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Hechler forcing and its relatives

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by

Justin Thomas Palumbo

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Abstract of the Dissertation

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by

Justin Thomas Palumbo

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor Itay Neeman, Chair

This thesis is divided into two main parts. In the first part, we focus on analyzing the properties of the single step extension of V via Hechler forcing $\mathbb{D}$, as well as two closely related forcing notions which we refer to as the tree Hechler forcing $\mathbb{D}_{\text{tree}}$ and the non-decreasing Hechler forcing $\mathbb{D}_{\text{nd}}$.

In Section 2 we prove (jointly with Itay Neeman) that $\mathbb{D}$ and $\mathbb{D}_{\text{nd}}$ are actually equivalent as forcing notions. This resolves an open question dating back at least as far as 1992 ([BJS92]). We also prove that $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$ are distinct forcing notions, one consequence of which is the disproof of a conjecture of Brendle and Löwe (Conjecture 15 of [BL11]) stating that every subextension of a Hechler extension is either a Cohen extension or a Hechler extension.

We distinguish $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$ by proving two theorems about the asymptotic relationship between dominating reals and unbounded reals in the two extensions. In Section 3 we prove that in the extension by $\mathbb{D}_{\text{tree}}$ for every unbounded real x there is some real which is dominating over V but does not dominate x. In Section 4 we prove that this is not the case for $\mathbb{D}$, and construct an unbounded real in the Hechler extension which is dominated by every dominating real. To prove this last result we prove a representation theorem about the dominating reals in the

Hechler extension given by a Hechler real d: we show that for any dominating real $y \in V[d]$ there are monotonically non-decreasing $z_0, z_1 \in V \cap \omega^\omega$ which go to infinity so that y dominates $z_0 \circ d \circ z_1$. Since any real of the form $z_0 \circ d \circ z_1$ is itself dominating, this shows that in an appropriate sense d asymptotically generates all of the dominating reals in $V[d]$. This theorem allows us to answer a question of Laflamme [Laf94] about the asymptotic structure of bounded subsets of ω^ω.

In Section 5 we will prove the existence of a forcing extension which adds a dominating real, but does not contain any dominating real asymptotically generating the others. (To allow for the case where the ground model satisfies CH the argument we give — joint with Itay Neeman — will use large cardinals. We conjecture this shouldn't be necessary.)

In Section 6 under large cardinal hypotheses we confirm a conjecture of Brendle and Löwe (Conjecture 14 of [BL11]), which states that any subextension of the Hechler extension adds a dominating real or is equivalent to a Cohen extension.

In Section 7 we prove that the product of any two forcing notions which add a dominating real adds a Hechler real. This strengthens an unpublished result of Goldstern's stating that the product of any two forcings which add a dominating real adds a Cohen real.

In Section 8 we prove that every nontrivial subextension of the Hechler extension contains a real which is Cohen over V. Under large cardinal hypotheses there is a proof of this fact (which we sketch) that falls very quickly out of work of Shelah [GS93], and is likely folklore. Our main argument goes through in ZFC.

In Section 9 we construct an example of a subiteration of a well-founded non-linear iteration of Hechler forcing which does not completely embed into the full iteration. We discuss a problem involving cardinal characteristics of the continuum which motivates this example.

In the second part of the thesis, we study the interaction of polychromatic Ramsey theory and monochromatic Ramsey theory in a variety of settings. The main theme of our work in this part is that the monochromatic theory is strictly stronger than the polychromatic theory. We begin in Section 11 by looking at the rainbow Ramsey theorem as a choice principle over the theory ZF. We prove that some amount of choice is necessary to prove the rainbow Ramsey theorem. We also prove that the rainbow Ramsey theorem is not sufficiently strong as a choice principle to imply Ramsey's theorem, and we do this by showing that the rainbow Ramsey theorem holds in Cohen's standard model for the failure of AC. This entails proving that in the Cohen model every non-well-orderable set has an infinite subset bijecting with a subset of the adjoined Cohen reals. This fact is an analogue of a corresponding property of the Fraenkel model, due to Blass [Bla77] and may be of independent interest.

In Section 12 we prove a result (joint with Anush Tserunyan) which shows that rainbow Ramsey flavored infinite exponent partition relations conflict with the axiom of choice. This may be viewed as a generalization of the classical result of Erdös and Rado which says that under the axiom of choice Ramsey's theorem fails for infinite exponent partitions (Proposition 7.1 of [Kan03].)

In Section 13 we investigate the (countable) combinatorial power of the rainbow Ramsey theorem; to accomplish this we introduce rainbow Ramsey ultrafilters, a polychromatic analogue of the classical Ramsey ultrafilters. Every Ramsey ultrafilter is a rainbow Ramsey ultrafilter, yet consistently there are rainbow Ramsey ultrafilters which are not Ramsey. Thus in the context of ultrafilters the polychromatic theory is weaker. We will investigate the relationship between rainbow Ramsey ultrafilters and other well-known types of special ultrafilters which encapsulate various combinatorial principles on ω. Ramsey's theorem is sufficiently strong as a combinatorial principle so that any Ramsey ultrafilter falls into one of these special types; this is not so for the rainbow Ramsey theorem.

Constructing ultrafilters which are rainbow Ramsey but fail to have some other property requires building polychromatic sets with various special properties, properties for which one cannot generally find monochromatic sets. For example, we will (assuming MA) construct a rainbow Ramsey ultrafilter which is not rapid. To do this we must be able to build polychromatic sets whose enumerating functions do not grow too fast; the monochromatic theory is strong enough to enforce fast growth while the polychromatic theory is not. In a similar vein we will construct a rainbow Ramsey ultrafilter which is not discrete. This requires building polychromatic subsets of $\mathbb{Q}$ which have high Cantor-Bendixson rank, something not in general possible in the monochromatic theory. However we will prove that every rainbow Ramsey ultrafilter is nowhere dense. One aspect of the proof involves showing that there are 2-bounded colorings for which polychromatic sets are necessarily nowhere dense. We also show that there may exist weakly selective ultrafilters which are not rainbow Ramsey.

We close the section on rainbow Ramsey ultrafilters by showing that, unlike any other class of special ultrafilters considered in the literature (that we are aware of), the class of rainbow Ramsey ultrafilters is not closed downward under the Rudin-Blass order.

Finally, in Section 14 we give several cardinal characteristics of the continuum new characterizations in the spirit of polychromatic Ramsey theory.

The dissertation of Justin Thomas Palumbo is approved.

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To Taktin

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Publications

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Part I

Hechler forcing

1 Introduction

In this part of the thesis we study Hechler forcing and its variants. Hechler forcing is the most basic method for adding a dominating real to the universe, a technique which is now ubiquitous in the study of the set theory of the reals. Three variations of Hechler forcing have been considered in the literature. Notationally little distinction has been made between them; all three have been commonly referred to as Hechler forcing and designated by the symbol $\mathbb{D}$. In this paper we refer to them in words as the original Hechler forcing, the non-decreasing Hechler forcing and the tree Hechler forcing and symbolically we use $\mathbb{D}$, $\mathbb{D}_{\text{nd}}$ and $\mathbb{D}_{\text{tree}}$, respectively.

Brendle, Judah and Shelah [BJS92] used a rank analysis of $\mathbb{D}_{\text{nd}}$ originally due to Baumgartner and Dordal [BD85] to analyze the combinatorial consequences of forcing with $\mathbb{D}_{\text{nd}}$. They showed that in $V^{\mathbb{D}_{\text{nd}}}$ there is a MAD family of size ω_1 and a Luzin set of size 2^ω. The existence of the latter implies that $\text{non}(\mathcal{M}) = \omega_1$ and $\text{cov}(\mathcal{M}) = 2^\omega$ and thus completely determines Cichoń's diagram of cardinal characteristics. They also showed how one can modify the rank analysis of $\mathbb{D}_{\text{nd}}$ to analyze $\mathbb{D}$ and prove that such objects exist in $V^{\mathbb{D}}$ as well. There is also a rank analysis for $\mathbb{D}_{\text{tree}}$ (see the definitions just before Theorem 12 in [BL11].) The rank analysis for $\mathbb{D}_{\text{tree}}$ is simpler than for either $\mathbb{D}$ or $\mathbb{D}_{\text{nd}}$ and it is not hard to see that the same Brendle, Judah and Shelah arguments go through for $V^{\mathbb{D}_{\text{tree}}}$ as well.

Not only do all three forcings have the same effect on the standard cardinal characteristics of the continuum, but as we will see it is not difficult to prove that forcing with any one of them produces a generic for each of the others. It is only natural then to ask if all three forcings are in fact the same. In this paper we will show that while $\mathbb{D}$ and $\mathbb{D}_{\text{nd}}$ are equivalent from the forcing point of view (and thus we may safely use the term Hechler forcing for both), $\mathbb{D}_{\text{tree}}$ is different.

To accomplish this we compare the unbounded and dominating reals in $V^{\mathbb{D}}$ and $V^{\mathbb{D}_{\text{tree}}}$ and prove two results distinguishing the asymptotics of dominating and unbounded reals in the two extensions.

Theorem 1.1. Let d a be $\mathbb{D}_{\text{tree}}$-generic real over V. Then for any unbounded real $x \in \omega^\omega \cap V[d]$ there is some dominating real $y \in \omega^\omega \cap V[d]$ so that x is not eventually dominated by y.

Theorem 1.2. Let d be a $\mathbb{D}$-generic real over V. Then there is an unbounded real $x \in \omega^\omega \cap V[d]$ so that for every dominating real $y \in \omega^\omega \cap V[d]$ we have that x is eventually dominated by y.

Thus the two forcings are not equivalent. We will derive Theorem 1.2 from the following theorem, which we consider the main result of this part of the thesis. Let $\omega^{\nearrow\omega}$ denote the set of all monotonically nondecreasing members of ω^ω which limit to infinity. Note that whenever y is a dominating real and $z \in V \cap \omega^{\nearrow\omega}$ then $z \circ y$ and $y \circ z$ are also dominating. The next result shows that when adding a Hechler real this is in some sense the only way to get dominating reals.

Theorem 1.3. Let d be a $\mathbb{D}_{\text{nd}}$-generic real over V and let $y \in \omega^\omega \cap V[d]$ be dominating. Then there are $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ so that y eventually dominates $z_0 \circ d \circ z_1$.

In the later sections in this part of the thesis we will examine issues related to the notion of subforcing. We will prove that under large cardinals any non-trivial non-Cohen subextension of a Hechler extension adds a dominating real. We will also show that the product of any two forcing notions which add a dominating real must add a Hechler real and that any non-trivial subextension of a Hechler extension contains a Cohen real.

Our notation and terminology are mostly standard. We use ω^ω to refer to the set of all functions on the natural numbers, and often we will call elements of ω^ω reals. We use $\leq^*$ to refer to the preorder of eventual domination on ω^ω. This means that we have

$$x \leq^* y \Leftrightarrow (\forall^\infty n) x(n) \leq y(n).$$

A dominating real in a generic extension is a real $y \in \omega^\omega$ for which for all $f \in V \cap \omega^\omega$ we have $f \leq^* y$. An unbounded real in a generic extension is a real $x \in \omega^\omega$ for which for all $f \in V \cap \omega^\omega$ we have $x \not\leq^* f$.

