(\mathbf{f}_u(x)) = x$. Let $y = \mathbf{f}_u(x)$, and let $z = \mathbf{f}_u(y)$. We will show that $z = x$. Assume the contrary. Thus, $z \neq x$. Hence, there exists some $i \in \{1, 2, \ldots, n\}$ such that $z_i \neq x_i$. Consider this i. Hence, either $z_i > x_i$ or $z_i < x_i$. We WLOG assume that $z_i > x_i$ (since the proof in the case of $z_i < x_i$ is identical, except that all inequality signs are reversed). But Theorem 3.11 **(c)** yields

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right).$$

Likewise, Theorem 3.11 **(c)** (applied to y and z instead of x and y) yields

$$(u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) = (u_i + z_i)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right)$$

(since $z = \mathbf{f}_u(y)$). Combining these two equalities, we find

$$\begin{aligned}(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) &= \left(u_i + \underbrace{z_i}_{>x_i}\right)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right) \\ &> (u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}\right).\end{aligned}$$

Cancelling the positive number $u_i + x_i$ from this inequality, we obtain $\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}} > \frac{1}{u_{i+1}} + \frac{1}{z_{i+1}}$. Hence, $\frac{1}{x_{i+1}} > \frac{1}{z_{i+1}}$, so that $z_{i+1} > x_{i+1}$. Thus, from $z_i > x_i$, we have obtained $z_{i+1} > x_{i+1}$. The same reasoning (but applied to $i + 1$ instead of i) now yields $z_{i+2} > x_{i+2}$ (since $z_{i+1} > x_{i+1}$). Proceeding in the same way, we successively obtain $z_{i+3} > x_{i+3}$ and $z_{i+4} > x_{i+4}$ and $z_{i+5} > x_{i+5}$ and so on. Hence,

$$\underbrace{z_i}_{>x_i} \underbrace{z_{i+1}}_{>x_{i+1}} \cdots \underbrace{z_{i+n-1}}_{>x_{i+n-1}} > x_i x_{i+1} \cdots x_{i+n-1}. \tag{3.3}$$

But Theorem 3.11 **(b)** yields

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 .$$

Also, Theorem 3.11 **(b)** (applied to y and z instead of x and y) yields

$$z_1 z_2 \cdots z_n \cdot y_1 y_2 \cdots y_n = (u_1 u_2 \cdots u_n)^2$$

(since $z = \mathbf{f}_u(y)$). Comparing these two equalities, we find $y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = z_1 z_2 \cdots z_n \cdot y_1 y_2 \cdots y_n$, so that

$$x_1 x_2 \cdots x_n = z_1 z_2 \cdots z_n. \tag{3.4}$$

But Lemma 3.7 yields $z_i z_{i+1} \cdots z_{i+n-1} = z_1 z_2 \cdots z_n$ and $x_i x_{i+1} \cdots x_{i+n-1} = x_1 x_2 \cdots x_n$. In light of these two equalities, we can rewrite (3.3) as $z_1 z_2 \cdots z_n > x_1 x_2 \cdots x_n$. This, however, contradicts (3.4). This contradiction shows that our assumption was false, thus concluding our proof of $z = x$. Now, recall that $\mathbf{f}_u(x) = y$. Hence, $\mathbf{f}_u(\mathbf{f}_u(x)) = \mathbf{f}_u(y) = z = x$, as we wanted to prove. Hence, Theorem 3.11 **(a)** is proved again.

We shall return to the birational involution $\mathbf{f}_u$ in Subsection 5.1, where we will pose several questions about its meaning and uniqueness properties.

4. Proof of the main theorem

We shall now slowly approach the proof of Theorem 2.3 through a litany of auxiliary results.

4.1. From the life of snakes

Recall the conventions introduced in Section 1 and in Convention 2.1. Let us next introduce some further notations.

Definition 4.1.

(a) Let $\mathcal{L}$ denote the ring $\mathbf{k}\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ of Laurent polynomials in the n indeterminates $x_1, x_2, \ldots, x_n$ over $\mathbf{k}$. Clearly, the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$ is a subring of $\mathcal{L}$.

(b) We let x_Π denote the monomial $x_1 x_2 \cdots x_n \in \mathbf{k}[x_1, x_2, \ldots, x_n] \subseteq \mathcal{L}$.

If $f \in \Lambda$ is a symmetric function[4], and if $a_1, a_2, \ldots, a_n$ are n elements of a commutative $\mathbf{k}$-algebra A, then $f(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)$ means the result of substituting the values $a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots$ for $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, x_{n+3}, \ldots$ in f. This is a well-defined element of A, and is denoted by $f(a_1, a_2, \ldots, a_n)$. It is called the *evaluation* of f at $a_1, a_2, \ldots, a_n$.

For any symmetric function $f \in \Lambda$, the evaluation

$$f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$$

is a polynomial in $\mathbf{k}[x_1, x_2, \ldots, x_n]$ and thus a Laurent polynomial in $\mathcal{L}$. Moreover, for any symmetric function $f \in \Lambda$, the evaluation

$$f\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) = f\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}, 0, 0, 0, \ldots\right)$$

is a Laurent polynomial in $\mathcal{L}$ as well.

Convention 4.2. For the rest of Section 4, let us agree to the following notation: If γ is an n-tuple (of any objects), then we let γ_i denote the i-th entry of γ whenever $i \in \{1, 2, \ldots, n\}$. Thus, each n-tuple γ satisfies $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Definition 4.3.

(a) A *snake* means an n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers (not necessarily nonnegative) such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$.

(b) A snake λ is said to be *nonnegative* if it belongs to $\mathbb{N}^n$ (that is, if all its entries are nonnegative). Thus, a nonnegative snake is the same as a partition having length $\leqslant n$. In other words, a nonnegative snake is the same as a partition $\lambda \in \operatorname{Par}[n]$.

(c) If $\lambda \in \mathbb{Z}^n$ is an n-tuple, and d is an integer, then $\lambda + d$ denotes the n-tuple $(\lambda_1 + d, \lambda_2 + d, \ldots, \lambda_n + d) \in \mathbb{Z}^n$ (which is obtained from λ by adding d to each entry), whereas $\lambda - d$ denotes the n-tuple $\lambda + (-d) = (\lambda_1 - d, \lambda_2 - d, \ldots, \lambda_n - d) \in \mathbb{Z}^n$.

(d) If $\lambda \in \mathbb{Z}^n$, then $\lambda^\vee$ denotes the n-tuple $(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \in \mathbb{Z}^n$.

(e) We regard $\mathbb{Z}^n$ as a $\mathbb{Z}$-module in the obvious way. Thus, if $\lambda \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^n$ are two n-tuples of integers, then

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_n + \mu_n),$$
$$\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_n - \mu_n).$$

(f) We let ρ denote the nonnegative snake $(n-1, n-2, \ldots, 2, 1, 0)$. Thus,

$$\rho_i = n - i \qquad \text{for each } i \in \{1, 2, \ldots, n\}. \tag{4.1}$$

Example 4.4. In this example, let $n = 3$.

[4] or, more generally, any formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that is of bounded degree

(a) The four 3-tuples $(3,1,0)$, $(2,2,1)$, $(1,0,-1)$ and $(-1,-2,-5)$ are examples of snakes.

(b) The first two of these four snakes (but not the last two) are nonnegative.

(c) We have $(5,3,1)+3=(8,6,4)$ and $(5,3,1)-3=(2,0,-2)$.

(d) We have $(5,2,2)^{\vee}=(-2,-2,-5)$.

(e) We have $(2,1,2)+(3,4,5)=(5,5,7)$.

(f) We have $\rho=(2,1,0)$.

Note that what we call a "snake" here is called a "staircase of height n" in Stembridge's work [22], where he uses these snakes to index finite-dimensional polynomial representations of the group $\mathrm{GL}_n(\mathbb{C})$. We avoid calling them "staircases", as that word has since been used for other things (in particular, ρ is often called "the n-staircase" in the jargon of combinatorialists).

The notations introduced in Definition 4.3 have the following properties:

Proposition 4.5.

(a) *If λ is a snake, and d is an integer, then $\lambda+d$ and $\lambda-d$ are snakes as well.*

(b) *If λ is a snake, then $\lambda^{\vee}$ is a snake as well.*

(c) *We have $(\lambda+\mu)+d=(\lambda+d)+\mu$ for any $\lambda\in\mathbb{Z}^n$, $\mu\in\mathbb{Z}^n$ and $d\in\mathbb{Z}$.*

(d) *We have $\lambda+(d+e)=(\lambda+d)+e$ for any $\lambda\in\mathbb{Z}^n$, $d\in\mathbb{Z}$ and $e\in\mathbb{Z}$.*

(e) *We have $(\lambda+d)-d=(\lambda-d)+d=\lambda$ for any $\lambda\in\mathbb{Z}^n$ and $d\in\mathbb{Z}$.*

Proof of Proposition 4.5. Completely straightforward. $\square$

Let us now assign a Laurent polynomial a_λ to each $\lambda\in\mathbb{Z}^n$:

Definition 4.6. Let $\lambda\in\mathbb{Z}^n$ be any n-tuple. Then, we define the Laurent polynomial

$$a_\lambda := \sum_{w\in\mathfrak{S}_n} (\operatorname{sign} w)\, x_{w(1)}^{\lambda_1} x_{w(2)}^{\lambda_2} \cdots x_{w(n)}^{\lambda_n} \in \mathcal{L},$$

where $\mathfrak{S}_n$ is the symmetric group of the set $\{1,2,\ldots,n\}$ (and where $\operatorname{sign} w$ denotes the sign of a permutation w). This Laurent polynomial a_λ is called the *alternant* corresponding to the n-tuple λ.

(The "a" in the notation "a_λ" has nothing to do with the a in Theorem 2.3.)

Example 4.7. We have

$$\begin{aligned} a_{(5,3,2)} &= \sum_{w\in\mathfrak{S}_3} (\operatorname{sign} w)\, x_{w(1)}^5 x_{w(2)}^3 x_{w(3)}^2 \\ &= x_1^5x_2^3x_3^2 + x_2^5x_3^3x_1^2 + x_3^5x_1^3x_2^2 - x_1^5x_3^3x_2^2 - x_2^5x_1^3x_3^2 - x_3^5x_2^3x_1^2. \end{aligned}$$

The sum in Definition 4.6 is the same kind of sum that appears in the definition of a determinant. Therefore, we can rewrite the alternant as follows:

Proposition 4.8. *Let $\lambda \in \mathbb{Z}^n$ be an n-tuple. Then, the alternant $a_\lambda \in \mathcal{L}$ satisfies*

$$a_\lambda = \det\left(\left(x_j^{\lambda_i}\right)_{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n}\right) = \det\left(\left(x_i^{\lambda_j}\right)_{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n}\right).$$

Thus, in particular, the alternant a_ρ corresponding to the snake

$$\rho = (n-1, n-2, \ldots, 2, 1, 0) = (n-1, n-2, \ldots, n-n)$$

satisfies

$$a_\rho = \det\left(\left(x_i^{n-j}\right)_{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n}\right) = \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)$$

(by the classical formula for the Vandermonde determinant).

We recall a standard concept from commutative algebra: An element a of a commutative ring A is said to be *regular* if it has the property that every $x \in A$ satisfying $ax = 0$ must satisfy $x = 0$. (Thus, regular elements are the same as elements that are not zero-divisors, if one does not require zero-divisors to be nonzero.)

