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The Interaction Between Weak Variants of Square and Other Combinatorial Principles in Set Theory

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The Interaction Between Weak Variants of Square and Other Combinatorial Principles in Set Theory

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by

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Abstract of the Dissertation

The Interaction Between Weak Variants of Square and Other Combinatorial Principles in Set Theory

by

John Susice

Doctor of Philosophy in Mathematics

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Professor Itay Neeman, Chair

We investigate weak variants of Jensen's square principle $\square_\kappa$ and show that there are a variety of set-theoretic principles which although inconsistent with $\square_\kappa$, are nonetheless consistent with one of its weak variants.

It is well-known that $\square_{\omega_1}$ is inconsistent with Chang's Conjecture. Sakai, however, showed that $\square_{\omega_1,2}$ is compatible with Chang's Conjecture, assuming the existence of a measurable cardinal [Sak13]. In light of this, he posed the question of the exact consistency strength of this conjunction. We answer this question by pushing down Sakai's large cardinal hypothesis to an ω_1-Erdős cardinal, which is optimal due to work of Silver and Donder [DJK81].

Shelah and Stanley showed that for κ uncountable, $\square_\kappa$ implies the existence of a non-special κ^+-Aronszajn tree [SS88]. We show that this result is best possible in the sense that for any regular κ, $\square_{\kappa,2}$ is consistent with "all κ^+-Aronszajn trees are special" (assuming the existence of a weakly compact cardinal). Moreover, by employing methods of Golshani and Hayut [GH16], we are able to establish this consistency result simultaneously for all regular κ from the existence of class many supercompact cardinals.

Finally, we introduce a weak variant $R_2^*(\aleph_2, \aleph_1)$ of the reflection principle $R_2(\aleph_2, \aleph_1)$ introduced by Rinot and show that unlike Rinot's principle our weak variant is consistent with $\square(\omega_2)$ (though still inconsistent with $\square_{\omega_1}$).

The dissertation of John Susice is approved.

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2019

To Mom

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Vita

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CHAPTER 1

Introduction

The study of square sequences is an integral component of modern set theory. Square sequences were first introduced by Jensen during the course of his research on the fine structure of L [Jen72]. They are defined as follows:

Definition 1.1. Suppose that κ is some cardinal. A $\square_\kappa$ sequence is a sequence $\vec{C} = \langle C_\alpha \colon \alpha < \kappa^+ \rangle$ such that the following hold:

- If $\alpha < \kappa^+$ is a limit ordinal then α C_α is a club subset of α.
- If $\alpha < \kappa^+$ is a successor ordinal, say $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.
- (Coherence) If $\alpha \in \operatorname{Lim} C_\beta$ then $C_\alpha = C_\beta \cap \alpha$.
- The order type of each C_α is $\leq \kappa$.

We say that $\square_\kappa$ holds if a $\square_\kappa$ sequence exists.

Observe that such a sequence cannot be *threaded*, i.e. there is no club $C \subseteq \kappa^+$ such that $C \cap \alpha = C_\alpha$ for all $\alpha \in \operatorname{Lim} C$. For this reason, square principles are considered to be canonical examples of incompactness. Moreover, not only are they strong witnesses to incompactness in and of themselves, they also serve as crucial ingredient in the construction of many other combinatorial objects which exhibit strong non-compactness properties.

For example, recall the definition of definition of non-reflecting stationary set:

Definition 1.2. Suppose that κ is a regular uncountable cardinal and $S \subseteq \kappa$ is stationary. We say that S is *non-reflecting* if for all $\alpha < \kappa$ of uncountable cofinality, $S \cap \alpha$ is non-stationary.

Theorem 1.3 ([Jen72]). *Suppose that $\square_\kappa$ holds. Then there is a non-reflecting stationary subset of κ^+ consisting only of points of countable cofinality.*

Proof. Let $\vec{C} = \langle C_\alpha \colon \alpha < \kappa^+\rangle$ be a $\square_\kappa$ sequence. For $\alpha < \kappa^+$ of cofinality ω, let

$$f(\alpha) = \operatorname{otp} C_\alpha$$

Observe that $f(\alpha) \leq \kappa < \alpha$ if $\kappa < \alpha < \kappa^+$, so by Fodor's Lemma there is stationary S consisting of ordinals of countable cofinality such that $f \restriction S$ is constant, i.e. $\operatorname{otp} C_\alpha = \operatorname{otp} C_\beta$ for all $\alpha, \beta \in S$. We claim that S is non-reflecting.

Supposing otherwise, choose $\gamma < \kappa^+$ of uncountable cofinality such that $S \cap \gamma$ is stationary in γ. Since C_γ is club in γ, by assumption we may take two points $\alpha < \beta$ which both lie in $S \cap \operatorname{Lim} C_\gamma$. By coherence,

$$C_\alpha = C_\gamma \cap \alpha$$
$$C_\beta = C_\gamma \cap \beta$$

and so in particular $\operatorname{otp} C_\alpha < \operatorname{otp} C_\beta$, contradicting $\alpha, \beta \in S$. $\square$

One catalyst for the recent interest in the tension between compactness and incompactness in combinatorial set theory has been the proliferation of new weak square principles in recent years. Foremost among these have been the "Schimmerling Square Principles" $\square_{\kappa,\lambda}$ (which are intermediate in strength between $\square_\kappa$ and weak square $\square^*_\kappa$) and "Round Bracket Square" $\square(\kappa)$.

Definition 1.4 ([Sch95]). Suppose that κ is an infinite cardinal and λ is a (potentially finite) cardinal. A $\square_{\kappa,\lambda}$ sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha \in \operatorname{Lim} \kappa^+\rangle$ such that:

- If $\alpha < \kappa^+$ is a limit ordinal, $\mathcal{C}_\alpha$ is a collection of club subsets of α.
- If $\alpha < \kappa^+$ is a successor ordinal, say $\alpha = \bar{\alpha} + 1$, then $\mathcal{C}_\alpha = \{\{\bar{\alpha}\}\}$.
- (Coherence) If $C \in \mathcal{C}_\beta$ and $\alpha \in \operatorname{Lim} C$, then $C \cap \alpha \in \mathcal{C}_\alpha$.

- If $C \in \mathcal{C}_\alpha$, then $\operatorname{otp} C \leq \kappa$.

We say that $\square_{\kappa,\lambda}$ holds if there exists a $\square_{\kappa,\lambda}$ sequence.

Observe that $\square_\kappa$ is $\square_{\kappa,1}$ and $\square^*_\kappa$ is $\square_{\kappa,\kappa}$. The principle $\square_{\kappa,\lambda}$ decreases in strength as λ increases, and separation results for these principles were proved in unpublished work by Jensen as well as in [CFM01].

Definition 1.5 ([Jen72])**.** Suppose that κ is some infinite cardinal. A $\square(\kappa)$ sequence is a sequence $\vec{C} = \langle C_\alpha \colon \alpha < \kappa \rangle$ such that:

- If $\alpha < \kappa$ is a limit ordinal, C_α is club in α.
- If $\alpha < \kappa$ is a successor ordinal, say $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.
- (Coherence) If $\beta < \kappa$ and $\alpha \in \operatorname{Lim}(C_\beta)$, then $C_\beta \cap \alpha = C_\alpha$.
- $\vec{C}$ cannot be *threaded*, i.e. there is no club D in κ such that $D \cap \alpha = C_\alpha$ for all $\alpha \in \operatorname{Lim}(D)$.

Note that $\square_\kappa$ implies $\square(\kappa)$.

In Chapter 2 we review the basic properties of posets which force the principles $\square_{\mu,2}$, $\square(\kappa)$ as well as other posets which thread these sequences. These posets will be used extensively in the subsequent chapters to establish the various consistency results.

In Chapter 3 we establish the consistency of "$\square_{\omega_1,2}$ + Chang's Conjecture" from an ω_1-Erdős cardinal and also answer another question of Sakai's concerning the inconsistency of square principles with higher Chang's Conjectures. Work in this chapter is joint with Itay Neeman.

In Chapter 4 we force the consistency of "$\square_{\kappa,2}$ + All κ^+-Aronszajn trees are special" from a weakly compact cardinal, and show how the methods of Golshani and Hayut may be used to establish a global result.

Finally, in Chapter 5 we introduce our weak variant $R_2^*(\aleph_2, \aleph_1)$ of Rinot's reflection principle $R_2(\aleph_2, \aleph_1)$ and obtain the compatibility of this principle with $\square(\omega_2)$ as well as its incompatibility with $\square_{\omega_1}$.

CHAPTER 2

Forcing to Add $\square_{\mu,2}$ and $\square(\kappa)$

In this chapter we describe two basic posets that we will use to force the weak variants of square we are concerned with in this thesis. We also introduce threading posets associated with these sequences and demonstrate how the composition of a poset for adding a square sequence and another for threading it may be absorbed into a Levy Collapse. The latter observation will prove essential when we seek to lift elementary embeddings in the following chapters.

The essential ingredients in the aforementioned absorption results are the following well-known lemmas, which we will invoke without comment.

Lemma 2.1 (Solovay, see e.g. [Kan09], [Cum10]). *Let $\mu \leq \lambda$ be regular cardinals, and suppose that $\mathbb{P}$ is a $< \mu$-closed poset with $|\mathbb{P}| = \lambda$ which forces $|\lambda| = \mu$. Then $\mathbb{P}$ is forcing equivalent to* $\mathrm{Col}(\mu, \lambda)$.

Corollary 2.2. Suppose that κ is an inaccessible cardinal and $\mu < \kappa$ is regular. Suppose moreover that $\mathbb{P}$ is a separative $< \mu$-closed poset with $|\mathbb{P}| < \kappa$. Then $\mathbb{P} \times \mathrm{Col}(\mu, < \kappa) \simeq \mathrm{Col}(\mu, < \kappa)$.

2.1 The Cummings-Schimmerling Poset

The first poset we describe is one introduced by Cummings and Schimmerling [CS02] which, given cardinals $\mu < \kappa$ with μ regular and κ inaccessible, will simultaneously collapse κ to become μ^+ and add a $\square_{\mu,2}$ sequence.

We will denote this poset by $\mathbb{P}(\mu, < \kappa)$. A conditiion p in $\mathbb{P}(\mu, < \kappa)$ is a function such that:

1. The domain of p is a closed set of ordinals below κ of cardinality $< \mu$.

2. If $\alpha \in \operatorname{dom} p$ is a successor ordinal, say $\alpha = \bar{\alpha} + 1$, then the unique element of $p(\alpha)$ is $\{\bar{\alpha}\}$.

3. If $\alpha \in \operatorname{dom} p$ is a limit ordinal with cofinality $< \mu$ then $1 \leq |p(\alpha)| \leq 2$ and each element of $p(\alpha)$ is a club subset of α with order type $< \mu$.

4. If $\alpha \in \operatorname{dom} p$ is a limit ordinal with cofinality $\geq \mu$ then $p(\alpha) = \{C\}$, where C is some closed subset of α with order type $< \mu$ such that $\max C = \sup(\operatorname{dom} p \cap \alpha)$.

5. (Coherence) If $\alpha \in \operatorname{dom} p$, $C \in p(\alpha)$, and $\beta \in C$, then $\beta \in \operatorname{dom} p$. If moreover $\beta \in \operatorname{Lim} C$, then $C \cap \beta \in p(\beta)$.

The ordering of the poset is defined by $q \leq p$ iff $\operatorname{dom} q \supseteq \operatorname{dom} p$ and:

(a) $q(\alpha) = p(\alpha)$ for all $\alpha \in \operatorname{dom} p$ of cofinality $< \mu$.

(b) If α is of cofinality $\geq \mu$, $p(\alpha) = \{C\}$, and $q(\alpha) = \{D\}$, then $C = D \cap (\max C + 1)$.

The Cummings-Schimmerling poset will not be $< \mu$-closed but will still be sufficiently closed so as to not add bounded subset of μ:

Definition 2.3. Suppose that ν is some ordinal. A poset $\mathbb{P}$ is said to be *ν-strategically closed* if Player II has a winning strategy in the following game $G(\mathbb{P}, \nu)$ of length ν:

I	p_1		p_3	$\cdots$		$p_{\omega+1}$		$\cdots$
II		p_2		$\cdots$	p_ω		$p_{\omega+2}$	$\cdots$

In this game the two players alternate building a descending chain $\langle p_\xi \colon 1 \leq \xi < \nu \rangle$ with Player II playing at all even ordinals (including limits) and Player II loses if he is unable to make a legal move.

Suppose that μ is some cardinal. We say that $\mathbb{P}$ is *$< \mu$-strategically closed* if it is ν-strategically closed for all $\nu < \mu$.

Lemma 2.4. *Suppose that $\mu < \kappa$ are cardinals with μ regular and κ inaccessible. Then $\mathbb{P}(\mu, < \kappa)$ is $< \mu$-strategically closed and κ-Knaster. Moreover, in the generic extension by $\mathbb{P}(\mu, < \kappa)$, $\kappa = \mu^+$ and $\square_{\mu,2}$ holds.*

Proof. We prove strategic closure. Fix $\nu < \mu$. We define a winning strategy for Player II in the game $G(\mathbb{P}(\mu, < \kappa), \nu)$. Suppose that $\xi < \nu$ is even and $\langle p_\zeta \colon 1 \leq \zeta < \xi \rangle$ have already been played.

If ξ is a successor ordinal then Player II plays an arbitrary extension p_ξ of $p_{\xi-1}$ such that $\sup(\operatorname{dom}(p_\xi))$ is strictly greater than $\sup(\operatorname{dom}(p_{\xi-1}))$. Now suppose that ξ is a nonzero limit ordinal and let

$$D = \bigcup_{1 \leq \zeta < \xi} \operatorname{dom}(p_\xi)$$

We define a condition p_ξ with domain $\bar{D} = D \cup \operatorname{Lim}(D)$ which extends all $\langle p_\zeta \colon 1 \leq \zeta < \xi \rangle$. First, if $\alpha \in D$ with $\operatorname{cf}(\alpha) < \mu$, we let $p_\xi(\alpha) = p_\zeta(\alpha)$ for any $1 \leq \zeta < \xi$ such that $\alpha \in \operatorname{dom}(p_\zeta)$. If $\alpha \in D$ with $\operatorname{cf}(\alpha) \geq \mu$ then for each $1 \leq \zeta < \xi$ let C_ζ be the unique element of $p_\alpha(\zeta)$ and let $p_\xi(\alpha) = \{C\}$, where

$$C = \bigcup_{\zeta < \xi} C_\zeta \cup \left\{ \sup \bigcup_{\zeta < \xi} C_\zeta \right\}$$

If $\alpha \in \operatorname{Lim} D \setminus D$ is below $\sup D$ then choose $\beta \in D$ least such that $\alpha < \beta$. Note that β must be a limit ordinal by conditions (2), (5) in the definition of $\mathbb{P}(\mu, < \kappa)$. We claim $\operatorname{cf}(\beta) \geq \mu$.

Suppose otherwise and let E be an element of $p_\xi(\beta)$ (note that $p_\xi(\beta)$ has already been defined). If α is not a limit point of E, then let γ be the least element of E above α. Then $\alpha < \gamma < \beta$, and by condition (5) in the definition of $\mathbb{P}(\mu, < \kappa)$ we have $\gamma \in D$, contradicting choice of β.

Thus $\operatorname{cf}(\beta) \geq \mu$ as desired, and so if we let E be the unique element of $p_\xi(\beta)$ (again, this has already been defined) then clause (4) in the definition of $\mathbb{P}(\mu, < \kappa)$ guarantees $\max(E) = \alpha$, and we may define $p_\xi(\alpha) = \{E\}$.

Finally, if $\alpha = \sup D$, let

$$p_\xi(\alpha) = \{\max(\mathrm{dom}(p_\zeta)) \colon 1 \le \zeta < \xi\}$$

It should be clear that p_ξ as defined above is a condition in $\mathbb{P}(\mu, <\kappa)$ and the strategy described is a winning strategy for Player II in $G(\mathbb{P}(\mu, <\kappa), \nu)$.

The rest of the lemma may be proved exactly as in [CS02]. □

Note that an argument similar to the one above will show that if $\mu = \aleph_1$ then $\mathbb{P}(\mu, <\kappa)$ is in fact countably closed.

In the proof of our consistency results it will be crucial that for $\mu < \kappa_0 < \kappa_1$ with μ regular and κ_0, κ_1 inacessible, $\mathbb{P}(\mu, <\kappa_0)$ may be viewed as a factor of $\mathbb{P}(\mu, <\kappa_1)$. In order to precisely state the necessary factorization result, we first define two auxilliary posets:

Suppose that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \colon \alpha < \mu^+ \rangle$ is a $\square_{\mu,2}$-sequence. We let $\mathbb{T} = \mathbb{T}_{\vec{\mathcal{C}}}$ be the poset of closed bounded $C \subseteq \mu^+$ of order-type $< \mu$ such that C threads $\langle \mathcal{C}_\alpha \colon \alpha \le \max C \rangle$ in the sense that $C \cap \alpha \in \mathcal{C}_\alpha$ for all $\alpha \in \mathrm{Lim}\, C$. For $C, D \in \mathbb{T}$, we set $D \le C$ if and only if D is an end-extension of C.

Finally, if G is the generic added by $\mathbb{P}(\mu, <\kappa_0)$, then $\mathbb{Q} = \mathbb{Q}_{\mu,\kappa_0,\kappa_1,G}$ is the poset defined in $V[G]$ by setting $q \in \mathbb{Q}$ iff:

(1) The domain of q is a set of limit ordinals in the interval (κ_0, κ_1) of size $< \mu$.

(2) If $\alpha \in \mathrm{dom}\, q$ has cofinality $< \mu$ then $1 \le |q(\alpha)| \le 2$ and each element of $q(\alpha)$ is a club subset of α with order type $< \mu$.

(3) If $\alpha \in \mathrm{dom}\, q$ has cofinality $\ge \mu$ then $q(\alpha) = \{C\}$, where C is a club subset of α with order type $< \mu$ such that $\max C \ge \sup(\mathrm{dom}\, q \cap \alpha)$.

(4) (Coherence) If $\alpha \in \mathrm{dom}\, q$, $C \in q(\alpha)$, and $\beta \in \mathrm{Lim}\, C$, then:

 (A) If $\beta > \kappa_0$, then $\beta \in \mathrm{dom}\, q$ and $C \cap \beta \in q(\beta)$.

 (B) If $\beta < \kappa_0$, then $C \cap \beta \in \mathcal{C}_\beta$, where $\langle \mathcal{C}_\beta \colon \beta < \kappa_0 \rangle$ is $\bigcup G$.

For two elements $p, q \in \mathbb{Q}_{\mu,\kappa_0,\kappa_1,G}$, we set $p \leq q$ iff:

(1) $\operatorname{dom} q \subseteq \operatorname{dom} p$.

(2) For all $\alpha \in \operatorname{dom} q$:

   (a) If α has cofinality $< \mu$ then $p(\alpha) = q(\alpha)$.