We use $\mathbb{C}$ to denote Cohen forcing, whose conditions we will take to come from either $2^{<\omega}$ or $\omega^{<\omega}$ as the situation demands. When $\mathbb{P}$ is a ccc notion of forcing we abuse notation somewhat and let $V^{\mathbb{P}} \cap \omega^\omega$ denote the collection of nice names for reals. For two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ we will use $\mathbb{P} \equiv \mathbb{Q}$ to denote forcing equivalence which means 1) for any G a $\mathbb{P}$-generic filter over V there is some $H \in V[G]$ which is a $\mathbb{Q}$-generic filter over V and for which $V[G] = V[H]$ and 2) vice versa: for any H a $\mathbb{Q}$-generic filter over V there is some $G \in V[H]$ which is a $\mathbb{P}$-generic filter over V and for which $V[G] = V[H]$.

Our main reference both for forcing and for set theory in general is Kunen's text [Kun83]. For forcing as it relates to the set theory of the reals in particular we refer the reader to the monograph [BJ95].

2 Notions of Hechler forcing

In this section we introduce Hechler forcing and the variations which we will be studying in the rest of this part of the thesis. We will also prove that $\mathbb{D}$ and $\mathbb{D}_{\text{nd}}$ are forcing equivalent. All three forcing notions are σ-centered partial orderings adding a dominating real, and each consists of two parts: a stem giving a finite approximation of the real being added, and a commitment restricting the possible values the real may take beyond the stem.

We start with the definition of Hechler forcing.

Definition 2.1. Hechler forcing is the forcing notion $\mathbb{D}$ defined as follows. Conditions p in $\mathbb{D}$ are pairs $p = \langle s, f \rangle$ where $s \in \omega^{<\omega}$ and $f \in \omega^\omega$. We call s the stem of p and write $s = \mathrm{stem}(p)$. We refer to f as the commitment of p. The ordering on $\mathbb{D}$ is given by

$$\langle s', f' \rangle \leq \langle s, f \rangle \Leftrightarrow s \subseteq s', (\forall n) f(n) \leq f'(n) \text{ and } (\forall n \in |s'| \setminus |s|) f(n) \leq s'(n).$$

Notice that $\mathbb{D}$ is ccc and furthermore that it is σ-centered. By partitioning the elements of $\mathbb{D}$ based on their stems we arrive at a decomposition $\mathbb{D} = \bigcup_{n<\omega} \mathbb{P}_n$ so that any finitely many conditions in $\mathbb{P}_n$ are mutually compatible.

Suppose that $G \subseteq \mathbb{D}$ is a generic filter. Defining $d = \bigcup \{s : \langle s, f \rangle \in G\}$ yields a dominating real. (The term 'dominate' is the reason for the symbol $\mathbb{D}$ used to denote Hechler forcing. In some of the older literature 'Hechler forcing' is instead referred to as 'dominating forcing'.) Given such a d defined from G, we may in turn define G from d as the collection of all $\langle s, f \rangle \in \mathbb{D}$ such that $s \subseteq d$ and $(\forall n) f(n) \leq d(n)$. Any real from which a $\mathbb{D}$-generic filter can be so defined is referred to as a $\mathbb{D}$-generic real, or a Hechler real. Similar terminology will apply to the variants of Hechler forcing that we define later on.

Hechler forcing was introduced by Hechler in [Hec74] to analyze the structure of cofinal subsets of $(\omega^\omega, \leq^*)$. In that paper Hechler used nonlinear iterations of $\mathbb{D}$ to prove that for any σ-directed partially ordered set P there is a generic extension in which P is isomorphic to a cofinal subset of $(\omega^\omega, \leq^*)$. These iterations will be described in Section 9.

The following theorem of Truss shows that $\mathbb{D}$ is fundamental amongst those forcing notions which add a dominating real: adding a Cohen real over a model that contains a dominating real will always produce a Hechler real over the ground model.

Theorem 2.2 (Truss, [Tru77])**.** Suppose that d is a dominating real over V, and $c \in \omega^\omega$ is a real which is $\mathbb{C}$-generic over $V[d]$. Then $d + c$ is $\mathbb{D}$-generic over V.

The first variant of Hechler forcing we consider is the following.

Definition 2.3 (Baumgartner and Dordal, [BD85])**.** Non-decreasing Hechler forcing $\mathbb{D}_{\mathrm{nd}}$ is the subposet of $\mathbb{D}$ consisting of conditions p whose stem is monotonically nondecreasing as a sequence of natural numbers.

The non-decreasing Hechler forcing was first used by Baumgartner and Dordal in [BD85] where among other things they showed that by iterating $\mathbb{D}_{\mathrm{nd}}$ over a model of CH one obtains a model where the splitting number $\mathfrak{s}$ is strictly less than the bounding number $\mathfrak{b}$. By work of Judah and Shelah [IS88] the same holds for the iteration of any Suslin ccc forcing notion adding a dominating real. We will discuss this in Section 9.

Though the difference in the definitions of $\mathbb{D}$ and $\mathbb{D}_{\mathrm{nd}}$ is slight and one often appears in arguments where the other would serve just as well, the two have occasionally been treated as separate entities, as in [BJS92]. Intuitively there should be little difference but whether the two are actually equivalent was an open question. See for example the discussion after definition 3.1.9 in [BJ95].

The two forcing extensions are in fact the same. This theorem is joint with Itay Neeman.

Theorem 2.4. $\mathbb{D} \equiv \mathbb{D}_{\mathrm{nd}}$.

Proof. The proof comes in two steps. First we will prove that $\mathbb{D}_{\rm nd} * \mathbb{C} \equiv \mathbb{D}$, and then we will prove that $\mathbb{D}_{\rm nd} * \mathbb{C} \equiv \mathbb{D}_{\rm nd}$.

Suppose d is a $\mathbb{D}$-generic real over V. Define the real $d_{\rm nd}$ by

$$d_{\rm nd}(n) = \min\{d(k) : k \geq n\}.$$

Then $d_{\rm nd}$ is a $\mathbb{D}_{\rm nd}$-generic real over V. Let $d' = d - d_{\rm nd}$. Now while d' is a Cohen real over V it is not quite true that it is Cohen over $V[d_{\rm nd}]$. This is because whenever $d_{\rm nd}(n) \neq d_{\rm nd}(n+1)$ we have $d(n) = d_{\rm nd}(n)$. But this is the only barrier. Let A be the set $\{n : d_{\rm nd}(n) = d_{\rm nd}(n+1)\}$. Then $d' \restriction A$ is a Cohen real over $V[d]$ (where for Cohen forcing we use the forcing consisting of sequences of natural numbers with domain a finite subset of A.) Furthermore $V[d] = V[d_{\rm nd}][d' \restriction A]$.

Going the other way, suppose d_0 is a $\mathbb{D}_{\rm nd}$-generic real over V. Let A be the set $\{n : d_0(n) = d_0(n+1)\}$ and suppose c is generic over $V[d_0]$ for the forcing consisting of sequences of natural numbers with domain a finite subset of A. Letting c_0 agree with c on A and take the value 0 outside of A, we have that $d = d_0 + c_0$ is a $\mathbb{D}$-generic real. Since $d_0 = d_{\rm nd}$ we have $V[d] = V[d_0][c]$. Thus $\mathbb{D}_{\rm nd} * \mathbb{C} \equiv \mathbb{D}$.

It remains to show $\mathbb{D}_{\rm nd} * \mathbb{C} \equiv \mathbb{D}_{\rm nd}$. Towards that end suppose that d is a $\mathbb{D}_{\rm nd}$-generic real over V. Let $\{r_k : k \in \omega\} \subseteq \omega$ enumerate the range of d in increasing order. Let $I_k(d)$ be the interval on which d takes value r_k. Let $c \in 2^\omega$ be defined so that $c(k)$ is equal to the parity of the length of the interval $I_k(d)$. We define d_0 to be the nondecreasing real with the same range as d but for which $I_k(d_0)$ has half the length (rounded up) of $I_k(d)$. Then it is straightforward to check that d_0 is a $\mathbb{D}_{\rm nd}$-generic real and that c is a Cohen real over $V[d_0]$. Also $V[d] = V[d_0][c]$.

This process is reversible. Given a $\mathbb{D}_{\rm nd}$-generic real d_0 and a Cohen real $c \in 2^\omega$

let d be the nondecreasing real with the same range as d_0 and for which the length of $I_k(d)$ is equal to twice the length of $I_k(d_0)$ minus $c(k)$. Then d is $\mathbb{D}$-generic over V and $V[d] = V[d_0][c]$. This completes the proof. $\square$

The next variant of Hechler forcing is the tree Hechler forcing. The tree Hechler forcing $\mathbb{D}_{\text{tree}}$ is a special case of the forcings made up of trees branching into a filter that were considered by Groszek [Gro87]. The forcing $\mathbb{D}_{\text{tree}}$ has only been isolated relatively recently in the literature. For example it was used by Brendle and Löwe [BL11] to obtain by iteration a model where $\Delta^1_2(\mathbb{D})$ holds and $\Delta^1_2(\mathbb{E})$ fails.

Definition 2.5. The tree Hechler forcing $\mathbb{D}_{\text{tree}}$ is the poset whose conditions are trees $T \subseteq \omega^{<\omega}$ with a distinguished stem $s = \text{stem}(T)$ so that for all t in T either s extends t or t extends s and so that whenever t in T extends s we have $(\forall^\infty n) t \frown n \in T$. The forcing is ordered by inclusion: $T' \leq T$ exactly when $T' \subseteq T$.

Note that the set of conditions $T \in \mathbb{D}_{\text{tree}}$ in which each $s \in T$ is eventually strictly increasing is a dense subforcing of $\mathbb{D}_{\text{tree}}$. The collection of $T \in \mathbb{D}_{\text{tree}}$ with all $s \in T$ strictly increasing is forcing equivalent to $\mathbb{D}_{\text{tree}}$ and we may identify the two forcing notions as necessary.

Now we compare the forcings $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$. The next proposition shows that each is a subforcing of the other.

Proposition 2.6. Forcing with $\mathbb{D}_{\text{tree}}$ adds a $\mathbb{D}$-generic real, and forcing with $\mathbb{D}$ adds a $\mathbb{D}_{\text{tree}}$-generic real.

Proof. That forcing with $\mathbb{D}$ adds a $\mathbb{D}_{\text{tree}}$-generic real was observed by Brendle and Löwe in [BL11]. Given d a $\mathbb{D}$-generic real let $N \in \omega$ be such that $N \leq n$ implies

$n < d(n)$. Define d' by setting $d'(n) = d(n)$ for $n \leq N$ and recursively setting $d'(n+1) = d(d'(n))$ for $N \leq n$. Then d' is a $\mathbb{D}_{\text{tree}}$-generic real over V.

For the other direction let d be a $\mathbb{D}_{\text{tree}}$-generic real over V. Take d' to be defined by letting $d'(n)$ take the value of half that of $d(n)$, rounded down. It is not difficult to check that d' is also a tree Hechler real over V. Now define $c \in 2^\omega$ by setting $c(n)$ equal to the parity of $d(n)$. Then c is Cohen over $V[d']$. By Truss's theorem $d' + c$ is $\mathbb{D}$-generic over V. $\square$

We will show that $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$ are not forcing equivalent, despite the fact that each of the two forcings adds a generic real for the other. Thus these forcings witness the failure of a natural Cantor-Bernstein theorem for forcing notions. Following our result another such example was produced by Joel David Hamkins, based on a conversation with Arthur Apter. If one takes $\mathbb{P}$ to be the forcing to add a Cohen subset of ω_2 and $\mathbb{S}$ to be the forcing to add a stationary nonreflecting subset of ω_2, then together $\mathbb{P}$ and $\mathbb{P} * \mathbb{S}$ give such an example. The reader may find more details at [Ham].

We now give some notation and terminology for stems consistent with that introduced in [BL11]. We will be using the same terminology for $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$; which forcing notion we mean will be clear from context.

