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ORIGINAL PAPER

Optimal parameter selection for constraint-following control for mechanical systems based on Stackelberg game

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Abstract This paper proposes an optimal parameter design of control scheme for mechanical systems by adopting the Stackelberg game theory. The goal of the control is to drive the mechanical system to follow the prescribed constraints. The system uncertainty is (possibly fast) time-varying and bounded. A β-measure is defined to gauge the performance. A robust control is proposed to render the β-measure uniformly ultimately bounded. This control scheme is based on feasible design parameters (i.e., parameters within prescribed range), and the choices of these parameters may not be unique. For optimal (unique) parameter selection, a Stackelberg game is formulated. By taking the control design parameters as the players, for each player, a cost function is built with the consideration of the performance cost, the time cost and the control cost. To follow, the Stackelberg strategy is then carried out via backward induction, which results in the choice of the optimal parameters.

Keywords Mechanical systems · Uncertainty · Constraint-following control · Robust control · Stackelberg game

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1 Introduction

A *system-task-specific* control design strategy is employed in this paper. It refers to a kind of control design strategy that is specifically explored for particular dynamical system with particular control task. By this, we mean the control design takes the characteristics of the *system* and *task* into consideration, and perhaps even taking the advantage of them, such that the resulting controller is more targeted and can render better control performance.

This is in contrast to the more popular *universal* control design strategy, whose existence has been as long as the control subject. By this, we mean the control design is intended to be as broad as possible, encompassing as many varieties of system characteristics as possible. This design conception gave birth to the successful establishments such as *Pontryagin's principle*, *Bellman's dynamic programming*, H_∞, *sliding mode control*, and *adaptive control* practicably cover all major accomplishments in control in the past seven decades. In this recognition, a research work is considered superior to the previous work if it is based on less stringent assumptions, which in turn means it is applicable to a broader class of systems. This can be viewed as a *mathematical triumph*.

The *system-task-specific* control design strategy, on the other hand, has been less addressed in the past. One might assert that, given the ultimate goal of any control theory to be its successful application to a physical system, a *physical triumph* may be considered as another option (but not so much as a replacement). When more specifics of the system characteristics and tasks are used in the design, it is generally anticipated that the control is in more coherence with the said framework. As an outstanding representative of system-task-specific control, a new approach of constraint-following control is introduced by Chen and his cooperators [1–7], which guides the design of control strategy in this paper.

We consider the control for mechanical systems, whose tasks are prescribed by (holonomic and/or nonholonomic) constraints [8–10]. The goal is to design a feasible control which renders the mechanical system to follow the constraints. In this consideration, the Lagrange's virtual work principle-based setting fits right into the control design [11,12]. We then add additional consideration that system uncertainty is in the presence, which is unknown, time-varying, and bounded. The only available information of the uncertainty is its possible bound. The desired control should guarantee prescribed performance even in the presence of the uncertainty. Compared to previous methods of uncertainty management (e.g., [13–19]), this paper shows an alternative way. The control design is proposed, which is facilitated by three feasible design parameters. That is, the choices of the three parameters only need to be within prescribed range, hence may not be unique.

While there are many choices of the design parameters, one may anticipate an optimal trio. By the nature of the problem, the most fitting and realistic way of choosing the optimal parameters is via the Stackelberg game, which is a leader–follower game. In this game competition, the three players (three parameters) are selected sequentially via backward induction, rather than simultaneously (as in Nash or Pareto) [20–23]. This consideration allows a *full* communication (as opposed to *no* communication, such as in Nash or Pareto) between the players (the parameters) for their choices, which in turn should enhance the system performance.

The paper possesses three significant contributions. First, a constraint following task is formulated for the motion control of mechanical systems, for which a robust control with three tunable parameters is proposed to render the constraint-following error (i.e., the β-measure) to be uniformly bounded and uniformly ultimately bounded. Second, for the seeking of optimal design parameters, an optimization problem oriented by a three-player Stackelberg game is formulated. Based on performance analysis, cost functions are constructed in a static way, and then this problem is proven to be valid and tractable by solving the Stackelberg strategy. Third, it is proved that the Stackelberg strategy (i.e., the optimal design parameters) can be determined by the way of backward induction, such that this optimization problem is successfully solved. This paper is among the first endeavors that combine constraint following and three-player Stackelberg game for control design of uncertain mechanical systems.

2 Stackelberg game

2.1 What is Stackelberg game?

Consider a game with N players, the rules of the game then impose the following mappings

$$J_i(\cdot) : \prod_{i=1}^{N} D_i \to \mathbf{R} \quad i = 1, 2, ..., N, \tag{1}$$

where D_i and $J_i(\cdot)$ are, respectively, the decision set and cost function for player i. Note that the player, decision set, and cost function are the three main elements of a game.

Each player naturally desires to attain the smallest possible cost to himself, such that the ideal decision $d^* \in \prod_{i=1}^{N} D_i$ to all the players satisfies

$$J_i(d^*) \le J_i(d) \quad \forall d \in \prod_{i=1}^{N} D_i. \tag{2}$$

It shows that each player's cost is simultaneously minimized by the decision N-tuple d^*. However, in general, such utopia does not exist, and the players should make their decisions according to the specific rules of the game that they enter in. In a *Stackelberg game*, the players strive to minimize everyone's cost *sequentially* rather than *simultaneously* [24].

2.2 Three-player Stackelberg game

As a special case, a three-player Stackelberg game is shown in Fig. 1, in which player 1 (as the leader) makes his or her decision first, Player 2 (as the follower 1) then chooses a decision according to player 1's decision, and finally Player 3 (as the follower 2) chooses a decision according to player 1 and 2's decisions. The details are summarized as:

Step 1: if player 1 chooses a decision d_1 from the feasible set D_1, then player 2 chooses a decision $d_2 = \Omega_2(d_1)$;
Step 2: if players 1 to 2 choose decision $d_{1,2}$, then player 3 chooses a decision $d_3 = \Omega_3(d_1, d_2)$ from the feasible set D_3;
Step 3: based on $(d_1, d_2 = \Omega_2(d_1), d_3 = \Omega_3(d_1, d_2))$, player 1 chooses a decision d_1^* by minimizing the cost function $J_1(d_1, d_2, d_3)$;
Step 4: Based on d_1^*, player 2 chooses a decision $d_2^* = \Omega_2(d_1^*)$ that minimizes the cost function $J_2(d_1^*, d_2, d_3)$;
Step 5: Based on $d_{1,2}^*$, player 3 chooses a decision $d_3^* = \Omega_3(d_1^*, d_2^*)$ that minimizes the cost function $J_3(d_1^*, d_2^*, d_3)$.

Remark 1 As a salient feature of three-player Stackelberg game, the leader make his or her practical decision d_1^* based on an anticipation of the followers' reactions (i.e., $\Omega_2(d_1)$ and $\Omega_3(d_1, d_2)$) to decision d_1 that the leader might take.

2.3 Stackelberg strategy of three-player game

The *Stackelberg strategy* of a three-player game can be solved by *backward induction*, for which we reach to following definition.

Definition 1 When player 3 gets the decisions at Step 2, he or she will face the following problem: for $d_{1,2}$ previously chosen by players 1 to 2

$$\min_{d_3 \in D_3} J_3(d_1, d_2, d_3). \tag{3}$$

Assume that, for each $d_{1,2} \in D_{1,2}$, player 3's optimization problem has a unique solution $\Omega_3(d_1, d_2)$. Since player 1 can solve 2 to 3's problems as well as 2 to 3 can, he or she can anticipate the functions $\Omega_2(d_1)$ and $\Omega_3(d_1, d_2)$, so player 1's problem at the Step 3 amounts to

$$\min_{d_1 \in D_1} J_1(d_1, \Omega_2(d_1), ..., \Omega_3(d_1, \Omega_2(d_1))). \tag{4}$$

Assume that this optimization problem for player 1 also has a unique solution d_1^*. By substituting d_1^* into $d_2 = \Omega_2(d_1)$, player 2 can obtain his or her decision $d_2^* = \Omega_2(d_1^*)$, and then by substituting $d_{1,2}^*$ into $d_3 = \Omega_3(d_1, d_2)$, player 3 can obtain his or her decision $d_3^* = \Omega_3(d_1^*, d_2^*)$. The *backward induction* approach resulted optimal decision set (d_1^*, d_2^*, d_3^*) is the *Stackelberg strategy*.

Remark 2 According to above presentations, the Stackelberg game theory is instructive for multi-parameter optimal design problem by taking the decisions d_i and the cost functions J_i, respectively, as the design parameter and the objective function, hence, is chosen to guide the optimal design in this work.

3 Constraint analysis of uncertain mechanical system

Consider a mechanical system described as (see [25, 26]):

$$S(z(t), d(t), t)\ddot{z}(t) + K(z(t), \dot{z}(t), d(t), t)\dot{z}(t) + G(z(t), d(t), t) + D(z(t), \dot{z}(t), d(t), t) = \tau(t). \tag{5}$$

Here $z, \dot{z}, \ddot{z} \in \mathbf{R}^n$ are, respectively, the coordinate, the velocity and the acceleration, $t \in \mathbf{R}$ is the independent variable, and $\tau \in \mathbf{R}^n$ is the control input. $d \in U \subset \mathbf{R}^p$ presents the (possibly fast) time-varying uncertainty with the (possible) compact bound $U \subset \mathbf{R}^p$. $S(z, d, t) > 0$, $K(z, \dot{z}, d, t)\dot{z}$, $G(z, d, t)$

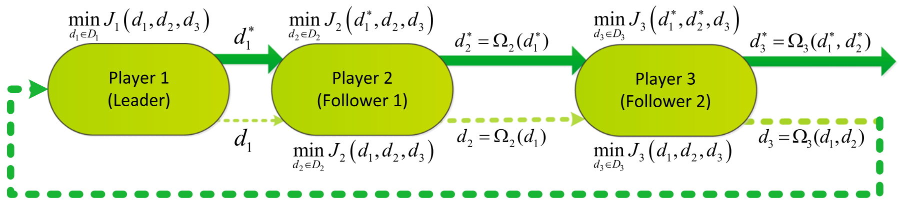


Fig. 1 Three-player Stackelberg game

and $D(z, \dot{z}, d, t)$ are, respectively, the inertia matrix, the Coriolis/centrifugal force, the gravitational force, and the friction force or/and other external disturbances. In general, formula (5) gives the generalized coordinate form of the dynamics model of uncertain mechanical systems (such as mechanical arms, robots, and autonomous vehicles), where z represents the generalized coordinate. Based on such dynamics model, an universal motion control and optimization design method for *general* (rather than *specific* structural/functional) mechanical systems is investigated in this paper, that is to say, the dynamic model described as (5) provides the model basis for later controller design. By this, for any machine, as long as we can obtain its dynamic model in the form of (5), the control and optimization method proposed in this paper will be effective. This just reflects the effectiveness and universality of the method proposed in this paper.

The *first-order* form constraints which the system needs to follow can be expressed as

$$\sum_{i=1}^{n} \alpha_{li}(z, t)\, \dot{z}_i = c_l(z, t), \quad l = 1, ..., m, \tag{6}$$

where $\dot{z}_i$ is the ith component of $\dot{z}$, $1 \le m \le n$, both $\alpha_{li}(\cdot)$ and $c_l(\cdot)$ are C^1 in z and t. They can be shown in matrix form as

$$\alpha(z, t)\, \dot{z} = c(z, t), \tag{7}$$

where $\alpha = [\alpha_{li}]_{m \times n}$, $c = [c_1\ c_2\ \cdots\ c_m]^T$. By differentiation, it can be shown in the *second-order* form as

$$\alpha(z, t)\, \ddot{z} = b(\dot{z}, z, t), \tag{8}$$

where $b = [b_1\ b_2\ \cdots\ b_m]^T$. Here, the constraint (8) is *consistent* and may be holonomic and/or nonholonomic. $\alpha(z, t)$ is of full rank for each (z, t) and $\text{rank}(\alpha(z, t)) \ge 1$.

