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NAMING AN INDISCERNIBLE SEQUENCE IN NIP THEORIES

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Abstract. In this short note we show that if we add predicate for a dense complete indiscernible sequence in a dependent theory then the result is still dependent. This answers a question of Baldwin and Benedikt and implies that every unstable dependent theory has a dependent expansion interpreting linear order.

Introduction

Let T be an NIP theory in a language L. Consider a model M and a small indiscernible sequence I indexed by a dense complete linear order (small means that M is $|I|^+$-saturated). We consider the language L_P with a unary predicate P added for the sequence I, and let $T_P = Th(M, I)$.

Definition 1. We say that an L_P-formula is *bounded* if it is of the form

$$(Q_1 x_1 \in P)(Q_2 x_2 \in P)...(Q_n x_n \in P)\phi(x_1, ..., x_n, \bar{y}),$$

where ϕ is an L-formula and each Q_i is either $\exists$ or $\forall$.

In [BB00] Baldwin and Benedikt prove the following.

Theorem 2. *Assume T is NIP.*

1) For each dense complete indiscernible sequence I and formula $\phi(\bar{a}, y)$, there is some $\bar{c} \in I$ such that for every $\bar{b} \in I$, the truth value of $\phi(\bar{a}, \bar{b})$ is totally determined by the quantifier-free order type of $\bar{b}$ over $\bar{c}$.

2) Every formula in T_P is equivalent to a bounded one.

From this follows :

3) $Th(M, I) \equiv Th(M, J)$ if and only if $EM(I) = EM(J)$.

4) P is stably embedded and the L_P-induced structure (traces on P of all L_P-definable sets with parameters from M) is that of a pure linear order.

Remark 3. Point 1) is the Theorem 5.2 there, but for a simplified proof see [Adl08], Section 3. 2) is Theorem 3.3, 3) is Theorem 8.1, 4) is Corollary 3.6.

They prove that if T is stable then T_P is stable as well, and ask whether T_P is always dependent when T was. In the next section we answer this question positively. Throughout the paper we assume Martin's Axiom (MA).

Dependence of T_P

First a trivial combinatorial observation.

Lemma 4. *Let $h_k : \omega \to I$, $k \leq m$ be monotone functions, $h_0(n) < ... < h_m(n)$ for all n and let $a_1, ..., a_n \in I$. Then for some $p \leq 2nm + 1$ both $(h_k(p))_{k \leq m}$ and $(h_k(p+1))_{k \leq m}$ have the same order type over $a_1, ..., a_n$.*

Proof. Suppose not. Then for each $p \leq 2nm+1$ there is some $i \leq n, j \leq m$ with $h_i(p) < a_j, h_i(p+1) \geq a_j$ or $h_i(p) > a_j, h_i(p+1) \leq a_j$ or $h_i(p) = a_j, h_i(p+1) \neq a_j$, and by monotonicity for every pair of i, j there can be only up to two such p - a contradiction. $\square$

Next a crucial technical lemma.

Proposition 5. *1) Assume P is ordered by some L_p-definable "$<$". Let $I = (b_i)_{i<\omega}$ be an L_P-indiscernible sequence and $E \subset \omega$ the set of even numbers. Assume that $f : E \to P(\mathbb{M})$, $n < \omega$ even and $\phi(x_1, ..., x_n; y_1, ..., y_n) \in L_p$ such that for any sequence $k_1 < k_2 < ... < k_n \in E$ we have*
$\models \phi(b_{k_1}, ..., b_{k_n}; f(k_1), ..., f(k_n))$.

Then there is $g : \omega \to P(\mathbb{M})$ extending f and $l_1, ..., l_n \in \omega$ with $l_i \equiv i \ (mod 2)$ and $\models \phi(b_{l_1}, ..., b_{l_n}; g(l_1), ..., g(l_n))$.

2) Same claim but assuming that the L_P-induced structure on P is just the equality.

Proof. 1) Since by Theorem 2 the L_P-induced structure on P is just that of linear order by compactness there is some $m < \omega$ such that given any $(a_1, ..., a_n) \in \mathbb{M}$ there are some $(c_1 < ... < c_m) \in P(\mathbb{M})$ such that for any $(d_1, ..., d_n) \in P(\mathbb{M})$ the truth value of $\phi(a_1, ..., a_n; d_1, ..., d_n)$ is totally determined by the order type of $\bar{d}$ over $\bar{c}$.

Now for each $k \leq m$ let $h_k : \mathbb{M}^n \to P$ be the L_P-definable function sending $(a_1, ..., a_n)$ to the corresponding c_k (W.l.o.g. we assume there is a constant ρ in P. If for some $\bar{a}$ there are $k' < k$ alternations we let $h_j(\bar{a}) = \rho$ for $j > k'$).