First we consider $\mathbb{D}$ (and $\mathbb{D}_{\text{nd}}$.) For a condition $p = \langle s, f \rangle$ and $t \in \omega^{<\omega}$ we write $t \leq p$ to mean

$$s \subseteq t \text{ and } (\forall n \in |t| \setminus |s|) t(n) \geq f(n).$$

We say that $s \in \omega^{<\omega}$ forces a formula φ if there exists some commitment f for which $\langle s, f \rangle \Vdash \phi$. Let $A \subseteq \omega^{<\omega}$. We will say that s favors A if for every choice of commitment f there is some $t \in A$ so that $t \leq \langle s, f \rangle$. We say that s favors φ if s favors the set $\{t \in \omega^{<\omega} : t \text{ forces } \varphi\}$. Notice that s favors φ exactly when s does not force $\neg\varphi$.

Our terminology for $\mathbb{D}_{\text{tree}}$ is similar. We write $t \leq T$ to mean $\text{stem}(T) \subseteq t$ and $t \in T$. We say s forces φ when there is $T \in \mathbb{D}_{\text{tree}}$ with $\text{stem}(T) = s$ and $T \Vdash \varphi$. We say that s favors A if for every $T \in \mathbb{D}$ with $\text{stem}(T) = s$ there is $t \leq T$ with $t \in A$. When $T \in \mathbb{D}_{\text{tree}}$ and $\text{stem}(T) \subseteq t$, write T_t for the tree with $\text{stem}(T_t) = t$ containing exactly the initial segments of t and the extensions of t in T.

Since any two conditions with the same stem are compatible any condition with stem forcing φ may be strengthened to a condition forcing φ.

3 Unbounded and dominating reals in the tree Hechler extension

Our goal in this section is to prove Theorem 1.1. The following easy proposition characterizing the unbounded reals in a generic extension gives the motivation for our method. We leave the proof to the reader.

Proposition 3.1. Let $\mathbb{P}$ be an arbitrary notion of forcing, and let $\dot{x} \in V^{\mathbb{P}} \cap \omega^\omega$. Then

$$\Vdash_{\mathbb{P}} \text{“}\dot{x} \text{ is unbounded”} \iff (\forall p \in \mathbb{P})(\exists^\infty n)(\forall i) p \nVdash \dot{x}(n) \leq i.$$

In order to prove Theorem 1.1 we give a strengthening of Proposition 3.1 for the case where $\mathbb{P} = \mathbb{D}_{\text{tree}}$. We give a characterization of the unbounded reals in the tree Hechler extension expressed using stems rather than outright conditions.

Lemma 3.2. Fix $\dot{x} \in V^{\mathbb{D}_{\text{tree}}}$. Set $A = \{t \in \omega^{<\omega} | (\exists n \geq |t|)(\forall i) t \text{ favors } i < \dot{x}(n)\}$. Then

$$\Vdash_{\mathbb{D}_{\text{tree}}} \text{“}\dot{x} \text{ is unbounded”} \iff \text{every } s \in \omega^{<\omega} \text{ favors } A.$$

Proof. First we go from right to left. Let z be a real in the ground model. Suppose for contradiction that there is some $T \Vdash_{\mathbb{D}_{\text{tree}}} (\forall n \geq N)\dot{x}(n) \leq z(n)$. By strengthening T as necessary we may assume that $s = \text{stem}(T)$ has length greater than N. Since s favors A by further strengthening T if necessary we may assume that s belongs to A. But now there is some $n \geq |s| \geq N$ so that $(\forall i)$ s favors $i < \dot{x}(n)$. Take $i = z(n)$. We may extend T to T' with $\text{stem}(T')$ forcing $z(n) < \dot{x}(n)$. That is a contradiction.

The left to right implication is more involved. We argue by contrapositive. Suppose there is some s which does not favor A. Then we can find a tree T with $\text{stem}(T) = s$ for which $t \leq T$ implies $t \notin A$. To simplify notation we will assume that $\text{stem}(T) = \varnothing$ and $T = \omega^{<\omega}$; the simplification does little to change the argument.

Now by assumption, every $s \in \omega^{<\omega}$ fails to belong to A. That means there is a function $v : \omega^{<\omega} \times \omega \to \omega$, and for every s and n with $n \geq |s|$ some tree $T^{s,n}$ with $\text{stem}(T^{s,n}) = s$ such that

$$T^{s,n} \Vdash \dot{x}(n) \leq v(s,n).$$

Claim 1. There exists $U \in \mathbb{D}_{\text{tree}}$ with $\text{stem}(U) = \varnothing$ such that

$$(\forall s \leq U)(\forall n \geq |s|)(\forall^\infty m)U_{s\frown m} \subseteq T^{s,n}.$$

Proof of Claim 1. We use a fusion argument. We define a sequence of trees $\{U^l | l \in \omega\}$ such that

1. $\text{stem}(U^l) = \varnothing$, $U^{l+1} \subseteq U^l$
2. $l < j$, $s \in U^l$ with $|s| \leq l$ implies $s \in U^j$
3. for all $s \in U^{l+1}$ with $|s| = l$ we have $(\forall n)(\forall^\infty m)U^{l+1}_{s\frown m} \subseteq T^{s,n}$.

Then we can take $U = \cap_{l<\omega} U^l$. Start with $U^0 = \omega^{<\omega}$. Supposing U^l is defined, let $s \in U^l$ with $|s| = l$. For each $n \geq |s|$ there is some $i(n)$ so that $m \geq i(n)$ implies $s \frown m \in T^{s,n}$. Then define $U^{l+1}_{s\frown m}$ to be the intersection of $U^l_{s\frown m}$ with each $T^{s,n}$ for $n, i(n) \leq m$. Then $m \geq i(n), n$ will imply $U^{l+1}_{s\frown m} \subseteq T^{s,n}$. □

Now fix U as in the claim and let $c : \omega^{<\omega} \times \omega \to \omega$ be such that

$$(\forall s \leq U)(\forall n \geq |s|)(\forall m \geq c(s,n)) U_{s\frown m} \subseteq T^{s,n}.$$

By further extending U we may assume that for every $s \in \omega^{<\omega}$, we have that $U_s \subseteq T^{s,n}$ whenever $n \leq \max(\mathrm{ran}(s))$.

Define $f \in \omega^\omega$ so that whenever $|s|, \max(\mathrm{ran}(s)) \leq n$ and $m < c(s,n)$ we have $v(s \frown m, n) \leq f(n)$. Let $g \in \omega^\omega$ be such that $v(s,n) \leq g(n)$ whenever $|s|, \max(\mathrm{ran}(s)) \leq n$. We claim $U \Vdash \dot{x} \leq^* \max(f,g)$.

We work now in an arbitrary generic extension $V[G]$ with $U \in G$. Let d be the corresponding tree Hechler real. Then G is exactly the set of members of $\mathbb{D}_{\mathrm{tree}}$ through which d is a branch and in particular $U_{d\restriction k} \in G$ for all k. Let x be the evaluation of $\dot{x}$ via G. Let $n \in \omega$ with $d(k) < n \leq d(k+1)$. For sufficiently large k we have $k \leq d(k)$ so by taking n sufficiently large we may assume that $k < n$. We show $x(n) \leq \max\{f(n), g(n)\}$.

Claim 2. $x(n) \leq v(d \restriction k+2, n)$.

Proof of Claim 2. This is because $T^{d\restriction k+2,n}$ belongs to G, which follows from our assumption that $U_s \subseteq T^{s,n}$ whenever $n \leq \max(\mathrm{ran}(s))$. □

Now we split into two cases. In the first case, if $T^{d\restriction k+1,n}$ belongs to G then $x(n) \leq v(d \restriction k+1, n) \leq g(n)$. In the second case, if $T^{d\restriction k+1,n} \notin G$ then we claim that $v(d \restriction k+2, n) \leq f(n)$ which by Claim 2 will give $x(n) \leq f(n)$. Since

$T^{d\restriction k+1,n} \notin G$ it follows that $U_{d\restriction k+2} \not\subseteq T^{d\restriction k+1,n}$ (because $U_{d\restriction k+2} \in G$.) It follows by definition of c that $d(k+1) < c(d \restriction k+1, n)$. Then the definition of f gives $v(d \restriction k+2, n) \leq f(n)$ as required. $\square$

Armed with Lemma 3.2 we can prove Theorem 1.1.

Proof of Theorem 1.1. Fix $\dot{x} \in V^{\mathbb{D}_{\text{tree}}} \cap \omega^\omega$ with $\Vdash$ "$\dot{x}$ is unbounded". Taking A as in Lemma 3.2 we know that every $s \in \omega^{<\omega}$ favors A. Let $\phi : A \to \omega$ satisfy $\phi(t) \geq |t|$ and

$$(\forall i) t \text{ favors } i < \dot{x}(\phi(t)).$$

We let d be a tree Hechler real over V and work in $V[d]$.

Define d' by

$$d'(k) = \begin{cases} d(n) & \text{where } n \text{ is least such that } k = \phi(d \restriction n), \text{ if such an } n \text{ exists} \\ d(k) & \text{if no such } n \text{ exists} \end{cases}$$

Claim 1. $\Vdash d'$ is dominating.

Proof of Claim 1. For any ground model real f and any $T \in \mathbb{D}$ we can extend to T' with $\text{stem}(T) = \text{stem}(T')$ such that $\text{stem}(T') \subseteq s$ and $s \frown m \in T'$ implies $m \geq f(\phi(s))$. $\square$

Now observe that if s favors φ then $(\exists^\infty m) s \frown m$ favors φ. This allows us to define $z \in V \cap \omega^{\nearrow\omega}$ such that

$$(\forall s \in A)(\exists^\infty m) s \frown m \text{ favors } \dot{x}(\phi(s)) \geq z(m).$$

Then, because z belongs to the ground model it follows that $z \circ d'$ is dominating. Thus the theorem will be proved given the following claim.

Claim 2. $\Vdash (\exists^\infty k) z(d'(k)) \leq \dot{x}(k)$.

Proof of Claim 2. Fix N and T. We want to find $k \geq N$ and $U \leq T$ such that $U \Vdash z(d'(k)) \leq \dot{x}(k)$. Let $s = \text{stem}(T)$. We may assume that $|s| \geq N$ and that $j \geq i \geq |s|$ implies $d(i) \leq d(j)$. Since s favors A we may also assume that $s \in A$. Now pick m such that $s \frown m \in T$ and $s \frown m$ favors $\dot{x}(\phi(s)) \geq z(m)$. Since $s \frown m \in T$ there is some $U \leq T$ such that $\text{stem}(U) = s \frown m$ and also

$$U \Vdash \dot{x}(\phi(s))) \geq z(m).$$

Now taking $l = |s|$ we have

$$U \Vdash \dot{x}(\phi(d \restriction l)) = \dot{x}(\phi(s)) \geq z(m) = z(d(l)) \geq z(d'(\phi(d \restriction l)).$$

And $\phi(d \restriction l) \geq l \geq N$. So $k = \phi(d \restriction l)$ satisfies the claim. (We've implicitly used the assumption that d is strictly increasing but this assumption does no harm, since as was earlier remarked the collection of $T \in \mathbb{D}_{\text{tree}}$ with all $s \in T$ strictly increasing is forcing equivalent to $\mathbb{D}_{\text{tree}}$.) $\square$

$\square$

4 Unbounded and dominating reals in the standard Hechler extension

4.1 Proof of Theorem 3

Our objective in this section is to prove Theorem 1.3. Let us note that although we have seen that $\mathbb{D}$ and $\mathbb{D}_{\text{nd}}$ are equivalent as forcing notions, nonetheless the direct analogue of Theorem 1.3 for $\mathbb{D}$ is not true. For example, suppose that d is a $\mathbb{D}$-generic real and let $d_0 \in V[d] \cap \omega^\omega$ satisfy

$$(\forall n) d_0(2n) = d_0(2n+1) = \min\{d(2n), d(2n+1)\}.$$

Then d_0 is a dominating real but for any $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ we have $z_0 \circ d \circ z_1 \not\leq^* d_0$.