   (b) If α has cofinality $\geq \mu$, $p(\alpha) = \{C\}$, and $q(\alpha) = \{D\}$, then C is an end-extension of D.

Lemma 2.5. *Suppose that $\mu < \kappa_0 < \kappa_1$ are cardinals with μ regular and κ_0, κ_1 inaccessible, and $\dot{G}$ is the canonical name for the $\mathbb{P}(\mu, < \kappa_0)$-generic. Then if we let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}}$, $\dot{\mathbb{Q}} = \check{\mathbb{Q}}_{\mu,\kappa_0,\kappa_1,\dot{G}}$, there is an isomorphism between a dense subset of $\mathbb{P}(\mu, < \kappa_1)$ and a dense subset of $\mathbb{P}(\mu, < \kappa_0) * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$. In particular these two forcings are equivalent, and so informally we may view them as being equal.*

Proof. As in [CS02]. □

Lemma 2.6. *Suppose that $\mu < \kappa$ are cardinals with μ regular and κ inaccessible. Let $\mathbb{P} = \mathbb{P}(\mu, < \kappa)$, let $\dot{G}$ be the canonical name for the $\mathbb{P}$-generic, and let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}}$. Then there is a dense subset of $\mathbb{P} * \dot{\mathbb{T}}$ which is $< \mu$-closed and so in particular $\mathbb{P} * \dot{\mathbb{T}}$ is forcing equivalent to $\operatorname{Col}(\mu, \kappa)$.*

Proof. Let D be the dense set of conditions in $\mathbb{P} * \dot{\mathbb{T}}$ of the form $(p, \check{t})$ which are *flat* in the sense that $\max(\operatorname{dom} p) = \max t$. We claim that D is as desired. To see this, suppose that $\nu < \mu$ is a limit ordinal and let $\langle (p_\xi, \check{t}_\xi) \colon \xi < \nu \rangle$ be a descending sequence of conditions in D. We find a lower bound p^* for $\langle p_\xi \colon \xi < \nu \rangle$ as in the limit case of Lemma 2.4, except that we set

$$p^*\big(\sup_{\xi<\nu} (\max(\operatorname{dom} p_\xi))\big) = \{t^*\}$$

where

$$t^* = \bigcup_{\xi<\nu} t_\xi$$

Then $(p^*, \check{t}^*) \in D$ is our desired lower bound. Since D is $< \mu$-closed, $|D| = \kappa$, and D forces $|\kappa| = \mu$, D is forcing equivalent to $\mathrm{Col}(\mu, \kappa)$ by a well-known result due to Solovay (see, e.g. Lemma 2.3 of [Sak13]). □

2.2 Forcing to Add $\square(\kappa)$

We now describe the forcing introduced by Sakai [Sak13] which, given uncountable regular κ, introduces a $\square(\kappa)$ sequence.

Definition 2.7 ([Sak13])**.** Suppose that κ is an uncountable regular κ. Then $\mathbb{S} = \mathbb{S}(\square(\kappa))$ is defined as follows: Conditions in $\mathbb{S}$ are sequences $p = \langle C_\alpha \colon \alpha \leq \delta \rangle$ $(\delta < \kappa)$ such that

(i) If $\alpha \leq \delta$ is limit, then C_α is a club subset of α.

(ii) If $\alpha \leq \delta$ is a successor ordinal, say $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.

(iii) C_α threads $\langle C_\beta \colon \beta < \alpha \rangle$, i.e. if $\gamma \in \mathrm{Lim}(C_\alpha)$, then $C_\alpha \cap \gamma = C_\gamma$.

We refer to δ as the *height* of the condition p and write $\delta = \mathrm{height}(p)$ (note that strictly speaking, if we view p as a sequence, its length is $\delta + 1$ rather than δ).

Lemma 2.8 ([Sak13])**.** *Suppose that κ is an uncountable regular cardinal. Then $\mathbb{S}$ is κ-strategically closed.*

Proof. Consider the game $G = G(\mathbb{S}, \kappa)$ as described in Definition 2.3. We describe a winning strategy for Player II in this game. First, suppose that $\xi = \bar{\xi} + 1$ is an even successor ordinal and Player I has played $p_{\bar{\xi}}$ at stage $\bar{\xi}$. Then Player II chooses p_ξ which strictly extends $p_{\bar{\xi}}$. Next, suppose that ξ is a limit ordinal. Next, suppose that ξ is a limit ordinal and the players have played $\langle p_\zeta \colon 1 \leq \zeta < \xi \rangle$. Let

$$C_\xi = \{\mathrm{height}(p_\zeta) \colon 1 \leq \zeta < \xi\}$$

Then Player II plays

$$p_\xi = \bigcup_{1 \leq \zeta < \xi} p_\zeta \cup \{(\xi, C_\xi)\}$$

at stage ξ. By induction we may show that for each limit $\xi < \kappa$ the set C_ξ is club in height (p_ξ) and threads $\bigcup_{1 \leq \zeta < \xi} p_\zeta$. Therefore $p_\xi \in \mathbb{S}$ and p_ξ is a lower bound of $\langle p_\zeta \colon 1 \leq \zeta < \xi \rangle$, showing that this is a winning strategy as desired. $\square$

Lemma 2.9 ([Sak13])**.** *Let κ be an uncountable regular cardinal. Then $\mathbb{S}(\square(\kappa))$ forces $\square(\kappa)$.*

Proof. Let $\mathbb{S} = \mathbb{S}(\square(\kappa))$ and let G be $\mathbb{S}$-generic over V. We claim that $\bigcup G$ is a $\square(\kappa)$ sequence in $V[G]$. First note that for all $\alpha < \kappa$ $D_\alpha = \{p \in \mathbb{S} \colon \text{height}\,(p) \geq \alpha\}$ is dense in $\mathbb{S}$ by Lemma 2.8. Thus by genericity of G it remains only to verify that $\bigcup G$ may not be threaded in $V[G]$. Let $C \in V[G]$ be a club in κ and let $\dot{C}$ be a name such that $C = \dot{C}^G$ and $\Vdash_{\mathbb{S}}$ "$\dot{C}$ is a club." In order to show that C doesn't thread $\bigcup G$, it suffices by genericity to show that the following set is dense (in V):

$$E_{\dot{C}} = \left\{ p \in \mathbb{S} \colon p \Vdash \text{height}\,(\check{p}) \in \text{Lim}\,(\dot{C}) \wedge \dot{C} \cap \text{height}\,(\check{p}) \neq \check{p}(\text{height}\,(\check{p})) \right\}$$

To show this, work in V and fix $q \in \mathbb{S}$. Recursively construct a descending sequence $\langle q_n \colon n < \omega \rangle$ of conditions in $\mathbb{S}$ as well as an ascending sequence $\langle \alpha_n \colon n < \omega \rangle$ of ordinals in κ such that:

- $q_0 = q$.
- For all $n < \omega$, $q_{n+1} \Vdash \check{\alpha}_n \in \dot{C} \wedge \check{\alpha}_n > \text{height}\,(q_n)$.
- $\alpha_n < \text{height}\,(q_{n+1})$

Let $\alpha^* = \sup_{n<\omega} \alpha_n$, $D = \{\alpha_{2k} \colon k < \omega\}$. Then

$$p^* = \bigcup_{n<\omega} q_n \cup \{(\alpha^*, D)\}$$

is an element of $E_{\dot{C}}$ below q, as desired. $\square$

We now state the definition and basic properties of the threading forcing associated to these square sequences.

Definition 2.10 ([Sak13]). Suppose that κ is an uncountable regular cardinal and $\vec{C} = \langle C_\alpha \colon \alpha < \kappa \rangle$ is a $\square(\kappa)$ sequence. Then the poset for threading $\vec{C}$ is the poset $\mathbb{T} = \mathbb{T}_{\vec{C}}$ [1] whose conditions are closed bounded subsets D of κ such that if $\gamma \in \operatorname{Lim} D$, then $D \cap \gamma = C_\gamma$. The poset is ordered by end-extension.

We also have an analogue of Lemma 2.6 for the forcing $\mathbb{S}(\square(\kappa))$:

Lemma 2.11. *Suppose that κ is an uncountable regular cardinal, $\mathbb{S} = \mathbb{S}(\square(\kappa))$ is the poset for adding a $\square(\kappa)$ sequence, and $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}}$, where $\dot{G}$ is the canonical name for the $\mathbb{S}$-generic. Then $\mathbb{S} * \dot{\mathbb{T}}$ contains a $< \kappa$-closed dense subset.*

Proof. Let

$$D = \left\{ s * \check{t} \in \mathbb{S} * \dot{\mathbb{T}} \colon t \in V \wedge \operatorname{height}(s) = \max(t) \right\}$$

We claim that D is as desired. First we argue for density. With this in mind, let $p * \dot{q} \in \mathbb{S} * \dot{\mathbb{T}}$ be arbitrary. By Lemma 2.8 we may take $s \leq p$ and $r \in V$ such that $s \Vdash \dot{q} = \check{r}$. Extending s if necessary, we may assume without loss of generality that $\operatorname{height}(s) > \max(r)$. Let $t = r \cup \{\operatorname{height}(s)\}$. Then $s * \check{t}$ is in D and extends $p * \dot{q}$ as desired.

For $< \kappa$-closure, suppose that $\beta < \kappa$ is a limit ordinal and $\langle s_\alpha * \check{t}_\alpha \colon \alpha < \beta \rangle$ is a descending sequence of elements of D. If we let

$$\begin{aligned} t^* &= \bigcup_{\alpha < \beta} t_\alpha \\ s^* &= \bigcup_{\alpha < \beta} s_\alpha \cup \left\{ \left(\sup_{\alpha < \beta} \operatorname{height}(s_\alpha), t^* \right) \right\} \end{aligned}$$

then $s^* * \check{t}^*$ is the desired lower bound of this sequence. $\square$

[1] Recall that the forcing to thread a $\square_{\mu,2}$ sequence described in the previous section was also denoted by $\mathbb{T}$. We trust that no confusion will be introduced by employing the same symbol for both forcings, as in the following it should be clear from context which type of square sequence we're threading.

CHAPTER 3

The Consistency of "$\square_{\omega_1,2}$ + Chang's Conjecture" From an ω_1-Erdős Cardinal

3.1 Background

Chang's Conjecture is a model-theoretic principle asserting a strengthening of the Löwenheim-Skolem Theorem [Cha65]. Chang's Conjecture was originally shown to be consistent assuming the existence of a Ramsey cardinal by Silver (see [KM78]) and this assumption was later weakened to the existence of an ω_1-Erdős cardinal [DL89]. This result is best possible, since Chang's Conjecture implies that ω_2 is ω_1-Erdős in the core model [DJK81].

Chang's Conjecture is known to be incompatible with Jensen's square principle $\square_{\omega_1}$ (see [Tod07]) but was recently shown to be consistent with Schimmerling's square principle $\square_{\omega_1,2}$ by Sakai [Sak13], assuming the existence of a measurable cardinal. In light of this consistency upper bound, Sakai posed the following:

Question 3.1. What is the consistency strength of the conjunction of Chang's Conjecture with $\square_{\omega_1,2}$?

In Corollary 3.11 we show that the consistency of the given statement follows from the existence of an ω_1-Erdős cardinal, answering Sakai's question. Section 3.2.1 will cover some basic preliminaries, such as the definition of the relevant square principle and large cardinal. In Section 3.2.2 we describe our forcing poset. In Silver's consistency proof, he used what is now called a Silver forcing poset–a modification of the Levy Collapse forcing which allows larger supports [KM78]. Cummings and Schimmerling [CS02] have introduced another variant of the Levy Collapse forcing which collapses inaccessible κ to ω_2 while simultaneously adjoining a square sequence. Our forcing will be a hybrid of these two posets–in other words it will be a "Silverized" Cummings-Schimmerling poset.

Finally, in Section 3.2.3 we give the proof of our result, which is based on the methods of [Sak13] and [DL89].

In Section 3.3 we investigate the relation between weak square principles and model theoretic transfer properties (i.e., generalizations of Chang's Conjecture) of the form $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ for $\kappa \geq \aleph_1$. Sakai proved the following:

Theorem 3.2 (Sakai, [Sak13])**.** *Suppose that* $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$*, where* κ *is an uncountable cardinal and* λ *is a cardinal* $> \kappa$*. Moreover, suppose that either of the following holds:*

(I) $\lambda^{<\lambda} = \lambda$

(II) $\kappa < \aleph_{\omega_1}$*, and there are strictly more regular cardinals in the interval* $[\aleph_0, \kappa]$ *than in the interval* $(\kappa, \lambda]$*.*

Then $\square_{\lambda,\kappa}$ *fails.*

Although Theorem 3.2 imposes substantial constraints on the interaction of weak square principles and model theoretic transfer properties, there are many instances where it does not apply. For example, it does not answer the question of whether $(\aleph_4, \aleph_3) \twoheadrightarrow (\aleph_2, \aleph_1)$ is incompatible with $\square_{\omega_3,2}$ when $2^{\aleph_2} > \aleph_3$.

In light of these limitations, Sakai posed the following question:

Question 3.3 (Sakai, [Sak13])**.** Let κ be an uncountable cardinal and λ a cardinal $> \kappa$. Does $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ imply the failure of $\square_{\lambda,2}$?

We answer this question in the affirmative in Corollary 3.19 (in fact we obtain the failure of $\square_{\lambda,\omega}$ under these hypotheses and more under slightly stronger hypotheses–see Corollaries 3.20 and 3.21). Taking $\kappa = \aleph_1$, $\lambda = \aleph_3$ in this theorem shows that indeed $(\aleph_4, \aleph_3) \twoheadrightarrow (\aleph_2, \aleph_1)$ is incompatible with $\square_{\omega_3,2}$, regardless of the value of $2^{\aleph_2}$. Work in this chapter is joint with Itay Neeman.

3.2 The Consistency of Chang's Conjecture and $\square_{\omega_1,2}$ from an ω_1-Erdős Cardinal

3.2.1 Preliminaries

In the following, for any cardinal θ we denote by $H(\theta)$ the collection of all sets whose transitive closure has size $< \theta$. We frequently confuse a structure and its underlying set. I.e., if $\mathcal{M} = \langle M, \ldots \rangle$ is a structure and α is an ordinal, we write $\alpha \subseteq \mathcal{M}$ to mean $\alpha \subseteq M$. All structures we consider have at most countably many symbols in their signature.

Definition 3.4. *Chang's Conjecture* is the assertion that for any structure $\mathcal{N}$ with $\omega_2 \subseteq \mathcal{N}$, there exists $\mathcal{M} \preceq \mathcal{N}$ such that $|\mathcal{M}| = \aleph_1$ and $|\mathcal{M} \cap \omega_1| = \aleph_0$.

We observe that to verify Chang's Conjecture it suffices to verify it for models with underlying set $H(\omega_2)$:

Claim 3.5 (Folklore)**.** *Suppose that for all structures $\mathcal{H} = \langle H(\omega_2), \ldots \rangle$ there exists $\mathcal{M} \preceq \mathcal{H}$ of cardinality $\aleph_1$ such that $|\mathcal{M} \cap \omega_1| = \aleph_0$. Then Chang's Conjecture holds.*

Proof. This is a standard model-theoretic argument. Suppose that $\mathcal{N} = \langle N, R_1, R_2, \ldots \rangle$ is any structure with $\omega_2 \subseteq N$. We may assume without loss of generality that $|\mathcal{N}| = \aleph_2$. Let $\pi \colon N \to H(\omega_2)$ be any injection which is the identity on ω_2. Let $\mathcal{H} = \langle H(\omega_2), \tilde{N}, \tilde{R}_1, \tilde{R}_2, \ldots \rangle$, where $\tilde{N}$ is a predicate representing membership in $\pi[N]$ and $\tilde{R}_i$ is a predicate representing R_i in the natural way. By our assumption there is $\mathcal{M} \preceq \mathcal{H}$ of cardinality $\aleph_1$ such that $|\mathcal{M} \cap \omega_1| = \aleph_0$. Pulling back via π, we get the desired submodel of $\mathcal{N}$. $\square$

In order to obtain our consistency result, we will need to make use of a large cardinal hypothesis:

Definition 3.6. A cardinal κ is said to be ω_1-*Erdős* if for any partition $f \colon [\kappa]^{<\omega} \to 2$, there is $H \in [\kappa]^{\omega_1}$ which is homogeneous for f.

Lemma 3.7 (Silver). *If κ is ω_1-Erdős, then for any structure $\mathcal{M}$ with $\kappa \subseteq \mathcal{M}$, there is a set of indiscernibles $I \in [\kappa]^{\omega_1}$ for $\mathcal{M}$. Morever, if $\mathcal{M}$ has underlying set $H(\kappa)$ and includes among its predicates some $\lhd$ which is a well-ordering of its universe, we may take the set of indiscernibles I to be* remarkable *in the sense that for any $\gamma \in I$, $I \setminus \gamma$ is a set of indiscernibles for $\langle \mathcal{M}, (\delta)_{\delta<\gamma} \rangle$.*

Proof. See [Kan09]. □

3.2.2 The Poset

Our poset $\mathbb{P}$ is a "Silverized" version of the one described in Section 2.1 in the sense that we modify the poset described there to allow conditions with ω_1-sized support. We define $\mathbb{P} = \mathbb{P}_\kappa$ as follows: set $p \in \mathbb{P}$ iff p is a function so that

(1) The domain of p is a closed $\leq \omega_1$-sized set of limit ordinals less than κ.

(2) If $\operatorname{cf} \alpha = \omega$ and $\alpha \in \operatorname{dom} p$ then $1 \leq |p(\alpha)| \leq 2$ and each set in $p(\alpha)$ is a club subset of α with countable order type.

(3) If $\operatorname{cf} \alpha = \omega_1$ and $\alpha \in \operatorname{dom} p$ then $p(\alpha) = \{C\}$ where C is a club subset of α with order type ω_1.

(4) If $\operatorname{cf} \alpha \geq \omega_2$ then $p(\alpha) = \{C\}$ where C is a closed bounded subset of α with countable order type such that $\max C \geq \sup(\operatorname{dom} p \cap \alpha)$.

(5) If $\alpha \in \operatorname{dom} p$, $C \in p(\alpha)$ and $\beta \in \lim(C)$, then $\beta \in \operatorname{dom} p$ and $C \cap \beta \in p(\beta)$.

(6) The supremum of $\operatorname{otp} C$ taken over all $C \in p(\alpha)$, $\operatorname{cf} \alpha \geq \omega_2$, is strictly below ω_1.

For two elements $p, q \in \mathbb{P}_\kappa$, we set $p \leq q$ iff:

1. $\operatorname{dom} q \subseteq \operatorname{dom} p$

2. For all $\alpha \in \operatorname{dom} q$:

(a) If $\operatorname{cf}\alpha \in \{\omega, \omega_1\}$, then $p(\alpha) = q(\alpha)$.