Lemma 4.9. *The alternant a_ρ is a regular element of $\mathcal{L}$.*

Proof of Lemma 4.9. It is easy to see that every regular element of the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$ is also a regular element of $\mathcal{L}$. (Indeed, this is an easy consequence of the facts that $\mathbf{k}[x_1, x_2, \ldots, x_n]$ is a subring of $\mathcal{L}$ and that every element of $\mathcal{L}$ has the form $\dfrac{c}{x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}}$ for some $u_1, u_2, \ldots, u_n \in \mathbb{Z}$ and some polynomial $c \in \mathbf{k}[x_1, x_2, \ldots, x_n]$.)

On the other hand, it is well-known (see, e.g., [7, Corollary 4.4]) that the polynomial $\prod_{1\leqslant i<j\leqslant n} (x_i - x_j)$ is a regular element of the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$. In other words, a_ρ is a regular element of $\mathbf{k}[x_1, x_2, \ldots, x_n]$ (since we have $a_\rho = \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)$). Hence, a_ρ is also a regular element of $\mathcal{L}$ (by the previous paragraph). □

Lemma 4.9 shows that fractions of the form $\dfrac{u}{a_\rho}$ (where $u \in \mathcal{L}$) are well-defined if u is a multiple of a_ρ. (That is, there is never more than one $b \in \mathcal{L}$ that satisfies $a_\rho b = u$.)

We notice that the element $x_\Pi = x_1 x_2 \cdots x_n$ of $\mathcal{L}$ is invertible.

Lemma 4.10. *Let $\lambda \in \mathbb{Z}^n$ be any n-tuple, and let $d \in \mathbb{Z}$. Then, $a_{\lambda+d} = x_\Pi^d a_\lambda$.*

Proof of Lemma 4.10. This follows easily from the definitions of a_λ and $a_{\lambda+d}$. □

Lemma 4.11. *Let λ be a snake. Then, $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$.*

Proof of Lemma 4.11. Our proof will consist of two steps:

Step 1: We will prove Lemma 4.11 in the particular case when λ is nonnegative.

Step 2: We will use Lemma 4.10 to derive the general case of Lemma 4.11 from this particular case.

We will use this strategy again further on; we shall refer to it as the *right-shift strategy*.

Here are the details of the two steps:

Step 1: Let us prove that Lemma 4.11 holds in the particular case when λ is nonnegative.

Indeed, let us assume that λ is nonnegative. We must show that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$.

We know that λ is a nonnegative snake, thus a partition of length $\leqslant n$. Hence, [10, Corollary 2.6.7] shows that $s_\lambda(x_1, x_2, \ldots, x_n) = \dfrac{a_{\lambda+\rho}}{a_\rho}$. Thus, $a_{\lambda+\rho} = a_\rho \cdot s_\lambda(x_1, x_2, \ldots, x_n)$. This shows that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$ (since we have $s_\lambda(x_1, x_2, \ldots, x_n) \in \mathbf{k}[x_1, x_2, \ldots, x_n] \subseteq \mathcal{L}$). Thus, Lemma 4.11 is proved under the assumption that λ is nonnegative. This completes Step 1.

Step 2: Let us now prove Lemma 4.11 in the general case.

The snake λ may or may not be nonnegative. However, there exists some integer d such that the snake $\lambda + d$ is nonnegative (for example, we can take $d = -\lambda_n$). Consider this d.

The snake $\lambda + d$ is nonnegative; thus, we can apply Lemma 4.11 to $\lambda + d$ instead of λ (because in Step 1, we have proved that Lemma 4.11 holds in the particular case when λ is nonnegative). Thus we conclude that $a_{(\lambda+d)+\rho}$ is a multiple of a_ρ in $\mathcal{L}$. In other words, there exists some $u \in \mathcal{L}$ such that $a_{(\lambda+d)+\rho} = a_\rho u$. Consider this u. But Proposition 4.5 **(c)** yields $(\lambda+\rho)+d = (\lambda+d)+\rho$, and thus $a_{(\lambda+\rho)+d} = a_{(\lambda+d)+\rho} = a_\rho u$.

Lemma 4.10 (applied to $\lambda+\rho$ instead of λ) yields $a_{(\lambda+\rho)+d} = x_\Pi^d a_{\lambda+\rho}$. Since the element x_Π of $\mathcal{L}$ is invertible, we thus obtain

$$a_{\lambda+\rho} = \underbrace{\left(x_\Pi^d\right)^{-1}}_{=x_\Pi^{-d}} \underbrace{a_{(\lambda+\rho)+d}}_{=a_\rho u} = x_\Pi^{-d} a_\rho u = a_\rho \cdot x_\Pi^{-d} u.$$

Hence, $a_{\lambda+\rho}$ is a multiple of a_ρ. This completes the proof of Lemma 4.11. □

Definition 4.12. Let λ be a snake. We define an element $\overline{s}_\lambda \in \mathcal{L}$ by $\overline{s}_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$. (This is well-defined, because Lemma 4.11 shows that $a_{\lambda+\rho}$ is a multiple of a_ρ in $\mathcal{L}$, and because Lemma 4.9 shows that the fraction $\dfrac{a_{\lambda+\rho}}{a_\rho}$ is uniquely defined.)

It makes sense to refer to the elements $\overline{s}_\lambda$ just defined as "*Schur Laurent polynomials*". In fact, as the following lemma shows, they are identical with the Schur polynomials $s_\lambda(x_1, x_2, \ldots, x_n)$ when the snake λ is nonnegative:

Lemma 4.13. *Let* $\lambda \in \operatorname{Par}[n]$. *Then,*

$$\overline{s}_\lambda = s_\lambda(x_1, x_2, \ldots, x_n).$$

Proof of Lemma 4.13. We know that λ is a partition of length $\leqslant n$ (since $\lambda \in \operatorname{Par}[n]$). Hence, λ is a nonnegative snake. Furthermore, since λ is a partition of length $\leqslant n$, we can apply [10, Corollary 2.6.7] and obtain $s_\lambda(x_1, x_2, \ldots, x_n) = \dfrac{a_{\lambda+\rho}}{a_\rho} = \overline{s}_\lambda$ (since $\overline{s}_\lambda$ was defined to be $\dfrac{a_{\lambda+\rho}}{a_\rho}$). This proves Lemma 4.13. □

The Schur Laurent polynomials $\overline{s}_\lambda$ appear in Stembridge's [22], where they are named s_λ. (The equivalence of our definition with his follows from [22, Theorem 7.1].)

The following lemma is an analogue of Lemma 4.10 for Schur Laurent polynomials:

Lemma 4.14. *Let $\lambda \in \mathbb{Z}^n$ be any snake, and let $d \in \mathbb{Z}$. Then, $\overline{s}_{\lambda+d} = x_\Pi^d \overline{s}_\lambda$.*

Proof of Lemma 4.14. This follows easily by applying Lemma 4.10 to $\lambda + \rho$ instead of λ. □

Lemma 4.15. *Let $\mu, \nu \in \operatorname{Par}[n]$. Then,*

$$\overline{s}_\mu \overline{s}_\nu = \sum_{\lambda \in \operatorname{Par}[n]} c_{\mu,\nu}^\lambda \overline{s}_\lambda.$$

Proof of Lemma 4.15. It is well-known (see, e.g., [10, Exercise 2.3.8(b)]) that if λ is a partition having length $> n$, then

$$s_\lambda(x_1, x_2, \ldots, x_n) = 0. \tag{4.2}$$

We have $\mu \in \operatorname{Par}[n]$. Hence, Lemma 4.13 (applied to $\lambda = \mu$) yields the equality $\overline{s}_\mu = s_\mu(x_1, x_2, \ldots, x_n)$. Likewise, $\overline{s}_\nu = s_\nu(x_1, x_2, \ldots, x_n)$. Multiplying these two equalities, we obtain

$$\begin{aligned} \overline{s}_\mu \overline{s}_\nu &= s_\mu(x_1, x_2, \ldots, x_n) \cdot s_\nu(x_1, x_2, \ldots, x_n) \\ &= \sum_{\lambda \in \operatorname{Par}} c_{\mu,\nu}^\lambda s_\lambda(x_1, x_2, \ldots, x_n) \end{aligned} \tag{4.3}$$

(where the last equality sign follows by substituting $0, 0, 0, \ldots$ for $x_{n+1}, x_{n+2}, x_{n+3}, \ldots$ in (1.1)). But the sum on the right hand side of (4.3) can be split into two sums: one collecting all addends with $\lambda \in \operatorname{Par}[n]$, and one collecting all remaining addends. The second of these sums is 0, because if $\lambda \in \operatorname{Par}$ satisfies $\lambda \notin \operatorname{Par}[n]$, then λ has length $> n$ and therefore satisfies $s_\lambda(x_1, x_2, \ldots, x_n) = 0$ (by (4.2)), so the corresponding addend vanishes. Thus, only the first sum survives. Hence, (4.3) simplifies to

$$\overline{s}_\mu \overline{s}_\nu = \sum_{\lambda \in \operatorname{Par}[n]} c_{\mu,\nu}^\lambda \underbrace{s_\lambda(x_1, x_2, \ldots, x_n)}_{\substack{=\overline{s}_\lambda \\ \text{(by Lemma 4.13)}}} = \sum_{\lambda \in \operatorname{Par}[n]} c_{\mu,\nu}^\lambda \overline{s}_\lambda.$$ □

Lemma 4.16. *The family $(\overline{s}_\lambda)_{\lambda \in \{\text{snakes}\}}$ of elements of $\mathcal{L}$ is* $\mathbf{k}$*-linearly independent.*

Proof of Lemma 4.16. Let us define a *strict snake* to be an n-tuple $\alpha \in \mathbb{Z}^n$ of integers satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. It is easy to see that the map

$$\begin{aligned} \{\text{snakes}\} &\to \{\text{strict snakes}\}, \\ \lambda &\mapsto \lambda + \rho \end{aligned} \tag{4.4}$$

is a bijection.

It is also easy to see that any two strict snakes α and β satisfy

$$\left(\text{the coefficient of } x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \text{ in } a_\alpha\right) = \delta_{\alpha,\beta}, \tag{4.5}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta of α and β (that is, the integer 1 if $\alpha = \beta$, or the integer 0 otherwise). From this, it is easily seen that the family $(a_\alpha)_{\alpha\in\{\text{strict snakes}\}}$ is $\mathbf{k}$-linearly independent. But this family $(a_\alpha)_{\alpha\in\{\text{strict snakes}\}}$ is just a re-indexing of the family $(a_{\lambda+\rho})_{\lambda\in\{\text{snakes}\}}$ (since the map (4.4) is a bijection). Hence, the latter family $(a_{\lambda+\rho})_{\lambda\in\{\text{snakes}\}}$ must be $\mathbf{k}$-linearly independent, too. Therefore, the family $\left(\dfrac{a_{\lambda+\rho}}{a_\rho}\right)_{\lambda\in\{\text{snakes}\}}$ is also $\mathbf{k}$-linearly independent (since any $\mathbf{k}$-linear dependence relation between the $\dfrac{a_{\lambda+\rho}}{a_\rho}$ would yield a corresponding $\mathbf{k}$-linear dependence relation between the $a_{\lambda+\rho}$). But this latter family is precisely the family $(\overline{s}_\lambda)_{\lambda\in\{\text{snakes}\}}$ (by the definition of $\overline{s}_\lambda$). Hence, the family $(\overline{s}_\lambda)_{\lambda\in\{\text{snakes}\}}$ is $\mathbf{k}$-linearly independent. $\square$

Lemma 4.16 is actually part of a stronger claim: The family $(\overline{s}_\lambda)_{\lambda\in\{\text{snakes}\}}$ is a basis of the $\mathbf{k}$-module of symmetric Laurent polynomials in $x_1, x_2, \ldots, x_n$. We shall not need this, however, so we omit the proof (which follows easily from Lemma 4.14 and the analogous result for symmetric polynomials, which is well-known).