Therefore we will exclusively be working with the poset $\mathbb{D}_{\text{nd}}$ and thus we will only be concerned with stems s which are nondecreasing. For the rest of this section when we refer to finite sequences of naturals we shall always mean nondecreasing ones, even when not explicitly stated. Let $\omega^{\nearrow<\omega}$ be the collection of such sequences, and let $\omega^{\nearrow m}$ be the collection of nondecreasing sequences of naturals of length m.

To motivate we start with the following simple proposition about dominating reals in $V^{\mathbb{D}_{\text{nd}}}$.

Proposition 4.1. Let $\dot{y} \in V^{\mathbb{D}_{\text{nd}}} \cap \omega^\omega$ and let A be the subset of $\omega^{<\omega}$ given by $A = \{t | (\forall^\infty n)(\forall i) t \text{ forces } i \leq \dot{y}(n)\}$. Then

$$\Vdash_{\mathbb{D}_{\text{nd}}} \text{“}\dot{y} \text{ is dominating ”} \implies \text{ every } s \text{ favors } A.$$

Proof. Argue by contrapositive; if some s does not favor A then we can find some $f \in \omega^\omega$ such that $t \leq \langle s, f\rangle$ implies $t \notin A$. For each such t we have $(\exists^\infty n)(\exists i)t$ favors $\dot{y}(n) < i$. This allows us to define a function $z \in \omega^\omega$ so that for each $t \notin A$ we have $(\exists^\infty n)t$ favors $\dot{y}(n) < z(n)$. Thus

$$\langle s, f\rangle \Vdash (\exists^\infty n)\dot{y}(n) < z(n).$$

We have $\not\Vdash_{\mathbb{D}_{\text{nd}}}$ “$\dot{y}$ is dominating ”, as desired. □

For the rest of this section we let $\dot{y} \in V^{\mathbb{D}_{\text{nd}}} \cap \omega^\omega$ and take A to be defined as in Proposition 4.1. Let $\phi : A \to \omega$ be defined so that that $\phi(s)$ equals the least N such that

$$(\forall n \geq N)(\forall i)s \text{ forces } i \leq \dot{y}(n).$$

We extend ϕ to a function $\phi : \omega^{\nearrow<\omega} \to \omega \cup \{\infty\}$ by letting $\phi(s) = \infty$ when no such N exists.

Our strategy for characterizing when $\dot{y}$ is a dominating real is to analyze the growth of the function ϕ. Supposing for example that $\dot{y}$ were of the form $z_0 \circ \dot{d} \circ z_1$ for some $z_0, z_1 \in \omega^{\nearrow\omega}$, it is not hard to see we would have that $\phi(s)$ is a function of the *length* of s. It turns out that this is essentially an exact characterization of the dominating reals.

Definition 4.2. Fix $q \in \mathbb{D}$. We say that ϕ is *length bounded below* q if there is some function $\psi \in \omega^\omega$ so that whenever $s \leq q$ we have $\phi(s) \leq \psi(|s|)$.

We are now ready to give several characterizations of the dominating reals in $V^{\mathbb{D}_{\mathrm{nd}}}$. Let $B \subseteq \omega^{\nearrow<\omega}$ be the collection

$$\{s | (\exists m)(\exists \{t_l : l \in \omega\} \subseteq \omega^{\nearrow m}) \lim_{l<\omega} t_l(0) = \infty \text{ and } \lim_{l<\omega} \phi(s \frown t_l) = \infty\}.$$

The definition of B is motivated in part by the Baumgartner-Dordal rank analysis of $\mathbb{D}_{\mathrm{nd}}$. For someone hoping that ϕ is everywhere length bounded B is a bad set and in order for $\dot{y}$ to be a dominating real we must mostly be able to avoid it.

Lemma 4.3. The following are equivalent:

1. $\Vdash$ "$\dot{y}$ is dominating"

2. $(\forall p)(\exists q \leq p)(\forall t \leq q) t \notin B$.

3. $(\forall p)(\exists q \leq p)$ ϕ is length bounded below q.

4. $(\forall p)(\exists q \leq p)(\exists z_0, z_1 \in \omega^{\nearrow\omega}) q \Vdash z_0 \circ \dot{d} \circ z_1 \leq^* \dot{y}$.

Notice that (1) implies (4) gives Theorem 1.3.

Proof. That (4) implies (1) is clear.

We show (1) implies (2). For each $s \in B$ fix a witnessing sequence $\{t^s_l : l \in \omega\}$. Then we may define a function $z \in \omega^\omega$ such that

$$(\forall s \in B)(\forall N)(\exists n, l > N) s \frown t^s_l \text{ favors } \dot{y}(n) < z(n).$$

Suppose now that (2) failed and there was some p so that $(\forall q \leq p)(\exists s \leq q) s \in B$. We claim that

$$p \Vdash (\exists^\infty n)\dot{y}(n) < z(n).$$

If not then there is some $q \leq p$ with $q \Vdash (\forall n \geq N_0) z(n) \leq \dot{y}(n)$. Write $q = \langle t, f \rangle$. There is $s \in B$ with $s \leq q$. Since $s \in B$ we may take $l, n \in \omega$ so that $s \frown t^s_l$ favors $\dot{y}(n) < z(n)$ and l, n are large enough that $n \geq N_0$, $s \frown t^s_l \leq q$. Since $s \frown t^s_l$ favors $\dot{y}(n) < z(n)$ we may further extend q to force $\dot{y}(n) < z(n)$, a contradiction.

Next we show that (2) implies (3). Fix a condition $p \in \mathbb{D}_{\mathrm{nd}}$. Taking $q \leq p$ as given by (2), write $q = \langle s, f \rangle$. We will define an $r \leq q$ so that ϕ is length bounded below r. In particular we construct functions ψ, $f' \in \omega^\omega$ such that $s \frown t \leq \langle s, \max\{f, f'\} \rangle$ implies $\phi(s \frown t) \leq \psi(|s \frown t|)$. Start by setting $\psi(|s|)$ equal to $\phi(s)$. Before we proceed further let us note that when $t \notin B$ it follows that for every m there is some N, L so that if $t \in \omega^{\nearrow m}$ with $t(0) \geq L$ then $\phi(s \frown t) \leq N$.

Fix $m \in \omega$. We define $\psi(|s|+m+1)$, $f'(|s|+m)$. To do so we recursively define a finite set $S_m \subseteq \omega^{\nearrow \leq m}$, and we simultaneously define $L_t, N_t \in \omega$ for each $t \in S_m$. We will make sure that $t \in S_m$ implies $s \frown t \notin B$. Start by placing $\varnothing \in S_m$. Now suppose that $t \in S_m$. Since $s \frown t \notin B$ there is $L_t, N_t \in \omega$ such that whenever $u \in \omega^{\nearrow < \omega}$ with $|u| = m + 1 - |t|$ and $u(0) \geq L_t$, $\phi(s \frown t \frown u) \leq N_t$. If $|t| < m$ put $s \frown t \frown i \in S_m$ whenever $i < L_t$ and $s \frown t \frown i \notin B$. That completes our definition of S_m. Let $\psi(|s| + m + 1) = \max_{t \in S_m} N_t$ and $f'(|s| + m) = \max_{t \in S_m} L_t$.

Let us check that this works. Suppose $s \frown t \leq \langle s, \max\{f, f'\}\rangle$, and say $|t| = m + 1$. Notice that $t \upharpoonright 0 = \varnothing \in S_m$. Take k as large as possible with $t \upharpoonright k \in S_m$. First suppose $k < m$. Since $s \frown t \upharpoonright k + 1 \notin B$ by definition of S_m we must have that $t(k) \geq L_{t \upharpoonright k}$. From this we can infer the inequalities $\phi(s \frown t \upharpoonright k \frown t \upharpoonright [k+1, m]) \leq N_t \leq \psi(|s| + m + 1)$. Now suppose $k = m$. Since $t(m) \geq f'(|s| + m) \geq L_{t \upharpoonright k}$ we have

$$\phi(s \frown t \upharpoonright k \frown t(m)) \leq N_{t \upharpoonright k} \leq \psi(|s| + m + 1)$$

as needed.

Finally we show that (3) implies (4). Fix $p \in \mathbb{D}_{\mathrm{nd}}$ and let $q \leq p$ with ϕ length bounded below q. Let $\psi \in \omega^\omega$ witness the bound. We may assume without loss of generality that ψ is a strictly increasing function. Whenever $t \leq q$ and $n \geq \psi(|t|)$ we have for every $i \in \omega$ some commitment $f^t_{n,i}$ such that

$$\langle t, f^t_{n,i}\rangle \Vdash i \leq \dot{y}(n).$$

Now say $q = \langle s, f\rangle$. Our goal is to construct z_0, z_1 and h so that

$$(*)\ \langle s, \max\{f, h\}\rangle \Vdash (\forall^\infty n) z_0(\dot{d}(z_1(n))) \leq \dot{y}(n).$$

We let $z_1 \in \omega^\omega$ be defined by having $z_1(n) = l$ whenever $\psi(l) \leq n < \psi(l+1)$. To define h and z_0 we will make use of the following simple proposition whose proof we leave to the reader.

Proposition 4.4. Let $\mathcal{G}$ be a countable subset of ω^ω. Then there is a $z \in \omega^{\nearrow \omega}$ so that for all $g \in \mathcal{G}$ we have

$$(\forall^\infty m) g(z(m)) \leq m.$$

Fix $n, j \in \omega$ with $|s| \leq l$ where $l = z_1(n)$. We define a finite set $S_n(j) \subseteq \omega^{\nearrow \leq l}$ by recursion. We will guarantee that $t \in S_n(j)$ implies $t \leq q$. In particular

$f^t_{n,j}$ will be defined for $t \in S_n(j)$. Start by putting s in $S_n(j)$. Then, whenever $t \in S_n(j)$ place u in $S_n(j)$ if $u \leq q$, $t \subseteq u$, $|u| \leq l$ and $u(|u|-1) < f^t_{n,j}(|u|-1)$. Since we have restricted our attention to nondecreasing sequences there are only finitely many options for u. Now define $g_{n,k}$ by

$$g_{n,k}(j) = \max\{f^t_{n,j}(k) : t \in S_n(j)\}.$$

Let $\mathcal{G}$ be the collection

$$\{g_{n,k} : z_1(n) \leq k\}.$$

Apply Proposition 4.4 to $\mathcal{G}$ to obtain z_0. By the defining property of z_0 for each k the set

$$X_{n,k} = \{m : (\exists t \in S_n(z_0(m))m < f^t_{n,z_0(m)}(k)\}$$

is finite. Let $h \in \omega^\omega$ with $f^t_{n,z_0(m)}(k) \leq h(k)$ whenever $m \in X_k$, $z_1(n) \leq k$ and $t \in S_n(z_0(m))$. Then h satisfies

$$(\dagger)\ (\forall t \in S_n(z_0(m)))m < f^t_{n,z_0(m)}(k) \Rightarrow f^t_{n,z_0(m)}(k) \leq h(k)$$

whenever $z_1(n) \leq k$.

We complete the proof by checking that $(*)$ holds. Let d be a $\mathbb{D}_{\mathrm{nd}}$-generic real so that $\langle s, \max\{f,h\}\rangle$ belongs to the corresponding generic filter G. Fix $n \geq \psi(|s|)$ and let $l = z_1(n)$.