(b) If $\operatorname{cf}\alpha \geq \omega_2$, $p(\alpha) = \{C\}$ and $q(\alpha) = \{D\}$, then C is an end-extension of D in the sense that $D = C \cap (\max(D) + 1)$.

Lemma 3.8. *Suppose that κ is inaccessible. Then $\mathbb{P} = \mathbb{P}_\kappa$ is κ-c.c. and countably closed, and collapses κ to $\aleph_2$ while adding a $\square_{\omega_1,2}$-sequence.*

Proof. The proof is very similar to that of the corresponding result in [CS02]. The fact that $\mathbb{P}$ is κ-c.c. follows from a standard Δ-system argument. If we can show that $\mathbb{P}$ is countably closed, then the second conclusion follows immediately. So suppose that $\langle p_n \colon n < \omega\rangle$ is a decreasing sequence of conditions.

Let X be the set of $\alpha \in \bigcup_{n<\omega} \operatorname{dom} p_n$ such that the value of $p_n(\alpha)$ does not eventually stabilize and let

$$Y = \{\sup_{n<\omega} \max p_n(\alpha) \colon \alpha \in X\}$$

Observe that $Y \cap \left(\bigcup_{n<\omega} \operatorname{dom} p_n\right) = \emptyset$, since if $\alpha \in X$, then the fact that $\max(p_n(\alpha)) \geq \sup(\operatorname{dom} p_n \cap \alpha)$ for every n gives

$$\sup_{n<\omega} \max p_n(\alpha) \notin \bigcup_{n<\omega} \operatorname{dom} p_n$$

We will define a condition p_ω with domain $\left(\bigcup_{n<\omega} \operatorname{dom} p_n\right) \cup Y$ which is a lower bound for $\langle p_n \colon n < \omega\rangle$. First, if $\alpha \in \bigcup_{n<\omega} \operatorname{dom} p_n \setminus X$, let $p_\omega(\alpha)$ be the eventual value of the sequence $\langle p_n(\alpha) : n < \omega\rangle$. If $\alpha \in X$, then set

$$p_\omega(\alpha) = \bigcup_{n<\omega} p_n(\alpha) \cup \{\sup_{n<\omega} \max p_n(\alpha)\}$$

Finally, if $\alpha \in Y$, then $\alpha = \sup_{n<\omega} \max p_n(\beta)$ for a unique $\beta \in X$, and we set

$$p_\omega(\alpha) = \bigcup_{n<\omega} p_n(\beta) \cup \{\sup_{n<\omega} \max p_n(\beta)\}$$

for this β. We refer to the condition p_ω defined above as the *canonical lower bound* of $\langle p_n \colon n < \omega\rangle$. $\square$

We also define a *threading poset* for a given $\square_{\omega_1,2}$-sequence. Supposing that $\vec{\mathcal{C}} = \langle C_\alpha \colon \alpha < \omega_2 \rangle$ is a such a sequence, we let $\mathbb{T} = \mathbb{T}_{\vec{\mathcal{C}}}$ be the poset of closed bounded subsets C of ω_2 of countable order type such that C threads $\langle \mathcal{C}_\alpha \colon \alpha \leq \max C \rangle$ in the sense that $C \cap \alpha \in \mathcal{C}_\alpha$ for all α which are limit points of C.

If $C, D \in \mathbb{T}$, then we set $C \leq D$ if and only if C is an end-extension of D.

Finally, suppose that $\mu < \kappa$ are two inaccessible cardinals. If G is the generic added by $\mathbb{P}_\mu$, then $\mathbb{Q} = \mathbb{Q}_{\mu,\kappa,G}$ is the poset in $V[G]$ defined by setting $q \in \mathbb{Q}$ iff:

(a) $\operatorname{dom} q$ is a closed $\leq \omega_1$-sized set of limit ordinals in the interval (μ, κ).

(b) If $\operatorname{cf} \alpha = \omega$ and $\alpha \in \operatorname{dom} q$, then $1 \leq |q(\alpha)| \leq 2$ and each element of $q(\alpha)$ is a club with countable order type.

(c) If $\operatorname{cf} \alpha = \omega_1$ and $\alpha \in \operatorname{dom} q$ then $q(\alpha) = \{C\}$ where C is a club subset of α with order type ω_1.

(d) If $\operatorname{cf} \alpha \geq \omega_2$, then $q(\alpha) = \{C\}$ where C is a closed bounded subset of α with countable order type such that $\max C \geq \sup(\operatorname{dom} q \cap \alpha)$.

(e) If $\alpha \in \operatorname{dom} q$, $C \in q(\alpha)$, and $\beta \in \lim C$, then:

  (A) If $\beta > \mu$, then $\beta \in \operatorname{dom} q$ and $C \cap \beta \in q(\beta)$.

  (B) If $\beta < \mu$, then $C \cap \beta \in \mathcal{C}_\beta$, where $\langle \mathcal{C}_\beta \colon \beta < \mu \rangle$ is $\bigcup G$.

(f) The supremum of $\operatorname{otp} C$ taken over all $C \in q(\alpha)$, $\operatorname{cf} \alpha \geq \omega_2$, is strictly below ω_1.

For two elements $p, q \in \mathbb{Q}_{\mu,\kappa}$, we set $p \leq q$ iff:

(1) $\operatorname{dom} q \subseteq \operatorname{dom} p$

(2) For all $\alpha \in \operatorname{dom} q$:

  (a) If $\operatorname{cf} \alpha \in \{\omega, \omega_1\}$, then $p(\alpha) = q(\alpha)$.

(b) If $\operatorname{cf} \alpha \geq \omega_2$, $p(\alpha) = \{C\}$, $q(\alpha) = \{D\}$, then C is an end-extension of D.

Claim 3.9. *Suppose that μ, κ are inaccessible cardinals with $\mu < \kappa$, and $\dot{G}$ is the canonical name for the $\mathbb{P}_\mu$-generic. Then if we let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}}$, $\dot{\mathbb{Q}} = \check{\mathbb{Q}}_{\mu,\kappa,\dot{G}}$, there is an isomorphism between a dense subset of $\mathbb{P}_\kappa$ and a dense subset of $\mathbb{P}_\mu * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$. In particular these two forcings are equivalent, so informally we may view them as being equal.*

Proof. As in [CS02]. □

3.2.3 The Proof

Theorem 3.10. *Suppose that κ is an ω_1-Erdős cardinal. Let $\mathbb{P} = \mathbb{P}_\kappa$. Then for any $\mathbb{P}$-generic G, $V[G]$ satisfies Chang's Conjecture.*

Corollary 3.11. The existence of an ω_1-Erdős cardinal is equiconsistent with "Chang's Conjecture plus $\square_{\omega_1,2}$."

Proof of Corollary 3.11. By Theorem 3.10 and Lemma 3.8 an ω_1-Erdős cardinal suffices for the consistency of Chang's Conjecture plus $\square_{\omega_1,2}$. By [DJK81], the consistency of Chang's Conjecture implies that of the existence of of an ω_1-Erdős cardinal. □

Proof of Theorem 3.10. Suppose that G is a $\mathbb{P}$-generic over V. Then $\omega_2^{V[G]} = \kappa$ and $(H(\kappa))^{V[G]} = H(\kappa)[G]$. Let $\mathcal{H} = \langle H(\kappa), \in, \dot{R} \rangle$, which we view as a name for a structure $\mathcal{H}[G]$ with underlying set $H(\kappa)[G]$ and predicate $R = \dot{R}^G \subseteq H(\kappa)[G]$.

We seek a condition $p^* \in \mathbb{P}$ and a name $\dot{\mathcal{A}}$ for an elementary substructure $\mathcal{A}$ of $\mathcal{H}[G]$ such that p^* forces $|\dot{\mathcal{A}}| = \aleph_1$, $|\mathcal{A} \cap \omega_1| = \aleph_0$. With this in mind, let $I = \{\iota_\alpha \colon \alpha < \omega_1\}$ be a collection of remarkable indiscernibles for $\mathcal{H}$. For each $\alpha < \omega_1$, let $I_\alpha = \{\iota_\delta \colon \delta < \omega\alpha\}$ be the set of the first $\omega\alpha$ indiscernibles and let $\gamma_\alpha = \iota_{\omega\alpha}$. Let $\mathcal{M}_\alpha$ be the Skolem Hull of I_α in $\mathcal{H}$.

We construct a sequence $\langle p^*_\alpha \colon 1 \leq \alpha < \omega_1 \rangle$ by induction on α so that:

(a) If $1 \leq \alpha < \beta < \omega_1$ then $p^*_\beta \leq p^*_\alpha$.

(b) p^*_α is a master condition for $\mathbb{P}$ over $\mathcal{M}_\alpha$.

(c) p^*_α is an element of $\mathbb{P}_{\gamma_\alpha}$.

We begin with the base case $\alpha = 1$. Consider the set $\mathbb{P} \cap \mathcal{M}_1 = (\mathbb{P}_{\mathrm{ON}})^{\mathcal{M}_1}$, which is a proper class in $\mathcal{M}_1$. Observe that since $\mathcal{M}_1$ is elementary in $\mathcal{H}$, $\mathcal{M}_1$ satisfies "$\mathbb{P}$ has the $<$-ON chain condition." In other words, $\mathcal{M}_1$ believes that every antichain in $\mathbb{P}$ is a set. For each antichain A in $\mathcal{M}_1$, let $A^{\downarrow} = \{p \in \mathbb{P} \colon (\exists q \in A)\, p \leq q\}$ be the downwards closure of A. Let $\{A_i \colon i < \omega\}$ enumerate the collection of all maximal antichains which are elements of $\mathcal{M}_1$. By induction we may construct a descending sequence $\{r_i \colon i < \omega\}$ of elements of $\mathbb{P}$ such that $r_i \in A_i^{\downarrow} \cap \mathcal{M}_1$. Let $p^*_1 \in \mathbb{P}$ be the canonical lower bound for the sequence $\{r_i \colon i < \omega\}$. Then p^*_1 is a master condition for $\mathbb{P}$ over $\mathcal{M}_1$ and is an element of $\mathbb{P}_{\gamma_1}$, as desired.

Next suppose that α is limit. Choose a sequence $\langle \alpha_n \colon n < \omega \rangle$ cofinal in α, and let p^*_α be the canonical lower bound for $\langle p^*_{\alpha_n} \colon n < \omega \rangle$. It should be clear that properties (a)-(c) are satisfied, since $\mathbb{P} \cap \mathcal{M}_\alpha = \bigcup_{n<\omega} (\mathbb{P} \cap \mathcal{M}_{\alpha_n})$, and $\mathbb{P} \cap \mathcal{M}_\alpha = (\mathbb{P}_{\mathrm{ON}})^{\mathcal{M}_\alpha}$ has the $<$-ON chain condition in $\mathcal{M}_\alpha$.

Finally we consider the case where $\alpha = \bar{\alpha} + 1$ is a successor ordinal. We distinguish between the case where $\bar{\alpha}$ is limit and where $\bar{\alpha}$ is itself a successor ordinal, considering first the latter. Since $p^*_{\bar{\alpha}}$ was chosen to be a master condition for $\mathbb{P}$ over $\mathcal{M}_{\bar{\alpha}}$, we have

$$p^*_{\bar{\alpha}} \Vdash \mathcal{M}_{\bar{\alpha}}[\dot{G}] \preceq \mathcal{H}[\dot{G}] \wedge \mathrm{ON} \cap \mathcal{M}_{\bar{\alpha}}[\dot{G}] = \mathrm{ON} \cap \mathcal{M}_{\bar{\alpha}}$$

Consider $\mathcal{M}_\alpha$. By remarkability of the indiscernibles which generate $\mathcal{M}_\alpha$, we have $H(\gamma_{\bar{\alpha}}) \cap \mathcal{M}_\alpha = \mathcal{M}_{\bar{\alpha}}$ and $\mathbb{P}_{\gamma_{\bar{\alpha}}} \cap \mathcal{M}_\alpha = \mathbb{P} \cap \mathcal{M}_{\bar{\alpha}}$. Moreover, $p^*_{\bar{\alpha}}$ is a master condition for the forcing $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ over the model $\mathcal{M}_\alpha$, since $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ has the $\gamma_{\bar{\alpha}}$-c.c. and therefore every antichain of $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ in $\mathcal{M}_\alpha$ is as element of $H(\gamma_{\bar{\alpha}}) \cap \mathcal{M}_\alpha = \mathcal{M}_{\bar{\alpha}}$. So if we let $\dot{G}_{\gamma_{\bar{\alpha}}}$ be the canonical name for the $\mathbb{P}_{\gamma_{\bar{\alpha}}}$-generic, then

$$p^*_{\bar{\alpha}} \Vdash \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}] \preceq \mathcal{H}[\dot{G}_{\gamma_{\bar{\alpha}}}] \wedge \mathrm{ON} \cap \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}] = \mathrm{ON} \cap \mathcal{M}_\alpha$$

Working in V, let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{G}_{\gamma_{\bar{\alpha}}}}$ be the canonical name for the threading forcing associated to $G_{\gamma_{\bar{\alpha}}}$. Let $\{\dot{B}_i \colon i < \omega\}$ enumerate all names in $\mathcal{M}_\alpha$ which are forced by $p^*_{\bar{\alpha}}$ to be maximal antichains of $\dot{\mathbb{T}}$. By induction we may construct a descending sequence $\left\{\check{t}_i \colon i < \omega\right\}$ of "check-names" (by which we mean canonical names for elements of V) for elements of $\mathbb{T} = \dot{\mathbb{T}}^{G_{\gamma_{\bar{\alpha}}}}$ such that

$$p^*_{\bar{\alpha}} \Vdash \check{t}_i \in \dot{B}_i^{\downarrow} \cap \mathcal{M}_\alpha[\dot{G}_{\gamma_{\bar{\alpha}}}]$$

where $\dot{B}_i^{\downarrow}$ is a name for the downwards closure of $B_i = \dot{B}_i^{G_{\gamma_{\bar{\alpha}}}}$ in $\mathbb{T}$. Observe that we may take canonical names for elements of V $\check{t}_i$ rather than merely arbitrary names $\dot{t}_i$ since $p^*_{\bar{\alpha}}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}}$ over $\mathcal{M}_\alpha$.

Still working in V, we let

$$t = \bigcup_{i<\omega} t_i$$

$$p^{**}_{\bar{\alpha}} = p^*_{\bar{\alpha}} \cup \{(\sup(\mathcal{M}_{\bar{\alpha}} \cap \kappa), \{t\})\}$$

Then $p^*_{\bar{\alpha}} * \check{t}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}} * \dot{\mathbb{T}}$ over $\mathcal{M}_\alpha$. Proceeding as above, we may find $\dot{q} \in \dot{\mathbb{Q}} = \mathbb{Q}_{\gamma_{\bar{\alpha}}, \mathrm{ON}, \dot{G}_{\gamma_{\bar{\alpha}}}}$, such that $p^*_{\bar{\alpha}} * \check{t} * \dot{q}$ is a master condition for $\mathbb{P}_{\gamma_{\bar{\alpha}}} * \dot{\mathbb{T}} * \dot{\mathbb{Q}}$ over $\mathcal{M}_\alpha$. Thus if we set $p^*_\alpha = p^{**}_{\bar{\alpha}} * \check{t} * \dot{q}$, we may view p^*_α as a master condition for $\mathbb{P}$ over $\mathcal{M}_\alpha$ which extends $p^*_{\bar{\alpha}}$. We note that $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa)) = \{t\}$.

For nonzero limit $\bar{\alpha}$, the construction is exactly as above, except we modify $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa))$ to be $\{t, F\}$, where t is a master condition for the threading poset associated to the generic for $\mathbb{P}_{\gamma_\alpha}$ (as above) and $F = \{\sup(\mathcal{M}_\delta \cap \kappa) \colon \delta < \bar{\alpha}\}$, rather than merely taking $p^*_\alpha(\sup(M_{\bar{\alpha}} \cap \kappa))$ to be $\{t\}$.

Observe that this is the only place in the proof where we use the allowed "two-ness" of the square sequence. Moreover, in adding F we preserve the coherence property since its initial segments of limit length were put on the square sequence at earlier successor of limit stages.

Finally, at the end of the construction we set

$$p^* = \bigcup_{\alpha<\omega_1} p^*_\alpha \cup \big\{(\sup((\bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha) \cap \kappa), F^*)\big\}$$

where $F^* = \{\sup(\mathcal{M}_\alpha \cap \kappa) \colon \alpha < \kappa\}$. The construction ensures that this is a condition in $\mathbb{P} = \mathbb{P}_\kappa$. In particular, successor of limit stages ensure that the initial segments of limit length of F^* appear on the square sequence, and so when adding F^* there is no danger of violating coherence. Moreover, p^* is a master condition for $\mathbb{P}$ over $\mathcal{M} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Thus p^* forces that $\mathcal{M}[G]$ is the desired elementary submodel of $\mathcal{H}[G]$. □

3.3 Chang's Conjecture vs. Squares

In this section we concern ourselves with generalizations of Chang's Conjectures to higher cardinals.

Definition 3.12. Suppose that $\tau \leq \kappa < \lambda$ are cardinals. We write $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ if for every structure $\mathcal{N}$ with $\lambda^+ \subseteq \mathcal{N}$, there exists $\mathcal{M} \preceq \mathcal{N}$ such that $|\mathcal{M}| = \kappa^+$ and $|\mathcal{M} \cap \lambda| = \kappa$.

Similarly, we write $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ if for every structure $\mathcal{N}$ with $\lambda^+ \subseteq \mathcal{N}$, there exists $\mathcal{M} \preceq \mathcal{N}$ such that $|\mathcal{M}| = \kappa^+$, $|\mathcal{M} \cap \lambda| = \kappa$, and $\tau \subseteq \mathcal{M}$.

Observe that Chang's Conjecture is equivalent to $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ and that $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ is equivalent to $(\lambda^+, \lambda) \twoheadrightarrow_\omega (\kappa^+, \kappa)$ for any infinite cardinals $\kappa < \lambda$. Moreover, we also have:

Lemma 3.13. *Suppose that $\tau \leq \kappa < \lambda$ are infinite cardinals and there are at most τ many cardinals between κ and λ. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ implies $(\lambda^+, \lambda) \twoheadrightarrow_\kappa (\kappa^+, \kappa)$.*

Proof. The lemma is implicit in [Sak13]. Specifically, the conclusion of the lemma holds by following the argument of Case (2) of Lemma 4.15 in [Sak13]. □

Lemma 3.14. *Suppose that $\tau \leq \kappa < \lambda$ are infinite cardinals such that $\lambda^\tau = \lambda$. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$.*

Proof. Take $B = \tau$ in Case (1) of Lemma 4.15 in [Sak13]. $\square$

In the argument below we make use of the following claim without comment:

Claim 3.15. *Suppose that for all sufficiently large θ and all structures $\mathcal{H} = \langle H(\theta), \in, \dots \rangle$ there exists $\mathcal{M} \preceq \mathcal{H}$ such that $|\mathcal{M} \cap \lambda^+| = \kappa^+$, $|\mathcal{M} \cap \lambda| = \kappa$, and $\tau \subseteq \mathcal{M}$. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$.*

The proof is entirely analogous to that of Claim 3.5.

