

# UC Irvine

## Recent Work

### Title

Optimality of Simple Layered Superposition Coding in the 3 User MISO BC with Finite Precision CSIT

### Permalink

<https://escholarship.org/uc/item/41h6v54h>

### Authors

Gholami Davoodi, Arash  
Jafar, Syed A

### Publication Date

2018-01-12

### Copyright Information

This work is made available under the terms of a Creative Commons Attribution-NonCommercial-NoDerivatives License, available at <https://creativecommons.org/licenses/by-nc-nd/4.0/>

# Optimality of Simple Layered Superposition Coding in the 3 User MISO BC with Finite Precision CSIT

Arash Gholami Davoodi and Syed A. Jafar  
Center for Pervasive Communications and Computing (CPCC)  
University of California Irvine, Irvine, CA 92697  
*Email: {gholamid, syed}@uci.edu*

## Abstract

We study the  $K = 3$  user multiple input single output (MISO) broadcast channel (BC) with  $M = 3$  antennas at the transmitter and 1 antenna at each receiver, from the generalized degrees of freedom (GDoF) perspective, under the assumption that the channel state information at the transmitter (CSIT) is limited to finite precision. In particular, our goal is to identify a parameter regime where a simple layered superposition (SLS) coding scheme achieves the entire GDoF region. With  $\alpha_{ij}$  representing the channel strength parameter for the link from the  $j^{\text{th}}$  antenna of the transmitter to the  $i^{\text{th}}$  receiver, we prove that SLS is GDoF optimal without the need for time-sharing if  $\max(\alpha_{ki}, \alpha_{im}) \leq \alpha_{ii}$  and  $\alpha_{ki} + \alpha_{im} \leq \alpha_{ii} + \alpha_{km}$  for all  $i, k \in [3], m \in [M]$ . The GDoF region under this condition is a convex polyhedron. The result generalizes to arbitrary  $M \geq 3$ .

## 1 Introduction

Capacity characterizations of broadcast and interference channels are among the most important open problems in network information theory. Especially significant for wireless networks are the  $K$  user Gaussian interference channel (IC) and the corresponding  $K$  user MISO BC that is obtained by allowing full cooperation between all the transmitters of the  $K$  user interference channel. Macroscopic insights into the performance limits of wireless networks can be obtained through generalized degrees of freedom (GDoF) studies. However, these studies often lead to sophisticated but fragile schemes such as dirty paper coding, zero forcing, and interference alignment. Robustness and simplicity are paramount in practice. For robust insights it is desirable to restrict the channel state information at the transmitter(s) (CSIT) to finite precision. At the same time, the need for simplicity motivates the search for broad regimes where simple coding schemes are optimal. For the  $K$  user interference channel, this approach is exemplified by recent studies that have found broad regimes where simple schemes such as orthogonal access [1, 2] or treating interference as noise (TIN) [3] are optimal in a GDoF sense. Notably, these insights have already found use in information-theoretically inspired scheduling algorithms [4, 5]. In contrast, for the corresponding  $K$  user MISO BC, much less is known about the optimality of simple schemes under finite precision CSIT. This is the motivation for our work.

Our goal is to identify broad regimes where simple<sup>1</sup> layered superposition (SLS) coding schemes are GDoF-optimal for the  $K$  user MISO BC under finite precision CSIT. By simple layered su-

---

<sup>1</sup>There is no non-trivial regime where TIN is GDoF optimal in the  $K$  user MISO BC under finite precision CSIT [6]. SLS is therefore the natural choice for the simplest scheme of interest.

perposition coding schemes we mean the following. In the  $K$  user MISO BC there are  $K$  independent messages, one for each receiver. Let us partition each message into several independent sub-messages, intended to be decoded by various subsets of users that must always include the desired user of the original message (cf. Han-Kobayashi scheme for the interference channel [7]). These sub-messages are independently coded. Each transmit antenna sends a weighted sum (*superposition*) of these independent codewords. The weights assigned to the codewords are primarily for power control. In the GDoF sense, the codewords transmitted from an antenna are mapped to various partitions (*layers*) of the signal dimension according to power levels (cf. ADT deterministic models [8]). Furthermore, we restrict the codebook design to single-letter<sup>2</sup> Gaussian (*simple*) codebooks, over the input random variables corresponding to one channel use. This is the class of coding schemes that we call simple layered superposition, or SLS in short, in this work.

The possibility that SLS could be GDoF-optimal in the  $K$  user MISO BC over a potentially large regime under finite precision CSIT is intriguing. For example, consider the  $K = 2$  user case. Reference [6] has shown that SLS achieves the entire GDoF region of the 2 user MISO BC under finite precision CSIT for *all* choices of channel strength parameters. The optimality of SLS remains unexplored for  $K \geq 3$ . As the next step forward, in this work we focus primarily on the  $K = 3$  user MISO BC setting with finite precision CSIT. The main technical challenge is two-fold. First, we apply recent generalizations of the aligned image sets [12, 13, 14, 15, 16] argument to generate an outer bound. Then, we prove that in the appropriate parameter regime, the bound is achievable by SLS. Our main result, reported in Theorem 1, identifies a broad parameter regime where SLS achieves the entire GDoF region. Indeed, this parameter regime is significantly larger than the parameter regime where the GDoF optimality of TIN was established for the corresponding  $K$  user IC in [3]. A direct representation of the GDoF region in this regime is also presented, which eliminates all power control and rate partitioning variables, automatically optimizing over all such choices within the scope of SLS. In this parameter regime, the GDoF region shows a surprising duality property, i.e., it remains unchanged if the roles of all transmit and receive antennas are switched.

*Notation:* For  $n \in \mathbb{N}$ , we use the notation  $[n] = \{1, 2, \dots, n\}$  and  $X^{[n]} = \{X(1), X(2), \dots, X(n)\}$ . The cardinality of a set  $A$  is denoted as  $|A|$ . If  $A$  is a set of random variables, then  $H(A)$  refers to the joint entropy of the random variables in  $A$ . Conditional entropies, mutual information and joint and conditional probability densities of sets of random variables are similarly interpreted. The notation  $f(x) = o(g(x))$  denotes that  $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{|g(x)|} = 0$ . We define  $(x)^+ = \max(x, 0)$ . The transpose of a matrix  $M$  is represented by  $M^\dagger$ .

## 2 Definitions

The following definitions are needed for aligned image sets arguments.

**Definition 1** (Bounded Density Channel Coefficients [12]) *Define a set of real valued random variables,  $\mathcal{G}$  such that the magnitude of each random variable  $g \in \mathcal{G}$  is bounded away from infinity,  $|g| \leq \Delta < \infty$ , for some positive constant  $\Delta$ , and there exists a finite positive constant  $f_{\max}$ , such that for all finite cardinality disjoint subsets  $\mathcal{G}_1, \mathcal{G}_2$  of  $\mathcal{G}$ , the joint probability density function of all random variables in  $\mathcal{G}_1$ , conditioned on all random variables in  $\mathcal{G}_2$ , exists and is bounded above by  $f_{\max}^{|\mathcal{G}_1|}$ . Without loss of generality we will assume that  $f_{\max} \geq 1, \Delta \geq 1$ .*

<sup>2</sup>This rules out space-time rate-splitting schemes [9, 10, 11].

**Definition 2** Define a set of real valued random variables,  $\mathcal{H}$  where each random variable  $h \in \mathcal{H}$  is bounded away from infinity,  $|h| \leq \Delta < \infty$ .

**Definition 3 (Power Levels)** An integer valued random variable  $X$  has power level not more than  $\lambda$  if it takes values over alphabet  $\mathcal{X}_\lambda$ ,

$$\mathcal{X}_\lambda \triangleq \{0, 1, 2, \dots, \bar{P}^\lambda - 1\} \quad (1)$$

where  $\bar{P}^\lambda$  is a compact notation for  $\lfloor \sqrt{P^\lambda} \rfloor$ .

Note that if  $X \in \mathcal{X}_\lambda$ , then it is also true that  $X \in \mathcal{X}_{\lambda+\epsilon}$  for all  $\epsilon > 0$ .

**Definition 4** For  $X \in \mathcal{X}_\lambda$ , and  $0 \leq \lambda_1 \leq \lambda$ , define the random variable  $(X)_{\lambda_1}^\lambda$  as,

$$(X)_{\lambda_1}^\lambda \triangleq \left\lfloor \frac{X}{\bar{P}^{\lambda_1}} \right\rfloor \quad (2)$$

In words, we say that  $(X)_{\lambda_1}^\lambda$  is comprised of the power levels above  $\lambda_1$  of  $X \in \mathcal{X}_\lambda$ . Also note that if  $X \in \mathcal{X}_\lambda$  then  $(X)_{\lambda_1}^\lambda = (X)_{\lambda_1}^{\lambda+\epsilon}$  for all  $\epsilon > 0$ .

**Definition 5** For  $x_1, x_2, \dots, x_k \in \mathcal{X}_\lambda$ , define the notations  $L_j^b(x_i, 1 \leq i \leq k)$  and  $L_j(x_i, 1 \leq i \leq k)$  as,

$$L_j^b(x_1, x_2, \dots, x_k) = \sum_{1 \leq i \leq k} \lfloor g_{ji} x_i \rfloor \quad (3)$$

$$L_j(x_1, x_2, \dots, x_k) = \sum_{1 \leq i \leq k} \lfloor h_{ji} x_i \rfloor \quad (4)$$

for distinct random variables  $g_{ji} \in \mathcal{G}$ , and  $h_{ji} \in \mathcal{H}$ . The subscript  $j$  is used to distinguish among various linear combinations. We refer to the  $L^b$  functions as bounded density linear combinations.

### 3 System Model

While in this section we define the system model for arbitrary  $K, M$ , note that our focus is primarily on the  $K = 3$  user MISO BC with  $M = 3$  antennas at the transmitter as shown in Fig. 1. The

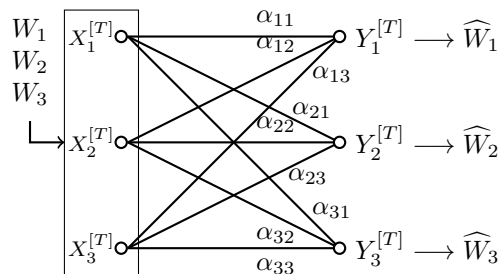


Figure 1:  $K = 3$  user MISO BC with  $M = 3$  antennas at the transmitter.

channel is defined by the following input-output equation over  $T$  channel uses,  $t \in [T]$ .

$$Y_k(t) = \sum_{m=1}^M \sqrt{P^{\alpha_{km}}} G_{km}(t) X_m(t) + Z_k(t). \quad (5)$$

Over the  $t^{\text{th}}$  channel use,  $Y_k(t)$  is the signal observed by the  $k^{\text{th}}$  receiver (user),  $k \in [K]$ ,  $Z_k(t)$  is the zero mean unit variance additive white Gaussian noise (AWGN),  $X_m(t)$  is the symbol sent from the  $m^{\text{th}}$  transmit antenna,  $m \in [M]$ ,  $G_{km}(t)$  are random variables representing the fading coefficient for the channel between the  $m^{\text{th}}$  antenna of the transmitter and the  $k^{\text{th}}$  receiver,  $\alpha_{km}$  is the channel strength parameter for the same channel, and  $P$  is the nominal power parameter that is allowed to approach infinity in the GDoF limit while the  $\alpha_{km}$  parameters are held fixed. The transmitted signals  $X_m(t)$  are each subject to unit power constraint. All symbols take real values.

### 3.1 CSIT and CSIR

The channel coefficients are distinct random variables from the bounded density set, i.e.,  $G_{km}(t) \in \mathcal{G}$ ,  $\forall k \in [K], m \in [M], t \in [T]$ . Channel state information at the receivers (CSIR) is assumed to be perfect, while the CSIT is limited to finite precision. Thus, the transmitter is only aware of the joint probability density functions of the channel fading coefficients  $G_{km}(t)$  and not the actual realizations of the channel coefficients. The receivers know all channel realizations.

### 3.2 GDoF

Achievable rates  $R_i(P)$  and capacity region  $\mathcal{C}(P)$  are defined in the standard Shannon-theoretic sense. The GDoF region is defined as

$$\begin{aligned} \mathcal{D} = & \{(d_1, d_2, \dots, d_K) : \exists (R_1(P), R_2(P), \dots, R_K(P)) \in \mathcal{C}(P), \\ & \text{s.t. } d_k = \lim_{P \rightarrow \infty} \frac{R_k(P)}{\frac{1}{2} \log(P)}, \forall k \in [K]\}. \end{aligned} \quad (6)$$

### 3.3 Simple Layered Superposition (SLS) Coding

Let us partition User  $k$ 's message as  $W_k = (\bar{W}_{\{S\}} : S \subset [K], k \in S)$ . Here,  $\bar{W}_{\{k\}}$  acts as a private sub-message to be decoded only by user  $k$  while  $\bar{W}_{\{S\}}$  for  $|S| > 1$  acts a common sub-message to be decoded by each User  $j$ , such that  $j \in S$ . For instance, when  $K = 3$ ,  $W_1 = \{\bar{W}_{\{1\}}, \bar{W}_{\{12\}}, \bar{W}_{\{13\}}, \bar{W}_{\{123\}}\}$ . The message  $\bar{W}_S$  carries  $d_S$  DoF. Messages  $\bar{W}_{\{1\}}, \bar{W}_{\{2\}}, \bar{W}_{\{1,2\}}, \dots, \bar{W}_{[K]}$  are encoded according to independent Gaussian codebooks into  $X_{\{1\}}, X_{\{2\}}, X_{\{1,2\}}, \dots, X_{[K]}$  with powers  $P^{-\lambda_{\{1\}}}, P^{-\lambda_{\{2\}}}, P^{-\lambda_{\{1,2\}}}, \dots, P^{-\lambda_{[K]}}$ , respectively, such that,

$$\sum_{S \subset [K], k \in S} P^{-\lambda_S} \leq 1, \forall k \in [K]. \quad (7)$$

The transmitted and received signals are,

$$X_m = \sum_{S \subset [K]} \sqrt{P^{-\gamma_{m,S}}} X_S, \forall m \in [M], \quad (8)$$

$$Y_k = \sum_{m \in [M]} \sqrt{P^{\alpha_{km}}} G_{km} X_m + Z_k, \forall k \in [K], \quad (9)$$

where  $\lambda_S$ ,  $\gamma_{k,S}$  and  $d_S$  are some arbitrary non-negative numbers depending on  $S$  which should be optimized for each point in the GDoF region separately. Note that power control is integral to SLS.

## 4 Main Result

**Theorem 1** *In the  $K = 3$  user MISO BC with  $M = 3$  transmit antennas defined in Section 3, if the following conditions are satisfied for all  $i, k \in [3], m \in [M]$ ,*

$$\max(\alpha_{im}, \alpha_{ki}) \leq \alpha_{ii}, \quad (10)$$

$$\alpha_{ki} + \alpha_{im} \leq \alpha_{ii} + \alpha_{km}, \quad (11)$$

then simple layered superposition (SLS) coding achieves the whole GDoF region, which is described as follows.

$$\mathcal{D} = \left\{ (d_1, d_2, d_3) \in \mathbb{R}_+^3, \text{ such that } \forall \text{ distinct } i, k \in [3], \right. \\ \left. \begin{aligned} d_i &\leq \alpha'_{ii}, \\ d_i + d_k &\leq \min(\alpha'_{ii} + \delta_{ki}, \alpha'_{kk} + \delta_{ik}), \\ d_1 + d_2 + d_3 &\leq \delta \end{aligned} \right\}, \quad (12)$$

where

$$\alpha'_{ii} = \max_{m \in [M]} \alpha_{im}, \quad (13)$$

$$\delta = \min_{\{i,j,k\}=[3]} \min \left( \alpha'_{ii} + \delta_{ji} + \delta_{kj}, \frac{\alpha'_{ii} + \alpha'_{kk} + \delta_{ij} + \delta_{ji} + \delta_{jk} + \delta_{ki}}{2} \right), \quad (14)$$

$$\delta_{ij} = \max_{m \in [M]} (\alpha_{im} - \alpha_{jm})^+, \forall i, j \in [3], i \neq j. \quad (15)$$

The following remarks are in order.