Claim 1. For $k \geq l, |s|$ and $t \in S_n(z_0(d(l)))$ we have $f^t_{n,z_0(d(l))}(k) \leq d(k)$.

Proof of Claim 1. We split into two cases. Suppose for the first case that $f^t_{n,z_0(d(l))}(k) \leq d(l)$ holds. Then we are done since $l \leq k$ and d is nondecreasing. In the second case $d(l) < f^t_{n,z_0(d(l))}(k)$. But then by $(\dagger)$ we have

$$f^t_{n,z_0(d(l))}(k) \leq h(k) \leq d(k).$$

We have $h(k) \leq d(k)$ since $|s| \leq k$ and $\langle s, h\rangle$ belongs to G. □

Now take $l_0 \leq l$ to be as large as possible so that $d \restriction l_0$ belongs to $S_n(z_0(d(l))$.

Claim 2. For $k \geq l_0$ we have $f^{d \restriction l_0}_{n, z_0(d(l))}(k) \leq d(k)$.

Proof of Claim 2. If not there is some violating $k \geq l_0$. By Claim 1 we know $k < l$. We have that q belongs to G and so $d \restriction k+1 \leq q$. We also have $d(k) < f^{d \restriction l_0}_{n, z_0(d(l))}(k)$. Thus by the definition of $S_n(z_0(d(l)))$ we find that $d \restriction k+1 \in S_n(z_0(d(l)))$ which is contrary to the maximality of l_0. □

By Claim 2 (and the fact that $z_1(n) = l$) we have

$$\langle d \restriction l_0, f^{d \restriction l_0}_{n, z_0(d(z_1(n)))} \rangle \in G.$$

Since this condition forces that $z_0(d(z_1(n))) \leq \dot{y}(n)$ we are done. □

4.2 Proof of Theorem 2

Using Theorem 1.3 we can now prove Theorem 1.2.

Proof. Let d be a $\mathbb{D}_{\mathrm{nd}}$-generic real. Our goal is to to produce an unbounded real x in $V[d]$ which is eventually dominated by every dominating real. Fix $n \in \omega$. Let k be least with $d(k) \geq n$. Then we set $x(n) = i$ where i is large as possible so that

$$(\forall j \in [k, k+i]) d(k) = d(j).$$

An easy density argument shows that x is indeed unbounded. To show that x is eventually dominated by every dominating real, it is enough by Theorem 1.3 to show that $x \leq^* z_0 \circ d \circ z_1$ for every $z_0, z_1 \in V \cap \omega^{\nearrow \omega}$.

Fix such z_0 and z_1 and let $f \in V \cap \omega^\omega$ satisfy

$$(1)\ (\forall n) n < f(z_0(n)) \text{ and } (2)\ (\forall n) n < f(z_1(n)).$$

We claim then that for any $s \in \omega^{\nearrow<\omega}$ we have

$$\langle s, f\rangle \Vdash (\forall^\infty n) x(n) < z_0(d(z_1(n)))$$

which will complete the proof.

Assume instead that $\langle s, f\rangle$ belongs to the generic filter G corresponding to d and yet $z_0(d(z_1(n))) \leq x(n)$ holds for infinitely many n. We know

$$(\forall n \geq |s|) f(n) \leq d(n).$$

We also know that $z_0(d(z_1(n)))$ is dominating and thus for sufficiently large n we have $z_1(n) \leq z_0(d(z_1(n)))$. Fix an n with $|s| \leq n$, $|s|, z_1(n) \leq z_0(d(z_1(n)))$ and $z_0(d(z_1(n))) \leq x(n)$. Let k be least with $n \leq d(k)$. By (2) $n \leq d(z_1(n))$ and thus $k \leq z_1(n)$. By assumption $x(n)$ is larger than or equal to $z_0(d(z_1(n)))$ and by the definition of x we have that d is fixed on the interval $[k, k + x(n)]$ and therefore

$$d(k) = d(z_1(n)) = d(z_0(d(z_1(n)))) = d(x(n)).$$

But applying (1) with $d(z_1(n))$ in place of n we also get

$$d(z_1(n)) < f(z_0(d(z_1(n)))) \leq d(z_0(d(z_1(n))))$$

which brings us to a contradiction. □

4.3 Consequences

In this subsection we mention some consequences of the other work from this section. Let d be a $\mathbb{D}_{\mathrm{nd}}$-generic real, and let $\mathcal{D}$ be the collection of dominating reals in $V[d]$.

Corollary 4.5. The structures $(V \cap \omega^\omega, \leq^*)$ and $(\mathcal{D}, {}^*\!\geq)$ are cofinally isomorphic.

Proof. From Theorem 1.3 we have that the set $\{z \circ d \circ z : z \in V \cap \omega^{\nearrow\omega}\}$ is cofinal in $(\mathcal{D}, {}^*\!\geq)$. In V there is a cofinal mapping $z \mapsto z'$ from ω^ω to $\omega^{\nearrow\omega}$ such that

$$z_0 \leq^* z_1 \Leftrightarrow z_0' \,{}^*\!\geq z_1'.$$

For $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ we also have

$$z_0 \leq^* z_1 \Leftrightarrow z_0 \circ d \circ z_0 \leq^* z_1 \circ d \circ z_1.$$

(The right to left direction uses the genericity of d). The corollary follows. $\square$

An interesting and immediate consequence of Corollary 4.5 is the following.

Corollary 4.6. Let $\{d_n : n \in \omega\} \in V[d]$ be a countable collection of dominating reals. Then there is a single dominating real d^* such that $d^* \leq^* d_n$ for every $n \in \omega$.

In the terminology of Laflamme [Laf94] Corollary 4.6 says that $V \cap \omega^\omega$ has uncountable upperbound. In the cited paper Laflamme makes the following definitions. Let $\mathcal{F} \subseteq \omega^\omega$ be a bounded family of functions. Then $\mathcal{F}^\downarrow \subseteq \omega^\omega$ is the set of functions dominating $\mathcal{F}$. (So if $\mathcal{F} = V \cap \omega^\omega$ then $\mathcal{F}^\downarrow = \mathcal{D}$.)

$$\mathfrak{b}(\mathcal{F}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{F} \text{ is unbounded in } \mathcal{F}\}$$

$$\mathfrak{d}(\mathcal{F}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{F} \text{ is dominating in } \mathcal{F}\}$$

$$\mathfrak{b}^\downarrow(\mathcal{F}) = \min\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{F}^\downarrow \text{ is unbounded in } (\mathcal{F}^\downarrow, {}^*\!\geq)\}$$

We see then that working in $V[d]$ we have that $\mathfrak{b}(V \cap \omega^\omega) = \mathfrak{b}^V$, $\mathfrak{d}(V \cap \omega^\omega) = \mathfrak{d}^V$ and $\mathfrak{b}^\downarrow(V \cap \omega^\omega) = \mathfrak{b}^V$. In section 4 of his paper Laflamme constructed several ZFC examples of bounded families of $\mathcal{F}$ to achieve various values of $\mathfrak{b}(\mathcal{F})$, $\mathfrak{d}(\mathcal{F})$ and $\mathfrak{b}^\downarrow(\mathcal{F})$. In each of his constructions one of these three parameters is countable. Since for any regular uncountable cardinals $\kappa \leq \lambda$ one may find a ground model

V with $\mathfrak{b}^V = \kappa$ and $\mathfrak{d}^V = \lambda$, our corollary gives for any such κ, λ the consistency of the existence of a bounded family $\mathcal{F}$ with $\mathfrak{b}(\mathcal{F}) = \kappa$, $\mathfrak{d}(\mathcal{F}) = \lambda$ and $\mathfrak{b}^{\downarrow}(\mathcal{F}) = \kappa$. Laflamme also specifically asked whether one could consistently obtain a family with $\mathfrak{b}^{\downarrow}(\mathcal{F}) = \mathfrak{b}$ and $\mathfrak{b}(\mathcal{F}) = \mathfrak{b}$. He showed that consistently there is no such family. Since $V[d]$ satisfies $\mathfrak{b} = \omega_1$ ([BJS92]), by starting with a model V with $\mathfrak{b}^V = \omega_1$ we find that in $V[d]$ such a family does exist.

Now we turn to some recent work of Brendle and Löwe. In their paper [BL11] the authors were concerned with building models containing many Hechler generic reals but no eventually different reals. One consequence of their work is the following dichotomy theorem for reals in $\mathbb{D}_{\text{tree}}$. The authors originally stated their result as holding for 'Hechler reals', a term the authors use as a catch-all. The proof they gave was for $\mathbb{D}_{\text{tree}}$.

Theorem 4.7 (Brendle and Löwe,[BL11])**.** Let d be a $\mathbb{D}_{\text{tree}}$-generic real and suppose $x \in V[d] \cap \omega^\omega$. Then either

1. x is dominating, or
2. x is not eventually different over V (that is, there is some $f \in V \cap \omega^\omega$ such that $(\exists^\infty n) f(n) = x(n)$).

Using characterization (2) from Lemma 4.3 we get the same dichotomy for reals in $\mathbb{D}$.

Corollary 4.8. Let d be a $\mathbb{D}$-generic real and let $x \in V[d] \cap \omega^\omega$. Then either

1. x is dominating, or
2. x is not eventually different over V.

Proof. By Theorem 2.4 we may work with d a $\mathbb{D}_{\mathrm{nd}}$-generic real instead. Let $\dot{x} \in V^{\mathbb{D}} \cap \omega^\omega$ and suppose

$$p \Vdash \text{“}\dot{x} \text{ is not dominating”}.$$

Using a version of Lemma 4.3 relativized to $\mathbb{D}_{\mathrm{nd}}$ restricted to conditions below p, there is $q \leq p$ such that

$$(\forall r \leq q)(\exists s \leq r) s \in B.$$

For each $s \in B$ let $\{t_l^s : l \in \omega\}$ be a witnessing sequence.

Now notice that

$$(\forall i) t \text{ forces } \dot{x}(n) \geq i \Leftrightarrow (\forall i) t \text{ forces } \dot{x}(n) \neq i.$$

Thus if $\phi(t) > N$ that means there exists some $n \geq N$ and i such that t favors $\dot{x}(n) = i$. So we may define a function $y \in \omega^\omega$ so that

$$(\forall s \in B)(\forall N)(\exists n, l > N) s \frown t_l^s \text{ favors } \dot{x}(n) = y(n).$$

Then

$$q \Vdash (\exists^\infty n) \dot{x}(n) = y(n).$$

□

Also in [BL11] the authors conjectured (Conjecture 15) that given a Hechler real d and a new real x in $V[d]$ either $V[x]$ is equivalent to a Hechler extension of V or $V[x]$ is equivalent to a Cohen extension of V. The authors there use the term 'Hechler real' as a catch-all, and so their conjecture has several interpretations. Our results show that whether one interprets the term 'Hechler real' in their conjecture to mean $\mathbb{D}_{\mathrm{tree}}$-generic real or interprets it to mean $\mathbb{D}$-generic real the conjecture is false. This is because (by Proposition 2.6) forcing with $\mathbb{D}$ and $\mathbb{D}_{\mathrm{tree}}$

each add reals generic for the other, but a $\mathbb{D}$-generic extension is not the same as a $\mathbb{D}_{\text{tree}}$-generic extension.

We do not know if the following trichotomous reinterpretation of their conjecture holds.

Conjecture 4.9. Let d be a real which is either $\mathbb{D}$-generic or $\mathbb{D}_{\text{tree}}$-generic, and let $x \in V[d] \cap \omega^\omega$ be a new real. Then exactly one of the following holds

1. $V[x]$ is equivalent to an extension of V by $\mathbb{D}$,
2. $V[x]$ is equivalent to an extension of V by $\mathbb{D}_{\text{tree}}$, or
3. $V[x]$ is a equivalent to an extension of V by $\mathbb{C}$.