Lemma 3.16 (Folklore)**.** *Suppose that $\kappa < \lambda$ are infinite cardinals and θ is a sufficiently large regular cardinal. Let M be an elementary substructure of $\langle H(\theta), \in \rangle$ such that $|M \cap \lambda^+| = \kappa^+$ and $|M \cap \lambda| = \kappa$. Then the order type of $M \cap \lambda^+$ is κ^+.*

Proof. Suppose otherwise for a contradiction. Since $|M \cap \lambda^+| = \kappa^+$, the order type of $M \cap \lambda^+$ must be strictly greater than κ^+. Let α be the κ^+ element of $M \cap \lambda^+$. Observe that $\alpha \geq \lambda$ (since there are only κ many elements of M below λ) and hence by elementarity $\lambda = |\alpha|$ is an element of M. Applying elementarity again, there is $f \in M$ which is a bijection from α to λ. In particular,

$$f``(M \cap \alpha) \subseteq M \cap \lambda$$

which is a contradiction since the left hand side has cardinality κ^+ (since f is a bijection) whereas the right hand side has cardinality κ. $\square$

Lemma 3.17 (Folklore)**.** *Suppose that M is an elementary substructure of $\langle H(\theta), \in \rangle$ for some sufficiently large θ and $\alpha \in M$. Letting $\mu = \operatorname{cf} \alpha$, if $f \in M$ is an increasing function from μ into α whose range is cofinal in α, then*

$$\sup (f``(M \cap \mu)) = \sup (M \cap \alpha)$$

Proof. Clearly $\sup(f``(M \cap \mu)) \leq \sup(M \cap \alpha)$, since $f \in M$ and $f\colon \mu \to \alpha$. For equality, suppose for a contradiction that

$$\sup(f``(M \cap \mu)) < \sup(M \cap \alpha)$$

and choose $\beta \in M \cap \alpha$ such that $\beta > \sup(f``(M \cap \alpha))$. By elementarity

$$M \models (\exists \xi \in \mu)(f(\xi) > \beta)$$

and so choosing $\xi_0 \in M \cap \mu$ to witness the existential statement above we have:

$$\sup(f``(M \cap \mu)) < \beta < f(\xi_0)$$

an obvious contradiction. □

Theorem 3.18. *Suppose that $\kappa < \lambda$ are uncountable cardinals and $\tau \leq \kappa$ is infinite. Suppose moreover that $\square_{\lambda,\tau}$ holds. Then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ fails.*

Corollary 3.19. Suppose that $\kappa < \lambda$ are uncountable cardinals and $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ holds. Then $\square_{\lambda,\omega}$ fails.

Proof. Immediate from the theorem and the fact that $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ is equivalent to $(\lambda^+, \lambda) \twoheadrightarrow_\omega (\kappa^+, \kappa)$. □

Corollary 3.20. Suppose that $\kappa < \lambda$ are uncountable cardinals and there are at most countably many cardinals between κ and λ. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\kappa}$.

Proof. This follows immediately from Theorem 3.18 and Lemma 3.13 by taking $\tau = \omega$. □

Observe that the same argument shows that if there are at most τ many cardinals between κ and λ then $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\kappa}$.

Corollary 3.21. Suppose that $\kappa < \lambda$ are uncountable cardinals and $\tau \leq \kappa$ is some infinite cardinal with $\lambda^\tau = \lambda$. Then $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$ implies the failure of $\square_{\lambda,\tau}$.

Proof. Immediate from Theorem 3.18 and Lemma 3.14. □

Proof of Theorem 3.18. Suppose for a contradiction that $\square_{\lambda,\tau}$ held in conjunction with $(\lambda^+, \lambda) \twoheadrightarrow_\tau (\kappa^+, \kappa)$, and let $\vec{\mathcal{C}} = \langle \mathcal{C}_\xi \colon \xi < \lambda^+ \rangle$ be a $\square_{\lambda,\tau}$ sequence. Choose M an elementary substructure of $\langle H(\theta), \in, \vec{\mathcal{C}} \rangle$ (for sufficiently large θ) such that $|M \cap \lambda^+| = \kappa^+$, $|M \cap \lambda| = \kappa$, and $\tau \subseteq M$.

Fix a club $C^* \in \mathcal{C}_{\sup(M \cap \lambda^+)}$. By Lemma 3.16, we may choose a club D in $\sup(M \cap \lambda^+)$ of ordertype κ^+. We assume moreover that D consists only of limits of ordinals in M.

Claim 3.22. *For all sufficiently large $\alpha \in C^*$, the ordertype of $C^* \cap \alpha$ is not an element of M.*

Proof. These ordertypes are distinct elements of λ, and since $|M \cap \lambda| = \kappa$, at most κ of them can belong to M. Since the cofinality of $\sup C^* = \sup(M \cap \lambda^+)$ is κ^+, the result follows immediately. □

Claim 3.23. *For all sufficiently large $\alpha \in \operatorname{Lim} C^*$, $\alpha \notin M$.*

Proof. Choose $\alpha \in \operatorname{Lim} C^*$ and note that $C^* \cap \alpha \in \mathcal{C}_\alpha$. If $\alpha \in M$, then $\mathcal{C}_\alpha \subseteq M$ (since $|\mathcal{C}_\alpha| \leq \tau$ and $\tau \subseteq M$) and so in particular $C^* \cap \alpha \in M$, giving $\operatorname{otp}(C^* \cap \alpha) \in M$. By Claim 3.22, this may happen for only boundedly many $\alpha \in C^*$. □

For each α below $\sup(M \cap \lambda^+)$, let $\alpha^\uparrow$ denote the least element of M which is $\geq \alpha$.

Claim 3.24. *For all sufficiently large $\alpha \in \operatorname{Lim}(C^* \cap D)$, $\alpha^\uparrow$ is strictly greater than α.*

Proof. Immediate from Claim 3.23. □

Now define:

$$Z = \left\{ \mu \leq \lambda \colon \mu = \operatorname{cf}(\alpha^\uparrow) \text{ for unboundedly many } \alpha \text{ in } \operatorname{Lim}(C^* \cap D) \right\}$$

Claim 3.25. $|Z| \leq \kappa$.

Proof. For each $\alpha \in \mathrm{Lim}\,(C^* \cap D)$, $\alpha^\uparrow$ is an element of M below λ^+, and therefore its cofinality is an element of $M \cap (\lambda + 1)$, which has cardinality κ. □

Claim 3.26. *There is $\mu \in Z$ with $\mu \geq \kappa^+$.*

Proof. By Claim 3.25, it is enough to find unboundedly many $\alpha \in \mathrm{Lim}\,(C^* \cap D)$ such that $\mathrm{cf}\,(\alpha^\uparrow) \geq \kappa^+$.

Fix any $\alpha \in \mathrm{Lim}\,(C^* \cap D)$ large enough for Claim 3.24, with $\mathrm{cf}\,(\alpha) = \kappa$. Observe that there are unboundedly many such α since the ordertype of $C^* \cap D$ is κ^+. By choice of α, $\sup(M \cap \alpha^\uparrow) = \alpha < \alpha^\uparrow$. Then:

$$\kappa = \mathrm{cf}\,\alpha < \mathrm{cf}\,\alpha^\uparrow$$

by Lemma 3.17. □

Claim 3.27. $|Z| \geq 2$*.*

Proof. By Claim 3.25, it suffices to find disjoint $A_1, A_2 \subseteq \mathrm{Lim}\,(C^* \cap D)$ such that A_1, A_2 are unbounded and for any $\alpha_1 \in A$, $\alpha_2 \in A_2$, we have $\mathrm{cf}\,(\alpha_1^\uparrow) \neq \mathrm{cf}\,(\alpha_2^\uparrow)$.

To do so, choose distinct regular $\eta_1, \eta_2 \leq \kappa$. Observe that this is possible since κ is uncountable. Now let

$$A_1 = \{\alpha \in \mathrm{Lim}\,(C^* \cap D)\colon\ \mathrm{cf}\,\alpha = \eta_1\}$$
$$A_2 = \{\alpha \in \mathrm{Lim}\,(C^* \cap D)\colon\ \mathrm{cf}\,\alpha = \eta_2\}$$

Clearly A_1, A_2 are disjoint and unbounded. Moreover, for any $\alpha_1 \in A_1$, $\alpha_2 \in A_2$, we have $\mathrm{cf}\,(\alpha_1^\uparrow) \neq \mathrm{cf}\,(\alpha_2^\uparrow)$ by Lemma 3.17. □

Now to prove the theorem:

Fix distinct $\mu_1, \mu_2 \in Z$ with $\mu_1 > \mu_2$ and $\mu_1 \geq \kappa^+$. Fix $\alpha_1, \alpha_2 \in \mathrm{Lim}\,(C^* \cap D)$, large enough for Claims 3.22 and 3.23 and with $\alpha_1 < \alpha_2$, so that $\mathrm{cf}\,(\alpha_1^\uparrow) = \mu_1$ and $\mathrm{cf}\,(\alpha_2^\uparrow) = \mu_2$. Fix $E \in M$ cofinal in $\alpha_2^\uparrow$ of ordertype μ_2.

Let

$$U = \left\{ \sup\left(C \cap \alpha_1^{\uparrow}\right) + 1 \colon C \in \bigcup_{\xi \in E} \mathcal{C}_\xi \text{ with } \sup\left(C \cap \alpha_1^{\uparrow}\right) < \alpha_1^{\uparrow} \right\}$$

Claim 3.28. $\sup(C^* \cap \alpha_1^{\uparrow}) + 1 \in U$.

Proof. Note first that $C^* \cap \alpha_1^{\uparrow}$ is bounded in $\alpha_1^{\uparrow}$, since otherwise we would have $\alpha_1^{\uparrow} \in \operatorname{Lim} C^*$, and as $\alpha_1^{\uparrow} > \alpha_1$ is an element of M this would contradict choice of α_1. Now since E is club in $\alpha_2^{\uparrow}$ and belongs to M, we have $\alpha_2 = \sup(M \cap \alpha_2^{\uparrow}) \in E$, where the equality follows from Lemma 3.17. Since $C^* \cap \alpha_2 \in \mathcal{C}_{\alpha_2}$, and since $C^* \cap \alpha_2 \cap \alpha_1^{\uparrow} = C^* \cap \alpha_1^{\uparrow}$ is bounded in $\alpha_1^{\uparrow}$, it follows by definition that $\sup(C^* \cap \alpha_1^{\uparrow}) + 1$ is in U. □

We have $U \in M$ by elementarity and since the parameters used are in M. U has cardinality $< \mu_1$ by definition and since $\mu_1 > \max(\mu_2, \kappa)$. Since $\operatorname{cf}(\alpha_1^{\uparrow}) = \mu_1$, it follows that U is bounded in $\alpha_1^{\uparrow}$.

Moreover, since $U \in M$ we have $\sup U \in M$, and since $\sup U < \alpha_1^{\uparrow}$ it follows that $\sup U \leq \alpha_1$. But this contradicts Claim 3.28, since $\alpha_1 \in C^*$ and therefore

$$\sup(C^* \cap \alpha_1^{\uparrow}) + 1 \geq \alpha_1 + 1 > \alpha_1$$

□

CHAPTER 4

The Consistency of "$\square_{\kappa,2}$ + $\mathsf{SATP}(\kappa^+)$"

4.1 Background

Given some infinite cardinal κ, a κ-*Aronszajn tree* is a tree of height κ without cofinal branches all of whose levels have cardinality $< \kappa$. By classical results of König and Aronszajn, respectively, there are no $\aleph_0$-Aronszajn trees but there are $\aleph_1$-Aronszajn trees.

Of particular interest to us are special Aronszajn trees. For any successor cardinal κ^+, we say that a κ^+-Aronszajn tree T is *special* if there exists a function $f \colon T \to \kappa$ such that if $x <_T y$ then $f(x) \neq f(y)$. Following [GH16], if there are Aronszajn trees of height κ^+ and all such trees are special, we say that the *Special Aronszajn Tree Property* holds at κ^+, and denote this by $\mathsf{SATP}(\kappa^+)$.

By a result of Baumgartner, Malitz, and Reinhardt [BMR70] the forcing axiom $\mathsf{MA}_{\aleph_1}$ implies $\mathsf{SATP}(\aleph_1)$. Laver and Shelah [LS81] showed that $\mathsf{SATP}(\aleph_2)$ is consistent assuming the existence of a weakly compact cardinal. The forcing which achieves this result is a Levy Collapse of κ to $\aleph_2$ followed by an iteration of length $\geq \kappa^+$ of posets which successively specialize all new κ-Aronszajn trees arising in the extension.

Golshani and Hayut [GH16] showed that under the same assumption it is consistent that $\mathsf{SATP}(\aleph_1)$ and $\mathsf{SATP}(\aleph_2)$ hold simultaneously, and achieved a global result by showing that it is consistent that $\mathsf{SATP}(\kappa^+)$ holds simultaneously for all regular κ, assuming the existence of a proper class of supercompact cardinals. This result is achieved by adapting the methods of [LS81] to specialize all possible names for trees of height κ^+ while anticipating the specialization of trees of height κ.

We show that the result of Laver and Shelah may be improved by establishing the consistency of $\mathsf{SATP}(\aleph_2)$ with $\square_{\omega_1,2}$. By a result of Shelah and Stanley [SS88] $\mathsf{SATP}(\aleph_2)$ is incompatible with $\square_{\omega_1}$, so this result is optimal. Our result is obtained by using an iteration similar to that of Laver-Shelah, with the exception that we use a poset of Cummings and Schimmerling [CS02]–which collapses weakly compact κ to $\aleph_2$ while adding a $\square_{\omega_1,2}$-sequence simultaneously–in place of the Levy Collapse.

Furthermore, we show that our methods are compatible with the anticipatory framework of Golshani and Hayut, and thus we are also able to obtain the analogous global result–namely the consistency of $\mathsf{SATP}(\kappa^+)$ plus $\square_{\kappa,2}$ for all regular κ.

4.2 Specializing Trees with Anticipation

In this section we review the methods of [GH16] for specializing trees while anticipating subsequent forcing.

First we introduce the modified Baumgartner forcing which specializes a single tree while anticipating a single subsequent forcing.

Definition 4.1 ([GH16])**.** Suppose that $\mu < \kappa$ are regular cardinals in V and $\mathbb{I}^2 * \dot{\mathbb{I}}^1$ is a κ-c.c. two-step iteration which forces $\kappa = \mu^+$. Suppose moreover that $\dot{T}$ is an $\mathbb{I}^2 * \dot{\mathbb{I}}^1$-name for a κ-Aronszajn tree, which we view as a subset of $\kappa \times \mu$. Then $\mathbb{B}_{\mu,\mathbb{I}^1}(\dot{T})$ is defined in $V^{\mathbb{I}^2}$ as the poset of partial functions $f\colon \kappa \times \mu \to \mu$ of size $< \mu$ such that if $s, t \in \operatorname{dom} f$ and $f(s) = f(t)$, then

$$\Vdash_{\mathbb{I}^1}^{V^{\mathbb{I}^2}} \check{s} \perp_{\dot{T}} \check{t}$$

The forcing is ordered by reverse inclusion.

If μ is understood (as it usually is) then we suppress the dependence on μ and write $\mathbb{B}_{\mathbb{I}^1}(\dot{T})$ in place of $\mathbb{B}_{\mu,\mathbb{I}^1}(\dot{T})$.

Lemma 4.2 ([GH16])**.** *Suppose $\mu < \kappa$ are regular cardinals and, $\mathbb{I}^2 * \dot{\mathbb{I}}^1$ is a κ-c.c. forcing* *which forces $\kappa = \mu^+$, $\dot{T}$ is an $\mathbb{I}^2 * \mathbb{I}^1$-name for a κ-Aronszajn tree, and G is $\mathbb{I}^2$-generic. Then in $V[G]$ the following hold:*

(a) $\mathbb{B}_{\mathbb{I}^1}(\dot{T})$ is $< \mu$-closed.

(b) In the extension by the generic for $\mathbb{B}_{\mathbb{I}^1}(\dot{T})$ there is a function $F\colon \kappa \times \mu \to \mu$ which is a specializing function for the tree $\dot{T}[G][H]$ for any *$\mathbb{I}^1$-generic H.*

Now we describe the general form of iterations $\vec{\mathbb{I}}^2$ and $\vec{\mathbb{I}}^1$ such that $\vec{\mathbb{I}}^2$ specializes *all* κ-Aronszajn trees while anticipating forcing by $\vec{\mathbb{I}}^1$.

Definition 4.3. Suppose that $\mu < \kappa < \kappa^+ \leq \delta$ are regular cardinals in V, and the iterations

$$\vec{\mathbb{I}}^2 = \langle\langle \mathbb{I}^2_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^2_\gamma \colon \gamma < \delta\rangle\rangle$$
$$\dot{\vec{\mathbb{I}}}^1 = \langle\langle \dot{\mathbb{I}}^1_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^1_\gamma \colon \gamma < \delta\rangle\rangle$$

are as follows:

- $\mathbb{I}^2_1 = \mathbb{P}(\mu, < \kappa)$, the forcing which collapses κ to μ^+ while adding $\square_{\mu,2}$.
- $\mathbb{I}^2_\gamma$ is the iteration with $< \mu$-support of $\langle \dot{\mathbb{J}}^2_{\gamma'} \colon \gamma' < \gamma\rangle$. In other words, if γ is a limit ordinal of cofinality $\geq \mu$, then $\mathbb{I}^2_\gamma$ is the direct limit of $\langle \mathbb{I}^2_{\gamma'} \colon \gamma' < \gamma\rangle$, if γ is a limit ordinal of cofinality $< \mu$, then $\mathbb{I}^2_\gamma$ is the inverse limit of $\langle \mathbb{I}^2_{\gamma'} \colon \gamma' < \gamma\rangle$, and if $\gamma = \bar{\gamma} + 1$ is a successor ordinal then $\mathbb{I}^2_\gamma = \mathbb{I}^2_{\bar{\gamma}} * \dot{\mathbb{J}}^2_{\bar{\gamma}}$.
- Each $\dot{\mathbb{I}}^1_\gamma$ is an $\mathbb{I}^2_\gamma$-name for a μ-c.c. poset.
- $\dot{\mathbb{J}}^2_\gamma$ is a name for the poset $\mathbb{B}_{\mathbb{I}^1_\gamma}(\dot{T}_\gamma)$, where $\dot{T}_\gamma$ is an $\mathbb{I}^2_\gamma * \dot{\mathbb{I}}^1_\gamma$-name for a κ-Aronszajn tree, chosen according to some appropriate bookkeeping function.

Then we refer to $\vec{\mathbb{I}}^2$ as an "iteration which collapses κ to μ^+, adds $\square_{\mu,2}$ and specializes all κ-Aronszajn trees while anticipating the subsequent iteration $\vec{\mathbb{I}}^1$" (or some similar locution for the sake of brevity).