1. The result of Theorem 1 generalizes to  $M > 3$  transmit antennas. The converse proof of Theorem 1, provided in Section 5, allows  $M \geq 3$ , and since the achievability proof, presented in Section 6, utilizes only the first three transmit antennas, it applies to  $M \geq 3$  as well, simply by switching off the remaining antennas. Note that if (10), (11) are satisfied, then the GDoF region in (12) does not depend on  $\alpha_{km}$  for  $m > 3$ .
2. The converse proof of Theorem 1 shows that the region described by (12) is a valid outer bound on the GDoF region for all  $\alpha_{ij}$  values. The parameter regime identified by (10) and (11) is the regime where the outer bound is tight, and is achieved by SLS. In this parameter regime,  $\alpha'_{ii} = \alpha_{ii}$  and  $\delta_{ij} = \alpha_{ii} - \alpha_{ji}$ . Condition (11) is illustrated in Fig. 2.
3. Subject to conditions (10) and (11), the GDoF region shows a surprising duality property. Specifically, the GDoF region remains unchanged if the roles of transmitters and receivers are switched, i.e., if  $\alpha_{ij}$  and  $\alpha_{ji}$  values are switched. The top of Fig. 3 shows an example of a 3

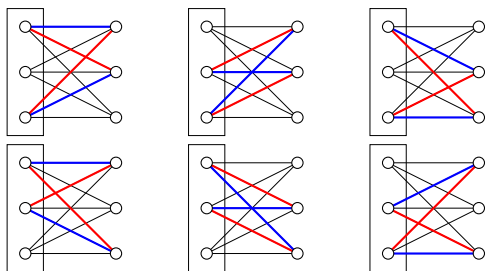


Figure 2: The six conditions implied by (11) are illustrated. The sum of blue channel strengths ( $\alpha_{ij}$ ) must be greater than or equal to the sum of the red channel strengths in each case.

user MISO BC and its dual. It is easy to verify that conditions (10) and (11) are satisfied and the GDoF region (sketched at the bottom of Fig. 3) is the following.

$$\mathcal{D} = \{(d_1, d_2, d_3) : \\ 0 \leq d_1 \leq 1.2, \quad 0 \leq d_2 \leq 1.3, \quad 0 \leq d_3 \leq 1, \\ d_1 + d_2 \leq 1.4, \quad d_1 + d_3 \leq 1.3, \quad d_2 + d_3 \leq 1.4, \\ d_1 + d_2 + d_3 \leq 1.6\}$$

It is also easy to construct examples where such a duality does not hold and (10) and (11) are not satisfied. As a simple example, consider the case where all channel strength parameters  $\alpha_{ij} = 0$  except,  $\alpha_{12} = 1, \alpha_{21} = 2$ . Note that (10) is not satisfied because  $\max(\alpha_{12}, \alpha_{21}) = 2 > 0 = \alpha_{11}$ . For this example, the GDoF region is  $\{(d_1, d_2, d_3) : d_1 \leq 1, d_2 \leq 2, d_3 = 0\}$ , but in its dual (reciprocal) setting the GDoF region is  $\{(d_1, d_2, d_3) : d_1 \leq 2, d_2 \leq 1, d_3 = 0\}$ . Thus, the dual setting does not have the same GDoF region.

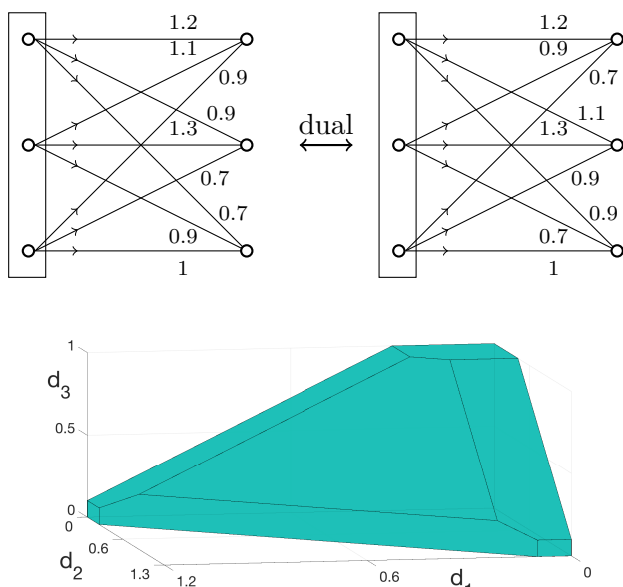


Figure 3: (Top): A 3-user MISO BC that satisfies the SLS-optimality conditions (10) and (11), and its dual channel where values of  $\alpha_{ij}$  and  $\alpha_{ji}$  are switched. The duality property implies that both have the same GDoF region. (Bottom): The GDoF region.

4. The GDoF region of a MISO BC does not depend on the labeling of transmit antennas, i.e., it remains the same if we substitute each  $\alpha_{km}$  with  $\alpha_{k\pi(m)}$  where  $\pi$  is any permutation on  $[M]$ . However, note that the conditions (10) and (11) do depend on the labeling of transmit antenna indices. Therefore, in order to determine if a given MISO BC setting satisfies the SLS optimality conditions (10) and (11), it is necessary to check these conditions for all permutations of transmit antenna indices. Furthermore, if the conditions are satisfied for one of these permutations, say  $\pi_1$ , and not for another permutation, say  $\pi_2$ , then the duality property described above is claimed for the labeling of transmit antenna according to  $\pi_1$ , but not for  $\pi_2$ .
5. It would be useful to consider as a special case of Theorem 1, a 3 user cyclic  $(1, a, b)$  MISO BC sketched in the left half of Fig. 4. In the parameter regime  $0 \leq a \leq b \leq 1$  and  $b - a \leq 1 - b$ , the GDoF region for this channel is achieved by SLS and is represented as follows.

$$\begin{aligned} \mathcal{D} = \{ & (d_1, d_2, d_3) : 0 \leq d_i \leq 1, d_i + d_j \leq 2 - b, \\ & d_1 + d_2 + d_3 \leq 3 - 2b, \forall i, j \in [3], i \neq j \} \end{aligned} \quad (16)$$

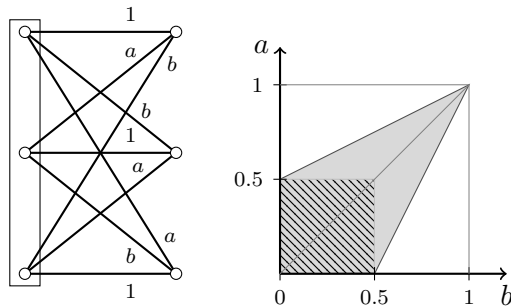


Figure 4: (Left): 3 user cyclic  $(1, a, b)$  MISO BC with channel strength levels  $(\alpha_{ij})$  shown for each link. (Right): Gray shaded region shows the regime where SLS is optimal in the 3 user cyclic  $(1, a, b)$  MISO BC. The slanted line pattern is the regime where TIN is optimal for the corresponding 3 user IC.

6. From [3], TIN is optimal in the 3 user IC if  $\max_{j \in [3], j \neq i} \alpha_{ij} + \max_{k \in [3], k \neq i} \alpha_{ki} \leq \alpha_{ii}, \forall i \in [3]$ . It is easily observed that the region of  $\alpha_{ij}$  where SLS is optimal in the 3 user MISO BC is larger than the one where TIN is optimal in the corresponding 3 user IC. For instance, as shown on the right half of Fig. 4, in the 3 user cyclic  $(1, a, b)$  MISO BC, the region in the  $(a, b)$  plane where TIN is optimal has an area of  $1/4$  while the region where SLS is optimal has an area of  $1/2$ .
7. For a challenging example outside the parameter regime identified by (10) and (11), consider the three user cyclic  $(1, 2, 2)$  MISO BC where the condition (11) is not satisfied. From Theorem 1, the best sum GDoF bound for this channel is equal to 4. However, this bound is not tight because we are able to establish a tighter bound of  $\frac{15}{4}$ , see Appendix A. Thus, (12) does not describe the GDoF region when conditions (10), (11) are not satisfied.



## 5 Proof of Theorem 1: Converse

The bounds  $d_i \leq \alpha'_{ii}$  follow from the single user bounds. For the remaining bounds, the first step in the converse proof is the transformation into a deterministic setting such that a GDoF outer bound on the deterministic setting is also a GDoF outer bound on the original setting. This step is identical to [17]. To avoid repetition, let us start our proof after this step.

### 5.1 Deterministic Model

The following input-output relationship holds in the deterministic model,

$$\bar{Y}_k(t) = \sum_{m=1}^M \left[ G_{km}(t) \lfloor \bar{P}^{\alpha_{km} - \lambda_m} \bar{X}_m(t) \rfloor \right] \quad (17)$$

for all  $k \in [3], t \in [T]$ , where  $\forall m \in [M], t \in [T]$ , we have

$$\bar{X}_m(t) \in \mathcal{X}_{\lambda_m}, \quad (18)$$

$$\lambda_m \triangleq \max_{k \in [3]} \alpha_{km}, \quad (19)$$

$$\lambda \triangleq \max_{m \in [M]} \lambda_m. \quad (20)$$

Thus, the signal from the  $m^{\text{th}}$  transmit antenna,  $\bar{X}_m$ , has power level no more than  $\lambda_m$ , which is the highest power level with which  $\bar{X}_m$  can be heard by any receiver  $k$ ,  $k \in [3]$ . Furthermore,  $\lambda$  is the maximum of all  $\lambda_m$ , so that for all  $m \in [M]$ , we can also write that  $\bar{X}_m(t) \in \mathcal{X}_\lambda$ . Note that (17) can be equivalently<sup>3</sup> expressed as follows.

$$\bar{Y}_k(t) = \sum_{m=1}^M \left[ G_{km}(t) (\bar{X}_m(t))_{\lambda_m - \alpha_{km}}^\lambda \right]. \quad (21)$$

### 5.2 A Key Lemma and an Observation

To invoke the aligned image sets argument, we need the following lemma from [18].

**Lemma 1** ([18], Lemma 1) *Define the two random variables  $\bar{\mathbf{U}}_1$  and  $\bar{\mathbf{U}}_2$  as,*

$$\bar{\mathbf{U}}_1 = \left( U_{11}^{[T]}, U_{12}^{[T]}, \dots, U_{1N}^{[T]} \right) \quad (22)$$

$$\bar{\mathbf{U}}_2 = \left( U_{21}^{[T]}, U_{22}^{[T]}, \dots, U_{2N}^{[T]} \right) \quad (23)$$

where for all  $n \in [N], t \in [T]$ ,  $U_{1n}(t)$  and  $U_{2n}(t)$  are defined as,

$$U_{1n}(t) = L_{1n}^b(t) \left( (\bar{V}_1(t))_{\eta - \lambda_{11}}^\eta, (\bar{V}_2(t))_{\eta - \lambda_{12}}^\eta, \dots, (\bar{V}_M(t))_{\eta - \lambda_{1M}}^\eta \right), \quad (24)$$

$$U_{2n}(t) = L_{2n}^b(t) \left( (\bar{V}_1(t))_{\eta - \lambda_{21}}^\eta, (\bar{V}_2(t))_{\eta - \lambda_{22}}^\eta, \dots, (\bar{V}_M(t))_{\eta - \lambda_{2M}}^\eta \right). \quad (25)$$

---

<sup>3</sup>From (17) and (2), we have  $\bar{Y}_k(t) = \sum_{m=1}^M \left[ G_{km}(t) \left\lfloor \frac{\bar{X}_m(t)}{\bar{P}^{\lambda_m - \alpha_{km}}} \right\rfloor \right] = \sum_{m=1}^M \left[ G_{km}(t) (\bar{X}_m(t))_{\lambda_m - \alpha_{km}}^{\lambda_m} \right] = \sum_{m=1}^M \left[ G_{km}(t) (\bar{X}_m(t))_{\lambda_m - \alpha_{km}}^\lambda \right]$ .

The  $\bar{V}_m(t) \in \mathcal{X}_\eta$ ,  $m \in [M]$ , are all independent of  $\mathcal{G}$ , and  $\forall m \in [M]$ ,  $0 \leq \lambda_{1m}, \lambda_{2m} \leq \eta$ . Without loss of generality,  $(\lambda_{1m} - \lambda_{2m})^+$  are sorted in descending order, i.e.,  $(\lambda_{1m} - \lambda_{2m})^+ \geq (\lambda_{1m'} - \lambda_{2m'})^+$  if  $1 \leq m < m' \leq M$ . For any acceptable<sup>4</sup> random variable  $W$ , if  $N \leq M$ , then we have,

$$H(\bar{\mathbf{U}}_1 | W, \mathcal{G}) - H(\bar{\mathbf{U}}_2 | W, \mathcal{G}) \leq T \left( \sum_{m=1}^N (\lambda_{1m} - \lambda_{2m})^+ \right) \log \bar{P} + T o(\log \bar{P}). \quad (26)$$

Lemma 1 may be intuitively understood as follows. Consider a transmitter with  $M$  antennas, with transmit symbol  $\bar{V}_m$  originating at the  $m^{\text{th}}$  antenna,  $m \in [M]$ . The transmitted signals have power levels no more than  $\eta$ . Consider 2 receivers,  $\bar{\mathbf{U}}_1, \bar{\mathbf{U}}_2$ , each equipped with  $N$  receive antennas, that see different bounded density linear combinations of the  $M$  transmitted symbols, scaled by channels of different strengths, so that the  $n^{\text{th}}$  receive antenna of the  $k^{\text{th}}$  receiver,  $k \in [2]$ , sees only the power levels above  $\eta - \lambda_{km}$  of the transmitted signal  $\bar{V}_m$ . If the CSIT is limited to finite precision, CSIR is perfect, and  $N \leq M$ , then the greatest difference in entropies that can be created between the two receivers in the GDoF sense is no more than the sum of the  $N$  largest terms of the pairwise differences between strengths of signals seen at the two receivers from the same transmit antenna. The random variable  $W$  generalizes this statement to conditional entropies provided that the bounded density character of the linear combinations is maintained even after conditioning on  $W$ .

Now consider the specialization of Lemma 1 to the system model in this paper. Our transmitter has  $M$  antennas, each receiver has  $N = 1$  antenna, all transmitted signals  $\bar{V}_m = \bar{X}_m, m \in [M]$ , have power levels no more than  $\eta = \lambda$ , and the  $k^{\text{th}}$  receiver sees only the power levels above  $\lambda_m - \alpha_{km}$  from  $\bar{X}_m$ , so that

$$\eta - \lambda_{km} = \lambda_m - \alpha_{km} \quad (27)$$

$$\Rightarrow \lambda_{km} = \eta - \lambda_m + \alpha_{km} \quad (28)$$

$$= \lambda - \lambda_m + \alpha_{km}. \quad (29)$$

Furthermore,  $M > N$ , the CSIT is limited to finite precision, and the CSIR is perfect. Therefore, for any  $k_1, k_2 \in [3]$ , and for any acceptable  $W$ , from Lemma 1, we conclude,

$$H(\bar{Y}_{k_1}^{[T]} | W, \mathcal{G}) - H(\bar{Y}_{k_2}^{[T]} | W, \mathcal{G}) \leq T \max_{m \in [M]} (\alpha_{k_1 m} - \alpha_{k_2 m})^+ \log \bar{P} + T o(\log \bar{P}). \quad (30)$$

where we used the fact that based on (29), we have  $(\lambda_{k_1 m} - \lambda_{k_2 m})^+ = (\alpha_{k_1 m} - \alpha_{k_2 m})^+$ .

### 5.3 Proof of bound: $d_1 + d_2 + d_3 \leq \alpha'_{11} + \delta_{21} + \delta_{32}$

Suppressing  $o(T)$  and  $o(\log(P))$  terms that are inconsequential for GDoF,

$$TR_1 \leq I(\bar{Y}_1^{[T]}; W_1 | \mathcal{G}) \quad (31)$$

$$TR_2 \leq I(\bar{Y}_2^{[T]}; W_2 | W_1, \mathcal{G}) \quad (32)$$

$$TR_3 \leq I(\bar{Y}_3^{[T]}; W_3 | W_1, W_2, \mathcal{G}) \quad (33)$$

<sup>4</sup>Let  $\mathcal{G}(Z) \subset \mathcal{G}$  denote the set of all bounded density channel coefficients that appear in  $\bar{\mathbf{U}}_1, \bar{\mathbf{U}}_2$ .  $W$  is acceptable if conditioned on any  $\mathcal{G}_o \subset (\mathcal{G}/\mathcal{G}(Z)) \cup \{W\}$ , the channel coefficients  $\mathcal{G}(Z)$  satisfy the bounded density assumption. For instance, any random variable  $W$  independent of  $\mathcal{G}$  can be utilized in Lemma 1.

Summing over (31), (32) and (33), we have,

$$\begin{aligned} \sum_{k=1}^3 TR_k &\leq H(\bar{Y}_1^{[T]} | \mathcal{G}) + H(\bar{Y}_2^{[T]} | W_1, \mathcal{G}) - H(\bar{Y}_1^{[T]} | W_1, \mathcal{G}) \\ &\quad + H(\bar{Y}_3^{[T]} | W_1, W_2, \mathcal{G}) - H(\bar{Y}_2^{[T]} | W_1, W_2, \mathcal{G}) \end{aligned} \quad (34)$$

$$\leq H(\bar{Y}_1^{[T]} | \mathcal{G}) + (\delta_{21} + \delta_{32})T \log \bar{P} \quad (35)$$

$$\leq (\alpha'_{11} + \delta_{21} + \delta_{32})T \log \bar{P} \quad (36)$$

(35) follows from (30), and for (36) we use the fact that  $|\bar{Y}_1(t)| \leq M\Delta\bar{P}^{\alpha'_{11}}$  and that the entropy of any discrete random variable is bounded by the logarithm of the cardinality of its support. From (36) we obtain the GDoF bound  $d_1 + d_2 + d_3 \leq \alpha'_{11} + \delta_{21} + \delta_{32}$ . Similarly, the bound  $d_1 + d_2 \leq \alpha'_{11} + \delta_{21}$  follows by summing (31) and (32),

$$TR_1 + TR_2 \leq H(\bar{Y}_1^{[T]} | \mathcal{G}) + H(\bar{Y}_2^{[T]} | W_1, \mathcal{G}) - H(\bar{Y}_1^{[T]} | W_1, \mathcal{G}) \quad (37)$$

$$\leq (\alpha'_{11} + \delta_{21})T \log \bar{P}. \quad (38)$$

#### 5.4 Proof of bound: $d_1 + d_2 + d_3 \leq (\alpha'_{11} + \alpha'_{33} + \delta_{12} + \delta_{21} + \delta_{31} + \delta_{23})/2$

$$TR_1 \leq I(\bar{Y}_1^{[T]}; W_1 | \mathcal{G}) \quad (39)$$

$$TR_1 \leq I(\bar{Y}_1^{[T]}; W_1 | W_2, \mathcal{G}) \quad (40)$$

$$TR_2 \leq I(\bar{Y}_2^{[T]}; W_2 | \mathcal{G}) \quad (41)$$

$$TR_2 \leq I(\bar{Y}_2^{[T]}; W_2 | W_3, \mathcal{G}) \quad (42)$$

$$TR_3 \leq I(\bar{Y}_3^{[T]}; W_3 | \mathcal{G}) \quad (43)$$

$$TR_3 \leq I(\bar{Y}_3^{[T]}; W_3 | W_1, W_2, \mathcal{G}) \quad (44)$$

Using the fact that  $I(A; B) \leq I(A; B | C)$  if  $B$  and  $C$  are independent of each other, we have

$$H(\bar{Y}_2^{[T]} | \mathcal{G}) - H(\bar{Y}_2^{[T]} | W_2, W_3, \mathcal{G}) \leq H(\bar{Y}_2^{[T]} | W_1, \mathcal{G}) \quad (45)$$

Moreover, applying (30) we have,

$$H(\bar{Y}_1^{[T]} | \mathcal{G}) \leq T\alpha'_{11} \log \bar{P} \quad (46)$$

$$H(\bar{Y}_3^{[T]} | \mathcal{G}) \leq T\alpha'_{33} \log \bar{P} \quad (47)$$

$$H(\bar{Y}_2^{[T]} | W_1, \mathcal{G}) - H(\bar{Y}_1^{[T]} | W_1, \mathcal{G}) \leq T\delta_{21} \log \bar{P} \quad (48)$$

$$H(\bar{Y}_1^{[T]} | W_2, \mathcal{G}) - H(\bar{Y}_2^{[T]} | W_2, \mathcal{G}) \leq T\delta_{12} \log \bar{P} \quad (49)$$

$$H(\bar{Y}_3^{[T]} | W_1, W_2, \mathcal{G}) - H(\bar{Y}_1^{[T]} | W_1, W_2, \mathcal{G}) \leq T\delta_{31} \log \bar{P} \quad (50)$$

$$H(\bar{Y}_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) \leq T\delta_{23} \log \bar{P} \quad (51)$$

Summing over ((39)-(51)), the bound  $d_1 + d_2 + d_3 \leq (\alpha'_{11} + \alpha'_{33} + \delta_{12} + \delta_{21} + \delta_{31} + \delta_{23})/2$  is obtained.