Recall Definition 4.3 **(d)**. Our next lemma connects the Laurent polynomials $\overline{s}_\lambda$ and $\overline{s}_{\lambda^\vee}$ for every snake λ; it is folklore (see [10, Exercise 2.9.15(d)] for an equivalent version), but we have not seen it stated in this exact form in the literature.

Lemma 4.17. *Let λ be a snake. Then,*

$$\overline{s}_{\lambda^\vee} = \overline{s}_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

Here, of course, $\overline{s}_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right)$ means the result of substituting the elements x_1^{-1}, $x_2^{-1}, \ldots, x_n^{-1}$ for $x_1, x_2, \ldots, x_n$ in the Laurent polynomial $\overline{s}_\lambda \in \mathcal{L}$.

Proof of Lemma 4.17. This follows by fairly straightforward manipulation of determinants, using the definitions of $\overline{s}_\lambda$ and $\overline{s}_{\lambda^\vee}$. Again, we refer to [9] for the details. $\square$

4.2. h_k^+, h_k^- and the Pieri rule

Definition 4.18. Let $k \in \mathbb{Z}$. Then, we define two Laurent polynomials $h_k^+ \in \mathcal{L}$ and $h_k^- \in \mathcal{L}$ by

$$h_k^+ = h_k(x_1, x_2, \ldots, x_n) \qquad \text{and}$$
$$h_k^- = h_k\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right).$$

Note that if $k \in \mathbb{Z}$ is negative, then $h_k^+ = 0$ (since $h_k = 0$) and $h_k^- = 0$ (similarly).

We begin by describing h_k^+ as a Schur Laurent polynomial:

Lemma 4.19. *Let $k \in \mathbb{N}$. Then, the partition (k) is a nonnegative snake (when regarded as the n-tuple $(k, 0, 0, \ldots, 0)$), and satisfies*

$$\overline{s}_{(k)} = h_k^+.$$

Proof of Lemma 4.19. It is well-known that $s_{(k)} = h_k$. Substituting $0, 0, 0, \ldots$ for the variables $x_{n+1}, x_{n+2}, x_{n+3}, \ldots$ on both sides of this equality, we obtain $\overline{s}_{(k)} = h_k^+$. This proves Lemma 4.19. $\square$

Next, we need to know what happens when a Schur Laurent polynomial $\overline{s}_\lambda$ is multiplied by some h_k^+. We will answer this question using the classical *first Pieri rule*. To state the answer, we introduce some more notation:

Definition 4.20. Let $\lambda \in \mathbb{Z}^n$. Then, we define the *size* $|\lambda|$ of λ to be the integer $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Definition 4.21. Let $\lambda, \mu \in \mathbb{Z}^n$. Then, we write that $\mu \rightharpoonup \lambda$ if and only if we have

$$\mu_1 \geqslant \lambda_1 \geqslant \mu_2 \geqslant \lambda_2 \geqslant \ldots \geqslant \mu_n \geqslant \lambda_n. \tag{4.6}$$

In other words, we write that $\mu \rightharpoonup \lambda$ if and only if we have

$$\begin{aligned} &(\mu_i \geqslant \lambda_i \text{ for each } i \in \{1, 2, \ldots, n\}) \qquad \text{and} \\ &(\lambda_i \geqslant \mu_{i+1} \text{ for each } i \in \{1, 2, \ldots, n-1\}). \end{aligned}$$

The following two propositions are easily proved:

Proposition 4.22.

(a) *If $\lambda, \mu \in \mathbb{Z}^n$, then $|\lambda + \mu| = |\lambda| + |\mu|$.*

(b) *If $\lambda \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$, then $|\lambda + d| = |\lambda| + nd$.*

(c) *If $\lambda \in \mathbb{Z}^n$, then $|\lambda^\vee| = -|\lambda|$.*

Proposition 4.23. *Let $\lambda, \mu \in \mathbb{Z}^n$.*

(a) *If $\mu \rightharpoonup \lambda$, then both λ and μ are snakes.*

(b) *We have $\mu \rightharpoonup \lambda$ if and only if $\lambda^\vee \rightharpoonup \mu^\vee$.*

(c) *Let $d \in \mathbb{Z}$. Then, we have $\mu \rightharpoonup \lambda$ if and only if $\mu + d \rightharpoonup \lambda + d$.*

We can now state the Pieri rule in the form we need:

Proposition 4.24. *Let λ be a snake. Let $k \in \mathbb{Z}$. Then,*

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu| - |\lambda| = k}} \overline{s}_\mu. \tag{4.7}$$

Proof of Proposition 4.24. We follow the same right-shift strategy as we did in our proof of Lemma 4.11. Thus, our proof shall consist of two steps:

Step 1: We will prove Proposition 4.24 in the particular case when λ is nonnegative.

Step 2: We will use Lemma 4.14 to derive the general case of Proposition 4.24 from this particular case.

Here are some details on the two steps (again, more can be found in [9]):

Step 1: Let us prove that Proposition 4.24 holds in the particular case when λ is nonnegative. Indeed, let us assume that λ is nonnegative. We must prove the equality (4.7).

If $k < 0$, then both sides of this equality are 0 (indeed, the sum on the right hand side is empty, since $\mu \rightharpoonup \lambda$ implies $|\mu| - |\lambda| \geqslant 0$). Thus, the equality (4.7) holds if $k < 0$. Therefore, for the rest of Step 1, we WLOG assume that $k \geqslant 0$.

Note that λ is a partition of length $\leqslant n$ (since λ is a nonnegative snake). In other words, $\lambda \in \operatorname{Par}[n]$.

We will use some standard notations concerning partitions. Specifically:

- The *size* $|\mu|$ of a partition $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ is defined to be $\mu_1 + \mu_2 + \mu_3 + \cdots \in \mathbb{N}$.
- If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ and $\beta = (\beta_1, \beta_2, \beta_3, \dots)$ are two partitions, then we will write $\alpha \subseteq \beta$ if and only if each $i \in \{1, 2, 3, \dots\}$ satisfies $\alpha_i \leqslant \beta_i$.
- If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ and $\beta = (\beta_1, \beta_2, \beta_3, \dots)$ are two partitions, then we say that α/β *is a horizontal strip* if they satisfy

$$\beta \subseteq \alpha \text{ and } \beta_i \geqslant \alpha_{i+1} \text{ for every } i \in \{1, 2, 3, \dots\}.$$

  (This is not the usual definition of a "horizontal strip", but it is equivalent to that definition; the equivalence follows from [10, Exercise 2.7.5(a)].)
- If α and β are two partitions, and if $k \in \mathbb{N}$, then we say that α/β *is a horizontal k-strip* if α/β is a horizontal strip satisfying $|\alpha| - |\beta| = k$.

The following is easy to see:

Claim 1: We have

$$\begin{aligned} &\{\text{partitions } \mu \in \operatorname{Par}[n] \text{ such that } \mu/\lambda \text{ is a horizontal } k\text{-strip}\} \\ &= \{\text{snakes } \mu \text{ such that } \mu \rightharpoonup \lambda \text{ and } |\mu| - |\lambda| = k\}. \end{aligned}$$

[*Proof of Claim 1:* Unravel the definitions and recall that partitions in $\operatorname{Par}[n]$ are the same as nonnegative snakes. We leave the details to the reader.]

From the first Pieri rule ([10, (2.7.1)][5], applied to k instead of n), we obtain

$$s_\lambda h_k = \sum_{\substack{\lambda^+ \in \operatorname{Par}; \\ \lambda^+/\lambda \text{ is a horizontal } k\text{-strip}}} s_{\lambda^+} = \sum_{\substack{\mu \in \operatorname{Par}; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} s_\mu$$

(here, we have renamed the summation index λ^+ as μ).

[5] This also appears in [16, Theorem 5.3], in [21, Theorem 7.15.7] and in [4, Theorem 9.3].

Evaluating both sides of this equality at $x_1, x_2, \ldots, x_n$, we find

$$\left(s_\lambda h_k\right)\left(x_1, x_2, \ldots, x_n\right) = \sum_{\substack{\mu \in \operatorname{Par}; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} s_\mu\left(x_1, x_2, \ldots, x_n\right) = \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} s_\mu\left(x_1, x_2, \ldots, x_n\right)$$

(where the last equality sign follows from (4.2) by a similar argument as in the proof of Lemma 4.15 above). Comparing this with

$$\left(s_\lambda h_k\right)\left(x_1, x_2, \ldots, x_n\right) = \underbrace{s_\lambda\left(x_1, x_2, \ldots, x_n\right)}_{\substack{=\overline{s}_\lambda \\ \text{(by Lemma 4.13)}}} \cdot \underbrace{h_k\left(x_1, x_2, \ldots, x_n\right)}_{=h_k^+} = \overline{s}_\lambda \cdot h_k^+ = h_k^+ \cdot \overline{s}_\lambda,$$

we obtain

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} \underbrace{s_\mu\left(x_1, x_2, \ldots, x_n\right)}_{\substack{=\overline{s}_\mu \\ \text{(by Lemma 4.13,} \\ \text{applied to } \mu \text{ instead of } \lambda)}} = \sum_{\substack{\mu \in \operatorname{Par}[n]; \\ \mu/\lambda \text{ is a horizontal } k\text{-strip}}} \overline{s}_\mu = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu$$

(where we used Claim 1 to rewrite the summation sign). This proves (4.7). Thus, Proposition 4.24 is proved under the assumption that λ is nonnegative. This completes Step 1.

Step 2: We now need to prove Proposition 4.24 in the general case.

The idea is to find an integer d such that the snake $\lambda + d$ is nonnegative (for example, $d = -\lambda_n$), and apply Proposition 4.24 to $\lambda + d$ instead of λ (which we can do, since Step 1 has already covered this case). This yields

$$h_k^+ \cdot \overline{s}_{\lambda+d} = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda+d;\ |\mu|-|\lambda+d|=k}} \overline{s}_\mu = \sum_{\substack{\mu+d \text{ is a snake;} \\ \mu+d \rightharpoonup \lambda+d;\ |\mu+d|-|\lambda+d|=k}} \overline{s}_{\mu+d}$$

(here, we substituted $\mu + d$ for μ in the sum). The conditions under the summation sign on the right hand side can be simplified using Proposition 4.22 **(b)** and Proposition 4.23 **(c)**, and the addends $\overline{s}_{\mu+d}$ can be rewritten as $x_\Pi^d \overline{s}_\mu$ using Lemma 4.14. Thus, the equality simplifies to

$$h_k^+ \cdot \overline{s}_{\lambda+d} = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} x_\Pi^d \overline{s}_\mu = x_\Pi^d \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu.$$

Since Lemma 4.14 yields $\overline{s}_{\lambda+d} = x_\Pi^d \overline{s}_\lambda$, we can rewrite this as

$$h_k^+ \cdot x_\Pi^d \overline{s}_\lambda = x_\Pi^d \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu.$$

We can cancel x_{Π}^{d} from this equality (since $x_{\Pi} \in \mathcal{L}$ is invertible), and thus obtain

$$h_k^+ \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu .$$

This proves Proposition 4.24. □

Using Lemma 4.17, we can "turn Proposition 4.24 upside down", obtaining the following analogous result for h_k^- instead of h_k^+:

Proposition 4.25. *Let λ be a snake. Let $k \in \mathbb{Z}$. Then,*

$$h_k^- \cdot \overline{s}_\lambda = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda \rightharpoonup \mu;\ |\lambda|-|\mu|=k}} \overline{s}_\mu . \tag{4.8}$$

Proof of Proposition 4.25. It is easy to see that $(\lambda^\vee)^\vee = \lambda$. Likewise, $(\mu^\vee)^\vee = \mu$ for any snake μ. Hence, the map $\{\text{snakes}\} \to \{\text{snakes}\}$, $\mu \mapsto \mu^\vee$ is inverse to itself, and thus is a bijection. It is also easy to see (using Proposition 4.22 **(c)**) that every snake μ satisfies

$$|\lambda^\vee| - |\mu^\vee| = |\mu| - |\lambda| . \tag{4.9}$$

Comparing the definitions of h_k^- and h_k^+ easily yields

$$h_k^- = h_k^+ \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) .$$

Also, Lemma 4.17 yields $\overline{s}_{\lambda^\vee} = \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right)$. Multiplying these two equalities, we obtain

$$\begin{aligned}
h_k^- \cdot \overline{s}_{\lambda^\vee} &= h_k^+ \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \cdot \overline{s}_\lambda \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\
&= \left(h_k^+ \cdot \overline{s}_\lambda\right) \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \\
&= \sum_{\substack{\mu \text{ is a snake;} \\ \mu \rightharpoonup \lambda;\ |\mu|-|\lambda|=k}} \overline{s}_\mu \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right) \qquad \text{(by Proposition 4.24)} \\
&= \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu^\vee;\ |\lambda^\vee|-|\mu^\vee|=k}} \overline{s}_\mu \left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right)
\end{aligned}$$

(here, we used Proposition 4.23 **(b)** to replace the "$\mu \rightharpoonup \lambda$" under the summation sign by "$\lambda^\vee \rightharpoonup \mu^\vee$", and we used (4.9) to replace the "$|\mu| - |\lambda|$" under the summation sign by "$|\lambda^\vee| - |\mu^\vee|$").