Remark 3 The constraints as (6)–(8) actually denote the servo (or control) constraints that come from the desired control tasks (i.e., desired system performance or motion pattern), such as optimal performance, stabilization, robustness, and trajectory tracking. By formulating the desired control tasks as a set of servo constraints as (6)–(8), a proper control is proposed to drive the concerned system to follow such servo constraints (in the sense of uniform boundedness and uniform ultimate boundedness) approximately in this paper. By this, a problem of approximate constraint following is arisen and addressed.

4 Robust control design: approximate constraint-following

A control τ is expected to render the system (5) to be approximate constraint-following. This system can be decomposed into nominal and uncertain portions with $S(z, d, t) = \bar{S}(z, t) + \Delta S(z, d, t)$, $K(z, \dot{z}, d, t) = \bar{K}(z, \dot{z}, t) + \Delta K(z, \dot{z}, d, t)$, $G(z, d, t) = \bar{G}(z, t) + \Delta G(z, d, t)$, and $D(z, d, t) = \bar{D}(z, t) + \Delta D(z, d, t)$. Here $\bar{S} > 0$, the functions $\bar{(\cdot)}$ and $\Delta(\cdot)$ are continuous, and denote the nominal and uncertain portions, respectively. Let $W := \bar{S}^{-1}$, $\Delta W := S^{-1} - \bar{S}^{-1}$, $E := \bar{S}S^{-1} - I$; hence, $\Delta W = WE$.

Theorem 1 (Udwadia and Kalaba [27]) *Consider the mechanical system* (5) *without uncertainty and the constraint* (8). *The constraint force*

$$Q_c(\dot{z}, z, t) = \bar{S}^{1/2}(z, t) \left(\alpha(z, t)\, \bar{S}^{-1/2}(z, t)\right)^{+} \left[b(\dot{z}, z, t) + \alpha(z, t)\, \bar{S}^{-1}(z, t) \left(\bar{K}(z, \dot{z}, t)\, \dot{z} + \bar{G}(z, t) + \bar{D}(z, t)\right)\right] \tag{9}$$

observes the Lagrange's form of d'Alembert's principle [28] *and renders the system to meet the constraint.*

Remark 4 By taking the whole system (5) as nominal and uncertain two parts, a partial controller equal to constraint force Q_c (9) can be applied on the nominal part to follow the constraint (8). If we can design a compensation controller to handle the uncertain part, the whole system will be controlled. This paves the way for later control design. In this sense, constraint force Q_c (9) plays a role of bridge in control design of this paper.

Let

$$\vartheta\left(z, \dot{z}, t\right) := \alpha\left(z, t\right) \dot{z} - c\left(z, t\right). \tag{10}$$

It is used to measure the degree to which the constraints are followed. Based on decomposition, we obtain ϑ-dynamics as

$$\begin{aligned}\dot{\vartheta} &= \alpha \ddot{z} - b \\ &= \alpha\left[S^{-1}\left(-K\dot{z} - G - D\right) + S^{-1}\tau\right] - b \\ &= \alpha[W\left(-\bar{K}\dot{z} - \bar{G} - \bar{D}\right) + W\tau \\ &\quad + W\left(-\Delta K\dot{z} - \Delta G - \Delta D\right) \\ &\quad + \Delta W\left(-K\dot{z} - G - D + \tau\right)] - b.\end{aligned} \tag{11}$$

Assumption 1 There exist a constant $\kappa \in (0, \infty)$ and a function $f(\kappa, \cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m$ such that (i) the function $f(\kappa, \cdot)$ is in the range space of α, (ii) there are a Lyapunov function $V(\cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}_+$, and strictly increasing functions $\gamma_i(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\begin{aligned}&\gamma_i(0) = 0, \\ &\lim_{r\to\infty} \gamma_i(r) = \infty, \quad i = 1, 2, 3,\end{aligned} \tag{12}$$

such that for all $(\kappa, \vartheta, z, \dot{z}, t) \in (0, \infty) \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$,

$$\gamma_1(\|\vartheta\|) \le V(\vartheta, t) \le \gamma_2(\|\vartheta\|), \tag{13}$$

$$\frac{\partial^T V(\vartheta, t)}{\partial t} + \frac{\partial V(\vartheta, t)}{\partial \vartheta} f(\kappa, \vartheta, t) \le -\kappa\gamma_3(\|\vartheta\|). \tag{14}$$

Choose

$$\begin{aligned}p_1\left(\kappa, \vartheta, z, \dot{z}, t\right) = \bar{S}^{1/2}\left(z, t\right)&\left(\alpha\left(z, t\right)\bar{S}^{-1/2}\left(z, t\right)\right)^+ \\ &\left[f\left(\kappa, \vartheta, t\right) + b\left(\dot{z}, z, t\right)\right. \\ &+ \alpha\left(z, t\right)\bar{S}^{-1}\left(z, t\right)\left(\bar{K}\left(z, \dot{z}, t\right)\dot{z}\right. \\ &\left.\left. + \bar{G}\left(z, t\right) + \bar{D}\left(z, t\right)\right)\right].\end{aligned} \tag{15}$$

Note that, the partial controller p_1 as (15) is a derivative of constraint force Q_c as (9). The only difference between them is that the partial controller p_1 has an extra part of function $f(\kappa, \vartheta, t)$. By this, constraint force Q_c enters the controlled system indirectly through the partial controller p_1 in the control process.

Remark 5 The main purpose of Assumption 1 is to let the nominal systems $\bar{S}\ddot{z} + \bar{K}\dot{z} + \bar{G} + \bar{D} = \tau$ to be uniformly asymptotically stable at the origins $z = 0$ under the action of the partial controller $\tau = p_1$. For this, it should date back to the original source of the core ideas of the controller design in this paper, that is, the *boundedness control theory* which was proposed by Corless and Leitmann (see Ref. [29]). According to it, to let the uncontrolled nominal systems to be uniformly asymptotically stable at the origin is one of the most important premise for control design. By this, Assumption 1 is arisen. In recent years, the *boundedness control theory* has matured, and many works (such as Refs. [1–10]) have been explored based on it.

Theorem 2 *Subject to Assumption 1, the control* $\tau = p_1(\kappa, \vartheta, z, \dot{z}, t)$ *renders* [14]

$$\alpha[W\left(-\bar{K}\dot{z} - \bar{G} - \bar{D}\right) + W\tau] - b = f. \tag{16}$$

Remark 6 Due to the uncertainty, the constraint force as (9) is not enough to render $\alpha\dot{z} = c$; hence, an *approximate constraint-following* task is considered.

Assumption 2 (1) There exists a possibly unknown constant $\rho_E > -1$ such that for all $(z, t) \in \mathbf{R}^n \times \mathbf{R}$,

$$\frac{1}{2}\min_{d\in U} \lambda_m\left(E\left(z, d, t\right) + E^T\left(z, d, t\right)\right) \ge \rho_E. \tag{17}$$

(2) There exist a possibly unknown scalar ζ, and a known function $\rho(\cdot) : (0, \infty) \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that for all $(\kappa, \vartheta, z, \dot{z}, t) \in (0, \infty) \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, $d \in U$,

$$\begin{aligned}&\|E(z, d, t)(-K(z, \dot{z}, d, t)\dot{z} - G(z, d, t) \\ &\quad - D(z, d, t) + p_1(\kappa, \vartheta, z, \dot{z}, t)) \\ &\quad - (\Delta K(z, \dot{z}, d, t)\dot{z} + \Delta G(z, d, t) \\ &\quad + \Delta D(z, d, t))\| \le \zeta\rho(\kappa, \vartheta, z, \dot{z}, t).\end{aligned} \tag{18}$$

Here $\lambda_m(\cdot)$ represents the minimum eigenvalue of the concerned matrix.

Remark 7 The main purpose of Assumption 2(1) is to keep the effect of uncertainty on the possible deviation of S from $\bar{S}$ to be within a certain threshold, which is unidirectional (i.e., not bounded in one direction), such that the uncertainty existing in the mass matrix S is robust and controllable. This just reflects a general premise of robust control design. The main purpose of Assumption 2(2) is to find an appropriate parameter ζ along with the function $\rho(\cdot)$ to measure the comprehensive influences of uncertainty on system dynamics, based on which a robust control is subsequently designed. As the comprehensive uncertainty measure ζ is calculated by taking the uncertainty bounds but not the actual influences, so we actually consider the maximum (most conservative) influence of the uncertainty in the following control design. This just reflects the general way (as shown in Refs. [1–10]) of uncertainty handling in robust control design.

Now the following control is proposed:

$$\tau(t) = p_1(\kappa, \vartheta(t), z(t), \dot{z}(t), t) + p_2(s, \gamma, \kappa, \vartheta(t), z(t), \dot{z}(t), t), \tag{19}$$

with p_1 as (15) and p_2 as

$$p_2(s, \gamma, \kappa, \vartheta, z, \dot{z}, t) = -\gamma W(z,t)\alpha^T(z,t)\frac{\partial V(\vartheta,t)}{\partial \vartheta} \left\| W(z,t)\alpha^T(z,t)\frac{\partial V(\vartheta,t)}{\partial \vartheta}\right\|^s \rho^{s+2}(\kappa, \vartheta, z, \dot{z}, t), \tag{20}$$

where the scalars $s, \gamma, \kappa > 0$ are constant design parameters and will be optimally designed later based on Stackelberg game theory.

Theorem 3 *Subject to Assumptions 1–2, the control (19) renders the performance of uniform boundedness and uniform ultimate boundedness for the system (5).*

Proof Choose $V(\cdot)$ that subjects to Assumption 1 as the Lyapunov function candidate. Its derivative is given as

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \vartheta}\dot{\vartheta}. \tag{21}$$

Based on Assumption 1 and by (16), we have

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \vartheta}\left\{\alpha\left[W\left(-\bar{K}\dot{z} - \bar{G} - \bar{D}\right) + Wp_1\right] - b\right\}\\ &= \frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial \vartheta} f\\ &\le -\kappa\gamma_3(\|\vartheta\|).\end{aligned} \tag{22}$$

Define $\epsilon^T := \frac{\partial^T V}{\partial \vartheta}\alpha W$, such that $\epsilon = W\alpha^T\frac{\partial V}{\partial \vartheta}$, and $\epsilon \in \mathbf{R}^m$. Next, by (18) and with $\Delta W = WE$,

$$\begin{aligned}&\frac{\partial^T V}{\partial \vartheta}\alpha[\Delta W(-K\dot{z} - G - D + p_1) + W(-\Delta K\dot{z} - \Delta G - \Delta D)]\\ &= \overbrace{\frac{\partial^T V}{\partial \vartheta}\alpha W}^{=\epsilon^T} [E(-K\dot{z} - G - D + p_1) + (-\Delta K\dot{z} - \Delta G - \Delta D)]\\ &= \epsilon^T\,[E(-K\dot{z} - G - D + p_1) + (-\Delta K\dot{z} - \Delta G - \Delta D)]\\ &\le \|\epsilon\|\,\|[E(-K\dot{z} - G - D + p_1) + (-\Delta K\dot{z} - \Delta G - \Delta D)]\|\\ &\le \zeta\,\|\epsilon\|\,\rho.\end{aligned} \tag{23}$$

By (20) and with $\Delta W = WE$,

$$\begin{aligned}\frac{\partial^T V}{\partial \vartheta}\alpha(W + \Delta W)p_2 = {}&\frac{\partial^T V}{\partial \vartheta}\alpha W \left(-\gamma W\alpha^T\frac{\partial V}{\partial \vartheta}\left\|W\alpha^T\frac{\partial V}{\partial \vartheta}\right\|^s\rho^{s+2}\right)\\ &+ \frac{\partial^T V}{\partial \vartheta}\alpha WE \left(-\gamma W\alpha^T\frac{\partial V}{\partial \vartheta}\left\|W\alpha^T\frac{\partial V}{\partial \vartheta}\right\|^s\rho^{s+2}\right).\end{aligned} \tag{24}$$