## 6 Proof of Theorem 1: Achievability

Since SLS is a simple achievable scheme, it is not difficult to characterize its achievable GDoF region.<sup>5</sup> Recall that SLS allows arbitrary power control, as well as arbitrary partitioning of sub-messages across arbitrary decoding subsets of users. These choices are represented by auxiliary variables. In terms of these auxiliary variables a description of the SLS achievable GDoF region is straightforward. However, note that our GDoF outer bound does not involve any auxiliary variables, i.e., it represents a direct characterization of the GDoF region *optimized* over all auxiliary variables. Eliminating the auxiliary variables from the achievable regions, and then proving that the union of those achievable regions matches the outer bound is the key technical challenge for proving the achievability result of Theorem 1. What is required is essentially a Fourier-Motzkin (FM) elimination, but the number of variables is large enough to make a direct application of the FM algorithm prohibitively complex. Recall that in [3] the elimination of auxiliary power control variables was accomplished by the use of the Potential Theorem, in order to find a direct characterization of the achievable region of TIN. For SLS the potential theorem seems less useful due to the added complexity of layered rate-partitioning on top of power control. We will need a bit more tedious reasoning to navigate through this challenge. As it turns out, we need 12 different specializations of SLS schemes. We will present two of them, leading to achievable GDoF regions labeled  $\hat{\mathcal{D}}_{123}$  and  $\hat{\mathcal{F}}_{123}$ . The remaining 10 cases are obtained from these two by switching indices. We start with  $\hat{\mathcal{D}}_{123}$ .

### 6.1 $\hat{\mathcal{D}}_{123}$

For this achievable scheme, we consider the parameter regime where

$$\max_{k,m \in [3], k \neq m} \alpha_{km} \leq \min(\alpha_{11}, \alpha_{22}). \quad (52)$$

#### 6.1.1 SLS Coding

Consider four non-negative values  $\lambda, \lambda', \gamma, \gamma'$ , and five independent messages  $\bar{W}_{\{1\}}, \bar{W}_{\{2\}}, \bar{W}_{\{3\}}, \bar{W}_{\{1,2\}}, \bar{W}_{\{1,2,3\}}$ , carrying non-negative values of  $d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  GDoF, respectively. The messages  $\bar{W}_{\{1\}}, \bar{W}_{\{2\}}, \bar{W}_{\{3\}}, \bar{W}_{\{1,2\}}, \bar{W}_{\{1,2,3\}}$  are encoded into independent Gaussian codebooks  $X_{\{1\}}, X_{\{2\}}, X_{\{3\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  with powers,

$$E|X_{\{1,2,3\}}|^2 = 1 - 2P^{-\lambda} \quad (53)$$

$$E|X_{\{1,2\}}|^2 = P^{-\lambda} \quad (54)$$

$$E|X_{\{1\}}|^2 = P^{-\lambda-\lambda'} \quad (55)$$

$$E|X_{\{2\}}|^2 = P^{-\lambda-\lambda'} \quad (56)$$

$$E|X_{\{3\}}|^2 = P^{-\lambda} \quad (57)$$

---

<sup>5</sup> Note that, when conditions (10) and (11) are true the GDoF region given in (12) does not depend on channel strengths of the links associated with the  $m^{\text{th}}$  antenna for all  $m > 3$  and will remain the same if we remove all the transmit antennas except the first 3. Therefore, it is sufficient to derive the achievability for the 3 user MISO BC where only the first three antennas are present.

The transmitted and received signals are,

$$X_1 = \bar{P}^{-\gamma'} (X_{\{1,2,3\}} + X_{\{1,2\}} + X_{\{1\}}) \quad (58)$$

$$X_2 = X_{\{1,2,3\}} + X_{\{1,2\}} + X_{\{2\}} \quad (59)$$

$$X_3 = X_{\{1,2,3\}} + X_{\{3\}} \quad (60)$$

$$Y_k = \sum_{m=1}^3 \sqrt{P^{\alpha_{km}}} G_{km} X_m + Z_k, \forall k \in [3] \quad (61)$$

This SLS coding is illustrated in Figure 5.

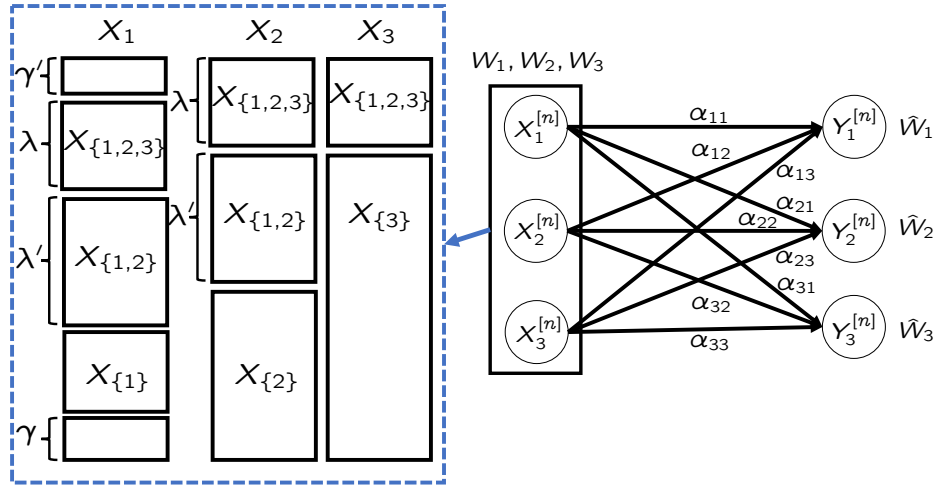


Figure 5: SLS coding for  $\hat{\mathcal{D}}_{123}$ .

### 6.1.2 Decoding

The decoding proceeds as follows.

1. At the first receiver,  $X_{\{1,2,3\}}$ ,  $X_{\{1,2\}}$ ,  $X_{\{1\}}$  are decoded sequentially with successive interference cancellation while treating  $X_{\{2\}}$  and  $X_{\{3\}}$  as Gaussian noise.
2. At the second receiver,  $X_{\{1,2,3\}}$ ,  $X_{\{1,2\}}$ ,  $X_{\{2\}}$  are decoded sequentially with successive interference cancellation while treating  $X_{\{1\}}$  and  $X_{\{3\}}$  as noise.
3. At the third receiver,  $X_{\{1,2,3\}}$ ,  $X_{\{3\}}$  are decoded sequentially with successive interference cancellation while treating  $X_{\{1\}}$ ,  $X_{\{2\}}$  and  $X_{\{1,2\}}$  as noise.

### 6.1.3 Achievable Region $\mathcal{D}_{123}$

As shown in Appendix B, the following GDoF region is achievable.

$$\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') = \left\{ (d_1, d_2, d_3) : \right.$$

$$d_1 = d_{\{1\}} + \mu_1 d_{\{1,2\}} + \xi_1 d_{\{1,2,3\}} \quad (62)$$

$$d_2 = d_{\{2\}} + \mu_2 d_{\{1,2\}} + \xi_2 d_{\{1,2,3\}} \quad (63)$$

$$d_3 = d_{\{3\}} + \xi_3 d_{\{1,2,3\}} \quad (64)$$

$$\mu_1 + \mu_2 = 1 \quad (65)$$

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad (66)$$

$$d_{\{1\}} \leq \alpha_{11} - \lambda - \lambda' - \gamma - \gamma' \quad (67)$$

$$d_{\{2\}} \leq \alpha_{22} - \lambda - \lambda' \quad (68)$$

$$d_{\{3\}} \leq \alpha_{33} - \lambda \quad (69)$$

$$d_{\{1,2\}} \leq \lambda' \quad (70)$$

$$d_{\{1,2,3\}} \leq \lambda \quad (71)$$

$$0 \leq \mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}} \quad (72)$$

$$\left. \right\}$$

for all choices of  $\lambda, \lambda', \gamma, \gamma'$  such that

$$\lambda + \lambda' + \gamma + \gamma' \leq \alpha_{11} \quad (73)$$

$$\lambda + \lambda' \leq \alpha_{22} \quad (74)$$

$$\lambda \leq \alpha_{33} \quad (75)$$

$$\alpha_{12} \leq \lambda + \lambda' + \gamma \quad (76)$$

$$\alpha_{13} \leq \lambda + \gamma \quad (77)$$

$$\alpha_{21} \leq \lambda + \lambda' + \gamma' \quad (78)$$

$$\alpha_{23} \leq \lambda \quad (79)$$

$$\alpha_{31} \leq \lambda + \gamma' \quad (80)$$

$$\alpha_{32} \leq \lambda \quad (81)$$

$$0 \leq \lambda, \lambda', \gamma, \gamma' \quad (82)$$

Note that this achievable region (which is one of 12 different regions) involves 14 auxiliary random variables that do not appear in the outer bound, namely,  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, \lambda, \lambda', \gamma, \gamma', d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$ . The union over the regions corresponding to all feasible choices of these 14 auxiliary variables is also achievable. Furthermore, there are 12 such regions and their union gives us the overall achievable region. To show that the overall achievable region matches the outer bound we will need to eliminate the auxiliary variables. In the next step, we eliminate  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  from  $\mathcal{D}_{123}$  to obtain the simplified region  $\bar{\mathcal{D}}_{123}$ .

#### 6.1.4 Achievable Region $\bar{\mathcal{D}}_{123}$

As shown in Appendix C, elimination of  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  gives us the following equivalent region  $\bar{\mathcal{D}}_{123}$  which retains only 4 auxiliary variables  $\lambda, \lambda', \gamma, \gamma'$ .

$$\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') = \left\{ (d_1, d_2, d_3) \in \mathbb{R}_+^3 : \right.$$

$$d_1 \leq \alpha_{11} - \gamma - \gamma', \quad (83)$$

$$d_2 \leq \alpha_{22}, \quad (84)$$

$$d_3 \leq \alpha_{33}, \quad (85)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \lambda - \lambda' - \gamma - \gamma', \quad (86)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \lambda - \gamma - \gamma', \quad (87)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \lambda, \quad (88)$$

$$\left. d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\lambda - \lambda' - \gamma - \gamma' \right\} \quad (89)$$

such that  $\lambda, \lambda', \gamma, \gamma'$  satisfy conditions (73) to (82).

#### 6.1.5 Achievable Region $\hat{\mathcal{D}}_{123}$

As shown in Appendix D, the union of the regions  $\bar{\mathcal{D}}_{123}$  over all possible choices of  $\lambda, \lambda', \gamma, \gamma'$  gives us the following region  $\hat{\mathcal{D}}_{123}$ .

$$\hat{\mathcal{D}}_{123} = \left\{ (d_1, d_2, d_3) \in \mathbb{R}_+^3 : \right.$$

$$d_1 \leq \alpha_{11}, \quad (90)$$

$$d_2 \leq \alpha_{22}, \quad (91)$$

$$d_3 \leq \alpha_{33}, \quad (92)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (93)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), \quad (94)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), \quad (95)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{32}, \alpha_{23}), \\ \alpha_{13} + \alpha_{21}, \\ \alpha_{12} + \alpha_{31}, \\ \alpha_{13} + \alpha_{31} \end{array} \right\} \quad (96)$$

$$\left. \right\}$$

## 6.2 $\hat{\mathcal{F}}_{123}$

Assume that

$$\max_{k,m \in [3], k \neq m} \alpha_{km} \leq \min(\alpha_{11}, \alpha_{22}). \quad (97)$$

### 6.2.1 SLS Coding

Similar to 6.1, consider four non-negative values  $\lambda, \lambda', \gamma, \gamma'$ , and five independent messages  $\bar{W}_{\{1\}}, \bar{W}_{\{2\}}, \bar{W}_{\{3\}}, \bar{W}_{\{1,2\}}, \bar{W}_{\{1,2,3\}}$  each carrying non-negative values of  $d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  GDoF, respectively. The messages  $\bar{W}_{\{1\}}, \bar{W}_{\{2\}}, \bar{W}_{\{3\}}, \bar{W}_{\{1,2\}}, \bar{W}_{\{1,2,3\}}$  are encoded into independent Gaussian codebooks  $X_{\{1\}}, X_{\{2\}}, X_{\{3\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  with powers,

$$E|X_{\{1,2,3\}}|^2 = 1 - 2P^{-\lambda} \quad (98)$$

$$E|X_{\{1,2\}}|^2 = P^{-\lambda} \quad (99)$$

$$E|X_{\{1\}}|^2 = P^{-\lambda-\lambda'} \quad (100)$$

$$E|X_{\{2\}}|^2 = P^{-\lambda-\lambda'} \quad (101)$$

$$E|X_{\{3\}}|^2 = P^{-\lambda} \quad (102)$$

The transmitted and received signals are,

$$X_1 = X_{\{1,2,3\}} + X_{\{1,2\}} + X_{\{1\}} \quad (103)$$

$$X_2 = \bar{P}^{-\gamma'}(X_{\{1,2,3\}} + X_{\{1,2\}} + X_{\{2\}}) \quad (104)$$

$$X_3 = X_{\{1,2,3\}} + X_{\{3\}} \quad (105)$$

$$Y_k = \sum_{j=1}^3 \sqrt{P^{\alpha_{kj}}} G_{kj} X_j + Z_k, \forall k \in [3] \quad (106)$$

This SLS coding is illustrated in Figure 6.

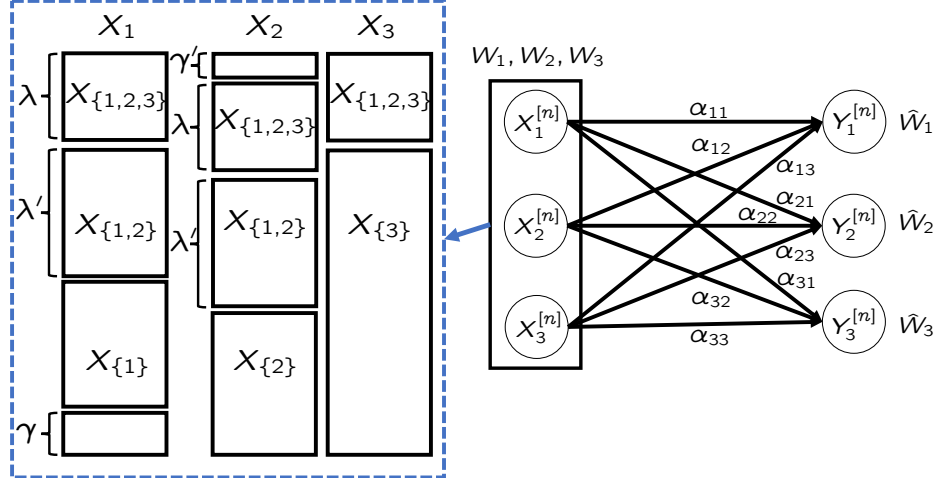


Figure 6: SLS coding for  $\hat{\mathcal{F}}_{123}$

### 6.2.2 Decoding

The decoding proceeds similar to 6.1.2.