We have :

(*) for any $(a_1, ..., a_n) \in \mathbb{M}, (d_1, ..., d_n), (d'_1, ..., d'_n) \in P$,
$\bar{d}$ and $\bar{d}'$ have the same order type over $(h_k(\bar{a}))_{\leq m} \implies \phi(\bar{a}; \bar{d}) \equiv \phi(\bar{a}; \bar{d}')$

From this by L_P-indiscernibility of I :

(**) for any $b_1, ..., b_n$ and $b'_1, ..., b'_n$ increasing sequences from I and $\bar{d}, \bar{d}' \in P$,
$\bar{d}$ has the same order type over $(h_k(\bar{b}))_{\leq m}$ as $\bar{d}'$ over $(h_k(\bar{b}'))_{\leq m} \implies \phi(\bar{b}; \bar{d}) \equiv \phi(\bar{b}'; \bar{d}')$.

Choose some $(0 < l_2 < ... < l_{n-2} < l_n) \in E^{n/2}$ with $l_{2(i+1)} - l_{2i} > 2mn + 1$. Define $h'_k : \omega \to P(\mathbb{M})$ by $h'_k(p) = h_k(b_p, b_{l_2}, b_{l_2+p}, ..., b_{l_{n-2}+p}, b_{l_n})$.

By L_P-indiscernibility of I the h'_k's are monotonic (at least in the interval $[1, 2mn+1]$ which is all that matters). Thus by Lemma 4 we find some (w.l.o.g. odd) $p_0 \leq 2mn+1$ such that $(h'_k(p_0))_{\leq m}$ has the same order type as $(h'_k(p_0+1))_{\leq m}$ over $f(l_2), ..., f(l_n)$. And again by L_P-indiscernibility and density of P we can find some $g(p_0), g(l_2+p_0), ..., g(l_{n-2}+p_0) \in P(\mathbb{M})$ such that $g(p_0), f(l_2), ..., g(l_{n-2}+p_0), f(l_n)$ has the same order-type over $(h'_k(p_0))_{\leq m}$ as $f(p_0+1), f(l_2), ..., f(l_{n-2}+p_0+1), f(l_n)$ over $(h'_k(p_0+1))_{\leq m}$, and so by (**)

$\models \phi(b_{p_0}, b_{l_2}, ..., b_{l_{n-2}+p_0}, b_{l_n}; g(p_0), f(l_2), ..., g(l_{n-2}+p_0), f(l_n))$ and we are done.

2) Analogously. $\square$

This gives us a Ramsey-like result on completing indiscernible sequences of triangles

Corollary 6. *Let $(a_i)_{i\in\omega} \in \mathbb{M}$, $(b_{2i})_{i\in\omega} \in P$ be given and $d \in \mathbb{M}$. Then there is some sequence $(a'_i b'_i)_{i\in\omega}$ L_P-indiscernible over d and such that for every $\psi \in L_P$*

(+) $\psi((a'_{2i}a'_{2i+1}b'_{2i})_{i<n},d) \implies \psi((a_{2k_i}a_{2k_i+1}b_{2k_i})_{i<n},d)$ *for some* $k_0 < k_1 < ... < k_{n-1} \in \omega$.

Proof. First by Ramsey find an L_P-indiscernible sequence $(a''_{2i}a''_{2i+1}b''_{2i})_{i<\omega}$ with property (+). Now let $I = (a''_i)_{i<\omega}$ and $f(a''_{2i}) = b''_{2i}$ and use Proposition 5 with compactness to conclude. □

Finally we are ready to prove our main result.

Theorem 7. *T_P is dependent.*

Proof. First note that by Theorem 2, the L_P-induced structure on P is equality or it is ordered by some L-formula (with parameters).

We prove by induction on the number of bounded quantifiers that all L_P-formulas are dependent, and since the set of formulas with NIP is closed under boolean combinations it is enough to consider adding single existential bounded quantifier to a dependent formula.

So assume $\phi(x;y) = (\exists z \in P)\psi(x,y,z)$ has IP where $\psi(x,y,z)$ is an L_P-formula. Then there is some L_p-indiscernible sequence $(a_i)_{i<\omega}$ and d such that $\phi(d,a_i)$ holds if and only if i is even, and so for $i = 2k$ let $b_i \in P$ be such that $\psi(d,a_i,b_i)$ holds. By Lemma 6 we find some sequence $(a'_ib'_i)_{i<\omega}$ which is L_P-indiscernible and (using (+)) still $\psi(d,a'_{2i},b'_{2i})$ and $\neg\psi(d,a'_{2i+1},b'_{2i+1})$ hold. But this means that $\psi(d;y,z)$ has infinite alternation - contradicting the inductive assumption. □

Question 8. *Assuming T is strongly-dependent, is T_P strongly-dependent ?*

Remark 9. Note however that unsurprisingly *dp*-minimality is not preserved in general after naming an indiscernible sequence. By [Goo09], Lemma 3.3, in an ordered *dp*-minimal group, there is no infinite definable nowhere-dense subset, but of course every small indiscernible sequence is like this.

Corollary 10. *Every unstable dependent theory has a dependent expansion interpreting an infinite linear order.*

Proof. Just take a small indiscernible sequence that is not an indiscernible set, mark it by a predicate and use Theorem 7. □

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