By a direct algebra, we can show that

$$\begin{aligned}&\frac{\partial^T V}{\partial \vartheta}\alpha W\left(-\gamma W\alpha^T\frac{\partial V}{\partial \vartheta}\left\|W\alpha^T\frac{\partial V}{\partial \vartheta}\right\|^s\rho^{s+2}\right)\\ &= -\gamma\epsilon^T\epsilon\,\|\epsilon\|^s\,\rho^{s+2}\\ &= -\gamma\|\epsilon\|^{s+2}\rho^{s+2}.\end{aligned} \tag{25}$$

By Assumption 2(1) and adopting the Rayleigh's principle [30], we have

$$\frac{\partial^T V}{\partial \vartheta}\alpha W E\left(-\gamma W\alpha^T\frac{\partial V}{\partial \vartheta}\left\|W\alpha^T\frac{\partial V}{\partial \vartheta}\right\|^s\rho^{s+2}\right)$$
$$=-\frac{1}{2}\gamma\overbrace{\frac{\partial^T V}{\partial \vartheta}\alpha W}^{=\epsilon^T}(E+E^T)\overbrace{W\alpha^T\frac{\partial V}{\partial \vartheta}}^{=\epsilon}\left\|W\alpha^T\frac{\partial V}{\partial \vartheta}\right\|^s\rho^{s+2}$$
$$=-\frac{1}{2}\gamma\epsilon^T(E+E^T)\epsilon\left\|\epsilon\right\|^s\rho^{s+2}$$
$$\leq-\frac{1}{2}\gamma\epsilon^T\lambda_m(E+E^T)\epsilon\left\|\epsilon\right\|^s\rho^{s+2}$$
$$\leq-\gamma\rho_E\left\|\epsilon\right\|^{s+2}\rho^{s+2}. \tag{26}$$

Combining (25) and (26) yields

$$\frac{\partial^T V}{\partial \vartheta}\alpha(W+\Delta W)p_2\leq-\gamma(1+\rho_E)\left\|\epsilon\right\|^{s+2}\rho^{s+2}. \tag{27}$$

With (22)–(27), we have

$$\dot V\leq-\kappa\gamma_3(\|\vartheta\|)+\zeta\overbrace{\|\epsilon\|\rho}^{=:x}-\gamma(1+\rho_E)\left\|\epsilon\right\|^{s+2}\rho^{s+2}$$
$$=-\kappa\gamma_3(\|\vartheta\|)+\zeta x-\gamma(1+\rho_E)x^{s+2} \tag{28}$$

Recalling $\rho>0$, define a function

$$f(x):=\zeta x-\gamma(1+\rho_E)x^{s+2}, \tag{29}$$

with $x>0$. Its first-order and second-order derivatives can be obtained as

$$f^{'}(x)=\zeta-\gamma(1+\rho_E)(s+2)x^{s+1} \tag{30}$$

and

$$f^{''}(x)=-\gamma(1+\rho_E)(s+2)(s+1)x^s<0. \tag{31}$$

When $f^{'}(x)=0$, we have

$$x=\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{1}{s+1}}; \tag{32}$$

hence, the maximum of f can be obtained as

$$f_{\max}=\gamma(1+\rho_E)(s+1)\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}, \tag{33}$$

such that we have

$$\dot V\leq-\kappa\gamma_3(\|\vartheta\|)+\gamma(1+\rho_E)(s+1)\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}. \tag{34}$$

This in turn means that $\dot V$ is negative definite for all ϑ such that

$$-\kappa\gamma_3(\|\vartheta\|)+\gamma(1+\rho_E)(s+1)\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}<0, \tag{35}$$

that is

$$\gamma_3(\|\vartheta\|)>\frac{\gamma(1+\rho_E)(s+1)}{\kappa}\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}=:h(\zeta,s,\gamma,\kappa). \tag{36}$$

Here ζ is bounded since the bounding set of uncertainty is compact. The uniform boundedness performance follows Chen and Leitmann [31]. That is, given any $r>0$ with $\|\vartheta_0\|\leq r$, where $\vartheta_0=\vartheta(t_0)$ and t_0 is the initial time, there is a $d(r)$ given by

$$d(r)=\begin{cases}(\gamma_1^{-1}\circ\gamma_2)(r), & \text{if } r>R,\\ (\gamma_1^{-1}\circ\gamma_2)(R), & \text{if } r\leq R.\end{cases} \tag{37}$$
$$R=\gamma_3^{-1}(h). \tag{38}$$

such that $\|\vartheta(t)\|\leq d(r)$ for all $t\geq t_0$. Uniform ultimate boundedness also follows. That is, given any $\overline{d}$ with

$$\overline{d}>(\gamma_1^{-1}\circ\gamma_2)(R), \tag{39}$$

we have $\|\vartheta(t)\|\leq\overline{d},\ \forall t\geq t_0+T(\overline{d},r)$, with

$$T(\overline{d},r)=\begin{cases}0, & \text{if } r\leq\overline{R},\\ \frac{\gamma_2(r)-\gamma_1(\overline{R})}{\gamma_3(\overline{R})-h}, & \text{otherwise},\end{cases} \tag{40}$$
$$\overline{R}=(\gamma_2^{-1}\circ\gamma_1)(\overline{d}). \tag{41}$$

The performance is guaranteed. □

Remark 8 As mentioned in the Introduction, this paper employs an system-task-specific control design strategy. Especially, the controlled system refers to the uncertain mechanical system described as (5), the control task refers to the constraint described as (7) and (8), and the specific controller refers to the robust controller described as (19).

Remark 9 The design parameters s, γ, κ will affect the size of the finite entering time T, the ultimate boundedness ball $\bar{d}$ and the control input τ. Inspired by this, an optimal design problem with multiple objectives and multiple design parameters to yield out the optimal design parameters s^*, γ^* and κ^* is arisen. From the view of game theory, by taking s, γ and κ as the players, the seeking of optimal design parameters s^*, γ^* and κ^* can be formulated as a three-player game.

5 Optimal design based on three-player Stackelberg game

We consider a three-player Stackelberg game, in which the players are s, γ and κ; the decision sets, for s is $D_1 = (0, \infty)$, for γ is $D_2 = (0, \infty)$ and for κ is $D_3 = (0, \infty)$; the cost functions will be formulated as follows.

5.1 Cost functions

For a comprehensive index of system performance, we first explore more on performance of the concerned mechanical system. First, with the performance of uniform boundedness and uniform ultimate boundedness shown as (37)–(41), $(\gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))$ can be seen as the upper bound of the steady state performance. Ideally, this upper bound is equal to zero; however, it is hardly to reach, and so the error is almost inevitable in practical issues. In this regard, $(\gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))$ can serve as the error of the system performance that desired to be the smaller, the better. Inspired by this, we denote a measure for *performance cost*:

$$\eta_\infty(\zeta, s, \gamma, \kappa) := (\gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1})\,(h(\zeta, s, \gamma, \kappa))\,. \tag{42}$$

Second, define $\overline{d} = (\gamma_1^{-1} \circ \gamma_2)(R) + e$, with $e \geq 0$ a constant. Taking it and (38) into (41) yields

$$\overline{R} = \gamma_3^{-1}(h(\zeta, s, \gamma, \kappa)) + (\gamma_2^{-1} \circ \gamma_1)(e). \tag{43}$$

By introducing (43), the finite entering time T as (40) can be rewritten as: if $r \leq \overline{R}$, $T(\kappa, \gamma) = 0$; if $r > \overline{R}$

$$T(\kappa, \gamma) = \frac{\gamma_2(r) - (\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1)(e) - (\gamma_1 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))}{(\gamma_3 \circ \gamma_2^{-1} \circ \gamma_1)(e)}. \tag{44}$$

It shows that $-(\gamma_1 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))$ serves as a measure of finite entering time that desired to be the closer to $-\gamma_2(r) + (\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1)(e)$, the better. Inspired by this, we denote a measure for *time cost*:

$$\eta_T(\zeta, s, \gamma, \kappa) := (\gamma_1 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa)). \tag{45}$$

Third, the control presented as (19) shows that s, γ and κ serve as measures of control effort that desired to be the smaller, the better. Inspired by this, we denote three measures for *control cost*: s^2 (for player s), γ^2 (for player γ), and κ^2 (for player κ).

We now propose the following cost functions, respectively, for s, γ and κ:

$$J_1(s, \gamma, \kappa) := \eta_\infty^2(\zeta, s, \gamma, \kappa) - \eta_T^2(\zeta, s, \gamma, \kappa) + s^2 =: \underbrace{J_{11}(s, \gamma, \kappa)}_{\text{performance cost}} + \underbrace{J_{12}(s, \gamma, \kappa)}_{\text{time cost}} + \underbrace{J_{13}(s)}_{\text{control cost}}, \tag{46}$$

$$J_2(s, \gamma, \kappa) := \eta_\infty^2(\zeta, s, \gamma, \kappa) - \eta_T^2(\zeta, s, \gamma, \kappa) + \gamma^2 =: \underbrace{J_{21}(s, \gamma, \kappa)}_{\text{performance cost}} + \underbrace{J_{22}(s, \gamma, \kappa)}_{\text{time cost}} + \underbrace{J_{23}(\gamma)}_{\text{control cost}}, \tag{47}$$

$$J_3(s, \gamma, \kappa) := \eta_\infty^2(\zeta, s, \gamma, \kappa) - \eta_T^2(\zeta, s, \gamma, \kappa) + \kappa^2 =: \underbrace{J_{31}(s, \gamma, \kappa)}_{\text{performance cost}} + \underbrace{J_{32}(s, \gamma, \kappa)}_{\text{time cost}} + \underbrace{J_{33}(\kappa)}_{\text{control cost}}. \tag{48}$$

We notice that $\eta_\infty(\zeta, s, \gamma, \kappa)$ and $\eta_T(\zeta, s, \gamma, \kappa)$ depend on ζ. The exact value of ζ can be calculated with (18) based on the value of the bound of uncertainty. We would like to emphasize that, in the control and the cost functions, all the parameters (including the model parameters, the design parameters, the uncertain parameters) are bounded, and all the functions (including the Lyapunov function $V(\cdot)$, the auxiliary functions $f(\cdot)$, $\gamma_{1,2,3}(\cdot)$, the constructed function $h(\cdot)$) with bounded variable are also bounded; hence, the control and the cost functions are finite, and the control policies are admissible.

Remark 10 It can be seen that the specific form of the cost functions can be further determined with specific functions $\gamma_{1,2,3}(\cdot)$. As the choices of the functions $\gamma_{1,2,3}(\cdot)$ are not unique, the specific form of the cost function cannot be uniquely determined here. However, we know that $\gamma_{1,2,3}(\cdot)$ are strictly increasing, such that the effect of strategies on cost functions can be clearly analyzed. By this, instead of the specific form of the cost functions, we focus on their generalized form as (46)–(48). In this way, not only the mathematical expression is simpler, but also the subsequent parameter optimization design is more universal and flexible.