### 6.2.3 Achievable Region $\mathcal{F}_{123}$

As shown in Appendix E, the following GDoF region is achievable.

$$\mathcal{F}_{123}(\lambda, \lambda', \gamma, \gamma') = \left\{ (d_1, d_2, d_3) : \right.$$

$$d_1 = d_{\{1\}} + \mu_1 d_{\{1,2\}} + \xi_1 d_{\{1,2,3\}}, \quad (107)$$

$$d_2 = d_{\{2\}} + \mu_2 d_{\{1,2\}} + \xi_2 d_{\{1,2,3\}}, \quad (108)$$

$$d_3 = d_{\{3\}} + \xi_3 d_{\{1,2,3\}}, \quad (109)$$

$$\mu_1 + \mu_2 = 1, \quad (110)$$

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad (111)$$

$$d_{\{1\}} \leq \alpha_{11} - \lambda - \lambda' - \gamma, \quad (112)$$

$$d_{\{2\}} \leq \alpha_{22} - \lambda - \lambda' - \gamma', \quad (113)$$

$$d_{\{3\}} \leq \alpha_{33} - \lambda, \quad (114)$$

$$d_{\{1,2\}} \leq \lambda', \quad (115)$$

$$d_{\{1,2,3\}} \leq \lambda, \quad (116)$$

$$0 \leq \mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}} \quad (117)$$

$$\left. \right\} \quad (118)$$

for all choices of  $\lambda, \lambda', \gamma, \gamma'$  such that

$$\lambda + \lambda' + \gamma \leq \alpha_{11} \quad (119)$$

$$\lambda + \lambda' + \gamma' \leq \alpha_{22} \quad (120)$$

$$\lambda \leq \alpha_{33} \quad (121)$$

$$\alpha_{12} \leq \lambda + \lambda' + \gamma + \gamma' \quad (122)$$

$$\alpha_{13} \leq \lambda + \gamma \quad (123)$$

$$\alpha_{21} \leq \lambda + \lambda' \quad (124)$$

$$\alpha_{23} \leq \lambda \quad (125)$$

$$\alpha_{31} \leq \lambda \quad (126)$$

$$\alpha_{32} \leq \lambda + \gamma' \quad (127)$$

$$0 \leq \lambda, \lambda', \gamma, \gamma' \quad (128)$$

Similar to 6.1.3, this achievable region involves 14 auxiliary random variables that do not appear in the outer bound, namely,  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, \lambda, \lambda', \gamma, \gamma', d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$ . The union over the regions corresponding to all feasible choices of these 14 auxiliary variables is also achievable. In the next step, we eliminate  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  from  $\mathcal{F}_{123}$  to obtain the simplified region  $\bar{\mathcal{F}}_{123}$ .

### 6.2.4 Achievable Region $\bar{\mathcal{F}}_{123}$

Similar to (6.2.4), elimination of  $\mu_1, \mu_2, \xi_1, \xi_2, \xi_3, d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}$  gives us the following equivalent region  $\bar{\mathcal{F}}_{123}$  which retains only 4 auxiliary variables  $\lambda, \lambda', \gamma, \gamma'$ <sup>6</sup>.

$$\bar{\mathcal{F}}_{123}(\lambda, \lambda', \gamma, \gamma') = \left\{ (d_1, d_2, d_3) \in \mathbb{R}_+^3 : \right.$$

$$d_1 \leq \alpha_{11} - \gamma, \quad (129)$$

$$d_2 \leq \alpha_{22} - \gamma', \quad (130)$$

$$d_3 \leq \alpha_{33}, \quad (131)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \lambda - \lambda' - \gamma - \gamma', \quad (132)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \lambda - \gamma, \quad (133)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \lambda - \gamma', \quad (134)$$

$$\left. d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\lambda - \lambda' - \gamma - \gamma' \right\} \quad (135)$$

such that  $\lambda, \lambda', \gamma, \gamma'$  satisfy conditions (119) to (123).

### 6.2.5 Achievable Region $\hat{\mathcal{F}}_{123}$

Similar to 6.1.5, the union of the regions  $\bar{\mathcal{F}}_{123}$  over all possible choices of  $\lambda, \lambda', \gamma, \gamma'$  gives us the following region  $\hat{\mathcal{F}}_{123}$ .

$$\hat{\mathcal{F}}_{123} = \left\{ (d_1, d_2, d_3) \in \mathbb{R}_+^3 : \right.$$

$$d_1 \leq \alpha_{11}, \quad (136)$$

$$d_2 \leq \alpha_{22}, \quad (137)$$

$$d_3 \leq \alpha_{33}, \quad (138)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (139)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{23}, \alpha_{31}, \alpha_{13}), \quad (140)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{31}, \alpha_{32}), \quad (141)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{31}, \alpha_{23}), \\ \alpha_{13} + \alpha_{21}, \\ \alpha_{32} + \alpha_{21}, \\ \alpha_{32} + \alpha_{13}, \\ \frac{\alpha_{12} + \alpha_{13} + \alpha_{32} + \alpha_{21}}{2} \end{array} \right\} \quad (142)$$

$$\left. \right\}$$

The equivalence of  $\hat{\mathcal{F}}_{123} = \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{F}}_{123}(\lambda, \lambda', \gamma, \gamma')$  is proved similar to the equivalence of  $\hat{\mathcal{D}}_{123} = \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ .

<sup>6</sup>The proof follows similar to Appendix C.

### 6.3 All 12 Achievable GDoF Regions

By symmetry, switching the indices, e.g.,  $(1, 2, 3) \rightarrow (2, 3, 1)$  in ((90) – (96)) and ((136) – (142)), ten other achievable regions are obtained. Therefore, the following region is achievable.

$$\mathcal{D}_a = \bigcup_{\{i,j,k\}=\{1,2,3\}} \left( \hat{\mathcal{D}}_{ijk} \cup \hat{\mathcal{F}}_{ijk} \right) \quad (143)$$

where for distinct values of  $\{i, j, k\} = \{1, 2, 3\}$ ,  $\hat{\mathcal{D}}_{ijk}$  and  $\hat{\mathcal{F}}_{ijk}$  are defined as follows.

$$\hat{\mathcal{D}}_{ijk} = \left\{ (d_i, d_j, d_k) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \right. \quad (144)$$

$$d_i + d_j \leq \alpha_{ii} + \alpha_{jj} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (145)$$

$$d_i + d_k \leq \alpha_{ii} + \alpha_{kk} - \max(\alpha_{jk}, \alpha_{kj}, \alpha_{ki}, \alpha_{ik}), \quad (146)$$

$$d_j + d_k \leq \alpha_{jj} + \alpha_{kk} - \max(\alpha_{jk}, \alpha_{kj}), \quad (147)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{jk}, \alpha_{kj}), \\ \alpha_{ik} + \alpha_{ki}, \\ \alpha_{ki} + \alpha_{ij}, \\ \alpha_{ji} + \alpha_{ik} \end{array} \right\} \quad (148)$$

if  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{ii}, \alpha_{jj})$ . Otherwise, we define  $\hat{\mathcal{D}}_{ijk} = \emptyset$ .

$$\hat{\mathcal{F}}_{ijk} = \left\{ (d_i, d_j, d_k) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \right. \quad (149)$$

$$d_i + d_j \leq \alpha_{ii} + \alpha_{jj} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (150)$$

$$d_i + d_k \leq \alpha_{ii} + \alpha_{kk} - \max(\alpha_{jk}, \alpha_{ki}, \alpha_{ik}), \quad (151)$$

$$d_j + d_k \leq \alpha_{jj} + \alpha_{kk} - \max(\alpha_{jk}, \alpha_{ki}, \alpha_{kj}), \quad (152)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{ki}, \alpha_{jk}), \\ \alpha_{ik} + \alpha_{ji}, \\ \alpha_{kj} + \alpha_{ji}, \\ \alpha_{kj} + \alpha_{ik}, \\ \frac{\alpha_{ij} + \alpha_{ik} + \alpha_{kj} + \alpha_{ji}}{2} \end{array} \right\} \quad (153)$$

if  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{ii}, \alpha_{jj})$ . Otherwise, we define  $\hat{\mathcal{F}}_{ijk} = \emptyset$ .

## 7 Achievability Matches the Outer Bound

Finally,  $\mathcal{D}_a$  is shown to produce region (12). Specifically, for each value of parameters  $\alpha_{ij}$ , we show that one of the 12 regions  $\hat{\mathcal{D}}_{ijk}, \hat{\mathcal{F}}_{ijk}, \forall \{i, j, k\} = \{1, 2, 3\}$  subsumes all others and matches (12). For example, in the 3 user MISO BC illustrated at the top of Fig. 3, it is not difficult to verify that the GDoF region (12) turns out to be identical to the region  $\hat{\mathcal{F}}_{123}$  described in ((136) – (142)). In this section, we prove that the GDoF region  $\mathcal{D}_a$  defined in (143) and  $\mathcal{D}$  defined in (12) are equivalent.

1.  $\mathcal{D}_a \subset \mathcal{D}$ .

In order to show that  $\mathcal{D}_a \subset \mathcal{D}$ , we prove  $\hat{\mathcal{D}}_{ijk} \subset \mathcal{D}$  and  $\hat{\mathcal{F}}_{ijk} \subset \mathcal{D}$  for any  $\{i, j, k\} = \{1, 2, 3\}$ .

For instance, consider the region  $\hat{\mathcal{D}}_{123}$ . Any tuple  $(d_1, d_2, d_3) \in \hat{\mathcal{D}}_{123}$  satisfies the inequalities ((90) – (96)). Comparing ((90) – (96)) and (12), it is verified that  $(d_1, d_2, d_3) \in \mathcal{D}$ . For instance from (93) we have,

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (154)$$

$$\leq \alpha_{11} + \alpha_{22} - \max(\alpha_{12}, \alpha_{21}) \quad (155)$$

Therefore, we conclude that  $\hat{\mathcal{D}}_{123} \subset \mathcal{D}$ . Similarly,  $\hat{\mathcal{D}}_{ijk} \subset \mathcal{D}$  and  $\hat{\mathcal{F}}_{ijk} \subset \mathcal{D}$  is concluded for any  $\{i, j, k\} = \{1, 2, 3\}$ .

2.  $\mathcal{D} \subset \mathcal{D}_a$ .

Without loss of generality assume  $\alpha_{12}$  is the largest of all cross links.

$$\alpha_{12} = \max_{l,m \in [3], l \neq m} \alpha_{lm}.$$

Therefore, from (10),  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{11}, \alpha_{22})$ . Consider the following three cases.

(a)  $\max(\alpha_{13}, \alpha_{31}) \leq \alpha_{23}$ . Consider any tuple  $(d_1, d_2, d_3) \in \mathcal{D}$ . From (12),  $\mathcal{D}$  is represented as

$$\mathcal{D} = \{(d_1, d_2, d_3) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \quad (156)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \alpha_{12}, \quad (157)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{13}, \alpha_{31}), \quad (158)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), \quad (159)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max\{\alpha_{12} + \alpha_{23}, \alpha_{32} + \alpha_{21}\} \quad (160)$$

On the other hand, from ((144) – (148)) as  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{11}, \alpha_{22})$ ,  $\hat{\mathcal{D}}_{213}$  is equal to,

$$\hat{\mathcal{D}}_{213} = \{(d_1, d_2, d_3) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \quad (161)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \alpha_{12}, \quad (162)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{13}, \alpha_{31}), \quad (163)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{13}, \alpha_{31}, \alpha_{23}, \alpha_{32}), \quad (164)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \begin{cases} \alpha_{12} + \max(\alpha_{13}, \alpha_{31}), \\ \alpha_{23} + \alpha_{32}, \\ \alpha_{32} + \alpha_{21}, \\ \alpha_{12} + \alpha_{23} \end{cases} \quad (165)$$

In this case,  $\mathcal{D} = \hat{\mathcal{D}}_{213}$  as  $\max(\alpha_{13}, \alpha_{31}) \leq \alpha_{23}$  and  $\alpha_{12}$  is the biggest one among all cross links.

(b)  $\max(\alpha_{23}, \alpha_{32}) \leq \alpha_{31}$ . From (12), ((90) – (96)) and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{11}, \alpha_{22})$ , we conclude that  $\mathcal{D} = \hat{\mathcal{D}}_{123}$ .

$$\mathcal{D} = \hat{\mathcal{D}}_{123} = \left\{ (d_1, d_2, d_3) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \right. \quad (166)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \alpha_{12}, \quad (167)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{31}, \alpha_{13}), \quad (168)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), \quad (169)$$

$$\left. d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max(\alpha_{13} + \alpha_{21}, \alpha_{12} + \alpha_{31}) \right\} \quad (170)$$

(c)  $\alpha_{31} \leq \max(\alpha_{23}, \alpha_{32}), \alpha_{23} \leq \max(\alpha_{13}, \alpha_{31})$ . In this case from (12),  $\mathcal{D}$  is represented as

$$\mathcal{D} = \{(d_1, d_2, d_3) : 0 \leq d_1 \leq \alpha_{11}, 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \quad (171)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \alpha_{12}, \quad (172)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \max(\alpha_{13}, \alpha_{31}), \quad (173)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), \quad (174)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \alpha_{12} + \max(\alpha_{23}, \alpha_{31}), \\ \alpha_{13} + \alpha_{21}, \\ \alpha_{32} + \alpha_{21}, \\ \alpha_{13} + \alpha_{32}, \\ \frac{\alpha_{12} + \alpha_{13} + \alpha_{32} + \alpha_{21}}{2} \end{array} \right. \quad (175)$$

Therefore,  $\mathcal{D} = \hat{\mathcal{F}}_{123}$  from ((136) – (142)) and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{11}, \alpha_{22})$ .

## 8 Conclusion

A broad regime of channel strength parameters is identified where simple layered superposition coding achieves the GDoF region of a 3 user MISO BC with  $M$  antennas at the transmitter, under finite precision CSIT. The parameter regime is larger than the corresponding regime for the 3 user IC where treating interference as noise (TIN) is shown to be GDoF-optimal, and reveals an interesting duality property in that the region remains unchanged if the roles of all transmit antennas and receive antennas are switched. Extensions to  $K \geq 4$  users for the MISO BC, as well as studies of GDoF regimes where SLS is optimal in other wireless networks are among the most interesting directions for future work. The combination of simplicity, robustness and information theoretic optimality imparts this research avenue the potential for both theoretical and practical impact.

## A Proof of Sum GDoF Bound of 15/4 in the Three User Cyclic (1, 2, 2) MISO BC

Consider the three user cyclic (1, 2, 2) MISO BC in Fig. 4. From the deterministic model in Section 5.1, the following input-output relationship holds,

$$\bar{Y}_k(t) = [G_{kk}(t) [\bar{P}^{-1} \bar{X}_k(t)]] + \sum_{m \in [3], m \neq k} [G_{km}(t) \bar{X}_m(t)] \quad (176)$$

$$= [G_{kk}(t) (\bar{X}_k(t))_1^2] + \sum_{m \in [3], m \neq k} [G_{km}(t) \bar{X}_m(t)] \quad (177)$$

where  $\bar{X}_k(t) \in \mathcal{X}_2$  for all  $k \in [3], t \in [T]$ . Define the random variables  $\bar{Y}'_k(t)$  as,

$$\bar{Y}'_1(t) = \sum_{m \in [3]} [G'_{1m}(t) \bar{X}_m(t)] \quad (178)$$

$$\bar{Y}'_2(t) = \sum_{m \in [3]} [G'_{2m}(t) \bar{X}_m(t)] \quad (179)$$

$$\bar{Y}'_3(t) = [G'_{33}(t) [\bar{P}^{-1} \bar{X}_3(t)]] + \sum_{m \in [3], m \neq 3} [G'_{3m}(t) \bar{X}_m(t)] \quad (180)$$

$$= [G'_{33}(t) (\bar{X}_3(t))_1^2] + \sum_{m \in [3], m \neq 3} [G'_{3m}(t) \bar{X}_m(t)] \quad (181)$$

where for all  $k, m \in [3]$ ,  $G'_{km}(t)$  are distinct random variables chosen from  $\mathcal{G}$  and are different from the random variables  $G_{km}(t), \forall k, m \in [3]$ . Writing Fano's inequality for all three users, we obtain the following bounds,<sup>7</sup>

$$TR_1 + TR_2 \leq I(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]}; W_1, W_2 | W_3, \mathcal{G}) \quad (185)$$

$$TR_3 \leq I(\bar{Y}_3^{[T]}; W_3 | \mathcal{G}) \quad (186)$$

From (186), we have,

$$\begin{aligned} 2TR_3 &\leq 2I(\bar{Y}_3^{[T]}; W_3 | \mathcal{G}) \\ &\leq 4T \log \bar{P} - 2H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) + T o(\log \bar{P}) \end{aligned} \quad (187)$$

$$= 4T \log \bar{P} - H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_3^{[T]} | W_3, \mathcal{G}) + T o(\log \bar{P}) \quad (188)$$

where (187) is true as similar to (36) we have  $H(\bar{Y}_3^{[T]} | \mathcal{G}) \leq 2T \log \bar{P} + T o(\log \bar{P})$ . In order to check whether (188) is true or not observe that,  $\bar{Y}'_3(t)$  is a bounded density copy of  $\bar{Y}_3(t)$ . So, we expect that

$$|H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_3^{[T]} | W_3, \mathcal{G})| \leq T o(\log \bar{P}) \quad (189)$$

<sup>7</sup>Suppressing  $o(T)$  terms for simplicity, we have

$$TR_1 + TR_2 \leq I(\bar{Y}_1^{[T]}; W_1 | \mathcal{G}) + I(\bar{Y}_2^{[T]}; W_2 | \mathcal{G}) \quad (182)$$

$$\leq I(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]}; W_1, W_2 | \mathcal{G}) \quad (183)$$

$$\leq I(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]}; W_1, W_2 | W_3, \mathcal{G}) \quad (184)$$

(183) and (184) follow from the facts that  $I(A; B) + I(D; C) \leq I(A, D; B, C)$  and  $I(A; B) \leq I(A; B | C)$  if  $B$  and  $C$  are independent of each other.

which is true<sup>8</sup> from (30). Summing over (185) and (188), we have,

$$\begin{aligned} & TR_1 + TR_2 + 2TR_3 \\ & \leq 4T \log \bar{P} + \left( H(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}_3^{[T]}, \bar{Y}'_3^{[T]} | W_3, \mathcal{G}) \right) + T o(\log \bar{P}) \end{aligned} \quad (192)$$

With the aid of Lemma 1, let us prove that

$$H(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}_3^{[T]}, \bar{Y}'_3^{[T]} | W_3, \mathcal{G}) \leq T \log \bar{P} + T o(\log \bar{P}) \quad (193)$$

(193) is proved in the following three steps.