Comparing this with

$$\sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu;\ |\lambda^\vee|-|\mu|=k}} \overline{s}_\mu = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu^\vee;\ |\lambda^\vee|-|\mu^\vee|=k}} \underbrace{\overline{s}_{\mu^\vee}}_{\substack{=\overline{s}_\mu\left(x_1^{-1},x_2^{-1},\ldots,x_n^{-1}\right) \\ \text{(by Lemma 4.17,} \\ \text{applied to } \mu \text{ instead of } \lambda)}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } \mu^\vee \text{ for } \mu \text{ in the sum,} \\ \text{since the map } \{\text{snakes}\} \to \{\text{snakes}\},\ \mu \mapsto \mu^\vee \\ \text{is a bijection}\end{array}\right)$$

$$= \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu^\vee;\ |\lambda^\vee|-|\mu^\vee|=k}} \overline{s}_\mu\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right),$$

we obtain

$$h_k^- \cdot \overline{s}_{\lambda^\vee} = \sum_{\substack{\mu \text{ is a snake;} \\ \lambda^\vee \rightharpoonup \mu;\ |\lambda^\vee|-|\mu|=k}} \overline{s}_\mu .$$

We have proved this equality for any snake λ. Thus, we can apply it to $\lambda^\vee$ instead of λ. We obtain

$$h_k^- \cdot \overline{s}_{(\lambda^\vee)^\vee} = \sum_{\substack{\mu \text{ is a snake;} \\ (\lambda^\vee)^\vee \rightharpoonup \mu;\ \left|(\lambda^\vee)^\vee\right|-|\mu|=k}} \overline{s}_\mu .$$

But because of $(\lambda^\vee)^\vee = \lambda$, this equality is precisely (4.8). Thus, Proposition 4.25 is proved. $\square$

4.3. Computing $\overline{s}_\alpha$

Convention 4.26. From now on, for the rest of Section 4, we assume that $n \geqslant 2$.

Our next goal is to obtain a simple formula for the Schur polynomial $\overline{s}_\alpha$, where α is as in Theorem 2.3. The first step is the following definition:

Definition 4.27. Let $a, b \in \mathbb{N}$. Then, $b \ominus a$ will denote the snake $(b, 0^{n-2}, -a)$. (This is indeed a well-defined snake, since $n \geqslant 2$ and since $b \geqslant 0 \geqslant -a$.)

Proposition 4.28. *Let $a, b \in \mathbb{Z}$. Then,*

$$h_a^- h_b^+ = \sum_{k=0}^{\min\{a,b\}} \overline{s}_{(b-k)\ominus(a-k)} . \tag{4.10}$$

Proof of Proposition 4.28. This is trivial when $\min\{a, b\} < 0$, and otherwise follows easily from Proposition 4.25 (applied to $\lambda = (b)$ and $k = a$). Details can be found in [9]. $\square$

Proposition 4.29. *Let $a, b \in \mathbb{N}$. Then,*

$$\overline{s}_{b\ominus a} = h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+ .$$

(Recall that every negative integer k satisfies $h_k^- = 0$ and $h_k^+ = 0$.)

Proof of Proposition 4.29. Apply Proposition 4.28 twice (once to a and b, and once to $a-1$ and $b-1$), and subtract. See [9] for the details. $\square$

Remark 4.30. The right hand side in Proposition 4.28 looks suspiciously like a determinant. This is no coincidence, and Proposition 4.28 can in fact be generalized to a determinantal formula for $\overline{s}_\lambda$ where λ is any snake of the form $(b_1, b_2, \ldots, b_q, 0^{n-p-q}, -a_p, -a_{p-1}, \ldots, -a_1)$. The latter formula can be obtained from an identity of Koike [13, Proposition 2.8] (see also [11, (6) and (10)]). See [9, §5.1] for some more details.

Corollary 4.31. *Let $a, b \in \mathbb{N}$. Define the partition $\alpha = (a+b, a^{n-2})$. Then, α is a nonnegative snake and satisfies*

$$\overline{s}_\alpha = x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right). \tag{4.11}$$

Proof of Corollary 4.31. It is easy to see that $\alpha = (b \ominus a) + a$ (regarded as snakes). Hence, Lemma 4.14 (applied to $\lambda = b \ominus a$ and $d = a$) yields

$$\overline{s}_\alpha = x_\Pi^a \overline{s}_{b\ominus a} = x_\Pi^a \cdot \left(h_a^- h_b^+ - h_{a-1}^- h_{b-1}^+\right) \qquad \text{(by Proposition 4.29)}.$$

This proves Corollary 4.31. $\square$

4.4. The sets $R_{\mu,a,b}(\gamma)$ and a formula for $h_a^- h_b^+ \overline{s}_\mu$

We shall next aim for a formula for $h_a^- h_b^+ \overline{s}_\mu$ (for a snake μ and integers $a, b \in \mathbb{Z}$), which will be obtained in a straightforward way by applying Propositions 4.24 and 4.25. We will need the following definition:

Definition 4.32. Let $\mu, \gamma \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$. Then, $R_{\mu,a,b}(\gamma)$ shall denote the set of all snakes ν satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b.$$

Lemma 4.33. *Let $\mu, \gamma \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$. Assume that γ is not a snake. Then, $|R_{\mu,a,b}(\gamma)| = 0$.*

Proof of Lemma 4.33. Let $\nu \in R_{\mu,a,b}(\gamma)$. We shall obtain a contradiction.

Indeed, $\nu \in R_{\mu,a,b}(\gamma)$ means that ν is a snake satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b$$

(by the definition of $R_{\mu,a,b}(\gamma)$). Thus, in particular, we have $\gamma \rightharpoonup \nu$. Hence, Proposition 4.23 **(a)** (applied to γ and ν instead of μ and λ) yields that both ν and γ are snakes. Hence, γ is a snake. This contradicts the fact that γ is not a snake.

We thus have obtained a contradiction for each $\nu \in R_{\mu,a,b}(\gamma)$. Hence, there exists no $\nu \in R_{\mu,a,b}(\gamma)$. In other words, $|R_{\mu,a,b}(\gamma)| = 0$. $\square$

Lemma 4.34. *Let μ be a snake. Let $a, b \in \mathbb{Z}$. Then,*

$$h_a^- h_b^+ \overline{s}_\mu = \sum_{\gamma \text{ is a snake}} \left| R_{\mu,a,b}\left(\gamma\right) \right| \overline{s}_\gamma .$$

Proof of Lemma 4.34. Proposition 4.25 (with the letters λ, k and μ renamed as μ, a and ν) says that

$$h_a^- \cdot \overline{s}_\mu = \sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a}} \overline{s}_\nu . \tag{4.12}$$

Proposition 4.24 (with the letters λ, k and μ renamed as ν, b and γ) says that

$$h_b^+ \cdot \overline{s}_\nu = \sum_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoondown \nu;\ |\gamma| - |\nu| = b}} \overline{s}_\gamma \tag{4.13}$$

for each snake ν.

Now,

$$\begin{aligned} h_a^- h_b^+ \overline{s}_\mu &= h_b^+ \cdot \underbrace{h_a^- \cdot \overline{s}_\mu}_{\substack{= \sum\limits_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a}} \overline{s}_\nu \\ \text{(by (4.12))}}} = \sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a}} \underbrace{h_b^+ \cdot \overline{s}_\nu}_{\substack{= \sum\limits_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoondown \nu;\ |\gamma| - |\nu| = b}} \overline{s}_\gamma \\ \text{(by (4.13))}}} \\ &= \underbrace{\sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a}} \sum_{\substack{\gamma \text{ is a snake;} \\ \gamma \rightharpoondown \nu;\ |\gamma| - |\nu| = b}}}_{= \sum\limits_{\gamma \text{ is a snake}} \sum\limits_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a; \\ \gamma \rightharpoondown \nu;\ |\gamma| - |\nu| = b}}} \overline{s}_\gamma = \sum_{\gamma \text{ is a snake}} \underbrace{\sum_{\substack{\nu \text{ is a snake;} \\ \mu \rightharpoondown \nu;\ |\mu| - |\nu| = a; \\ \gamma \rightharpoondown \nu;\ |\gamma| - |\nu| = b}}}_{\substack{= \sum\limits_{\nu \in R_{\mu,a,b}(\gamma)} \\ \text{(by the definition} \\ \text{of } R_{\mu,a,b}(\gamma))}} \overline{s}_\gamma \\ &= \sum_{\gamma \text{ is a snake}} \underbrace{\sum_{\nu \in R_{\mu,a,b}(\gamma)} \overline{s}_\gamma}_{= \left| R_{\mu,a,b}(\gamma) \right| \overline{s}_\gamma} = \sum_{\gamma \text{ is a snake}} \left| R_{\mu,a,b}\left(\gamma\right) \right| \overline{s}_\gamma . \end{aligned}$$

This proves Lemma 4.34. ☐

Corollary 4.35. *Let $\mu \in \operatorname{Par}[n]$. Let $a, b \in \mathbb{N}$. Define the partition $\alpha = (a + b, a^{n-2})$. Then, every $\lambda \in \mathbb{Z}^n$ satisfies*

$$c_{\alpha,\mu}^{\lambda} = \left| R_{\mu,a,b}\left(\lambda - a\right) \right| - \left| R_{\mu,a-1,b-1}\left(\lambda - a\right) \right| . \tag{4.14}$$

Here, we understand $c_{\alpha,\mu}^{\lambda}$ to mean 0 if λ is not a partition (i.e., if λ is not a nonnegative snake).

Proof of Corollary 4.35. Every snake γ satisfies

$$\overline{s}_{\gamma + a} = x_{\Pi}^{a} \overline{s}_\gamma \tag{4.15}$$

(by Lemma 4.14, applied to γ and a instead of λ and d).