5 Forcing extensions with no $\preceq$-least dominating real

Let $V[G]$ be some generic extension of the universe. Given $f, g \in V[G] \cap \omega^\omega$ we write $f \preceq g$ if there are $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ such that $z_0 \circ f \circ z_1 \leq^* g$. It is easy to see that $\preceq$ gives a preordering on ω^ω, and furthermore that $f \preceq g$ is equivalent to the existence of $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ such that $f \leq^* z_0 \circ g \circ z_1$. Theorem 1.3 tells us that in the model obtained by adding a nondecreasing Hechler real d we have that d is a $\preceq$-least dominating real: for any dominating real y in $V[d]$ we have $d \preceq y$.

So $V^{\mathbb{D}_{\text{nd}}}$ contains a $\preceq$-least dominating real, and by forcing equivalence so does $V^{\mathbb{D}}$. By suitably modifying the arguments from Section 4 one can also show that a $\preceq$-least dominating real is present in $V^{\mathbb{D}_{\text{tree}}}$. If d is a tree Hechler real (which we may assume is strictly increasing), then a $\preceq$-least real $d^\downarrow$ is defined by the equation

$$d^\downarrow(n) = d(k+1) \text{ if } n \in [d(k-1), d(k)).$$

The key difference in the argument for $\mathbb{D}_{\text{tree}}$ is that instead of bounding $\phi(s)$ by a function of the length $|s|$, one must be content to bound $\phi(s)$ by a function of $s \upharpoonright (|s|-1)$.

Need there always exist $\preceq$-least reals in a ccc extension adding a dominating real? The answer is no. Suppose V is a model of $\mathfrak{b} = \mathfrak{d} = \aleph_2$, so that V contains a dominating family $\{z_\alpha : \alpha < \omega_2\}$ well-ordered by $\leq^*$. Let $V[G]$ be an extension satisfying MA and $2^{\aleph_0} = \aleph_3$. There is no $\preceq$-least dominating real d in $V[G]$. If there were then $\{z_\alpha : \alpha < \omega_2\}$, $\{z'_\alpha \circ d \circ z'_\alpha : \alpha < \omega_2\}$ would be an (ω_2, ω_2)-gap in the sense of [Sch93], but Proposition 90 of that article shows that no such gap exists.

This simple argument does not work if the ground model satisfies CH. Together with Itay Neeman we found a construction to produce an appropriate ccc forcing extension over a model of CH. In fact, this contruction produces a model that not only has no $\preceq$-least dominating reals but also has no $\preceq$-minimal dominating reals; that is, no dominating reals y_0 such that whenever y is dominating and $y \preceq y_0$ holds then it follows that $y_0 \preceq y$. The disadvantage of the argument is that it uses large cardinals and a fair amount of technical overhead. The main "trick" used in the argument is rather nice and may be applicable in other situations, so we will include a proof.

The general idea of the construction is the natural one. We do an ω_1-length finite support iteration of ccc forcings which at each stage places a dominating real that lies below all the dominating reals added so far. The tricky part is in making sure that each iterand in the forcing is actually ccc. To show this we will use an absoluteness argument; this is where the large cardinal assumptions come in.

The forcing we will iterate is a slight modification of the Laver interpolation order given as Definition 13 in [Sch93]. Let $\mathcal{F}_0$, $\mathcal{F}_1$ be two subsets of ω^ω so that every member of $\mathcal{F}_0$ is dominated by every member of $\mathcal{F}_1$. Say that a real $h \in \omega^\omega$ interpolates $\mathcal{F}_0$ and $\mathcal{F}_1$ if $f_0 \leq^* h$ and $h \leq^* f_1$ for every $f_0 \in \mathcal{F}_0$ and every $f_1 \in \mathcal{F}_1$. The forcing $\mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ consists of conditions $\langle s, f_0, f_1 \rangle \in \mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ satisfying

1. $s \in \omega^{<\omega}, f_0 \in \mathcal{F}_0, f_1 \in \mathcal{F}_1$
2. $(\forall n \geq |s|) f_0(n) \leq f_1(n)$.

and is ordered by $\langle s', f_0', f_1' \rangle \leq \langle s, f_0, f_1 \rangle$ if

1. $s \subseteq s'$
2. $(\forall n \geq |s|) f_0(n) \leq f_0'(n), f_1'(n) \leq f_1(n)$
3. $(\forall n \in |s'| \setminus |s|) f_0(n) \leq s'(n) \leq f_1(n)$.

It is not hard to see that $\mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ adds a real interpolating $\mathcal{F}_0$ and $\mathcal{F}_1$. Unfortunately this forcing may collapse ω_1. The following proposition is very similar to Lemma 45 in [Sch93] and can be proved in an identical way.

Proposition 5.1. Let Q be a transitive model of ZFC containing $\mathcal{F}_0$ and $\mathcal{F}_1$. If there is a transitive model Q^* of ZFC with $Q \subseteq Q^*$ and $\omega_1^Q = \omega_1^{Q^*}$ and so that Q^* contains an interpolant of $\mathcal{F}_0$ and $\mathcal{F}_1$, then in Q the forcing $\mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ is ccc.

Given an ordinal α we let $\mathbb{P}_\alpha$ be the α-length finite support iterated forcing given by $\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ which we describe as follows. We take $\dot{\mathbb{Q}}_0$ to be Hechler forcing $\mathbb{D}$. Given $0 < \beta < \alpha$ with $\mathbb{P}_\beta$ already defined, we take $\dot{\mathbb{Q}}_\beta$ to be a $\mathbb{P}_\beta$-name for $\mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0$ is $V \cap \omega^\omega$ and $\mathcal{F}_1$ is the collection of dominating reals in $V^{\mathbb{P}_\beta}$.

Lemma 5.2. Assume there exists a sharp for ω many Woodin cardinals. Let $\alpha \leq \omega_1$. Then $\mathbb{P}_\alpha$ is ccc.

Proof. By induction on α. The base case is trivial and the limit case follows from the inductive assumption and the fact that the iteration has finite support.

Assume by induction that $\mathbb{P}_\beta$ is ccc for all $\beta \leq \alpha$. By identifying nice names for reals with reals we may view $\mathbb{P}_\alpha$ as a subset of ω^ω. Let G be $\mathbb{P}_\alpha$-generic over V. Let $\mathcal{F}_0$ be $V \cap \omega^\omega$ and $\mathcal{F}_1$ be the collection of dominating reals in $V[G]$. We need to show that $\dot{\mathbb{Q}}_\alpha[G] = \mathbb{Q}(\mathcal{F}_0, \mathcal{F}_1)$ is ccc.

Let M be a countable elementary submodel of a large rank initial segment of V with $\alpha \in M$, and let $\pi : M \to Q$ be the transitive collapse. Note that $\pi(\mathbb{P}_\alpha) = Q \cap \mathbb{P}_\alpha$. Because $\mathbb{P}_\alpha$ is ccc it follows that $G \cap Q$ is $\pi(\mathbb{P}_\alpha)$-generic over Q. Let $\bar{\mathcal{F}}_0 = Q[G] \cap \mathcal{F}_0$ (which is just $Q \cap \omega^\omega$) and let $\bar{\mathcal{F}}_1 = Q[G] \cap \mathcal{F}_1$ so that $\pi(\dot{\mathbb{Q}}_\alpha)[G] = \mathbb{Q}(\bar{\mathcal{F}}_0, \bar{\mathcal{F}}_1)$.

By elementarity we need only show that $\mathbb{Q}(\bar{\mathcal{F}}_0, \bar{\mathcal{F}}_1)$ is ccc in $Q[G]$. By Proposition 5.1 it is enough to find a transitive model Q^* extending $Q[G]$ with $\omega_1^{Q[G]} = \omega_1^{Q^*}$ and which contains a real interpolating $\bar{\mathcal{F}}_0$ and $\bar{\mathcal{F}}_1$. Let $\bar{d} \in V \cap \omega^\omega$ be $\mathbb{D}$-generic over Q. Then $\bar{d}$ is such an interpolant and so we just need to find an appropriate model Q^* containing it.

Now note that $\mathbb{P}_\alpha$ is a set of reals definable in $L(\mathbb{R})$ (from real parameters). Since $\mathbb{P}_\alpha$ is ccc the collection of maximal antichains of $\mathbb{P}_\alpha$ is also a set of reals definable in $L(\mathbb{R})$. We use $\mathbb{P}_\alpha^{Q[\bar{d}]}$ to refer to $\mathbb{P}_\alpha$ computed relative to $Q[\bar{d}]$. By the inductive hypothesis applied inside $Q[\bar{d}]$ we have that

$$Q[\bar{d}] \vDash \mathbb{P}_\alpha \text{ is ccc.}$$

If we show that G is $\mathbb{P}_\alpha^{Q[\bar{d}]}$-generic over $Q[\bar{d}]$ the proof will be complete, for then $Q[\bar{d}][G]$ can serve as the desired Q^*.

Claim. Suppose φ is a formula and $x \in Q[\bar{d}] \cap \omega^\omega$. Then

$$L(\mathbb{R}) \vDash \phi(x) \iff Q[\bar{d}] \vDash \text{“}L(\mathbb{R}) \vDash \varphi(x)\text{”}.$$

Proof. Here is where the large cardinal machinery comes in. We just give a sketch:

The theory of $L(\mathbb{R})$ with parameter x reduces to the theory of any model $(N;\mathcal{E})$, where $\mathcal{E}$ is a class of extenders rich enough to witness that N has a sharp for ω Woodin cardinals, $x \in N$, and $(N;\mathcal{E})$ is iterable for iteration trees using extenders from $\mathcal{E}$. (See section 4 of [Ste93] where full iterability is used, or section 3 of [Nee95] which uses ω-iterability.) It is therefore enough to find, in $Q[\bar{d}]$, a model of this kind which is ω-iterable in both $Q[\bar{d}]$ and V.

Let V_θ^Q be a rank initial segment of Q large enough to contain a sharp for ω Woodin cardinals. Working in $Q[\bar{d}]$ using the genericity iterations of [Nee95] one can find a countable model N, which embeds into V_θ^Q, and a generic extension $N[g]$ of N collapsing its first Woodin cardinal, so that $x \in N[g]$. Since N embeds into V_θ^Q, and since $Q[\bar{d}]$ is a small generic extension of Q, N is ω-iterable in $Q[\bar{d}]$. Since V_θ^Q embeds into a rank initial segment of V, so does N, and hence N is ω-iterable in V. The iterability transfers to $(N[g],\mathcal{E})$, where $\mathcal{E}$ consists of the natural extensions of extenders in N. $(N[g],\mathcal{E})$ is then ω-iterable in both $Q[\bar{d}]$ and V, contains x, and has a sharp for ω Woodin cardinals, as required. $\square$

Let $\mathcal{A} \in Q[\bar{d}]$ be such that

$$Q[\bar{d}] \models \text{“}\mathcal{A} \text{ is a maximal antichain of } \mathbb{P}_\alpha\text{”}.$$

As we observed above $\mathbb{P}_\alpha^{Q[\bar{d}]}$ is ccc from the point of view of $Q[\bar{d}]$ and so we may view $\mathcal{A}$ as an element of ω^ω. Applying the claim we have (in V) that $\mathcal{A}$ is a maximal antichain of $\mathbb{P}_\alpha$. Thus $G \cap \mathcal{A} \neq \varnothing$ and we are done. $\square$

Theorem 5.3. Suppose V is a model of ZFC which contains a sharp for ω many Woodin cardinals. Then $\mathbb{P}_{\omega_1}$ is a ccc forcing which adds a dominating real but no $\preceq$-minimal dominating real.