1. Consider the random variables  $\bar{Y}_1(t)$  and  $\bar{Y}'_1(t)$ .  $\bar{Y}_1(t)$  is a bounded density linear combination of  $[\bar{P}^{-1} \bar{X}_1(t)] = (\bar{X}_1(t))_1^2, \bar{X}_2(t), \bar{X}_3(t)$  while  $\bar{Y}'_1(t)$  is a bounded density linear combination of  $\bar{X}_1(t), \bar{X}_2(t), \bar{X}_3(t)$ . Now, compare the terms  $H(\bar{Y}_1^{[T]} | \bar{Y}_2^{[T]}, W_3, \mathcal{G})$  and  $H(\bar{Y}'_1^{[T]} | \bar{Y}_2^{[T]}, W_3, \mathcal{G})$ . Due to the bounded density assumption, we expect that

$$\begin{aligned} & H(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_1^{[T]}, \bar{Y}_2^{[T]} | W_3, \mathcal{G}) \\ & = H(\bar{Y}_1^{[T]} | \bar{Y}_2^{[T]}, W_3, \mathcal{G}) - H(\bar{Y}'_1^{[T]} | \bar{Y}_2^{[T]}, W_3, \mathcal{G}) \\ & \leq T o(\log \bar{P}) \end{aligned} \quad (194)$$

which is true from (30).

2. Similarly, we have

$$\begin{aligned} & H(\bar{Y}_1^{[T]}, \bar{Y}_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_1^{[T]}, \bar{Y}'_2^{[T]} | W_3, \mathcal{G}) \\ & = H(\bar{Y}_2^{[T]} | \bar{Y}'_1^{[T]}, W_3, \mathcal{G}) - H(\bar{Y}'_2^{[T]} | \bar{Y}'_1^{[T]}, W_3, \mathcal{G}) \\ & \leq T o(\log \bar{P}) \end{aligned} \quad (195)$$

where (195) follows from (30) similar to (194).

3. Now, let us prove the following inequality.

$$H(\bar{Y}'_1^{[T]}, \bar{Y}'_2^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_3^{[T]}, \bar{Y}'_3^{[T]} | W_3, \mathcal{G}) \leq T \log \bar{P} + T o(\log \bar{P}) \quad (196)$$

To apply Lemma 1, set  $l = 3$ ,  $N = 2$  and  $\eta = 2$ . The random variables  $\bar{U}_{11}^{[T]}, \bar{U}_{12}^{[T]}, \bar{U}_{21}^{[T]}, \bar{U}_{22}^{[T]}, \bar{V}_1^{[T]}, \bar{V}_2^{[T]}$  and  $\bar{V}_3^{[T]}$  are interpreted as  $\bar{Y}'_1^{[T]}, \bar{Y}'_2^{[T]}, \bar{Y}'_3^{[T]}, \bar{Y}'_3^{[T]}, \bar{X}_1^{[T]}, \bar{X}_2^{[T]}$  and  $\bar{X}_3^{[T]}$ , respectively.

Thus, from Lemma 1 we conclude (196) as  $(\lambda_{11} - \lambda_{21})^+ = 1$ ,  $(\lambda_{12} - \lambda_{22})^+ = (\lambda_{13} - \lambda_{23})^+ = 0$ .

(193) is concluded by summing (194), (195) and (196). By symmetry from (192) and (193) we have,

$$TR_1 + TR_2 + 2TR_3 \leq 5T \log \bar{P} + T o(\log \bar{P}) \quad (197)$$

$$TR_1 + 2TR_2 + TR_3 \leq 5T \log \bar{P} + T o(\log \bar{P}) \quad (198)$$

$$2TR_1 + TR_2 + TR_3 \leq 5T \log \bar{P} + T o(\log \bar{P}) \quad (199)$$

Summing (197), (198) and (199) and applying the GDoF limit, we conclude that  $d_1 + d_2 + d_3 \leq 15/4$ .

<sup>8</sup>Note that from (30) we have,

$$H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}'_3^{[T]} | W_3, \mathcal{G}) \leq T o(\log \bar{P}) \quad (190)$$

$$H(\bar{Y}'_3^{[T]} | W_3, \mathcal{G}) - H(\bar{Y}_3^{[T]} | W_3, \mathcal{G}) \leq T o(\log \bar{P}) \quad (191)$$

## B Achievability of $\mathcal{D}_{123}$

1.  $X_{\{1\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the first receiver treating  $X_{\{2\}}$  and  $X_{\{3\}}$  as noise.

- (a) The SINR for decoding  $X_{\{1,2,3\}}$  at the first receiver treating the other signals as white Gaussian noise is equal to

$$\begin{aligned} & \frac{P^{\alpha_{11}} P^{-\gamma'} (1 - 2P^{-\lambda}) |G_{11}|^2}{1 + P^{\alpha_{11}} P^{-\gamma'} P^{-\lambda} |G_{11}|^2 + P^{\alpha_{12}} P^{-\lambda} |G_{12}|^2 + P^{\alpha_{13}} P^{-\lambda} |G_{13}|^2} \\ & \approx P^{\min(\lambda, \alpha_{11} - \gamma', \lambda + \alpha_{11} - \gamma' - \alpha_{12}, \lambda + \alpha_{11} - \gamma' - \alpha_{13})} \end{aligned} \quad (200)$$

The codeword  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully if

$$d_{\{1,2,3\}} \leq \min(\lambda, \alpha_{11} - \gamma', \lambda + \alpha_{11} - \gamma' - \alpha_{12}, \lambda + \alpha_{11} - \gamma' - \alpha_{13}) \quad (201)$$

From (73) and (82) we have  $\lambda \leq \alpha_{11} - \gamma'$ . Adding (73) and (76) we have  $\alpha_{11} - \gamma' - \alpha_{12} \geq 0$ , and similarly, adding (73) and (77) we have  $\alpha_{11} - \gamma' - \alpha_{13} \geq \lambda' \geq 0$ . Therefore, the RHS of (201) is equal to  $\lambda$ . From (71) we have  $d_{\{1,2,3\}} \leq \lambda$ , therefore (201) holds and  $X_{\{1,2,3\}}$  is successfully decoded at Receiver 1.

- (b) After decoding the messages  $\bar{W}_{\{1,2,3\}}$ , the first receiver reconstructs the codeword  $X_{\{1,2,3\}}$  and subtracts its contribution from the received signal. The SINR for decoding  $X_{\{1,2\}}$  at the first receiver while treating the other signals as white Gaussian noise is equal to

$$\begin{aligned} & \frac{P^{\alpha_{11}} P^{-\gamma'} P^{-\lambda} |G_{11}|^2}{1 + P^{\alpha_{11}} P^{-\gamma'} P^{-\lambda - \lambda'} |G_{11}|^2 + P^{\alpha_{12}} P^{-\lambda - \lambda'} |G_{12}|^2 + P^{\alpha_{13}} P^{-\lambda} |G_{13}|^2} \\ & \approx P^{\min(\lambda', \alpha_{11} - \lambda - \gamma', \lambda' + \alpha_{11} - \gamma' - \alpha_{12}, \alpha_{11} - \gamma' - \alpha_{13})} \end{aligned} \quad (202)$$

The codeword  $X_{\{1,2\}}$  which carries  $d_{\{1,2\}}$  GDoF is decoded successfully if

$$d_{\{1,2\}} \leq \min(\lambda', \alpha_{11} - \lambda - \gamma', \lambda' + \alpha_{11} - \gamma' - \alpha_{12}, \alpha_{11} - \gamma' - \alpha_{13}) \quad (203)$$

From (73), (76), (77) it is easy to verify that the RHS of (203) is equal to  $\lambda'$ . However, from (70) we have  $d_{\{1,2\}} \leq \lambda'$ , therefore (203) holds and  $X_{\{1,2\}}$  is successfully decoded at Receiver 1.

- (c) After decoding the messages  $\bar{W}_{\{1,2\}}$ , the first receiver reconstructs the codeword  $X_{\{1,2\}}$  and subtracts its contribution from the received signal. The SINR for decoding  $X_{\{1\}}$  at the first receiver while treating the other signals as white Gaussian noise is equal to

$$\begin{aligned} & \frac{P^{\alpha_{11}} P^{-\gamma'} P^{-\lambda - \lambda'} |G_{11}|^2}{1 + P^{\alpha_{12}} P^{-\lambda - \lambda'} |G_{12}|^2 + P^{\alpha_{13}} P^{-\lambda} |G_{13}|^2} \\ & \approx P^{\min(\alpha_{11} - \lambda - \lambda' - \gamma', \alpha_{11} - \alpha_{12} - \gamma', \alpha_{11} - \alpha_{13} - \gamma' - \lambda')} \end{aligned}$$

The message  $X_{\{1\}}$  which carries  $d_{\{1\}}$  GDoF is decoded successfully if

$$d_{\{1\}} \leq \min(\alpha_{11} - \lambda - \lambda' - \gamma - \gamma', \alpha_{11} - \alpha_{12} - \gamma', \alpha_{11} - \alpha_{13} - \gamma' - \lambda') \quad (204)$$

From (76) and (77), we conclude that the RHS of (204) is equal to  $\alpha_{11} - \lambda - \lambda' - \gamma - \gamma'$ . However, from (67) we have  $d_{\{1\}} \leq \alpha_{11} - \lambda - \lambda' - \gamma - \gamma'$ , therefore (204) holds and  $X_{\{1\}}$  is successfully decoded at Receiver 1.



2.  $X_{\{2\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the second receiver treating  $X_{\{1\}}$  and  $X_{\{3\}}$  as noise.

(a) The SINR for decoding  $X_{\{1,2,3\}}$  at the second receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{22}}(1 - 2P^{-\lambda})|G_{22}|^2}{1 + P^{\alpha_{21}}P^{-\gamma'}P^{-\lambda}|G_{21}|^2 + P^{\alpha_{22}}P^{-\lambda}|G_{22}|^2 + P^{\alpha_{23}}P^{-\lambda}|G_{23}|^2} \approx P^\lambda \quad (205)$$

(205) follows as  $\max(\alpha_{ji}, \alpha_{ij}) \leq \alpha_{ii}$  is true for all  $i, j, k \in [3]$  from (10). Therefore, the message  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully at the second receiver.

(b) After decoding the messages  $\bar{W}_{\{1,2,3\}}$ , the second receiver reconstructs the codeword  $X_{\{1,2,3\}}$  and subtracts its contribution from the received signal. The SINR for decoding  $X_{\{1,2\}}$  at the second receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{22}}P^{-\lambda}|G_{22}|^2}{1 + P^{\alpha_{21}}P^{-\gamma'}P^{-\lambda-\lambda'}|G_{21}|^2 + P^{\alpha_{22}}P^{-\lambda-\lambda'}|G_{22}|^2 + P^{\alpha_{23}}P^{-\lambda}|G_{23}|^2} \approx P^{\min(\lambda', \alpha_{22}-\lambda, \alpha_{22}+\lambda'+\gamma'-\alpha_{21}, \alpha_{22}-\alpha_{23})} \quad (206)$$

The message  $X_{\{1,2\}}$  which carries  $d_{\{1,2\}}$  GDoF is decoded successfully if

$$d_{\{1,2\}} \leq \min(\lambda', \alpha_{22} - \lambda, \alpha_{22} + \lambda' + \gamma' - \alpha_{21}, \alpha_{22} - \alpha_{23}) \quad (207)$$

Adding (74) and (81) we have  $\lambda' \leq \alpha_{22} - \alpha_{23}$ . From (74) and (78), the RHS of (207) is equal to  $\lambda'$ . Moreover, from (70) we have  $d_{\{1,2\}} \leq \lambda'$ , therefore (207) holds and  $X_{\{1,2\}}$  is successfully decoded at Receiver 2.

(c) After decoding the messages  $\bar{W}_{\{1,2\}}$ , the second receiver reconstructs the codeword  $X_{\{1,2\}}$  and subtracts its contribution from the received signal. SINR for decoding  $X_{\{2\}}$  at the second receiver is equal to

$$\frac{P^{\alpha_{22}}P^{-\lambda-\lambda'}|G_{22}|^2}{1 + P^{\alpha_{21}}P^{-\gamma'}P^{-\lambda-\lambda'}|G_{21}|^2 + P^{\alpha_{23}}P^{-\lambda}|G_{23}|^2} \approx P^{\min(\alpha_{22}-\lambda-\lambda', \alpha_{22}-\alpha_{21}+\gamma', \alpha_{22}-\alpha_{23}-\lambda')} \quad (208)$$

Thus, the message  $X_{\{2\}}$  which carries  $d_{\{2\}}$  GDoF is decoded successfully if

$$d_{\{2\}} \leq \min(\alpha_{22} - \lambda - \lambda', \alpha_{22} - \alpha_{21} + \gamma', \alpha_{22} - \alpha_{23} - \lambda') \quad (209)$$

From (78) and (79), we conclude that the RHS of (209) is equal to  $\alpha_{22} - \lambda - \lambda'$ . However, from (68) we have  $d_{\{2\}} \leq \alpha_{22} - \lambda - \lambda'$ , therefore (209) holds and  $X_{\{2\}}$  is successfully decoded at Receiver 2.

3.  $X_{\{3\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the third receiver treating  $X_{\{1\}}, X_{\{2\}}$  and  $X_{\{1,2\}}$  as noise.

- (a) The SINR for decoding  $X_{\{1,2,3\}}$  at the third receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{33}}(1 - 2P^{-\lambda})|G_{33}|^2}{1 + P^{\alpha_{31}}P^{-\gamma'}P^{-\lambda}|G_{31}|^2 + P^{\alpha_{32}}P^{-\lambda}|G_{32}|^2 + P^{\alpha_{33}}P^{-\lambda}|G_{33}|^2} \approx P^\lambda \quad (210)$$

where (210) follows as from (10) we have  $\max(\alpha_{ji}, \alpha_{ij}) \leq \alpha_{ii}$  for all  $i, j, k \in [3]$ . Therefore, the message  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully at the third receiver.

- (b) Finally, the third receiver decodes  $X_{\{3\}}$  treating  $X_{\{1\}}, X_{\{2\}}$  as noise with SINR equal to,

$$\frac{P^{\alpha_{33}}P^{-\lambda}|G_{33}|^2}{1 + P^{\alpha_{31}}P^{-\gamma'}P^{-\lambda}|G_{31}|^2 + P^{\alpha_{32}}P^{-\lambda}|G_{32}|^2} \approx P^{\min(\alpha_{33}-\lambda, \alpha_{33}-\alpha_{31}+\gamma', \alpha_{33}-\alpha_{32})} \quad (211)$$

Therefore, the message  $X_{\{3\}}$  which carries  $d_{\{3\}}$  GDoF is decoded successfully if

$$d_{\{3\}} \leq \min(\alpha_{33} - \lambda, \alpha_{33} - \alpha_{31} + \gamma', \alpha_{33} - \alpha_{32}) \quad (212)$$

From (80) and (81), we conclude that the RHS of (212) is equal to  $\alpha_{33} - \lambda$ . However, from (69) we have  $d_{\{3\}} \leq \alpha_{33} - \lambda$ , therefore (212) holds and  $X_{\{3\}}$  is successfully decoded at Receiver 3.

## C $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') = \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$

Consider some arbitrary quadruple  $(\lambda, \lambda', \gamma, \gamma')$  satisfying ((73)–(82)) and the regions  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  and  $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  given in ((62) – (71)) and ((83) – (89)). In order to show  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') = \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ , let us prove  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  and  $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  separately as follows.

