Title

Towards Analysis on Fractals: Piecewise C^1-Fractal Curves, Spectral Triples, and the Gromov-Hausdorff Propinquity

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Towards Analysis on Fractals: Piecewise C^1-Fractal Curves, Spectral Triples, and the Gromov-Hausdorff Propinquity

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of the requirements for the degree of

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To my son AJ.

ABSTRACT OF THE DISSERTATION

Towards Analysis on Fractals: Piecewise C^1-Fractal Curves, Spectral Triples, and the Gromov-Hausdorff Propinquity

by

Therese-Marie Basa Landry

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, March 2022
Dr. Michel Lapidus, Chairperson

Many important physical processes can be described by differential equations. The solutions of such equations are often formulated in terms of operators on smooth manifolds. A natural question is to determine whether differential structures defined on fractals can be realized as a metric limit of differential structures on their approximating finite graphs. One of the fundamental tools of noncommutative geometry is Alain Connes' spectral triple. Because spectral triples generalize differential structure, they open up promising avenues for extending analytic methods from mathematical physics to fractal spaces. The Gromov-Hausdorff distance is an important tool of Riemannian geometry and building on the earlier work of Marc Rieffel, Frederic Latremoliere introduced a generalization of the Gromov-Hausdorff distance that was recently extended to spectral triples. The class of piecewise C^1-fractal curves was first characterized by Michel Lapidus and Jonathan Sarhad as a generalized setting for the spectral triple construction developed by Christensen, Ivan, and Lapidus in the context of the Sierpinski gasket. We provide an analytic framework for the metric approximation of the Lapidus-Sarhad spectral triple on a piecewise C^1-fractal curve

by spectral triples defined on an approximating sequence of finite graphs which exhibit properties motivated by the setting of the Sierpinski gasket.

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Chapter 1

Introduction

Classical geometry relies on curves and surfaces that appear locally Euclidean. In contrast, fractals are infinite objects often characterized by self-similarity– the repetition of a base pattern across a boundless set of scales. Mandelbrot coined the term ”fractal” to describe rough or fragmented geometric shapes or processes. More than a century ago, curves now classified as fractals were generated by mathematicians as examples of objects that exhibit pathological geometric behavior. Such examples include the Weierstrass function, the Sierpinski gasket, and the Koch curve. Scientists have successfully modified fractal patterns to model many diverse natural phenomena such as the bronchial tubes of a lung, the canopy of a tree, the network of blood vessels in the human body, the pathway of a lightning bolt, and the distribution of noise in data transmission over a communications channel [26, 27]. Because fractal structure in nature has self-similarity over an extended but finite scale range, advancement in the theory of finite approximations of fractals can lead to a better understanding of how fractal structures arise and evolve in nature.

Many important physical processes can be described by differential equations. The solutions of such equations are often formulated in terms of operators on smooth manifolds. One motivation for the development of analysis on fractals is the extension of these methods from mathematical physics to fractal spaces. Many fractal curves can be approximated by simpler structures like finite graphs. A natural question, and the goal of this doctoral thesis project, as well as the subject of a collaboration with my PhD advisor, Michel Lapidus, and Frédéric Latrémolière, is to determine whether differential structures defined on fractals can be realized as a metric limit of differential structures on their approximating finite graphs [41]. Such an advancement would set the stage for the definition of operators on fractals that suitably generalize their classical counterparts and grant access to analysis via differentiable methods.

The emergence of architectures in information theory and signal processing at the quantum scale necessitates theoretical advances in both fractal geometry and noncommutative geometry [40] [16]. At every level of detection, the length of gaps between noise in a signal transmission is fractal [27]. At the quantum mechanical level, the order of measurement affects the outcome, and as a consequence, coordinates, which completely determine a system in classical geometry, do not commute. In particular, development of a noncommutative fractal geometry would enhance the capability of mathematicians to describe important physical spaces. Many important advancements in the definition and study of noncommutative fractal geometry are due to the efforts of Michel Lapidus, some of which are in collaboration [11, 17, 18, 41, 19]. This doctoral thesis project, which owes its conception, evolution, and fruitful resolution to his direction, increases the range of tools from noncommutative geometry which can be used to describe and understand fractals.

At the quantum scale, the wave function of a particle, but not its path in space, can be studied. Riemannian methods often rely on smooth paths to encode the geometry of a space. Noncommutative geometry generalizes analysis on manifolds by replacing this requirement with operator algebraic data. From this perspective, the topology of a space X determines, and is determined by, the C^*-algebra $C(X)$. When X is a compact Hausdorff space, $C(X)$ is a C^*-algebra. The space of bounded linear operators on a Hilbert space is another fundamental example of a C^*-algebra. Gelfand, Naimark, and Segal discovered that the category of unital commutative C–algebras is dual to that of compact Hausdorff spaces. Classical spaces correspond to commutative C^*-algebras, and noncommutative spaces to noncommutative C^*-algebras.

Much of the foundation of noncommutative geometry comes from Alain Connes' efforts to adapt classical tools from topology and Riemannian geometry to the operator algebraic setting. Connes formalized the operator algebraic elements essential to his approach in the definition of a spectral triple (see Definition 9). Because spectral triples generalize differential structure, they open up promising avenues for extending analytic methods from mathematical physics to fractal spaces [17, 18]. Since spaces that do not have paths or smooth structure often still admit many kinds of functions, fractals and Riemannian manifolds can be studied on the same rigorous footing when viewed as noncommutative spaces.

Noncommutative geometry can be used to give a rigorous functional analytic framework for models in high energy physics [4, 7, 21, 35, 36]. These same "point-free" techniques have also been used to study the geometry of classically pathological spaces like fractals. Michel Lapidus established a research program to stimulate advancement in noncommutative fractal geometry, where methods from noncommutative geometry are used to study fractals as generalized manifolds [17, 18]. Since the geodesic distance encodes the geometry of a manifold, recovery of this intrinsic metric is crucial to the development of analysis on fractals. Lapidus and his collaborators have built spectral triples that all recover the geodesic distance and in some instances also the Minkowski and complex fractal dimensions of the space [11, 19].

The Sierpinski gasket belongs to a class of fractal curves that can be suitably approximated by finite graphs. The Sierpinski gasket is also an important test case for much work in analysis on fractals. Michel Lapidus, together with Erik Christensen and Cristina Ivan, developed a spectral triple for the Sierpinski gasket that recovers the Hausdorff dimension, the geodesic metric, and the $\log_2 3$-dimensional Hausdorff measure [11]. The aforementioned class of fractals– that is, piecewise C^1-fractal curves– was first characterized by Michel Lapidus and Jonathan Sarhad as a generalized setting for the spectral triple construction developed by Christensen, Ivan, and Lapidus in the context of the Sierpinski gasket [19]. My collaboration in [41] with Lapidus and Latrémolière provides an analytic framework for the metric approximation of the Lapidus-Sarhad spectral triple on a piecewise C^1-fractal curve by spectral triples defined on an approximating sequence of finite graphs which exhibit properties motivated by the setting of the Sierpinski gasket.

The class of piecewise C^1-fractal curves also includes the harmonic gasket, which is a self-affine set that is homeomorphic to the Sierpinski gasket and path-connected via C^1 curves. Via the efforts of Jun Kigami and Shigeo Kusuoka, there exists a fractal version of Riemannian geometry in the setting of the harmonic gasket – more precisely, formulas for energy and geodesic distance involving measurable analogs to Riemannian metric, Riemannian gradient, and Riemannian volume [15]. In contrast, paths on the Sierpinski gasket are only piecewise C^1. Intrinsic metrics like the geodesic distance on a space play an important role in Riemannian geometry. The Lapidus-Sarhad spectral triple recovers the geodesic distance on a piecewise C^1-fractal curve [19]. Because the Sierpinski gasket cannot even be viewed as a classical manifold in the topological sense, the ability of the Lapidus-Sarhad spectral triple to recover the geodesic metric in this and broader settings is an important advancement in the development of generalized notions of manifolds that include fractal spaces.

A piecewise C^1-fractal curve can be written as the closure of a countable union of parameterized curves. Hence different subsequences of a parameterization define different sequences of finite graph approximations. The Lapidus-Sarhad spectral triple on a piecewise C^1-fractal curve is a countable direct sum of spectral triples where each summand corresponds to a curve in the fractal. The Gromov-Hausdorff distance is an important tool of Riemannian geometry, and building on the earlier work of Marc Rieffel, Frédéric Latremoliere introduced a generalization of the Gromov-Hausdorff distance that was recently extended to spectral triples in a form called the spectral propinquity [20, 23]. Our work determines conditions under which finite subsets of these summands yield spectral triples that approximate a spectral triple on the piecewise C^1-fractal curve, thereby facilitating metric approximation of these spectral triples and their underlying noncommutative geometric structures by the spectral propinquity. My thesis describes these conditions in the definition of an approximation sequence for a piecewise C^1-fractal curve (see Definition 5). My thesis also shows that these finite direct sums of spectral triples recover the geodesic distance for the corresponding finite graph in an approximation sequence for a piecewise C^1-fractal curve (see Theorem 27).

The Lapidus-Sarhad spectral triple and the approximating spectral triples described above are all distinguished in that they each recover the geodesic distances on their respective spaces. Intrinsic metrics like the geodesic distance on a space play an important role in Riemannian geometry. The application of the spectral propinquity in this setting not only respects the convergence of the coinciding classical structures but also adds new information by metrizing a notion of closeness for spectral triples also supported on the same spaces. The spectral propinquity can be viewed as an extension of Hausdorff distance up to the level of spectral triples. Chapter 2 begins with an exposition of Hausdorff distance as a tool for describing fractality. Lapidus and Sarhad's formulation of piecewise C^1-fractal curves is introduced. This framework is enhanced with the notion of an approximation sequence. The Gromov-Hausdorff distance is given as both a classical analogue for the basis of the spectral propinquity metric and a means to describe geodesic metric structure for piecewise C^1-fractal curves. Chapter 3 is a development of some themes in noncommutative fractal geometry and falls within the much wider scope of Michel Lapidus' research program where that subject is established. With the aim of understanding fractality, this account details the metric perspective in the sense of Rieffel, as well as the Riemannian angle advanced by Connes. As a metric on spectral triples of a certain class, Latremoliere's spectral propinquity is introduced as a tool for studying spectral triples on fractals. Chapter 4 applies the elements of noncommutative fractal geometry given in Chapter 3 to the metric approximation of Lapidus-Sarhad spectral triples on piecewise C^1-fractal curves. In service of this goal, metric approximation of each of several underlying noncommutative structures is obtained. The properties encoded in the notion of an approximation sequence play a major role in multiple estimates. Chapter 5 concludes with the description of several directions in noncommutative fractal geometry for current and future investigations.

Chapter 2

Hausdorff Distance, Iterated Function Systems, and Fractals

A crumpled sheet of paper looks like a plane when viewed close enough. A tangled piece of string likewise seems like a straight line at a precise enough level of detail. Classical geometry relies on curves and surfaces that appear locally flat at large enough levels of magnification. Fractals often appear pathological in this setting.

A fractal can sometimes be precisely described as the limit of the infinite iteration of a simple rule applied to a space. Mathematicians have identified special properties with which to define such iterated functions and the settings in which they give rise to fractals. Many fractal curves which can be characterized in this way also satisfy the criteria codified by Lapidus and Sarhad in the definition of a piecewise C^1-fractal curve. Important examples of such fractals include the Sierpinski gasket and the harmonic gasket. The class of piecewise C^1-fractal curve is broader than the usual framework through which these fractals are viewed, thereby making possible the description of more complicated examples of fractality.

2.1 Hausdorff Distance

Fractal curves like the Sierpinski gasket can be viewed as subsets of $\mathbb{R}^2$. When equipped with the Euclidean distance d, $(\mathbb{R}^2, d)$ is a metric space. Sets in a metric space can be compared using the Hausdorff distance. To define the Hausdorff distance for sets in $(\mathbb{R}^2, d)$, let A be a subset of $\mathbb{R}^2$ and r be some real number greater than 0. Then the *open r-neighborhood* around A is

$$N_r(A) = \{y : d(x,y) < r, \forall x \in A\}.$$

Definition 1. *Let A, B be subsets of a metric space. The* ***Hausdorff distance*** *between A and B is*

$$Haus_d(A,B) = inf\{r > 0, B \subseteq N_r(A), A \subseteq N_r(B)\}.$$

Many fractals can be viewed as an increasing union of sets with a simpler structure. In the case of piecewise C^1-fractal curves, these sets will be supplied by finite graphs. These fractal curves have finite approximations in the Hausdorff distance by such graphs. Both kinds of sets are also compact metric spaces when equipped with their respective geodesic distances. An extension of the Hausdorff distance can be applied to the consideration of convergence for such spaces.

The Hausdorff distance can be used to define a metric on the subsets of a metric space. Trivial Hausdorff distance between two subsets of a metric space implies one set is dense in the other. This metric is an extended pseudo-metric if no qualifications are put on these subsets. For instance, the Hausdorff distance between a point and a line as subsets of $(\mathbb{R}^2, d)$ is infinite. In contrast, the Hausdorff distance between any two non-empty, compact subsets of $(\mathbb{R}^2, d)$ is finite. More precisely, let (X, d) be a metric space. Denote the set of non-empty compact subsets of X by $\mathcal{H}(X)$. When equipped with the Euclidean distance d, $(\mathbb{R}^2, d)$ is a complete metric space. Consequently, $(\mathcal{H}(\mathbb{R}^2), \mathrm{Haus}_d)$ is also a complete metric space [12].

2.2 Iterated Function Systems

Some fractals can be represented as limits of Cauchy sequences in $(\mathcal{H}(\mathbb{R}^2), d_H)$. To construct such sequences in a more general setting, let (X, d) denote a metric space.

Definition 2. *A map $T : X \to X$ is called a* ***contraction mapping*** *if there exists $k \in (0, 1)$ such that for all x and y in X,*

$$d(T(x), T(y)) \leqslant k\, d(x, y).$$

An ***iterated function system (IFS)*** *is a finite collection of contraction mappings $\{T_s\}_{s=1}^m$ from the space X to itself.*

In fact, iterated functions systems exhibit an important property when they are defined on a complete metric space.

Theorem 1. *[12] Suppose (X, d) is a complete metric space. Let $\{F_s\}_{s=1}^m$ be an iterated function system in (X, d). Set $F : \mathcal{H}(X) \to \mathcal{H}(X)$ equal to*

$$F(A) := \bigcup_{s=1}^{m} F_s(A).$$

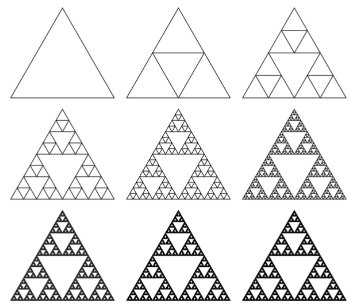

Figure 2.1: Finite Graph Approximations of the Sierpinski Gasket

Then F admits a unique fixed point. In other words, there exists a unique non-empty compact subset of X that is invariant for $\{F_s\}_{s=1}^{m}$.

Many important fractals can be realized as the unique fixed point of an iterated function system on a complete metric space. In particular, $\{F^n(A)\}_{n\in\mathbb{N}}$ is a Cauchy sequence when A and F are defined as in the above theorem. Fractal curves like the Sierpinski gasket can be represented as the unique fixed point of such a Cauchy sequence. For the case of the Sierpinski gasket, the unit equilateral triangle is a finite subgraph. In fact, this fractal can be approximated in the Hausdorff distance by the finite iterates of an affine transformation applied to a unit equilateral triangle. The properties exhibited by this approximating sequence of finite graphs will later be generalized to the setting of piecewise C^1-fractal curves in the notion of an approximation sequence for such fractals.

The Sierpinski gasket is both a classical example of a nowhere differentiable planar curve and an important test case for the theory of analysis on fractals. Let p_1, p_2, and p_3 each represent a vertex in a unit equilateral triangle and

$$F_s(x) := \frac{1}{2}(x - p_s) + p_s$$

for $s \in \{1,2,3\}$. Analytically, the Sierpinski gasket, SG, can be defined as the unique nonempty compact subset of $\mathbb{R}^2$ such that

$$SG = \bigcup_{s=1}^{3} F_s(SG).$$

The system of contraction mappings defined above can also be used to obtain finite graph approximations of SG. Let SG_0 signify the initial unit equilateral triangle, n any positive integer, $t = (t_1, \cdots, t_k)$ a word of length $|t| = k$ letters in $\{1,2,3\}$ for $1 \leqslant i \leqslant k$, $F_t(x) = F_{t_k} \circ \cdots \circ F_{t_1}(x)$, and

$$SG_n := \bigcup_{|t| \leqslant n} F_t(SG_0).$$

Then $\{SG_n\}_{n \geqslant 0}$ is an increasing sequence of graphs and SG can also be viewed as the closure of the limit of this sequence. Moreover, SG can be decomposed into n-cells – that is,

$$SG = \bigcup_{|t|=n} F_t(SG)$$

with

$$\bigcap_{|t|=n} F_t(SG_0) = \bigcup_{|t|=n} F_t(\{p_1, p_2, p_3\}) = V_n,$$

where V_n denotes the vertices in the level n approximation to SG, which will be denoted by SG_n. In analogy with Euclidean neighborhoods on a classical manifold, n-cells in SG can be seen as graph neighborhoods on the fractal. Because the lengths of the edges in an n-cell in SG are bounded by 2^{-n}, arbitrarily small regions of the fractal can be considered. Similarly,

$$V_* := \bigcup_{n \geqslant 0} V_n$$

is dense in SG and coincides with the vertices of SG. Fractals like SG carry an intrinsic metric and are examples of compact length spaces [19]. More precisely,

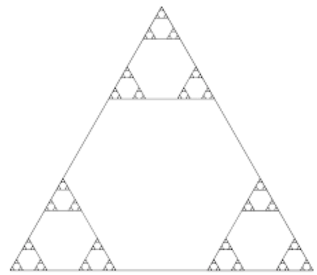

Figure 2.2: The Stretched Sierpinski Gasket

Definition 3. *Let (M, d) be a compact metric space. The* ***induced intrinsic metric*** *$d_I = d_I(x, y)$ is defined as the infimum of the d-induced lengths of (continuous) paths from x to y. When there is no path from x to y, then $d_I(x, y)$ is defined to be infinite. If $d(x, y) = d_I(x, y)$ for all x and y in M, then (M, d) is called a* ***compact length space*** *and the metric d is said to be* ***intrinsic****.*

Although SG is not differentiable at its vertices, each point in V_* can be connected to each point in SG by a minimizing geodesic that is a countable concatenation of edges in SG, hence piecewise rectifiable and C^1 [19]. Because SG cannot even be viewed as a classical manifold in the topological sense, the ability of the Lapidus-Sarhad spectral triple to recover the geodesic metric in this and the wider setting of piecewise C^1-fractal curves is an important advancement in the development of analysis on fractals via noncommutative geometry.

The Sierpinski gasket and the stretched Sierpinski gasket are both examples of fractals that can each be viewed as the unique fixed point of some iterated function system.

To obtain the stretched Sierpinski gasket of parameter α, $0 < \alpha < \frac{1}{3}$, set $p_4 = \frac{1}{2}(p_2 + p_3)$, $p_5 = \frac{1}{2}(p_1 + p_3)$, and $p_6 = \frac{1}{2}(p_1 + p_2)$. If $G_{\alpha,i} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$G_{\alpha,i}(x) := \frac{1-\alpha}{2}(x - p_i) + p_i \text{ for } i = 1, 2, 3,$$

$$G_{\alpha,i}(x) := T_i(x - p_i) + p_i \text{ for } i = 4, 5, 6,$$

where

$$T_4 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, T_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, T_6 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix},$$

then SG_α can be written as the unique nonempty compact subset of $(\mathbb{R}^2, d)$ such that

$$SG_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(SG_\alpha).$$

One way to obtain SG_α from SG is by replacing each vertex, or branching point in V_n, $n \geqslant 1$, with an interval of length $\alpha(\frac{1-\alpha}{2})^{n-1}$. The set of such intervals is dense in SG_α [30]. Recall that SG_0 is a unit equilateral triangle with vertices at $V_0 = \{p_1, p_2, p_3\}$. For $n = 1$, these intervals coincide with the images of SG_0 under the contractive affine maps in the iterated function system – that is, $G_{\alpha,i}$ for $i = 4, 5, 6$. Let

$$e_i := G_{\alpha,i+1}(SG_0) \text{ for } i = 1, 2, 3.$$

Then

$$I_1 := \{e_1, e_2, e_3\}$$

is the set of intervals in SG_α of length α. Each interval in I_1 can be associated to a vertex in V_1 via this correspondence. To describe the remaining intervals, as well as other components of SG_α, only the contractive similarities in the iterated function system– that is, $G_{\alpha,i}$ for $i = 1, 2, 3$– are needed. As in the case of SG, given t in $\{1, 2, 3\}^k$, $G_{\alpha,t}(x)$ is taken to be $G_{\alpha,t_k} \circ \cdots \circ G_{\alpha,t_1}(x)$. Thus for $n \geqslant 2$, each interval of length $\alpha(\frac{1-\alpha}{2})^{n-1}$ can now be described by $G_{\alpha,t}(e_i)$ for some t in $\{1, 2, 3\}^{n-1}$ and e_i in I_1. If

$$I_n := \{G_{\alpha,t}(e) : |t| = n - 1, e \in I_1\}$$

for $n \geqslant 2$ and

$$I_* := \bigcup_{n \geqslant 1} I_n,$$

then

$$SG_\alpha = \overline{I_*}.$$

Patricia Alonso Ruiz and Uta Freiburg showed that SG_α converges to SG for the Hausdorff distance when α goes to zero [39]. Part of their argument relies on bounds for the Hausdorff distance between the vertices of SG_α and the set of vertices of SG. In contrast to SG, the vertices of SG_α are not dense in SG_α. To describe the vertices of SG_α, set W_0 equal to V_0,

$$W_n := \{G_{\alpha,t}(p) : |t| = n, p \in W_0\},$$

and

$$W_* := \bigcup_{n \geqslant 0} W_n.$$

For $n \geqslant 1$, each vertex in W_n can be viewed as the endpoint of an interval in I_n. In the case of $n = 1$, $G_{\alpha,(3)}(p_2)$ and $G_{\alpha,(2)}(p_3)$ are the endpoints of e_1, $G_{\alpha,(3)}(p_1)$ and $G_{\alpha,(1)}(p_3)$ are the endpoints of e_2, and $G_{\alpha,(2)}(p_1)$ and $G_{\alpha,(1)}(p_2)$ are the endpoints of e_3. As a consequence, q

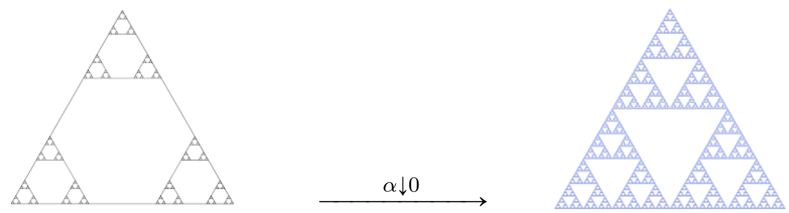

Figure 2.3: Hausdorff Convergence of the Stretched Sierpinski Gasket of Parameter α to the Sierpinski Gasket for $\alpha \to 0$

and r in W_n are the endpoints of the same interval e in I_n if there exists p_i and p_j in W_0, e_k in I_1, and t in $\{1, 2, 3\}^{n-1}$ such that

$$G_{\alpha,t} \circ G_{\alpha,(j)}(p_i) = q,$$

$$G_{\alpha,t} \circ G_{\alpha,(i)}(p_j) = r,$$

and

$$G_{\alpha,t}(e_k) = e.$$

When α goes to zero, the length of e goes to zero. In particular,

$$|q - F_t \circ F_{(j)}(p_i)| = |G_{\alpha,(j,t_1,\cdots,t_{n-1})}(p_i) - F_{(j,t_1,\cdots,t_{n-1})}(p_i)| \xrightarrow{\alpha\downarrow 0} 0,$$

as does the difference between r and $F_{(i,t_1,\cdots,t_{n-1})}(p_j)$. Since $F_{(j,t_1,\cdots,t_{n-1})}(p_i)$ and $F_{(i,t_1,\cdots,t_{n-1})}(p_j)$ describe the same vertex in V_n, the edge in I_n described by $G_{\alpha,(t_1,\cdots,t_{n-1})}(e_k)$ can be associated to this vertex when comparing SG_α and SG using the Hausdorff distance. The Hausdorff distance is therefore not only an important geometric tool for describing settings in which fractality can arise and finite approximations for fractals, but also for metric comparison between fractals.

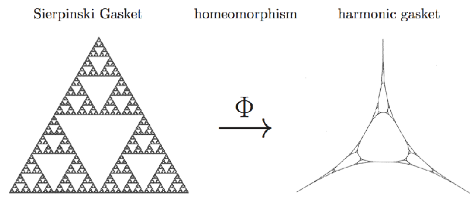

Figure 2.4: The Sierpinski Gasket and the Harmonic Sierpinski Gasket, [19]

2.3 Analysis on Fractals and Piecewise C^1-Fractal Curves

The harmonic gasket, HG, is a self-affine fractal that is homeomorphic to SG. In contrast, SG is self-similar. While paths in SG are piecewise C^1, paths in HG are C^1. Jun Kigami combined a formulation of the geodesic metric on HG with Kusuoka's measurable generalization of Riemannian structure on SG [15]. Via these efforts, there exist formulas for energy and geodesic distance on HG involving measurable analogues to Riemannian metric, Riemannian gradient, and Riemannian volume. Since HG is obtained from SG using the space of harmonic functions, HG can be viewed as SG in harmonic coordinates. Like SG, HG is also an important fractal in the theory of analysis on fractals.