Proof. From Lemma 5.2 we get that $\mathbb{P}_{\omega_1}$ is ccc. From this it follows that any dominating real d added by $\mathbb{P}_{\omega_1}$ is added at some countable stage $\mathbb{P}_\alpha$. With z_0, z_1 varying over $V \cap \omega^{\nearrow\omega}$, every real of the form $z_0 \circ d \circ z_1$ belongs to $V^{\mathbb{P}_\alpha}$ and so the forcing $\dot{\mathbb{Q}}_\alpha$ adds a dominating real h below them all. Thus d is not a minimal dominating real for the preordering $\preceq$ in $V^{\mathbb{P}_\alpha}$. □

6 A Hechler subextension dichotomy that holds under large assumptions

In this small section we show that a weaker conjecture made in [BL11] does hold, assuming that one is willing to make some large cardinal assumptions. In that article the authors conjecture (Conjecture 14) that if d is a Hechler real over V and $x \in V[d]$ is a new real then either there is a dominating real over V in $V[x]$, or $V[x]$ is equivalent to a Cohen extension of V. By piecing together results of Bartoszyński, Shelah and Zapletal we will prove the following.

Theorem 6.1. Suppose there is a proper class of Woodin cardinals. Let d be $\mathbb{D}$-generic or $\mathbb{D}_{\text{tree}}$-generic over V, and let $x \in V[d]$ be a new real. Either $V[x]$ contains a dominating real over V, or $V[x]$ is a Cohen extension.

The first step of the argument is to note that every eventually different real in $V[d]$ is dominating. This follows from Theorem 4.7 in the case where d is $\mathbb{D}_{\text{tree}}$-generic and from Corollary 4.8 in the case where d is $\mathbb{D}$-generic. Hence it is enough to show that if $V[x]$ is not a Cohen extension then $V[x]$ contains an eventually different real.

The second step is to note that Bartoszyński's characterization of $\mathbf{non}(\mathcal{M})$ (see [BJ95], Theorem 2.4.7) shows that forcing with $\mathbb{P}$ adds an eventually different real exactly when forcing with $\mathbb{P}$ makes the collection of ground model reals meager. Hence it is enough to show that if $V[x]$ is not a Cohen extension then in $V[x]$ the collection of ground model reals is meager.

We use notation similar to that in Zapletal's text [Zap08]. Let $\mathcal{I}$ be a σ-ideal on a Polish space X. (We only need to consider the case where X is Baire space ω^ω.) Then $\mathbb{P}_\mathcal{I}$ denotes the σ-ideal of Borel subsets of X which do not belong to $\mathcal{I}$.

We are going to apply a theorem of Zapletal's about forcing notions of the form $\mathbb{P}_\mathcal{I}$ where $\mathcal{I}$ is universally Baire. For our purposes, all that needs to be known about universally Baire sets is that said theorem applies to them, and that assuming there is a proper class of Woodin cardinals, any set of reals in $L(\mathbb{R})$ is universally Baire. Consult page 7 of [Zap08] for a discussion of the relevance of the notion of universal Baireness in this context.

We need a small lemma which allows us to represent reals $x \in V[d]$ in a more convenient way. We invite the reader to compare this lemma with Propositions 2.1.6 and 2.1.8 in [Zap08].

Lemma 6.2. Let $V[G]$ be a forcing extension via some ccc forcing notion $\mathbb{P}$, and let $x \in V[G] \cap \omega^\omega$ be a new real. There is a σ-ideal $\mathcal{I}$ on ω^ω in V and a generic $H \subseteq \mathbb{P}_\mathcal{I}$ so that $V[x] = V[H]$. Furthermore, if $\mathbb{P}$ belongs to $L(\mathbb{R})$ then so does $\mathcal{I}$, so that $\mathcal{I}$ is universally Baire assuming the existence of a proper class of Woodin cardinals.

Proof. Let $\dot{x}$ be a $\mathbb{P}$-name for the real x. Define $\mathcal{I}$ to be the collection of Borel sets B coded in V for which $\Vdash_\mathbb{P} \dot{x} \notin B$. We easily see that $\mathcal{I}$ is a σ-ideal, and that $\mathcal{I}$ belongs to $L(\mathbb{R})$ if $\mathbb{P}$ does.

Now let $H \subseteq \mathbb{P}_\mathcal{I}$ be defined to be all of the $B \in \mathbb{P}_\mathcal{I}$ with $x \in B$. Since H is definable from x we see that $V[H] \subseteq V[x]$. But x is also definable from H as the unique real such that for each n the set $N_{x\restriction n}$ belongs to H. (Here $N_s \subseteq \omega^\omega$ denotes the basic open neighborhood of all reals extending s).

It just remains to show that H is actually a $\mathbb{P}_\mathcal{I}$-generic filter. This is where we use that $\mathbb{P}$ is ccc. Let $\mathcal{A} \subseteq \mathbb{P}_\mathcal{I}$ be a maximal antichain. We must show that $H \cap \mathcal{A}$ is not empty. We first claim that $\mathcal{A}$ is countable (and indeed, that $\mathbb{P}_\mathcal{I}$ is ccc). To see this, fix $B \in \mathcal{A}$. Since $B \notin \mathcal{I}$ there is some $p_B \in \mathbb{P}$ with $p_B \Vdash \dot{x} \in B$. If $B \neq B'$ belong to $\mathcal{A}$ then $B \cap B' \in \mathcal{I}$ and hence $\Vdash \dot{x} \notin B \cap B'$ and hence p_B and $p_{B'}$ are incompatible. Since $\mathbb{P}$ is ccc, $\mathcal{A}$ must be countable.

So enumerate $\mathcal{A} = \{B_n : n \in \omega\}$. Then $B = \bigcup_{n<\omega} B_n$ is a Borel set, and maximality of $\mathcal{A}$ guarantees that $\omega^\omega \setminus B \in \mathcal{I}$ so that $\Vdash \dot{x} \in B$. In particular, x belongs to B_n for some n, so that some B_n belongs to H and hence $H \cap \mathcal{A}$ is not empty. $\square$

By the lemma, we may think of any $x \in V[d]$ as the generic for an appropriate $\mathbb{P}_\mathcal{I}$, and assuming the existence of a proper class of Woodin cardinals we may assume that $\mathcal{I}$ is universally Baire. The proof of Theorem 6.1 is now completed by citing the following theorem, proved by Shelah, and then later and independently by Zapletal. We state the theorem in Zapletal's language.

Theorem 6.3 (Shelah [She04], Zapletal [Zap08] Corollary 3.5.7)**.** Suppose that $\mathcal{I}$ is a universally Baire ccc ideal such that $\mathbb{P}_\mathcal{I}$ does not make the collection of ground model reals meager. Then $\mathbb{P}_\mathcal{I}$ is equivalent to Cohen forcing.

7 The product of two forcings adding a dominating real adds a Hechler real.

In [FV07] Farah and Veličković mention the following unpublished result due to Goldstern.

Theorem 7.1 (Goldstern)**.** If $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions adding a dominating real then the product $\mathbb{P} \times \mathbb{Q}$ adds a Cohen real.

This section is dedicated to proving the following strengthening of that result.

Theorem 7.2. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing, each of which adds a dominating real. Then the product $\mathbb{P} \times \mathbb{Q}$ adds a $\mathbb{D}$-generic real.

Before going into the details let us give a sketch of the proof. Suppose x and y are dominating reals added by $\mathbb{P}$ and $\mathbb{Q}$, respectively. We replace x and y by iterated versions, so that the reals are not just dominating but in fact *strongly dominating*. This will guarantee that the digits $x(n)$ and $y(n)$ are independent to such an extent that we get a Cohen real c by considering the characteristic function of the relation $x(n) < y(n)$. In fact, by a careful analysis we will show that c is not only Cohen over V but is in fact Cohen over $V[x]$ and hence we may finish by appealing to Truss's Theorem 2.2.

We start by defining what we mean by a strongly dominating real.

Definition 7.3. A real $d \in V[G] \cap \omega^\omega$ is strongly dominating over V if for every function $f : \omega^{<\omega} \to \omega$ belonging to V we have $(\forall^\infty n) f(d \restriction n) \leq d(n)$.

Notice that whenever a forcing notion adds a dominating real, it also adds a strongly dominating real. Indeed, if $d \in V[G] \cap \omega^\omega$ is a dominating real, then any real d' which satisfies the formula $(\forall n) d'(n+1) = d(d'(n))$ will be strongly dominating.

Fix a forcing notion $\mathbb{P}$ which adds a dominating real. Let $\dot{x} \in V^{\mathbb{P}} \cap \omega^\omega$ name a strongly dominating real. For each $p \in \mathbb{P}$ let $T_p^{\dot{x}}$ be the *tree of possibilities for* $\dot{x}$, defined to be the collection of all the $s \in \omega^{<\omega}$ for which some $q \leq p$ forces that s is an initial segment of $\dot{x}$. Recall now that a Laver tree $T \subseteq \omega^{<\omega}$ is a tree with a distinguished stem $s = \mathrm{stem}(T)$ so that every member of T is either an initial segment of s or an extension of s, and furthermore, whenever $t \in T$ with $s \subseteq t$ there are infinitely many n with $s \frown n$ also in T.

Lemma 7.4. If $x \in V^{\mathbb{P}} \cap \omega^\omega$ is forced to be strongly dominating, then for any $p \in \mathbb{P}$ there is some Laver tree $T \subseteq T_p^{\dot{x}}$.

Proof. We perform a rank analysis on $T_p^{\dot{x}}$. For $s \in \omega^{<\omega}$ we will, if possible, assign an ordinal α to s, which we call the rank of s and write $\mathrm{rk}(s) = \alpha$. This is done as follows. If $s \notin T_p^{\dot{x}}$ then we set $\mathrm{rk}(s) = 0$. If $(\exists^\infty n)\mathrm{rk}(s \frown n) \geq \alpha$ then we declare that $\mathrm{rk}(s) > \alpha$. We define $\mathrm{rk}(s) = \alpha$ if α is the smallest possible ordinal subject to these requirements, if such an ordinal exists. Otherwise $\mathrm{rk}(s) > \alpha$ for all α. In this case we say that s has infinite rank and write $\mathrm{rk}(s) = \infty$. A node $s \in T_p^{\dot{x}}$ has infinite rank precisely when there is a Laver tree T with stem s with $T \subseteq T_p^{\dot{x}}$. So to prove the lemma we must show that $T_p^{\dot{x}}$ contains a node s with $\mathrm{rk}(s) = \infty$.

If not, then for each $s \in T_p^{\dot{x}}$ there is some N such that for all $n \geq N$ we have $\mathrm{rk}(s \frown n) < \mathrm{rk}(s)$. Define $f : \omega^{<\omega} \to \omega$ by letting $f(s)$ be this N. Now let G be a $\mathbb{P}$-generic filter containing p, and let $x = \dot{x}[G]$. Each $x \restriction n$ belongs to $T_p^{\dot{x}}$. But also x is strongly dominating and so we have $x(n) > f(x \restriction n)$ for all sufficiently large n. That means that $(\forall^\infty n)\mathrm{rk}(x \restriction n+1) < \mathrm{rk}(x \restriction n)$. This brings us to a strictly decreasing sequence of ordinals and hence a contradiction. □

Proof of Theorem 7.2. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions which add dominating reals and let $\dot{x} \in V^{\mathbb{P}} \cap \omega^\omega$ and $\dot{y} \in V^{\mathbb{Q}} \cap \omega^\omega$ name strongly dominating reals. For two sequences s, t of natural numbers of the same (and possibly countably infinite) length, we let $\phi_<(s,t)$ be the binary sequence of the same length, given by $\phi_<(s,t)(n) = 1$ when $s(n) < t(n)$ and $\phi_<(s,t)(n) = 0$ when $s(n) \geq t(n)$. Then we take $\dot{c} \in V^{\mathbb{P}\times\mathbb{Q}} \cap 2^\omega$ to name canonically the real $\phi_<(\dot{x}, \dot{y})$. We will show that $\dot{c}$ in fact names a real which is Cohen over $V^{\mathbb{P}}$. Since $\mathbb{P}$ adds a dominating real, $\mathbb{P} \times \mathbb{Q}$ contains a $\mathbb{D}$-generic real, by Truss's theorem.