But α is a nonnegative snake; thus, $\alpha \in \operatorname{Par}[n]$. Hence, Lemma 4.15 (applied to α and μ instead of μ and ν) yields

$$\begin{aligned}
\overline{s}_\alpha \overline{s}_\mu &= \sum_{\lambda \in \operatorname{Par}[n]} c^\lambda_{\alpha,\mu} \overline{s}_\lambda = \sum_{\substack{\lambda \text{ is a snake;} \\ \lambda \text{ is nonnegative}}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda \\
&\qquad \left(\text{since the partitions } \lambda \in \operatorname{Par}[n] \text{ are precisely the nonnegative snakes}\right) \\
&= \sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda
\end{aligned}$$

(where the last equality sign is owed to the fact that we understand $c^\lambda_{\alpha,\mu}$ to mean 0 if λ is not a nonnegative snake). Hence,

$$\begin{aligned}
\sum_{\lambda \text{ is a snake}} c^\lambda_{\alpha,\mu} \overline{s}_\lambda &= \overline{s}_\alpha \overline{s}_\mu = x^a_\Pi \cdot \left(h^-_a h^+_b - h^-_{a-1} h^+_{b-1}\right) \overline{s}_\mu \qquad (\text{by } (4.11)) \\
&= x^a_\Pi \cdot \underbrace{h^-_a h^+_b \overline{s}_\mu}_{\substack{= \sum_{\gamma \text{ is a snake}} \left|R_{\mu,a,b}(\gamma)\right| \overline{s}_\gamma \\ \text{(by Lemma 4.34)}}} - x^a_\Pi \cdot \underbrace{h^-_{a-1} h^+_{b-1} \overline{s}_\mu}_{\substack{= \sum_{\gamma \text{ is a snake}} \left|R_{\mu,a-1,b-1}(\gamma)\right| \overline{s}_\gamma \\ \text{(by Lemma 4.34,} \\ \text{applied to } a-1 \text{ and } b-1 \\ \text{instead of } a \text{ and } b)}} \\
&= x^a_\Pi \cdot \sum_{\gamma \text{ is a snake}} \left|R_{\mu,a,b}(\gamma)\right| \overline{s}_\gamma - x^a_\Pi \cdot \sum_{\gamma \text{ is a snake}} \left|R_{\mu,a-1,b-1}(\gamma)\right| \overline{s}_\gamma \\
&= \sum_{\gamma \text{ is a snake}} \left(\left|R_{\mu,a,b}(\gamma)\right| - \left|R_{\mu,a-1,b-1}(\gamma)\right|\right) \underbrace{x^a_\Pi \overline{s}_\gamma}_{\substack{= \overline{s}_{\gamma+a} \\ \text{(by (4.15))}}} \\
&= \sum_{\gamma \text{ is a snake}} \left(\left|R_{\mu,a,b}(\gamma)\right| - \left|R_{\mu,a-1,b-1}(\gamma)\right|\right) \overline{s}_{\gamma+a}.
\end{aligned}$$

We can compare coefficients on both sides of this equality (since Lemma 4.16 shows that the family $(\overline{s}_\lambda)_{\lambda \in \{\text{snakes}\}}$ of elements of $\mathcal{L}$ is $\mathbf{k}$-linearly independent), and thus conclude that

$$c^\lambda_{\alpha,\mu} = \left|R_{\mu,a,b}(\lambda - a)\right| - \left|R_{\mu,a-1,b-1}(\lambda - a)\right| \qquad \text{for every snake } \lambda.$$

This proves (4.14) in the case when λ is a snake.

However, it is easy to see that (4.14) also holds in the case when λ is not a snake[6]. Thus, (4.14) always holds. This proves Corollary 4.35. □

4.5. The map $\mathbf{f}_\mu$

Convention 4.36. For the whole Subsection 4.5, we shall use Convention 3.6 (not only for n-tuples $a \in \mathbb{K}^n$, but for any n-tuples a). This convention does not conflict with Convention 4.2, because both conventions define γ_i in the same way when γ is an n-tuple and $i \in \{1, 2, \ldots, n\}$.

[6]Indeed, if $\lambda \in \mathbb{Z}^n$ is not a snake, then $\lambda - a$ is not a snake either, and thus the equality (4.14) boils down to $c^\lambda_{\alpha,\mu} = 0 - 0$ (by Lemma 4.33); but this is true, since we have defined $c^\lambda_{\alpha,\mu}$ to be 0 if λ is not a nonnegative snake.

Convention 3.6 does conflict with our old convention (from Section 1) to identify partitions with finite tuples: Indeed, if we let γ be the n-tuple $(1, 1, \ldots, 1)$, then Convention 3.6 yields $\gamma_{n+1} = \gamma_1 = 1$ when we regard γ as an n-tuple, but we get $\gamma_{n+1} = 0$ if we regard γ as a partition. We shall resolve this conflict by agreeing **not to identify partitions with finite tuples in Subsection 4.5**. (Thus, in particular, we will not identify a nonnegative snake $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n$ with its corresponding partition $(\mu_1, \mu_2, \ldots, \mu_n, 0, 0, 0, \ldots) \in \operatorname{Par}[n]$.)

Let us now apply the results of Section 3. The abelian group $(\mathbb{Z}, +, 0)$ of integers is totally ordered (in the usual way). Thus, Example 3.3 (applied to $(\mathbb{A}, *, e) = (\mathbb{Z}, +, 0)$) shows that there is a semifield $(\mathbb{Z}, \min, +, 0)$ (that is, a semifield with ground set $\mathbb{Z}$, addition $\min$, multiplication $+$ and unity 0), called the *min tropical semifield* of $(\mathbb{Z}, +, 0)$. We have the following little dictionary between various operations on this semifield $(\mathbb{Z}, \min, +, 0)$ and familiar operations on integers:

- For any $a, b \in \mathbb{Z}$, the sum $a + b$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer $\min\{a, b\}$.
- If $r \in \mathbb{N}$, and if $a_0, a_1, \ldots, a_r \in \mathbb{Z}$, then the sum $\sum_{k=0}^{r} a_k$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is $\min\{a_0, a_1, \ldots, a_r\} = \min\{a_k \mid k \in \{0, 1, \ldots, r\}\}$.
- For any $a, b \in \mathbb{Z}$, the product ab understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer $a + b$.
- The unity of the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer 0.
- For any $a, b \in \mathbb{Z}$, the quotient $\frac{a}{b}$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is precisely the difference $a - b$ understood with respect to the integer ring $\mathbb{Z}$.
- For any $a \in \mathbb{Z}$, the square a^2 understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the product $2a$ understood with respect to the integer ring $\mathbb{Z}$.
- For any $a \in \mathbb{Z}$, the reciprocal $\frac{1}{a}$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the integer $-a$ understood with respect to the integer ring $\mathbb{Z}$.
- If $r \in \mathbb{N}$, and if $a_1, a_2, \ldots, a_r$ are any r integers, then the product $\prod_{k=1}^{r} a_k$ understood with respect to the semifield $(\mathbb{Z}, \min, +, 0)$ is the sum $\sum_{k=1}^{r} a_k$ understood with respect to the integer ring $\mathbb{Z}$.

Thus, applying Definition 3.8 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and renaming u, x, $t_{r,j}$ and y as μ, γ, $\tau_{r,j}$ and η), we obtain the following:

Definition 4.37. Fix any n-tuple $\mu \in \mathbb{Z}^n$. We define a map $\mathbf{f}_\mu : \mathbb{Z}^n \to \mathbb{Z}^n$ as follows: Let $\gamma \in \mathbb{Z}^n$ be an n-tuple. For each $j \in \mathbb{Z}$ and $r \in \mathbb{N}$, define an element $\tau_{r,j} \in \mathbb{Z}$ by

$$\begin{aligned}\tau_{r,j} = \min \{&(\gamma_{j+1} + \gamma_{j+2} + \cdots + \gamma_{j+k}) + (\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+r}) \\ &\mid \ k \in \{0, 1, \ldots, r\}\}.\end{aligned}$$

Define $\eta \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{n-1,i-1}) - (\gamma_{i+1} + \tau_{n-1,i+1}) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Set $\mathbf{f}_\mu(\gamma) = \eta$.

Applying Theorem 3.11 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and renaming u, x and y as μ, γ and η), we thus obtain the following (using our above dictionary):

Theorem 4.38. *Fix any n-tuple $\mu \in \mathbb{Z}^n$.*

(a) *The map $\mathbf{f}_\mu$ is an involution (i.e., we have $\mathbf{f}_\mu \circ \mathbf{f}_\mu = \mathrm{id}$).*

(b) *Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,*

$$(\eta_1 + \eta_2 + \cdots + \eta_n) + (\gamma_1 + \gamma_2 + \cdots + \gamma_n) = 2(\mu_1 + \mu_2 + \cdots + \mu_n).$$

(c) *Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,*

$$\min\{\mu_i, \gamma_i\} + \min\{-\mu_{i+1}, -\gamma_{i+1}\} = \min\{\mu_i, \eta_i\} + \min\{-\mu_{i+1}, -\eta_{i+1}\}$$

for each $i \in \mathbb{Z}$.

(d) *Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then,*

$$\sum_{i=1}^{n} (\min\{\mu_i, \gamma_i\} - \gamma_i) = \sum_{i=1}^{n} (\min\{\mu_i, \eta_i\} - \mu_i).$$

We obtain the following corollaries from Theorem 4.38:

Corollary 4.39. *Fix any n-tuple $\mu \in \mathbb{Z}^n$. Let $\gamma \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ be such that $\eta = \mathbf{f}_\mu(\gamma)$. Then:*

(a) *We have $|\eta| - |\mu| = |\mu| - |\gamma|$.*

(b) *We have*

$$\min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\} = \max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\}$$

for each $i \in \{1, 2, \ldots, n-1\}$.

(c) *We have*

$$\sum_{i=1}^{n} (\mu_i - \min\{\mu_i, \eta_i\} + \min\{\mu_i, \gamma_i\}) = \sum_{i=1}^{n} \gamma_i.$$

(d) *We have* $\gamma = \mathbf{f}_\mu(\eta)$.

Proof of Corollary 4.39. **(a)** Theorem 4.38 **(b)** yields

$$(\eta_1 + \eta_2 + \cdots + \eta_n) + (\gamma_1 + \gamma_2 + \cdots + \gamma_n) = 2(\mu_1 + \mu_2 + \cdots + \mu_n).$$

In view of the equalities $|\eta| = \eta_1 + \eta_2 + \cdots + \eta_n$ and $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ and $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$, we can rewrite this as $|\eta| + |\gamma| = 2|\mu|$. Equivalently, $|\eta| - |\mu| = |\mu| - |\gamma|$. This proves Corollary 4.39 **(a)**.

(b) Let $i \in \{1, 2, \ldots, n-1\}$. Then, Theorem 4.38 **(c)** yields

$$\min\{\mu_i, \gamma_i\} + \min\{-\mu_{i+1}, -\gamma_{i+1}\} = \min\{\mu_i, \eta_i\} + \min\{-\mu_{i+1}, -\eta_{i+1}\}.$$

In view of the equalities $\min\{-\mu_{i+1}, -\gamma_{i+1}\} = -\max\{\mu_{i+1}, \gamma_{i+1}\}$ and $\min\{-\mu_{i+1}, -\eta_{i+1}\} = -\max\{\mu_{i+1}, \eta_{i+1}\}$, we can rewrite this as

$$\min\{\mu_i, \gamma_i\} - \max\{\mu_{i+1}, \gamma_{i+1}\} = \min\{\mu_i, \eta_i\} - \max\{\mu_{i+1}, \eta_{i+1}\}.$$

Equivalently,

$$\min\{\mu_i, \eta_i\} - \min\{\mu_i, \gamma_i\} = \max\{\mu_{i+1}, \eta_{i+1}\} - \max\{\mu_{i+1}, \gamma_{i+1}\}.$$

This proves Corollary 4.39 **(b)**.