### C.1 $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$

Consider some arbitrary triple  $(d_1, d_2, d_3) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ . From ((62) – (71)) there exists some tuple  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  where

$$d_{\{1\}} \leq \alpha_{11} - \lambda - \lambda' - \gamma - \gamma' \quad (213)$$

$$d_{\{2\}} \leq \alpha_{22} - \lambda - \lambda' \quad (214)$$

$$d_{\{3\}} \leq \alpha_{33} - \lambda \quad (215)$$

$$d_{\{1,2\}} \leq \lambda' \quad (216)$$

$$d_{\{1,2,3\}} \leq \lambda \quad (217)$$

$$d_1 = d_{\{1\}} + \mu_1 d_{\{1,2\}} + \xi_1 d_{\{1,2,3\}} \quad (218)$$

$$d_2 = d_{\{2\}} + \mu_2 d_{\{1,2\}} + \xi_2 d_{\{1,2,3\}} \quad (219)$$

$$d_3 = d_{\{3\}} + \xi_3 d_{\{1,2,3\}} \quad (220)$$

$$0 \leq \mu_1, \mu_2, \xi_1, \xi_2, \xi_3 \quad (221)$$

$$\mu_1 + \mu_2 = 1 \quad (222)$$

$$\xi_1 + \xi_2 + \xi_3 = 1 \quad (223)$$

Now, we claim that  $(d_1, d_2, d_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ . In order to prove the claim, it is sufficient to check the following bounds.

$$0 \leq d_1 \leq \alpha_{11} - \gamma - \gamma', 0 \leq d_2 \leq \alpha_{22}, 0 \leq d_3 \leq \alpha_{33}, \quad (224)$$

$$d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \lambda - \lambda' - \gamma - \gamma', \quad (225)$$

$$d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \lambda - \gamma - \gamma', \quad (226)$$

$$d_2 + d_3 \leq \alpha_{22} + \alpha_{33} - \lambda, \quad (227)$$

$$d_1 + d_2 + d_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\lambda - \lambda' - \gamma - \gamma' \quad (228)$$

which are true from ((213) – (223)) as follows.

$$d_1 \leq d_{\{1\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \leq \alpha_{11} - \gamma - \gamma' \quad (229)$$

$$d_2 \leq d_{\{2\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \leq \alpha_{22} \quad (230)$$

$$d_3 \leq d_{\{3\}} + d_{\{1,2,3\}} \leq \alpha_{33} \quad (231)$$

$$d_1 + d_2 \leq d_{\{1\}} + d_{\{2\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \quad (232)$$

$$\leq (\alpha_{11} - \lambda - \lambda' - \gamma - \gamma') + (\alpha_{22} - \lambda - \lambda') + \lambda' + \lambda \quad (233)$$

$$= \alpha_{11} + \alpha_{22} - \lambda - \lambda' - \gamma - \gamma' \quad (234)$$

$$d_1 + d_3 \leq d_{\{1\}} + d_{\{3\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \quad (235)$$

$$\leq (\alpha_{11} - \lambda - \lambda' - \gamma - \gamma') + (\alpha_{33} - \lambda) + \lambda' + \lambda \quad (236)$$

$$= \alpha_{11} + \alpha_{33} - \lambda - \gamma - \gamma' \quad (237)$$

$$d_2 + d_3 \leq d_{\{2\}} + d_{\{3\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \quad (238)$$

$$\leq (\alpha_{22} - \lambda - \lambda') + (\alpha_{33} - \lambda) + \lambda' + \lambda \quad (239)$$

$$= \alpha_{22} + \alpha_{33} - \lambda \quad (240)$$

$$d_1 + d_2 + d_3 \leq d_{\{1\}} + d_{\{2\}} + d_{\{3\}} + d_{\{1,2\}} + d_{\{1,2,3\}} \quad (241)$$

$$\leq (\alpha_{11} - \lambda - \lambda' - \gamma - \gamma') + (\alpha_{22} - \lambda - \lambda') + (\alpha_{33} - \lambda) + \lambda' + \lambda \quad (242)$$

$$= \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\lambda - \lambda' - \gamma - \gamma' \quad (243)$$

Therefore, for any triple  $(d_1, d_2, d_3) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  we conclude that  $(d_1, d_2, d_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ . Thus,  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ .

## C.2 $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ and $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ are convex sets

The set  $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  is a convex polyhedron by definition.<sup>9</sup> In fact, since it is bounded, it is a convex polytope [19].

Let us consider the set  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  and two arbitrary members of it, e.g.,  $(d_1, d_2, d_3)$  and  $(d'_1, d'_2, d'_3)$ . From ((62) – (71)), there exists  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  where ((62)–(71)) are satisfied for  $(d_1, d_2, d_3)$  and there exists  $(d'_{\{1\}}, d'_{\{2\}}, d'_{\{3\}}, d'_{\{1,2\}}, d'_{\{1,2,3\}}, \mu'_1, \mu'_2, \xi'_1, \xi'_2, \xi'_3)$  where ((62)–(71)) are satisfied for  $(d'_1, d'_2, d'_3)$ . Consider  $0 < \zeta < 1$ . Let us prove that  $(\zeta d_1 + (1-\zeta)d'_1, \zeta d_2 + (1-\zeta)d'_2, \zeta d_3 + (1-\zeta)d'_3) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ . In order to do so, we derive the variables  $(d''_{\{1\}}, d''_{\{2\}}, d''_{\{3\}}, d''_{\{1,2\}}, d''_{\{1,2,3\}}, \mu''_1, \mu''_2, \xi''_1, \xi''_2, \xi''_3)$  satisfying ((62) – (71)) for the point

<sup>9</sup>A convex polyhedron in  $\mathbb{R}^n$  is defined as  $\{x \mid Ax \leq b\}$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^{m \times 1}$ .

$(\zeta d_1 + (1 - \zeta)d'_1, \zeta d_2 + (1 - \zeta)d'_2, \zeta d_3 + (1 - \zeta)d'_3)$  as follows.

$$d''_{\{k\}} = \zeta d_{\{k\}} + (1 - \zeta)d'_{\{k\}}, \forall k \in [3] \quad (244)$$

$$d''_{\{1,2\}} = \zeta d_{\{1,2\}} + (1 - \zeta)d'_{\{1,2\}} \quad (245)$$

$$d''_{\{1,2,3\}} = \zeta d_{\{1,2,3\}} + (1 - \zeta)d'_{\{1,2,3\}} \quad (246)$$

$$\mu''_1 = \frac{\zeta \mu_1 d_{\{1,2\}} + (1 - \zeta) \mu'_1 d'_{\{1,2\}}}{d''_{\{1,2\}}} \quad (247)$$

$$\xi''_1 = \frac{\zeta \xi_1 d_{\{1,2,3\}} + (1 - \zeta) \xi'_1 d'_{\{1,2,3\}}}{d''_{\{1,2,3\}}} \quad (248)$$

$$\xi''_2 = \frac{\zeta \xi_2 d_{\{1,2,3\}} + (1 - \zeta) \xi'_2 d'_{\{1,2,3\}}}{d''_{\{1,2,3\}}} \quad (249)$$

$$\mu''_2 = 1 - \mu''_1 \quad (250)$$

$$\xi''_3 = 1 - \xi''_1 - \xi''_2 \quad (251)$$

Note that, the variables  $(d''_{\{1\}}, d''_{\{2\}}, d''_{\{3\}}, d''_{\{1,2\}}, d''_{\{1,2,3\}}, \mu''_1, \mu''_2, \xi''_1, \xi''_2, \xi''_3)$  satisfy ((62) – (71)). For instance,

$$\begin{aligned} d''_1 &= \zeta d_1 + (1 - \zeta)d'_1 \\ &= (\zeta d_{\{1\}} + (1 - \zeta)d'_{\{1\}}) + (\zeta \mu_1 d_{\{1,2\}} + (1 - \zeta) \mu'_1 d'_{\{1,2\}}) + (\zeta \xi_1 d_{\{1,2,3\}} + (1 - \zeta) \xi'_1 d'_{\{1,2,3\}}) \\ &= d''_{\{1\}} + \mu''_1 d''_{\{1,2\}} + \xi''_1 d''_{\{1,2,3\}} \end{aligned} \quad (252)$$

Therefore,  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  is a convex set.

### C.3 $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$

Consider a feasible quadruple  $(\lambda, \lambda', \gamma, \gamma')$ . It is sufficient to show that all the corner points of the convex polytope  $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  reside in the convex set  $\mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ .<sup>10</sup>

1.  $d_1 = \alpha_{11} - \gamma - \gamma'$ . Consider the hyperplane  $d_1 = \alpha_{11} - \gamma - \gamma'$  and the set of all the points contained in this hyperplane which satisfy all the other inequalities ((83) – (89)), i.e.,

$$\mathbb{S}_1 = \{(d_1, d_2, d_3); d_1 = \alpha_{11} - \gamma - \gamma', 0 \leq d_2 \leq \alpha_{22} - \lambda - \lambda', 0 \leq d_3 \leq \alpha_{33} - \lambda\} \quad (253)$$

In order to check that  $\mathbb{S}_1 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ , it is sufficient to prove that for any  $(d_1, d_2, d_3) \in \mathbb{S}_1$  there exists some tuple  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  where ((62)–(71)) are satisfied. This is true by choosing  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  as follows.

$$\begin{aligned} &(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3) \\ &= (\alpha_{11} - \lambda - \lambda' - \gamma - \gamma', d_2, d_3, \lambda', \lambda, 1, 0, 1, 0, 0) \end{aligned} \quad (254)$$

<sup>10</sup>For any convex polytope  $A$  and a convex set  $B$ ,  $A \subset B$  if and only if the vertices of  $A$  are members of  $B$ . While the statement is obvious, a short proof is included in Appendix F for the sake of completeness.

2.  $d_2 = \alpha_{22}$ . Consider the hyperplane  $d_2 = \alpha_{22}$  and the set of all the points contained in this hyperplane which satisfy all the other inequalities ((83) – (89)), i.e.,

$$\mathbb{S}_2 = \{(d_1, d_2, d_3); 0 \leq d_1 \leq \alpha_{11} - \lambda - \lambda' - \gamma - \gamma', d_2 = \alpha_{22}, 0 \leq d_3 \leq \alpha_{33} - \lambda\} \quad (255)$$

Similarly, we show that  $\mathbb{S}_2 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  as for any  $(d_1, d_2, d_3) \in \mathbb{S}_2$  there exists some tuple  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  where ((62) – (71)) are satisfied. This is true by choosing  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  as follows.

$$\begin{aligned} & (d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3) \\ &= (d_1, \alpha_{22} - \lambda - \lambda', d_3, \lambda', \lambda, 0, 1, 0, 1, 0) \end{aligned} \quad (256)$$

3.  $d_3 = \alpha_{33}$ . Consider the hyperplane  $d_3 = \alpha_{33}$  and the hyperplane  $\mathbb{S}_3$  as follows.

$$\mathbb{S}_3 = \left\{ (d_1, d_2, d_3); d_3 = \alpha_{33}, 0 \leq d_1 \leq \alpha_{11} - \lambda - \gamma - \gamma', \right. \\ \left. 0 \leq d_2 \leq \alpha_{22} - \lambda, d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - 2\lambda - \lambda' - \gamma - \gamma' \right\} \quad (257)$$

In order to prove that  $\mathbb{S}_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ , it is sufficient to show that the line  $l_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  where  $l_3$  is defined as follows.<sup>11</sup>

$$l_3 = \left\{ (d_1, d_2, d_3); d_3 = \alpha_{33}, 0 \leq d_1 \leq \alpha_{11} - \lambda - \gamma - \gamma', \right. \\ \left. 0 \leq d_2 \leq \alpha_{22} - \lambda, d_1 + d_2 = \alpha_{11} + \alpha_{22} - 2\lambda - \lambda' - \gamma - \gamma' \right\} \quad (259)$$

Now, let us prove that  $l_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ , i.e., for any  $(d_1, d_2, d_3) \in l_3$  there exists some tuple  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  where ((62) – (71)) are satisfied. This is verified to be true by choosing  $(d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3)$  as follows.

$$\begin{aligned} & (d_{\{1\}}, d_{\{2\}}, d_{\{3\}}, d_{\{1,2\}}, d_{\{1,2,3\}}, \mu_1, \mu_2, \xi_1, \xi_2, \xi_3) \\ &= (\alpha_{11} - \lambda - \lambda' - \gamma - \gamma', \alpha_{22} - \lambda - \lambda', \alpha_{33} - \lambda, \lambda', \lambda, \frac{d_1 - d_{\{1\}}}{\lambda'}, \frac{d_2 - d_{\{2\}}}{\lambda'}, 0, 0, 1) \end{aligned} \quad (260)$$

4. It is trivial to verify that the corner point  $(0, 0, 0) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  by choosing  $d_{\{1\}} = d_{\{2\}} = d_{\{3\}} = d_{\{1,2\}} = d_{\{1,2,3\}} = 0$ .
5. Surprisingly, all the corner points are already considered in the previous cases. For instance

<sup>11</sup>To see why  $l_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  results in  $\mathbb{S}_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ , consider the tuples  $(d'_1, d'_2, d'_3)$  and  $(d_1, d_2, d_3)$  where  $0 \leq d'_i \leq d_i$  for any  $i \in [3]$ . From ((62) – (71)), we have

$$(d_1, d_2, d_3) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') \Rightarrow (d'_1, d'_2, d'_3) \in \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma') \quad (258)$$

Therefore, if  $l_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$  then we conclude that  $\mathbb{S}_3 \subset \mathcal{D}_{123}(\lambda, \lambda', \gamma, \gamma')$ .

consider the point  $s$  obtained from the intersection of the following three facets.<sup>12</sup>

$$d_1 + d_3 = \alpha_{11} + \alpha_{33} - \lambda - \gamma - \gamma' \quad (264)$$

$$d_2 + d_3 = \alpha_{22} + \alpha_{33} - \lambda \quad (265)$$

$$d_1 + d_2 + d_3 = \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\lambda - \lambda' - \gamma - \gamma' \quad (266)$$

From ((264) – (266)), we have  $d_3 = \alpha_{33} + \lambda'$  which contradicts the condition  $d_3 < \alpha_{33}$ .

$$\mathbf{D} \quad \hat{\mathcal{D}}_{123} = \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$$

Let us prove that  $\cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \hat{\mathcal{D}}_{123}$  and  $\hat{\mathcal{D}}_{123} \subset \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  separately.

$$\mathbf{D.1} \quad \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \hat{\mathcal{D}}_{123}$$

In order to prove that  $\cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \hat{\mathcal{D}}_{123}$ , it is sufficient to prove that  $\bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \subset \hat{\mathcal{D}}_{123}$  for any  $\lambda, \lambda', \gamma, \gamma'$  satisfying ((73) – (82)). In other words, we need to prove that (73)-(82) and (83)-(89) together imply (90)-(96). But this is easily verified as follows.

$$(82), (83) \Rightarrow (90) \quad (267)$$

$$(84) \Rightarrow (91) \quad (268)$$

$$(85) \Rightarrow (92) \quad (269)$$

$$(76), (77), (78), (79), (80), (81), (86) \Rightarrow (93) \quad (270)$$

$$(77), (79), (80), (81), (82), (87) \Rightarrow (94) \quad (271)$$

$$(79), (81), (88) \Rightarrow (95) \quad (272)$$

$$(76), (77), (78), (79), (80), (81), (82), (89) \Rightarrow (96) \quad (273)$$

$$\mathbf{D.2} \quad \hat{\mathcal{D}}_{123} \text{ and } \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \text{ are convex sets}$$

Similar to C.2,  $\hat{\mathcal{D}}_{123}$  is a convex polytope by definition. Next we have  $\cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ . Consider two members of it, e.g.,  $(\hat{d}_1, \hat{d}_2, \hat{d}_3)$  and  $(\bar{d}_1, \bar{d}_2, \bar{d}_3)$ .

1. As  $(\hat{d}_1, \hat{d}_2, \hat{d}_3) \in \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ , there exists  $(\hat{\lambda}, \hat{\lambda}', \hat{\gamma}, \hat{\gamma}')$  where  $(\hat{d}_1, \hat{d}_2, \hat{d}_3) \in \bar{\mathcal{D}}_{123}(\hat{\lambda}, \hat{\lambda}', \hat{\gamma}, \hat{\gamma}')$ ,

---

<sup>12</sup>We are considering the points not considered in the previous cases. Thus, we assume that

$$d_1 < \alpha_{11} - \gamma - \gamma', \quad (261)$$

$$d_2 < \alpha_{22}, \quad (262)$$

$$d_3 < \alpha_{33}. \quad (263)$$

i.e.,

$$\hat{d}_1 \leq \alpha_{11} - \hat{\gamma} - \hat{\gamma}' \quad (274)$$

$$\hat{d}_2 \leq \alpha_{22} \quad (275)$$

$$\hat{d}_3 \leq \alpha_{33}, \quad (276)$$

$$\hat{d}_1 + \hat{d}_2 \leq \alpha_{11} + \alpha_{22} - \hat{\lambda} - \hat{\lambda}' - \hat{\gamma} - \hat{\gamma}', \quad (277)$$

$$\hat{d}_1 + \hat{d}_3 \leq \alpha_{11} + \alpha_{33} - \hat{\lambda} - \hat{\gamma} - \hat{\gamma}', \quad (278)$$

$$\hat{d}_2 + \hat{d}_3 \leq \alpha_{22} + \alpha_{33} - \hat{\lambda}, \quad (279)$$

$$\hat{d}_1 + \hat{d}_2 + \hat{d}_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\hat{\lambda} - \hat{\lambda}' - \hat{\gamma} - \hat{\gamma}' \quad (280)$$

such that  $\hat{\lambda}, \hat{\lambda}', \hat{\gamma}, \hat{\gamma}'$  satisfy conditions (73) to (82).

2. Similarly, as  $(\bar{d}_1, \bar{d}_2, \bar{d}_3) \in \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ , there exists  $(\bar{\lambda}, \bar{\lambda}', \bar{\gamma}, \bar{\gamma}')$  where  $(\bar{d}_1, \bar{d}_2, \bar{d}_3) \in \bar{\mathcal{D}}_{123}(\bar{\lambda}, \bar{\lambda}', \bar{\gamma}, \bar{\gamma}')$ , i.e.,

$$\bar{d}_1 \leq \alpha_{11} - \bar{\gamma} - \bar{\gamma}' \quad (281)$$

$$\bar{d}_2 \leq \alpha_{22} \quad (282)$$

$$\bar{d}_3 \leq \alpha_{33}, \quad (283)$$

$$\bar{d}_1 + \bar{d}_2 \leq \alpha_{11} + \alpha_{22} - \bar{\lambda} - \bar{\lambda}' - \bar{\gamma} - \bar{\gamma}', \quad (284)$$

$$\bar{d}_1 + \bar{d}_3 \leq \alpha_{11} + \alpha_{33} - \bar{\lambda} - \bar{\gamma} - \bar{\gamma}', \quad (285)$$

$$\bar{d}_2 + \bar{d}_3 \leq \alpha_{22} + \alpha_{33} - \bar{\lambda}, \quad (286)$$

$$\bar{d}_1 + \bar{d}_2 + \bar{d}_3 \leq \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\bar{\lambda} - \bar{\lambda}' - \bar{\gamma} - \bar{\gamma}' \quad (287)$$

such that  $\bar{\lambda}, \bar{\lambda}', \bar{\gamma}, \bar{\gamma}'$  satisfy conditions (73) to (82).