Definition 4.4 ([GH16]). Suppose that $\mu < \kappa < \delta$ are regular cardinals and $\vec{\mathbb{I}}^2$ is an iteration of length δ which collapses κ to μ^+, adds $\square_{\mu,2}$, and anticipates the iteration $\dot{\vec{\mathbb{I}}}^1$ in the sense described above. Let $\gamma \leq \delta$ be some ordinal and suppose that M is an elementary substructure of $H(\theta)$ (θ sufficiently large) of cardinality κ such that $V_\kappa \cup M^{<\kappa} \cup \{\gamma\} \subseteq M$ and M contains all relevant parameters. Furthermore, let $\phi\colon \kappa \to M$ be a bijection and for all $\alpha < \kappa$ set $M_\alpha = \phi``\alpha$. We say that $\vec{\mathbb{I}}^2$, $\dot{\vec{\mathbb{I}}}^1$ are *suitable for* M, ϕ, γ if:

(1) For all $\bar{\gamma} \leq \gamma$, $\Vdash_{\mathbb{I}^2_{\bar{\gamma}}}$ "$\dot{\mathbb{I}}^1_{\bar{\gamma}}$ is μ-c.c."

(2) For all $\alpha < \kappa$ and $\bar{\gamma} \in M_\alpha \cap \gamma$, if:

  (a) $\mathbb{I}^2_{\bar{\gamma}} \cap M_\alpha$ is a regular subposet of $\mathbb{I}^2_{\bar{\gamma}} \cap M$.

  (b) $\Vdash_{\mathbb{I}^2_{\bar{\gamma}} \cap M}$ "$\dot{\mathbb{I}}^1_{\bar{\gamma}} \cap M_\alpha$ is a regular subiteration of $\dot{\mathbb{I}}^1_{\bar{\gamma}} \cap M$."

  (c) $\dot{T}_{\bar{\gamma}} \cap M_\alpha$ is an $(\mathbb{I}^2_{\bar{\gamma}} \cap M_\alpha) * (\dot{\mathbb{I}}^1_{\bar{\gamma}} \cap M_\alpha)$-name for an α-Aronszajn tree.

  Then forcing with $(\mathbb{I}^2_{\bar{\gamma}} \cap M) * (\dot{\mathbb{I}}^1_{\bar{\gamma}} \cap M)/G$, where G is generic for $(\mathbb{I}^2_{\bar{\gamma}} \cap M) * (\dot{\mathbb{I}}^1_{\bar{\gamma}} \cap M_\alpha)$, doesn't add any new branches to the tree named by $\dot{T}_{\bar{\gamma}} \cap M_\alpha$.

We will need to make use of the following basic lemma about forcings which don't add branches to trees:

Lemma 4.5 (Folklore, see [CF98], [KT79]). *Suppose that T is a κ-tree and $\mathbb{P}$ is a κ-Knaster poset. Then forcing with $\mathbb{P}$ doesn't add a branch to T.*

4.3 Obtaining $\square_{\omega_1,2}$ + SATP($\aleph_2$) + SATP($\aleph_1$)

Theorem 4.6. *Suppose that $\mu < \kappa < \kappa^+ \leq \delta$ are cardinals with μ, δ regular and κ weakly compact. Suppose moreover that*

$$\vec{\mathbb{I}}^2 = \langle\langle \mathbb{I}^2_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^2_\gamma \colon \gamma < \delta\rangle\rangle$$
$$\dot{\vec{\mathbb{I}}}^1 = \langle\langle \dot{\mathbb{I}}^1_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^1_\gamma \colon \gamma < \delta\rangle\rangle$$

are two iterations such that $\vec{\mathbb{I}}^2$ collapses κ to μ^+, adds $\square_{\mu,2}$, and specializes all κ-Aronszajn trees while anticipating $\vec{\mathbb{I}}^1$ (in the sense described in the previous section).

Finally, suppose that for all ordinals $\gamma \leq \delta$ there exists M elementary in $H(\theta)$ (θ sufficiently large) of cardinality κ such that $V_\kappa \cup M^{<\kappa} \cup \{\gamma\} \subseteq M$, M contains all relevant parameters, and $\vec{\mathbb{I}}^2$, $\dot{\vec{\mathbb{I}}}^1$ are suitable for M, ϕ, γ (for some fixed bijection $\phi\colon \kappa \to M$). Then the generic extension by $\mathbb{I}^2_\delta * \dot{\mathbb{I}}^1_\delta$ satisfies

$$\kappa = \mu^+ \wedge \square_{\mu,2} \wedge \mathsf{SATP}(\kappa) \wedge 2^\mu \geq \delta$$

The majority of the remainder of this section is devoted to giving a proof of this result. We follow closely the proof of the main theorem in [GH16].

Lemma 4.7. *For every $\gamma \leq \delta$, $\mathbb{I}^2_\gamma$ is $< \mu$ strategically closed.*

Proof. The forcing $\mathbb{I}^2_\gamma$ is a $< \mu$-strategically closed forcing (namely, $\mathbb{P}(\mu, < \kappa)$) followed by the $< \mu$-support iteration of $< \mu$-closed posets. □

Lemma 4.8. *For every $\gamma \leq \delta$, $\Vdash_{\mathbb{I}^2_\gamma}$ "$\dot{\mathbb{I}}^1_\gamma$ is μ-c.c." and $\Vdash_{\mathbb{I}^2_\delta}$ "$\dot{\mathbb{I}}^1_\gamma$ is μ-c.c."*

Proof. This is immediate from the definition of suitability of $\vec{\mathbb{I}}^2$, $\dot{\vec{\mathbb{I}}}^1$. □

Lemma 4.9. *For every $\gamma \leq \delta$, $\mathbb{I}^2_\gamma$ is κ-Knaster.*

Proof. By induction on γ. For the base case, we know $\mathbb{I}^2_1 \simeq \mathbb{P}(\mu, < \kappa)$ is κ-Knaster by Lemma 2.4. So suppose $\gamma \leq \delta$ and each $\mathbb{I}^2_{\gamma'}$ is κ-Knaster for all $\gamma' < \gamma$. We seek to show that $\mathbb{I}^2_\gamma$ is also κ-Knaster.

If γ is a limit ordinal and $\mu \leq \operatorname{cf} \gamma \neq \kappa$ this is immediate since any subset of $\mathbb{I}^2_\gamma$ of cardinality κ may be refined to a subset of $\mathbb{I}^2_{\gamma'}$ of cardinality κ for some $\gamma' < \gamma$.

If γ is a limit ordinal with $\operatorname{cf} \gamma = \kappa$ this follows from a Δ-system argument.

Thus suppose that either γ is a limit ordinal with $\operatorname{cf} \gamma < \mu$ or $\gamma = \bar{\gamma} + 1$ for some ordinal $\bar{\gamma}$. Fix M as in the hypothesis of the theorem, and in either case fix an increasing sequence

$\{\gamma_i \colon i < \mathrm{cf}\,\gamma\}$ in M which is cofinal in γ (if $\gamma = \bar{\gamma} + 1$ is a successor ordinal we say its cofinality is 1 and we let $\gamma_0 = \bar{\gamma}$, so in this case $\{\gamma_i \colon i < 1\}$ is cofinal in γ).

Let R be a subset of V_κ which encodes both M and ϕ (where ϕ is the bijection from the hypothesis of Theorem 4.6). Fix a $< \kappa$-complete normal filter $\mathcal{F}$ on κ which extends the club filter and satisfies

$$\{\alpha < \kappa \colon (V_\alpha, \in, R \cap V_\alpha) \models \psi\} \in \mathcal{F}$$

for each formula ψ which is Π^1_1 over V_κ. For all $\alpha < \kappa$ set $M_\alpha = \phi``\alpha$.

Claim 4.10 ([LS81], [GH16]). *Assume that $\bar{\gamma} \in \gamma \cap M$, and for all $\bar{\bar{\gamma}} \in \bar{\gamma} \cap M$ we have that $\mathbb{I}^2_{\bar{\bar{\gamma}}}$ is κ-Knaster and $\dot{T}_{\bar{\bar{\gamma}}}$ is an $\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}$-name for a κ-Aronszajn tree. Then there exists $X = X_{\bar{\gamma}} \in \mathcal{F}$ such that for all $\alpha \in X$ and $\bar{\bar{\gamma}} \in \bar{\gamma} \cap M_\alpha$:*

1. *α is inaccessible.*

2. *$M_\alpha \cap \kappa = \alpha$.*

3. *$M_\alpha^{<\alpha} \subseteq M_\alpha$.*

4. *$\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M_\alpha$ is a regular subposet of $\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M$ and is α-c.c.*

5. *$\mathbb{I}^1_{\bar{\bar{\gamma}}} \cap M_\alpha$ is equivalent to an $\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M_\alpha$-name.*

6. *$(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M_\alpha$ is a regular subposet of $(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M$.*

7. *$(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M_\alpha$ forces that $T_{\bar{\bar{\gamma}}} \cap (\alpha \times \mu)$ is an α-Aronszajn tree.*

Proof. Let $X = X_{\bar{\gamma}}$ be the set of all $\alpha < \kappa$ that satisfy these requirements for all $\bar{\bar{\gamma}} \in \bar{\gamma} \cap M_\alpha$. The claim follows immediately from a Π^1_1 reflection argument together with the fact that $\mathcal{F}$ extends the club filter. In (7) we make use of the fact that $(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M$ forces that $T_{\bar{\bar{\gamma}}}$ is a κ-Aronszajn tree, which follows from observing that $(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M$ is a regular subposet of $\mathbb{I}^2_{\bar{\gamma}} * \dot{\mathbb{I}}^1_{\bar{\gamma}}$ (this is itself a consequence of the fact that $\mathbb{I}^2_{\bar{\gamma}} * \dot{\mathbb{I}}^1_{\bar{\gamma}}$ has the κ-c.c., by the inductive hypothesis). $\square$

Definition 4.11 ([LS81])**.** A condition $p \in \mathbb{I}^2_\gamma$ is said to be *determined* if there is in V a sequence $\langle x_\xi \colon 1 \leq \xi < \gamma \rangle$ such that for all $1 \leq \xi < \gamma$, $p \restriction \xi \Vdash_{\mathbb{I}^2_\xi} p\,(\xi) = \check{x}_\xi$.

As in [LS81], we may easily observe that the set of determined conditions is dense in $\mathbb{I}^2_\gamma$.

Definition 4.12 ([GH16])**.** Suppose that $p \in \mathbb{I}^2_\gamma \cap M$ is some condition and $\alpha < \kappa$. Write $p = \langle p(\xi) \colon \xi < \gamma \rangle$. Then $p \restriction M_\alpha$ denotes the condition $\langle p'(\xi) \colon \xi < \gamma \rangle$, where $p'(\xi)$ is the trivial condition if $\xi \notin M_\alpha$ and $p'(\xi) = p(\xi) \cap M_\alpha$ otherwise. We say that p is *α-compatible* if $p \restriction M_\alpha$ forces that p is a determined condition in $(\mathbb{I}^2_\gamma \cap M)/(G_{\mathbb{I}^2_\gamma} \cap M_\alpha)$.

Claim 4.13. *Let $X = X_{\bar{\gamma}}$ be as in the previous claim. Then for every $\alpha \in X$, $\bar{\bar{\gamma}} \in (\bar{\gamma}+1) \cap M_\alpha$, $\bar{p} \in \mathbb{I}^2_\gamma \cap M_\alpha$, α-compatible $p^L, p^R \in \mathbb{I}^2_\gamma \cap M$ with $\bar{p} = p^L \restriction M_\alpha = p^R \restriction M_\alpha$, and every pair $(\dot{x}^L, \dot{x}^R)$ of $(\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M) * (\mathbb{I}^1_{\bar{\bar{\gamma}}} \cap M_\alpha)$-names for nodes in $T_{\bar{\bar{\gamma}}}$ above level α, there are α-compatible conditions $p^L_*, p^R_* \in \mathbb{I}^2_\gamma \cap M$, $\bar{p}_* \in \mathbb{I}^2_\gamma \cap M_\alpha$, and a sequence*

$$\langle r_\eta, \xi_\eta, \check{x}^L_\eta, \check{x}^R_\eta \colon \eta < \vartheta \rangle$$

(for some $\vartheta < \mu$) in M_α such that:

(a) $p^L_* \leq p^L$, $p^R_* \leq p^R$ and $\bar{p}_* = p^L_* \restriction M_\alpha = p^R_* \restriction M_\alpha$.

(b) For all $\eta < \vartheta$ $\bar{p}_* \Vdash_{\mathbb{I}^2_\gamma \cap M_\alpha} r_\eta \in \dot{\mathbb{I}}^1_\gamma \cap M_\alpha$.

(c) For all $\eta < \vartheta$, $\xi_\eta < \alpha$ and x^L_η, x^R_η are elements of $\{\xi_\eta\} \times \mu$ with $x^L_\eta \neq x^R_\eta$.

(d) $(p^L_* \restriction \bar{\bar{\gamma}}, r_\eta \restriction \bar{\bar{\gamma}}) \Vdash \check{x}^L_\eta \leq \dot{x}^L$ and $(p^R_* \restriction \bar{\bar{\gamma}}, r_\eta \restriction \bar{\bar{\gamma}}) \Vdash \check{x}^R_\eta \leq \dot{x}^R$.

(e) $\bar{p}_* \Vdash_{\mathbb{I}^2_\gamma \cap M_\alpha} \{\dot{r}_\eta \colon \eta < \vartheta\}$ is a maximal antichain in $\dot{\mathbb{I}}^1_\gamma$.

Proof. Suppose $\alpha \in X$, $\bar{\bar{\gamma}} \in (\bar{\gamma}+1) \cap M_\alpha$, and fix names $\dot{x}^L$, $\dot{x}^R$ for nodes in $T_{\bar{\bar{\gamma}}}$ of level $\geq \alpha$ and conditions $\bar{p}$, p^L, p^R as in the statement of the claim. It follows from the choice of α that for any $(\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M) * (\dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}} \cap M_\alpha)$-generic G the branches in $T_{\bar{\bar{\gamma}}} \restriction \alpha$ below $\dot{x}^L$, $\dot{x}^R$ are not in $V[G \cap (\mathbb{I}^2_{\bar{\bar{\gamma}}} * \mathbb{I}^1_{\bar{\bar{\gamma}}}) \cap M_\alpha]$.

Subclaim 4.14. For any pair (s^L, t), (s^R, t) of conditions in $(\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$ such that s^L, s^R are α-compatible and $s^L \restriction M_\alpha = s^R \restriction M_\alpha$, there is another pair (q^L, r), (q^R, r) of conditions in $(\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$ such that:

- $(q^L, r) \leq (s^L, t)$ and $(q^R, r) \leq (s^R, t)$
- $(q^L \restriction \bar{\bar{\gamma}}, r \restriction \bar{\bar{\gamma}})$, $(q^R \restriction \bar{\bar{\gamma}}, r \restriction \bar{\bar{\gamma}})$ force incompatible values for the branches below $\dot{x}^L$ and $\dot{x}^R$
- q^L, q^R are α-compatible
- $q^L \restriction M_\alpha = q^R \restriction M_\alpha$.

Proof. This is done exactly as in [GH16]. We give the proof for the convenience of the reader. Suppose the opposite for the sake of a contradiction, and consider pairs (s^L, t), (s^R, t) witnessing the negation. Let H be $(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M_\alpha$-generic with $(s^L \restriction \bar{\bar{\gamma}}) \restriction M_\alpha = (s^R \restriction \bar{\bar{\gamma}}) \restriction M_\alpha \in H$ and J_i be $(\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap M)/(\mathbb{I}^2_{\bar{\bar{\gamma}}} \cap H)$-mutually generic with $(s^i \restriction \bar{\bar{\gamma}}, t) \in J_i$ (for $i \in \{L, R\}$).

If K_i is any $[(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M]/(H * J_i)$- generic ($i \in \{L, R\}$) then in $V[H][J_i][K_i]$ there is a branch b^i in the tree $T_{\bar{\bar{\gamma}}} \cap (\alpha \times \mu)$ consisting of nodes which lie below x^i. Moreover, by condition (2) of Definition 4.4, we have $b^i \in V[H][J_i]$ (note, however, that $V[H][J_i]$ may not recognize that all nodes in b^i are below x^i, or even that $T_{\bar{\bar{\gamma}}}$ itself is a tree). Nonetheless, by condition (1) of Definition 4.4 there exists $\mu^i_0 < \mu$ and a collection $\{\check{b}^i_\xi : \xi < \mu^i_0\}$ of names for elements of $V[H][J_i]$ which are cofinal branches through $T_{\bar{\bar{\gamma}}} \cap (\alpha \times \mu)$ such that in $V[H][J_i]$ the following holds:

$$\Vdash_{[(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M]/(H * J_i)} \left(\exists \xi < \mu^i_0\right) \left(\check{b}^i_\xi = \dot{b}^i\right)$$

where $\dot{b}^i$ is the canonical name for the branch b^i described above.

Moreover, by the assumption of the subclaim, there must exist $\xi^L < \mu^L_0$ and $\xi^R < \mu^R_0$ such that $b^L_{\xi^L} = b^R_{\xi^R}$. Denoting this common value by b, we have

$$b \in V[H][J_L] \cap V[H][J_R]$$

and since J_L, J_R were chosen to be mutually generic we have $b \in V[H]$. But this is a contradiction since $(\mathbb{I}^2_{\bar{\bar{\gamma}}} * \dot{\mathbb{I}}^1_{\bar{\bar{\gamma}}}) \cap M_\alpha$ forces $T_{\bar{\bar{\gamma}}} \cap (\alpha \times \mu)$ to be an α-Aronszajn tree. □

Invoking this claim, we may find a pair of conditions (p_0^L, r_0), (p_0^R, r_0) in $(\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$ with $p_0^L \leq p^L$, $p_0^R \leq p^R$, and $p_0^L \restriction M_\alpha = p_0^R \restriction M_\alpha$ together with $\xi_0 < \alpha$ and elements x_0^L, x_0^R in $\{\xi_0\} \times \mu$ such that

$$(p_0^L \restriction \bar{\bar{\gamma}}, r_0 \restriction \bar{\bar{\gamma}}) \Vdash \check{x}_0^L \leq \dot{x}^L$$

$$(p_0^R \restriction \bar{\bar{\gamma}}, r_0 \restriction \bar{\bar{\gamma}}) \Vdash \check{x}_0^R \leq \dot{x}^R$$

Furthermore, we may assume that if we let t_0^L be the unique element of $p_0^L(0)(\alpha)$ and t_0^R be the unique element of $p_0^R(0)(\alpha)$ then

$$\big(p_0^L \restriction M_\alpha\big)(0) * \check{t}_0^L,$$

$$\big(p_0^R \restriction M_\alpha\big)(0) * \check{t}_0^R$$

are flat conditions in $\mathbb{P}(\mu, < \alpha) * \dot{\mathbb{T}}_\alpha$, where $\dot{\mathbb{T}}_\alpha = \check{\mathbb{T}}_{\bigcup \dot{G}_{\mathbb{P}(\mu, <\alpha)}}$.