(c) We have

$$\begin{aligned}
&\sum_{i=1}^{n} \underbrace{(\mu_i - \min\{\mu_i, \eta_i\} + \min\{\mu_i, \gamma_i\})}_{=\min\{\mu_i,\gamma_i\}-(\min\{\mu_i,\eta_i\}-\mu_i)} \\
&= \sum_{i=1}^{n} (\min\{\mu_i, \gamma_i\} - (\min\{\mu_i, \eta_i\} - \mu_i)) \\
&= \sum_{i=1}^{n} \min\{\mu_i, \gamma_i\} - \underbrace{\sum_{i=1}^{n} (\min\{\mu_i, \eta_i\} - \mu_i)}_{\substack{=\sum_{i=1}^{n}(\min\{\mu_i,\gamma_i\}-\gamma_i) \\ \text{(by Theorem 4.38 (d))}}} \\
&= \sum_{i=1}^{n} \min\{\mu_i, \gamma_i\} - \sum_{i=1}^{n} (\min\{\mu_i, \gamma_i\} - \gamma_i) = \sum_{i=1}^{n} \gamma_i.
\end{aligned}$$

This proves Corollary 4.39 **(c)**.

(d) Theorem 4.38 **(a)** shows that $\mathbf{f}_\mu \circ \mathbf{f}_\mu = \mathrm{id}$. But recall that $\eta = \mathbf{f}_\mu(\gamma)$. Applying the map $\mathbf{f}_\mu$ to both sides of this equality, we obtain

$$\mathbf{f}_\mu(\eta) = \mathbf{f}_\mu(\mathbf{f}_\mu(\gamma)) = (\mathbf{f}_\mu \circ \mathbf{f}_\mu)(\gamma) = \gamma$$

(since $\mathbf{f}_\mu \circ \mathbf{f}_\mu = \mathrm{id}$). This proves Corollary 4.39 **(d)**. $\square$

We are now ready to prove the key lemma:

Lemma 4.40. *Fix any n-tuple $\mu \in \mathbb{Z}^n$. Let $\gamma \in \mathbb{Z}^n$. Let $a, b \in \mathbb{Z}$. Then,*

$$\left|R_{\mu,b,a}\left(\mathbf{f}_\mu\left(\gamma\right)\right)\right| = \left|R_{\mu,a,b}\left(\gamma\right)\right|.$$

Proof of Lemma 4.40. Define $\eta \in \mathbb{Z}^n$ by $\eta = \mathbf{f}_\mu\left(\gamma\right)$. We must therefore prove that $\left|R_{\mu,b,a}\left(\eta\right)\right| = \left|R_{\mu,a,b}\left(\gamma\right)\right|$.

We know that $R_{\mu,a,b}\left(\gamma\right)$ is the set of all snakes ν satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b.$$

Likewise, $R_{\mu,b,a}\left(\eta\right)$ is the set of all snakes ν satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = b \qquad \text{and} \qquad \eta \rightharpoonup \nu \qquad \text{and} \qquad |\eta| - |\nu| = a.$$

Now, fix $\nu \in R_{\mu,a,b}\left(\gamma\right)$. Thus, ν is a snake satisfying the four conditions

$$\mu \rightharpoonup \nu \qquad \text{and} \qquad |\mu| - |\nu| = a \qquad \text{and} \qquad \gamma \rightharpoonup \nu \qquad \text{and} \qquad |\gamma| - |\nu| = b$$

(by the definition of $R_{\mu,a,b}\left(\gamma\right)$).

We define an n-tuple $\zeta \in \mathbb{Z}^n$ by setting

$$\zeta_i = \min\left\{\mu_i, \eta_i\right\} - \min\left\{\mu_i, \gamma_i\right\} + \nu_i \qquad \text{for each } i \in \left\{1, 2, \ldots, n\right\}.$$

We shall prove that $\zeta \in R_{\mu,b,a}\left(\eta\right)$. First, we will show some auxiliary claims[7]:

Claim 1: We have $\min\left\{\mu_i, \eta_i\right\} \geqslant \zeta_i$ for each $i \in \left\{1, 2, \ldots, n\right\}$.

[*Proof of Claim 1:* Let $i \in \left\{1, 2, \ldots, n\right\}$. Then, $\mu \rightharpoonup \nu$ yields $\mu_i \geqslant \nu_i$, while $\gamma \rightharpoonup \nu$ yields $\gamma_i \geqslant \nu_i$. Combining these two inequalities, we obtain $\min\left\{\mu_i, \gamma_i\right\} \geqslant \nu_i$, so that $\nu_i - \min\left\{\mu_i, \gamma_i\right\} \leqslant 0$. But the definition of ζ_i yields $\zeta_i - \min\left\{\mu_i, \eta_i\right\} = \nu_i - \min\left\{\mu_i, \gamma_i\right\} \leqslant 0$, so that $\min\left\{\mu_i, \eta_i\right\} \geqslant \zeta_i$. This proves Claim 1.]

Claim 2: We have $\zeta_i \geqslant \max\left\{\mu_{i+1}, \eta_{i+1}\right\}$ for each $i \in \left\{1, 2, \ldots, n-1\right\}$.

[*Proof of Claim 2:* Similar to Claim 1.]

Claim 3: The n-tuple ζ is a snake and satisfies $\mu \rightharpoonup \zeta$ and $\eta \rightharpoonup \zeta$.

[*Proof of Claim 3:* Both statements $\mu \rightharpoonup \zeta$ and $\eta \rightharpoonup \zeta$ follow easily from Claim 1 and Claim 2. Hence, Proposition 4.23 **(a)** shows that ζ is a snake.]

Claim 4: We have $|\mu| - |\zeta| = b$ and $|\eta| - |\zeta| = a$.

[7] Again, see [9] for details.

[*Proof of Claim 4:* We have $|\mu| = \sum_{i=1}^{n} \mu_i$ and $|\zeta| = \sum_{i=1}^{n} \zeta_i$ (by the definition of ζ). Subtracting these two equalities from one another, we find

$$\begin{aligned}
|\mu| - |\zeta| &= \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \zeta_i = \sum_{i=1}^{n} \left(\mu_i - \underbrace{\zeta_i}_{\substack{=\min\{\mu_i,\eta_i\}-\min\{\mu_i,\gamma_i\}+\nu_i \\ \text{(by the definition of } \zeta)}} \right) \\
&= \sum_{i=1}^{n} \underbrace{\left(\mu_i - \left(\min\{\mu_i,\eta_i\} - \min\{\mu_i,\gamma_i\} + \nu_i\right)\right)}_{=(\mu_i - \min\{\mu_i,\eta_i\}+\min\{\mu_i,\gamma_i\})-\nu_i} \\
&= \underbrace{\sum_{i=1}^{n} \left(\mu_i - \min\{\mu_i,\eta_i\} + \min\{\mu_i,\gamma_i\}\right)}_{\substack{=\sum_{i=1}^{n}\gamma_i \\ \text{(by Corollary 4.39 \textbf{(c)})}}} - \sum_{i=1}^{n} \nu_i = \underbrace{\sum_{i=1}^{n} \gamma_i}_{=|\gamma|} - \underbrace{\sum_{i=1}^{n} \nu_i}_{=|\nu|} \\
&= |\gamma| - |\nu| = b.
\end{aligned}$$

Furthermore,

$$|\mu| - |\gamma| = \underbrace{(|\mu| - |\nu|)}_{=a} - \underbrace{(|\gamma| - |\nu|)}_{=b} = a - b$$

and

$$|\eta| - |\zeta| = \underbrace{(|\eta| - |\mu|)}_{\substack{=|\mu|-|\gamma| \\ \text{(by Corollary 4.39 \textbf{(a)})}}} + \underbrace{(|\mu| - |\zeta|)}_{=b} = \underbrace{|\mu| - |\gamma|}_{=a-b} + b = a.$$

Thus, Claim 4 is proven.]

Claim 3 and Claim 4 show that ζ is a snake satisfying the four conditions

$$\mu \rightharpoonup \zeta \qquad \text{and} \qquad |\mu| - |\zeta| = b \qquad \text{and} \qquad \eta \rightharpoonup \zeta \qquad \text{and} \qquad |\eta| - |\zeta| = a.$$

In other words, $\zeta \in R_{\mu,b,a}(\eta)$ (by the definition of $R_{\mu,b,a}(\eta)$).

Forget that we fixed ν. Thus, for each $\nu \in R_{\mu,a,b}(\gamma)$, we have constructed a $\zeta \in R_{\mu,b,a}(\eta)$. Let us denote this ζ by $\widetilde{\nu}$. We thus have defined a map

$$R_{\mu,a,b}(\gamma) \to R_{\mu,b,a}(\eta),$$
$$\nu \mapsto \widetilde{\nu}.$$

Let us denote this map by $\mathbf{g}_{\gamma,a,b}$. Its definition shows that

$$\left(\mathbf{g}_{\gamma,a,b}(\nu)\right)_i = \widetilde{\nu}_i = \min\{\mu_i,\eta_i\} - \min\{\mu_i,\gamma_i\} + \nu_i \tag{4.16}$$

for each $\nu \in R_{\mu,a,b}(\gamma)$ and each $i \in \{1, 2, \ldots, n\}$.

However, from $\eta = \mathbf{f}_\mu(\gamma)$, we obtain $\gamma = \mathbf{f}_\mu(\eta)$ (by Corollary 4.39 **(d)**). The relation between γ and η is thus symmetric. Hence, in the same way as we defined a map $\mathbf{g}_{\gamma,a,b}$:

$R_{\mu,a,b}(\gamma) \to R_{\mu,b,a}(\eta)$, we can define a map $\mathbf{g}_{\eta,b,a} : R_{\mu,b,a}(\eta) \to R_{\mu,a,b}(\gamma)$ (by repeating the above construction of $\mathbf{g}_{\gamma,a,b}$ with b, a, η and γ taking the roles of a, b, γ and η, respectively). The resulting map $\mathbf{g}_{\eta,b,a}$ satisfies

$$(\mathbf{g}_{\eta,b,a}(\nu))_i = \min\{\mu_i, \gamma_i\} - \min\{\mu_i, \eta_i\} + \nu_i \tag{4.17}$$

for each $\nu \in R_{\mu,b,a}(\eta)$ and each $i \in \{1, 2, \ldots, n\}$.

Now it is easy to see (using (4.16) and (4.17)) that the two maps $\mathbf{g}_{\gamma,a,b}$ and $\mathbf{g}_{\eta,b,a}$ are mutually inverse. Hence, these two maps are bijections. Therefore, $|R_{\mu,a,b}(\gamma)| = |R_{\mu,b,a}(\eta)| = |R_{\mu,b,a}(\mathbf{f}_\mu(\gamma))|$ (since $\eta = \mathbf{f}_\mu(\gamma)$). This proves Lemma 4.40. □

Having learned a lot about the map $\mathbf{f}_\mu$, let us now connect it to the map φ defined in Theorem 2.3. For this, we shall use the following lemma:

Lemma 4.41. *Fix any n-tuple $\mu \in \mathbb{Z}^n$. Let $\nu \in \mathbb{Z}^n$ be an n-tuple. For each $j \in \mathbb{Z}$, let*

$$\tau_j = \min\{(\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}) + (\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}) \\ \mid k \in \{0, 1, \ldots, n-1\}\}.$$

Let $\eta \in \mathbb{Z}^n$ be such that

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}) \qquad \textit{for each } i \in \{1, 2, \ldots, n\}.$$

Then, $\mathbf{f}_\mu(\nu) = \eta$.