The theory of harmonic functions on SG is an important development in the theory of analysis on fractals. As in the classical case, harmonic functions on SG are energy minimizers. To define harmonic functions on SG, recall that SG can be viewed as the closure of an increasing union of finite graphs. Each nth level approximation of SG from this vantage point is composed of 3^n equilateral triangles with sides of length 2^{-n}. If f and g are real-valued functions on SG_n, then the energy on SG_n is given by

$$E_n(f,g) = \Sigma_{x\sim_n y}(f(x)-f(y))(g(x)-g(y)),$$

where $x \sim_n y$ signifies that x and y are connected by a single edge in SG_n. An extension of f to the vertices of SG_{n+1} that minimizes $E_{n+1}(f) := E_{n+1}(f, f)$ is defined as the harmonic extension of f to those vertices. A real-valued function f defined on the vertices of SG_n which, given its values on the vertices of SG_0, minimizes $E_k(f)$ for each $k = 1, 2, ..., n$ is called a *harmonic function*.

Harmonic functions on SG play a critical role in the characterization of the Laplacian on SG. A real-valued function defined on V_0 can be uniquely extended to a real-valued harmonic function on SG_n for any n and hence to V_*. Since V_* is dense in SG, the function can also be extended to SG via continuity. If V_0 is viewed as the boundary of SG, then, in continued analogy with classical harmonic theory on a manifold, harmonic functions on SG can be said to be uniquely determined by their boundary values. Working in this setting, Kigami used the renormalized energy on SG_n to define a Laplacian on SG and its relationship with harmonic functions on SG [15]. More precisely, let

$$\mathcal{E}(f, g) := \lim_{n\to\infty} \left(\frac{5}{3}\right)^n E_n(f, g),$$

μ be a probability measure on SG, $f \in \mathrm{dom}(\mathcal{E})$, and $u \in C(SG)$. Note that $g \in \mathrm{dom}(\mathcal{E})$ if $\mathcal{E}(g, g) < \infty$. Consider Δ_μ, the Laplacian on SG with respect to the measure μ, i.e. $\Delta_\mu f = u$ if

$$\mathcal{E}(f, g) = -\int_{SG} ug \, d\mu$$

for all $g \in \mathrm{dom}(\mathcal{E})$. If f is harmonic, then $f \in \mathrm{dom}(\Delta_\mu)$ and $\Delta_\mu f = 0$. Conversely, if $f \in \mathrm{dom}(\Delta_\mu)$ and $\Delta_\mu f = 0$, then f is harmonic. In particular, Kigami's Laplacian on SG is a well-developed example of a differential operator on a fractal that suitably generalizes key properties from the Riemannian manifold setting.

To construct a homeomorphism between SG and HG, let R_j, denote, for each $j \geqslant 1$, continuous injective functions to edges in SG_n such that

$R_j : [0,1] \to$ the edges in the graph SG_0 for $j = 1,2,3,$

$R_j : [0,2^{-1}] \to$ the edges in the graph SG_1 for $j = 4,5,\cdots,12$

$R_j : [0,2^{-2}] \to$ the edges in the graph SG_2 for $j = 13,14,\cdots,39$

$\vdots$

$R_j : [0,2^{-n}] \to$ the edges in the graph SG_n for $j = 1+\sum_{i=1}^{n} 3^i, 2+\sum_{i=1}^{n} 3^i, \cdots, 3^{n+1}+\sum_{i=1}^{n} 3^i,$

$\vdots$

Suppose each R_j is parametrized by arclength. Each curve will be mapped to HG as follows. Let $\{p_1,p_2,p_3\}$ denote the set of vertices of SG_0 and V_* that of SG. Note that V_* is dense in SG. For each $j = 1,2,3$, let the function $h_j : V_0 \to \mathbb{R}^3$ be given by $h_j(p_k) = \delta_j(k)$ for $k = 1,2,3$. Extend h_j harmonically to V_* and by continuity to SG. Then $\Phi : SG \to \mathbb{R}^3$ is defined by

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right),$$

and

$$HG := \Phi(SG).$$

Set $l_j = L(\Phi(R_j))$. After a reparametrization,

$$\Phi(R_j) : [0,l_j] \to HG$$

for every $j \in \mathbb{N}$. Moreover,

$$HG = \overline{\bigcup_{j \geqslant 1} \Phi(R_j)}.$$

The fact that Φ is a homeomorphism between SG and HG when both of these spaces is equipped with the topology induced by the restriction of the Euclidean metric was also shown by Kigami [15]. In particular,

$$HG = \bigcup_{s=1}^{3} \Phi \circ F_s \circ \Phi^{-1}(HG).$$

A set of contractive affine maps $\{H_s\}_{s=1}^{3}$ exists for which HG is the unique fixed point and $\Phi \circ F_s = H_s \circ \Phi$ [15].

To build spectral triples on fractals, Lapidus and Sarhad identified a class of fractal curves that includes both HG and SG. The verification that each fractal belongs to this class is supplied by Proposition 2 and Proposition 3 of [19].

Definition 4 ([19])**.** *A* ***piecewise*** C^1***-fractal curve*** *is a compact length space* $X \subseteq \mathbb{R}^n$ *that satisfies the axioms below. Let* $L(\gamma)$ *denote the length of the continuous curve* γ *parametrized by its arclength.*

Axiom 1. $X = \overline{R}$ *where* $R = \bigcup_{j \geqslant 1} R_j$ *and* R_j *for each* $j \in \mathbb{N}$ *is a rectifiable* C^1 *curve with* $L(R_j) \to 0$ *as* $j \to \infty$*.*

Axiom 2. *There exists a dense subset* $\mathcal{B} \subset X$ *such that for each* $p \in \mathcal{B}$ *and* $q \in X$*, one of the minimizing geodesics from* p *to* q *can be given as a countable (or finite) concatenation of the* R_j'*s.*

The countable concatenation of R_j's mentioned in Axiom 2 is understood to begin with $p \in B$ as the initial endpoint of some R_j. As a consequence, Axiom 2 implies that B

is a subset of the collection of the endpoints of the R_j curves [19]. In particular, the set of endpoints of the R_j curves in a parameterization of a piecewise C^1-fractal curve is, as in the case of V_* in SG, also dense. When this collection of endpoints satisfies the additional conditions detailed below, the Lapidus-Sarhad spectral triple on the piecewise C^1-fractal curve can be metrically approximated by spectral triples on finite graphs (see Theorem 32).

Definition 5. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j\in\mathbb{N}}$. An **approximation sequence of X compatible with** $(R_j)_{j\in\mathbb{N}}$ is a strictly increasing function $B : \mathbb{N} \to \mathbb{N}$ such that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $n \geqslant N$, and letting*

- $X_n = \bigcup_{j=1}^{B(n)} R_j$,
- V_n *denote the set of the endpoints of the curves* $R_1, \cdots, R_{B(n)}$,
- V_* *signify* $\bigcup_{n\geqslant 0} V_n$,
- d_n *be the geodesic distance on* X_n,

the following properties hold:

(1) $Haus_{d_n}(V_n, X_n) < \epsilon$,

(2) the restriction of d_∞ *to* $V_n \times V_n$ *is* d_n.

The sequence of nth level approximations of SG given by $\{SG_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite subgraphs of SG. The geometric properties exhibited by this sequence and the notation used in their description supplied the prototype for the notion of an approximation sequence for a piecewise C^1-fractal curve.

Lemma 1. *Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \sum_{j=0}^{n} 3^{j+1}$. Then B defines an approximation sequence of SG compatible with the parameterization $(R_j)_{j\in\mathbb{N}}$.*

Proof. Fix a choice of positive ϵ. Then there exists $N \in \mathbb{N}$ such that $2^{-n} < \epsilon$ for all $n \geqslant N$. Choose some such n. If $B(n) = \sum_{j=0}^{n} 3^{j+1}$, then $SG_n = \bigcup_{j=1}^{B(n)} R_j$ coincides with the nth level approximation of SG described earlier by $\bigcup_{|t|\leqslant n} F_t(SG_0)$. In particular, each $F_t(SG_0)$ is an equilateral triangle in SG with sides of length $2^{-|t|}$. When $|t| \leqslant n$, the vertices of such triangles belong to V_n. Since paths in SG_n are composed of edges belonging to such triangles, $Haus_{d_n}(V_n, SG_n) < \epsilon$.

To determine the restriction of d_∞ to $V_n \times V_n$, recall that SG_n can be decomposed into n-cells – that is,

$$SG_n = \bigcup_{|t|=n} F_t(SG_0).$$

Each n-cell is an equilateral triangle with sides of length 2^{-n}. The set of vertices of all such triangles in SG_n coincides with V_n. If two n-cells intersect, they intersect only at a vertex. Consequently, any path connecting points in distinct n-cells must pass through a vertex of each of these n-cells. Let p and q be distinct points in V_n. If p and q belong to the same n-cell, then they are also vertices of an edge belonging to an equilateral triangle with sides of length 2^{-n}. Hence $d_\infty(p,q) = d_n(p,q) = 2^{-n}$. Next suppose that p and q belong to distinct n-cells and that γ is a path in SG_∞ not contained in SG_n. Then γ contains some point c in $SG_\infty \backslash SG_n$. Because

$$SG = \bigcup_{|t|=n} F_t(SG),$$

c belongs to $F_{t'}(SG)$ for some $t' \in \{1,2,3,\}^n$. Since $F_{t'}(SG)$ lies in the convex hull of $F_{t'}(SG_0)$, γ must pass through two vertices of the triangle described by $F_{t'}(SG_0)$. As $\gamma \cap F_{t'}(SG)$ is longer than the straight edge connecting these two vertices, γ cannot be a geodesic in SG. Therefore any geodesic in SG connecting points in V_n must also be a geodesic in SG_n. ■

The identification of an approximation sequence for a piecewise C^1-fractal curve given a particular parameterization requires some characterization of the geodesics in that space. Since each R_j-curve is a straight line segment in $(\mathbb{R}^2, \mathrm{d})$, it is the minimizing geodesic in SG between its endpoints. For HG, each $\Phi \circ R_j$-curve corresponds to a harmonic edge. As harmonic edges in HG are not straight line segments in $(\mathbb{R}^2, d)$, additional argument is needed to show each $\Phi \circ R_j$-curve is the minimizing geodesic in HG between its endpoints. In the proof of [19, Proposition 3], Lapidus and Sarhad provide such verification. Furthermore, paths in SG when viewed with the parameterization $\{R_j\}_{j \in \mathbb{N}}$ can only enter or exit an n-cell via the endpoints of such curves. As a consequence of the homeomorphism with SG, paths in HG when viewed with the parameterization $\{\Phi \circ R_j\}_{\mathbb{R}}$ exhibit the same property. For both SG and HG, it then suffices to consider geodesics within n-cells. These replacements are made explicit in the demonstration that B is also an approximation sequence for HG but compatible with the parameterization $\{\Phi \circ R_j\}_{j \in \mathbb{N}}$.

Lemma 2. *Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \sum_{j=0}^{n} 3^{j+1}$. Then B defines an approximation sequence of HG compatible with the parameterization $(\Phi \circ R_j)_{j \in \mathbb{N}}$.*

Proof. Fix a choice of positive ϵ. Since HG is a piecewise C^1-fractal curve, there exists $N' \in \mathbb{N}$ such that $L(\Phi \circ R_j) < \epsilon$ for all $j \geqslant N'$. Choose some $n \geqslant N = N' + 1$. If $B(n) = \sum_{j=0}^{n} 3^{j+1}$, then

$$HG_n = \bigcup_{j=1}^{B(n)} \Phi \circ R_j = \Phi(SG_n) = \bigcup_{|t|=n} \Phi \circ F_t(SG_0) = \bigcup_{j=B(n-1)+1}^{B(n)} \Phi \circ R_j,$$

where

$$B(n-1)+1 > B(n-1) \geqslant B(N') > N'$$

implies $L(\Phi \circ R_j) < \epsilon$ for all $j \geqslant B(n-1)+1$. Furthermore, every point in HG_n lies on some $\Phi \circ R_j$ for $B(n-1)+1 \leqslant j \leqslant B(n)$. Hence $\text{Haus}_{d_n}(V_n, HG_n) < \epsilon$.

Now consider the restriction of d_∞ to $V_n \times V_n$. As a consequence of the homeomorphism with SG, HG_n can be decomposed into n-cells each described by $\Phi \circ F_t(SG_0)$ for some $t \in \{1,2,3\}^n$. Each n-cell is determined by three $\Phi \circ R_j$-curves, each of which intersects each other only at their endpoints. The homeomorphism with SG implies that if two n-cells in HG_n intersect, they intersect only at the endpoints of these $\Phi \circ R_j$-curves. Every point in V_n is also an endpoint of one of these defining $\Phi \circ R_j$ harmonic edges for some n-cell. In particular, each n-cell in HG_n contains three points in V_n. Consequently, any path connecting points in distinct n-cells must pass through at least one of these points in V_n for each of these n-cells. Let p and q be distinct points in V_n. If p and q belong to the same n-cell, then they are also endpoints of one of the defining $\Phi \circ R_j$ harmonic edges for that cell. As shown in the proof of [19, Proposition 3], that $\Phi \circ R_j$ harmonic edge is the minimizing geodesic in HG between p and q. Next suppose that p and q belong to distinct n-cells and that γ is a path in HG_∞ not contained in HG_n. Then γ contains some point c in $HG_\infty \backslash HG_n$. Because

$$HG = \Phi(SG) = \bigcup_{|t|=n} \Phi \circ F_t(SG),$$

c belongs to $\Phi \circ F_{t'}(SG)$ for some $t' \in \{1,2,3,\}^n$. Since $\Phi \circ F_{t'}(SG) \backslash \Phi F_{t'}(SG_0)$ lies in the interior of $\Phi \circ F_{t'}(SG_0)$, γ must pass through two of the points of V_n belonging to the n-cell described by $\Phi \circ F_{t'}(SG_0)$. In particular, $\gamma \cap F_{t'}(SG)$ cannot coincide with the harmonic edge $\Phi \circ R_j$ connecting these two points. As this $\Phi \circ R_j$-curve is the minimizing geodesic in HG between these two points, γ is longer than any path connecting p and q which also passes through $\Phi \circ F_{t'}(SG_0)$ but instead along one of its defining $\Phi \circ R_j$ curves. Therefore any geodesic in HG connecting points in V_n must also be a geodesic in HG_n. ■

Since HG is self-affine rather than self-similar, the lengths of $\Phi \circ R_j$-curves which coincide with harmonic edges in HG_n may differ. This self-affine property leads to different bounds on n than in the case of SG_n to guarantee that $\mathrm{Haus}_{d_n}(V_n, HG_n)$ is bounded by some choice of positive ϵ. More importantly, neither self-similarlity nor self-affinity is required for classification as a piecewise C^1-fractal curve. Thus the framework afforded by this class of parameterized curves makes possible the description of more general fractal behavior and structures.

Both SG and HG are fundamental examples in the theory of analysis on fractals. They are also each fractals that can be realized by an iterated function system. However, not all such fractals qualify as piecewise C^1-fractal curves. For instance, the stretched Sierpinski gasket can be realized by an iterated function system but does not satisfy Axiom 2 in the criteria for a piecewise C^1-fractal curve. Unlike SG and HG, the set of vertices of SG_α is not dense in SG_α. As a potential model for heat and wave propagation in branching media, the stretched Sierpinski gasket is a subject of active research for mathematicians working in the theory of analysis on fractals [30]. Since SG_α converges for the Hausdorff distance to SG when α goes to zero, SG_α would be a natural prototype for the definition of a class of *almost piecewise* C^1*-fractal curves*. Such an extension of the piecewise C^1-fractal curve framework would be useful in the development of generalized manifolds that can be a basis for analysis on many kinds of fractals.

2.4 Fractals, Geodesic Distance, and Gromov-Hausdorff Distance

The space of compact metric spaces can be equipped with a metric. Fractals that are compact for a given metric belong to this space. As compact length spaces, piecewise C^1-fractal curves are compact metric spaces when equipped with the geodesic distance. In particular, questions of approximation can be considered for their respective metric structures. These metric structures can be shown to be induced by Lapidus-Sarhad spectral triples. Metric approximations of these spectral triples will be obtained via an extension of a noncommutative generalization of the following metric.

Definition 6. *The* ***Gromov-Hausdorff distance*** *between two compact metric spaces* (X, d_X) *and* (Y, d_Y) *is*

$$GH(\,(X, d_X), (Y, d_Y)\,) = \inf_{T_X : X \to Z,\, T_Y : Y \to Z \text{ are isometries}} Haus_{d_Z}(T_X(X), T_Y(Y)).$$

The Gromov-Hausdorff distance between two compact metric spaces is always well-defined. To see that a third metric space always exists which admits isometric embeddings of two compact metric spaces, let (X, d_X) and (Y, d_Y) both signify non-empty compact metric spaces. Then $X \coprod Y$ becomes a metric space if equipped with

$$d_{X\coprod Y}(p,q) = \begin{cases} d_X(p,q) & \text{if } p,q \in X, \\ d_Y(p,q) & \text{if } p,q \in Y, \\ \max\{\operatorname{diam}(X), \operatorname{diam}(Y)\} & \text{otherwise.} \end{cases}$$

By construction, the canonical inclusions for X and Y into $X \coprod Y$ are isometries. Since the isometric images of these maps are non-empty compact subsets of $(X \coprod Y, d_{X\coprod Y})$, $GH(\,(X, d_X), (Y, d_Y)\,)$ is also always finite. Furthermore,

Theorem 2. *[9, Theorem 7.3.30] The Gromov-Hausdorff distance defines a complete metric on the space of isometry classes of compact metric spaces.*

The geometry of a Riemannian manifold can be recovered from its geodesic distance [14, p.388]. A development of fractals in the context of generalized manifolds should therefore be built with the metric structure determined by the geodesic distance. A natural approach to such a goal would be to construct such generalizations on simpler spaces like finite graphs and show that they approximate those on a fractal. Since Lapidus-Sarhad spectral triples are defined on piecewise C^1-fractal curves, this class of fractals can be viewed as a type of fractal "manifold." Finite subgraphs of piecewise C^1-fractal curves are also compact metric spaces when equipped with their respective geodesic distances. The geodesic distance between two points in such a subgraph may differ from the geodesic distance when viewed as points in the piecewise C^1-fractal curve. Therefore, a finite subgraph of a piecewise C^1-fractal endowed with the geodesic distance may not be a subset of the piecewise C^1-fractal curve endowed with the geodesic distance. In contrast, this finite subgraph equipped with the restriction of the Euclidean distance can be viewed as a subset of the piecewise C^1-fractal curve equipped with the restriction of the Euclidean distance. Recall the notation used in the definition of an approximation sequence for a piecewise C^1-fractal curve. The Hausdorff distance requirements outlined in that definition will be used to show (X, d_∞) can be approximated in the Gromov-Hausdorff distance by (X_n, d_n). Classical compact metric spaces also have well-developed noncommutative analogues. This same suite of geometric properties for the Hausdorff distance will additionally be a basis for demonstrating the Lapidus-Sarhad spectral triple on X can likewise be approximated in a quantum metric by spectral triples on finite graphs.

Chapter 3

Tools from Noncommutative Geometry

To build a framework for fractals that supports a suitable generalization of analysis on manifolds, tools from noncommutative geometry will be used to capture the geometry of a fractal. Fundamentally, a fractal cannot locally resemble Euclidean space. Through the lens of noncommutative geometry, a fractal can be viewed via an algebra of functions on that space. Since Riemannian manifolds and fractals both admit continuous functions, the functional analytic perspective afforded by noncommutative geometry places both objects on the same footing. Topological, metric, and differential structures will be defined and studied for certain fractals using noncommutative notions first formulated by Connes, Rieffel, and Latremoliere. In particular, the selection of techniques detailed here can also be viewed as building blocks for Lapidus' program for the development of a noncommutative fractal geometry.

3.1 C^*-Algebras

Since many fractals are compact Hausdorff spaces, topological properties of such fractals can be recovered from algebraic properties of their spaces of continuous, complex-valued functions. Classical Riemannian methods often rely on smooth paths to encode the geometry of a space and noncommutative geometry generalizes analysis on manifolds by replacing this requirement with operator theoretic data. Coordinates are given by functions on the underlying space. Whether X is a fractal or a Riemannian manifold, $C(X)$, when equipped appropriately, is a fundamental example of the following function space.

Definition 7. *A **C*-algebra** A is a Banach algebra equipped with a map $^* : A \to A$ that for all $\lambda \in \mathbb{C}$ and $a, b \in A$ satisfies*

- $(\lambda(a+b))^* = \overline{\lambda}(a^* + b^*)$,
- $(a^*)^* = a$,
- $(ab)^* = b^*a^*$.

*This map is called the **adjoint** of A. Furthermore, all $a \in A$ satisfy the C^***-identity**, that is,*

$$||aa^*||_A = ||a||_A^2.$$

*A subalgebra $B \subseteq A$ that is closed with respect to the norm and the adjoint of A is called a **C*-subalgebra** of A. When A is unital, $B \subseteq A$ is said to be a **unital C*-subalgebra** when B contains the multiplicative identity of A.*

If equipped with the supremum norm and given pointwise conjugation as the adjoint operation, then every function in $C(X)$ exhibits the C^* identity when X is a compact

Hausdorff space. Moreover, the constant 1 function plays the role of the multiplicative identity for $C(X)$ in this setting and pointwise operations for functions are commutative. Since functions in $C(X)$ may be unbounded when X is a locally compact Hausdorff space, restriction to functions in $C(X)$ which vanish at infinity, or $C_0(X)$, yields C^*-algebraic structure that is commutative but not necessarily unital. In fact, $C(X)$ and $C_0(X)$ coincide if X is a compact Hausdorff space.

The complex numbers with complex conjugation as the adjoint and modulus as the norm is another example of a unital commutative C^*-algebra. In particular, any multiplicative linear functional, or *character*, on $C(X)$, respects the adjoint operation. Morphisms between unital C^*-algebras are given by unital **-homomorphisms*– that is, algebra homomorphisms that are unit- and *-preserving. The study of C^*-algebras begins with the following result for unital commutative C^*-algebras.

Theorem 3 (Gelfand-Naimark Theorem)**.** *Any unital commutative C^*-algebra A is *-isomorphic to the C^*-algebra $C(X)$ for some compact Hausdorff space X.*

When equipped with the weak*-topology, the space of characters on A, or $\hat{A}$, supplies the compact Hausdorff space provided by the Gelfand-Naimark Theorem. The topology of $\hat{A}$ is a consequence of the Banach algebra structure of A [10, Theorem I.2.5, Corollary I.2.6]. If δ_a denotes evaluation at a, then δ_a is a weak*-continuous complex-valued function on $\hat{A}$. The *Gelfand transform of* A, or $\Gamma_A : A \to C(\hat{A})$, which is given by

$$\Gamma_A(a) = \delta_a,$$

will be shown to be a *-isomorphism when A is a unital commutative C^*-algebra. The commutativity condition also guarantees that $\hat{A}$ is non-empty.

Theorem-Definition 1. *Let A be a unital C^*-algebra. For $a \in A$, the **spectrum** of a is given by*

$$\sigma(a) := \sup\{\, \lambda \in \mathbb{C} : a - \lambda I_A \text{ is not invertible} \},$$

*and the **spectral radius** by*

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$

*If $a \in A$ and $a^*a = aa^*$, then a is called **normal** and $r(a)$ coincides with $||a||$.*

Proof. Since a C^*-algebra is also a Banach algebra,

$$r(a) = \lim_{n \to \infty} ||a^n||^{\frac{1}{n}}$$

holds for any $a \in A$ [10, Proposition I.2.3]. Suppose first that a is self-adjoint. Then the C^*-identity yields

$$||a^2|| = ||aa^*|| = ||a||^2,$$

hence induction gives

$$||a^{2^n}|| = ||a||^{2^n}$$

for every $n \geqslant 1$. As a consequence,

$$r(a) = \lim_{n \to \infty} ||a^{2^n}||^{\frac{1}{2^n}} = \lim_{n \to \infty} \left(||a||^{2^n} \right)^{\frac{1}{2^n}} = ||a||.$$

For the weaker assumption that a commutes with a^*, note that

$$(aa^*)^2 = a(a^*a)a^* = a(aa^*)a^* = (aa)(aa)^* = a^2(a^2)^*$$

implies

$$(aa^*)^n = a^n(a^n)^*$$

for every $n \geqslant 2$, hence

$$r(aa^*) = \lim_{n\to\infty} \left(||aa^*||^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(||(aa^*)^n||\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(||a^n(a^n)^*||\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(||a^n||^2\right)^{\frac{1}{n}} = \left(\lim_{n\to\infty} ||a^n||^{\frac{1}{n}}\right)^2 = r(a)^2.$$

Since aa^* is always self-adjoint,

$$||a||^2 = ||aa^*|| = r(aa^*) = r(a)^2.$$

where the first equality is again a consequence of the C^*-identity. ■

Corollary 1. *If A is a unital commutative C^*-algebra, then $\hat{A}$ is non-empty.*

Proof. In a unital commutative C^*-algebra, every element commutes with its adjoint. Thus $r(a) = ||a||$ for every $a \in A$. Since multiplicative linear functionals are nonzero, fix some nonzero choice of $a \in A$. Then

$$\{\lambda \in \mathbb{C} : a - \lambda I_A \text{ is not invertible}\} = \{\varphi(a) : \varphi \in \hat{A}\}$$

as a consequence of the unital commutative Banach algebra structure of A [10, Corollary I.2.8]. In particular, there exists $\varphi \in \hat{A}$ such that $|\varphi(a)| = r(a) = ||a|| \neq 0$. ■

The algebra of $n \times n$ complex matrices, or $M_n(\mathbb{C})$, when given the operator norm and the conjugate transpose as the adjoint, is a unital noncommutative C^*-algebra. More precisely, the multiplicative identity of $M_n(\mathbb{C})$ is the n-dimensional identity matrix I_n. To see that $\widehat{M_n(\mathbb{C})}$ is empty, let e_{ij} denote the matrix with 1 for the ij-entry and 0 otherwise. If $i \neq j$, then $(e_{ij})^2 = 0$ for $i \neq j$, hence $\varphi(e_{ij})_{ij}) = \varphi((e_{ij})^2) = 0$ for all $\varphi \in \widehat{M_n(\mathbb{C})}$. Since $e_{ii} = e_{ij}e_{ji}$ for such matrices, $\varphi(e_{ii}) = 0$ for all $\varphi \in \widehat{M_n(\mathbb{C})}$ and $i = 1, 2, \cdots, n$. However,

$\varphi(a) = \varphi(aI_n) = \varphi(a)\varphi(I_n)$ for all $\varphi \in \widehat{M_n(\mathbb{C})}$ and $a \in M_n(\mathbb{C})$ implies

$$1 = \varphi(I_n) = \varphi(e_{11}) + \cdots + \cdots \varphi(e_{nn}) = 0,$$

which is a contradiction. Thus a noncommutative C^*-algebra may have no characters.