Now, consider the point  $d = (\zeta \hat{d}_1 + (1 - \zeta) \bar{d}_1, \zeta \hat{d}_2 + (1 - \zeta) \bar{d}_2, \zeta \hat{d}_3 + (1 - \zeta) \bar{d}_3)$ . We claim that  $d \in \check{\mathcal{D}}_{123}(\check{\lambda}, \check{\lambda}', \check{\gamma}, \check{\gamma}')$  where  $\check{\lambda}, \check{\lambda}', \check{\gamma}, \check{\gamma}'$  are defined as <sup>13</sup>,

$$\check{\lambda} = \zeta \hat{\lambda} + (1 - \zeta) \bar{\lambda} \quad (289)$$

$$\check{\lambda}' = \zeta \hat{\lambda}' + (1 - \zeta) \bar{\lambda}' \quad (290)$$

$$\check{\gamma} = \zeta \hat{\gamma} + (1 - \zeta) \bar{\gamma} \quad (291)$$

$$\check{\gamma}' = \zeta \hat{\gamma}' + (1 - \zeta) \bar{\gamma}' \quad (292)$$

This is verified by checking ((83) – (89)). For instance, we check the inequalities  $\zeta \hat{d}_1 + (1 - \zeta) \bar{d}_1 + \zeta \hat{d}_2 + (1 - \zeta) \bar{d}_2 \leq \alpha_{11} + \alpha_{22} - \check{\lambda} - \check{\lambda}' - \check{\gamma} - \check{\gamma}'$  and  $\zeta \hat{d}_1 + (1 - \zeta) \bar{d}_1 + \zeta \hat{d}_2 + (1 - \zeta) \bar{d}_2 + \zeta \hat{d}_3 + (1 - \zeta) \bar{d}_3 \leq$

<sup>13</sup>Note that,  $\check{\lambda}, \check{\lambda}', \check{\gamma}, \check{\gamma}'$  satisfy conditions (73) to (82). For instance, (73) is verified as,

$$\begin{aligned} & \check{\lambda} + \check{\lambda}' + \check{\gamma} + \check{\gamma}' \\ &= \zeta(\hat{\lambda} + \hat{\lambda}' + \hat{\gamma} + \hat{\gamma}') + (1 - \zeta)(\bar{\lambda} + \bar{\lambda}' + \bar{\gamma} + \bar{\gamma}') \\ &\leq \zeta \alpha_{11} + (1 - \zeta) \alpha_{11} = \alpha_{11} \end{aligned} \quad (288)$$

All the other conditions (73) to (82) are also true as they are linear combinations of  $\check{\lambda}, \check{\lambda}', \check{\gamma}, \check{\gamma}'$ .

$\alpha_{11} + \alpha_{22} + \alpha_{33} - 2\check{\lambda} - \check{\lambda}' - \check{\gamma} - \check{\gamma}'$  as follows.

$$\begin{aligned} & \zeta \hat{d}_1 + (1 - \zeta) \bar{d}_1 + \zeta \hat{d}_2 + (1 - \zeta) \bar{d}_2 \\ &= \zeta(\hat{d}_1 + \hat{d}_2) + (1 - \zeta)(\bar{d}_1 + \bar{d}_2) \end{aligned} \quad (293)$$

$$\leq \zeta(\alpha_{11} + \alpha_{22} - \hat{\lambda} - \hat{\lambda}' - \hat{\gamma} - \hat{\gamma}') + (1 - \zeta)(\alpha_{11} + \alpha_{22} - \bar{\lambda} - \bar{\lambda}' - \bar{\gamma} - \bar{\gamma}') \quad (294)$$

$$= \alpha_{11} + \alpha_{22} - \check{\lambda} - \check{\lambda}' - \check{\gamma} - \check{\gamma}' \quad (295)$$

$$\begin{aligned} & \zeta \hat{d}_1 + (1 - \zeta) \bar{d}_1 + \zeta \hat{d}_2 + (1 - \zeta) \bar{d}_2 + \zeta \hat{d}_3 + (1 - \zeta) \bar{d}_3 \\ &= \zeta(\hat{d}_1 + \hat{d}_2 + \hat{d}_3) + (1 - \zeta)(\bar{d}_1 + \bar{d}_2 + \bar{d}_3) \end{aligned} \quad (296)$$

$$\leq \zeta(\alpha_{11} + \alpha_{22} + \alpha_{33} - 2\hat{\lambda} - \hat{\lambda}' - \hat{\gamma} - \hat{\gamma}') + (1 - \zeta)(\alpha_{11} + \alpha_{22} + \alpha_{33} - 2\bar{\lambda} - \bar{\lambda}' - \bar{\gamma} - \bar{\gamma}') \quad (297)$$

$$= \alpha_{11} + \alpha_{22} + \alpha_{33} - 2\check{\lambda} - \check{\lambda}' - \check{\gamma} - \check{\gamma}' \quad (298)$$

Therefore, as  $d \in \bar{\mathcal{D}}_{123}(\check{\lambda}, \check{\lambda}', \check{\gamma}, \check{\gamma}')$ , we conclude that  $d \in \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$ . This proves that the set  $\cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  is convex.

### D.3 $\hat{\mathcal{D}}_{123} \subset \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$

In this section, let us use the compact notation

$$\hat{\mathcal{D}}_{123} \triangleq \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \quad (299)$$

It is sufficient to show that all the corner points of the convex polytope  $\hat{\mathcal{D}}_{123}$  reside in the convex set  $\hat{\mathcal{D}}'_{123}$ .

1. consider the tuples  $(d'_1, d'_2, d'_3)$  and  $(d_1, d_2, d_3)$  where  $0 \leq d'_i \leq d_i$  for any  $i \in [3]$ . We claim that, if  $(d_1, d_2, d_3) \in \hat{\mathcal{D}}'_{123}$  then  $(d'_1, d'_2, d'_3) \in \hat{\mathcal{D}}'_{123}$ . This is true as from ((83) – (89)), we have

$$(d_1, d_2, d_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \Rightarrow (d'_1, d'_2, d'_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \quad (300)$$

Therefore, as  $\hat{\mathcal{D}}'_{123} \triangleq \cup_{\lambda, \lambda', \gamma, \gamma'} \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  we have

$$(d_1, d_2, d_3) \in \hat{\mathcal{D}}'_{123} \Rightarrow \exists(\lambda, \lambda', \gamma, \gamma') \text{ s.t. } (d_1, d_2, d_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \quad (301)$$

$$\Rightarrow (d'_1, d'_2, d'_3) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma') \Rightarrow (d'_1, d'_2, d'_3) \in \hat{\mathcal{D}}'_{123} \quad (302)$$

Therefore, if  $(d_1, d_2, d_3) \in \hat{\mathcal{D}}'_{123}$  then  $(d'_1, d'_2, d'_3) \in \hat{\mathcal{D}}'_{123}$ .

2.  $d_1 = \alpha_{11}$ . Consider the hyperplane  $d_1 = \alpha_{11}$  and the set of all the points contained in this hyperplane which satisfy all the other inequalities ((90) – (96)), i.e.,

$$\begin{aligned} \mathcal{S}_1 &= \{(d_1, d_2, d_3); d_1 = \alpha_{11}, 0 \leq d_2 \leq \alpha_{22} - \max_{l, m \in [3], l \neq m} \alpha_{lm}, \\ & 0 \leq d_3 \leq \alpha_{33} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13})\} \end{aligned} \quad (303)$$

Consider the following corner point.

$$A = (d_1 = \alpha_{11}, d_2 = \alpha_{22} - \max_{l, m \in [3], l \neq m} \alpha_{lm}, d_3 = \alpha_{33} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13})) \quad (304)$$

$\mathcal{S}_1 \subset \hat{\mathcal{D}}'_{123}$  since  $A \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  where,<sup>14</sup>

$$(\lambda, \lambda', \gamma, \gamma') = (\max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), \max_{l, m \in [3], l \neq m} \alpha_{lm} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), 0, 0) \quad (305)$$

<sup>14</sup>This is true from (301) and (302).



3.  $d_2 = \alpha_{22}$ . The hyperplane  $d_2 = \alpha_{22}$  is represented as,

$$\mathcal{S}_2 = \left\{ (d_1, d_2, d_3); d_2 = \alpha_{22}, 0 \leq d_1 \leq \alpha_{11} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \right. \\ \left. 0 \leq d_3 \leq \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), d_1 + d_3 \leq \alpha_{11} + \alpha_{33} - \alpha \right\} \quad (306)$$

where  $\alpha$  is equal to

$$\alpha = \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{32}, \alpha_{23}), \\ \alpha_{13} + \alpha_{21}, \\ \alpha_{12} + \alpha_{31}, \\ \alpha_{13} + \alpha_{31} \end{array} \right\} \quad (307)$$

In order to show that  $\mathcal{S}_2 \subset \hat{\mathcal{D}}'_{123}$ , it is sufficient to show that the two corner points  $B$  and  $C$  belong to the set  $\hat{\mathcal{D}}'_{123}$  where  $B$  and  $C$  are equal to,<sup>15</sup>

$$B = (d_1 = \alpha_{11} - \alpha + \max(\alpha_{23}, \alpha_{32}), d_2 = \alpha_{22}, d_3 = \alpha_{33} - \max(\alpha_{23}, \alpha_{32})) \quad (308)$$

$$C = (d_1 = \alpha_{11} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, d_2 = \alpha_{22}, d_3 = \alpha_{33} + \max_{l,m \in [3], l \neq m} \alpha_{lm} - \alpha) \quad (309)$$

(a) In order to prove that  $B \in \hat{\mathcal{D}}'_{123}$ , it is sufficient to show that  $B \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  for a quadruple  $(\lambda, \lambda', \gamma, \gamma')$  satisfying ((73)–(82)). Let us show how the variables  $(\lambda, \lambda', \gamma, \gamma')$  are derived. First of all, note that as  $B \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  and as  $(\lambda, \lambda', \gamma, \gamma')$  satisfies ((73)–(82)), we have

$$d_2 = \alpha_{22}, d_3 = \alpha_{33} - \max(\alpha_{23}, \alpha_{32}), (88) \Rightarrow \lambda \leq \max(\alpha_{32}, \alpha_{23}) \quad (310)$$

$$(79), (81) \Rightarrow \lambda \geq \max(\alpha_{32}, \alpha_{23}) \quad (311)$$

$$d_1 + d_2 + d_3 = \alpha_{11} + \alpha_{22} + \alpha_{33} - \alpha, (89) \Rightarrow 2\lambda + \lambda' + \gamma + \gamma' \leq \alpha \quad (312)$$

$$(76), (77), (78), (79), (80), (81), (307) \Rightarrow 2\lambda + \lambda' + \gamma + \gamma' \geq \alpha \quad (313)$$

Therefore, we conclude that

$$\lambda = \max(\alpha_{32}, \alpha_{23}) \quad (314)$$

$$2\lambda + \lambda' + \gamma + \gamma' = \alpha \quad (315)$$

Next,  $(\gamma, \gamma', \lambda')$  satisfying ((73)–(82)), (314) and (315) are obtained as follows.

i. If  $\alpha_{13} + \alpha_{21} = \alpha$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{13} - \max(\alpha_{32}, \alpha_{23}), \alpha_{31} - \max(\alpha_{32}, \alpha_{23}), \alpha_{21} - \alpha_{31}) \quad (316)$$

ii. If  $\alpha_{13} + \alpha_{31} = \alpha$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{13} - \max(\alpha_{32}, \alpha_{23}), \alpha_{31} - \max(\alpha_{32}, \alpha_{23}), 0) \quad (317)$$

iii. If  $\alpha_{12} + \alpha_{31} = \alpha$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{13} - \max(\alpha_{32}, \alpha_{23}), \alpha_{31} - \max(\alpha_{32}, \alpha_{23}), \alpha_{12} - \alpha_{13}) \quad (318)$$

---

<sup>15</sup>This is true from (301) and (302).

iv. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} + \lambda = \alpha$  and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{12}$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{12} - \max(\alpha_{21}, \alpha_{32}, \alpha_{23}), 0, \max(\alpha_{21}, \alpha_{32}, \alpha_{23}) - \lambda) \quad (319)$$

v. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} + \lambda = \alpha$  and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{13}$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{13} - \max(\alpha_{32}, \alpha_{23}), 0, 0) \quad (320)$$

vi. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} + \lambda = \alpha$  and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{21}$ , then

$$(\gamma, \gamma', \lambda') = (0, \alpha_{21} - \max(\alpha_{12}, \alpha_{23}, \alpha_{32}), \max(\alpha_{12}, \alpha_{23}, \alpha_{32}) - \lambda) \quad (321)$$

vii. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} + \lambda = \alpha$  and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{31}$ , then

$$(\gamma, \gamma', \lambda') = (0, \alpha_{31} - \max(\alpha_{32}, \alpha_{23}), 0) \quad (322)$$

viii. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} + \lambda = \alpha$  and  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \max(\alpha_{23}, \alpha_{32})$ , then

$$(\gamma, \gamma', \lambda') = (0, 0, 0) \quad (323)$$

(b)  $C \in \hat{\mathcal{D}}'_{123}$  since  $C \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  for a quadruple  $(\lambda, \lambda', \gamma, \gamma')$  where

$$\lambda = \alpha - \max_{l,m \in [3], l \neq m} \alpha_{lm} \quad (324)$$

and  $(\gamma, \gamma', \lambda')$  is represented as follows.

i. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{12}$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{12} - \max(\alpha_{21}, \alpha_{32}, \alpha_{23}), 0, \max(\alpha_{21}, \alpha_{32}, \alpha_{23}) - \lambda) \quad (325)$$

ii. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{21}$ , then

$$(\gamma, \gamma', \lambda') = (0, \alpha_{21} - \max(\alpha_{12}, \alpha_{32}, \alpha_{23}), \max(\alpha_{12}, \alpha_{32}, \alpha_{23}) - \lambda) \quad (326)$$

iii. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{31}$ , then

$$(\gamma, \gamma', \lambda') = (0, \alpha_{31} - \lambda, 0) \quad (327)$$

iv. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \alpha_{13}$ , then

$$(\gamma, \gamma', \lambda') = (\alpha_{13} - \lambda, 0, 0) \quad (328)$$

v. If  $\max_{l,m \in [3], l \neq m} \alpha_{lm} = \max(\alpha_{23}, \alpha_{32})$ , then

$$(\gamma, \gamma', \lambda') = (0, 0, 0) \quad (329)$$

4.  $d_3 = \alpha_{33}$ . The hyperplane  $d_3 = \alpha_{33}$  is represented as,

$$\mathcal{S}_3 = \left\{ (d_1, d_2, d_3) : d_3 = \alpha_{33}, d_1 \leq \alpha_{11} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), \right. \quad (330)$$

$$\left. d_2 \leq \alpha_{22} - \max(\alpha_{23}, \alpha_{32}), d_1 + d_2 \leq \alpha_{11} + \alpha_{22} - \alpha \right\} \quad (331)$$

Let us consider the two corner points  $D$  and  $E$ ,

$$D = (d_1 = \alpha_{11} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), d_2 = \alpha_{22} + \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) - \alpha, d_3 = \alpha_{33})$$

$$E = (d_1 = \alpha_{11} - \alpha + \max(\alpha_{23}, \alpha_{32}), d_2 = \alpha_{22} - \max(\alpha_{23}, \alpha_{32}), d_3 = \alpha_{33}) \quad (332)$$

(a)  $D \in \hat{\mathcal{D}}'_{123}$  since  $D \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  for the following quadruple  $(\lambda, \lambda', \gamma, \gamma')$ .

i. If  $\max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) = \alpha_{13}$ , then

$$(\lambda, \gamma, \gamma', \lambda') = (\max(\alpha_{23}, \alpha_{32}, \alpha_{31}), \alpha_{13} - \lambda, 0, \alpha - \alpha_{13} - \lambda) \quad (333)$$

ii. If  $\max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) = \alpha_{31}$ , then

$$(\lambda, \gamma, \gamma', \lambda') = (\max(\alpha_{23}, \alpha_{32}, \alpha_{13}), 0, \alpha_{31} - \lambda, \alpha - \alpha_{31} - \lambda) \quad (334)$$

iii. If  $\max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) = \max(\alpha_{23}, \alpha_{32})$ , then

$$(\lambda, \gamma, \gamma', \lambda') = (\max(\alpha_{23}, \alpha_{32}), 0, 0, \alpha - 2\lambda) \quad (335)$$

Note that,  $\max_{l,m \in [3], l \neq m} \alpha_{lm} \leq \min(\alpha_{11}, \alpha_{22})$  is assumed in 6.1.

(b)  $E \in \hat{\mathcal{D}}'_{123}$  since  $E \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  for the quadruple  $(\lambda, \lambda', \gamma, \gamma')$  given in ((316) – (323)).

5. Consider the point  $F$  obtained from the intersection of the following three facets.

$$d_1 + d_2 = \alpha_{11} + \alpha_{22} - \max_{l,m \in [3], l \neq m} \alpha_{lm}, \quad (336)$$

$$d_1 + d_3 = \alpha_{11} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}), \quad (337)$$

$$d_1 + d_2 + d_3 = \alpha_{11} + \alpha_{22} + \alpha_{33} - \alpha \quad (338)$$

$F \in \hat{\mathcal{D}}'_{123}$  as  $F \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  for the following quadruple  $(\lambda, \lambda', \gamma, \gamma')$ .

$$\lambda = \alpha - \max_{l,m \in [3], l \neq m} \alpha_{lm} \quad (339)$$

$$\lambda' = \max_{l,m \in [3], l \neq m} \alpha_{lm} - \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) \quad (340)$$

$$\gamma = \max(\alpha_{13} - \lambda, \alpha_{12} - \lambda - \lambda', 0) \quad (341)$$

$$\gamma' = \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{23}, \alpha_{32}, \alpha_{31}, \alpha_{13}) - \alpha - \gamma \quad (342)$$

6. Trivially, the corner point  $(0, 0, 0) \in \bar{\mathcal{D}}_{123}(\lambda, \lambda', \gamma, \gamma')$  by choosing  $\lambda = \lambda' = \gamma = \gamma' = \max_{i,j \in [3]} \alpha_{ij}$ .