Proof of Lemma 4.41. Lemma 4.41 is obtained (using our above dictionary) when we apply Lemma 3.13 to $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ (and rename u, x, q_j and z as μ, ν, τ_j and η). □

We can now connect the map $\mathbf{f}_\mu$ with the map φ from Theorem 2.3:

Lemma 4.42. *Let $a, b \in \mathbb{N}$. Fix any n-tuple $\mu \in \mathbb{Z}^n$. Define a map $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ as in Theorem 2.3. Then,*

$$\varphi(\omega) = \mathbf{f}_\mu(\omega - a) + b \qquad \textit{for each } \omega \in \mathbb{Z}^n.$$

Proof of Lemma 4.42. Let $\omega \in \mathbb{Z}^n$.

Define an n-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$ by

$$\nu_i = \omega_i - a \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Thus, $\nu = \omega - a$.

For each $i \in \mathbb{Z}$, we let $i\#$ denote the unique element of $\{1, 2, \ldots, n\}$ congruent to i modulo n. (This is the same notation that was used in Convention 3.6.)

For each $j \in \mathbb{Z}$, set

$$\tau_j = \min\Big\{\big(\nu_{(j+1)\#} + \nu_{(j+2)\#} + \cdots + \nu_{(j+k)\#}\big) \\ + \big(\mu_{(j+k+1)\#} + \mu_{(j+k+2)\#} + \cdots + \mu_{(j+n-1)\#}\big) \\ \mid k \in \{0, 1, \ldots, n-1\}\Big\}. \tag{4.18}$$

Define an n-tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{Z}^n$ by setting

$$\eta_i = \mu_{i\#} + \left(\mu_{(i-1)\#} + \tau_{(i-1)\#}\right) - \left(\nu_{(i+1)\#} + \tau_{(i+1)\#}\right) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

The definition of φ then yields

$$\varphi(\omega) = (\eta_1 + b, \eta_2 + b, \ldots, \eta_n + b) = \eta + b. \tag{4.19}$$

Our plan is now to show that $\mathbf{f}_\mu(\nu) = \eta$ by applying Lemma 4.41; but in order to do so, we need to show that the assumptions of Lemma 4.41 are satisfied.

We shall do this piece by piece. First, we make the following two claims, which both follow from Convention 3.6:

Claim 1: We have $\nu_{p\#} = \nu_p$ for each $p \in \mathbb{Z}$.

Claim 2: We have $\mu_{p\#} = \mu_p$ for each $p \in \mathbb{Z}$.

The next claim is an easy consequence of Claims 1 and 2:

Claim 3: For each $j \in \mathbb{Z}$, we have

$$\tau_j = \min\{(\nu_{j+1} + \nu_{j+2} + \cdots + \nu_{j+k}) + (\mu_{j+k+1} + \mu_{j+k+2} + \cdots + \mu_{j+n-1}) \\ \mid k \in \{0, 1, \ldots, n-1\}\}.$$

The next claim is an easy fact from elementary number theory:

Claim 4: We have $(p\# + q)\# = (p+q)\#$ for any $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$.

Using Claim 4, we easily obtain the following:

Claim 5: We have $\tau_{p\#} = \tau_p$ for each $p \in \mathbb{Z}$.

Now, let $i \in \{1, 2, \ldots, n\}$. Then, the definition of η yields

$$\begin{aligned} \eta_i &= \underbrace{\mu_{i\#}}_{\substack{=\mu_i \\ \text{(by Claim 2)}}} + \left(\underbrace{\mu_{(i-1)\#}}_{\substack{=\mu_{i-1} \\ \text{(by Claim 2)}}} + \underbrace{\tau_{(i-1)\#}}_{\substack{=\tau_{i-1} \\ \text{(by Claim 5)}}}\right) - \left(\underbrace{\nu_{(i+1)\#}}_{\substack{=\nu_{i+1} \\ \text{(by Claim 1)}}} + \underbrace{\tau_{(i+1)\#}}_{\substack{=\tau_{i+1} \\ \text{(by Claim 5)}}}\right) \\ &= \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}). \end{aligned}$$

Now, forget that we fixed i. We thus have proved that

$$\eta_i = \mu_i + (\mu_{i-1} + \tau_{i-1}) - (\nu_{i+1} + \tau_{i+1}) \qquad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Combining this with Claim 3, we conclude that the assumptions of Lemma 4.41 are satisfied. Hence, Lemma 4.41 yields $\mathbf{f}_\mu(\nu) = \eta$. In view of $\nu = \omega - a$, this rewrites as $\mathbf{f}_\mu(\omega - a) = \eta$. Hence, (4.19) rewrites as

$$\varphi(\omega) = \mathbf{f}_\mu(\omega - a) + b.$$

This proves Lemma 4.42. □

4.6. The finale

Now, let us again use the convention (from Section 1) by which we identify partitions with finite tuples (and therefore identify partitions in $\operatorname{Par}[n]$ with nonnegative snakes). This is no longer problematic, since we are not using Convention 3.6 any more.

Lemma 4.43. *Let $a, b \in \mathbb{N}$. Define the two partitions $\alpha = (a+b, a^{n-2})$ and $\beta = (a+b, b^{n-2})$. Fix any partition $\mu \in \operatorname{Par}[n]$. Consider the map $\mathbf{f}_\mu : \mathbb{Z}^n \to \mathbb{Z}^n$ defined in Definition 4.37. Then, for any $\lambda \in \mathbb{Z}^n$, we have*

$$c_{\alpha,\mu}^{\lambda+a} = c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}.$$

Here, we understand $c_{\alpha,\mu}^{\lambda+a}$ to mean 0 if $\lambda + a$ is not a partition, and likewise we understand $c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}$ to mean 0 if $\mathbf{f}_\mu(\lambda) + b$ is not a partition.

Proof of Lemma 4.43. Let $\lambda \in \mathbb{Z}^n$. Corollary 4.35 (applied to $\lambda + a$ instead of λ) yields

$$c_{\alpha,\mu}^{\lambda+a} = \left|R_{\mu,a,b}(\lambda)\right| - \left|R_{\mu,a-1,b-1}(\lambda)\right|. \tag{4.20}$$

On the other hand, $\beta = (b + a, b^{n-2})$. Hence, Corollary 4.35 (applied to b, a, β and $\mathbf{f}_\mu(\lambda) + b$ instead of a, b, α and λ) yields

$$c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b} = \underbrace{\left|R_{\mu,b,a}(\mathbf{f}_\mu(\lambda))\right|}_{\substack{=\left|R_{\mu,a,b}(\lambda)\right| \\ \text{(by Lemma 4.40,} \\ \text{applied to } \gamma=\lambda)}} - \underbrace{\left|R_{\mu,b-1,a-1}(\mathbf{f}_\mu(\lambda))\right|}_{\substack{=\left|R_{\mu,a-1,b-1}(\lambda)\right| \\ \text{(by Lemma 4.40,} \\ \text{applied to } \lambda,\ a-1 \text{ and } b-1 \\ \text{instead of } \gamma,\ a \text{ and } b)}} = \left|R_{\mu,a,b}(\lambda)\right| - \left|R_{\mu,a-1,b-1}(\lambda)\right|.$$

Comparing this with (4.20), we find $c_{\alpha,\mu}^{\lambda+a} = c_{\beta,\mu}^{\mathbf{f}_\mu(\lambda)+b}$. This proves Lemma 4.43. □

We are now ready to prove Theorem 2.3:

Proof of Theorem 2.3. The map $\mathbf{f}_\mu$ is an involution (by Theorem 4.38 **(a)**), thus a bijection.

Let $\mathbf{a}^- : \mathbb{Z}^n \to \mathbb{Z}^n$ be the bijection that sends each $\omega \in \mathbb{Z}^n$ to $\omega - a$.

Let $\mathbf{b}^+ : \mathbb{Z}^n \to \mathbb{Z}^n$ be the bijection that sends each $\omega \in \mathbb{Z}^n$ to $\omega + b$.

Now, Lemma 4.42 can be restated as follows:

$$\varphi = \mathbf{b}^+ \circ \mathbf{f}_\mu \circ \mathbf{a}^-.$$

Hence, φ is a bijection (since $\mathbf{b}^+$, $\mathbf{f}_\mu$ and $\mathbf{a}^-$ are bijections). This proves Theorem 2.3 **(a)**.

(b) Let $\omega \in \mathbb{Z}^n$. Then, Lemma 4.42 yields $\varphi(\omega) = \mathbf{f}_\mu(\omega - a) + b$. But Lemma 4.43 (applied to $\lambda = \omega - a$) yields

$$c_{\alpha,\mu}^{\omega} = c_{\beta,\mu}^{\mathbf{f}_\mu(\omega-a)+b} = c_{\beta,\mu}^{\varphi(\omega)} \qquad (\text{since } \mathbf{f}_\mu(\omega - a) + b = \varphi(\omega)).$$

This proves Theorem 2.3 **(b)**. □

5. Final remarks

5.1. Questions on $\mathbf{f}_u$

We shall now pose several questions about the birational involution $\mathbf{f}_u$ studied in Section 3. Convention 3.4, Convention 3.5 and Convention 3.6 will be used throughout Subsection 5.1.

Most of our questions are attempts at seeing the involution $\mathbf{f}_u$ from different directions. The first one is inspired by what is now known as the "*toggle approach*" to dynamical combinatorics (see, e.g., [19]), but is really an application of the age-old "divide and conquer" paradigm to complicated maps:

Question 5.1. Is there an equivalent definition of $\mathbf{f}_u$ as a composition of toggles? (A *toggle* here means a birational map $\mathbb{K}^n \to \mathbb{K}^n$ that changes only one entry of the n-tuple. An example for a birational map that can be defined as a composition of toggles is *birational rowmotion* – see, e.g., [5]. Cluster mutations, as in the theory of cluster algebras, are another example of toggles.)

Another set of questions concern the *uniqueness* of $\mathbf{f}_u$. While we defined the map $\mathbf{f}_u$ explicitly, all we have then used are the properties listed in Theorem 3.11. Thus, it is a natural question to ask whether these properties characterize $\mathbf{f}_u$ uniquely. A pointwise version of this question can be asked as well: Given $x \in \mathbb{K}^n$ and $y \in \mathbb{K}^n$ satisfying some of the equalities in parts **(b)**, **(c)** and **(d)** of Theorem 3.11, does it follow that $y = \mathbf{f}_u(x)$? (Keep in mind that u is fixed.)

Of course, the answers depend on which equalities we require. Let us first ask what happens if we require the equalities from Theorem 3.11 **(c)** only:

Question 5.2. Given $x, y \in \mathbb{K}^n$ satisfying

$$(u_i + x_i)\left(\frac{1}{u_{i+1}} + \frac{1}{x_{i+1}}\right) = (u_i + y_i)\left(\frac{1}{u_{i+1}} + \frac{1}{y_{i+1}}\right) \tag{5.1}$$

for all $i \in \mathbb{Z}$. Does it follow that $y = \mathbf{f}_u(x)$ or $y = x$?

Note that the "or $y = x$" part is needed here, since $y = x$ is obviously a solution to the equations (5.1).

The following example shows that the answer to Question 5.2 is "no" if $\mathbb{K}$ is the min tropical semifield $(\mathbb{Z}, \min, +, 0)$ of the totally ordered abelian group $\mathbb{Z}$.

Example 5.3. Let $k, g \in \mathbb{N}$ with $g \geqslant k$. Let $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 3$ and $u = (0, 0, g)$ and $x = (1, 2, 0)$. Set $y = (k+1, 2, k)$ (where the "+" sign in "$k+1$" stands for addition of integers, not addition in $\mathbb{K}$). Then, the equations (5.1) hold in $\mathbb{K}$ for all $i \in \mathbb{Z}$. (Restated in terms of standard operations on integers, this is saying that

$$\min\{u_i, x_i\} + \min\{-u_{i+1}, -x_{i+1}\} = \min\{u_i, y_i\} + \min\{-u_{i+1}, -y_{i+1}\}$$

for all $i \in \mathbb{Z}$.) This is straightforward to verify, and shows that for a given x there can be an arbitrarily high (finite) number of $y \in \mathbb{K}^n$ satisfying the equations (5.1) for all $i \in \mathbb{Z}$. (Incidentally, this number is always finite when $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$; however, this does not generalize to arbitrary $\mathbb{K}$.)