Whether a C^*-algebra is commutative or noncommutative, the C^*-identity encodes the norm in the adjoint. Algebraic properties of *-homomorphisms can therefore induce analytical properties for such maps.

Theorem 4. *Let A and B be unital C^*-algebras. If $\pi : A \to B$ is a unital *-homomorphism, then π is contractive, hence continuous. Moreover, π is an isometry when π is an injective *-homomorphism.*

Proof. For $a \in A$ and $\lambda \in \mathbb{C}$, $\pi(a - \lambda 1_A) = \pi(a) - \lambda 1_B$ is invertible if $\pi - \lambda 1_A$ is invertible. Therefore

$$||\pi(a)||_B^2 = ||\pi(a)\pi(a)^*||_B = r(\pi(a)\pi(a)^*) \leqslant r(aa^*) = ||aa^*||_A = ||a||_A^2,$$

where the first and last equalities follow from the C^*-identity, the second and penultimate equalities from the previous theorem, and the inequality from the definition of the spectral radius of an element. If π is also injective, then $\pi^{-1} : \pi(A) \to A$ is also a *-homomorphism, hence contractive. More precisely,

$$||a||_A = ||\pi^{-1}(\pi(a))||_A \leqslant ||\pi(a)||_B.$$

Therefore π injective implies π is an isometry between A and $\pi(A)$. ■

The Gelfand transform of a commutative C^*-algebra is a *-homomorphism of A into $C(\hat{A})$ [10, Theorem I.3.1]. The surjectivity aspect of the fact that this map is also a

*-isomorphism between these two spaces in the unital setting follows from the consequent continuity of Γ_A, whereas the injectivity will be shown to rely on the condition that A is commutative.

Proof. *(of the Gelfand-Naimark Theorem)* To apply the Stone-Weierstrass Theorem [8, Chapter IV, Theorem 8.1], note that arguments analgalous to those applied earlier in the setting of $M_n(\mathbb{C})$ may be used to show $\varphi(1_A)$ evaluates to 1 for all $\varphi \in \hat{A}$. Thus Γ_A takes 1_A to the constant 1 function on $\hat{A}$. The Banach algebra structure of A guarantees that it is complete with respect to its norm. Since Γ_A is a unital *-homomorphism from A into $C(\hat{A})$ and hence a continuous map, $\Gamma_A(A) \subseteq C(\hat{A})$ is closed. More precisely, $\Gamma_A(A)$ is a unital C^*-subalgebra of $C(\hat{A})$. If φ_1 and φ_2 are distinct characters on A, then φ_1 and φ_2 do not coincide on some $a \in A$. Therefore $\Gamma_A(A)$ separates the points of $\hat{A}$. As a result, $\Gamma_A(A) = C(\hat{A})$. Next consider whether Γ_A is injective. Suppose that a_1 and a_2 are distinct elements of A. If $\Gamma_A(a_1) = \Gamma_A(a_2)$, then $\delta_{a_1}(\varphi) = \delta_{a_2}(\varphi)$ for all $\varphi \in \hat{A}$. Since A is commutative, the previous Theorem-Definition and a result about unital commutative Banach algebras used in the proof of its corollary give

$$\begin{aligned}||a_1 - a_2||_A = r(a_1 - a_2) &= \sup\{|\lambda| : \lambda \in \mathbb{C}, (a_1 - a_2) - \lambda I_A \text{ is not invertible}\} \\ &= \sup\{|\varphi(a_1 - a_2)| : \varphi \in \hat{A}\} \\ &= ||\Gamma_A(a_1 - a_2)||_\infty = ||\Gamma_A(a_1) - \Gamma_A(a_2)||_\infty = 0,\end{aligned}$$

or $a_1 - a_2 = 0$, as desired. ■

In the setting of a compact Hausdorff space X, the Gelfand transform of X is a homeomorphism of X to $\widehat{C(X)}$ [28, Theorem 2.1.15]. Furthermore, a homeomorphism between two compact Hausdorff spaces extends to a *-isomorphism between their associated

C^*-algebras and vice versa [7, page 87]. Thus to identify a compact Hausdorff X such that $C(X)$ and a given unital commutative C^*-algebra A differ up to *-isomorphism, it suffices to determine $\widehat{A}$. In the case of $\mathbb{C}$, $\varphi(\lambda) = \varphi(\lambda \cdot 1) = \lambda\varphi(1) = \lambda$ for all $\lambda \in \mathbb{C}$ and $\varphi \in \widehat{\mathbb{C}}$. In particular, $\widehat{\mathbb{C}}$ is homeomorphic to the compact Hausdorff space $X = \{x\}$, hence $C(\widehat{\mathbb{C}})$ is *-isomorphic to $C(\{x\})$. Because $\mathbb{C}$ is *-isomorphic to $C(\widehat{\mathbb{C}})$ by the Gelfand-Naimark Theorem, $\mathbb{C}$ is *-isomorphic to $C(\{x\})$. More generally, the category of unital commutative C^*-algebras with unital *-homomorphisms is dual to the category of compact Hausdorff spaces with continuous maps and this correspondence is called *Gelfand duality.* Any homeomorphism invariant of the compact Hausdorff X can therefore be reframed as an algebraic invariant of the C^*-algebra $C(X)$. For example, X is a totally disconnected compact metric space if and only if $C(X)$ is a unital commutative approximately finite dimensional algebra (see section 3 of this chapter for a definition). Since any totally disconnected compact metric space is homeomorphic to a subset of the Cantor set, this fractal and its topology can be canonically associated with such algebras. Furthermore, C^*-algebras within the framework of Gelfand duality remain to be identified for other fractals like the Sierpinski gasket. Such an investigation could therefore begin to form the basis for a classification program of C^*-algebras on fractal spaces.

Given a Hilbert space H, the set of bounded linear operators from H to H, $B(H)$, with pointwise defined addition and composition as multiplication, is an important example of a noncommutative C^*-algebra. For any T in $B(H)$, the adjoint T^* is given by the unique T^* in $B(H)$ such that for all x and y in H, $\langle Tx, y\rangle_H = \langle x, T^*y\rangle_H$. When equipped with the operator norm, $B(H)$ exhibits the C^*-identity as a consequence of the Cauchy-

Schwartz Inequality and the submultiplicativity property of a Banach space norm. Another example of noncommutative C^*-algebraic structure emerges in $M_n(\mathbb{C})$ when provided with the operator norm and the conjugate transpose as the adjoint. If H is finite-dimensional, then $B(H)$ is *-isomorphic to $M_n(\mathbb{C})$. Correspondingly,

Definition 8. *A **representation** of a C^*-algebra A on a Hilbert space H is a *-homorphism,*

$$\pi : A \to B(H).$$

*If π is also injective, then π is called **faithful**.*

With this distinction in hand, the theory of noncommutative geometry builds upon the following theorem.

Theorem 5 (Gelfand-Naimark-Segal Theorem)**.** *Every C^*-algebra A has a faithful representation on a Hilbert space H as an operator norm-closed *-algebra of $B(H)$.*

Noncommutative C^*-algebras can be formally viewed as duals to noncommutative topologies or geometries. Commutative topologies or geometries correspond to commutative C^*-algebras and many spaces that are considered well-behaved from the perspective of Riemannian geometry fall into this category.

3.2 Spectral Triples

Alain Connes initiated a program to adapt classical tools from topology and Riemannian geometry to the operator algebraic setting. For example, Connes showed that the geodesic distance on a compact spin Riemannian manifold M can be recovered from the C^*-algebra $C(M)$, the Hilbert space H of L^2-spinor fields, and a differential operator D called the Dirac operator, via

$$d_g(x,y) = \sup\{|f(x) - f(y)| : f \in C(M), ||[D,f]||_{B(H)} \leqslant 1\}.$$

Since none of the arguments required for this result rely on the commutativity of $C(M)$, this formula remains valid for noncommutative C^*-algebras. As a consequence, this noncommutative formulation of intrinsic distance on M allows for consideration of other, possibly noncommutative, C^*-algebras on M in the role of $C(M)$. Connes also discovered that the pairing of the C^*-algebra with the Hilbert space yields only information about the dimension of M, as does knowledge of the Hilbert space in combination with that of the Dirac operator. All three sources of operator theoretic data are therefore necessary to recover the geometry of M. Connes formalized the essential operator algebraic elements needed to build a noncommutative metric geometry beyond the prototypical setting of M in the definition of a *spectral triple*. Following the convention in [11],

Definition 9. *Let A be a unital C^*-algebra. An **unbounded Fredholm module** (H, D) over A consists of a Hilbert space H together with a unital representation π of A into $B(H)$ and an unbounded, self-adjoint operator D on H such that*

(a) the set

$$\{a \in A \text{ for which } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator on } H\}$$

is dense in A,

(b) the operator $(I + D^2)^{-1}$ is compact.

If the underlying representation π is faithful, then $(\mathcal{A}, \mathcal{H}, D)$ is called a ***spectral triple****, and D a* ***Dirac operator****.*

While the choice of C^*-algebra A categorizes the space as commutative or noncommutative and the Gelfand-Naimark-Segal Theorem guarantees the existence of a suitable representation π on some Hilbert space H, the choice of Dirac operator D determines the differential structure. If, given a in A, differentiation of a is viewed as formation of the operator $[D, \pi(a)]$, then the dense set in A described in condition (a) is analogous to the dense set of C^1 functions in $C(X)$. The compact resolvent condition ensures that the eigenvalues of D exhibit properties that allow for the extraction of geometric information like measure and dimension from spectral data. More precisely, an operator $T \in B(H)$ is compact if the image of the unit ball of H under T has compact closure in the norm topology of H [5]. In fact, the set of compact operators $\mathcal{K}(H)$ is a norm closed ideal in $B(H)$ [31, p.107]. Many uses of compact operators in noncommutative geometry rely on the following property.

Theorem 6 ([28, Theorem 1.4.11])**.** *If $T \in B(H)$ is compact, then $\sigma(T)$ is countable and each non-zero point of $\sigma(T)$ is an isolated point.*

For instance, the presence of a nonzero accumulation point in the spectrum of a Dirac operator would be a necessary but not sufficient requirement for the following quantity to be finite. As in [19],

Definition 10. *Let* (A, H, D) *be a spectral triple. If* $Tr((I + D^2))^{\frac{-p}{2}}$ *is finite for some positive real number* p*, then the spectral triple is called* ***p-summable*** *or just* ***finitely summable****. The number* ∂_{SG}*, given by*

$$\partial_{ST} = \inf\{p > 0 : tr(D^2 + I)^{\frac{-p}{2}}, \infty\},$$

is called the ***spectral dimension*** *of the spectral triple.*

Since Hausdorff dimension is an important tool for detecting fractality, noncommutative fractal geometers are especially interested in finding Dirac operators which encode the Hausdorff dimensions of fractals in their asymptotics.

Because spectral triples generalize differential structure, they open up promising avenues for extending analytic methods from mathematical physics to fractal spaces. The Laplacian operator plays a critical role in the formulation of many important differential equations such as those that model heat dissipation or wave propagation. The definition of a Laplacian on a space requires the choice of a measure and a spectral triple induces a measure via the operator algebraic tool called the Dixmier Trace and denoted by Tr_ω (see Chapter 4 in [7] for a precise definition). For the setting of a compact spin Riemannian manifold M detailed at the beginning of this section, let v be the Riemannian volume measure of M, d the dimension of M, and $c(d) = 2^{d-[d/2]}\,\pi^{d/2}\,\Gamma(\frac{d}{2} + 1)$. Connes showed that for any $f \in C(M)$,

$$\int_M f\,dv = c(d)\,Tr_\omega(\pi(f)|D|^{-d}).$$

More generally, a desired measure μ can be recovered via a spectral triple (A, H, D) with representation π when (A, H, D) can be chosen so that the map $Tr_\omega(\pi(f)|D|^{-s})$, where s is the spectral dimension, is a nontrivial positive linear functional on $\mathcal{A}$ that induces a measure that differs from μ by a multiplicative constant. An example of a spectral triple developed by Christensen, Ivan, and Lapidus which yields the $\log_2 3$-dimensional Hausdorff measure on the Sierpinski gasket will be discussed in the next chapter.

Progress in noncommutative fractal geometry can lead to new insights about fractality. In the originating setting of M, the Riemannian metric determines the Dirac operator and the removal of the spin structure eliminates this uniqueness. Since the metric of a Riemannian manifold can be recovered from the geodesic distance, Connes' reformulation implies that a Dirac operator can under suitable conditions dictate a Riemannian metric. As a consequence, Connes' operator algebraic reframing of geodesic distance generated the discovery that the Dirac operator defines the geometry of a Riemannian manifold. Development of a noncommutative fractal geometry is therefore motivated by the exploration of new ways to describe, understand, and even define fractals. The development of noncommutative geometry due to Connes and outlined in this section can also be more precisely described as *noncommutative Riemannian geometry.* Since compact Riemannian manifolds can also be viewed as compact metric spaces, they can be examined using tools from metric geometry. A theory that may be termed *noncommutative metric geometry* was developed by Rieffel and is the subject of the next section. In analogy with the classical case and with the aim of laying the foundation for new insights about fractality, methods from noncommutative metric geometry will also be applied in concert with those from noncommutative Riemannian geometry to study piecewise C^1-fractal curves.

3.3 Quantum Compact Metric Spaces

When a compact Hausdorff space X is also a compact metric space, $\widehat{C(X)}$ can be equipped with a metric that encodes the metric on X. Piecewise C^1-fractal curves are compact metric spaces with respect to the geodesic distance. Via the work of Lapidus with Christensen, Ivan, and Sarhad, the geodesic metric on such a fractal can also be captured using the noncommutative differential structure of a spectral triple [11, 19]. Extending this toolkit to include techniques from noncommutative metric geometry will allow for meaningful metric approximation of piecewise C^1-fractal curves as noncommutative Riemannian manifolds.

To recover the metric structure of a compact metric space (X, d) from $C(X)$, set L_d equal to the Lipschitz seminorm on $C(X)$ associated to d- that is, for every f in $C(X)$,

$$L_d(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\right\}.$$

For any $\varphi_1, \varphi_2 \in \widehat{C(X)}$, let mk_{L_d} be defined by

$$\mathrm{mk}_{L_d}(\varphi_1, \varphi_2) = \sup\{|\varphi_1(f) - \varphi_2(f)| : f \in \mathrm{dom}(L_d), L_d(f) \leqslant 1\}.$$

When $\widehat{C(X)}$ is endowed with mk_{L_d}, the Gelfand transform of X becomes an isometry from (X, d) onto $(\widehat{C(X)}, \mathrm{mk}_{L_d})$.

Theorem 7. *If (X, d) is a compact metric space, then $(\widehat{C(X)}, mk_{L_d})$ is also a compact metric space. In particular, (X, d) and $(\widehat{C(X)}, mk_{L_d})$ are isometric.*

Proof. Fix a choice of $p \in X$. For any $q \in X \backslash \{p\}$ and $f \in C(X)$ such that $L_d(f) \leqslant 1$,

$$|\delta_p(f) - \delta_q(f)| = |f(p) - f(q)| \leqslant L_d(f)\, d(p, q) \leqslant d(p, q),$$

hence

$$\mathrm{mk}_{L_d}(\delta_p, \delta_q) \leqslant d(p, q).$$

To see that $\mathrm{mk}_{L_d}(\delta_p, \delta_q)$ achieves this upper bound, observe that $f_p(w) := d(p, w) \in C(X)$. Given any distinct $q, w \in X$, the triangle inequality yields

$$|d(p, w) - d(p, q)| \leqslant d(q, w).$$

In particular, $L_d(f_p) \leqslant 1$ and

$$|\delta_p(f_p) - \delta_q(f_p)| = |d(p, p) - d(p, q)| = d(p, q),$$

as desired. ■

Recall that without the condition of commutativity, $\widehat{A}$ may be empty. The following elements of a C^*-algebra are used to identify linear functionals suitable for the role of $\widehat{C(X)}$ in the more general setting of unital C^*-algebras.

Definition 11. *An element a of a C^*-algebra A is called* ***positive*** *if a is self-adjoint and $\sigma(a)$ is contained in the non-negative real line* $[0, \infty)$.

Every self-adjoint element in a C^*-algebra can be written as the difference of two positive elements [10, Corollary I.4.2]. The self-adjoint elements of the C^*-algebras $\mathbb{C}$ are the real numbers and the positive elements are the non-negative real numbers. Linear functionals that take positive elements to positive elements therefore preserve the self-adjoint property.

Definition 12. *A linear functional ψ on a C-algebra A is called* ***positive*** *if $\psi(a)$ is positive in $\mathbb{C}$ whenever a is positive in A. A positive linear functional on a C^*-algebra that is also of norm 1 is called a* ***state****.*

States on a unital C^*-algebra can always be identified in the following way.

Theorem 8 ([28, Theorem 3.3.1, Corollary 3.3.4])**.** *Let ψ be a linear functional on a unital C^*-algebra A. Then ψ is a state if and only if ψ is bounded and $\psi(1_A) = ||\psi|| = 1$.*

If ψ is a unital representation of a unital C^*-algebra A on a Hilbert H and x is a unit vector in H, then $\psi(a) := \langle \pi(a)x, x \rangle_H$ is a bounded linear functional on A as a consequence of the Cauchy Schwartz Inequality, the contractive property of *-homomorphisms, and the linearity of the inner product in the first component. In particular,

$$\psi(a) \leqslant ||\pi(a)x||_H \leqslant ||\pi(a)||_{B(H)} \leqslant ||a||_A,$$

hence $||\psi|| \leqslant 1$. Since $\psi(1_A) = 1$, ψ is a state on A. For the unital commutative case, the Riesz Representation Theorem gives a bijection between Borel probability measures on a compact Hausdorff space X and states on $C(X)$ [8, Theorem 5.7]. In fact, the Hahn-Banach Theorem can be used to show that for every self-adjoint element a in a unital C^*-algebra A, there exists a state ψ such that $|\psi(a)| = ||a||_A$ [5, Theorem 1.7.2]. Since bb^* is always self-adjoint given any $b \in A$, states, unlike characters, on a C^*-algebra always exist. Because of the identification with probability measures when A is commutative, states can be viewed as the noncommutative analogues of probability measures.

Definition 13. *The **state space** of a C^*-algebra A, or $S(A)$, is the set of positive linear functionals on A of norm 1.*

As a consequence of the previous theorem, the state space of a unital C^*-algebra is a subset of the dual space. When equipped with the weak*-topology, the state space exhibits the following properties.

Theorem 9. *If A is a unital C^*-algebra, then $S(A)$ is weak*-compact and convex.*

Proof. By the previous theorem and the definition of $S(A)$,

$$
\begin{aligned}
S(A) &= \{\psi \in \widehat{A} : ||\psi|| \leqslant 1, \psi(1_A) = 1\} \\
&= \{\psi \in \widehat{A} : ||\psi|| \leqslant 1\} \cap \{\delta_{1_A}(\psi) = \psi(1_A) = 1\} \\
&= \{\psi \in \widehat{A} : ||\psi|| \leqslant 1\} \cap \delta_{1_A}^{-1}(\{1\})
\end{aligned}
$$

To see that $S(A)$ is weak*-compact, note that the definition of the weak*-topology guarantees that δ_{1_A} is continuous. Hence $\delta_{1_A}^{-1}(\{1\})$ is weak*-closed. Since the Banach Alaoglu Theorem gives that the unit ball in $\widehat{A}$ is weak*-compact [8, Chapter 3, Theorem 3.1], $S(A)$ is weak*-compact. To verify that $S(A)$ is weak*-convex, the previous theorem will again be applied. For any $\psi_1, \psi_2 \in S(A)$ and $\alpha \in [0,1]$, observe that $\alpha\psi_1 + (1-\alpha)\psi_2$ is likewise a bounded linear functional that evaluates to 1 at 1_A. The previous theorem implies

$$
1 = \alpha\psi_1(1_A) + (1-\alpha)\psi_2(1_A) = \Big(\alpha\psi_1 + (1-\alpha)\psi_2\Big)(1_A) \leqslant ||\alpha\psi_1 + (1-\alpha)\psi_2||
$$

$$
\leqslant ||\alpha\psi_1|| + ||(1-\alpha)\psi_2|| = |\alpha|||\psi_1|| + |1-\alpha|||\psi_2|| = 1.
$$

Since $||\alpha\psi_1 + (1-\alpha)\psi_2||$ is also 1, the previous theorem also gives that $\alpha\psi_1 + (1-\alpha)\psi_2$ is in $S(A)$. ■

A state is called *pure* if is an extreme point of $S(A)$. The set of pure states on A is denoted by $P(A)$. When A is also commutative and so *-isomorphic to $C(X)$ for some compact Hausdorff space X, $P(A)$ coincides with the set of Dirac measures for each point in X. Unlike $\hat{A}$ sometimes in the general case, $P(A)$ is never empty.

Theorem 10 (Krein-Milman)**.** *If S is a non-empty compact convex subset of a locally convex space B, then the set of extreme points of S is non-empty and S is the closed convex hull of the set of extreme points of S.*

The two previous theorems together imply that $S(A)$ is the weak*-closure of convex combinations of states in $P(A)$. A metric that metrizes the weak*-topology on $S(A)$ would therefore allow for metric approximations of $S(A)$ by finite sums in $P(A)$. The existence of such a metric is one of Rieffel's requirements for the quantum counterpart of a compact metric space.

Definition 14 ([22])**.** *Let A be a unital C^*-algebra. If L is a seminorm defined on a dense subspace $dom(L)$ of $sa(A)$ such that*

(a) $\{a \in dom(L) : L(a) = 0\} = \mathbb{R}1_A$,

(b) L is lower semi-continuous with respect to $||\cdot||_A$,

(c) the Monge-Kantarovich distance associated to L, that is, the metric defined for all φ, ψ in $S(A)$ by

$$mk_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in dom(L), L(a) \leqslant 1\},$$

metrizes the weak-topology on* $S(A)$*,*

(d) for all $a, b \in dom(L)$*,*

$$\max\left\{L\Big(\frac{ab+ba}{2}\Big), L\Big(\frac{ab-ba}{2i}\Big)\right\} \leqslant L(a)||b||_A + ||a||_A L(b),$$

then (A, L) *is called a* ***quantum compact space*** *and* L *a* ***Lip-norm****.*

In the classical case where $(C(X), L_d)$ comes from the compact metric space (X, d), L_d vanishes precisely on scalar multiples of the identity and mk_{L_d} induces the weak*-topology on the set of regular Borel probability measures. Because d can be recovered in this setting from the restriction of mk_{L_d} to $P(A)$, the Krein-Milman theorem gives a means for extending d to the whole state space. Furthermore, L_d coincides with $L_{\mathrm{mk}_{L_d}}$: $sa(C(X)) \to \mathbb{R}_{\geqslant 0}$ given by

$$L_{\mathrm{mk}_{L_d}}(f) = \sup\left\{\frac{|\varphi(f) - \psi(f)|}{\mathrm{mk}_{L_d}(\varphi, \psi)} : \varphi, \psi \in S(C(X)), \varphi \neq \psi\right\}.$$

Without the additional lower semi-contininuity condition of L on $sa(A)$, $L_{\mathrm{mk}_L} \leqslant L$. In [33], Rieffel determines that L coincides with L_{mk_L} when this semi-continuity property is also present. While the Leibniz inequality connects the Lipschitz seminorm in the setting of $(C(X), L_d)$ with the underlying multiplication of functions in that algebra, the analogue of this inequality detailed in the last condition bounds the seminorm component of a quantum compact metric space with the multiplicative structure of the C^*-algebra. For the general setting beyond that of $(C(X), L_d)$, Rieffel also identified alternate characterizations of Lip-norms.