Assumption 3 Let $\bar{h}(s, \gamma, \kappa) := h(\zeta, s, \gamma, \kappa)$. There exist a twice continuously differentiable function $g(\cdot) : (0, \infty) \to (0, \infty)$ such that

$$\eta_\infty^2(\zeta, s, \gamma, \kappa) - \eta_T^2(\zeta, s, \gamma, \kappa) =: g\left(\bar{h}(s, \gamma, \kappa)\right). \tag{49}$$

Furthermore, $g_1(\cdot) : (0, \infty) \to (0, \infty)$ and $g_2(\cdot) : (0, \infty) \to \mathbf{R}_+$ where

$$g_1(x) := \frac{\partial g(x)}{\partial x},\ g_2(x) := \frac{\partial^2 g(x)}{\partial x^2}. \tag{50}$$

From Assumption 3, we can see $g_1(\cdot)$ is non-decreasing, in addition, $g(\cdot)$ is increasing. Then the cost function is rewritten as

$$J_1(s, \gamma, \kappa) = g\left(\bar{h}(s, \gamma, \kappa)\right) + s^2, \tag{51}$$

$$J_2(s, \gamma, \kappa) = g\left(\bar{h}(s, \gamma, \kappa)\right) + \gamma^2, \tag{52}$$

$$J_3(s, \gamma, \kappa) = g\left(\bar{h}(s, \gamma, \kappa)\right) + \kappa^2. \tag{53}$$

Remark 11 From (46) to (48), the cost functions have no relationship with the system dynamics, hence, are *static*. However, most of the past studies on differential games oriented optimal design problems are in a *dynamic* way, in which the cost function changes along with the system situations; hence, it is difficult to obtain the optimal solution. For this, instead of the past *dynamic* way, a *static* (easier) way is developed in this paper, in which the cost function does not depend on the system situations such that it is easier to obtain the optimal solution. The cost functions contain three parts of *performance cost* $J_{11}(s, \gamma, \kappa)/J_{21}(s, \gamma, \kappa)/J_{31}(s, \gamma, \kappa)$, *time cost* $J_{12}(s, \gamma, \kappa)/J_{22}(s, \gamma, \kappa)/J_{32}(s, \gamma, \kappa)$ and *control cost* $J_{13}(s)/J_{23}(\gamma)/J_{33}(\kappa)$.

5.2 Statement of optimal design problem

We assign player s as the leader, γ as the first follower, and player κ as the second follower in the concerned Stackelberg game. Recalling Definition 1, the optimal design problem is equivalent to the following problem: First, find $\kappa = \Omega_3(s, \gamma)$ that solves

$$\min_{\kappa \in D_3} : J_3(s, \gamma, \kappa); \tag{54}$$

Second, substitute $\kappa = \Omega_3(s, \gamma)$ into $J_2(s, \gamma, \kappa)$ and find $\gamma = \Omega_2(s)$ that solves

$$\min_{\gamma \in D_2} : J_2(s, \gamma, \Omega_2(s, \gamma)); \tag{55}$$

Third, substitute $\gamma = \Omega_2(s)$ and $\kappa = \Omega_3(s, \gamma)$ into $J_1(s, \gamma, \kappa)$ and find $s = s^*$ that solves

$$\min_{s \in D_1} : J_1\left(s, \Omega_2(s), \Omega_3(s, \Omega_2(s))\right); \tag{56}$$

Finally, substitute $s = s^*$ into $\gamma = \Omega_2(s)$ to obtain $\gamma^* = \Omega_2(s^*)$, and then substitute $s = s^*$ and $\gamma = \gamma^*$ into $\kappa = \Omega_3(s, \gamma)$ to obtain $\kappa^* = \Omega_3(s^*, \gamma^*)$; thereout, the Stackelberg strategy $(s^*, \gamma^*, \kappa^*)$ is achieved.

Remark 12 It is obvious that the problem formulated above is with triple objectives and triple design parameters, and the uncertainty considered is not necessary fuzzy. This shows the advantage of the Stackelberg game theory over the past methods in related studies (such as [1–4]), which merely handle optimal design problem with single objective, single design parameter, and fuzzy uncertainty.

5.3 Solution by Stackelberg strategy

To proceed to the Stackelberg strategy, we first focus on player κ's reaction (i.e., $\kappa = \Omega_3(s, \gamma)$) to arbitrary decisions chosen by players s and γ. Take the first-order derivative of $J_3(s, \gamma, \kappa)$ with respect to κ

$$\frac{\partial J_3(s,\gamma,\kappa)}{\partial \kappa} = g_1\left(\bar{h}(s,\gamma,\kappa)\right)\frac{\partial \bar{h}(s,\gamma,\kappa)}{\partial \kappa} + 2\kappa. \tag{57}$$

The stationary condition (a *necessary* condition) yields

$$g_1\left(\bar{h}(s,\gamma,\kappa)\right)\frac{\partial \bar{h}(s,\gamma,\kappa)}{\partial \kappa} + 2\kappa = 0. \tag{58}$$

We then explore the *sufficient* condition. Taking the second- order derivative of $J_3(s,\gamma,\kappa)$ with respect to κ, we have

$$\frac{\partial^2 J_3(s,\gamma,\kappa)}{\partial \kappa^2} = g_2\left(\bar{h}(s,\gamma,\kappa)\right)\left(\frac{\partial \bar{h}(s,\gamma,\kappa)}{\partial \kappa}\right)^2 + g_1\left(\bar{h}(s,\gamma,\kappa)\right)\frac{\partial^2 \bar{h}(s,\gamma,\kappa)}{\partial \kappa^2} + 2. \tag{59}$$

If the solution $\kappa = \Omega_3(s,\gamma) > 0$ to (58) exists and leads to $\partial^2 J_3(s,\gamma,\kappa)/\partial\kappa^2 > 0$, it will globally minimize the cost function J_3.

Second, we focus on player γ's reaction (i.e., $\gamma = \Omega_2(s)$) to arbitrary decisions chosen by player s. Since player γ can solve player κ's problem as well as player κ can solve it, player γ can anticipate $\kappa = \Omega_3(s,\gamma)$, with which its cost function can be rewritten as $J_2(s,\gamma,\Omega_3(s,\gamma))$. Take the first-order derivative of $J_2(s,\gamma,\Omega_3(s,\gamma))$ with respect to γ

$$\frac{\partial J_2(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma} = g_1\left(\bar{h}(s,\gamma,\Omega_3(s,\gamma))\right)\frac{\partial \bar{h}(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma} + 2\gamma. \tag{60}$$

The stationary condition (a *necessary* condition) yields

$$g_1\left(\bar{h}(s,\gamma,\Omega_3(s,\gamma))\right)\frac{\partial \bar{h}(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma} + 2\gamma = 0. \tag{61}$$

We then explore the *sufficient* condition. Taking the second- order derivative of $J_2(s,\gamma,\Omega_3(s,\gamma))$ with respect to γ, we have

$$\frac{\partial^2 J_2(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma^2} = g_2\left(\bar{h}(s,\gamma,\Omega_3(s,\gamma))\right) \times \left(\frac{\partial \bar{h}(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma}\right)^2 + g_1\left(\bar{h}(s,\gamma,\Omega_3(s,\gamma))\right)\frac{\partial^2 \bar{h}(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma^2} + 2. \tag{62}$$

If the solution $\gamma = \Omega_2(s) > 0$ to (61) exists and leads to $\partial^2 J_2(s,\gamma,\Omega_3(s,\gamma))/\partial\gamma^2 > 0$, it will globally minimize the cost function J_2.

Third, we focus on player s's practical decision (i.e., $s = s^*$). Since player s can solve player γ and κ's problems as well as player γ and κ can solve them, player s can anticipate $\Omega_2(s)$ and $\Omega_3(s,\Omega_2(s))$, with which its cost function can be rewritten as $J_1(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))$. Take the first-order derivative of $J_1(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))$ with respect to s

$$\frac{\partial J_1(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s} = g_1\left(\bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))\right)\frac{\partial \bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s} + 2s. \tag{63}$$

The stationary condition (a *necessary* condition) yields

$$g_1\left(\bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))\right)\frac{\partial \bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s} + 2s = 0. \tag{64}$$

We then explore the *sufficient* condition. Taking the second- order derivative of $J_2(s,\gamma,\Omega_3(s,\gamma))$ with respect to s, we have

$$\frac{\partial^2 J_2(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s^2} = g_2\left(\bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))\right)\left(\frac{\partial \bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s}\right)^2 + g_1\left(\bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))\right)\frac{\partial^2 \bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))}{\partial s^2} + 2. \tag{65}$$

If the solution $s = s^* > 0$ to (64) exists and leads to $\partial^2 J_1(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))/\partial s^2 > 0$, it will globally minimize the cost function J_1.

Finally, taking $s = s^*$ into $\gamma = \Omega_2(s)$ yields $\gamma^* = \Omega_2(s^*)$, and taking $s = s^*$ and $\gamma = \gamma^*$ into $\kappa = \Omega_3(s, \gamma)$ yields $\kappa^* = \Omega_3(s^*, \gamma^*)$. Therefore, based on above analysis from (57) to (65), we have the following theorem.

Theorem 4 *Subject to Assumption 3. If the solutions $s > 0$ to (64), $\gamma = \Omega_2(s) > 0$ to (61), and $\kappa = \Omega_3(s, \gamma) > 0$ to (58) exist (i.e., meet the necessary conditions) and lead to $\partial^2 J_1(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))/\partial s^2 > 0$, $\partial^2 J_2(s, \gamma, \Omega_3(s, \gamma))/\partial \gamma^2 > 0$ and $\partial^2 J_3(s, \gamma, \kappa)/\partial \kappa^2 > 0$ (i.e., meet the sufficient conditions), they result in the Stackelberg strategy $(s^*, \gamma^*, \kappa^*)$ of the proposed Stackelberg game.*

Proof The proof is given by (57)–(65). □

Remark 13 The development in Theorem 4 to obtain the Stackelberg strategy (as shown in Fig. 2) is based on Definition 1. It is able to show the choice of s in (64), γ in (61) and κ in (58) is indeed minimal, through necessary conditions of (64), (61) and (58) as well as sufficient conditions of $\partial^2 J_1(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))/\partial s^2 > 0$, $\partial^2 J_2(s, \gamma, \Omega_3(s, \gamma))/\partial \gamma^2 > 0$ and $\partial^2 J_3(s, \gamma, \kappa)/\partial \kappa^2 > 0$. By this, for the seeking of the optimal solution, not only the *necessary* conditions but also the *sufficient* conditions (i.e., the *convexity*) of the cost functions J_i, respectively, in s, γ, κ are verified.

Remark 14 Note that, as early as 1978, Professor Leitmann of University of California at Berkeley gave a detailed discussion about the *non-uniqueness* of the Stackelberg strategy [32]. It can be seen that the Stackelberg strategy may be not unique sometimes and some special analysis and comparison are needed to find out the relatively optimal solution. By taking the optimal design parameters s^*, γ^*, κ^* into the robust control (19), the ϑ-measure of the mechanical system can render uniform boundedness and uniform ultimate boundedness, and the optimization problem oriented by Stackelberg game described as in Sect. 5.2 is globally solved.