7. Note that all the corner points are already considered in the previous cases.<sup>16</sup> For instance, consider the point  $s$  obtained from the intersection of the following three facets.

$$d_1 + d_2 = \alpha_{11} + \alpha_{22} - \max_{l,m \in [3], l \neq m} \alpha_{lm} \quad (346)$$

$$d_2 + d_3 = \alpha_{22} + \alpha_{33} - \max(\alpha_{23}, \alpha_{32}) \quad (347)$$

$$d_1 + d_2 + d_3 = \alpha_{11} + \alpha_{22} + \alpha_{33} - \max \left\{ \begin{array}{l} \max_{l,m \in [3], l \neq m} \alpha_{lm} + \max(\alpha_{32}, \alpha_{23}), \\ \alpha_{13} + \alpha_{21}, \\ \alpha_{12} + \alpha_{31}, \\ \alpha_{13} + \alpha_{31} \end{array} \right\} \quad (348)$$

<sup>16</sup>We are considering the points not considered in the previous cases. Thus, we assume that

$$d_1 < \alpha_{11} \quad (343)$$

$$d_2 < \alpha_{22} \quad (344)$$

$$d_3 < \alpha_{33} \quad (345)$$

From ((346) – (348)), we have  $\alpha_{22} \leq d_2$  which contradicts the condition  $d_2 < \alpha_{22}$ . Thus, all the corner points are already considered in the previous cases.

## E Achievability of $\mathcal{F}_{123}$

As the derivation of ((119) – (128)) is similar to the derivation of ((73) – (82)) in Appendix B, we briefly go over it.

1.  $X_{\{1\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the first receiver treating  $X_{\{2\}}$  and  $X_{\{3\}}$  as noise.

- (a) The SINR for decoding  $X_{\{1,2,3\}}$  at the first receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{11}}(1 - 2P^{-\lambda})|G_{11}|^2}{1 + P^{\alpha_{11}}P^{-\lambda}|G_{11}|^2 + P^{\alpha_{12}}P^{-\gamma'}P^{-\lambda}|G_{12}|^2 + P^{\alpha_{13}}P^{-\lambda}|G_{13}|^2} \approx P^{\min(\alpha_{11}, \lambda, \lambda + \alpha_{11} + \gamma' - \alpha_{12}, \lambda + \alpha_{11} - \alpha_{13})} \quad (349)$$

The codeword  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully if

$$d_{\{1,2,3\}} \leq \min(\alpha_{11}, \lambda, \lambda + \alpha_{11} + \gamma' - \alpha_{12}, \lambda + \alpha_{11} - \alpha_{13}) \quad (350)$$

which is true as we have  $\lambda \leq \alpha_{11}$  from (119). From (116) we have  $d_{\{1,2,3\}} \leq \lambda$ , therefore (350) holds and  $X_{\{1,2,3\}}$  is successfully decoded at Receiver 1.

- (b) After decoding the messages  $\bar{W}_{\{1,2,3\}}$ , the first receiver reconstructs the codeword  $X_{\{1,2,3\}}$  and subtracts its contribution from the received signal. The SINR for decoding  $X_{\{1,2\}}$  at the first receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{11}}P^{-\lambda}|G_{11}|^2}{1 + P^{\alpha_{11}}P^{-\lambda-\lambda'}|G_{11}|^2 + P^{\alpha_{12}}P^{-\gamma'}P^{-\lambda-\lambda'}|G_{12}|^2 + P^{\alpha_{13}}P^{-\lambda}|G_{13}|^2} \approx P^{\min(\lambda', \alpha_{11} - \lambda, \lambda' + \alpha_{11} + \gamma' - \alpha_{12}, \alpha_{11} - \alpha_{13})} \quad (351)$$

Therefore,  $X_{\{1,2\}}$  which carries  $d_{\{1,2\}}$  GDoF is decoded successfully if

$$d_{\{1,2\}} \leq \min(\lambda', \alpha_{11} - \lambda, \lambda' + \alpha_{11} + \gamma' - \alpha_{12}, \alpha_{11} - \alpha_{13}) \quad (352)$$

which is true from (115), (119), (122) and (123). Therefore,  $X_{\{1,2\}}$  is successfully decoded at Receiver 1.

- (c) After decoding the messages  $\bar{W}_{\{1,2\}}$ , the first receiver reconstructs the codeword  $X_{\{1,2\}}$  and subtracts its contribution from the received signal. SINR for decoding  $X_{\{1\}}$  is equal to,

$$\frac{P^{\alpha_{11}}P^{-\lambda-\lambda'}|G_{11}|^2}{1 + P^{\alpha_{12}}P^{-\gamma'}P^{-\lambda-\lambda'}|G_{12}|^2 + P^{\alpha_{13}}P^{-\lambda}|G_{13}|^2} \approx P^{\min(\alpha_{11} - \lambda - \lambda', \alpha_{11} - \alpha_{12} + \gamma', \alpha_{11} - \alpha_{13} - \lambda')}$$

Therefore,  $X_{\{1\}}$  which carries  $d_{\{1\}}$  GDoF is decoded successfully if

$$d_{\{1\}} \leq \min(\alpha_{11} - \lambda - \lambda', \alpha_{11} - \alpha_{12} + \gamma', \alpha_{11} - \alpha_{13} - \lambda') \quad (353)$$

which is true from (112), (119), (122) and (123). Therefore,  $X_{\{1\}}$  is successfully decoded at Receiver 1.

2.  $X_{\{2\}}, X_{\{1,2\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the second receiver treating  $X_{\{1\}}$  and  $X_{\{3\}}$  as noise.

(a) The SINR for decoding  $X_{\{1,2,3\}}$  at the second receiver treating the other signals as noise is equal to

$$\begin{aligned} & \frac{P^{\alpha_{22}} P^{-\gamma'} (1 - 2P^{-\lambda}) |G_{22}|^2}{1 + P^{\alpha_{21}} P^{-\lambda} |G_{21}|^2 + P^{\alpha_{22}} P^{-\gamma'} P^{-\lambda} |G_{22}|^2 + P^{\alpha_{23}} P^{-\lambda} |G_{23}|^2} \\ & \approx P^{\min(\alpha_{22} - \gamma', \lambda, \lambda + \alpha_{22} - \gamma' - \alpha_{21}, \lambda + \alpha_{22} - \gamma' - \alpha_{23})} \end{aligned} \quad (354)$$

Therefore,  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully if

$$d_{\{1,2,3\}} \leq \min(\alpha_{22} - \gamma', \lambda, \lambda + \alpha_{22} - \gamma' - \alpha_{21}, \lambda + \alpha_{22} - \gamma' - \alpha_{23}) \quad (355)$$

which is true from (116), (120), (124) and (125). Therefore,  $X_{\{1,2,3\}}$  is successfully decoded at Receiver 2.

(b) After decoding the messages  $\bar{W}_{\{1,2,3\}}$ , the second receiver reconstructs the codeword  $X_{\{1,2,3\}}$  and subtracts its contribution from the received signal. The SINR for decoding  $X_{\{1,2\}}$  at the second receiver treating the other signals as noise is equal to

$$\begin{aligned} & \frac{P^{\alpha_{22}} P^{-\gamma'} P^{-\lambda} |G_{22}|^2}{1 + P^{\alpha_{21}} P^{-\lambda - \lambda'} |G_{21}|^2 + P^{\alpha_{22}} P^{-\gamma'} P^{-\lambda - \lambda'} |G_{22}|^2 + P^{\alpha_{23}} P^{-\lambda} |G_{23}|^2} \\ & \approx P^{\min(\lambda', \alpha_{22} - \gamma' - \lambda, \lambda' + \alpha_{22} - \gamma' - \alpha_{21}, \alpha_{22} - \gamma' - \alpha_{23})} \end{aligned} \quad (356)$$

Therefore,  $X_{\{1,2\}}$  which carries  $d_{\{1,2\}}$  GDoF is decoded successfully if

$$d_{\{1,2\}} \leq \min(\lambda', \alpha_{22} - \gamma' - \lambda, \lambda' + \alpha_{22} - \gamma' - \alpha_{21}, \alpha_{22} - \gamma' - \alpha_{23}) \quad (357)$$

which is true from (115), (120), (124) and (125). Therefore,  $X_{\{1,2\}}$  is successfully decoded at Receiver 2.

(c) After decoding the messages  $\bar{W}_{\{1,2\}}$ , the second receiver reconstructs the codeword  $X_{\{1,2\}}$  and subtracts its contribution from the received signal. SINR for decoding  $X_{\{1\}}$  is equal to,

$$\begin{aligned} & \frac{P^{\alpha_{22}} P^{-\gamma'} P^{-\lambda - \lambda'} |G_{22}|^2}{1 + P^{\alpha_{21}} P^{-\lambda - \lambda'} |G_{21}|^2 + P^{\alpha_{23}} P^{-\lambda} |G_{23}|^2} \\ & \approx P^{\min(\alpha_{22} - \lambda - \lambda' - \gamma', \alpha_{22} - \alpha_{21} - \gamma', \alpha_{22} - \alpha_{23} - \lambda' - \gamma')} \end{aligned} \quad (358)$$

Therefore,  $X_{\{2\}}$  which carries  $d_{\{2\}}$  GDoF is decoded successfully if

$$d_{\{2\}} \leq \min(\alpha_{22} - \lambda - \lambda' - \gamma', \alpha_{22} - \alpha_{21} - \gamma', \alpha_{22} - \alpha_{23} - \lambda' - \gamma') \quad (359)$$

which is true from (113), (120), (124) and (125). Therefore,  $X_{\{2\}}$  is successfully decoded at Receiver 2.

3.  $X_{\{3\}}, X_{\{1,2,3\}}$  are decoded with successive interference cancellation at the third receiver treating  $X_{\{1\}}, X_{\{2\}}$  and  $X_{\{1,2\}}$  as noise.

- (a) The SINR for decoding  $X_{\{1,2,3\}}$  at the third receiver treating the other signals as noise is equal to

$$\frac{P^{\alpha_{33}}(1 - 2P^{-\lambda})|G_{33}|^2}{1 + P^{\alpha_{31}}P^{-\lambda}|G_{31}|^2 + P^{\alpha_{32}}P^{-\gamma'}P^{-\lambda}|G_{32}|^2 + P^{\alpha_{33}}P^{-\lambda}|G_{33}|^2} \approx P^\lambda \quad (360)$$

(360) follows from (10), i.e.,  $\max(\alpha_{im}, \alpha_{ki}) \leq \alpha_{ii}$  is true for all  $i, k, m \in [3]$ . Therefore,  $X_{\{1,2,3\}}$  which carries  $d_{\{1,2,3\}}$  GDoF is decoded successfully as from (116) we have  $d_{\{1,2,3\}} \leq \lambda$ .

- (b) Finally, the third receiver decodes  $X_{\{3\}}$  treating  $X_{\{1\}}, X_{\{2\}}$  as noise with SINR equal to,

$$\frac{P^{\alpha_{33}}P^{-\lambda}|G_{33}|^2}{1 + P^{\alpha_{31}}P^{-\lambda}|G_{31}|^2 + P^{\alpha_{32}}P^{-\gamma'}P^{-\lambda}|G_{32}|^2} \approx P^{\min(\alpha_{33}-\lambda, \alpha_{33}-\alpha_{31}, \alpha_{33}-\alpha_{32}+\gamma')} \quad (361)$$

Therefore,  $X_{\{3\}}$  which carries  $d_{\{3\}}$  GDoF is decoded successfully if

$$d_{\{3\}} \leq \min(\alpha_{33} - \lambda, \alpha_{33} - \alpha_{31}, \alpha_{33} - \alpha_{32} + \gamma') \quad (362)$$

which is true from (114), (126) and (127). Therefore,  $X_{\{3\}}$  is successfully decoded at Receiver 3.

## F Convex Polyhedron

**Lemma 2** Consider a compact convex polyhedron  $A$  and a convex set  $B$ . Define  $\mathcal{U}$  as the set of all vertices of  $A$ . Then, the following statement is true.

$A \subset B$  if and only if  $\mathcal{U} \subset B$ .

### F.1 Proof of Lemma 2

If  $A \subset B$ , then  $\mathcal{U} \subset B$  as  $\mathcal{U} \subset A$ . So, let us prove the converse part i.e.,  $A \subset B$  if  $\mathcal{U} \subset B$ . Note that,  $A$  is the convex hull of  $\mathcal{U}$  as it is a compact convex polyhedron. On the other hand, the convex hull of a given set  $\mathcal{U}$  is defined as the set of all convex combinations of points in  $\mathcal{U}$  (the union of all simplices with points in  $\mathcal{U}$ ). Consider  $m \in A$ . Let us prove that  $m \in B$ . As  $m \in A$ , there exist the coefficients  $0 \leq c_v$  where

$$m = \sum_{v \in \mathcal{U}} c_v v \quad \text{s.t.} \quad \sum_{v \in \mathcal{U}} c_v = 1 \quad (363)$$

As  $\mathcal{U} \subset B$ , we infer that  $\sum_{v \in \mathcal{U}} c_v v \in B$  as  $B$  is a convex set. Thus,  $m \in B$  is concluded.

## References

- [1] H. Maleki and S. A. Jafar, "Optimality of orthogonal access for one-dimensional convex cellular networks," *IEEE communications letters*, vol. 17, no. 9, pp. 1770–1773, 2013.
- [2] X. Yi, H. Sun, S. Jafar, and D. Gesbert, "Fractional coloring (orthogonal access) achieves all-unicast capacity (dof) region of index coding (tim) if and only if network topology is chordal," *under preparation*, 2015.

- [3] C. Geng, N. Naderializadeh, S. Avestimehr, and S. Jafar, “On the Optimality of Treating Interference as Noise,” *ArXiv:1305.4610*, 2013.
- [4] N. Naderializadeh and A. S. Avestimehr, “Itlinq: A new approach for spectrum sharing in device-to-device communication systems,” *Selected Areas in Communications, IEEE Journal on*, vol. 32, no. 6, pp. 1139–1151, 2014.
- [5] X. Yi and G. Caire, “Optimality of treating interference as noise: A combinatorial perspective,” *IEEE Trans. Inform. Theory*, vol. 62, no. 8, pp. 4654 – 4673, Aug 2016.
- [6] A. G. Davoodi and S. A. Jafar, “Transmitter Cooperation under Finite Precision CSIT: A GDoF Perspective,” *IEEE Transactions on Information Theory*, 2016.
- [7] T. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” *IEEE Trans. Inform. Theory*, vol. 27, no. 1, pp. 49–60, Jan 1981.
- [8] A. Avestimehr, S. Diggavi, and D. Tse, “Wireless network information flow: A deterministic approach,” *IEEE Trans. on Inf. Theory*, vol. 57, pp. 1872–1905, 2011.
- [9] R. Tandon, S. A. Jafar, S. Shamai, and H. V. Poor, “On the synergistic benefits of alternating CSIT for the MISO BC,” *IEEE Transactions on Information Theory*, Submitted August 2012, available online: <http://arxiv.org/abs/1208.5071>.
- [10] R. Borzoo, C. Hao, and B. Clerckx, “DoF analysis of the  $k$ -user MISO broadcast channel with hybrid CSIT,” *IEEE International Conference (ICC)*, June 2015.
- [11] C. Hao, B. Rassouli, and B. Clerckx, “Achievable DoF regions of MIMO networks with imperfect CSIT,” <http://arxiv.org/abs/1603.07513>, vol. abs/1603.07513, 2016.
- [12] A. G. Davoodi and S. A. Jafar, “Aligned image sets under channel uncertainty: Settling conjectures on the collapse of degrees of freedom under finite precision CSIT,” *IEEE Transactions on Information Theory*, vol. 62, no. 10, pp. 5603–5618, 2016.
- [13] A. G. Davoodi, B. Yuan, and S. A. Jafar, “GDoF of the MISO BC: Bridging the gap between finite precision and perfect CSIT,” *arXiv preprint arXiv:1602.02203*, 2016.
- [14] A. G. Davoodi and S. A. Jafar, “Aligned image sets and the generalized degrees of freedom of symmetric MIMO interference channel with partial CSIT,” *arXiv preprint arXiv:1705.00769*, 2017.
- [15] —, “Sum-set inequalities from aligned image sets: Instruments for robust GDoF bounds,” *arXiv preprint arXiv:1703.01168*, 2017.
- [16] —, “Network coherence time matters – aligned image sets and the degrees of freedom of interference networks with finite precision CSIT and perfect CSIR,” *arXiv preprint arXiv:1705.02775*, May 2017.
- [17] —, “Generalized Degrees of Freedom of the Symmetric  $K$ -User Interference Channel under Finite Precision CSIT,” *IEEE Transactions on Information Theory*, vol. 63, no. 10, pp. 6561–6572, 2017.

- [18] —, “ $K$ -user symmetric  $M \times N$  MIMO interference channel under finite precision CSIT: A GDoF perspective,” *arXiv preprint arXiv:1711.00044*, Nov 2017.
- [19] A. Schrijver, *Combinatorial optimization: polyhedra and efficiency*. Springer, 2003, vol. 24.