Theorem 11 ([29, 36, 33])**.** *Let* A *be a unital* C^**-algebra and* L *a seminorm on* $sa(A)$ *such that* L *is lower semi-continuous with respect to* $||\cdot||_A$*, the set* $dom(L) = \{a \in sa(A) : L(a) <$

$\infty\}$ *is dense in* $sa(A)$*, and* $\{a \in dom(L) : L(a) = 0\} = \mathbb{R}1_A$*. The following are equivalent:*

1. (A, L) *is a quantum compact metric space;*

2. *the metric* mk_L *is bounded and there exists* $r \in \mathbb{R}$ *such that* r *is positive and the set*

$$\{a \in sa(A) : L(a) \leqslant 1, ||a||_A \leqslant r\}$$

is totally bounded in A *for* $||\cdot||_A$*;*

3. *the set*

$$\{a + \mathbb{R}1_A \in sa(A)/\mathbb{R}1_A : a \in sa(A), L(a) \leqslant 1\}$$

is totally bounded in $sa(A)/\mathbb{R}$ *for* $||\cdot||_{sa(A)/\mathbb{R}}$*;*

4. *there exists a state* $\psi \in S(A)$ *such that the set*

$$\{a \in sa(A) : L(a) \leqslant 1, \psi(a) = 0\}$$

is totally bounded in A *for* $||\cdot||_A$*;*

5. *for all* $\psi \in S(A)$ *the set*

$$\{a \in sa(A) : L(a) \leqslant 1, \psi(a) = 0\}$$

is totally bounded in A *for* $||\cdot||_A$*.*

For $(C(X), L_d)$, the self-adjoint elements are the real-valued functions. Recall the definition of f_p and the demonstration that $L_d(f_p) \leqslant 1$ from the first proof of this section. Since $f_p \in C(X, \mathbb{R})$ and $f_p(q)$ is nonzero for any $q \in X$ that is distinct from p, dom(L_d)

separates the points of X. The denseness of $\mathrm{dom}(L_d)$ in $C(X, R)$ then follows from the Stone-Weierstrass Theorem [8, Chapter IV, Theorem 8.1]. In both this example and the general setting, any element a belonging to a C^*-algebra yields the self-adjoint elements $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$. Continuous linear functionals which agree on a dense subspace of the set of self-adjoint elements therefore agree on the whole C^*-algebra. Since $\mathrm{dom}(L_d)$ separates the points of $S(C(X))$ and $f \in \mathrm{dom}(L_d)$ implies $L_d\left(\frac{f}{\max\{1, L_d(f)\}}\right) \leqslant 1$, distance zero for mk_{L_d} implies two states coincide. To see that mk_{L_d} is bounded when d is bounded, let p denote some fixed point in X and f a function in $C(X)$ such that $L_d(f) \leqslant 1$. Then for any distinct φ and ψ in $S(C(X))$,

$$\begin{aligned}
|\varphi(f) - \psi(f)| &= |\varphi(f) - f(p)\varphi(1_{C(X)}) + f(p)\psi(1_{C(X)}) - \psi(f)| \\
&= |\varphi(f - f(p)1_{C(X)}) - \psi(f - f(p)1_{C(X)})| \\
&\leqslant ||\varphi - \psi||\, ||f - f(p)1_{C(X)}|| \\
&\leqslant (||\varphi|| + ||\psi||) \sup\{|f(p) - f(q)| : q \in X\} \\
&\leqslant 2\, L_d(f) \operatorname{diam}(X)
\end{aligned}$$

which is bounded when X is a compact metric space. Furthermore, the Arzela-Ascoli Theorem guarantees that for any fixed choice of positive r, the set

$$\{f \in sa(C(X)) : L_d(a) \leqslant 1, ||f||_{C(X)} \leqslant r\}$$

is totally bounded in $C(X)$ for $|| \cdot ||_{C(X)}$ [8, Chapter VI, Theorem 3.8]. Hence the previous theorem not only establishes that all algebras of Lipschitz functions over classical compact metric spaces are quantum compact metric spaces, but also extends several of their key properties to the noncommutative setting. Because of the role of the Arzela-Ascoli Theorem in the application of this theorem to the classical case, this set of equivalences can be viewed as a *noncommutative Arzela-Ascoli Theorem.*

The space of quantum compact metric spaces contains finite-dimensional examples. Recall that seminorms exhibit both homogeneity and subadditivity. Since a Lip-norm also vanishes on scalar multiples of the identity for the C^*-algebra, the only seminorm that equips $\mathbb{C}$ with quantum compact metric space structure is the zero seminorm. In fact, the following seminorms always induce quantum compact metric space structure for finite-dimensional C^*-algebras.

Theorem 12. *If A is a unital finite-dimensional C^*-algebra and L is a lower semi-continuous seminorm on $sa(A)$ with domain that is a dense, unital subspace of $sa(A)$ that vanishes only on $\mathbb{R}1_A$, then (A, L) is a quantum compact metric space. In particular, $\mathbb{C} \oplus \mathbb{C}$ equipped with the seminorm $Q_\epsilon(w, z) = \frac{1}{\epsilon}|z - w|$ is a quantum compact metric space for every $\epsilon > 0$.*

Proof. Given a C*-algebra A and a seminorm L with these properties, (A, L) is a quantum compact metric space if and only if the set

$$\{a + \mathbb{R}1_A \in sa(A)/\mathbb{R}1_A : a \in (L), L(a) \leqslant 1\}$$

is totally bounded in $sa(A)/\mathbb{R}1_A$ for $||\cdot||_{sa(A)/\mathbb{R}1_A}$. Let $\tilde{L}$ denote the quotient seminorm of L on $sa(A)/\mathbb{R}1_A$ and $B_{\tilde{L}}$ the unit ball with respect to $\tilde{L}$. Because L vanishes only on $\mathbb{R}1_A$, $\tilde{L}$ is a norm on $sa(A)/\mathbb{R}1_A$. Moreover, A finite-dimensional and dom(L) a unital subspace of $sa(A)$ implies $sa(A)/\mathbb{R}1_A$ equipped with $||\cdot||_{sa(A)/\mathbb{R}1_A}$ is a finite-dimensional normed vector space. Equivalently, the unit ball in $sa(A)/\mathbb{R}1_A$ is compact with respect to $||\cdot||_{sa(A)/\mathbb{R}1_A}$. Since all norms on a finite-dimensional vector space are equivalent, $B_{\tilde{L}}$ is compact, hence totally bounded in $sa(A)/\mathbb{R}1_A$ for $||\cdot||_{sa(A)/\mathbb{R}1_A}$.

Now consider $(\mathbb{C}\oplus\mathbb{C}, Q_\epsilon)$ for some fixed choice of positive ϵ. The C*-algebra $\mathbb{C}\oplus\mathbb{C}$ is unital and finite-dimensional. The seminorm Q_ϵ vanishes on (w, z) if and only if $z = w$, that is, if and only if (w, z) is a scalar multiple of $1_{\mathbb{C}\oplus\mathbb{C}}$. Since $sa(\mathbb{C}\oplus\mathbb{C})$ coincides with $\mathbb{R}\oplus\mathbb{R}$, the only self-adjoint elements of $\mathbb{C}\oplus\mathbb{C}$ on which Q_ϵ vanishes are precisely $\mathbb{R}1_{\mathbb{C}\oplus\mathbb{C}}$. Because the choice of positive ϵ was arbitrary, $(\mathbb{C}\oplus\mathbb{C}, Q_\epsilon)$ is a quantum compact metric space for every $\epsilon > 0$. ■

The theorem above is well-known in the folklore of the noncommutative metric geometry community and Konrad Aguilar is gratefully acknowledged for the communication of this result and its proof. The first part of the theorem can also be used to identify Lip-norms for matrix algebras. Fix a choice of positive natural number n. The self-adjoint elements of the *full matrix algebra* $M_n(\mathbb{C})$ are the $n \times n$ matrices that are normal and have only real eigenvalues. Let Tr_n denote the trace of a matrix in $M_n(C)$, $\pi_{1,n} : \mathbb{C} \to M_n(C)$ the map given by

$$\pi_{1,n}(c) = \begin{bmatrix} c & 0 & \cdots & 0 & 0 \\ 0 & c & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & c \end{bmatrix},$$

and L the seminorm on $M_n(\mathbb{R})$ defined by

$$\left|\left|a - \pi_{1,n}\left(\frac{1}{n}Tr_n(a)\right)\right|\right|_{M_n(\mathbb{C})}.$$

In particular, $\pi_{1,n}\left(\frac{1}{n}Tr_n(a)\right) \in \mathbb{R}I_n$ for any $a \in sa(M_n(\mathbb{C}))$. Moreover, $||a||_{M_n(\mathbb{C})}$ coincides with $r(a)$ for such matrices, hence $\mathrm{dom}(L)$ is $sa(M_n(\mathbb{C}))$. Since $||\cdot||_{M_n(\mathbb{C})}$ vanishes only on the zero matrix, L vanishes only on $\mathbb{R}I_n$. Let a and b signify matrices in $\mathrm{dom}(L)$. As shown for the more general context considered in [2],

$$\max\left\{L\Big(\frac{ab+ba}{2}\Big), L\Big(\frac{ab-ba}{2i}\Big)\right\} \leqslant 2(L(a)||b||_{M_n(\mathbb{C})} + ||a||_{M_n(\mathbb{C})}L(b)).$$

Therefore $(M_n(\mathbb{C}), L)$ is a said to be a *(2, 0)-quasi-Leibniz quantum compact metric space.*

Based on earlier work with Latremoliere in [2], Aguilar and Brooker demonstrated in [1] that a seminorm like L is a (2, 0)-quasi-Leibniz Lip-norm for a full matrix algebra because it is an instance of a particular kind of map from a C^*-algebra to one of its C^*-subalgebras.

Definition 15. *A* ***conditional expectation*** $\mathbb{E}(\cdot|B) : A \to B$ *onto* B*, where* A *is a* C^**-algebra and* B *is a* C^**-subalgebra of* A*, is a linear positive map of norm 1 such that for all* $b, c \in B$ *and* $a \in A$*,*

$$\mathbb{E}(bac|B) = b\mathbb{E}(a|B)c.$$

By the Tomiyama Theorem [6, Theorem 1.5.10], a projection from a C^*-algebra to a C^*-subalgebra that is contractive is also a conditional expectation. To see that $\pi_{1,n}\Big(\frac{1}{n}Tr_n(a)\Big)$ is a conditional expectation, recall that *-homomorphisms are contractive and Tr_n is linear. Note that the range of $\pi_{1,n}\Big(\frac{1}{n}Tr_n(a)\Big)$ is $\mathbb{C}I_n$. Because $\pi_{1,n}\Big(\frac{1}{n}Tr_n(a)\Big)$ preserves the conjugate transpose operation and fixes matrices in $\mathbb{C}I_n$, this map is likewise a conditional expectation. By construction, $\pi_{1,n}\Big(\frac{1}{n}Tr_n(a)\Big)$ is, in addition, trace-preserving, hence L is a (2, 0)-quasi-Leibniz Lip-norm also as a consequence of [1, Theorem 2.3, Lemma 2.7]. For the broader framework studied by Aguilar and Latremoliere in [2], (2, 0)-quasi-Leibniz Lip-norms are constructed using conditional expectations on a class of C^*-algebras that can be built from a sequence of finite-dimensional C^*-algebras.

Definition 16. *A C^*-algebra is called* ***approximately finite dimensional*** *or* ***AF*** *if it can be written as the norm closure of an increasing union of finite-dimensional C^*-subalgebras A_n. When A is unital, A_0 must coincide with complex scalar multiples of 1_A.*

The conditional expectations defined on AF-algebras by Aguilar and Latremoliere rely on the existence of a *faithful tracial state*. More precisely, a state ψ on a C^*-algebra A is said to be *tracial* if for all a and b in A, $\psi(ab) = \psi(ba)$ and *faithful* if $\psi(a^*a) = 0$ implies $a = 0$. An example of a faithful tracial state on $M_n(\mathbb{C})$ is $\frac{1}{n}Tr_n$. In contrast, $\mathcal{K}(H)$ does not admit a tracial state when H is infinite-dimensional [28, Remark 6.2.2]. If A is a unital AF-algebra, then A has a faithful tracial state if its only norm-closed ideals are 0 and itself [28, Theorem 6.1.3,Remark 6.2.3,Remark 6.2.4].

Theorem 13 ([2])**.** *Let A be a unital AF-algebra for which there exists a faithful tracial state λ, A_0 the set of complex scalar multiples of 1_A, and $\cup_{n\in\mathbb{N}} A_n$ an increasing union of finite-dimensional C^*-algebras such that $A = \overline{\cup_{n\in\mathbb{N}} A_n}^{||\cdot||_A}$. Also, let $\mathbb{E}_n : A \to A_n$ be the unique conditional expectation with $\lambda \circ \mathbb{E}_n = \lambda$. Set $(\beta_n)_{n\in\mathbb{N}}$ in $(0,\infty)^{\mathbb{N}}$, with limit 0 at infinity. If, for all $a \in sa(A)$,*

$$L(a) := \sup\left\{\frac{||a - \mathbb{E}_n(a)||_A}{\beta_n} : n \in \mathbb{N}\right\},$$

then (A, L) is a (2,0)-quasi-Leibniz quantum compact metric space.

Recall that through the lens of Gelfand duality, $C(X)$ is *-isomorphic to a unital commutative AF-algebra if and only if X is a totally disconnected compact metric space. Unital AF-algebras like $C(X)$ when X is the Cantor set have natural candidates for approximating finite-dimensional C^*-algebras. Distance zero for a metric on noncommutative structures associated to C^*-algebras that respects Gelfand duality should imply the two underlying C^*-algebras are *-isomorphic. Such a metric exists for quantum compact metric spaces.

3.4 The Dual Gromov-Hausdorff Propinquity Metric

When (X, d) is a piecewise C^1-fractal curve equipped with its geodesic distance, (X, d) can be written as a Gromov-Hausdorff distance limit of an increasing sequence of finite graphs each equipped with their respective geodesic distances. As a consequence of Theorem 11, each of these compact metric spaces can be associated to a quantum compact metric space. As with classical compact metric spaces, the space of quantum compact metric spaces can be equipped with a metric. Since this space contains finite-dimensional examples and this metric is complete, the existence and identification of finite-dimensional metric approximations for a given quantum compact metric space can be considered. The definition of this metric requires a noncommutative analogue of isometry that takes into account the additional structure given to the C^*-algebra by the Lip-norm.

Definition 17 ([20]). *Let (A_j, L_j) be a quantum compact metric space for $j \in \{1, 2\}$. If (A, L) is a quantum compact metric space such that for each $j \in \{1, 2\}$, there exists a *-epimorphism $\pi_j : A \to A_j$ such that for every $a \in dom(L_j)$,*

$$L_j(a) = \inf\{L(b) : \pi_j(b) = a\},$$

*then (A, L, π_1, π_2) is called a **tunnel** from (A_1, L_1) to (A_2, L_2).*

The conditions formalized in the definition of a tunnel are sufficient to ensure that composition of every state in $S(A_j)$ with π_j gives an isometry from $(S(A_j), \mathrm{mk}_{L_j})$ onto its image in $(S(A), \mathrm{mk}_L)$ [34]. A number can be assigned to each tunnel so that a metric comparison of two quantum compact metric spaces can be obtained via the lenses of their respective state spaces.

Definition 18 ([22])**.** *Let $\tau = (A, L, \pi_1, \pi_2)$ be a tunnel from (A_1, L_1) to (A_2, L_2). The* ***extent*** *of τ, $\chi(\tau)$, is given by*

$$\chi(\tau) = \max_{j\in\{1,2\}} Haus_{mk_L}\{S(A), \{\varphi \circ \pi_j, \varphi \in S(A_j)\}\}.$$

Definition 19 ([20, 22])**.** *The* ***dual Gromov-Hausdorff propinquity****, or* ***dual propinquity****, between two quantum compact metric spaces (A_1, L_1) and (A_2, L_2) is given by*

$$\Lambda^*((A_1, L_1), (A_2, L_2)) = \inf\{\, \chi(\tau) : \tau \text{ is a tunnel from } (A_1, L_1) \text{ to } (A_2, L_2)\,\}.$$

Recall that the *-operation encodes the norm of a C^*-algebra and the Lip-norm determines a metric on the state space that metrizes the weak*-topology. An appropriate notion of equivalence between quantum compact metric spaces should therefore respect both the *- and the Lip-norm structures.

Definition 20 ([21, 41])**.** *Let (A_1, L_1) and (A_2, L_2) be quantum compact metric spaces. If $\pi : A_1 \to A_2$ is a *-isomorphism such that $L_2 \circ \pi = L_1$, then π is called a* ***full quantum isometry****.*

Theorem 14 ([20])**.** *The dual propinquity is a complete metric, up to full quantum isometry, on the class of quantum compact metric spaces.*

Moreover, the Gromov-Hausdorff topology on compact metric spaces can be recovered from the dual propinquity topology.

Theorem 15 ([20])**.** *If* (X_1, d_1) *and* (X_2, d_2) *are compact metric spaces, then*

$$\Lambda^*(\,(C(X_1), L_{d_1}), (C(X_2), L_{d_2})\,) \leqslant GH(\,(X_1, d_1), (X_2, d_2)\,).$$

Complete metric space structure makes measurement and approximation possible on a set. Such ideas are essential for understanding how mathematical models can simulate a physical system. Rieffel developed quantum compact metric spaces to give a mathematically formal framework for models found in quantum physics [35]. Examples of such models include full matrix algebras and $C(S^2)$, which is the commutative C^*-algebra of continuous, complex-valued functions on the sphere. When enriched with the appropriate Lip-norms, $C(S^2)$ has finite-dimensional approximations in the dual propinquity by full matrix algebras [37]. Any unital AF-algebra for which there exists a faithful tracial state also can be approximated in the dual propinquity by finite-dimensional C^*-algebras. The required quantum compact metric space structures are constructed using the class of Lip-norms built by Aguilar and Latrémolière for such an AF-algebra and described in the previous section [2]. More importantly, Aguilar and Latremoliere's results on finite-dimensional approximations of AF-algebras in the dual propinquity apply to $C(X)$ when X is the Cantor set $\mathcal{C}$ [2, 1]. Since $\mathcal{C}$ becomes a compact metric space if given the restriction of the Euclidean distance, any approximation results for this space in the Gromov-Hausdorff distance can also be used to build approximation results in the dual propinquity. Similarly,

Theorem 16. *Let X be a piecewise C^1-fractal curve, with parameterization $\{R_j\}_{j\in\mathbb{N}}$ and $B(n)$ an approximation sequence of X compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the geodesic distance on X by d_∞ and the geodesic distance on $X_n = \bigcup_{j=1}^{B(n)} R_j$ by d_n. Then*

$$\lim_{n\to\infty} \Lambda^*(\,(C(X), L_{d_\infty}), (C(X_n), L_{d_n})\,) = 0.$$

Proof. Given any fixed choice of $\epsilon > 0$, the definition of approximation sequence for a piecewise C^1-fractal curve guarantees there exists $N_1 \in \mathbb{N}$ such that for $n > N_1$, $\mathrm{Haus}_{d_n}(V_n, X_n) < \epsilon$. Furthermore, $GH(\,(V_n, d_\infty), (V_n, d_n)\,) = 0$ because by assumption, the restriction of d to $V_n \times V_n$ is d_n. Since V_* is dense in (X, d_∞) and $V_* = \bigcup_{n\geqslant 0} V_n$, there also exists $N_2 \in \mathbb{N}$ such that when $n > N_2$, $\mathrm{Haus}_{d_\infty}(X, V_n) < \epsilon$. Hence $n > \max\{N_1, N_2\}$, together with the previous theorem, yields

$$\begin{aligned}
\Lambda^*(\,(C(X), L_{d_\infty}), (C(X_n), L_{d_n})\,) &\leqslant GH(\,(X, d_\infty), (X_n, d_n)\,)\\
&\leqslant GH(\,(X, d_\infty), (V_n, d_\infty)\,) + GH(\,(V_n, d_\infty), (V_n, d_n)\,)\\
&\qquad + GH(\,(V_n, d_n), (X_n, d_n)\,)\\
&\leqslant \mathrm{Haus}_{d_\infty}(X, V_n) + GH(\,(V_n, d_\infty), (V_n, d_n)\,)\\
&\qquad + \mathrm{Haus}_{d_n}(V_n, X_n) < 2\epsilon,
\end{aligned}$$

as desired. ■

Application of the dual propinquity and its extensions to the study of fractals makes possible an enlarged understanding of these objects as noncommutative spaces. The same C^*-algebra can support non-equivalent Lip-norms. As a consequence, quantum compact metric space structures that are not fully quantum isometric can be defined on the same C^*-algebra. In the case of $\mathcal{C}$, $C(\mathcal{C})$ can be equipped with the Lip-norm associated to a classical metric on this fractal. Aguilar and Lopez show in [3] that this Lip-norm differs from the Lip-norm constructed as a consequence of Aguilar and Latremoliere's work on AF-algebras. They demonstrate that the corresponding Monge-Kantarovich metrics agree on $P(C(\mathcal{C}))$ but not on all of $S(C(\mathcal{C}))$. In particular, the enriched perspective of a fractal through the lens of a state space exhibits promise in capturing key aspects of fractality absent in purely point-based representations. For example, recall that the roughness, or complexity, of a fractal can be encoded its Hausdorff dimension and this quantity is often a non-integer quantity when associated to a fractal. Common approximations of fractals have integer dimension and are based on a view of this fractal as a set of points. Hence algebras of functions defined on finite approximations of fractals can lead to new insights about how to capture fractality through approximations on simpler structures.

In the noncommutative setting of the Gromov-Hausdorff propinquity, not all Hausdorff distance equivalent sets for a fractal support unital C^*-algebraic structure. For instance, V_* and $\bigcup_{n\geqslant 1} SG_n$ are both dense in SG. When each equipped with d_∞, both spaces are isometric with (SG, d_∞). Unlike $\mathcal{C}$, V_* is not compact. Consequently, $C(V_*)$ is not a unital C^*-algebra. Furthermore, $C(V_*)$ cannot be a component for a quantum compact metric space for any Lip-norm. Thus the hypotheses of Theorem 15 do not apply as in the case of Theorem 16. Recall that X is a totally disconnected compact metric space if and only if $C(X)$ is a unital commutative AF algebra. If $C(V_*)$ and $C(SG)$ were *-isomorphic as C^*-algebras, then $C(SG)$ could be represented as an AF-algebra. In particular, $C(SG)$ could be equipped with a Lip-norm L so that it is finitely approximable in the dual propinquity via the work of Aguilar and Latremoliere [2]. Thus the topology of a fractal is encoded in its admissable quantum compact metric space structures. Investigations into these quantum compact metric spaces could be the basis for new insights about the fractal. As a complete metric on quantum compact metric spaces, the dual propinquity would be a useful tool in such a study.

3.5 Spectral Propinquity

As the setting for the construction of Lapidus-Sarhad spectral triples, piecewise C^1-fractal curves can be viewed as a class of fractal-type "manifold." Many important examples of piecewise C^1-fractal curves, such as the Sierpinski gasket, are Hausdorff distance limits of increasing unions of finite graphs. An analytic framework for spectral triples will be given that enables consideration of whether spectral triples on such fractals can also be metrically approximated by spectral triples on approximating sets with simpler structures. An understanding of such questions would set the stage in noncommutative geometry for the definition of operators on fractals that suitably generalize their counterparts on classical manifolds.

Spectral triples grant access to analysis on fractals via tools from noncommutative Riemannian geometry. Methods from noncommutative metric geometry can also be applied to the development of spectral triples on fractals when quantum compact metric space structure is present.

Definition 21 ([23]). *A* ***metric spectral triple*** (A, H, D) *is a spectral triple such that the Monge-Kantarovich metric associated to* L_D *metrizes the weak***-topology on* $S(A)$*.*

For a metric spectral triple, all components are at work in the definition of a metric on the state space of the underlying C^*-algebra. Geometric information like measure and dimension can be also extracted from a spectral triple using the data given by the spectrum of the Dirac operator component. In particular, a metric on metric spectral triples should encode equivalence that requires the induced quantum compact metric spaces to be fully quantum isometric. Since unitarily equivalent operators share the same spectrum, such equivalence should also include this condition for the Dirac operator components. To see why the spectrum of an operator is invariant under conjugation with a unitary, suppose (A_1, H_1, D_1) and (A_2, H_2, D_2) are spectral triples and there exists a unitary $U : H_1 \to H_2$ such that

$$UD_1U^* = D_2.$$

Then for any $\lambda \in \mathbb{C}$,

$$D_2 - \lambda I = UD_1U^* - \lambda I = UD_1U^* - \lambda IUU^* = U(D_1 - \lambda I)U^*,$$

hence $D_2 - \lambda I$ is invertible if and only if $D_1 - \lambda I$ is invertible. In fact, Latrémolière's analytic framework for metric spectral triples detects both properties [23]. His metric on metric spectral triples is therefore a natural choice for the study of metric approximations of spectral triples on fractals.

A metric spectral triple supports a wealth of noncommutative geometric structures. When a spectral triple induces a quantum compact metric space, it also carries a

- a Hilbert space with an extra norm defined on some dense subspace, given by the the graph norm of the Dirac operator
- a *-representation of the quantum compact metric space on the Hilbert space,
- a group action of R on the Hilbert space obtained by exponentiating i times the Dirac operator.

A notion for convergence of metric spectral triples requires a form of convergence for each of these elements. Convergence of quantum compact metric spaces will be given by the dual propinquity. For Hilbert spaces with the graph norms of the Dirac operators of spectral triples, convergence will be defined by the *modular propinquity* [24]. The *metrical propinquity* will determine convergence of the *-representations in spectral triples [23]. By including covariant quantities with those obtained from each of these propinquity metrics, the *spectral propinquity* will quantify convergence of all of the corresponding noncommutative structures in addition to that of the actions of $\mathbb{R}$ on the Hilbert spaces obtained from the Dirac operators [23].

3.5.1 Convergence of Modules

Module structures in noncommutative geometry give rise to quantum analogues of vector bundles [41]. For a smooth manifold, a tangent space can be associated to each point. The corresponding set of vectors belonging to all such tangent spaces defines the *tangent* *bundle* of a smooth manifold. In differential geometry, vector bundles generalize tangent bundles. One module structure from noncommutative geometry that can be obtained from a spectral triple comes from that of the Hilbert space.

Definition 22 ([6, 24]). *A **Hilbert module** $(M, \langle\cdot,\cdot\rangle_M)$ **over a C^*-algebra** B, or **Hilbert B-module**, is a a right B-module with a map*

$$\langle\cdot,\cdot\rangle_M : M \times M \to B$$

such that

(1) $\langle\cdot,\cdot\rangle_M$ is linear in the second variable,

(2) for every $\xi, \eta \in M$ and $b \in B$, $\langle b\xi, \eta\rangle = \langle\xi, \eta\rangle b$,

(3) for every $\xi, \eta \in M$, $\langle\xi, \eta\rangle = \langle\eta, \xi\rangle^*$,

(4) for every $\xi \in M$, $\langle\xi, \xi\rangle \geqslant 0$,

(5) for every $\xi \in M$, $\langle\xi, \xi\rangle = 0$ if and only if $\xi = 0$,

(6) M is complete with respect to the norm given for every $\xi \in M$ by $||\xi||_M = \sqrt{||\langle\xi, \xi\rangle_M||_B}$.

Every Hilbert space is an example of a Hilbert module over $\mathbb{C}$. This Hilbert $\mathbb{C}$-module can be endowed with the graph norm of the Dirac operator. When the spectral triple is also a metric spectral triple, another module structure comes from the action of the induced quantum compact metric space made possible by its *-representation on the Hilbert space.