5.4 Special case

For the mechanical system (5), we consider

$$\begin{aligned}\gamma_1(\|\vartheta\|) &= a_1 \|\vartheta\|^2,\\ \gamma_2(\|\vartheta\|) &= a_2 \|\vartheta\|^2,\\ \gamma_3(\|\vartheta\|) &= a_3 \|\vartheta\|^2.\end{aligned} \tag{66}$$

Here $a_{1,2,3} > 0$ are constants subjected to $a_2 > a_1^2$. With appropriately selected coefficients $a_{1,2,3}$, the conditions of (12)–(14) in Assumption 1 can be satisfied; hence, $\gamma_{1,2,3}(\cdot)$ are indeed K_∞ functions here. We then have

$$\begin{aligned}\eta_\infty(\zeta, s, \gamma, \kappa) &= (\gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))\\ &= \sqrt{\frac{a_2}{a_1 a_3} h(\zeta, s, \gamma, \kappa)},\end{aligned} \tag{67}$$

$$\begin{aligned}\eta_T(\zeta, s, \gamma, \kappa) &= (\gamma_1 \circ \gamma_3^{-1})(h(\zeta, s, \gamma, \kappa))\\ &= \sqrt{\frac{a_1}{a_3} h(\zeta, s, \gamma, \kappa)},\end{aligned} \tag{68}$$

such that

$$\begin{aligned}&\eta_\infty^2(\zeta, s, \gamma, \kappa) - \eta_T^2(\zeta, s, \gamma, \kappa)\\ &= \frac{a_2}{a_1 a_3} h(\zeta, s, \gamma, \kappa) - \frac{a_1}{a_3} h(\zeta, s, \gamma, \kappa)\\ &= \frac{(a_2 - a_1^2)}{a_1 a_3} h(\zeta, s, \gamma, \kappa)\\ &=: g\left(\bar{h}(s, \gamma, \kappa)\right).\end{aligned} \tag{69}$$

For the function $g(\cdot)$, we derive

$$g_1(x) = \frac{a_2 - a_1^2}{a_1 a_3}, \quad g_2(x) = 0. \tag{70}$$

By this, we obtain the cost functions for player s, γ, κ, respectively, as

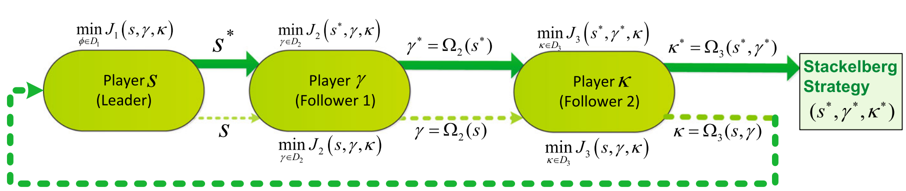


Fig. 2 Solution of the Proposed Stackelberg Game

$$\begin{aligned}J_1(s,\gamma,\kappa) &= \eta_\infty^2(\zeta,s,\gamma,\kappa) - \eta_T^2(\zeta,s,\gamma,\kappa) + s^2\\ &= g\left(\bar{h}(s,\gamma,\kappa)\right) + s^2,\\ J_2(s,\gamma,\kappa) &= \eta_\infty^2(\zeta,s,\gamma,\kappa) - \eta_T^2(\zeta,s,\gamma,\kappa) + \gamma^2\\ &= g\left(\bar{h}(s,\gamma,\kappa)\right) + \gamma^2,\\ J_3(s,\gamma,\kappa) &= \eta_\infty^2(\zeta,s,\gamma,\kappa) - \eta_T^2(\zeta,s,\gamma,\kappa) + \kappa^2\\ &= g\left(\bar{h}(s,\gamma,\kappa)\right) + \kappa^2. \end{aligned} \tag{71}$$

First, we focus on $\kappa = \Omega_3(s,\gamma)$. With (36), we have

$$\bar{h}(s,\gamma,\kappa) = \frac{\gamma(1+\rho_E)(s+1)}{\kappa} \left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}. \tag{72}$$

For its derivative, we have

$$\frac{\partial \bar{h}(s,\gamma,\kappa)}{\partial \kappa} = -\frac{\gamma(1+\rho_E)(s+1)}{\kappa^2} \left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}. \tag{73}$$

Introducing it and $g_1(\cdot)$ as (70) into (58) yields

$$\frac{a_2-a_1^2}{a_1a_3}\left\{-\frac{\gamma(1+\rho_E)(s+1)}{\kappa^2}\left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{s+1}}\right\} + 2\kappa = 0. \tag{74}$$

It renders the solution $\kappa = \Omega_3(s,\gamma)$ as

$$\Omega_3(s,\gamma) = \left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{\frac{1}{3}} \left[\gamma(1+\rho_E)(s+1)\right]^{\frac{1}{3}} \left[\frac{\zeta}{\gamma(1+\rho_E)(s+2)}\right]^{\frac{s+2}{3(s+1)}}. \tag{75}$$

Second, we focus on $\gamma = \Omega_2(s)$. Taking $\Omega_3(s,\gamma)$ into (72), yields

$$\bar{h}(s,\gamma,\Omega_3(s,\gamma)) = \left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{-\frac{1}{3}} \left[(1+\rho_E)(s+1)\right]^{\frac{2}{3}} \left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}} \gamma^{-\frac{2}{3(s+1)}}. \tag{76}$$

For its derivative, we have

$$\frac{\partial \bar{h}(s,\gamma,\Omega_3(s,\gamma))}{\partial \gamma} = -\frac{2}{3}\left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{-\frac{1}{3}} (1+\rho_E)^{\frac{2}{3}}(s+1)^{-\frac{1}{3}} \left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}} \gamma^{-\frac{3s+5}{3(s+1)}}. \tag{77}$$

Introducing it and $g_1(\cdot)$ as (70) into (61) and with some reductions, we have

$$-\frac{4}{3}\left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{\frac{2}{3}} (1+\rho_E)^{\frac{2}{3}}(s+1)^{-\frac{1}{3}} \left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}} \gamma^{-\frac{3s+5}{3(s+1)}} + 2\gamma = 0. \tag{78}$$

It renders the solution $\gamma = \Omega_2(s)$ as

$$\Omega_2(s) = \left\{\frac{2}{3}\left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{\frac{2}{3}} (1+\rho_E)^{\frac{2}{3}}(s+1)^{-\frac{1}{3}} \left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}}\right\}^{\frac{3(s+1)}{6s+8}}. \tag{79}$$

Third, we focus on $s = s^*$. Taking $\gamma = \Omega_2(s)$ into (76) yields

$$\begin{aligned}&\bar{h}(s,\Omega_2(s),\Omega_3(s,\Omega_2(s)))\\ &= \left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{-\frac{1}{3}} \left[(1+\rho_E)(s+1)\right]^{\frac{2}{3}} \left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}}\\ &\times \left\{\frac{2}{3}\left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{\frac{2}{3}} (1+\rho_E)^{\frac{2}{3}}(s+1)^{-\frac{1}{3}}\right.\end{aligned}$$

$$\left[\frac{\zeta}{(1+\rho_E)(s+2)}\right]^{\frac{2(s+2)}{3(s+1)}}\Bigg\}^{-\frac{1}{3s+4}}$$

$$= \left(\frac{2}{3}\right)^{-\frac{1}{3s+4}} \left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{-\frac{s+2}{3s+4}} (1+\rho_E)^{-\frac{2}{3s+4}} \zeta^{\frac{2(s+2)}{3s+4}} (s+1)^{\frac{2s+3}{3s+4}} (s+2)^{-\frac{2(s+2)}{3s+4}}. \tag{80}$$

For its derivative, we first have

$$\begin{aligned}
&\ln \bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))\\
&= -\frac{1}{3s+4}\ln\left(\frac{2}{3}\right) - \frac{s+2}{3s+4}\ln\left(\frac{a_2-a_1^2}{2a_1a_3}\right)\\
&\quad - \frac{2}{3s+4}\ln(1+\rho_E) + \frac{2(s+2)}{3s+4}\ln\zeta\\
&\quad + \frac{2s+3}{3s+4}\ln(s+1) - \frac{2(s+2)}{3s+4}\ln(s+2).
\end{aligned} \tag{81}$$

Taking the derivative of the both sides with respect to s yields

$$\begin{aligned}
&\frac{1}{\bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))}\frac{\partial \bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))}{\partial s}\\
&= \frac{3}{(3s+4)^2}\ln\left(\frac{2}{3}\right) + \frac{2}{(3s+4)^2}\ln\left(\frac{a_2-a_1^2}{2a_1a_3}\right)\\
&\quad + \frac{6}{(3s+4)^2}\ln(1+\rho_E) - \frac{4}{(3s+4)^2}\ln\zeta\\
&\quad - \frac{1}{(3s+4)^2}\ln(s+1) + \frac{2s+3}{(3s+4)\,(s+1)}\\
&\quad + \frac{4}{(3s+4)^2}\ln(s+2) - \frac{2}{3s+4} =: y(s),
\end{aligned} \tag{82}$$

such that

$$\begin{aligned}
&\frac{\partial \bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))}{\partial s}\\
&= \bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))y(s).
\end{aligned} \tag{83}$$

Introducing it and $g_1(\cdot)$ as (70) into (64), we have

$$\frac{a_2-a_1^2}{a_1a_3}\bar{h}(s, \Omega_2(s), \Omega_3(s, \Omega_2(s)))y(s) + 2s = 0. \tag{84}$$

It renders the solution $s = s^*$, for which we do numerical analysis by selecting $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $\zeta = 6\sqrt{6}$ and $\rho_E = 0$, which will be discussed and adopted in the simulation in Sect. 7. By numerical analysis, we obtain $s = s^* = 0.7566$, taking it into (79) yields $\gamma^* = \Omega_2(s^*) = 1.3395$, and then taking $s^* = 0.7566$ and $\gamma^* = 1.3395$ into (75) yields $\kappa^* = \Omega_3(s^*, \gamma^*) = 2.1744$.

Finally, it needs to verify the sufficient conditions as in Theorem 4. First, with $a_{1,2,3} > 0, a_2 > a_1^2, 1+\rho_E > 0$, $\zeta > 0$, $s^*, \gamma^*, \kappa^* > 0$ and (73), we have

$$\frac{\partial^2 \bar{h}(s^*, \gamma^*, \kappa^*)}{\partial\,(\kappa^*)^2} = \frac{2\gamma^*(1+\rho_E)(s^*+1)}{(\kappa^*)^3} \left[\frac{\zeta}{\gamma^*(1+\rho_E)(s^*+2)}\right]^{\frac{s^*+2}{s^*+1}} > 0. \tag{85}$$

Taking it and $g_{1,2}(\cdot)$ as (70) into (59), we have

$$\frac{\partial^2 J_3(s^*, \gamma^*, \kappa^*)}{\partial (\kappa^*)^2} = \frac{a_2-a_1^2}{a_1a_3}\frac{\partial^2 \bar{h}(s^*, \gamma^*, \kappa^*)}{\partial\,(\kappa^*)^2} + 2 > 0. \tag{86}$$

Second, with $a_{1,2,3} > 0$, $a_2 > a_1^2$, $1+\rho_E > 0$, $\zeta > 0$, $s^*, \gamma^* > 0$ and (77), we have

$$\begin{aligned}
&\frac{\partial^2 \bar{h}(s^*, \gamma^*, \Omega_3(s^*, \gamma^*))}{\partial\,(\gamma^*)^2}\\
&= \frac{2}{9}\left(\frac{a_2-a_1^2}{2a_1a_3}\right)^{-\frac{1}{3}}(1+\rho_E)^{\frac{2}{3}}(s^*+1)^{-\frac{4}{3}}(3s^*+5)\\
&\quad \left[\frac{\zeta}{(1+\rho_E)(s^*+2)}\right]^{\frac{2(s^*+2)}{3(s^*+1)}}(\gamma^*)^{-\frac{6s^*+8}{3(s^*+1)}}\\
&> 0.
\end{aligned} \tag{87}$$

Taking it and $g_{1,2}(\cdot)$ as (70) into (62), we have

$$\begin{aligned}
&\frac{\partial^2 J_2(s^*, \gamma^*, \Omega_3(s^*, \gamma^*))}{\partial(\gamma^*)^2}\\
&= \frac{a_2-a_1^2}{a_1a_3}\frac{\partial^2 \bar{h}(s^*, \gamma^*, \Omega_3(s^*, \gamma^*))}{\partial\,(\gamma^*)^2} + 2 > 0.
\end{aligned} \tag{88}$$

Third, with (89), we have

$$\frac{\partial^2 \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*)))}{\partial (s^*)^2} = \frac{\partial \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*)))}{\partial s^*} y(s^*) + \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*))) \frac{\partial y(s^*)}{\partial s^*} = \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*))) \left(y^2(s^*) + \frac{\partial y(s^*)}{\partial s^*} \right). \tag{89}$$

With $a_{1,2,3} > 0$, $a_2 > a_1^2$, $1 + \rho_E > 0$, $\zeta > 0$, $s^* > 0$, $\bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*))) > 0$. For $\partial y(s^*)/\partial s^*$, we do numerical analysis with the resulting $s^* = 0.7566$ and then obtain $\partial y(s^*)/\partial s^* = 0.1673 > 0$, such that $\partial^2 \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*)))/\partial (s^*)^2 > 0$. Taking it and $g_{1,2}(\cdot)$ as (70) into (65), we have

$$\frac{\partial^2 J_2(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*)))}{\partial (s^*)^2} = \frac{a_2 - a_1^2}{a_1 a_3} \frac{\partial^2 \bar{h}(s^*, \Omega_2(s^*), \Omega_3(s^*, \Omega_2(s^*)))}{\partial (s^*)^2} > 0. \tag{90}$$

Therefore, by Theorem 4, the resulting set $(s^*, \gamma^*, \kappa^*)$ =(0.7566, 1.3395, 2.1744) is the Stackelberg strategy of the proposed three-player Stackelberg game.