However, the answer to Question 5.2 is "yes" if $\mathbb{K} = \mathbb{Q}_+$ and, more generally, if the semifield $\mathbb{K}$ embeds into an integral domain:

Proposition 5.4. *Assume that there is an integral domain $\mathbb{L}$ such that the semifield $\mathbb{K}$ is a subsemifield of $\mathbb{L}$ (in the sense that $\mathbb{K} \subseteq \mathbb{L}$ and that the operations $+$ and $\cdot$ of $\mathbb{K}$ are restrictions of those of $\mathbb{L}$, whereas the unity of $\mathbb{K}$ is the unity of $\mathbb{L}$). Let $x \in \mathbb{K}^n$. Then, the only n-tuples $y \in \mathbb{K}^n$ satisfying the equations (5.1) for all $i \in \mathbb{Z}$ are $y = \mathbf{f}_u(x)$ and $y = x$.*

Another avatar of the uniqueness question is the following:

Question 5.5. Given $x, y \in \mathbb{K}^n$ satisfying both (5.1) for all $i \in \mathbb{Z}$ and

$$y_1 y_2 \cdots y_n \cdot x_1 x_2 \cdots x_n = (u_1 u_2 \cdots u_n)^2 . \tag{5.2}$$

Does it follow that $y = \mathbf{f}_u(x)$?

The answer to this question is definitely "yes" when $\mathbb{K} = \mathbb{Q}_+$, by essentially the same argument that we used in Remark 3.16. Again, however, the answer is "no" when $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$. For example, if $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 4$ and $u = (2, 1, 1, 0)$ and $x = (1, 1, 1, 1)$, then the two n-tuples $(1, 1, 1, 1)$ and $(2, 2, 0, 0)$ both can be taken as y in Question 5.5, but clearly cannot both equal $\mathbf{f}_u(x)$. (On the other hand, if $\mathbb{K} = (\mathbb{Z}, \min, +, 0)$ and $n = 3$, then the answer is "yes" again; this can be shown by an unenlightening yet not particularly arduous case analysis.)

An even stronger version of Question 5.5 holds when $\mathbb{K} = \mathbb{Q}_+$:

Proposition 5.6. *Assume that $\mathbb{K} = \mathbb{Q}_+$. Let $x, y \in \mathbb{K}^n$. Assume that (5.1) holds for all $i \in \{1, 2, \ldots, n-1\}$, and assume that (5.2) holds. Then, $y = \mathbf{f}_u(x)$.*

Another question concerns Lemma 3.12:

Question 5.7. What is the "real meaning" of some of the more complicated parts of Lemma 3.12? In particular, Lemma 3.12 **(g)** reminds of the Plücker relation for minors of a $2 \times m$-matrix; can it be viewed that way (at least when $\mathbb{K}$ is a subsemifield of a field)?

5.2. The birational R-matrix connection

In this section, we shall connect the map $\mathbf{f}_u$ from our Definition 3.8 with the *birational R-matrix* η defined in [14, §6] and studied further (e.g.) in [3].

We fix a positive integer n and a semifield $\mathbb{K}$. We shall use Convention 3.4 and Convention 3.6. Let us recall the definition of the birational R-matrix η (no relation to the η in Theorem 2.3):

Definition 5.8. We define a map $\eta : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \times \mathbb{K}^n$ as follows: Let $a \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$ be two n-tuples. For any $i \in \mathbb{Z}$, define an element $\kappa_i(a, b) \in \mathbb{K}$ by

$$\kappa_i(a, b) = \sum_{j=i}^{i+n-1} \underbrace{b_{i+1} b_{i+2} \cdots b_j}_{= \prod_{p=i+1}^{j} b_p} \cdot \underbrace{a_{j+1} a_{j+2} \cdots a_{i+n-1}}_{= \prod_{p=j+1}^{i+n-1} a_p} .$$

Define $a' \in \mathbb{K}^n$ and $b' \in \mathbb{K}^n$ by setting

$$a'_i = \frac{a_{i-1}\kappa_{i-1}(a,b)}{\kappa_i(a,b)} \qquad \text{and} \qquad b'_i = \frac{b_{i+1}\kappa_{i+1}(a,b)}{\kappa_i(a,b)} \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

Set $\eta(a,b) = (a',b')$.

The map η we just defined is known as a *birational R-matrix*; related maps have previously appeared in the literature ([1, Lemma 8.6], [23, Definition 2.1], [6, Proposition 3.1]). In particular, the map R from [6, Proposition 3.1] is equivalent to η (at least up to technical issues of where it is defined[8]). Indeed, it is not hard to see that the map η from Definition 5.8 becomes the map R from [6, Proposition 3.1] if we set $x_i = b_{i+1}$ and $y_i = a_i$ and $x'_i = b'_i$ and $y'_i = a'_{i+1}$ (that is, if we define x_i, y_i, x'_i, y'_i this way, then the equalities [6, (8), (9) and (10)] are satisfied, so that we have $R(x,y) = (x',y')$ where R is as defined in [6, Proposition 3.1]). This birational R-matrix R has its origins in the theory of geometric crystals and total positivity. A related map is the transformation $(x,a) \mapsto (y,b)$ in [17, §2.2] (see also [24]).

Now, we shall see that the map η is intimately related to our map $\mathbf{f}_u$ (even though $\mathbf{f}_u$ transforms a single n-tuple x into a single n-tuple y using the fixed n-tuple u, while η takes a pair of two n-tuples to another such pair). In order to state this relation, we define some more notation:

Definition 5.9. If $a \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$ are two n-tuples, then we define two new n-tuples $ab \in \mathbb{K}^n$ and $\frac{a}{b} \in \mathbb{K}^n$ by setting

$$(ab)_i = a_i b_i \qquad \text{and} \qquad \left(\frac{a}{b}\right)_i = \frac{a_i}{b_i} \qquad \text{for each } i \in \{1,2,\ldots,n\}.$$

We can now express the map $\mathbf{f}_u$ from Definition 3.8 through the map η from Definition 5.8 as follows:

Theorem 5.10. *Let $u \in \mathbb{K}^n$ and $x \in \mathbb{K}^n$ be two n-tuples. Let $(a',b') = \eta(u,x)$. Then,*

$$\mathbf{f}_u(x) = u\frac{a'}{b'}.$$

Proof. Straightforward comparison of definitions. □

We finish by stating two (easily verified) "gauge-invariances" properties of $\mathbf{f}_u$ and η:

Proposition 5.11. *Let $g, u, x \in \mathbb{K}^n$. Then, $\mathbf{f}_{gu}(gx) = g\mathbf{f}_u(x)$.*

Proposition 5.12. *Let $g, a, b \in \mathbb{K}^n$. Let $(a',b') = \eta(a,b)$. Then, $(ga', gb') = \eta(ga, gb)$.*

[8]Namely: We have defined our map η as a literal map $\mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \times \mathbb{K}^n$ for any semifield $\mathbb{K}$, whereas [6, Proposition 3.1] defines R as a birational map $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$. Neither of these two settings generalizes the other, but it is not hard to transfer identities from one to the other (as long as they are *subtraction-free*, i.e., no minus signs appear in them).

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References

[1] Alexander Braverman, David Kazhdan, *Gamma-functions of representations and lifting*, arXiv:math/9912208v4.

[2] Emmanuel Briand, Mercedes Rosas, *The 144 symmetries of the Littlewood–Richardson coefficients of* SL_3, arXiv:2004.04995v1.

[3] Sunita Chepuri, Feiyang Lin, *Symmetric Group Action of the Birational* R*-matrix*, arXiv:2011.10128v1.

[4] Eric S. Egge, *An Introduction to Symmetric Functions and Their Combinatorics*, Stud. Math. Libr. **91**, AMS 2019.
See `http://www.ericegge.net/cofsf/` for errata.

[5] David Einstein, James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, arXiv:1310.5294v3.

[6] Pavel Etingof, *Geometric Crystals and Set-Theoretical Solutions to the Quantum Yang–Baxter Equation*, Comm. Algebra **31**, no. 4, pp. 1961–1973, 2003, arXiv:math/0112278v1.

[7] Darij Grinberg, *Regular elements of a ring, monic polynomials and "lcm-coprimality"*, 5 August 2019.
`https://www.cip.ifi.lmu.de/~grinberg/algebra/regpol.pdf`

[8] Darij Grinberg, *The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients [detailed version]*, detailed version of the present paper. Also available as an ancillary file at arXiv:2008.06128v4.

[9] Darij Grinberg, *The Pelletier–Ressayre hidden symmetry for Littlewood–Richardson coefficients*, arXiv version of the present paper. Also available as arXiv:2008.06128v4.

[10] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 27 July 2020, arXiv:1409.8356v7. Also available as Oberwolfach Preprints OWP-2020-14 and OWP-2020-15. [9]

[11] A. M. Hamel, R. C. King, *Extended Bressoud–Wei and Koike skew Schur function identities*, J. Combin. Theory Ser. A **118** (2011) pp. 545–557.

[9] See `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` for the current version of these notes; note, however, that its numbering may at some point diverge from our references.

[12] Anatol N. Kirillov, *Introduction to Tropical Combinatorics*, in: Physics and Combinatorics 2000 (Proceedings of the Nagoya 2000 International Workshop on Physics and Combinatorics, Nagoya University, August 21-26, 2000), pp. 82–149, World Scientific, Singapore, 2001.

[13] Kazuhiko Koike, *On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters*, Adv. Math. **74**, Issue 1, March 1989, pp. 57–86.

[14] Thomas Lam, Pavlo Pylyavskyy, *Total positivity in loop groups I: whirls and curls*, Advances in Mathematics **230**, Issue 3, 20 June 2012, pp. 1222–1271. A preprint is available at arXiv:0812.0840v3.

[15] Ian G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Math. Monogr., 2nd edition, Oxford Science Publications 1995.

[16] Anthony Mendes, Jeffrey Remmel, *Counting with Symmetric Functions*, Dev. Math. **43**, Springer 2015.

[17] Masatoshi Noumi, Yasuhiko Yamada, *Tropical Robinson–Schensted–Knuth correspondence and birational Weyl group actions*, arXiv:math-ph/0203030v2.

[18] Maxime Pelletier, Nicolas Ressayre, *Some unexpected properties of Littlewood–Richardson coefficients*, arXiv:2005.09877v3.

[19] Tom Roby, *Dynamical Algebraic Combinatorics and the Homomesy Phenomenon*, in: A. Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali, eds., Recent Trends in Combinatorics (IMA Volume in Mathematics and its Applications), Springer, 2016.

[20] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.1)*, 2020.

[21] Richard Stanley, *Enumerative Combinatorics, volume 2*, First edition, Cambridge University Press 2001.
See `http://math.mit.edu/~rstan/ec/` for errata.

[22] John R. Stembridge, *Rational Tableaux and the Tensor Algebra of gl_n*, J. Combin. Theory Ser. A **46** (1987), pp. 79–120.

[23] Y. Yamada, *A birational representation of Weyl group, combinatorial R-matrix and discrete Toda equation*, in: A.N. Kirillov, N. Liskova (Eds.), Physics and Combinatorics 2000: Proceedings of the Nagoya 2000 International Workshop, World Scientific, 2001, pp. 305–319.

[24] Nikos Zygouras, *Some algebraic structures in the KPZ universality*, arXiv:1812.07204v2.