Definition 23 ([24, 25]). *A **metrical quantum vector bundle** (M, DN, B, L_B, A, L_A) is given by two quantum compact metric spaces (A, L_A) and (B, L_B), a Hilbert B-module*

$(M, \langle \cdot, \cdot \rangle_M)$ which also carries a left A-module structure, and a norm DN defined on a dense A-submodule, dom(DN), of M such that

- *for every $\xi \in dom(DN)$, $||\xi||_M \leqslant DN(\xi)$,*
- *$\{\xi \in M : DN(\xi) \leqslant 1\}$ is compact with respect to $|| \cdot ||_M$,*
- *for every $\omega, \eta \in M$, the **inner Leibniz inequality**,*

$$\max \left\{ L_B\Big(\frac{\langle \omega, \eta \rangle_M + \langle \omega, \eta \rangle_M^*}{2} \Big), L_B\Big(\frac{\langle \omega, \eta \rangle_M - \langle \omega, \eta \rangle_M^*}{2i} \Big) \right\} \leqslant 2DN(\omega)DN(\eta),$$

  holds,
- *for every $a \in dom(L_A)$ and $\xi \in dom(DN)$, the **modular Leibniz inequality**,*

$$DN(a\xi) \leqslant (||a||_A + L_A(a))DN(\xi),$$

  holds.

*The norm DN is called a **D-norm**. When $(M, DN, \mathbb{C}, 0, A, L_A)$ is a metrical quantum vector bundle, $(M, DN, \mathbb{C}, 0)$ is called a **metrized quantum vector bundle**.*

As shown in [23], every metric spectral triple gives rise to a metrical quantum vector bundle. To build a D-norm from the elements of a metric spectral triple (A, H, D), begin with the domain of the Dirac operator. For every $\xi \in \mathrm{dom}(D)$, let

$$DN(\xi) = ||\xi||_H + ||D\xi||_H.$$

Recall that $(\mathbb{C}, 0)$ is the only quantum compact metric space with the complex numbers as the underlying C^*-algebra. Since H is a Hilbert space, one of the two required quantum compact metric spaces is supplied by $(\mathbb{C}, 0)$. The Lip-norm of the second quantum compact metric space is obtained from all three elements of the metric spectral triple. Set

$$\mathrm{dom}(L_D) = \{\, a \in sa(A), a\mathrm{dom}(D) \subseteq \mathrm{dom}(D), [D, a] \text{ is bounded } \},$$

and for every $a \in \mathrm{dom}(L_D)$, let

$$L_D(a) = ||\, [D, a]\, ||_{B(H)}.$$

Then

$$\mathrm{qvb}(A, H, D) = (H, DN, C, 0, A, L_D)$$

denotes the metrical quantum vector bundle given by this construction. Furthermore, $(H, DN, \mathbb{C}, 0)$ is a metrized quantum vector bundle.

Each quantum compact metric space component in a metrical quantum vector bundle is a source of module structure. A notion of equivalence between metrical quantum vector bundles will be built on a notion of morphism between Hilbert modules.

Definition 24 ([25])**.** *Let A_1 and A_2 be unital C^*-algebras. A* ***left module morphism*** *(Π, π) from a left A_1-module M_1 to a left A_2-module M_2 is a unital *-morphism $\pi : A_1 \to A_2$ and a linear map $\Pi : M_1 \to M_2$ such that for every $a \in A$ and $\omega \in M_1$,*

$$\Pi(a\omega) = \pi(a)\Pi(\omega).$$

The module morphism (Π, π) is said to be ***surjective*** *when both Π and π are surjective maps, and it is said to be an* ***isomorphism*** *when both Π and π are bijections.*

A ***right module morphism*** *is defined similarly.*

A ***Hilbert module morphism*** (Π, π) *from a Hilbert* A_1*-module* M_1 *to a Hilbert* A_2*-module* M_2 *is a module morphism where for every* $\omega, \eta \in M_1$*,*

$$\langle \Pi(\omega), \Pi(\eta) \rangle_{M_2} = \langle \omega, \eta \rangle_{M_1}.$$

Convergence for metrical quantum vector bundles in this framework will require convergence for the underlying metrized quantum vector bundles. Every metrized quantum vector bundle has quantum compact metric space and Hilbert module components. For metrized quantum vector bundles associated to metric spectral triples, the D-norm depends on both the Hilbert space norm and the action of the Dirac operator on the Hilbert space. In particular, the inner Leibniz inequality ties the Lip-norm to both structures. The definition of a metric between metrized quantum vector bundles will extend that of the dual propinquity metric between quantum compact metric spaces.

Definition 25 ([25])**.** *Let* $(M_j, DN_j, B_j, L_{B_j})$ *be a metrized quantum vector bundle for* $j \in \{1, 2\}$*. A* ***modular tunnel*** $(\mathbb{M}, (\Pi_1, \pi_1), (\Pi_2, \pi_2))$ *from* $(M_1, DN_1, B_1, L_{B_1})$ *to* $(M_2, DN_2, B_2, L_{B_2})$ *is given by*

- *a metrized quantum vector bundle* $\mathbb{M} = (M, DN, B, L_B)$*,*
- *a tunnel* (B, L_B, π_1, π_2) *from* (B_1, L_{B_1}) *to* (B_2, L_{B_2})*,*
- *surjective Hilbert module morphisms* (Π_j, π_j) *from* M *over* (B, L_B) *to* M_j *over* (B_j, L_{B_j}) *such that for every* $\omega \in M_j$*,*

$$DN_j(\omega) = \inf\{DN(\zeta) : \Pi_j(\zeta) = \omega\}$$

for each $j \in \{1, 2\}$*.*

The D-norms for two metrized quantum vector bundles are encoded in the quotient properties of the D-norm for the metrized quantum vector bundle component of a modular tunnel between them. Verification of the inner Leibniz inequality for this quantum vector bundle component would require consideration of both these D-norms. The elements of permissable modular tunnels between two metrized quantum vector bundles, together with the Hilbert module structures, are sufficient for full quantum isometry between the base quantum compact metric spaces to imply agreement between the D-norms.

Definition 26 ([25]). *Let $\mu = (\mathbb{M}, (\Pi_1, \pi_1), (\Pi_2, \pi_2))$ be a modular tunnel from $(\mathcal{M}_1, DN_1, B_1, L_{B_1})$ to $(\mathcal{M}_2, DN_2, B_2, L_{B_2})$ with $\mathbb{M} = (M, DN, B, L_B)$. The* ***extent*** *of μ, $\chi(\mu)$, is the extent of the tunnel (B, L_B, π_1, π_2).*

Definition 27 ([25]). *The* ***dual modular propinquity*** *between two metrized quantum vector bundles $\mathbb{M}_1 = (M_1, DN_1, B_1, L_{B_1})$ and $\mathbb{M}_2 = (M_2, DN_2, B_2, L_{B_2})$ is given by*

$$\Lambda^{*mod}(\mathbb{M}_1, \mathbb{M}_2) = \inf\{\chi(\mu) : \mu \text{ is a modular tunnel from } \mathbb{M}_1 \text{ to } \mathbb{M}_2\}.$$

Theorem 17 ([24, 25]). *If $(M_1, DN_1, A_1, L_{A_1})$ and $(M_2, DN_2, A_2, L_{A_2})$ are two metrized quantum vector bundles, then*

$$\Lambda^{*mod}((M_1, DN_1, B_1, L_{B_1}), (M_2, DN_2, B_2, L_{B_2})) = 0$$

if and only if there exists a Hilbert module morphism (Π, π) such that

- $L_{A_2} \circ \pi = L_{A_1}$,
- $DN_2 \circ \Pi = DN_1$.

Moreover, the dual modular propinquity is a complete metric on the class of metrized quantum vector bundles.

Spectral triples can have additional modular structures with respect to their C^*-algebra components. A representation for the C^*-algebra that sends elements of the C^*-algebra to bounded left multiplication operators on the Hilbert space is one example of how the Hilbert space can be viewed as a left module with respect to the C^*-algebra. For metrical quantum vector bundles of the form $qvb(A, H, D)$, the modular Leibniz inequality relates the action of the C^*-algebra on the Hilbert space to the norm and the Lip-norm of the induced quantum compact metric space, as well as to the Hilbert space norm and the action of the Dirac operator on the Hilbert space.

Definition 28 ([25])**.** *Let $(M_j, DN_j, B_j, L_{B_j}, A_j, L_{A_j})$ be a metrical quantum vector bundle for $j \in \{1, 2\}$. A **metrical tunnel** (μ, τ) from $(M_1, DN_1, B_1, L_{B_1}, A_j, L_{A_1})$ to $(M_2, DN_2, B_2, L_{B_2}, A_2, L_{A_2})$ is given by*

- *a modular tunnel $\mu = (\mathbb{M}, (\Pi_1, \pi_1), (\Pi_2, \pi_2))$ from $(M_1, DN_1, B_1, L_{B_1})$ to $(M_2, DN_2, B_2, L_{B_2})$ with $\mathbb{M} = (M, DN, B, L_B)$,*
- *a tunnel $\tau = (A, L_A, \theta_1, \theta_2)$ from (A_1, L_{A_1}) to (A_2, L_{A_2}),*
- *a left A-module structure for M such that for every $a \in dom(L_A)$ and $\omega \in dom(DN)$, the modular Leibniz inequality, that is,*

$$DN(a\omega) \leqslant (||a||_A + L_A(a))DN(\omega),$$

  holds,
- *left module morphisms (θ_j, Π_j) from the left A-module M to the left A_j-module M_j for each $j \in \{1, 2\}$.*

More generally, a metrical quantum vector bundle carries the action of a quantum compact space on a metrized quantum vector bundle. This quantum compact metric space will often differ from the base quantum compact metric space for the metrized quantum vector bundle. Every metrical tunnel between metrical quantum vector bundles is built on a modular tunnel between the underlying metrized quantum vector bundles. Verification of the inner Leibniz and modular Leibniz inequalities for the construction of a metrical tunnel requires working with the Lip-norms for each quantum compact metric space. Equivalence for metrical quantum vector bundles will rely on dual propinquity convergence for both sets of quantum compact metric spaces. In particular, quantum compact metric space structure grants access to an analytic framework for metrical quantum vector bundles.

Definition 29 ([25]). *Let* (μ, τ) *be a metrical tunnel from* $(M_1, DN_1, B_1, L_{B_1}, A_j, L_{A_1})$ *to* $(M_2, DN_2, B_2, L_{B_2}, A_2, L_{A_2})$. *The* ***extent*** *of* (μ, τ), $\chi(\mu, \tau)$, *is* $\max\{\chi(\mu), \chi(\tau)\}$.

Definition 30 ([25]). *Given two metrical quantum vector bundles* $\mathcal{QVB}_j = (M_j, DN_j, B_j, L_{B_j}, A_j, L_{A_j})$, $j \in \{1, 2\}$, *the* ***metrical propinquity*** *between* $\mathcal{QVB}_1$ *and* $\mathcal{QVB}_2$ *is*

$$\Lambda^{*met}(\mathcal{QVB}_1, \mathcal{QVB}_2) = \inf\{\chi(\mu, \tau) : (\mu, \tau) \text{ is a metrical tunnel from } \mathcal{QVB}_1 \text{ to } \mathcal{QVB}_2\}.$$

Theorem 18 ([25]). *If* $\mathcal{QVB}_1 = (M_1, DN_1, B_1, L_{B_1}, A_j, L_{A_1})$ *and* $\mathcal{QVB}_2 = (M_2, DN_2, B_2, L_{B_2}, A_2, L_{A_2})$ *are metrical quantum vector bundles, then*

$$\Lambda^{*met}(\mathcal{QVB}_1, \mathcal{QVB}_2) = 0,$$

if and only if there exists

- *a Hilbert module isomorphism* $(\pi, \Pi) : (M_1, \langle \cdot, \cdot \rangle_{(B_1, L_{B_1})}) \to (M_2, \langle \cdot, \cdot \rangle_{(B_2, L_{B_2})})$ *such that* $L_{B_2} \circ \pi = L_{B_1}$ *and* $DN_2 \circ \Pi = DN_1$,

- *a* **-isomorphism* $\theta : A_1 \to A_2$ *such that* $L_{A_2} \circ \theta = L_{A_1}$ *and* (θ, Π) *is a module morphism.*

Moreover, the metrical propinquity is a complete metric on the class of metrical quantum vector bundles.

The C^*-algebra in a metric spectral triple (A, H, D) can always be equipped with a Lip-norm via the construction described for $qvb(A, H, D)$. In this context,

Theorem 19 ([23]). *If* (A_1, H_1, D_1) *and* (A_2, H_2, D_2) *are metric spectral triples, then* $\Lambda^{*met}((A_1, H_1, D_1), (A_2, H_2, D_2)) = 0$ *if and only if there exists a unitary* $U : H_1 \to H_2$ *and a* **-isomorphism* $\theta : A_1 \to A_2$ *such that*

$$UD_1^2U^* = D_2^2,$$

and for every $a \in A_1$ *and* $\omega \in H_2$*,*

$$\theta(a)\omega = (UaU^*)\omega.$$

Moreover, θ *is also a full quantum isometry- that is,* $L_{D_2} \circ \theta = L_{D_1}$*.*

To define a metric on metric spectral triples that detects the stronger condition of unitary equivalence of the Dirac operators, the metrical propinquity will be extended to include covariant quantities.

3.5.2 Convergence of Spectral Triples

Since the Dirac operator in a spectral triple (A, H, D) is a self-adjoint operator on the Hilbert space, D can be used to construct a strongly continuous action of $\mathbb{R}$ on H by unitary operators [31, Proposition 5.3.13]. More precisely, for every $t \in \mathbb{R}$,

$$U(t) = exp(itD)$$

is a unitary operator on H. For this family of operators, $U(s + t) = U(s)U(t)$. Such families of operators are often used in quantum mechanics to represent the time evolution of a physical system. The final extension of the dual propinquity to metric spectral triples quantifies "closeness" for such induced actions of $\mathbb{R}$. This metric is a special case of a covariant version of the propinquity called the *covariant modular propinquity*. In the broader context addressed by that metric, maps on a certain class of monoids are used to define a distance on that class [23, 41]. For metric spectral triples, this class of monoids reduces to the single monoid $\mathbb{R}$. The kinds of maps on $\mathbb{R}$ required for application of this metric are each characterized by first setting a choice of $\varepsilon > 0$. A pair of maps $(\varsigma_1, \varsigma_2)$ from $\mathbb{R}$ to $\mathbb{R}$ is called an ε-*iso-iso* whenever for every $j, k \in \{1, 2\}$ and every $x, y, z \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right]$,

$$\Big| |\varsigma_j(x) + \varsigma_j(y) - z| - |(x + y) - \varsigma_j(z)| \Big| \leqslant \varepsilon$$

and $\varsigma_1(0) = \varsigma_2(0) = 0$. Such maps will be used to compare the dynamics of the Dirac operators belonging to metric spectral triples.

As in the case of the metrical propinquity, a metric on metric spectral triples will be built on tunnels between the underlying quantum compact metric spaces. Let (A_1, H_1, D_1) and (A_2, H_2, D_2) be metric spectral triples. Suppose there exists a tunnel $\tau =$

$(A, L_D, \theta_1, \theta_2)$ from (A_1, L_{D_1}) to (A_2, L_{D_2}) and a modular tunnel $\mu = (\mathbb{M}, (\Pi_1, \pi_1), (\Pi_2, \pi_2))$ from $(H_1, \mathrm{DN}, \mathbb{C}, 0)$ to $(H_2, \mathrm{DN}_2, \mathbb{C}, 0)$ with $\mathbb{M} = (M, \mathrm{DN}, B, L_B)$ such that (μ, τ) is a metrical tunnel from $qvb(A_1, H_1, D_1)$ to $qvb(A_2, H_2, D_2)$. If $(\varsigma_1, \varsigma_2)$ is an ε-*iso-iso* from $\mathbb{R}$ to $\mathbb{R}$ for some $\epsilon > 0$, then an ϵ-*covariant metrical tunnel* is given by $(\mu, \tau, \varsigma_1, \varsigma_2)$. Moreover, $(\mu, \varsigma_1, \varsigma_2)$ is an ε-*covariant tunnel* with an ε-*covariant modular reach* defined by

$$\rho_m((\mu, \varsigma_1, \varsigma_2)) = \max_{\{j,k\} \in \{1,2\}} \sup_{\xi \in H_j, DN_j(\xi) \leqslant 1} \inf_{\xi' \in H_k, DN_k(\xi') \leqslant 1} \sup_{|t| \leqslant \frac{1}{\varepsilon}} \sup_{\omega \in M, DN(\omega) \leqslant 1} \left| \langle U_j(t)\xi, \Pi_j(\omega) \rangle_{H_j} - \langle U_k(\varsigma_k(t))\xi', \Pi_k(\omega) \rangle_{H_k} \right|.$$

The ε-*metrical magnitude* of $(\mu, \tau, \varsigma_1, \varsigma_2)$ combines consideration of $\rho_m((\mu, \varsigma_1, \varsigma_2))$ with that of $\chi(\mu, \tau)$ and is determined by

$$\varrho((\mu, \tau, \varsigma_1, \varsigma_2)|\varepsilon) = \max \Big\{ \chi(\tau), \chi(\mu), \rho_m((\mu, \varsigma_1, \varsigma_2)) \Big\}.$$

An additional condition will ensure that these quantities yield a metric rather than an extended metric on metric spectral triples.

Definition 31 ([23])**.** *The **spectral propinquity** between two metric spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2) is given by*

$$\begin{aligned}
&\Lambda^{spec}((A_1, H_1, D_1), (A_2, H_2, D_2)) \\
&\quad = \max\Big\{\frac{\sqrt{2}}{2}, \inf\{\varepsilon > 0 : (\mu, \tau) \text{ is a metrical tunnel from } qvb(A_1, H_1, D_1) \text{ to} \\
&\qquad\qquad qvb(A_2, H_2, D_2), \\
&\qquad\qquad (\mu, \tau, \varsigma_1, \varsigma_2) \text{ is an } \varepsilon\text{-covariant metrical tunnel,} \\
&\qquad\qquad \varrho((\mu, \tau, \varsigma_1, \varsigma_2)|\varepsilon) \leqslant \varepsilon\}\Big\}.
\end{aligned}$$

Theorem 20 ([23])**.** *The spectral propinquity Λ^{spec} is a metric on the class of metric spectral triples, up to the following coincidence property: for any metric spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2),*

$$\Lambda^{spec}((A_1, H_1, D_1), (A_2, H_2, D_2)) = 0$$

*if and only if there exists a unitary map $U : H_1 \to H_2$ and a *-isomorphism $\theta : A_1 \to A_2$ such that*

$$UD_1U^* = D_2,$$

and for every $a \in A_1$ and $\omega \in H_2$,

$$\theta(a)\omega = (UaU^*)\omega.$$

In particular, θ is a full quantum isometry from $(A_1, \|\,[D_1, \pi_1(\cdot)\,]\,\|_{B(H_1)})$ to $(A_2, \|\,[D_2, \pi_2(\cdot)\,]\,\|_{B(H_2)})$.

The Lapidus-Sarhad spectral triple recovers the geodesic distance on a piecewise C^1-fractal curve [19]. Since the fractal equipped with this metric is a compact metric space, the Lapidus-Sarhad spectral triple can be shown to be a metric spectral triple. Their construction of this spectral triple relies on their piecewise C^1-fractal curve framework. The enrichment of this framework with the notion of an approximation sequence arose from identification of conditions that make direct application of the spectral propinquity possible. The application of the spectral propinquity to Lapidus-Sarhad spectral triples is a both a test case for this metric and a stepping stone towards new definitions of differential operators in noncommutative fractal geometry. Recall that the definition of a Laplacian on a space requires the choice of a measure and a spectral triple induces a measure via the Dixmier trace. An extension of the spectral propinquity to a setting that includes such operators would lay the foundation for the construction new Laplacians on fractals, as well as new ways to approximate more established Laplacians from the analysis on fractals literature with Laplacians on finite graphs. Such understandings would contribute to progress in the definition and study of a suitable notion of "fractal manifold." In the process of such development, new insights about Laplacians, differential structures, and fractality may be discovered.

Chapter 4

Metric Approximation of Spectral Triples on Piecewise C^1 Fractal Curves

Piecewise C^1-fractal curves naturally admit approximations in the Hausdorff distance by finite graphs. By definition, every piecewise C^1-fractal curve contains a countable set of parameterized curves that is dense in the fractal. Since this set of parameterized curves can be ordered by decreasing arclength, there exists at least one enumeration of these curves that respects this order. This enumeration can be used to define a sequence of finite graphs. Via the Lapidus-Sarhad construction, a metric spectral triple will be built for each finite graph. When such an enumeration exhibits the properties described in the definition of an approximation sequence compatible with that parameterization of the fractal curve, the corresponding sequence of metric spectral triples will be shown to converge in the spectral propinquity to a metric spectral triple on the piecewise C^1-fractal curve. Such convergence is an important step in the demonstration of the possibility of metric approximation of spectral triples on fractal spaces by spectral triples on simpler approximating spaces. Further study of the analytic framework used for such developments will advance understanding in the definition and study of generalized manifolds that include fractals.

4.1 Lapidus-Sarhad Spectral Triples

A parameterization for a piecewise C^1-fractal curve X is based on a sequence of rectifiable C^1 curves $(R_j)_{j\in\mathbb{N}}$. A Lapidus-Sarhad spectral triple is a direct sum of spectral triples for each R_j-curve. Each of these spectral triples is built from spectral triples for circles. To define a spectral triple for a circle in the complex plane centered at 0 and with radius $r > 0$, let

- AC_r denote the algebra of complex continuous $2\pi r$-periodic functions on the real line,
- $H_r := L^2([-\pi r, \pi r], (2\pi r)^{-1}\mathfrak{m})$, where $(2r)^{-1}\mathfrak{m}$ is the normalized Lebesgue measure on $[-\pi r, \pi r]$,
- $D_{C_r} = \overline{-i\frac{d}{dx}|_{\mathrm{span}(\phi_k^r)_{k\in\mathbb{Z}}}}$ with $\phi_k^r = \exp(\frac{ikx}{r})$, $k \in \mathbb{Z}$,
- π_{C_r} the representation that sends elements of AC_r to multiplication operators on H_r.

The sequence $(\phi_k^r)_{k\in\mathbb{Z}}$ is an orthonormal basis for H_r. These functions are also the eigenfunctions of D_{C_r}. In particular, D_{C_r} is self-adjoint with

$$\sigma(D_{C_r}) = \left\{\frac{k}{r} : k \in \mathbb{Z}\right\}.$$

By the Spectral Mapping Theorem [8, Chapter VIII, Theorem 2.7] and [31, Theorem 3.3.8], D_{C_r} has compact resolvent. If $f \in \mathrm{dom}(D_{C_r})$, then f is differentiable almost everywhere and $||f'||_{AC_r}$ is finite. In contrast, the period of the Weierstrass function can be adapted to produce a function that is in AC_r but not in $\mathrm{dom}(D_{C_r})$. For any $f \in AC_r$ and $g \in H_r$ with continuous derivatives,

$$[D_{C_r}, \pi_{C_r}(f)]\, g = \pi_{C_r}\Big(-i\frac{df}{dx}\Big)\, g = \pi_{C_r}(D_{C_r} f)\, g,$$

hence $[D_{C_r}, \pi_{C_r}(f)]$ is a densely defined operator and extends to the bounded operator $\pi_{C_r}(D_{C_r} f)$ on H_r. The set of $f \in AC_r$ with continuous derivatives can also be shown to be dense in AC_r. Specifically, the set of functions in AC_r of the form $\Sigma_{k=-n}^{n} c_k e^{\frac{ikx}{r}}$ is dense in AC_r by the Stone-Weierstrass Theorem [8, Chapter IV, Theorem 8.1]. As a consequence, the set

$$\{f \in AC_r : [D_{C_r}, \pi_{C_r}(f)] \text{ is densely defined and extends to a bounded operator on } H_r\}$$

is dense in AC_r. Together with H_r and D_{C_r}, the circle algebra AC_r with the representation π_{C_r} forms a spectral triple. In [11], the spectral triple given by $ST(C_r) = (AC_r, H_r, D_{C_r})$ is called the *natural spectral triple for the circle algebra* C_r.

Theorem 21. *The natural spectral triple for the circle algebra AC_r is a metric spectral triple.*

Proof. Verification that (AC_r, L_{AC_r}) is a quantum compact metric spaces requires first confirmation of certain properties for L_{AC_r}. Since

$$L_{AC_r}(f) = ||\, [D_{C_r}, \pi_{C_r}(f)]\, ||_{B(H_r)} = ||\pi_{C_r}(D_{C_r} f)||_{B(H_r)} = ||f'||_{L^\infty([-\pi r, \pi r])}$$

for all $f \in AC_r$, $f \in \mathrm{dom}(L_{AC_r})$ if and only if f is Lipschitz. Recall that the self-adjoint elements of AC_r are the real-valued functions. To see that Lipschitz functions are dense in the set of real-valued continuous $2\pi r$-periodic functions on the real line, note that this set contains the constant functions. Let f denote the sawtooth function defined by distance to the nearest odd integer multiple of πr. Given any two distinct real values, the period of f can be adapted so that its values differ at these two points. The desired denseness condition then follows as a consequence of the Stone-Weierstrass Theorem [8, Chapter IV, Theorem 8.1]. Moreover, the set of real-valued functions with trivial Lipchitz constant coincides with the set of real-valued constant functions. Because the Arzela-Ascoli Theorem [8, Chapter VI, Theorem 3.8] implies

$$\{a \in sa(AC_r) : L_{AC_r}(a) \leqslant 1, ||a||_{AC_r} \leqslant 1\}$$

is totally bounded in AC_r for $|| \cdot ||_{AC_r}$, (AC_r, L_{AC_r}) is a quantum compact metric space. ■

Furthermore,

Theorem 22 ([11, Theorem 2.4]). *The metric induced by $ST(C_r)$ coincides with the geodesic distance on C_r. More precisely, let d_{C_r} denote the geodesic distance on C_r. Then*

$$d_{C_r}(x, y) = \sup\{|f(x) - f(y)| \, : \, f \in AC_r, ||[D_{C_r}, \pi_{C_r}(f)]||_{B(H_r)} \leqslant 1\}.$$

A parameterization for a piecewise C^1-fractal curve is composed of countably many rectifiable C^1-curves. The Dirac operator belonging to a natural spectral triple for a circle always has the number zero in its spectrum. If a direct sum of such spectral triples for each curve in this parameterization is taken, then the operator obtained from the direct sum construction will not have compact resolvent. For each parameterized curve of length r, set

$$D_r = D_{C_{r/\pi}} + \frac{1}{2r} I.$$

Then

$$\sigma(D_r) = \left\{ \frac{\pi(2k+1)}{2r} : k \in \mathbb{Z} \right\},$$

the domains of definition for $D_{C_{r/\pi}}$ and D_r coincide, and for any $f \in AC_r$,

$$[D_r, \pi_{C_{r/\pi}}(f)] = [D_{C_{r/\pi}}, \pi_{C_{r/\pi}}(f)].$$

For each $j \in \mathbb{N}$, $R_j : [0, l_j] \to X$ is a continuous injective map with an image that is a curve of length l_j. Composition of this map with continuous functions on the interval $[0, l_j]$ yields a homomorphism of $C(X)$ onto $C([0, l_j])$. The corresponding continuous functions on the interval $[0, l_j]$ can then be taken to continuous functions on the double interval $[-l_j, l_j]$ via an injective homomorphism. More precisely, for every $f \in C(X)$ and $h \in H_{l_j}$, let

$$\pi_{l_j}(f)h(x) := f(R_j(|t|))h(x).$$

This representation of $C(X)$ as bounded operators on H_{l_j} can be used to build a faithful representation of $C(X)$ as bounded operators on $\bigoplus_{j \in \mathbb{N}} H_{l_j}$.