6 Design procedure

The optimal design procedure (as shown in Fig. 3) is summarized as follows.

Step 1: choose $V(\cdot)$, $f(\kappa, \cdot)$ and $\gamma_i(\cdot)$, $i = 1, 2, 3$ according to Assumption 1.
Step 2: determine the constraint matrix/vector α, b and the function $f(\kappa, \cdot)$ to design p_1 as in (15).
Step 3: choose ρ_E to meet Assumption 2(1), and determine ζ and $\rho(\cdot)$ by linear parameterizing to meet Assumption 2(2). Design p_2 as in (20) with predetermined α, $V(\cdot)$ and $\rho(\cdot)$.
Step 4: Determine $h(\zeta, s, \gamma, \kappa)$ as (36) with the previously determined ρ_E and ζ. Calculate $\eta_\infty(\zeta, s, \gamma, \kappa)$ (42) and $\eta_T(\zeta, s, \gamma, \kappa)$ (45), and formulate the cost functions with (46)–(48).

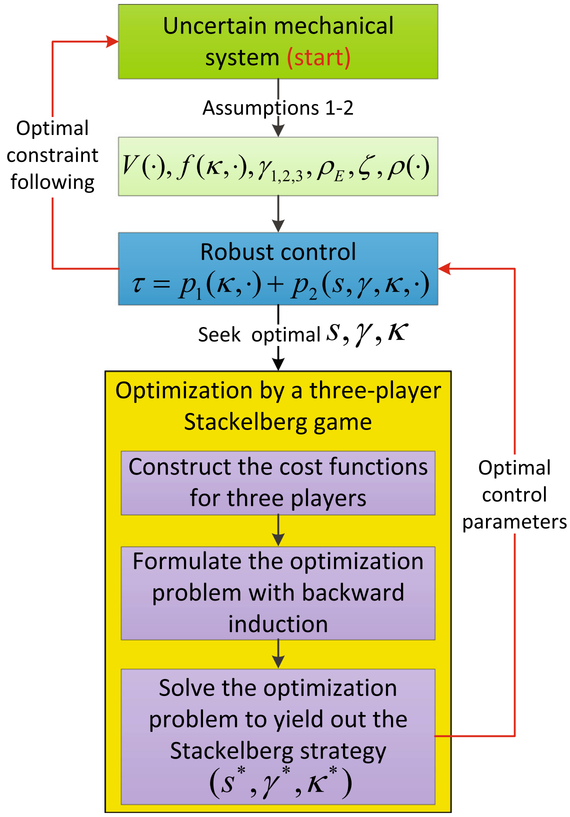


Fig. 3 Design procedure

Step 5: calculate the Stackelberg strategy $(s^*, \gamma^*, \kappa^*)$ with (58), (61) and (64) and the corresponding J_{min} with (46)–(48). The optimal robust control scheme is determined by taking predetermined $p_{1,2}$ and $(s^*, \gamma^*, \kappa^*)$ into (19).

Remark 15 In recently years, various optimal design methods based on game theory, such as potential heuristic algorithm [33,34], robust Stackelberg game [35], epsilon-generalized Nash equilibria [36], are proposed. They are all very practical optimization methods and each has its own features. Comparing with them, there are three major features (advantages) of the proposed method. First, it shows the first time that cast both constraint following and Stackelberg game into control framework for uncertain mechanical systems such that can render a twofold (guaranteed and optimal) performance. Second, it explores a *static* way of optimal design, in which the cost function does not depend on the system situations, so it is easier to obtain the optimal solution. Third, it can render an optimal balance between the control cost and the system performance under the disturbance of complex (even time-varying) uncertainty.

Remark 16 Even so many parameters appear in the proposed control method, but most of they are only used in the proof of the theoretical correctness and feasibility for the proposed method, so in practical application, except for the parameters used in the controller, most parameters do not need to be pre-designed. Therefore, in fact, only three parameters s, γ, κ and two functions $V(\cdot)$, $f(\cdot)$ need to be pre-designed in practical application. Here, the parameters s, γ, κ can be optimal designed by solving the proposed Stackelberg game, while the Lyapunov function $V(\cdot)$ and the function $f(\cdot)$ can be simply chosen as a quadratic-type one and a linear one.

7 Application to rotating rigid body

7.1 System model

We consider a rotating rigid body with the center of gravity O (as shown in Fig. 4). Its motion equation is [37]

$$\begin{aligned} S_1\ddot{\theta}_1(t) &= (S_2 - S_3)\,\dot{\theta}_2(t)\dot{\theta}_3(t) + \tau_1(t) + e_1(t),\\ S_2\ddot{\theta}_2(t) &= (S_3 - S_1)\,\dot{\theta}_3(t)\dot{\theta}_1(t) + \tau_2(t) + e_2(t),\\ S_3\ddot{\theta}_3(t) &= (S_1 - S_2)\,\dot{\theta}_1(t)\dot{\theta}_2(t) + \tau_3(t) + e_3(t), \end{aligned} \tag{91}$$

where $z = [\theta_1\ \theta_2\ \theta_3]^T$ is the angular displacement vector, $S_{1,2,3}$ are the principal moments of inertia, $e = [e_1\ e_2\ e_3]^T$ is the external disturbance, and $\tau = [\tau_1\ \tau_2\ \tau_3]^T$ is the torque inputs. It can be rewritten in the form of (5) with

$$\begin{aligned} S &= \begin{bmatrix} S_1 & 0 & 0\\ 0 & S_2 & 0\\ 0 & 0 & S_3 \end{bmatrix},\\ K\dot{z} &= \begin{bmatrix} (S_3 - S_2)\,\dot{\theta}_2\dot{\theta}_3\\ (S_1 - S_3)\,\dot{\theta}_3\dot{\theta}_1\\ (S_2 - S_1)\,\dot{\theta}_1\dot{\theta}_2 \end{bmatrix},\\ G &= 0, \quad D = [-e_1, -e_2, -e_3]^T. \end{aligned} \tag{92}$$

Suppose the moment of inertia $S_{1,2,3}$ and the external disturbance $e_{1,2,3}$ are uncertain: $S_{1,2,3} = \bar{S}_{1,2,3} + \Delta S_{1,2,3}(t)$, $e_{1,2,3} = \bar{e}_{1,2,3} + \Delta e_{1,2,3}(t)$, where $\bar{e}_{1,2,3} = 0$. Here $\Delta S_{1,2,3}(t)$ and $\Delta e_{1,2,3}$ are uncertainty, and

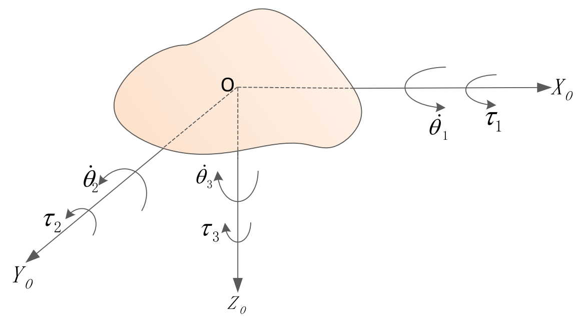


Fig. 4 Rotating rigid in body-fixed frame

their bounds are $\Delta\underline{S}_{1,2,3} \le \Delta S_{1,2,3} \le \Delta\overline{S}_{1,2,3}$ and $\Delta\underline{e}_{1,2,3} \le \Delta e_{1,2,3} \le \Delta\overline{e}_{1,2,3}$.

Rotating rigid control is common in stabilizing control of spacecraft and underwater vehicle (such as [38,39] and their bibliographies). We desire the rigid to be constrained by $\dot{\theta}_{1,2,3} = 0$, and then, the performance measure is $\vartheta = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3]^T$. By converting it into the first- and second-order forms, yields $\alpha = \text{Diag}\{1, 1, 1\}$, $c = b = [0, 0, 0]^T$.

7.2 Assumptions verification

For Assumption 1, we chose a Lyapunov function $V(\vartheta) = \vartheta^T P\vartheta$ and the function $f(\kappa, \vartheta) = -k\kappa\vartheta$, with $P \in \mathbf{R}^{m\times m}$, $P > 0$ and a positive constant k. Based on this, we further select $\gamma_1(\|\vartheta\|) = \lambda_m(P)\,\|\vartheta\|^2$, $\gamma_2(\|\vartheta\|) = \tilde{k}\lambda_M(P)\,\|\vartheta\|^2$, $\gamma_3(\|\vartheta\|) = 2k\lambda_m(P)\,\|\vartheta\|^2$, with $\tilde{k} > 1$.

For Assumption 2(1), recalling $W := \bar{S}^{-1}$, $\Delta W := S^{-1} - \bar{S}^{-1}$, we have $S^{-1} = W + \Delta W$. As $S,\ \bar{S} > 0$, thus $\left(S/\bar{S}\right) > 0$ and $\left(S/\bar{S}\right)^{-1} > 0$. We then have

$$E = W^{-1}\Delta W = \left(\frac{S}{\bar{S}}\right)^{-1} - 1 > -1. \tag{93}$$

Thus (17) is met.

For Assumption 2(2), with the numerical values shown later, we have

$$S^{-1} = \begin{bmatrix} \frac{1}{1+\Delta S_1} & 0 & 0\\ 0 & \frac{1}{2+\Delta S_2} & 0\\ 0 & 0 & \frac{1}{3+\Delta S_3} \end{bmatrix},\ \bar{S}^{-1} = W = \begin{bmatrix} 1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \tag{94}$$

$$K\dot{z} = \begin{bmatrix} (1 + \Delta S_3 - \Delta S_2)\,\dot{\theta}_2\dot{\theta}_3 \\ (-2 + \Delta S_1 - \Delta S_3)\,\dot{\theta}_3\dot{\theta}_1 \\ (1 + \Delta S_2 - \Delta S_1)\,\dot{\theta}_1\dot{\theta}_2 \end{bmatrix},$$

$$\bar{K}\dot{z} = \begin{bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ -2\dot{\theta}_3\dot{\theta}_1 \\ \dot{\theta}_1\dot{\theta}_2 \end{bmatrix}. \tag{95}$$

By these, we further have

$$\begin{aligned} E =&\bar{S}S^{-1} - I \\ =&\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{1+\Delta S_1} & 0 & 0 \\ 0 & \frac{1}{2+\Delta S_2} & 0 \\ 0 & 0 & \frac{1}{3+\Delta S_3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ =&-\begin{bmatrix} \frac{\Delta S_1}{1+\Delta S_1} & 0 & 0 \\ 0 & \frac{3+2\Delta S_2}{4+2\Delta S_2} & 0 \\ 0 & 0 & \frac{8+3\Delta S_3}{9+3\Delta S_3} \end{bmatrix}. \end{aligned} \tag{96}$$

Recalling p_1 as (15), with above calculations, we have

$$\begin{aligned} p_1 =& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} (-k\kappa\vartheta) + \begin{bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ -2\dot{\theta}_3\dot{\theta}_1 \\ \dot{\theta}_1\dot{\theta}_2 \end{bmatrix} \\ =& -\begin{bmatrix} \kappa\dot{\theta}_1 - \dot{\theta}_2\dot{\theta}_3 \\ 2\kappa\dot{\theta}_2 + 2\dot{\theta}_3\dot{\theta}_1 \\ 3\kappa\dot{\theta}_3 - \dot{\theta}_1\dot{\theta}_2 \end{bmatrix}, \end{aligned} \tag{97}$$

with $k = 1$. Let

$$\begin{aligned} \omega := &\|E(-K\dot{z} - G - D + p_1) \\ &-(\Delta K\dot{z} + \Delta G + \Delta D)\|. \end{aligned} \tag{98}$$