Proceeding inductively, suppose $\nu < \mu$ and we have defined the pairs $(p_\eta^L, \dot{r}_\eta)$, $(p_\eta^R, \dot{r}_\eta)$ in $(\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$, $\bar{p}_\eta$ in $\mathbb{I}^2_\gamma \cap M_\alpha$, and $\check{t}_\eta^L$, $\check{t}_\eta^R$ in $\dot{\mathbb{T}}_\alpha \cap M$ together with ξ_η and x_η^L, $x_\eta^R \in \{\xi_\eta\} \times \mu$, such that:

- The sequences $\langle p_\eta^L \colon \eta < \nu \rangle$ and $\langle p_\eta^R \colon \eta < \nu \rangle$ are decreasing and for each η p_η^L and p_η^R are α-compatible.

- $\bar{p}_\eta = p_\eta^L \restriction M_\alpha = p_\eta^R \restriction M_\alpha$.

- $\bar{p}_\eta \Vdash_{\mathbb{I}^2_\gamma \cap M_\alpha} \dot{r}_\eta \in \dot{\mathbb{I}}^1_\gamma \cap M_\alpha$.

- For $\eta_0 < \eta_1 < \nu$, $\bar{p}_{\eta_1} \Vdash_{\mathbb{I}^2_\delta \cap M_\alpha} \dot{r}_{\eta_0}, \dot{r}_{\eta_1}$ are incompatible.

- $\xi_n < \alpha$, x_η^L, $x_\eta^R \in \{\xi_\eta\} \times \mu$ and $x_\eta^L \neq x_\eta^R$.

- $(p_\eta^L \restriction \bar{\bar{\gamma}}, \dot{r}_\eta \restriction \bar{\bar{\gamma}}) \Vdash \check{x}_\eta^L \leq \dot{x}^L$.

- $(p_\eta^R \restriction \bar{\bar{\gamma}}, \dot{r}_\eta \restriction \bar{\bar{\gamma}}) \Vdash \check{x}_\eta^R \leq \dot{x}^R$.

- t^L_η is the unique element of $p^L_\eta(0)(\alpha)$.

- t^R_η is the unique element of $p^R_\eta(0)(\alpha)$.

- $(p^L_\eta \restriction M_\alpha)(0) * \check{t}^L_\eta$, $(p^R_\eta \restriction M_\alpha)(0) * \check{t}^R_\eta$ are flat conditions in $\mathbb{P}(\mu, < \alpha) * \dot{\mathbb{T}}_\alpha$, where $\dot{\mathbb{T}}_\alpha = \check{\mathbb{T}}_{\bigcup \dot{G}_{\mathbb{P}(\mu,<\alpha)}}$.

If ν is a successor ordinal let $q^L_\nu = p^L_{\nu-1}$, $q^R_\nu = p^R_{\nu-1}$. Otherwise, let

$$t^L_\nu = \bigcup_{\eta<\nu} t^L_\eta \cup \left\{ \sup \bigcup_{\eta<\nu} t^L_\eta \right\}$$
$$t^R_\nu = \bigcup_{\eta<\nu} t^R_\eta \cup \left\{ \sup \bigcup_{\eta<\nu} t^R_\eta \right\}$$

and let q^L_ν, q^R_ν be lower bounds of $\{p^L_\eta \colon \eta < \nu\}$, $\{p^R_\eta \colon \eta < \nu\}$ such that *both* t^L_ν, t^R_ν appear on the (approximations to) square sequences $q^L_\nu(0)$, $q^R_\nu(0)$. These lower bounds may be seen to exist by an argument similar to that used to prove strategic closure in Lemma 2.4. Namely, each initial segment of t^L_η, t^R_η of limit order type has already been placed on $p^L_\eta(0)$, $p^R_\eta(0)$ for some $\eta < \nu$, and therefore we may place t^L_ν, t^R_ν on $q^L_\nu(0)$, $q^R_\nu(0)$ without any danger of violating coherence. Observe that this is the part of the argument where we exploit the "two-ness" of the principle $\square_{\kappa,2}$ (and hence of the poset used to force it). Namely, we seek to ensure that p^L_*, p^R_* agree on M_α, and so must put both threads on both conditions. Finally, note that in either case (ν successor or limit) we have q^L_ν, q^R_ν are α-compatible conditions in $\mathbb{I}^2_\gamma \cap M$ and $q^L_\nu \restriction M_\alpha = q^R_\nu \restriction M_\alpha$. Let $\bar{q}_\nu = q^L_\nu \restriction M_\alpha = q^R_\nu \restriction M_\alpha$. If

$$\bar{q}_\nu \Vdash_{\mathbb{I}^2_\gamma \cap M_\alpha} \{\dot{r}_\eta \colon \eta < \nu\} \text{ is a maximal antichain}$$

we halt the construction and set $p^L_\nu = q^L_\nu$, $p^R_\nu = q^R_\nu$. Otherwise proceed exactly as when obtaining r_0, except now working below s_ν. Namely, find a condition s_ν forced to be incompatible with every r_η ($\eta < \nu$) and choose (p^L_ν, r_ν), (p^R_ν, r_ν), $\bar{p}_\nu$, $\xi_\nu < \alpha$, and $x^L_\nu, x^R_\nu \in \{\xi_\nu\} \times \mu$ such that:

- $(p^L_\nu, r_\nu), (p^R_\nu, r_\nu) \in (\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$.

- p^L_ν, p^R_ν are α-compatible.
- $(p^L_\nu, r_\nu) \leq (q^L_\nu, s_\nu)$ and $(p^R_\nu, r_\nu) \leq (q^R_\nu, s_\nu)$.
- $\bar{p}_\nu = p^L_\nu \upharpoonright M_\alpha = p^R_\nu \upharpoonright M_\alpha$.
- $(p^L_\nu \upharpoonright \bar{\bar{\gamma}}, \dot{r}_\nu \upharpoonright \bar{\bar{\gamma}}) \Vdash \check{x}^L_\nu \leq \dot{x}^L$.
- $(p^R_\nu \upharpoonright \bar{\bar{\gamma}}, \dot{r}_\nu \upharpoonright \bar{\bar{\gamma}}) \Vdash \check{x}^R_\nu \leq \dot{x}^R$.
- t^L_ν is the unique element of $p^L_\nu(0)(\alpha)$.
- t^R_ν is the unique element of $p^R_\nu(0)(\alpha)$.
- $(p^L_\nu \upharpoonright M_\alpha)(0) * \check{t}^L_\nu$, $(p^R_\nu \upharpoonright M_\alpha)(0) * \check{t}^R_\nu$ are flat conditions in $\mathbb{P}(\mu, <\alpha) * \dot{\mathbb{T}}_\alpha$, where $\dot{\mathbb{T}}_\alpha = \check{\mathbb{T}}_{\bigcup \dot{G}_{\mathbb{P}(\mu,<\alpha)}}$

By Lemma 4.8 this process terminates after $< \mu$ many steps. At its completion we get an ordinal $\vartheta < \mu$, descending sequences $\langle p^L_\eta \colon \eta \leq \vartheta\rangle$ and $\langle p^R_\eta \colon \eta \leq \vartheta\rangle$ of conditions in $\mathbb{I}^2_\gamma \cap M$, as well as sequences $\langle \bar{p}_\eta \colon \eta < \vartheta\rangle$, $\langle r_\eta \colon \eta < \vartheta\rangle$, and $\langle (\xi_\eta, \check{t}^L_\eta, \check{t}^R_\eta, \check{x}^L_\eta, \check{x}^R_\eta) \colon \eta < \vartheta\rangle$ such that:

- $(p^L_\eta, r_\eta), (p^R_\eta, r_\eta) \in (\mathbb{I}^2_\gamma \cap M) * (\dot{\mathbb{I}}^1_\gamma \cap M_\alpha)$.
- p^L_η, p^R_η are α-compatible.
- $\bar{p}_\eta = p^L_\eta \upharpoonright M_\alpha = p^R_\nu \upharpoonright M_\alpha$.
- $x^L_\eta, x^R_\eta \in \{\xi_\eta\} \times \mu$.
- $(p^L_\eta \upharpoonright \bar{\bar{\gamma}}, \dot{r}_\eta \upharpoonright \bar{\bar{\gamma}}) \Vdash \check{x}^L_\eta \leq \dot{x}^L$.
- $(p^R_\eta \upharpoonright \bar{\bar{\gamma}}, \dot{r}_\eta \upharpoonright \bar{\bar{\gamma}}) \Vdash \check{x}^R_\eta \leq \dot{x}^R$.
- t^L_η is the unique element of $p^L_\eta(0)(\alpha)$.
- t^R_η is the unique element of $p^R_\eta(0)(\alpha)$.
- $(p^L_\eta \upharpoonright M_\alpha)(0) * \check{t}^L_\eta$, $(p^R_\eta \upharpoonright M_\alpha)(0) * \check{t}^R_\eta$ are flat conditions in $\mathbb{P}(\mu, <\alpha) * \dot{\mathbb{T}}_\alpha$, where $\dot{\mathbb{T}}_\alpha = \check{\mathbb{T}}_{\bigcup \dot{G}_{\mathbb{P}(\mu,<\alpha)}}$

Finally, set $p^L_* = p^L_\vartheta$, $p^R_* = p^R_\vartheta$. Then this pair (p^L_*, p^R_*) together with $\langle r_\eta, \xi_\eta, \check{x}^L_\eta, \check{x}^R_\eta \colon \eta < \vartheta\rangle$ are as desired. □

Following [GH16], let us call the sequence $\langle (r_\eta, \xi_\eta, \check{x}^L_\eta, \check{x}^R_\eta) \colon \eta < \vartheta\rangle$ an α-separating witness for the nodes $\dot{x}^L$, $\dot{x}^R$ relative to p^L_*, p^R_*. We now continue with the proof of Lemma 4.9:

Claim 4.15. *There is $X \in \mathcal{F}$ such that for every condition $p \in \mathbb{I}^2_\gamma \cap M$ and $\alpha \in X$ there are conditions $p^L, p^R \le p$ (both in M) such that $p^L \restriction M_\alpha = p^R \restriction M_\alpha$ and for every $\gamma' \in \gamma \cap M_\alpha$, any pair of elements above α in $\operatorname{dom}(p^L(\gamma')) \times \operatorname{dom}(p^R(\gamma'))$ has an α-separating witness in M_α relative to $p^L \restriction \gamma$, $p^R \restriction \gamma$.*

We call such a pair (p^L, p^R) an *α-separating pair.*

Proof. Recall that we chose a sequence $\{\gamma_i \colon i < \operatorname{cf}\gamma\}$ cofinal in γ. For each $i < \operatorname{cf}\gamma$ let X_{γ_i} be as in Claim 4.13, and let $X = \bigcap_{i<\operatorname{cf}\gamma} X_{\gamma_i}$. This X suffices, as may be seen by applying Claim 4.13 $\operatorname{cf}\gamma$ many times and using the $<\mu$-strategic closure of $\mathbb{I}^2_\gamma$. □

Returning to the proof of the κ-c.c., let X be as in Claim 4.15 and let $\langle p_\alpha \colon \alpha < \kappa\rangle \in M$ be a sequence of conditions in $\mathbb{I}^2_\gamma$. For every $\alpha \in X$ we may extend p_α to an α-separating pair $(p^L_\alpha, p^R_\alpha) \in M$. Let $s_\alpha \in M_\alpha$ be the list of separating witnesses and let $\bar{p}_\alpha$ denote $p^L_\alpha \restriction M_\alpha = p^R_\alpha \restriction M_\alpha$.

The function $\alpha \mapsto (s_\alpha, \bar{p}_\alpha)$ is regressive, and so by normality of $\mathcal{F}$ there is a set Y which is positive with respect to this filter and a pair $(s^*, \bar{p}^*)$ such that for all $\alpha \in Y$ $(s_\alpha, \bar{p}_\alpha) = (s^*, \bar{p}^*)$. By further thinning we may assume that for every $\alpha_0, \alpha_1 \in Y$ with $\alpha_0 < \alpha_1$ we have $p^L_{\alpha_0}, p^R_{\alpha_1} \in M_{\alpha_1}$. Similarly, we may assume without loss of generality that

$$\left\{\operatorname{supp}(p^L_\alpha) \cup \operatorname{supp}(p^R_\alpha) \colon \alpha \in Y\right\}$$

is a Δ-system with root R.

We claim that for any $\alpha_0 < \alpha_1$ in Y, p_{α_0} is compatible with p_{α_1}, as witnessed by the condition q given by $q(\gamma') = p^L_{\alpha_0}(\gamma') \cup p^R_{\alpha_1}(\gamma')$ for every $\gamma' < \gamma$.

We must show that q so defined is a condition. Clearly $|\operatorname{dom}(q)| < \mu$ and so it remains only to show that $q \restriction \gamma'$ forces that $q(\gamma')$ is a condition in $\dot{\mathbb{J}}^2_{\gamma'}$ for all $\gamma' < \gamma$. We proceed by induction on γ'. For $\gamma' = 0$ $q(\gamma') \in \mathbb{P}(\mu, < \kappa)$, since $p^L_{\alpha_0}(\gamma')$, $p^R_{\alpha_1}(\gamma')$ have identical intersection with M_{α_0} and are disjoint above α_0.

Now assume $\gamma' > 0$ and $q \restriction \gamma'$ is a condition. Without loss of generality $\dot{T}_{\gamma'_0}$ is an $\mathbb{I}^2_{\gamma'} * \dot{\mathbb{I}}^1_{\gamma'}$-name for a κ-Aronszajn tree, as otherwise $\dot{\mathbb{J}}^2_{\gamma'}$ is a name for the trivial forcing. We may also assume $\gamma' \in R$, since otherwise $q(\gamma')$ is either $p^L_{\alpha_0}(\gamma')$ or $p^R_{\alpha_1}(\gamma')$.

In order to show that $q \restriction \gamma' \Vdash q(\gamma')$ is a condition we must show that if $\dot{x}^L, \dot{x}^R \in \operatorname{dom}(q(\gamma'))$ and $q \restriction \gamma' \Vdash q(\gamma')(\dot{x}^L) = q(\gamma')(\dot{x}^R)$, then

$$(q \restriction \gamma', \dot{1}_{\mathbb{I}^1_{\gamma'}}) \Vdash_{\mathbb{I}^2_{\gamma'} * \dot{\mathbb{I}}^1_{\gamma'}} \dot{x}^L \perp \dot{x}^R$$

Since $p^L_{\alpha_0} \restriction M_{\alpha_0} = \bar{p} = p^R_{\alpha_1} \restriction M_{\alpha_1}$, we may assume without loss of generality that both $\dot{x}^L$, $\dot{x}^R$ are names for nodes above level α_0. Letting $s^* = \langle r_\eta, \xi_\eta, \check{x}^L_\eta, \check{x}^R_\eta \colon \eta < \vartheta \rangle$ be our fixed separating witness, we have:

$$(p^L_{\alpha_0} \restriction \gamma', \dot{r}_\eta \restriction \gamma') \Vdash \check{x}^L_\eta \leq \dot{x}^L$$
$$(p^R_{\alpha_1} \restriction \gamma', \dot{r}_\eta \restriction \gamma') \Vdash \check{x}^R_\eta \leq \dot{x}^R$$

By the induction hypothesis $q \restriction \gamma'$ is a condition extending both $p^L_{\alpha_0} \restriction \gamma'$ and $p^R_{\alpha_0} \restriction \gamma'$ and so in particular, since $\check{x}^L_\eta \neq \check{x}^R_\eta$, for all $\eta < \vartheta$ we have

$$(q \restriction \gamma', \dot{r}_\eta \restriction \gamma') \Vdash \dot{x}^L \perp_{\dot{T}_{\gamma'}} \dot{x}^R$$

Since $\{\dot{r}_\eta \colon \eta < \vartheta\}$ is forced to be a maximal antichain, we have

$$(q \restriction \gamma', \dot{1}_{\mathbb{I}^2_{\gamma'}}) \Vdash \dot{x}^L \perp_{\dot{T}_{\gamma'}} \dot{x}^R$$

as desired. □

Remark. It behooves us to observe that the proof of Lemma 4.9 actually gives us something stronger–namely that for any $S \subseteq \delta$ such that $\mathbb{I}^2_\delta \restriction S$ is a regular subposet of $\mathbb{I}^2_\delta$, and for any $\mathbb{I}^2_\delta \restriction S$-generic K, the quotient $\mathbb{I}^2_\delta / K$ is κ-Knaster. The proof of this is almost identical to that of Lemma 4.9, but we must observe that every name $\dot{T}_\gamma$ for a κ-Aronszajn tree considered in the iteration remains a name for a κ-Aronszajn tree in $V[K]$. This is true since any such tree is κ-Aronszajn (in fact, special) in $V[K][L]$, where L is generic for $\mathbb{I}^2_\delta / K$.

From Lemmas 4.8 and 4.9 we have

Lemma 4.16. *For all $\gamma \leq \delta$ $\mathbb{I}^2_\gamma * \dot{\mathbb{I}}^1_\gamma$ has the κ-c.c.*

Lemma 4.17. *Let $\mathbb{I} = \mathbb{I}^2_\delta * \dot{\mathbb{I}}^1_\delta$ be as above and suppose that G is $\mathbb{I}$-generic over V. Then:*

(a) μ remains a cardinal in $V[G]$, $(\mu^+)^{V[G]} = \kappa$, and $(\mu^{++})^{V[G]} = (\kappa^+)^V$.

(b) $V[G] \models 2^\mu \geq \delta$.

(c) $V[G] \models \square_{\mu,2}$.

Now we are ready to complete the proof of Theorem 4.6.

Lemma 4.18. *Suppose that $X \in V[G]$ and $X \subseteq \kappa$. Then $X \in V[G_{\mathbb{I}^2_\gamma * \dot{\mathbb{I}}^1_\gamma}]$ for some $\gamma < \delta$.*

Proof. Immediate from Lemma 4.16. □

Lemma 4.19. *The poset $\mathbb{I}$ forces $\mathsf{SATP}(\kappa)$.*

Proof. Suppose that T is an κ-Aronszajn tree in $V[G_{\mathbb{I}}]$ and let $\dot{T}$ be a canonical name for it. Then for some $\gamma < \delta$ it is an $\mathbb{I}_\gamma$-name and $\dot{T} = \dot{T}_\gamma$. By construction of the forcing poset, $\Vdash_{\mathbb{I}_{\gamma+1}}$ "$\dot{T}$ is special" and since $V[G_{\mathbb{I}_{\gamma+1}}]$ and $V[G_{\mathbb{I}}]$ have the same cardinals T remains special in the latter generic extension. □

This concludes the proof of Theorem 4.6.