Theorem 23 ([11, 19])**.** *Let X be a piecewise C^1-fractal curve. Then $X = \overline{\bigcup_{j \geqslant 1} R_j}$, where R_j is a rectifiable C^1 curve of length l_j for each $j \in \mathbb{N}$. Set*

- $H_\infty := \bigoplus_{j \in \mathbb{N}} H_{l_j}$,

- $D_\infty := \bigoplus_{j\in\mathbb{N}} D_{l_j}$,
- $\pi_\infty = \bigoplus_{j\in\mathbb{N}} \pi_{l_j}$.

Then $ST(X) := (C(X), H_\infty, D_\infty)$ with representation π_∞ is a spectral triple for X.

A spectral triple defined on a piecewise C^1-fractal curve X by this construction is called a *Lapidus-Sarhad spectral triple on X*. Note that D_∞ is self-adjoint with

$$\sigma(D_\infty) = \bigcup_{j\in\mathbb{N}} \Big\{ \frac{(2k+1)\pi}{2l_j} : k \in \mathbb{Z} \Big\}.$$

In [19], Lapidus and Sarhad showed this spectral triple can be used to recover the geodesic distance on X. The lemma on which this result relies can also be used to show $ST(X)$ induces quantum compact metric space structure.

Theorem 24. *A Lapidus-Sarhad spectral triple on a piecewise C^1-fractal curve is a metric spectral triple.*

Proof. Since a piecewise C^1-fractal curve is also a compact length space, (X, d_∞) is a compact metric space. As shown in the previous chapter, algebras of Lipschitz functions over classical compact metric spaces are quantum compact metric spaces. In particular, $(C(X), L_{d_\infty})$ is a quantum compact metric space. Quantum compact metric space structure for $(C(X), L_{D_\infty})$ will be shown to be a consequence of that of $(C(X), L_{d_\infty})$. In [19, Lemma 3.5], Lapidus and Sarhad show that for all f in $\mathrm{dom}(D_\infty)$, $||\,\pi_\infty(D_\infty f)\,||_{B(H_\infty)} = L_{d_\infty}(f)$. Since $(C(X), L_{d_\infty})$ is a quantum compact metric space, L_{D_∞} is a seminorm on $sa(A)$ such that L_{D_∞} is lower semi-continuous with respect to $||\cdot||_{C(X)}$ and $\{f \in \mathrm{dom}(L_{D_\infty}) : L_{D_\infty}(f) = 0\} = \mathbb{R}1_{C(X)}$. Because the derivative of every Lipschitz continuous function is essentially bounded, $\mathrm{dom}(L_{d_\infty}) \subseteq \mathrm{dom}(L_{D_\infty})$. The set $\mathrm{dom}(L_{D_\infty}) = \{f \in sa(C(X)) : L_{D_\infty}(f) < \infty\}$ is therefore likewise dense in $sa(C(X))$. Furthermore,

$$\{f \in sa(C(X)) : L_{D_\infty}(f) \leqslant 1, \varphi(f) = 0\} \subseteq \{f \in sa(C(X)) : L_{d_\infty}(f) \leqslant 1, \varphi(f) = 0\}$$

is totally bounded in $C(X)$ for $||\cdot||_{C(X)}$. Hence $(C(X), L_{D_\infty})$ is a quantum compact metric space. As a consequence, $ST(X)$ is a metric spectral triple. ■

In fact, Lapidus and Sarhad used [19, Lemma 3.5] to recover the geodesic distance on X from $ST(X)$.

Theorem 25 ([19, Theorem 2])**.** *The metric induced by $ST(X)$ coincides with the geodesic distance on X. More precisely,*

$$d_\infty(x, y) = \sup\{|f(x) - f(y)| \ : \ f \in C(SG), ||[D_\infty, \pi_\infty(f)]||_{B(H_\infty)} \leqslant 1\}.$$

When X is the Sierpinski gasket, $ST(X)$ is the spectral triple developed by Lapidus, Christensen, and Ivan. In fact, $ST(SG)$ also recovers the Hausdorff dimension and the $\log_2 3$-dimensional Hausdorff measure [11].

4.2 Metric Approximation of Quantum Compact Metric Spaces Induced by Lapidus-Sarhad Spectral Triples

To build metric approximations in the spectral propinquity for a Lapidus-Sarhad spectral triple on X, suppose there exists an approximation sequence $B(n)$ of X compatible with the parameterization $(R_j)_{j \in N}$. The elements of a spectral triple on X_n can be given by the corresponding finite subsets of summands in a Lapidus-Sarhad spectral triple on X:

- $H_n = \bigoplus_{j=1}^{B(n)} H_{l_j}$,
- $D_n = \bigoplus_{j=1}^{B(n)} D_{l_j}$,
- $\pi_n = \bigoplus_{j=1}^{B(n)} \pi_{l_j}$.

For every $n \in \mathbb{N}$, H_n will be viewed as a subspace of H_∞ by identifying every $\eta = (\eta_1, \cdots, \eta_j, \cdots, \eta_n) \in H_n$ with

$$(\eta_j)_{j \in \mathbb{N}} = \begin{cases} \eta_j & \text{if } j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $ST(X_n)$ denote the spectral triple on X_n defined by $(C(X), H_n, D_n)$ with representation π_n. A spectral triple of this construction on X_n will be called a *Lapidus-Sarhad spectral triple on* X_n.

Theorem 26. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j \in \mathbb{N}}$. If there exists an approximation sequence $B(n)$ of X compatible with this parameterization, then the Lapidus-Sarhad spectral triple on X_n is a metric spectral triple.*

Proof. The following arguments are adapted from the Lapidus and Sarhad's proof of [19, Lemma 3.5]. Fix a choice of $n \in \mathbb{N}$. Note that since the geodesic distances differ on X and X_n, the Lipchitz constant with respect to d_∞ of a function in $C(X)$ may differ from the Lipchitz constant with respect to d_n of the restriction of that function to X_n. In particular, $||\, \pi_\infty(D_\infty g)\, ||_{B(H_\infty)} = L_{d_\infty}(g)$ for g in $\text{dom}(D_\infty)$ does not imply $||\, \pi_n(D_n g|_{X_n})\, ||_{B(H_n)} = L_{d_n}(g|_{X_n})$.

As in the case of $(C(X), L_{d_\infty})$, (X_n, d_n) is a compact metric space implies $(C(X_n), L_{d_n})$ is a quantum compact metric space. If L_{D_n} can be shown to coincide with L_{d_n} on $\mathrm{dom}(L_{D_n})$, then the same reasoning from the previous theorem can be applied to conclude that $ST(X_n)$ is a metric spectral triple. Suppose f is in $\mathrm{dom}(D_n)$. Then

$$\begin{aligned} \|\,\pi_n(D_n f)\,\|_{B(H_n)} &= \sup_{1\leqslant j\leqslant B(n)} \{\,\|\pi_j(D_j f)\|_{B(H_j)}\,\} = \sup_{1\leqslant j\leqslant B(n)} \{\,\|f'\|_{L^\infty(R_j)}\} \\ &\leqslant \sup_{1\leqslant j\leqslant B(n)} \Big\{ \sup_{p,q\in R_j} \Big\{ \frac{|f(p)-f(q)|}{d_n(p,q)} \Big\} \Big\} \leqslant L_{d_n}(f). \end{aligned}$$

To bound $L_{d_n}(f)$, choose any distinct p and q in X_n. Any geodesic in X_n between p and q passes through a sequence of vertices in X_n. Consecutive pairs of vertices in this sequence are also pairs of endpoints for the same R_j curve. Pick a geodesic connecting p and q. Let $\{p_k, p_{k+1}\}_{k=1}^m$ be the sequence of pairs of endpoints between p and q in this geodesic and $\{R_{j_k}\}_{k=1}^m$ the corresponding sequence of R_j-curves. Let R_{j_0} denote the R_j-curve containing p and $R_{j_{m+1}}$ the R_j curve containing q. Then

$$\begin{aligned} |f(p)-f(q)| &\leqslant |f(p)-f(p_1)| + \Big(\sum_{k=1}^m |f(p_{k+1})-f(p_k)|\Big) + |f(q)-f(p_{m+1})| \\ &\leqslant d_n(p,p_1)\|f'\|_{L^\infty(R_{j_0})} + \Big(\sum_{k=1}^m d_n(p_{k+1},p_k)\|f'\|_{L^\infty(R_{j_k})}\Big) + d_n(q,p_{m+1})\|f'\|_{L^\infty(R_{j_{m+1}})} \\ &= d_n(p,p_1)\,\|\pi_{l_{j_0}}(D_{l_{j_0}} f)\|_{B(H_{l_{j_0}})} + \Big(\sum_{k=1}^m d_n(p_{k+1},p_k)\,\|\pi_{l_{j_k}}(D_{l_{j_k}} f)\|_{B(H_{l_{j_k}})}\Big) \\ &\qquad + d_n(q,p_{m+1})\,\|\pi_{l_{j_{m+1}}}(D_{l_{j_{m+1}}} f)\|_{B(H_{l_{j_{m+1}}})} \\ &\leqslant \|\pi_n(D_n f)\|_{B(H_n)}\Big(d_n(p,p_1) + \Big(\sum_{k=1}^m d_n(p_{k+1},p_k)\Big) + d_n(q,p_{m+1})\Big) \\ &= \|\pi_n(D_n f)\|_{B(H_n)}\, d(p,q), \end{aligned}$$

hence

$$L_{d_n}(f) \leqslant ||\pi_n(D_n f)||_{B(H_n)},$$

as desired. ■

In the proof of the above theorem, $||[D_n, \pi_n(D_n f)]||_{B(H_n)}$ was shown to coincide with L_{d_n} on $\mathrm{dom}(D_n)$. As in Lapidus and Sarhad's proof of [19, Theorem 2] for the case of $ST(X)$, this result can be used to recover the geodesic distance on X_n from $ST(X_n)$.

Theorem 27 ([19, Theorem 2])**.** *The metric induced by $ST(X_n)$ coincides with the geodesic distance on X_n. More precisely,*

$$d_n(x,y) = \sup\{|f(x) - f(y)| \,:\, f \in C(SG_n), ||[D_n, \pi_n(f)]||_{B(H_n)} \leqslant 1\}.$$

Proof. In the proof of the previous theorem, $||[D_n, \pi_n(f)]||_{B(H_n)}$ was shown to coincide with L_{d_n} on $\mathrm{dom}(D_n)$. Thus for any $f \in C(SG_n)$ such that $||[D_n, \pi_n(f)]||_{B(H_n)} \leqslant 1$,

$$\frac{|f(x) - f(y)|}{d_n(x,y)} \leqslant L_{d_n}(f) = ||[D_n, \pi_n(f)]||_{B(H_n)} \leqslant 1,$$

hence $|f(x) - f(y)| \leqslant d_n(x,y)$, hence

$$\sup\{|f(x) - f(y)| \,:\, f \in C(SG_n), ||[D_n, \pi_n(f)]||_{B(H_n)} \leqslant 1\} \leqslant d_n(x,y).$$

To obtain the opposite inequality, note that $f_y(x) := d_n(x,y)$ is in $\mathrm{dom}(D_n)$ given any choice of $y \in X_n$. Then $||[D_n, \pi_n(f_y)]||_{B(H_n)} = L_{d_n}(f_y) = 1$ implies

$$d_n(x,y) = |d_n(x,y) - d_n(y,y)| = |f_y(x) - f_y(y)|$$
$$\leqslant \sup\{|f(x) - f(y)| \,:\, f \in C(SG_n), ||[D_n, \pi_n(f)]||_{B(H_n)} \leqslant 1\}.$$

■

Recall that

$$\lim_{n\to\infty} \Lambda^*(\,(C(X), L_{d_\infty}), (C(X_n), L_{d_n})\,) = 0.$$

The quantum compact metric space induced by the Lapidus-Sarhad spectral triple on X coincides with $(C(X), L_{d_\infty})$. Similarly, $(C(X_n), L_{D_n})$ is fully quantum isometric to $(C(X_n), L_{d_n})$. Although construction of explicit tunnels is not needed to show dual propinquity convergence of $\left\{(C(X), L_{D_n})\right\}_{n\in\mathbb{N}}$ to $(C(X), L_D)$, such tunnels are needed to show spectral propinquity convergence of Lapidus-Sarhad spectral triples on X_n to the Lapidus-Sarhad spectral triple on X. Calculation of the spectral propinquity requires calculation of the metrical propinquity for the canonically associated metrical quantum vector bundles. Metrical tunnels build on tunnels between underlying quantum compact metric spaces. Verification of the modular Leibniz inequality when building metrical tunnels requires checking bounds involving Lip-norms belonging to tunnels between the underlying quantum compact metric spaces. Since a Lip-norm must exhibit certain properties with respect to the self-adjoint elements of the C^*-algebra component of a quantum compact metric space, the McShane Extension Theorem will be used to obtain these needed conditions for C^*-algebras of the same form as $C(X)$ and $C(X_n)$.

Theorem 28 (McShane Extension Theorem, [**?**, Theorem 1.33])**.** *. Let X be a metric space, let X_0 be a nonempty subset of X, and let f_0 be a Lipschitz function from X_0 into $\mathbb{R}$. Then there is an extension $f : X \to \mathbb{R}$ which has the same Lipschitz constant. If f_0 is bounded, then $||f||_{C(X)} = ||f_0||_{C(X_0)}$.*

One requirement for the Lip-norm belonging to a tunnel is that this seminorm have particular quotient properties for the *-epimorphism components of that tunnel. The McShane Extension Theorem will play a role in that demonstration.

Theorem 29. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j\in\mathbb{N}}$. If there exists an approximation sequence $B(n)$ of X compatible with this parameterization, then*

$$\Lambda^*_{n\to\infty}(\,(C(X), L_{D_\infty}), (C(X_n), L_{D_n})\,) = 0.$$

Proof. Bounds on the dual propinquity between two quantum compact metric spaces can be obtained via with the construction of tunnels between these two spaces. To build tunnels between $(C(X), L_{D_\infty})$ and $(C(X_n), L_{D_n})$, choose an $\epsilon > 0$. If $f \in C(X_M)$ for some $M \in \mathbb{N} \cup \infty$ and $n \leqslant M$, let $f|_n$ denote the restriction of f to V_n. Since X is a piecewise C^1-fractal curve, there exists $N_1 \in \mathbb{N}$ such that if $n \geqslant N_1$, then

$$\mathrm{Haus}_{d_n}(X_n, V_n) < \epsilon.$$

The existence of an approximation sequence $B(n)$ for X guarantees there exists $N_2 \in \mathbb{N}$ such that if $n \geqslant N_2$, then

$$\mathrm{Haus}_{d_\infty}(X, V_n) < \epsilon.$$

One of the elements needed for the construction of a tunnel is a quantum compact metric space. Let N denote $\max\{N_1, N_2\}$. Fix some choice of $n \geqslant N$ and let A_n signify the unital C^*-algebra $C(X_n) \oplus C(X)$. Set $L_{n,\beta}$ equal to the seminorm on A_n given by

$$L_{n,\beta}(f, g) = \max\left\{L_{D_n}(f), L_{D_\infty}(g), \frac{1}{\beta}||\, g|_n - f|_n \,||_{C(V_n)}\right\}.$$

This seminorm will be shown to be a Lip-norm for A_n.

By construction, the seminorm $L_{n,\beta}$ is defined on a dense subspace of $sa(A_n)$. Furthermore, $L_{n,\beta}$ vanishes on $(f, g) \in A_n$ if and only if

$$L_{D_n}(f) = L_{D_\infty}(g) = ||\, g|_n - f|_n \,||_{C(V_n)} = 0.$$

Since the desired behavior of $L_{n,\beta}$ on A_n in the definition of a quantum compact metric space is specified only on a dense subspace of $sa(A_n)$, it suffices to check these conditions on $C(X_n, \mathbb{R}) \oplus C(X, \mathbb{R})$. Since $(C(X_n), L_{D_n})$ is a quantum compact metric space, the only elements of $sa(C(X_n))$ on which L_n vanishes coincides with the set of real-valued constant functions on X_n. Similarly, L_{D_∞} evaluates to zero precisely on $\mathbb{R}1_{C(X)}$. Moreover, $||\, g|_n - f|_n \,||_{C(V_n)}$ is zero only when f and g agree on V_n. In particular, $L_{n,\beta}$ vanishes exactly when f and g are constant functions on X_n and X that take on the same value in $\mathbb{R}$. Thus

$$\{(f, g) \in \mathrm{dom}(L_{n,\beta}) : L_{n,\beta}(f, g) = 0\}$$

is composed only of real-valued constant functions on A_n.

Next consider the semi-continuity condition for $L_{n,\beta}$. Note that L_{D_n} is lower semi-continuous with respect to $||\cdot||_{C(X_n)}$, as is $L_{\not{D}_\infty}$ with respect to $||\cdot||_{C(X)}$, via the quantum compact metric space structures of $(C(X_n), L_{D_n})$ and $(C(X), L_{\not{D}_\infty})$. Therefore, L_{D_n} and L_{D_∞} are both lower semi-continuous with respect to $||\cdot||_{A_n}$. Because $||\, f|_n \,||_{C(V_n)}$ is bounded by $||f||_{C(X_n)}$ for all $f \in C(X_n)$ and $||\, g|_n \,||_{C(V_n)}$ by $||g||_{C(X)}$ for all $g \in C(X)$,

$||\cdot||_{C(V_n)}$ is also lower semi-continuous with respect to $||\cdot||_{A_n}$. Hence $L_{n,\beta}$, as the point-wise maximum of three functions all lower semi-continuous with respect to $||\cdot||_{A_n}$, is likewise lower semi-continuous with respect to $||\cdot||_{A_n}$.

Given the verification of the previous two properties for $L_{n,\beta}$, the condition that $\mathrm{mk}_{L_{n,\beta}}$ metrizes the weak*-topology on $S(A_n)$ is now equivalent to the existence of a state $\psi \in S(A_n)$ such that the set

$$F_{\psi,0} := \{(f,g) \in \mathrm{dom}(L_{n,\beta}) : L_{n,\beta}(f,g) \leqslant 1, \psi(f,g) = 0\}$$

is totally bounded in A_n for $||\cdot||_{A_n}$. Let x_0 be some point in V_n, ψ_{x_0} the linear functional defined by evaluation at x_0, and $\theta_n : A_n \to C(X_n)$ the projection given by $\theta_n(f,g) = f$. Then $\psi_{x_0} \circ \theta_n$ is a state on A_n. Fix this choice of x_0. To show that $F_{\psi_{x_0}\circ\theta_n,0}$ is totally bounded in A_n for $||\cdot||_{A_n}$, a set containing $F_{\psi_{x_0}\circ\theta_n,0}$ will be shown to be totally bounded in A_n for $||\cdot||_{A_n}$. To build such a set, first consider

$$F^n_{\psi_{x_0},0} := \{f \in \mathrm{dom}(L_{D_n}) : L_{D_n}(f) \leqslant 1, \psi_{x_0}(f) = 0\}.$$

Since ψ_{x_0} is also in $S(C(X_n))$ and $(C(X_n), L_{D_n})$ is a quantum compact metric space, $F^n_{\psi_{x_0},0}$ is totally bounded in $C(X_n)$ for $||\cdot||_{C(X_n)}$. Next let $\theta_\infty : A_n \to C(X)$ be the projection given by $\theta_\infty(f,g) = g$ and consider whether

$$G^\infty_{\psi_{x_0},\beta} := \{g \in \mathrm{dom}(L_{D_\infty}) : L_{D_\infty}(g) \leqslant 1, |\psi_{x_0}(g)| \leqslant \beta\}.$$

is also totally bounded in $C(X_n)$ for $||\cdot||_{C(X_n)}$. Let $G^\infty_{\psi_{x_0},0}$ be defined similarly. Since ψ_{x_0} is likewise in $S(C(X))$ and $(C(X), L_{D_\infty})$ is a quantum compact metric space, $G^\infty_{\psi_{x_0},0}$ is totally bounded in $C(X)$ for $||\cdot||_{C(X)}$. Moreover, the lower semi-continuity of L_{D_∞} with respect to $||\cdot||_{C(X)}$ implies that $G^\infty_{\psi_{x_0},0}$ is also closed in $||\cdot||_{C(X)}$, hence compact. This compactness will be used to show compactness for $G^\infty_{\psi_{x_0},\beta}$ in the same norm. Suppose $\{g_n\}_{n\in\mathbb{N}}$ is a sequence in $G^\infty_{\psi_{x_0},\beta}$. Then $\{g_n(x_0)\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Such a sequence admits a convergent subsequence. Let $\{g_{t_1(n)}(x_0)\}_{n\in\mathbb{N}}$ denote this convergent subsequence and b the limit of this subsequence. Then $b \leqslant \beta$, $\{g_{t_1(n)} - g_{t_1(n)}(x_0)\}_{n\in\mathbb{N}}$ is a sequence of functions in $G^\infty_{\psi_{x_0},0}$, and $G^\infty_{\psi_{x_0},0}$ compact in $||\cdot||_{C(X)}$ implies $\{g_{t_1(n)} - g_{t_1(n)}(x_0)\}_{n\in\mathbb{N}}$ has a convergent subsequence for that norm. If $\{g_{t_2(t_1(n))} - g_{t_2(t_1(n))}(x_0)\}_{n\in\mathbb{N}}$ denotes this convergent subsequence and h the limit of this subsequence, then h vanishes at x_0. In particular, $h(x)+b$ is in $G^\infty_{\psi_{x_0},\beta}$ and $h(x)+b$ is the limit of $\{g_{t_2(t_1(n))}\}_{n\in\mathbb{N}}$. As the choice of sequence in $G^\infty_{\psi_{x_0},\beta}$ was arbitrary, $G^\infty_{\psi_{x_0},\beta}$ is compact for $||\cdot||_{C(X)}$. In consequence, $F^n_{\psi_{x_0},0} \times G^\infty_{\psi_{x_0},\beta}$ is totally bounded in A_n for $||\cdot||_{A_n}$. To see that $F^n_{\psi_{x_0},0}$ is a subset of $F^n_{\psi_{x_0},0} \times G^\infty_{\psi_{x_0},\beta}$, note that $L_{n,\beta}(f,g) \leqslant 1$ implies $L_{D_n}(f) \leqslant 1$, $L_{D_\infty}(g) \leqslant 1$ and $||\, f|_n - g|_n \,||_{C(V_n)} \leqslant \beta$. If $f(x_0) = \psi_{x_0}(f) = 0$, then

$$|\psi_{x_0}(g)| = |g(x_0)| = |f(x_0) - g(x_0)| \leqslant ||\, f|_n - g|_n \,||_{C(V_n)} \leqslant \beta.$$

Thus $\mathrm{mk}_{L_{n,\beta}}$ metrizes the weak*-topology on $S(A_n)$.