Taking (95)– (97) into (98) yields

$$\begin{aligned} \omega =& \|-EK\dot{z} + Ep_1 - \Delta K\dot{z} - \Delta D\| \\ =& \left\| \begin{bmatrix} \frac{\Delta S_2-\Delta S_3}{1+\Delta S_1}\dot{\theta}_2\dot{\theta}_3 + \frac{\Delta S_1}{1+\Delta S_1}\kappa\dot{\theta}_1 + \Delta e_1 \\ \frac{\Delta S_3-\Delta S_1}{4+2\Delta S_2}\dot{\theta}_3\dot{\theta}_1 + \frac{3+2\Delta S_2}{4+2\Delta S_2}\left(2\kappa\dot{\theta}_2\right) + \Delta e_2 \\ \frac{\Delta S_1-\Delta S_2}{9+3\Delta S_3}\dot{\theta}_1\dot{\theta}_2 + \frac{8+3\Delta S_3}{9+3\Delta S_3}\left(3\kappa\dot{\theta}_3\right) + \Delta e_3 \end{bmatrix} \right\| \\ \le& \underbrace{\left\| \begin{bmatrix} \frac{\Delta S_2-\Delta S_3}{1+\Delta S_1} & \frac{\Delta S_1}{1+\Delta S_1} & \Delta e_1 \\ \frac{\Delta S_3-\Delta S_1}{4+2\Delta S_2} & \frac{3+2\Delta S_2}{4+2\Delta S_2} & \Delta e_2 \\ \frac{\Delta S_1-\Delta S_2}{9+3\Delta S_3} & \frac{8+3\Delta S_3}{9+3\Delta S_3} & \Delta e_3 \end{bmatrix} \right\|}_{\omega_1} \underbrace{\left\| \begin{bmatrix} \dot{\theta}_2\dot{\theta}_3 & \dot{\theta}_3\dot{\theta}_1 & \dot{\theta}_1\dot{\theta}_2 \\ \kappa\dot{\theta}_1 & 2\kappa\dot{\theta}_2 & 3\kappa\dot{\theta}_3 \\ 1 & 1 & 1 \end{bmatrix} \right\|}_{\omega_2}. \end{aligned} \tag{99}$$

Recalling a property about the matrix norm: for matrix $X \in \mathbf{R}^{m\times n}$, $\frac{1}{\sqrt{m}}\|X\|_1 \le \|X\|_2 \le \sqrt{n}\|X\|_1$, we have

$$\omega_1 \le \sqrt{3}\max\{\rho_1, \rho_2, \rho_3\}, \tag{100}$$

with

$$\begin{aligned} \rho_1 =& \left\| \frac{\Delta S_2 - \Delta S_3}{1+\Delta S_1} \right\| + \left\| \frac{\Delta S_3 - \Delta S_1}{4+2\Delta S_2} \right\| + \left\| \frac{\Delta S_1 - \Delta S_2}{9+3\Delta S_3} \right\| \\ \le& \left\| \frac{\Delta \overline{S}_2 - \Delta \underline{S}_3}{1+\Delta \underline{S}_1} \right\| + \left\| \frac{\Delta \overline{S}_3 - \Delta \underline{S}_1}{4+2\Delta \underline{S}_2} \right\| + \left\| \frac{\Delta \overline{S}_1 - \Delta \underline{S}_2}{9+3\Delta \underline{S}_3} \right\| \\ =:& \overline{\rho}_1, \end{aligned} \tag{101}$$

$$\begin{aligned} \rho_2 =& \left\| \frac{\Delta S_1}{1+\Delta S_1} \right\| + \left\| \frac{3+2\Delta S_2}{4+2\Delta S_2} \right\| + \left\| \frac{8+3\Delta S_3}{9+3\Delta S_3} \right\| \\ \le& \left\| \frac{\Delta \overline{S}_1}{1+\Delta \underline{S}_1} \right\| + \left\| \frac{3+2\Delta \overline{S}_2}{4+2\Delta \underline{S}_2} \right\| + \left\| \frac{8+3\Delta \overline{S}_3}{9+3\Delta \underline{S}_3} \right\| \\ =:& \overline{\rho}_2 \end{aligned} \tag{102}$$

$$\begin{aligned} \rho_3 =& \|\Delta e_1\| + \|\Delta e_2\| + \|\Delta e_3\| \\ \le& \|\Delta \overline{e}_1\| + \|\Delta \overline{e}_2\| + \|\Delta \overline{e}_3\| \\ =:& \overline{\rho}_3. \end{aligned} \tag{103}$$

Taking $\overline{\rho}_{1,2,3}$ instead of $\rho_{1,2,3}$, we can obtain

$$\omega_1 \le \sqrt{3}\max\left\{\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3\right\}. \tag{104}$$

In the same way, we have

$$\begin{aligned} \omega_2 \le& \sqrt{3}\max\left\{\left\|\dot{\theta}_2\dot{\theta}_3\right\| + \kappa\left\|\dot{\theta}_1\right\| + 1, \left\|\dot{\theta}_3\dot{\theta}_1\right\|\right. \\ &\left. + 2\kappa\left\|\dot{\theta}_2\right\| + 1, \ \left\|\dot{\theta}_1\dot{\theta}_2\right\| + 3\kappa\left\|\dot{\theta}_3\right\| + 1\right\} \\ \le& \sqrt{3}\left(\left\|\dot{\theta}_2\dot{\theta}_3\right\| + \left\|\dot{\theta}_3\dot{\theta}_1\right\| + \left\|\dot{\theta}_1\dot{\theta}_2\right\|\right. \\ &\left. + \kappa\left\|\dot{\theta}_1\right\| + 2\kappa\left\|\dot{\theta}_2\right\| + 3\kappa\left\|\dot{\theta}_3\right\| + 3\right). \end{aligned} \tag{105}$$

Taking (104) and (105) into (99) yields

$$\begin{aligned} \omega \le& 3\max\left\{\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3\right\}\left(\left\|\dot{\theta}_2\dot{\theta}_3\right\| + \left\|\dot{\theta}_3\dot{\theta}_1\right\| + \left\|\dot{\theta}_1\dot{\theta}_2\right\|\right. \\ &\left. + \kappa\left\|\dot{\theta}_1\right\| + 2\kappa\left\|\dot{\theta}_2\right\| + 3\kappa\left\|\dot{\theta}_3\right\| + 3\right) \\ =:& \zeta\rho(\kappa, \dot{z}). \end{aligned} \tag{106}$$

For the uncertain moment of inertia $\Delta S_{1,2,3}(t)$ and the external disturbance $\Delta e_{1,2,3}(t)$, high frequency is considered by choosing the following: $\Delta S_1 = 0.1\sin(10t)$, $\Delta S_2 = 0.2\sin(10t)$, $\Delta S_3 = 0.3\sin(10t)$, $\Delta e_1 = \sin(10t)$, $\Delta e_2 = 2\sin(10t)$ and $\Delta e_3 = 3\sin(10t)$, then we can obtain $\zeta = 18$.

Finally, with the same analysis as (67)–(69), Assumption 3 is satisfied.

7.3 Optimal design parameters: Stackelberg strategy

For simulation, we select $P = I_3$, $k = 1$, $\tilde{k} = 2$ to render the same parameters $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $\rho_E = 0$ as in Sect. 5.4, and then choose $(s^*, \gamma^*, \kappa^*) = (0.7566, 1.3395, 2.1744)$. By this, according to (47), the minimum cost for player s, γ and κ are, respectively, given by

$$\begin{aligned} J_{min_1} &= \frac{\gamma^*(s^*+1)}{\kappa^*}\left[\frac{18}{\gamma^*(s^*+2)}\right]^{\frac{s^*+2}{s^*+1}} + (s^*)^2 \\ &= 10.0284, \\ J_{min_2} &= \frac{\gamma^*(s^*+1)}{\kappa^*}\left[\frac{18}{\gamma^*(s^*+2)}\right]^{\frac{s^*+2}{s^*+1}} + (\gamma^*)^2 \\ &= 11.2502, \\ J_{min_3} &= \frac{\gamma^*(s^*+1)}{\kappa^*}\left[\frac{18}{\gamma^*(s^*+2)}\right]^{\frac{s^*+2}{s^*+1}} + (\kappa^*)^2 \\ &= 14.1840. \end{aligned} \tag{107}$$

Figure 5 shows the relation between J_i $(i = 1, 2, 3)$, s, γ and κ. It shows that there is unique set $(s^*, \gamma^*, \kappa^*)$ to render the minimum performance index $J_{min_{1,2,3}}$. Note that, recalling the solving process of the proposed Stackelberg game shown in Fig. 2, player 1's decision $s = s^*$ is determined by minimizing the cost function $J_1(s, \gamma, \kappa)$ with determined predicted decision $\gamma = \Omega_2(s)$ of players 2 and determined predicted decision $\kappa = \Omega_3(s, \gamma)$ of player 3. In this process, the cost function J_1 is minimized by s alone rather than s, γ, κ simultaneously. By this, the minimum cost function J_{min1} indicated in Fig. 5a is actually not at the lowest point of the 3-D surface. Similarly, player 2's decision $\gamma = \gamma^*$ is determined by minimizing the cost function $J_2(s^*, \gamma, \kappa)$ with determined decision $s = s^*$ of player 1 and determined predicted decision $\kappa = \Omega_3(s, \gamma)$ of player 3, meanwhile, player 3's decision $\kappa = \kappa^*$ is determined by minimizing the cost function $J_3(s^*, \gamma*, \kappa)$ with determined decision $s = s^*$ of player 1 and determined decision $\gamma = \gamma^*$ of player 2. By this, the cost function J_2 and J_3 are, respectively, minimized by γ and κ rather than s, γ, κ simultaneously; hence, the minimum cost function J_{min2} and J_{min3} indicated in Fig. 5b, c are also not at the lowest point of the 3-D surface.

7.4 Simulation results

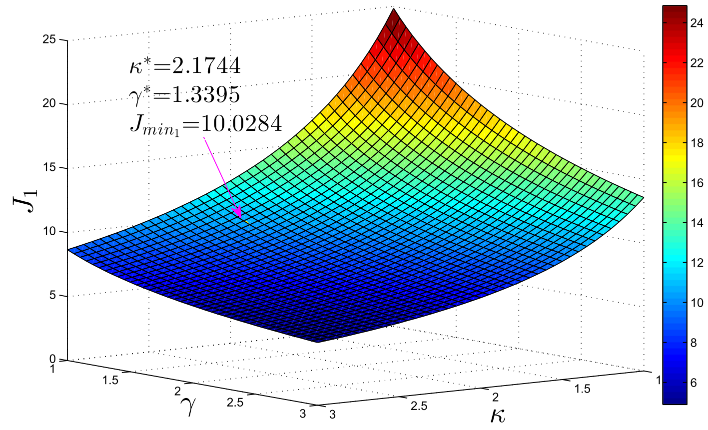
For simulation, we select $\bar{S}_1 = 1$, $\bar{S}_2 = 2$, $\bar{S}_3 = 3$ and the initial conditions $\theta_{1,2,3}(0) = 0$, $\dot{\theta}_{1,2,3}(0) = 0.5$. The area enclosed by $\|\vartheta\|$ and t is treated as the accumulative performance error for performance demonstrations. For comparison, the widely used LQR control is introduced. Let $x := [\dot{\theta}_1\ \dot{\theta}_2, \dot{\theta}_3]^T$, the motion equation of the rotating rigid (91) can be linearized as $\dot{x} = Ax + B\tau$, for which we consider Riccati equation $A^T P + PA - 2PBR^{-1}B^T P + Q = 0$, $Q, R > 0$, and then the LQR control can be determined as $\tau = -R^{-1}B^T Px$. For simulation, we choose $Q = R = I_3$. Simulation results are shown in Figs. 6, 7 and 8. They, respectively, present the comparison of the constraint history, the corresponding control input and the accumulative performance error with the proposed control (19) and the LQR control. With the proposed control (19), the constraints $\dot{\theta}_{1,2,3}$ approach to a desirable neighborhood close to 0 before $t_1 = 0.05$, $t_2 = 0.42$ and $t_3 = 1.05$, while, with almost the same control input, the LQR control does not result in any finite time settling and renders to much larger accumulative performance error.