Corollary 4.20. Suppose that there is a weakly compact cardinal. Then in some generic extension of V the following holds:

$$\square_{\omega_1,2} + \mathsf{SATP}(\aleph_2) + \mathsf{SATP}(\aleph_1)$$

Proof. Let κ be weakly compact and let

$$\vec{\mathbb{I}}^2 = \langle\langle \mathbb{I}^2_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^2_\gamma \colon \gamma < \delta\rangle\rangle$$
$$\dot{\vec{\mathbb{I}}}^1 = \langle\langle \dot{\mathbb{I}}^1_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}^1_\gamma \colon \gamma < \delta\rangle\rangle$$

be two iterations such that $\vec{\mathbb{I}}^2$ collapses κ to $\aleph_2$, adds $\square_{\omega_1,2}$, and specializes all κ-Aronszajn trees while anticipating $\dot{\vec{\mathbb{I}}}^1$–where $\dot{\vec{\mathbb{I}}}^1$ is an $\vec{\mathbb{I}}^2$-name for the Baumgartner forcing–i.e. a finite support iteration of posets of finite approximations to specializing functions of trees chosen according to some appropriate bookkeeping function. For each ordinal $\gamma \leq \delta$ let M be an arbitrary elementary substructure of $H(\theta)$ (θ sufficiently large) of cardinality κ such that $V_\kappa \cup M^{<\kappa} \cup \{\gamma\} \subseteq M$ and M contains all relevant parameters, and let $\phi\colon \kappa \to M$ be an arbitrary bijection. Then $\vec{\mathbb{I}}^2$, $\dot{\vec{\mathbb{I}}}^1$ are suitable for M, ϕ, γ by the remark after the proof of Lemma 4.9 together with [BMR70]. Therefore the result follows immediately from Theorem 4.6. □

4.4 The Global Result

Using the methods of [GH16] we are able to obtain our result simulaneously at ω many successive cardinals.

Theorem 4.21. *Let μ be an uncountable regular cardinal and assume that there are ω many supercompact cardinals above μ. Then there is a generic extension of V in which $\square_{\mu^{+n},2} + \mathsf{SATP}(\mu^{+n+1})$ holds for all $n \geq 0$. Furthermore, if $\mu = \aleph_1$, we may ensure that $\mathsf{SATP}(\aleph_1)$ holds as well.*

Proof. Let $\langle \kappa_n \colon n < \omega\rangle$ be an increasing sequence of indestructibly supercompact cardinals above μ and let $\delta = (\sup_{n<\omega} \kappa_n)^{++}$. In the following it will be convenient to write $\kappa_{-1} = \mu$. If $\mu = \aleph_1$, let $R = \omega \cup \{-1\}$, and otherwise let $R = \omega$. Let $h\colon \delta \setminus \{0\} \to R$ be a function such that for all $n \in R$, $h(\alpha) = n$ for unboundedly many $\alpha < \delta$. We define an iteration

$$\vec{\mathbb{I}} = \langle\langle \mathbb{I}_\gamma \colon \gamma \leq \delta\rangle, \langle \dot{\mathbb{J}}_\gamma \colon \gamma < \delta\rangle\rangle$$

as well as auxilliary forcings $\mathbb{I}_\gamma(>\kappa_n)$, $\dot{\mathbb{I}}_\gamma(\kappa_n)$, and $\dot{\mathbb{I}}_\gamma(<\kappa_n)$ for $n \in R$, $\gamma \leq \delta$ such that:

(a) $\mathbb{I}_\gamma \simeq \mathbb{I}_\gamma(>\kappa_n) * \dot{\mathbb{I}}_\gamma(\kappa_n) * \dot{\mathbb{I}}_\gamma(<\kappa_n)$

(b) $\mathbb{I}_\gamma(>\kappa_n)$ is $<\kappa_n$-strategically closed.

(c) For all $n \geq 0$, $\Vdash_{\mathbb{I}_\gamma(>\kappa_n)} \dot{\mathbb{I}}_\gamma(\kappa_n)$ is κ_n-c.c. and $<\kappa_{n-1}$- strategically closed, and for $n=-1$, $\Vdash_{\mathbb{I}_\gamma(>\kappa_n)} \dot{\mathbb{I}}_\gamma(\kappa_n)$ is c.c.c.

(d) For all $n \geq 0$, $\Vdash_{\mathbb{I}_\gamma(>\kappa_n)*\dot{\mathbb{I}}_\gamma(\kappa_n)} \dot{\mathbb{I}}_\gamma(<\kappa_n)$ is κ_{n-1}-c.c. and $<\mu$-closed.

Set $\mathbb{I}_1 = \prod_{n<\omega} \mathbb{P}(\kappa_{n-1}, <\kappa_n)$, where the product is taken with full support, and then let

- $\mathbb{I}_1(<\kappa_n) = \prod_{m<n} \mathbb{P}(\kappa_{m-1}, <\kappa_m)$ for $n \geq 0$ and is the trivial forcing for $n=-1$.
- $\mathbb{I}_1(\kappa_n) = \mathbb{P}(\kappa_{n-1}, <\kappa_n)$ for $n \geq 0$ and is the trivial forcing for $n=-1$.
- $\mathbb{I}_1(>\kappa_n) = \prod_{m>n} \mathbb{P}(\kappa_{m-1}, <\kappa_m)$, also taken with full support.

Now suppose that $2 \leq \gamma \leq \delta$ and we have already defined $\mathbb{I}_{\gamma'}$, $\mathbb{I}_{\gamma'}(>\kappa_n)$, $\dot{\mathbb{I}}_{\gamma'}(\kappa_n)$, and $\dot{\mathbb{I}}_{\gamma'}(<\kappa_n)$ for all $\gamma' < \gamma$ and $n \in R$. We define $\mathbb{I}_\gamma$, $\dot{\mathbb{I}}_\gamma(>\kappa_n)$, and $\dot{\mathbb{I}}_\gamma(<\kappa_n)$ as follows:

- If γ is a limit ordinal then $\mathbb{I}_\gamma$ is the set of all p with domain γ such that $p \restriction \gamma' \in \mathbb{I}_{\gamma'}$ for all $\gamma' < \gamma$, for all $n \geq 0$ we have $|\operatorname{supp}(p) \cap h^{-1}(n)| < \kappa_{n-1}$, and for $n=-1$ we have $|\operatorname{supp}(p) \cap h^{-1}(n)|$ finite.
- If $\gamma = \bar{\gamma}+1$ is a successor ordinal and $n = h(\bar{\gamma})$ then let $\dot{T}_{\bar{\gamma}}$ be an $\mathbb{I}_{\bar{\gamma}}$-name for a κ_n-Aronszajn tree chosen according to some bookkeeping function, and let

  $$\mathbb{I}_\gamma = \mathbb{I}_{\bar{\gamma}} * \dot{\mathbb{B}}_{\mathbb{I}_{\bar{\gamma}}(<\kappa_n)}(\dot{T}_{\bar{\gamma}})$$

  Observe that $\dot{\mathbb{B}}_{\mathbb{I}_{\bar{\gamma}}(<\kappa_n)}(\dot{T}_{\bar{\gamma}})$ is an $\mathbb{I}_{\bar{\gamma}}(>\kappa_{n-1})$-name (if $n=-1$ we mean here that it is an $\mathbb{I}_{\bar{\gamma}}$-name) and so may be viewed as an $\mathbb{I}_{\bar{\gamma}}(\kappa_n)$-name in the extension by $\mathbb{I}_{\bar{\gamma}}(>\kappa_n)$.

For $n \in R$ let

$$\mathbb{I}_\gamma(>\kappa_n) = \left\{ p \in \mathbb{I}_\gamma \colon p(0) \in \mathbb{I}_1(>\kappa_n) \wedge \operatorname{supp}(p) \setminus \{0\} \subseteq \bigcup_{m>n} h^{-1}(m) \right\}$$

Then $\mathbb{I}_\gamma(>\kappa_n)$ is a regular subforcing of $\mathbb{I}_\gamma$. We let $\dot{\mathbb{I}}_\gamma(\kappa_n)$ be an $\mathbb{I}_\gamma(>\kappa_n)$-name for the poset

$$\mathbb{I}_\gamma(\kappa_n) = \left\{p \in \mathbb{I}_\gamma \colon p(0) \in \mathbb{I}_1(\kappa_n) \wedge \operatorname{supp}(p) \setminus \{0\} \subseteq h^{-1}(n)\right\}$$

and let $\dot{\mathbb{I}}_\gamma(<\kappa_n)$ be an $\mathbb{I}_\gamma(>\kappa_n) * \dot{\mathbb{I}}_\gamma(\kappa_n)$-name for the poset

$$\mathbb{I}_\gamma(<\kappa_n) = \left\{p \in \mathbb{I}_\gamma \colon p(0) \in \mathbb{I}_1(<\kappa_n) \wedge \operatorname{supp}(p) \setminus \{0\} \subseteq \bigcup_{m<n} h^{-1}(m)\right\}$$

Observe that $\mathbb{I}_\gamma \simeq \mathbb{I}_\gamma(>\kappa_n) * \dot{\mathbb{I}}_\gamma(\kappa_n) * \dot{\mathbb{I}}_\gamma(<\kappa_n)$.

Lemma 4.22. *Let $G_{>\kappa_n}$ be generic for $\mathbb{I}_\gamma(>\kappa_n)$ and $\dot{G}_{\kappa_n}$ be an $\mathbb{I}_\gamma(>\kappa_n)$-name for the generic for $\dot{\mathbb{I}}_\gamma(\kappa_n)$. If $n \geq 0$, then $V[G_{>\kappa_n}] \models$ "$\mathbb{I}_\gamma(\kappa_n)$ is $<\kappa_{n-1}$-strategically closed" and $V[G_{>\kappa_n} * \dot{G}_{\kappa_n}] \models$ "$\dot{\mathbb{I}}_\gamma(<\kappa_n)$ is $<\mu$-strategically closed."*

Proof. Clearly

$$V\left[G_{>\kappa_n}\right] \models \mathbb{I}_\gamma(\kappa_n) \text{ is } <\kappa_{n-1}\text{-strategically closed}$$

since $\mathbb{I}_\gamma(\kappa_n)$ may be defined as an iteration with $<\kappa_{n-1}$-support in $V[G_{>\kappa_n}]$ where each iterand has the requisite closure. The fact that

$$V[G_{>\kappa_n} * \dot{G}_{\kappa_n}] \models \dot{\mathbb{I}}_\gamma(<\kappa_n) \text{ is } <\mu\text{-strategically closed.}$$

may be argued similarly. □

Lemma 4.23. *Let $G_{>\kappa_n}$, $\dot{G}_{\kappa_n}$ be as in the statement of Lemma 4.22. Then $V\left[G_{>\kappa_n}\right] \models$ "$\mathbb{I}_\gamma(\kappa_n)$ is κ_n-Knaster" and $V[G_{>\kappa_n} * \dot{G}_{\kappa_n}] \models$ "$\dot{\mathbb{I}}_\gamma(<\kappa_n)$ is κ_{n-1}-Knaster." Moreover, we actually have $V[G_{>\kappa_n}] \models$ "$\mathbb{I}_\gamma(\kappa_n)/L$ is κ_n-Knaster," for any L which is generic for a regular subiteration of $\mathbb{I}_\gamma(\kappa_n)$.*

Proof. We prove these simultaneously using induction on n. For each $m > n$ let G_{κ_m} denote the generic for $\mathbb{I}_\gamma(\kappa_m)$ and let $\dot{\mathbb{T}}(\kappa_m) = \check{\mathbb{T}}_{\bigcup \dot{G}_{\kappa_m}}$ be the $\mathbb{I}_\gamma(\kappa_m)$-name for the poset which threads $\bigcup G_{\kappa_m}$ with conditions of size $<\kappa_{m-1}$. An argument similar to that given in Lemma

2.6 tells us that $\mathbb{I}_\gamma(\kappa_m) * \dot{\mathbb{T}}(\kappa_m)$ is $< \kappa_{m-1}$-closed. Moreover, it is clear that this poset forces $|\kappa_m| = \kappa_{m-1}$ and has size $\leq \delta$. Let $\dot{\mathbb{T}}(> \kappa_n)$ be the $\mathbb{I}_\gamma(> \kappa_n)$-name for

$$\mathbb{T}(> \kappa_n) = \prod_{m>n} \mathbb{T}(\kappa_m)$$

where $\mathbb{T}(\kappa_m) = \dot{\mathbb{T}}(\kappa_m)[G_{\kappa_m}]$ and the product is taken with full support. Then $\mathbb{I}_\gamma(> \kappa_n) * \dot{\mathbb{T}}(> \kappa_n)$ is $< \kappa_n$-closed, forces $|\kappa_m| = \kappa_n$ for all $m > n$, and has cardinality $\leq \delta$, and so there is a regular embedding from $\mathbb{I}_\gamma(> \kappa_n) * \dot{\mathbb{T}}(> \kappa_n)$ into $\mathrm{Col}\,(\kappa_n, \delta)$–in fact we have $\mathrm{Col}\,(\kappa_n, \delta) \simeq (\mathbb{I}_\gamma(> \kappa_n) * \dot{\mathbb{T}}(> \kappa_n)) \times \mathrm{Col}\,(\kappa_n, \delta)$. Let $(G_{>\kappa_n} * \dot{H}_{>\kappa_n}) \times K_n$ be generic for the latter poset.

Then to show that $\mathbb{I}_\gamma(\kappa_n)$ is κ_n-Knaster in $V[G_{>\kappa_n}]$ it suffices to show that it satisfies this property in $V[(G_{>\kappa_n} * \dot{H}_{>\kappa_n}) \times K_n]$. But $(\mathbb{I}_\gamma(> \kappa_n) * \dot{\mathbb{T}}(> \kappa_n)) \times \mathrm{Col}\,(\kappa_n, \delta) \simeq \mathrm{Col}\,(\kappa_n, \delta)$ is $< \kappa_n$-directed closed and therefore κ_n is supercompact (and in particular weakly compact) in $V[(G_{>\kappa_n} * \dot{H}_{>\kappa_n}) \times K_n]$. Thus $\mathbb{I}_\gamma(\kappa_n)$ is κ_n-Knaster in this generic extension by Lemma 4.9 from the proof of Theorem 4.6. More precisely, we apply Lemma 4.9 to the pair $\vec{\mathbb{I}}_\gamma(\kappa_n)$, $\dot{\vec{\mathbb{I}}}_\gamma(< \kappa_n)$. Note that in order to do so we must have that this pair is *suitable* (with regard to sufficiently closed elementary substructures of $H(\theta)$) in the sense of Definition 4.4. But part (1) of this definition follows from the inductive hypothesis of the current lemma and part (2) follows from Lemma 4.5 in conjunction with the inductive hypothesis.

For the "moreover" part of the lemma, use the remark after the proof of Lemma 4.9 rather than Lemma 4.9 itself.

Finally, for the second part of the lemma, recall that by the inductive hypothesis we have

$$V[G_{>\kappa_{n-1}} * \dot{G}_{\kappa_{n-1}}] \models \text{“}\dot{\mathbb{I}}_\gamma(< \kappa_{n-1}) \text{ is } \kappa_{n-2}\text{-Knaster”}$$

Since $G_{>\kappa_{n-1}} = G_{>\kappa_n} * \dot{G}_{\kappa_n}$, $\mathbb{I}_\gamma(< \kappa_n) \simeq \mathbb{I}_\gamma(\kappa_{n-1}) * \dot{\mathbb{I}}_\gamma(< \kappa_{n-1})$, and $V[G_{>\kappa_{n-1}}] \models$ "$\dot{\mathbb{I}}_\gamma(\kappa_{n-1})$ is κ_{n-1}-Knaster" by the inductive hypothesis, we have

$$V[G_{>\kappa_n} * \dot{G}_{\kappa_n}] = V[G_{>\kappa_{n-1}}] \models \dot{\mathbb{I}}_\gamma(< \kappa_n) \text{ is } \kappa_{n-1}\text{-Knaster}$$

as desired. □

Now Theorem 4.21 follows immediately from Lemmas 4.22 and 4.23.

□

Finally we may use an Easton-support iteration of the ω-block posets given by Theorem 4.21 exactly as in [GH16] to obtain a global result:

Theorem 4.24. *Assume that there are class many supercompact cardinals with no inaccessible limit. Then there is a class forcing extension of* V *in which* $\square_{\kappa,2} + \mathsf{SATP}(\kappa^+)$ *holds for all regular* κ*.*

CHAPTER 5

A Weak Stationary Reflection Principle and $\square(\omega_2)$

5.1 Background and Basic Definitions

The goal of this chapter is to introduce the weak reflection principle $R_2^*(\aleph_2, \aleph_1)$ and show that it is compatible with $\square(\omega_2)$. To that end we begin in this section by introducing all relevant definitions and observing some basic properties of the relevant forcing posets. In Section 5.2 we give the proof of the consistency result, which uses ideas of [Mag82] together with strong closure properties of the poset to add $\square(\kappa)$ (as well as its composition with the associated threading poset). The proof proceeds by iterating shooting ω_1-closed unbounded subsets of reflection points for partitions of ω_2 into stationary sets (in a sense to be precisely described below).

Definition 5.1. Suppose that μ is a regular cardinal. We say that a set X of ordinals is μ-closed if whenever α is a limit point of X of cofinality μ, α is an element of X.

Definition 5.2. Suppose that X is a set of ordinals. Then the the *trace* of X is given by:

$$\mathrm{Tr}\,(X) = \{\alpha\colon\ \mathrm{cf}\,\alpha \geq \omega_1 \wedge X \cap \alpha \text{ is stationary in } \alpha\}$$

If X is a stationary subset of some cardinal κ and $\alpha \in \mathrm{Tr}\,(X)$, we refer to α as a *reflection point* of X and say X *reflects at* α.

For a cardinal κ and uncountable regular $\mu < \kappa$, let $E^\kappa_\mu = \{\alpha < \kappa\colon\ \mathrm{cf}\,\alpha = \mu\}$, $E^\kappa_{<\mu} = \{\alpha < \kappa\colon\ \mathrm{cf}\,\alpha < \mu\}$. Assaf Rinot introduced the following principle:

- $R_2(\kappa, \mu)$: For every partition $E^\kappa_{<\mu} = \bigcup_{i<\mu} S_i$ of $E^\kappa_{<\mu}$ there exists $j < \mu$ and a μ-closed set $C \subseteq E^\kappa_\mu$ such that for all $\alpha \in C$, $\bigcup_{i<j} S_i$ is stationary in α.

An easy argument shows this to be equivalent to the following:

- For every partition $E^{\kappa}_{<\mu} = \bigcup_{i<\mu} S_i$ of $E^{\kappa}_{<\mu}$ there exists $j < \mu$ and μ-closed $C \subseteq E^{\kappa}_{\mu}$ such that for all $\alpha \in C$ there exists $i < j$ with S_i stationary in α.