The final requirement for $L_{n,\beta}$ to qualify as a Lip-norm on A_n is that this seminorm satisfy the Leibniz inequality with respect to $||\cdot||_{A_n}$. For all $f_1, f_2 \in C(X_n)$ and $g_1, g_2 \in C(X)$,

$$\begin{aligned} L_{n,\beta}(f_1f_2, g_1g_2) &= \max\left\{L_{D_n}(f_1f_2), L_{D_\infty}(g_1g_2), \frac{1}{\beta}||\, f_1f_2|_n - g_1g_2|_n \,||_{C(V_n)}\right\} \\ &\leqslant \max\Big\{||f_1||_{C(X_n)}L_{D_n}(f_2) + L_{D_n}(f_1)||f_2||_{C(X_n)}, \\ &\qquad ||g_1||_{C(X)}L_{D_\infty}(g_2) + L_{D_\infty}(g_1)||g_2||_{C(X)}, \frac{1}{\beta}||\, f_1f_2|_n - g_1g_2|_n \,||_{C(V_n)}\Big\} \end{aligned}$$

$$\leqslant \max\Big\{||(f_1,g_1)||_{A_n}L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n},$$
$$\tfrac{1}{\beta}||\, f_1f_2|_n - f_1g_2|_n \,||_{C(V_n)} + \tfrac{1}{\beta}||\, f_1g_2|_n - g_1g_2|_n \,||_{C(V_n)}\Big\}$$

$$\leqslant \max\Big\{||(f_1,g_1)||_{A_n} L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n},$$

$$||f_1||_{C(X_n)}\tfrac{1}{\beta}||\, f_2|_n - g_2|_n\, ||_{C(V_n)} + \tfrac{1}{\beta}||\, f_1|_n - g_1|_n\, ||_{C(V_n)}||g_2||_{C(X)}\Big\}$$

$$\leqslant ||(f_1,g_1)||_{A_n} L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n}.$$

In particular,

$$L_{n,\beta}\Big(\frac{(f_1,g_1)(f_2,g_2)-(f_2,g_2)(f_1,g_1)}{2i}\Big) \leqslant \Big|\frac{1}{2i}\Big|\Big(L_{n,\beta}(f_1f_2,g_1g_2) + L_{n,\beta}(f_2f_1,g_2g_1)\Big)$$

$$\leqslant \frac{1}{2}\Big(||(f_1,g_1)||_{A_n} L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n}$$

$$+ ||(f_2,g_2)||_{A_n} L_{n,\beta}(f_1,g_1) + L_{n,\beta}(f_2,g_2)||(f_1,g_1)||_{A_n}\Big)$$

$$\leqslant ||(f_1,g_1)||_{A_n} L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n}.$$

Similarly, the same bound holds for $L_{n,\beta}\Big(\frac{(f_1,g_1)(f_2,g_2)+(f_2,g_2)(f_1,g_1)}{2}\Big)$, hence

$$\max\Big\{L_{n,\beta}\Big(\frac{(f_1,g_1)(f_2,g_2)+(f_2,g_2)(f_1,g_1)}{2}\Big), L_{n,\beta}\Big(\frac{(f_1,g_1)(f_2,g_2)-(f_2,g_2)(f_1,g_1)}{2i}\Big)\Big\}$$

$$\leqslant ||(f_1,g_1)||_{A_n} L_{n,\beta}(f_2,g_2) + L_{n,\beta}(f_1,g_1)||(f_2,g_2)||_{A_n},$$

thereby completing the verification that $(A_n, L_{n,\beta})$ is a quantum compact metric space.

The quotient properties of $L_{n,\beta}$ for θ_n and θ_∞ will next be examined. As in the definition of a Lip-norm, the desired quotient properties of θ_n and θ_∞ in the definition of a tunnel are specified only on a dense subspace of $sa(A_n)$. Consequently, it suffices to check these properties on $C(X_n,\mathbb{R}) \oplus C(X,\mathbb{R})$. To see that the quotient of $L_{n,\beta}$ for θ_n is L_{D_n}, let f be some function in $C(X_n,\mathbb{R})$ in the domain of L_{D_n}. Recall that for each $n \in \mathbb{N} \cup \infty$, L_{d_n} gives the same values as L_{D_n} on $\mathrm{dom}(L_{D_n})$. Since $B(n)$ is an approximation sequence of X, the restriction of d_∞ to $V_n \times V_n$ is d_n. As a consequence, $L_{D_n}(f) = L_{d_n}(f|_n)$. By the McShane Extension Theorem, there exists $g \in C(X,\mathbb{R})$ such that g and $f|n$ agree on V_n and

$L_{d_\infty}(g) = L_{d_n}(f|_n)$. In particular, $L_{D_\infty}(g) = L_{d_\infty}(g) = L_{D_n}(f)$ and $||\, g|_n - f|_n\, ||_{C(V_n)} = 0$, hence

$$L_{D_n}(f) = \inf\{\, L_{n,\beta}(h,k) : \theta_n(h,k) = f\,\}.$$

In the case of the quotient of $L_{n,\beta}$ for θ_∞, similar arguments can be applied to any $g \in C(X,\mathbb{R})$ except with the application of the McShane Extension Theorem to $g|_n$ to yield a function $f \in C(X_n,\mathbb{R})$ such that $L_{D_n}(f) = L_{D_\infty}(g)$ and f agrees with g on V_n. As projections are *-epimorphisms,

$$\tau_{n,\beta} = (A_n, L_{n,\beta}, \theta_n, \theta_\infty)$$

is a tunnel from $(C(X_n), L_{D_n})$ to $(C(X), L_{D_\infty})$.

Various bounds from the dual propinquity literature can be applied to the extent of tunnels of the same form as $\tau_{n,\beta}$. For quantum compact metric spaces like $(A_n, L_{n,\beta})$ built from a direct sum construction,

$$\chi(\tau_{n,\beta}) \leqslant \mathrm{Haus}_{mk_{L_{n,\beta}}}(S(C(X_n) \oplus C(X)), \overline{\mathrm{co}}(S(C(X_n)) \cup S(C(X)))$$

$$+\mathrm{Haus}_{mk_{L_{n,\beta}}}(\theta_n^*(S(C(X_n)), \theta_\infty^*(S(C(X))),$$

where $\overline{\mathrm{co}}(E)$ denotes the closure of the convex envelope of a set $E \subseteq S(C(X_n) \oplus C(X))$ [22]. Moreover, [20] gives that when the *-epimorphisms in tunnels with quantum compact metric spaces like $(A_n, L_{n,\beta})$ are projections to each of the summands,

$$\mathrm{Haus}_{mk_{L_{n,\beta}}}(S(C(X_n) \oplus C(X)), \overline{\mathrm{co}}(S(C(X_n)) \cup S(C(X))) = 0.$$

To determine

$$Haus_{mk_{L_{n,\beta}}}(\theta_n^*(S(C(X_n)), \theta_\infty^*(S(C(X)))),$$

begin by fixing some choice of φ in $S(C(X))$. By the Krein-Milman Theorem, there exists φ' in $S(C(X))$ such that $mk_{L_{D_\infty}}(\varphi, \varphi') < \beta$ and $\varphi' = \Sigma_{i=1}^{l} t_i \delta_{x_i}$ for some $l \in \mathbb{N}$ and $x_i \in X$ for $1 \leqslant i \leqslant l$. Because $\text{Haus}_{d_\infty}(X, V_n) < \epsilon$, there exists $v_1, \cdots, v_l \in V_n$ such that $d_\infty(x_i, v_i) < \epsilon$ for $1 \leqslant i \leqslant l$. In particular, $\varphi'' = \Sigma_{i=1}^{l} t_j \delta_{v_i}$ is in both $S(C(X_n))$ and $S(C(X))$, hence for all (f, g) in A_n with $L_{n,\beta}(f, g) \leqslant 1$,

$$
\begin{aligned}
|\varphi \circ \theta_\infty(f, g) - \varphi'' \circ \theta_n(f, g)| &= |\varphi(g) - \varphi''(f)| \\
&\leqslant |\varphi(g) - \varphi'(g)| + |\varphi'(g) - \varphi''(g)| + |\varphi''(g) - \varphi''(f)| \\
&< \beta + |\Sigma_{i=1}^{l} t_i \delta_{x_i}(g) - \Sigma_{i=1}^{l} t_i \delta_{v_i}(g)| + |\Sigma_{i=1}^{l} t_i \delta_{v_i}(g) - \Sigma_{i=1}^{l} t_i \delta_{v_i}(f)| \\
&\leqslant \beta + \Sigma_{i=1}^{l} t_i |g(v_i) - g(x_i)| + \Sigma_{i=1}^{l} t_i |g(v_i) - f(v_i)| \\
&\leqslant \beta + \Sigma_{i=1}^{l} t_i d_\infty(x_i, v_i) + \Sigma_{i=1}^{l} t_i || \, g|_n - f|_n \, ||_{C(V_n)} \\
&< 2\beta + \epsilon,
\end{aligned}
$$

where the second inequality follows from the conditions that $\text{mk}_{L_{D_\infty}}(\varphi, \varphi') < \beta$, the penultimate inequality as consequences of $L_{D_\infty}(g) \leqslant L_{n,\beta}(f, g) \leqslant 1$ and $L_{d_\infty}(g) = L_{D_\infty}(g)$, and the last inequality because $d_\infty(x_i, v_i) < \epsilon$ for $1 \leqslant i \leqslant l$ and $\frac{1}{\beta} || \, g|_n - f|_n \, ||_{C(V_n)} \leqslant L_{n,\beta}(f, g) \leqslant 1$. Similarly, the same bound can be achieved for an arbitrary ψ in $S(C(X_n))$ with an approximating ψ' composed of Dirac measures on X_n and for which $\text{mk}_{L_{D_n}}(\psi, \psi') < \beta$. In that context, the bound $\text{Haus}_{d_n}(X_n, V_n) < \epsilon$ implies the existence of $v_1, \cdots, v_m \in V_n$ such that $d_n(x_i, v_i) < \epsilon$ for $1 \leqslant i \leqslant m$. This set of vertices can also be used to build a state ψ'' from Dirac measures on V_n. Since $L_{D_\infty}(g) \leqslant L_{n,\beta}(f, g) \leqslant 1$ and $L_{d_n}(f) = L_{D_n}(f)$, this bound on

$d_n(x_i, v_i)$ can likewise be applied to $|\psi'(f) - \psi''(f)|$. Thus $\chi(\tau_{n,\beta}) < 2\beta + \epsilon$. As the choice of $\beta > 0$ was arbitrary,

$$\Lambda^*(\,(C(X), L_{D_\infty}), (C(X_n), L_{D_n})\,) \leqslant \chi(\tau_{n,\beta}) \leqslant \epsilon$$

■

4.3 Metric Approximation of Metrical Quantum Vector Bundles Associated to Lapidus-Sarhad Spectral Triples

Metrical tunnels are built from tunnels between their underlying quantum compact metric spaces. For metrical tunnels between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$, tunnels between $(C(X), L_{D_\infty})$ and $(C(X_n), L_{D_n})$ of the same construction as $\tau_{n,\beta}$ will be used. Another component of these metrical tunnels will be given by tunnels of the following form.

Lemma 3. *Let $\pi_n : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ denote projection to the first coordinate and $\pi_\infty : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ projection to the second coordinate. Then for any fixed choice of $\epsilon > 0$, $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)$ is a tunnel between $(\mathbb{C}, 0)$ and $(\mathbb{C}, 0)$. Moreover, $\chi(\,(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)\,) \leqslant \epsilon$.*

Proof. Fix a choice of $\epsilon > 0$. As shown in the previous chapter, $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon)$ is a quantum compact metric space. To see that the quotient of Q_ϵ for π_n is 0, take any $(w_0, w_0) \in \mathbb{C} \oplus \mathbb{C}$. Then

$$0 = (0)(w_0) = \inf\{\, Q_\epsilon(w, z) : \pi_n(w, z) = w_0 \,\}.$$

Similar arguments can be applied to show that the quotient of Q_ϵ for π_∞ is also 0. As projections are *-epimorphisms, $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)$ is a tunnel between $(\mathbb{C}, 0)$ and $(\mathbb{C}, 0)$.

To calculate the extent of $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)$, the state spaces of $(\mathbb{C}, 0)$ and $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon)$ will each be characterized. Recall that positive linear functionals are bounded and their norm coincides with their value at the identity. In the case of $\mathbb{C}$, linear functionals are also uniquely determined by their value at the identity. As a consequence, $S(\mathbb{C})$ is composed only of the identity function. For $\mathbb{C}\oplus\mathbb{C}$, linear functionals are uniquely determined by their values at $(1, 0)$ and $(0, 1)$, hence for every φ in $S(\mathbb{C} \oplus \mathbb{C})$,

$$1 = \varphi(1, 1) = \varphi(1, 0) + \varphi(0, 1).$$

For every t in $[0, 1]$, let φ_t denote the state in $S(\mathbb{C} \oplus \mathbb{C})$ that takes $(1, 0)$ to t. Then $\varphi_t(0, 1) = 1 - t$. In particular, $S(\mathbb{C} \oplus \mathbb{C})$ coincides with complex-valued functions on $\mathbb{C} \oplus \mathbb{C}$ in the set

$$\{\varphi_t(w, z) = tw + (1 - t)z \,:\, t \in [0, 1]\,\}.$$

Let $\mathrm{id}_{\mathbb{C}}$ signify the identity function on $\mathbb{C}$. The Hausdorff distance with respect to mk_{Q_ϵ} between $\mathrm{id}_{\mathbb{C}} \circ \pi_n$ and $S(\mathbb{C}\oplus\mathbb{C})$ can now be calculated. If (w, z) is in $\mathbb{C}\oplus\mathbb{C}$ with $Q_\epsilon(w, z) \leqslant 1$, then for every t in $[0,1]$,

$$\begin{aligned} |\,\varphi_t(w,z) - \mathrm{id}_{\mathbb{C}} \circ \pi_n(w,z)\,| &= |\,\varphi_t(w,z) - \varphi_1(w,z)\,| \\ &= |\,(tw + (1-t)z) - ((1)w + (1-1)z)\,| \\ &= |\,(1-t)z - (1-t)w\,| \\ &= (1-t)\,|z-w| \\ &\leqslant (1-t)\,\epsilon \\ &\leqslant \epsilon, \end{aligned}$$

hence

$$\mathrm{Haus}_{\mathrm{mk}_{Q_\epsilon}}(\{\psi \circ \pi_n : \psi \in S(\mathbb{C})\}, S(\mathbb{C} \oplus \mathbb{C})) \leqslant \epsilon.$$

Similarly, the same conditions yield

$$|\,\varphi_t(w,z) - \mathrm{id}_{\mathbb{C}} \circ \pi_\infty(w,z)\,| = |\,\varphi_t(w,z) - \varphi_0(w,z)\,| \leqslant t\epsilon \leqslant \epsilon.$$

■

A metrical quantum vector bundle carries Hilbert module structure via its metrized quantum vector bundle component. For metrical quantum vector bundles associated to metric spectral triples, that Hilbert module is always the Hilbert space from the metric spectral triple. Hilbert spaces are Hilbert modules over the quantum compact metric space $(\mathbb{C}, 0)$. For the case of quantum vector bundles that arise from Lapidus-Sarhad spectral triples, set for every $n \in \mathbb{N} \cup \infty$ and $\xi \in H_n$,

$$DN_n(\xi) = ||\xi||_{H_n} + ||D_n\xi||_{H_n}.$$

The metrized quantum vector bundle belonging to $qvb(C(X), H_\infty, D_\infty)$ is $(H_\infty, DN_\infty, \mathbb{C}, 0)$. Metric approximation in the metrical propinquity for $qvb(C(X), H_\infty, D_\infty)$ requires metric approximation in the dual modular propinquity for $(H_\infty, DN_\infty, \mathbb{C}, 0)$. Modular tunnels between $(H_\infty, DN_\infty, \mathbb{C}, 0)$ and $(H_n, DN_n, \mathbb{C}, 0)$ will be built using the quantum compact metric space $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon)$. The extent of these modular tunnels will therefore be determined by the extent of tunnels of the same form as $(\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)$.

Lemma 4. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j \in \mathbb{N}}$. If there exists an approximation sequence $B(n)$ of X compatible with this parameterization, then*

$$\Lambda^{*mod}_{n \to \infty}(\,(H_\infty, DN_\infty, \mathbb{C}, 0), (H_n, DN_n, \mathbb{C}, 0)\,) = 0.$$

Proof. As in the calculation of the dual propinquity between two quantum compact metric spaces, bounds on the dual modular propinquity between two metrized quantum vector bundles can be obtained via the construction of modular tunnels between these two spaces. To build modular tunnels between $(H_\infty, DN_\infty, \mathbb{C}, 0)$ and $(H_n, DN_n, \mathbb{C}, 0)$, choose an $\epsilon > 0$. Since X is a piecewise C^1-fractal curve, there exists $N \in \mathbb{N}$ such that if $j > B(N)$,

$$l_j < \frac{\pi \epsilon}{2}.$$

One of the elements needed for the construction of a modular tunnel is a metrized quantum vector bundle. Recall that H_n can be viewed as a subspace of H_∞ via the identification described in the previous section. Fix some choice of $n \geqslant N$. The Hilbert space $H_n \oplus H_\infty$ will be viewed as a Hilbert module over the C*-algebra $\mathbb{C} \oplus \mathbb{C}$ with action defined for every (w, z) in $\mathbb{C} \oplus \mathbb{C}$ and for every (η, ξ) in $H_n \oplus H_\infty$ by

$$(w, z) \cdot (\eta, \xi) = (w\eta, z\xi)$$

and inner product for every (η, ξ) and (η', ξ') in $H_n \oplus H_\infty$ by

$$\langle (\eta, \xi), (\eta', \xi') \rangle_{H_n \oplus H_\infty} = (\langle \eta, \eta' \rangle_{H_n}, \langle \xi, \xi' \rangle_{H_\infty}).$$

For the quantum compact metric space component of a metrized quantum vector bundle, take $(C \oplus C, Q_\epsilon)$. Let $DN_{n,\epsilon} : H_n \oplus H_\infty \to \mathbb{R}$ be given by

$$DN_{n,\epsilon}(\eta, \xi) = \max\left\{DN_n(\eta), DN_\infty(\xi), \frac{1}{\epsilon}||\xi - \eta||_{H_\infty}\right\}.$$

This norm will be shown to be a D-norm for the Hilbert $\mathbb{C}\oplus\mathbb{C}$-module $(H_n \oplus H_\infty, \langle\cdot,\cdot\rangle_{H_n\oplus H_\infty})$.

By construction, $DN_{n,\epsilon}$ is defined on a dense subspace of $H_n \oplus H_\infty$. Equipping $H_n \oplus H_\infty$ with Hilbert module structure over $\mathbb{C} \oplus \mathbb{C}$ yields for every $(\eta, \xi) \in \mathrm{dom}(DN_{n,\epsilon})$,

$$\begin{aligned}
||(\eta,\xi)||^2_{H_n\oplus H_\infty} &= ||\,\langle(\eta,\xi),(\eta,\xi)\rangle_{H_n\oplus H_\infty}\,||_{\mathbb{C}\oplus\mathbb{C}} \\
&= ||\,(\langle\eta,\eta\rangle_{H_n}, \langle\xi,\xi\rangle_{H_\infty})\,||_{\mathbb{C}\oplus\mathbb{C}} \\
&= \max\{\,||\eta||^2_{H_n}, ||\xi||^2_{H_\infty}\,\} \\
&\leqslant \max\{\,(DN_n(\eta))^2, (DN_\infty(\xi))^2\,\} \\
&\leqslant (DN_{n,\epsilon}(\eta,\xi))^2.
\end{aligned}$$

Now consider the unit ball with respect to $\mathrm{DN}_{n,\epsilon}$, that is,

$$B_{DN_{n,\epsilon}} := \{(\eta,\xi) \in H_n \oplus H_\infty : DN_{n,\epsilon}(\eta,\xi) \leqslant 1\}$$

$$\subseteq \{\eta \in H_n : DN_n(\eta) \leqslant 1\} \times \{\xi \in H_\infty : \mathrm{DN}_\infty(\xi) \leqslant 1\} := B_{DN_n} \times B_{DN_\infty}$$

The unit balls B_{DN_n} and B_{DN_∞} are each compact with respect to the Hilbert space norms for their respective domains [23]. Moreover, seminorms are lower semi-continuous, as are graph norms of Dirac operators. As the maximum of three lower semi-continuous functions, $DN_{n,\epsilon}$ is also lower semi-continuous. As a consequence, $B_{DN_{n,\epsilon}}$ is a closed subset of a compact set, hence compact.

The inner Leibniz inequality will next be checked. For every (η, ξ) and (η', ξ') in $H_n \oplus H_\infty$,

$$Q_\epsilon(\langle(\eta, \xi), (\eta', \xi')\rangle_{H_n \oplus H_\infty}) = Q_\epsilon((\langle \eta, \eta'\rangle_{H_n}, \langle \xi, \xi'\rangle_{H_\infty})) = \frac{1}{\epsilon}|\langle \xi, \xi'\rangle_{H_\infty} - \langle \eta, \eta'\rangle_{H_n}|$$

$$= \frac{1}{\epsilon}|\langle \xi, \xi'\rangle_{H_\infty} - \langle \eta, \eta'\rangle_{H_\infty}| = \frac{1}{\epsilon}|\langle \xi, \xi'\rangle_{H_\infty} - (\langle \eta', \xi\rangle^*_{H_\infty} + \langle \eta', \eta - \xi\rangle^*_{H_\infty})|$$

$$= \frac{1}{\epsilon}|\langle \xi, \xi'\rangle_{H_\infty} - \langle \xi, \eta'\rangle_{H_\infty} + \langle \xi - \eta, \eta'\rangle_{H_\infty}| = \frac{1}{\epsilon}|\langle \xi, \xi' - \eta'\rangle_{H_\infty} + \langle \xi - \eta, \eta'\rangle_{H_\infty}|$$

$$\leqslant \frac{1}{\epsilon}(|\langle \xi, \xi' - \eta'\rangle_{H_\infty}| + |\langle \xi - \eta, \eta'\rangle_{H_\infty}|)$$

$$\leqslant \frac{1}{\epsilon}||\xi||_{H_\infty}\, ||\xi' - \eta'||_{H_\infty} + \frac{1}{\epsilon}||\xi - \eta||_{H_\infty}\, ||\eta'||_{H_\infty}$$

$$\leqslant ||\xi||_{H_\infty}\, DN_{n,\epsilon}(\eta', \xi') + DN_{n,\epsilon}(\eta, \xi)\, ||\eta'||_{H_n}$$

$$\leqslant DN_{n,\epsilon}(\eta, \xi)\, DN_{n,\epsilon}(\eta', \xi') + DN_{n,\epsilon}(\eta, \xi)\, DN_{n,\epsilon}(\eta', \xi')$$

$$= 2DN_{n,\epsilon}(\eta, \xi)\, DN_{n,\epsilon}(\eta', \xi').$$

In particular,

$$Q_\epsilon\Big(\frac{\langle(\eta, \xi), (\eta', \xi')\rangle_{H_n \oplus H_\infty} + \langle(\eta, \xi), (\eta', \xi')\rangle^*_{H_n \oplus H_\infty}}{2}\Big)$$

$$\leqslant \frac{1}{2}\Big(Q_\epsilon(\langle(\eta, \xi), (\eta', \xi')\rangle_{H_n \oplus H_\infty}) + Q_\epsilon(\langle(\eta, \xi), (\eta', \xi')\rangle^*_{H_n \oplus H_\infty})\Big)$$

$$\leqslant \frac{1}{2}\Big(2DN_{n,\epsilon}(\eta, \xi)\, DN_{n,\epsilon}(\eta', \xi') + Q_\epsilon(\langle(\eta', \xi'), (\eta, \xi)\rangle_{H_n \oplus H_\infty})\Big)$$

$$\leqslant \frac{1}{2}\Big(2DN_{n,\epsilon}(\eta, \xi)\, \mathrm{DN}_{n,\epsilon}(\eta', \xi') + 2DN_{n,\epsilon}(\eta', \xi')\, DN_{n,\epsilon}(\eta, \xi)\Big)$$

$$= 2DN_{n,\epsilon}(\eta, \xi)\, DN_{n,\epsilon}(\eta', \xi').$$

Similarly, the same bound holds for $Q_\epsilon\Big(\frac{\langle(\eta,\xi),(\eta',\xi')\rangle_{H_n \oplus H_\infty} - \langle(\eta,\xi),(\eta',\xi')\rangle^*_{H_n \oplus H_\infty}}{2i}\Big)$. Therefore,

$$\max\Big\{Q_\epsilon\Big(\frac{\langle(\eta, \xi), (\eta', \xi')\rangle_{H_n \oplus H_\infty} + \langle(\eta, \xi), (\eta', \xi')\rangle^*_{H_n \oplus H_\infty}}{2}\Big),$$
$$Q_\epsilon\Big(\frac{\langle(\eta, \xi), (\eta', \xi')\rangle_{H_n \oplus H_\infty} - \langle(\eta, \xi), (\eta', \xi')\rangle^*_{H_n \oplus H_\infty}}{2i}\Big)\Big\}$$

$$\leqslant 2\mathrm{DN}_{n,\epsilon}(\eta,\xi)\,\mathrm{DN}_{n,\epsilon}(\eta',\xi'),$$

thereby completing the verification that $(H_n \oplus H_\infty, DN_{n,\epsilon}, \mathbb{C}\oplus\mathbb{C}, Q_\epsilon)$ is a metrized quantum vector bundle.

A modular tunnel also includes surjective module morphisms between Hilbert module structures. To define such morphisms, take the projections $\Pi_n : H_n \oplus H_\infty \to H_n$ given by $\Pi_n(\eta,\xi) = \eta$ and $\Pi_\infty : H_n \oplus H_\infty \to H_\infty$ specified by $\Pi_\infty(\eta,\xi) = \xi$. Let $\pi_n : \mathbb{C}\oplus\mathbb{C} \to \mathbb{C}$ denote the projection $\pi_n(w,z) = w$, and $\pi_\infty : \mathbb{C}\oplus\mathbb{C} \to \mathbb{C}$ signify the projection $\pi_\infty(w,z) = z$. By construction, (Π_n, Π_n) and (Π_∞, π_∞) are surjective Hilbert module morphisms. More precisely, π_n and π_∞ are *-morphisms and Π_n and Π_∞ are $\mathbb{C}$-linear maps such that for every (w,z) in $\mathbb{C}\oplus\mathbb{C}$ and for every (η,ξ) and (η',ξ') in $H_n \oplus H_\infty$,

$$\Pi_n((w,z)(\eta,\xi)) = \Pi_n(w\eta, z\xi) = w\eta = \pi_n(w,z)\Pi_n(\eta,\xi),$$

$$\Pi_\infty((w,z)(\eta,\xi)) = \Pi_\infty(w\eta, z\xi) = z\xi = \pi_\infty(w,z)\Pi_\infty(\eta,\xi),$$

and

$$\langle \Pi_n(\eta,\xi), \Pi_n(\eta',\xi')\rangle_{H_n} = \langle \eta,\eta'\rangle_{H_n} = \pi_n(\langle \eta,\eta'\rangle_{H_n}, \langle \xi,\xi'\rangle_{H_\infty}) = \pi_n(\langle (\eta,\xi),(\eta',\xi')\rangle_{H_n\oplus H_\infty}),$$

$$\langle \Pi_\infty(\eta,\xi), \Pi_\infty(\eta',\xi')\rangle_{H_\infty} = \langle \xi,\xi'\rangle_{H_\infty} = \pi_\infty(\langle \eta,\eta'\rangle_{H_n}, \langle \xi,\xi'\rangle_{H_\infty}) = \pi_\infty(\langle (\eta,\xi),(\eta',\xi')\rangle_{H_n\oplus H_\infty}).$$

Thus $(H_n \oplus H_\infty, \langle\cdot,\cdot\rangle_{H_n\oplus H_\infty})$, when viewed as a Hilbert module over $\mathbb{C}\oplus\mathbb{C}$, encodes the Hilbert module structures of $(H_n, \langle\cdot,\cdot\rangle_{H_n})$ and $(H_\infty, \langle\cdot,\cdot\rangle_{H_\infty})$ when each viewed as Hilbert modules over $\mathbb{C}$.