8 Conclusions

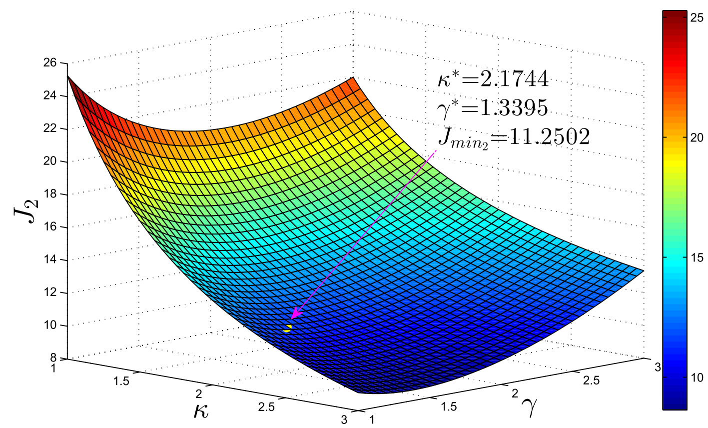
When the Nature drives a physical system, it always adopts a specific strategy, based on the characteristics of the system. For particle mass or rigid body, the strategy is the Newton's laws of motion, Lagrange's principle, or Hamilton's principle. For electromagnetic waves, the strategy is the Maxwell's equations. For fluid motions, the strategy is the Navier–Stokes equations. The *system-task-specific* control design strategy we adopt in this paper is motivated by this. By fully utilizing the characteristics of the mechanical system, the control scheme is based on the Lagrange's virtual work principle. The task is (holonomic or nonholonomic) constraint following. It is anticipated that the control is in coherence with the nature of the system's motion.

Once the control is proposed, which is based on three design parameters s, γ, and κ, the optimal choice of the parameters is facilitated by the Stackelberg strategy. The Stackelberg strategy, a leader–follower game, char-

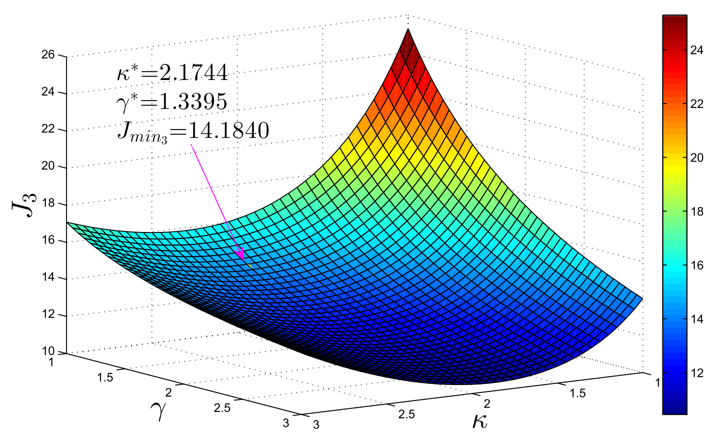
Fig. 5 Relation between $J_{1,2,3}$, κ and γ with $s = 0.7566$



(a) J_1

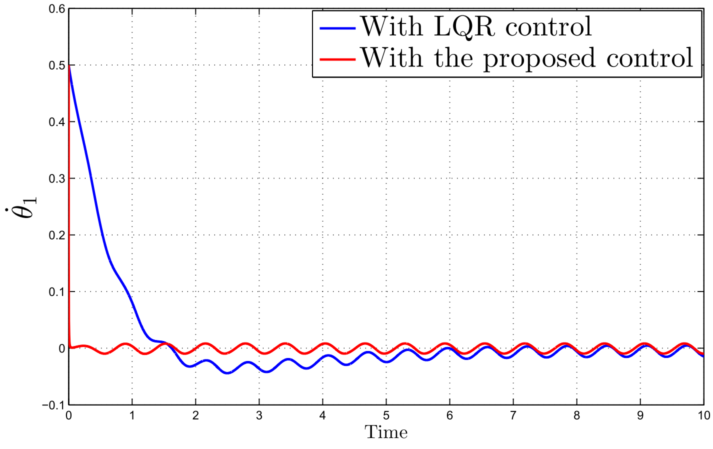


(b) J_2

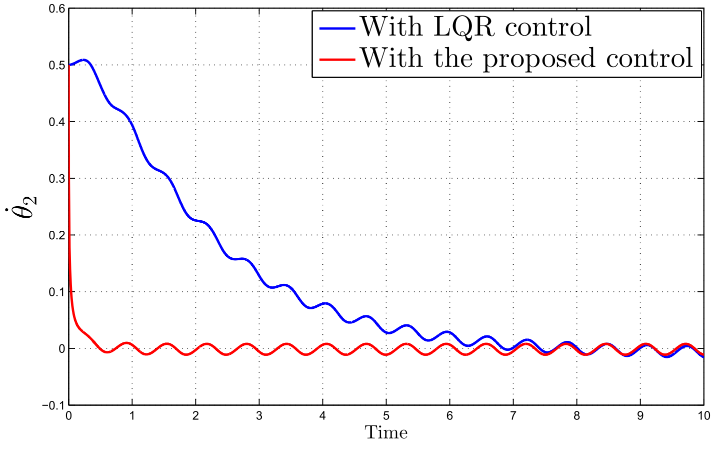


(c) J_3

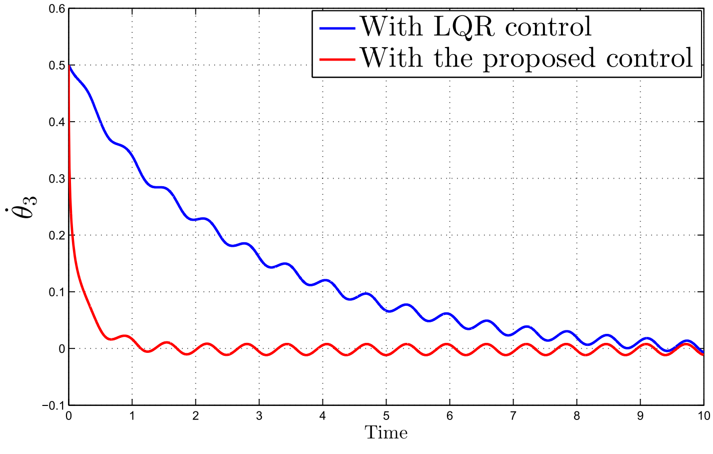
Fig. 6 System performance with the proposed control and LQR control



(a) $\dot{\theta}_1$

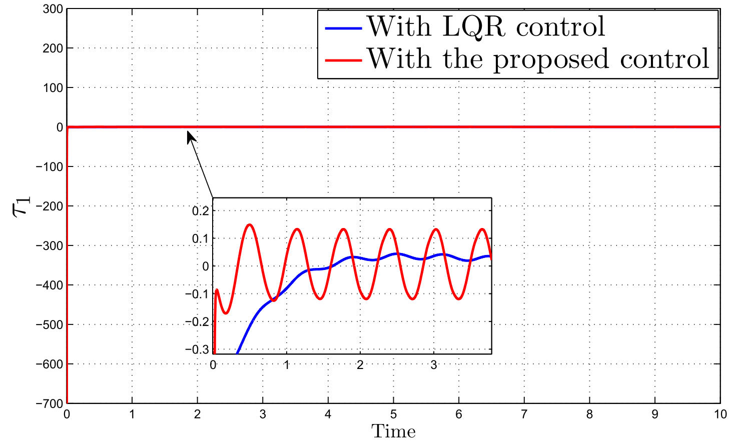


(b) $\dot{\theta}_2$

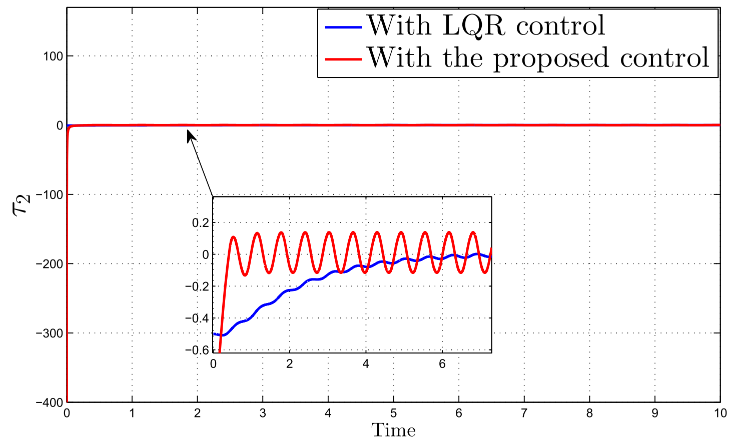


(c) $\dot{\theta}_3$

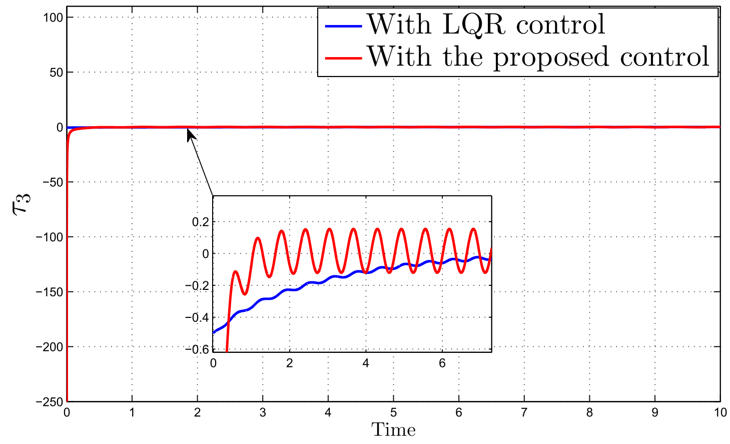
Fig. 7 Control input with the proposed control and LQR control



(a) τ_1



(b) τ_2



(c) τ_3

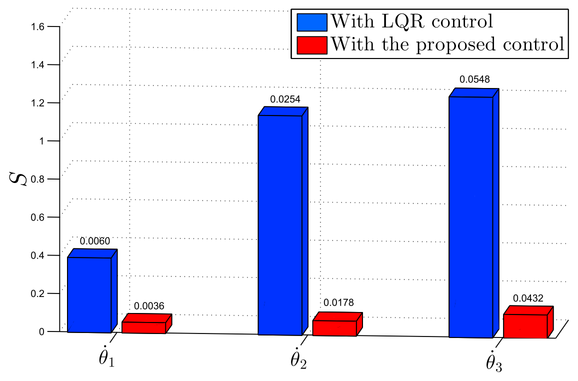


Fig. 8 Accumulative performance errors with the proposed control and LQR control

acterized by its *sequential* (as opposed to *simultaneous*) reasoning, is mostly aligned with what is actually operated in human society. For example, the competition in business often starts when one company (e.g., Coca-Cola) launches a new marketing strategy, then the next company (e.g., Pepsi) needs to follow suit by presenting its corresponding strategy. For another example, after the US Federal Reserve cuts the interest rate, the European Union needs to respond with its corresponding strategy.

This paper should be among the first ever endeavors to incorporate the characteristics of the Nature and human society into the motion control design for uncertain mechanical systems.

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Data availability All data are fully available without restriction.

Declarations

Conflict of interest The author(s) declared no potential conflicts of interest with respect to the research, authorship and/or publication of this article.

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