Assaf Rinot showed that $R_2(\aleph_2, \aleph_1)$ is inconsistent with $\square(\omega_2)$ [Rin16]. We repeat the argument for the reader's convenience. Suppose $\square(\omega_2)$ and $R_2(\aleph_2, \aleph_1)$ held simultaneously and let $\langle C_\alpha \colon \alpha < \omega_2 \rangle$ be a $\square(\omega_2)$-sequence. By [Rin14] we may assume without loss of generality that for all $j < \omega_2$ $\{\alpha \colon \min C_\alpha = j\}$ is stationary. Let S_i, $i < \omega_1$ be given by:

$$S_i = \begin{cases} \{\alpha < \omega_2 \colon \min C_\alpha = i\} & i \neq 0 \\ \{\alpha < \omega_2 \colon \min C_\alpha = 0 \vee \min C_\alpha \geq \omega_1\} & i = 0 \end{cases}$$

By $R_2(\aleph_2, \aleph_1)$, there exists $1 \leq j < \omega_1$ and ω_1-closed $D \subseteq E^{\omega_2}_{\omega_1}$ such that for every $\alpha \in D$ there exists $i < j$ such that S_i reflects at α. Using stationarity of S_j, fix $\alpha \in D \cap S_j$. Then $\min C_\alpha = j$ and so $\mathrm{Lim}\,(C_\alpha)$ is disjoint from S_i for all $i < j$, which contradicts stationarity of $\bigcup_{i<j} S_i$ in α.

We introduce a weaker reflection principle $R^*_2(\kappa, \mu)$ and show that although $R^*_2(\aleph_2, \aleph_1)$ is nontrivial (in the sense that it is not a consequence of ZFC) it is consistent with $\square(\omega_2)$. $R^*_2(\kappa, \mu)$ will be the following principle:

- For every partition $E^{\kappa}_{<\mu} = \bigcup_{i<\mu} S_i$ of $E^{\kappa}_{<\mu}$ there exists μ-closed $D \subseteq E^{\kappa}_{\mu}$ such that for all $\alpha \in D$ some S_i is stationary in α.

Note that the difference between $R^*_2(\kappa, \mu)$ and Rinot's principle $R_2(\kappa, \mu)$ is that there is no bound j on i.

Claim 5.3. *$R^*_2(\aleph_2, \aleph_1)$ is incompatible with $\square_{\omega_1}$ (and hence in particular is not a consequence of* ZFC*).*

Proof. Suppose that $\square_{\omega_1}$ holds and let $\langle C_\alpha \colon \alpha < \omega_2 \rangle$ witnesses this. I.e.,

- C_α is club in α for all limit $\alpha < \omega_2$.

- $\operatorname{otp}(C_\alpha) \leq \omega_1$ for all $\alpha < \omega_2$.
- (Coherence) If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$

Define S_i, $i \leq \omega_1$ by:

$$S_i = \{\alpha \colon \operatorname{otp} C_\alpha = i\}$$

If $R_2^*(\aleph_2, \aleph_1)$ were to hold, then in particular there would exist $\beta \in E^{\aleph_2}_{\aleph_1}$ and $i \leq \omega_1$ with S_i stationary in β. But this is impossible since $|\operatorname{Lim}(C_\beta) \cap S_i| \leq 1$ by coherence. □

In the following, for $\mu < \kappa$ we frequently confuse functions $f \colon E^\kappa_{<\mu} \to \mu$ with the associated partitions $E^\kappa_{<\mu} = \bigcup_{i<\mu} S^f_i$, where $S^f_i = \{\alpha < \kappa \colon f(\alpha) = i\}$.

Definition 5.4. If κ is some cardinal, $\mu < \kappa$ is regular uncountable and $f \colon E^\kappa_{<\mu} \to \mu$ is some partition of $E^\kappa_{<\mu}$ into μ many pieces, let $\mathbb{A}(f)$ denote the poset of all μ-closed bounded $D \subseteq E^\kappa_\mu$ such that for all $\alpha \in D$ there exists $i < \mu$ with $\alpha \in \operatorname{Tr}(S^f_i)$, ordered by end-extension.

We will later have occasion to use the following well-known lemma:

Lemma 5.5 ([Bau76], [Mag82])**.** *Suppose that κ is some uncountable regular cardinal and $S \subseteq \kappa$ is stationary. Suppose moreover that $\mathbb{P}$ is a countably closed forcing notion and G is $\mathbb{P}$-generic. Then S remains stationary in $V[G]$.*

5.2 The Proof

Theorem 5.6. *Suppose that there is a weakly compact cardinal. Then in some generic extension of V, $\square(\omega_2)$ holds in conjunction with $R_2^*(\aleph_2, \aleph_1)$.*

Proof. The idea of the proof is to collapse weakly compact κ to become $\aleph_2$, force to add $\square(\kappa)$, and then iterate club-shooting posets as in [Mag82].

Assume that in the ground model V we have $2^\kappa = \kappa^+$. Let $\mathbb{C} = \operatorname{Col}(\omega_1, <\kappa)$ and let $\dot{\mathbb{S}}$ be a name for the poset which adds $\square(\kappa)$ in $V^{\mathbb{C}}$, as in Definition 2.7. Working in $V^{\mathbb{C} * \dot{\mathbb{S}}}$, let

$$\vec{\mathbb{I}} = \langle \langle \mathbb{I}_\gamma \colon \gamma \leq \kappa^+ \rangle, \langle \dot{\mathbb{J}}_\gamma \colon \gamma < \kappa^+ \rangle \rangle$$

be an iteration with $\leq \omega_1$-support such that for each $\gamma < \kappa^+$ $\dot{\mathbb{J}}_\gamma$ is an $\mathbb{I}_\gamma$-name for $\mathbb{A}(\dot{f}_\gamma)$, where $\dot{f}_\gamma$ is some $\mathbb{I}_\gamma$-name for a partition of $E^\kappa_{\aleph_0}$ into $\aleph_1$ many pieces chosen according to some bookkeeping function for exhausting them (see [Mag82] for details). Let $\mathbb{I}$ denote $\mathbb{I}_{\kappa^+}$ and let $G * \dot{H} * \dot{I}$ be generic for $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}$. We claim that the generic extension by $G * \dot{H} * \dot{I}$ is as desired.

Lemma 5.7. $V[G * \dot{H}] \models$ *"for all $\gamma \leq \kappa^+$, $\mathbb{I}_\gamma$ is countably closed and satisfies the κ^+-c.c."*

Proof. Standard. □

Let $\dot{\mathbb{T}} = \check{\mathbb{T}}_{\bigcup \dot{H}}$ be a name for the poset in $V^{\mathbb{C} * \dot{\mathbb{S}}}$ which threads $\bigcup H$. The main claim of the proof is the following:

Lemma 5.8. *For all $\gamma < \kappa^+$ the following hold:*

(A) $V^{\mathbb{C}} \models$ *"forcing with $\mathbb{S} * \dot{\mathbb{I}}_\gamma * \dot{\mathbb{T}}$ doesn't introduce bounded subsets of κ". In other words, $(\kappa^{<\kappa})^{V^{\mathbb{C}*\dot{\mathbb{S}}*\dot{\mathbb{I}}_\gamma*\dot{\mathbb{T}}}} = (\kappa^{<\kappa})^{V^{\mathbb{C}}}$. In particular, $(\kappa^{<\kappa})^{V^{\mathbb{C}*\dot{\mathbb{S}}*\dot{\mathbb{I}}_\gamma}} = (\kappa^{<\kappa})^{V^{\mathbb{C}}}$.*

(B) $V^{\mathbb{C}*\dot{\mathbb{S}}*\dot{\mathbb{I}}_\gamma*\dot{\mathbb{T}}} \models$ *"There exist unboundedly many $\alpha \in E^\kappa_{\aleph_1}$ such that $\alpha \in \mathrm{Tr}\,(S_i^{\dot{f}_\gamma})$ for some $i < \omega_1$." Equivalently, $V^{\mathbb{C}*\dot{\mathbb{S}}*\dot{\mathbb{I}}_\gamma}$ satisfies the same.*

Proof. We proceed by induction on γ. For (A), suppose that $\dot{Y}$ is a $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma * \dot{\mathbb{T}}$-name for a bounded subset of κ. Let $\tilde{M}$ be an elementary substructure of $H(\theta)$ for sufficiently large θ such that

$$\gamma \cup (\kappa + 1) \cup \left\{\dot{Y}, \mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma * \dot{\mathbb{T}}\right\} \subseteq \tilde{M}$$

and $\tilde{M}$ has cardinality κ. Let M be the transitive collapse of $\tilde{M}$. Observe that $\dot{Y}$ and $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma$ are elements of M.

By weak compactness of κ there exist transitive N and an elementary embedding $j \colon M \to N$ with critical point κ. Letting G be M-generic for $\mathbb{C}$, we may lift j to an embedding

$$j \colon M\,[G] \to N\,[G^*]$$

where G^* is N-generic for $j(\mathbb{C}) = \mathrm{Col}(\omega_1, < j(\kappa))$. Arguments as in [Mag82] show that $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma$, $j \restriction \mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma$ are elements of N. By Lemmas 2.11, 2.2 the poset $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma * \dot{\mathbb{T}}$ embeds regularly into $j(\mathbb{C})$ and so there is a $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma * \dot{\mathbb{T}}$-generic of the form $G * \dot{H} * \dot{I} * \dot{J}$ (where G is as above) which is an element of $N[G^*]$.

Working in $N[G^*]$ and letting $\mathbb{S} = \dot{\mathbb{S}}^G$, $H = \dot{H}^{G^*}$, $J = \dot{J}^{G^*}$, we have that $\bigcup H \cup \{(\kappa, \{\bigcup J\})\} \in j(\mathbb{S})$ is a master condition for the embedding $j\colon M[G] \to N[G^*]$, and so we may lift this to an embedding

$$j\colon M[G * \dot{H}] \to N[G^* * \dot{H}^*]$$

where $\dot{H}^*$ names a generic for $j(\mathbb{S})$. Finally, let $\mathbb{I}_\gamma = \dot{\mathbb{I}}_\gamma^{G*\dot{H}}$, $I = \dot{I}^{G*\dot{H}}$, and let p^* be given by

$$p^* = \bigcup (j``I) \cup \{(j(\alpha), \check{\kappa}) : \alpha < \gamma\}$$

Note that $p^* \in N[G^* * \dot{H}^*]$, since $j \restriction \mathbb{I}_\gamma \in N[G^* * \dot{H}^*]$. We claim that p^* is a master condition for the embedding $j\colon M[G * \dot{H}] \to N[G^* * \dot{H}^*]$, i.e. that $p^* \in j(\mathbb{I}_\gamma)$ and $p^* \leq j(p)$ for all $p \in I$. Provided p^* is a condition in $j(\mathbb{I}_\gamma)$ the latter is immediate, and so we concentrate on proving p^* is in fact such a condition.

Let $\langle \tilde{\mathbb{I}}_\delta \colon \delta < j(\gamma)\rangle$ be $j(\langle \mathbb{I}_\delta \colon \delta < \gamma\rangle)$ and let $\langle \dot{\tilde{f}}_\delta \colon \delta < j(\gamma)\rangle$ be $j(\langle \dot{f}_\delta \colon \delta < \gamma\rangle)$. We show by induction on $\delta \leq j(\gamma)$ that $p^* \restriction \delta$ is in $\tilde{\mathbb{I}}_\delta$. If δ is a limit point or is not an element of $j``\gamma$ then this is immediate from the inductive hypothesis and the definition of $p^*$. Thus suppose that $\delta = \varepsilon + 1$ is a successor ordinal with $\delta$ (and hence $\varepsilon$) in $j``\gamma$. Write $\delta = j(\bar{\delta})$, $\varepsilon = j(\bar{\varepsilon})$. By the inductive hypothesis $p^* \restriction \varepsilon \in \tilde{\mathbb{I}}_\varepsilon$, so to show $p^* \restriction \delta \in \tilde{\mathbb{I}}_\delta$ we need only to show that $p^* \restriction \varepsilon \Vdash_{\tilde{\mathbb{I}}_\varepsilon} p^*(\varepsilon) \in \mathbb{A}(\dot{\tilde{f}}_\varepsilon)$.

With this in mind, choose some $\tilde{\mathbb{I}}_\varepsilon$-generic $I^*_\varepsilon \ni p^* \restriction \varepsilon$. We seek to show

$$N[G^*][H^*][I^*_\varepsilon] \models p^*(\varepsilon)[I^*_\varepsilon] \in \mathbb{A}(\dot{\tilde{f}}_\varepsilon)$$

By the inductive hypothesis $p^* \restriction \varepsilon \in \tilde{\mathbb{I}}_\varepsilon$ and by definition it extends $j(p \restriction \bar{\varepsilon})$ for each $p \in I$. Therefore $j\colon M[G][H] \to N[G^*][H^*]$ may be lifted to

$$j\colon M[G][H][I_{\bar{\varepsilon}}] \to N[G^*][H^*][I^*_\varepsilon]$$

where $I_{\bar{\varepsilon}} = I \cap \mathbb{I}_{\bar{\varepsilon}}$. By definition of p^* we have

$$p^*(\varepsilon)[I^*_\varepsilon] = \bigcup_{p \in I} j(p(\bar{\varepsilon})[I_{\bar{\varepsilon}}]) \cup \{\kappa\}$$
$$= \bigcup_{p \in I} p(\bar{\varepsilon})[I_{\bar{\varepsilon}}] \cup \{\kappa\}$$

since $j(p(\bar{\varepsilon})\,[I_{\bar{\varepsilon}}]) = p(\bar{\varepsilon})[I_{\bar{\varepsilon}}]$ for each $p \in I$. Since κ is the critical point of the elementary embedding j, we have $\tilde{f}_\varepsilon \restriction \kappa = f_{\bar{\varepsilon}}$, and since each $p(\bar{\varepsilon})[I_{\bar{\varepsilon}}]$ is an ω_1-closed subset of κ such that for all $\alpha \in p(\bar{\varepsilon})[I_{\bar{\varepsilon}}]$, $\alpha \in \mathrm{Tr}\,(S_i^{f_{\bar{\varepsilon}}})$ for some $i < \omega_1$, it remains only to show that κ itself is a point of reflection. In other words we must show that

$$N[G^*][H^*][I^*_{\bar{\varepsilon}}] \models (\exists i < \omega_1)\left(S_i^{f_{\bar{\varepsilon}}} \text{ is stationary in } \kappa\right)$$

Recall that H^*, I^*_ε are generic for the countably closed posets $j(\mathbb{S})$, $\tilde{\mathbb{I}}_\varepsilon = j(\mathbb{I}_{\bar{\varepsilon}})$ respectively. Therefore by Lemma 5.5, it suffices to show that

$$N[G^*] \models (\exists i < \omega_1)\left(S_i^{f_{\bar{\varepsilon}}} \text{ is stationary in } \kappa\right)$$

Recall that G^* is generic for $j(\mathbb{C}) = \mathrm{Col}\,(\omega_1, < j(\kappa))$. Moreover, $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_{\bar{\varepsilon}} * \dot{\mathbb{T}}$ is countably closed, and so

$$j(\mathbb{C}) \simeq \mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_{\bar{\varepsilon}} * \dot{\mathbb{T}} * \mathrm{Col}\,(\omega_1, < j(\kappa))$$

Therefore we may write $G^* \simeq G * \dot{H} * \dot{I}_{\bar{\varepsilon}} * \dot{J} * \dot{K}$, where $\dot{K}$ names another generic for $\mathrm{Col}\,(\omega_1, < j(\kappa))$. Since this poset is countably closed, by Lemma 5.5 again it suffices to show that

$$N[G][H][I_{\bar{\varepsilon}}][J] \models (\exists i < \omega_1)\left(S_i^{f_{\bar{\varepsilon}}} \text{ is stationary in } \kappa\right)$$

which is itself implied by

$$V[G][H][I_{\bar{\varepsilon}}][J] \models (\exists i < \omega_1)\left(S_i^{f_{\bar{\varepsilon}}} \text{ is stationary in } \kappa\right)$$

But the statement above is immediate since κ remains a cardinal in $V[G][H][I_{\bar{\varepsilon}}][J]$ by part (A) of the inductive hypothesis, and so long as κ remains a cardinal, in any partition of κ

into ω_1 many sets, at least one of the sets in the partition is stationary. This concludes the proof that p^* is a master condition, and therefore allows us to lift the embedding to

$$j\colon M[G][H][I] \to N[G^*][H^*][I^*]$$

Finally, it is easy to see that $\bigcup J$ is a $\mathbb{T}$-master condition for the embedding above, and so we may lift j one more time to

$$j\colon M[G][H][I][J] \to N[G^*][H^*][I^*][J^*]$$

for some $j(\mathbb{T})$-generic $J^* \ni \bigcup J$. Moreover, the argument above shows that this is witnessed by some master condition $m \in N[G^*]$. Now we are done by an argument as in [Mag82]. The key point is that $j(\dot{Y})[G^* * \dot{H}^* * \dot{I}^* * \dot{J}^*] = \dot{Y}[G * \dot{H} * \dot{I} * \dot{J}] \in N[G^*]$, and hence by elementarity $\dot{Y}[G * \dot{H} * \dot{I} * \dot{J}] \in M[G]$.

For (B), first note that it suffices to show that the desired result holds in $V^{\mathbb{C}*\dot{\mathbb{S}}*\dot{\mathbb{I}}_\gamma}$ by part (A). With this in mind, let $G * \dot{H} * \dot{I}$ be generic for $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_\gamma$, and let

$$j\colon M[G][H][I] \to N[G^*][H^*][I^*]$$

be a generic elementary embedding as in the proof of part (A). Furthermore, let $\beta < \kappa$ be arbitrary. We seek to show that

$$M[G][H][I] \models (\exists \alpha < \kappa)\,(\exists i < \omega_1)\left(\alpha > \beta \wedge \alpha \in \operatorname{Tr}\left(S^{\dot{f}_i^\gamma}\right)\right)$$

By elementarity it suffices to show

$$N[G^*][H^*][I^*] \models (\exists \alpha < j(\kappa))\,(\exists i < \omega_1)\left(\alpha > \beta \wedge \alpha \in \operatorname{Tr}\left(S^{\dot{\bar{f}}_i^{j(\gamma)}}\right)\right)$$

But the veracity of the latter statement may be witnessed by taking $\alpha = \kappa$, by an argument identical to that in part (A). □

Lemma 5.9. *The forcing $\mathbb{C} * \dot{\mathbb{S}} * \dot{\mathbb{I}}_{\kappa^+}$ has the κ^+-c.c.*

Proof. Standard, using the fact that we assumed $2^\kappa = \kappa^+$ in the ground model. □

Now Theorem 5.6 follows from Lemmas 5.7, 5.8, and 5.9. More precisely, Lemmas 5.7, 5.9 give preservation of ω_1 and all cardinals $\geq \kappa^+$ and Lemma 5.8 gives preservation of κ (which becomes $\aleph_2$) and shows that the posets $\mathbb{A}(f_\gamma)$ add closed *unbounded* sets witnessing $R_2^*(\aleph_2, \aleph_1)$.

$\square$

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