The quotient properties of $\mathrm{DN}_{n,\epsilon}$ for Π_n and Π_∞ will next be examined. Consider some choice of $\xi = (\xi_j)_{j\in\mathbb{N}}$ in H_∞ with ξ in the domain of DN_∞. With respect to the orthonormal basis $(\phi_k^{l_j/\pi})_{k\in\mathbb{Z}}$, each ξ_j can be written as $\xi_j = \sum\limits_{k\in\mathbb{Z}} t_{j,k}\,\phi_k^{l_j/\pi}$ with $(t_{j,k})_{k\in\mathbb{Z}}$ in

$\ell^2(\mathbb{Z})$ for all $j \in \mathbb{N}$, hence

$$\begin{aligned}||\not{D}_\infty \xi||^2_{H_\infty} &= \sum_{j\in\mathbb{N}} ||D_{l_j}\xi_j||^2_{H_{l_j}} = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left|\left| t_{j,k}\frac{(2k+1)\pi}{2l_j}\phi_k^{l_j/\pi}\right|\right|^2_{H_{l_j}} \\ &= \pi^2 \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \frac{1}{l_j^2}\left|t_{j,k}\right|^2\Big(k+\frac{1}{2}\Big)^2.\end{aligned}$$

Recall that for $j > B(N)$, $l_j < \frac{\pi\epsilon}{2}$. This lower bound on j yields

$$\begin{aligned}||\not{D}_\infty \xi||^2_{H_\infty} &\geqslant \pi^2 \sum_{j>B(N)}\sum_{k\in\mathbb{Z}} \frac{1}{l_j^2}\left|t_{j,k}\right|^2\Big(k+\frac{1}{2}\Big)^2 \\ &> \pi^2 \sum_{j>B(N)}\sum_{k\in\mathbb{Z}} \frac{4}{\pi^2\epsilon^2}\left|t_{j,k}\right|^2\Big(k+\frac{1}{2}\Big)^2 = \frac{4}{\epsilon^2}\sum_{j>B(N)}\sum_{k\in\mathbb{Z}}\left|t_{j,k}\right|^2\Big(k+\frac{1}{2}\Big)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j>B(N)} ||\xi_j||^2_{H_j} = \sum_{j>B(N)}\sum_{k\in\mathbb{Z}} |t_{j,k}|^2 &= 4\sum_{j>B(N)}\sum_{k\in\mathbb{Z}} |t_{j,k}|^2\frac{1}{4} \\ &\leqslant 4\sum_{j>B(N)}\sum_{k\in\mathbb{Z}} |t_{j,k}|^2\Big(k+\frac{1}{2}\Big)^2 < (\epsilon||D_\infty\xi||_{H_\infty})^2.\end{aligned}$$

To see that the quotient of $DN_{n,\epsilon}$ for Π_∞ is DN_∞, take η as the orthogonal projection of ξ to H_n. This choice of η gives

$$||\xi-\eta||^2_{H_\infty} = \sum_{j>B(N)} ||\xi_j||^2_{H_j} < (\epsilon||D_\infty\xi||_{H_\infty})^2 \leqslant (\epsilon(||\xi||_{H_\infty}+||D_\infty\xi||_{H_\infty}))^2 = (\epsilon DN_\infty(\xi))^2.$$

Since $\mathrm{DN}_n(\eta) \leqslant \mathrm{DN}_\infty(\xi)$,

$$\mathrm{DN}_\infty(\xi) = \inf\{\mathrm{DN}_{n,\epsilon}((\zeta_1,\zeta_2)) : \Pi_\infty((\zeta_1,\zeta_2)) = \xi\}.$$

In the case of the quotient of $\mathrm{DN}_{n,\epsilon}$ for Π_n, let η be an arbitrary vector in H_n viewed as a subspace of H_∞ and set ξ equal to η. Then $||\xi-\eta||_{H_\infty}$ is zero, $DN_\infty(\xi)$ coincides with $\mathrm{DN}_n(\eta)$, and as a consequence, $DN_{n,\epsilon}(\eta,\xi)$

is $\mathrm{DN}_n(\eta)$. Thus the quotient of $DN_{n,\epsilon}$ for Π_n is DN_n. Thus

$$\mu_{n,\epsilon} = ((H_n \oplus H_\infty, DN_{n,\epsilon}, \mathbb{C} \oplus \mathbb{C}, Q_\epsilon), (\Pi_n, \pi_n), (\Pi_\infty, \pi_\infty))$$

is a modular tunnel from $(H_n, DN_n, \mathbb{C}, 0)$ to $(H_\infty, DN_\infty, \mathbb{C}, 0)$. Consequently,

$$\Lambda^{*mod}((H_\infty, DN_\infty, \mathbb{C}, 0), (H_n, DN_n, \mathbb{C}, 0) \leqslant \chi(\mu_{n,\epsilon}) = \chi((\mathbb{C} \oplus \mathbb{C}, Q_\epsilon, \pi_n, \pi_\infty)) \leqslant \epsilon.$$

■

For $\epsilon' > \epsilon$, note that $(H_n \oplus H_\infty, DN_{n,\epsilon}, \mathbb{C} \oplus \mathbb{C}, Q_{\epsilon'})$ is also a modular tunnel between $(H_\infty, DN_\infty, \mathbb{C}, 0)$ and $(H_n, DN_n, \mathbb{C}, 0)$. If $\epsilon' < \epsilon$, then the penultimate inequality in the verification of the inner Leibniz inequality is no longer valid. Since $(H_n \oplus H_\infty, DN_{n,\epsilon}, \mathbb{C} \oplus \mathbb{C}, Q_{\epsilon'})$ cannot be a modular tunnel between $(H_\infty, DN_\infty, \mathbb{C}, 0)$ and $(H_n, DN_n, \mathbb{C}, 0)$ for $\epsilon' < \epsilon$, the smallest possible extent for a modular tunnel of the same construction as $\mu_{n,\epsilon}$ is ϵ. Modular tunnels like $\mu_{n,\epsilon}$ can be extended to metrical tunnels between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$.

Theorem 30. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j\in\mathbb{N}}$. If there exists an approximation sequence $B(n)$ of X compatible with this parameterization, then*

$$\Lambda^{*met}_{n\to\infty}(\, qvb(C(X), H_\infty, D_\infty), qvb(C(X_n), H_n, D_n)\,) = 0.$$

Proof. As in the calculation of the dual propinquity between two quantum compact metric spaces and the dual modular propinquity between two metrized quantum vector bundles, bounds on the metrical propinquity between two metrical quantum vector bundles can be obtained via the construction of metrical tunnels between these two spaces. To build metrical tunnels between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$, choose an $\epsilon > 0$. Since X is a piecewise C^1-fractal curve, there exists $N_1 \in N$ such that if $n \geqslant N_1$, then $\mathrm{Haus}_{d_n}(X_n, V_n) < \frac{\epsilon}{4}$. The existence of an approximation sequence $B(n)$ for X guarantees there exists $N_2 \in N$ such that if $n \geqslant N_2$, then $\mathrm{Haus}_{d_\infty}(X, V_n) < \epsilon$. Since $(H_\infty, \langle \cdot, \cdot \rangle_{H_\infty}, \mathbb{C}, 0)$ is the underlying metrized quantum vector bundle for $qvb(C(X), H_\infty, D_\infty)$ and $(H_n, \langle \cdot, \cdot \rangle_{H_n}, \mathbb{C}, 0)$ is that for $qvb(C(X), H_\infty, D_\infty)$, metrical tunnels will be built using modular tunnels of the form $\mu_{n,\epsilon}$. The previous lemma demonstrates there exists $N_3 \in \mathbb{N}$ such that if $n \geqslant N_3$, then $\chi(\mu_{n,\epsilon}) \leqslant \frac{\epsilon}{2} < \epsilon$. Fix some choice of $n \geqslant N := N_1 + N_2 + N_3$. The pairing $(\mu_{n,\epsilon}, \tau_{n,\epsilon/4})$ will be shown to be a metrical tunnel between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$.

To check the modular Leibniz inequality, recall that the Lip-norm properties of $L_{n,\epsilon/4}$ are specified on a dense subspace of $sa(A_n)$. Let $(f, g) \in C(X_n, \mathbb{R}) \oplus C(X, \mathbb{R})$. Set arbitrary choices of $\eta \in \mathrm{dom}(DN_n)$ and $\xi \in \mathrm{dom}(DN_\infty)$. Then

$$
\begin{aligned}
DN_{n,\epsilon}((f,g)(\eta,\xi) = DN_{n,\epsilon}(f\eta, g\xi) &= \max\left\{\mathrm{DN}_n(f\eta), \mathrm{DN}_\infty(g\xi), \frac{1}{\epsilon}||g\xi - f\eta||_{H_\infty}\right\} \\
&\leqslant \max\Big\{(||f||_{C(X_n)} + L_{D_n}(f)DN_n(\eta), \\
&\qquad (||g||_{C(X)} + L_{D_\infty}(g))DN_\infty(\xi), \\
&\qquad \frac{1}{\epsilon}||g\xi - f\eta||_{H_\infty}\Big\} \\
&\leqslant \max\Big\{(||(f,g)||_{A_n} + L_{n,\epsilon/4}(f,g))DN_{n,\epsilon}(\eta,\xi), \\
&\qquad \frac{1}{\epsilon}||g\xi - f\eta||_{H_\infty}\Big\}.
\end{aligned}
$$

Since H_n can be viewed as a subspace of H_∞, $\pi_\infty(g)\eta$ is well-defined. Let $g|_{X_n}$ denote the restriction of g to X_n. In particular,

$$\pi_\infty(g)\eta = g\eta = g|_{X_n}\eta = \pi_n(g|_{X_n})\eta.$$

Consequently,

$$\begin{aligned}
\frac{1}{\epsilon}||g\xi - f\eta||_{H_\infty} &\leqslant \frac{1}{\epsilon}\Big(||g\xi - g\eta||_{H_\infty} + ||g\eta - f\eta||_{H_\infty}\Big) \\
&\leqslant ||g||_{C(X)}\Big(\frac{1}{\epsilon}||\xi - \eta||_{H_\infty}\Big) + \frac{1}{\epsilon}||\, g|_{X_n}\eta - f\eta\,||_{H_n} \\
&\leqslant ||(f,g)||_{A_n} DN_{n,\epsilon}(\eta,\xi) + \frac{1}{\epsilon}||\, g|_{X_n} - f\,||_{C(X_n)}||\eta||_{H_n} \\
&\leqslant ||(f,g)||_{A_n} DN_{n,\epsilon}(\eta,\xi) + \frac{1}{\epsilon}||\, g|_{X_n} - f\,||_{C(X_n)} DN_{n,\epsilon}(\eta,\xi).
\end{aligned}$$

To show that $\frac{1}{\epsilon}||\, g|_{X_n} - f\,||_{C(X_n)}$ is bounded by $L_{n,\epsilon/4}(f,g)$, let x be some point in X_n. Then $n \geqslant N$ implies there exists $v \in V_n$ such that $d_n(x,v) < \frac{\epsilon}{4}$ and $d_\infty(x,v) < \frac{\epsilon}{4}$. Together with the definition of $L_{n,\epsilon/4}$, these inequalities yield

$$\begin{aligned}
\Big| g|_{X_n}(x) - f(x)\Big| &\leqslant \Big| g|_{X_n}(x) - g\,|_{X_n}(v)\Big| + \Big| g|_{X_n}(v) - f(v)\Big| + \Big| f(v) - f(x)\Big| \\
&\leqslant d_n(v,x)L_{d_n}(g|_{X_n}) + ||\, g|_n - f|_n\,||_{C(X_n)} + d_n(v,x)L_{d_n}(f) \\
&\leqslant d_n(v,x)L_{d_\infty}(g) + ||\, g|_n - f|_n\,||_{C(X_n)} + d_n(v,x)L_{D_n}(f) \\
&< \frac{\epsilon}{4}L_{D_\infty}(g) + \frac{\epsilon}{4}L_{n,\epsilon/4}(f,g) + \frac{\epsilon}{4}L_{D_n}(f) \\
&\leqslant \frac{\epsilon}{4}L_{n,\epsilon/4}(f,g) + \frac{\epsilon}{4}L_{n,\epsilon/4}(f,g) + \frac{\epsilon}{4}L_{n,\epsilon/4}(f,g) \\
&< \epsilon L_{n,\epsilon/4}(f,g).
\end{aligned}$$

The Lip-norm belonging to τ and the D-norm component coming from μ therefore together obey the modular Leibniz inequality.

A metrical tunnel also includes left module morphisms. Because the representation of A_n is as left multiplication operators on $H_n \oplus H_\infty$, (θ_n, Π_n) and $(\theta_\infty, \Pi_\infty)$ are left module morphisms. More precisely, π_n and π_θ are unital *-morphisms and Π_n and Π_∞ are linear maps such that for every (f,g) in A_n and every (η, ξ) in $H_n \oplus H_\infty$,

$$\Pi_n((f,g)(\xi,\eta)) = \Pi_n((f\eta, g\xi) = f\eta = \theta_n(f,g)\Pi_n(\xi,\eta),$$

$$\Pi_\infty((f,g)(\xi,\eta)) = \Pi_\infty((f\eta, g\xi) = f\eta = \theta_\infty(f,g)\Pi_\infty(\xi,\eta).$$

Thus $(H_n \oplus H_\infty$, when viewed as a left A_n-module, encodes the left-$C(X_n)$ module structure of H_n and the left-$C(X)$ module structure of H_∞. Furthermore, $(\mu_{n,\epsilon}, \tau_{n,\epsilon/4})$ is a metrical tunnel between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$. Recall from the calculation of the dual propinquity between $(C(X), L_{D_\infty})$ and $(C(X_n), L_{D_n})$ that $\chi(\tau_{n,\epsilon/4}) < \frac{3\epsilon}{4}$. Consequently,

$$\begin{aligned}\Lambda^{*met}(qvb(C(X), H_\infty, D_\infty), qvb(C(X_n), H_n, D_n)) &\leqslant \chi(\mu_{n,\epsilon}, \tau_{n,\epsilon/4}) = \max\{\chi(\mu_{n,\epsilon}), \chi(\tau_{n,\epsilon/4})\} \\ &\leqslant \max\left\{\epsilon, \frac{3\epsilon}{4}\right\} = \epsilon.\end{aligned}$$

■

4.4 Metric Approximation of Lapidus-Sarhad Spectral Triples

For a metric spectral triple like the Lapidus-Sarhad spectral triple on a piecewise C^1-fractal curve, the action of the Dirac operator on the Hilbert space can be captured by approximations in the spectral propinquity. These approximating spectral triples will be metric spectral triples defined on finite sub-graphs of the piecewise C^1-fractal curve. This sequence of finite sub-graphs also converges to the piecewise C^1-fractal curve in the Hausdorff distance. The corresponding Lapidus-Sarhad spectral triples will be shown to also converge in the spectral propinquity to the Lapidus-Sarhad spectral triple on the piecewise C^1-fractal curve when that sequence of finite graphs exhibits the geometric properties encoded in the definition of an approximation sequence for that fractal curve.

Theorem 31. *Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j \in \mathbb{N}}$. If there exists an approximation sequence $B(n)$ of X compatible with this parameterization, then*

$$\Lambda^{spec}_{n \to \infty}(\,(C(X), H_\infty, D_\infty), (C(X_n), H_n, D_n)\,) = 0.$$

Proof. Bounds on the spectral propinquity between two metric spectral triples can be obtained via the construction of ϵ-covariant metrical tunnels. To build an ϵ-covariant metrical tunnel between $(C(X), H_\infty, D_\infty$ and $(C(X_n), H_n, D_n)$, choose an $\epsilon > 0$. As shown in the calcuation of the metrical propinquity between $qvb(C(X), H_\infty, D_\infty)$ and $qvb(C(X_n), H_n, D_n)$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$, then $\chi(\mu_{n,\epsilon}, \tau_{n,\epsilon/4}) \leqslant \epsilon$. An ϵ-covariant metrical tunnel between $(C(X), H_\infty, D_\infty)$ and $(C(X_n), H_n, D_n)$ can be built from a metrical tunnel between $qvb(C(X), H_\infty, D_\infty$ and $qvb(C(X_n), H_n, D_n)$ if there exists an $\epsilon - iso - iso$ from $\mathbb{R}$ to $\mathbb{R}$. Consider $(id_\mathbb{R}, id_\mathbb{R})$ as a candidate for such a map. For every $x, y, z \in \left[\frac{1}{\epsilon}, \frac{1}{\epsilon}\right]$,

$$\Big|\,|id_\mathbb{R}(x) + id_\mathbb{R}(y) - z| - |(x+y) - id_\mathbb{R}(z)|\,\Big| \leqslant \epsilon.$$

Moreover, $id_\mathbb{R}(0) = id_\mathbb{R}(0) = 0$. Thus $(\mu_{n,\epsilon}, \tau_{n,\epsilon/4}, id_\mathbb{R}, id_\mathbb{R})$ is an ϵ-covariant metrical tunnel between $(C(X), H_\infty, D_\infty)$ and $(C(X_n), H_n, D_n)$. Moreover, $(\mu_{n,\epsilon}, id_\mathbb{R}, id_\mathbb{R})$ is an ϵ-covariant tunnel. For every $n \in \mathbb{N} \cup \infty$ and each $t \in \mathbb{R}$, let

$$U_n(t) = exp(itD_n).$$

Then the ϵ-covariant modular reach of $(\mu_{n,\epsilon}, id_{\mathbb{R}}, id_{\mathbb{R}})$ is given by

$$\rho_m((\mu_{n,\epsilon}, id_{\mathbb{R}}, id_{\mathbb{R}})) = \max_{\{j,k\}\in\{n,\infty\}} \sup_{\omega\in H_j, DN_j(\omega)\leqslant 1} \inf_{\omega'\in H_k, DN_k(\omega')\leqslant 1} \sup_{|t|\leqslant \frac{1}{\varepsilon}}$$

$$\sup_{(\eta,\xi)\in H_n\oplus H_\infty,\, DN_{n,\epsilon}(\eta,\xi)\leqslant 1} \left| \langle U_j(t)\eta, \Pi_j(\omega,\omega')\rangle_{H_j} - \langle U_k(t)\xi, \Pi_k(\omega,\omega')\rangle_{H_k} \right|.$$

To bound $\rho_m((\mu_{n,\epsilon}, id_{\mathbb{R}}, id_{\mathbb{R}}))$, recall that H_n can be viewed as a subspace of H_∞. Then for every $\eta \in H_n$, $D_n\eta = D_\infty\eta$. In particular, $U_n(t)\eta = U_\infty(t)\eta$. Furthermore, $DN_n(\omega) \leqslant 1$ implies $DN_\infty(\omega) \leqslant 1$. Thus for every $(\eta,\xi) \in H_n \oplus H_\infty$ with $DN_{n,\epsilon}(\eta,\xi) \leqslant 1$,

$$\begin{aligned}
&\left|\langle U_n(t)\eta, \Pi_n(\omega,\omega)\rangle_{H_n} - \langle U_\infty(t)\xi, \Pi_\infty(\omega,\omega)\rangle_{H_\infty}\right| \\
&= \left|\langle U_n(t)\eta, \Pi_n(\omega,\omega)\rangle_{H_\infty} - \langle U_\infty(t)\xi, \Pi_\infty(\omega,\omega)\rangle_{H_\infty}\right| \\
&= \left|\langle U_n(t)\eta, \omega\rangle_{H_\infty} - \langle U_\infty(t)\xi, \omega\rangle_{H_\infty}\right| \\
&= \left|\langle U_n(t)\eta - U_\infty(t)\xi, \omega\rangle_{H_\infty}\right| \\
&\leqslant ||\, U_n(t)\eta - U_\infty(t)\xi \,||_{H_\infty}\, ||\omega||_{H_\infty} \\
&\leqslant ||\, U_n(t)\eta - U_\infty(t)\xi \,||_{H_\infty} \\
&\leqslant ||\, U_n(t)\eta - U_\infty(t)\eta \,||_{H_\infty} + ||\, U_\infty(t)\eta - U_\infty(t)\xi \,||_{H_\infty} \\
&\leqslant ||\, U_\infty(t)\eta - U_\infty(t)\eta \,||_{H_\infty} + ||\, U_\infty(t)\eta - U_\infty(t)\xi \,||_{H_\infty} \leqslant 0 + ||\, \eta - \xi \,||_{H_\infty} \\
&\leqslant \epsilon DN_{n,\epsilon}(\eta,\xi) \leqslant \epsilon,
\end{aligned}$$

where the first inequality follows from the Cauchy Schwartz Inequality and the second inequality from the choice of the graph norm of the Dirac operator as the D-norm. Consequently,

$$\varrho(\,(\mu_{n,\epsilon},\tau_{n,\epsilon/4},id_{\mathbb{R}},id_{\mathbb{R}})\,) = \max\Big\{\chi(\tau_{n,\epsilon/4}),\chi(\mu_{n,\epsilon}),\rho_m((\mu,id_{\mathbb{R}},id_{\mathbb{R}}))\,\Big\}$$

$$= \Big\{\chi(\tau_{n,\epsilon/4},\mu_{n,\epsilon}),\rho_m((\mu_{n,\epsilon},id_{\mathbb{R}},id_{\mathbb{R}}))\,\Big\} \leqslant \max\Big\{\epsilon,\epsilon\Big\} = \epsilon.$$

Hence

$$\Lambda^{spec}(\,(C(X),H_\infty,D_\infty),(C(X_n),H_n,D_n)\,) \leqslant \epsilon,$$

as desired. ■

Recall that the Dirac operator defines the geometry of a Riemannian manifold. Since the spectral propinquity between two metric spectral triples with unitarily equivalent Dirac operators is trivial, this metric gives a notion of closeness for the actions of these operators on their respective Hilbert spaces. Therefore, if two metric spectral triples on a fractal curve are "close" for the spectral propinquity, then the two geometries determined on the fractal curve by these two metric spectral triples can also be viewed as "close."

Chapter 5

Conclusions

As the context for the construction of spectral triples, Lapidus and Sarhad's piecewise C^1-fractal curve framework is a crucial step towards the development of differential structures on fractals beyond the prototypical settings of the Sierpinski gasket and the harmonic gasket. As detailed in [41] and in this thesis, the application of the spectral propinquity to the metric approximation of Lapidus-Sarhad spectral triples is an important test case for demonstrating the possibility of combining elements from both noncommutative metric geometry and noncommutative Riemanninan geometry to the study of fractals. With a metric on spectral triples in hand, a natural direction for future work would be identification and study of a class of *almost piecewise C^1-fractal curves* for which the same construction yields a spectral triple. Although the stretched Sierpinski gasket is not a piecewise C^1-fractal curve, Andrea Arauza Rivera demonstrated in [38] that the Lapidus-Sarhad construction gives a spectral triple that recovers the geodesic distance. In [39], Patricia Alonso Ruiz and Uta Freiburg show that the stretched Sierpinski gasket converges to the

Sierpinski gasket for the Hausdorff distance when its defining parameter goes to zero. Comparison via the spectral propinquity of Lapidus-Sarhad spectral triples on both SG and SG_α for various value of α could yield insights about how to extend this spectral triple construction or build other spectral triples on other types of fractals. Such work could also inform the definition and study of generalized notions of manifolds that include fractal spaces.

Perceiving fractals through the lens of noncommutative geometry can lead to new expressions of the geometry of a fractal. Because of Gelfand duality, any homeomorphism invariant of the compact Hausdorff X can be reframed as an algebraic invariant of the C^*-algebra $C(X)$. For example, X is a totally disconnected compact metric space if and only if $C(X)$ is a unital commutative approximately finite dimensional algebra. Furthermore, C^*-algebras within the framework of this duality remain to be identified for other fractals like the Sierpinski gasket. Such an investigation could therefore begin to form the basis for a classification program of C^*-algebras on fractal spaces. Another avenue for exploration would be to study fractals through C^*-algebras that arise in dynamical settings. Since symbolic dynamics is an important tool for studying fractal sets, C^*-symbolic dynamical systems could be useful in the definition and study of noncommutative fractals. Since some fractals can be viewed as infinite graphs with self-similarity conditions and higher-rank graphs are a generalization of directed graphs, C^*-algebras associated to higher-rank graphs could be another promising source of noncommutative fractality.

Progress in noncommutative fractal geometry can lead to new insights about fractality. Expanding the formalism of fractal geometry to include the mathematical language of quantum theory would also give both mathematicians and physicists the tools to gain insights about quantum behaviors in solids and any new materials made possible by these phenomena. The 2016 Nobel Prize in Physics was awarded for work on Hofstadter's butterfly [13], which is a fractal that describes for theoretical condensed matter physicists the allowed energy levels for electrons confined to a crystalline atomic lattice as a function of the magnetic field applied to the system. Since many questions in noncommutative geometry are motivated by problems in quantum mechanics, the emergence of fractal patterns at the quantum level necessitates theoretical advances in both fractal geometry and noncommutative geometry. Development of a noncommutative fractal geometry is motivated by the exploration of new ways to describe, understand, and even define fractals. Since the dual Gromov-Hausdorff propinquity metric and its extensions are defined on various classes of noncommutative C^*-algebras, a closed class of quantum compact metric spaces or a complete class of metric spectral triples could be equipped with a finite collection of maps that are contractions for the corresponding propinquity metric. An advancement in this direction would then allow us to detect and examine examples of fractality that can only arise in a quantum setting. Given the wealth of natural phenomena where fractality has been observed, research in noncommutative fractal geometry enhances our ability to continue to meet new scientific and industrial challenges.

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