Fix $\langle p_0, q_0 \rangle \in \mathbb{P} \times \mathbb{Q}$, and let $\dot{D} \in V^{\mathbb{P}}$ name a dense subset of $\mathbb{C}$. We need to find $\langle p', q' \rangle \leq \langle p_0, q_0 \rangle$ with $\langle p', q' \rangle \Vdash \dot{c} \restriction n \in \dot{D}$ for some $n \in \omega$. Our plan is to strengthen p_0 to some p_1 so that the branches of $T^{\dot{x}}_{p_1}$ grow very quickly relative to the branches of $T^{\dot{y}}_{q_0}$. This way, when we strengthen p_1 to evaluate a member of $\dot{D}$, we will have the flexibility to select an appropriate branch through $T^{\dot{y}}_{q_0}$ so that $\dot{c}$ will match this member of $\dot{D}$.

Using Lemma 7.4 we get a Laver tree $T_0 \subseteq T^{\dot{y}}_{q_0}$. For each $s \in \omega^{<\omega}$ with $|s| \geq |\text{stem}(T_0)|$ we will define a function $f_s : \omega^{<\omega} \to \omega$. Fix such an s. Let $t(s) \in T_0$ with $|s| = |t(s)|$. We want f_s to satisfy the following property:

($\dagger$): If $s' = s \frown a$ and $(\forall n \in |s'| \setminus |s|) s'(n) \geq f_s(s' \restriction n)$, then $(\forall u \in 2^{|s'|\setminus|s|})(\exists b \in \omega^{|s'|\setminus|s|})$ with $t(s) \frown b \in T_0$ and $\phi_<(a,b) = u$.

Defining f_s to satisfy ($\dagger$) is a straightforward recursion. Notice first of all that we only have to worry about defining $f_s(s')$ for those $s' \supseteq s$ which recursively satisfy $s'(n) \geq f_s(s' \restriction n)$ for $n \in |s'| \setminus |s|$. So suppose $f_s(s')$ has recursively been defined on such an s'. Now we define $f_s(s' \frown m)$ for $m \geq f_s(s')$. Write $s' = s \frown a$. For each $u \in 2^{|s'|\setminus|s|}$, recursively using ($\dagger$) take $t'_u = t(s) \frown b_u \in T_0$ with $\phi(a, b_u) = u$. Then define $f_s(s' \frown n) \in \omega$ to be large enough so that for every $u \in 2^{|s'|\setminus|s|}$ there is an $i \in \omega$ with $t'_u \frown i \in T_0$ and $i < f_s(s' \frown m)$. To see that (†) holds, note that by taking $f_s(s' \frown m)$ large we have guaranteed that we may find an appropriate t' for any $u \frown 0$. To find an appropriate t' for any $u \frown 1$ we may use (†) at s' and the fact that T_0 is a Laver tree with t'_u extending the stem.

Now let $g : \omega^{<\omega} \to \omega$ be a function satisfying $g(s') \geq f_s(s')$ for all $s \subseteq s'$. Since $\dot{x}$ is forced to be strongly dominating, we may take $p_1 \leq p_0$ with

$$p_1 \Vdash (\forall n \geq N)\dot{x}(n) \geq g(\dot{x} \restriction n).$$

Let $s \in T^{\dot{x}}_{p_1}$ with $|s| \geq N$. Note that each $s' \in T^{\dot{x}}_{p_1}$ extending s has the property that $s'(n) \geq g(s' \restriction n) \geq f_s(s' \restriction n)$ and thus we may apply (†) to any such s'.

Let $v = \phi_<(s, t(s))$. Since there is some $p \leq p_1$ forcing s to be an initial segment of $\dot{x}$, and since $\dot{D}$ is forced to be a dense subset of $\mathbb{C}$, we may take $p_2 \leq p_1$ and $u \in 2^{<\omega}$ with $p_2 \Vdash s \subset \dot{x}$ and $p_2 \Vdash v \frown u \in \dot{D}$. The tree $T^{\dot{x}}_{p_2}$ is a subtree of $T^{\dot{x}}_{p_1}$ containing s (since $p_2 \Vdash s \subseteq \dot{x}$).

Let $s' = s \frown a \in T^{\dot{x}}_{p_2}$ with $|a| = |u|$. By (†) there is $t' = t(s) \frown b \in T_0 \subseteq T^{\dot{y}}_{q_0}$ with $\phi_<(a, b) = u$. Then $\phi_<(s', t') = v \frown u$. Take $p_3 \leq p_2$ with $p_3 \Vdash s' \subseteq \dot{x}$. Take $q_3 \leq q_0$ with $q_3 \Vdash t' \subseteq \dot{y}$. Then $\langle p_3, q_3 \rangle \Vdash v \frown u \subseteq \dot{c}$ and $\langle p_3, q_3 \rangle \Vdash v \frown u \in \dot{D}$. □

8 Every subextension of the Hechler extension contains a Cohen real

Our goal in this section is to prove the following theorem, which says that every subuniverse of the tree Hechler extension contains a Cohen real. A slight modification of the arguments here will prove that the same holds for the usual Hechler extension $\mathbb{D}$. One can also directly prove the result for $\mathbb{D}$ from the result for $\mathbb{D}_{\text{tree}}$.

This is because Proposition 2.6 implicitly constructs a complete embedding from $\mathbb{D}$ to $\mathbb{D}_{\text{tree}}$. Hence there is an appropriate forcing notion $\mathbb{Q}$ so that $\mathbb{D}_{\text{tree}}$ is forcing equivalent to $\mathbb{D} * \mathbb{Q}$, and thus every subuniverse of a Hechler extension may in turn be viewed as a subuniverse of a tree Hechler extension.

Theorem 8.1. Suppose d is a $\mathbb{D}_{\text{tree}}$-generic real, and $x \in V[d] \setminus V$. Then $V[x]$ contains a Cohen real over V.

The proof of Theorem 8.1 relies on the fact that σ-centered forcings add unbounded reals. This is a folklore fact, but for completeness we include a proof, which may be original. In fact, the argument we give proves a bit more. Recall that a poset $\mathbb{P}$ is σ-linked if it can be written as $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$ where for each $n < \omega$ any two members of $\mathbb{P}_n$ are compatible.

Lemma 8.2. If $\mathbb{P}$ is σ-centered, then forcing with $\mathbb{P}$ adds an unbounded real. If $\mathbb{P}$ is σ-linked, then forcing with $\mathbb{P}$ adds a real.

Proof. We claim that in either case we may assume that for any pair $m \neq n$ there are $p \in \mathbb{P}_m$ and $q \in \mathbb{P}_n$ which are incompatible. In the σ-linked case this is accomplished by repeatedly amalgamating together $\mathbb{P}_m$ and $\mathbb{P}_n$ with $\mathbb{P}_m \cup \mathbb{P}_n$ linked. In the case where $\mathbb{P} = \mathbb{P}_n$ is σ-centered we must first massage $\mathbb{P}$ into a more workable form.

First, by replacing $\mathbb{P}$ with $\text{RO}(\mathbb{P})$ we may assume that $\mathbb{P}$ is a complete Boolean algebra. Since the original poset will be a dense subset of $\text{RO}(\mathbb{P})$, this operation preserves σ-centeredness. Second, by adding to $\mathbb{P}_n$ the infimum $\inf(\{p_0, \dots p_n\})$ for each possible finite subset $\{p_0, \dots p_n\} \subseteq \mathbb{P}_n$, we may also assume that each $\mathbb{P}_n$ is not only centered, but *filtered*. This is, for any finite subset $\{p_0, \dots, p_n\} \subseteq \mathbb{P}_n$, a common strengthening of $\{p_0, \dots, p_n\}$ belongs to $\mathbb{P}_n$. And third (and finally), we may now repeatedly amalgmate pairs $\mathbb{P}_m$, $\mathbb{P}_n$ for which any conditions $p \in \mathbb{P}_m$

and $q \in \mathbb{P}_n$ are compatible into a single $\mathbb{P}_k$: we then repeat the second step of our massage to guarantee the centeredness of the resulting $\mathbb{P}_n$s.

Hence indeed we may assume that if $m \neq n$, then there exists $p_{m,n} \in \mathbb{P}_m$ and $p_{n,m} \in \mathbb{P}_n$ with $p_{m,n} \perp p_{n,m}$. We define the name $\dot{r} \in V^{\mathbb{P}} \cap \omega^\omega$ by declaring

$$\Vdash (\forall n) r(n) \text{ equals the least } m \text{ such that } p_{m,n} \in \dot{G}.$$

It is not hard to see that r is well-defined, for given any $p \in \mathbb{P}$, there is some m such that $p \in \mathbb{P}_m$. Then $p_{m,n}$ and p are compatible so p may be strengthened to q with $q \Vdash p_{m,n} \in \dot{G}$.

We claim that if each $\mathbb{P}_n$ is centered then $\Vdash \dot{r}$ is unbounded. Indeed, let $p \in \mathbb{P}$ and $f \in V \cap \omega^\omega$. Say that $p \in \mathbb{P}_n$. Using the fact that $\mathbb{P}_n$ is centered, we may strengthen p to q with $q \Vdash (\forall i \leq f(n)) p_{n,i} \in \dot{G}$. Then $q \Vdash (\forall i \leq f(n)) p_{i,n} \notin \dot{G}$. Hence $q \Vdash f(n) < \dot{r}(n)$.

Similarly one shows that if each $\mathbb{P}_n$ is linked then $\Vdash \dot{r} \notin V$. $\square$

If one is willing to step outside of ZFC and make some large cardinal assumptions then Theorem 8.1 can now be proven as follows. By Corollary 15.42 of [Jec03], we know that x is added by forcing with a forcing notion $\mathbb{P}$, a complete subalgebra of $\mathrm{RO}(\mathbb{D}_{\mathrm{tree}})$. Since $\mathbb{D}_{\mathrm{tree}}$ is σ-centered, it follows that $\mathbb{P}$ is as well. Hence $\mathbb{P}$ adds some real y. Consider a name $\dot{y} \in V^{\mathbb{D}_{\mathrm{tree}}}$. Let $\mathbb{Q}$ be the complete subalgebra of $\mathrm{RO}(\mathbb{D}_{\mathrm{tree}})$ generated by $\dot{y}$. Since $\mathbb{D}_{\mathrm{tree}}$ is ccc, we may think of $\dot{y}$ as a member of $\mathbb{R}$ and hence $\mathbb{Q}$ is definable in $L(\mathbb{R})$. A theorem of Shelah (Proposition 4.3 of [GS93]) says that sufficiently definable σ-centered forcings add Cohen reals. Examining the proof, what is really needed is that certain filters $\mathcal{F} \subseteq \mathcal{P}(\omega)$ which are definable from $\mathbb{Q}$ have the property of Baire. Under large cardinal assumptions all sets of reals in $L(\mathbb{R})$ have the property of Baire. Thus forcing with $\mathbb{Q}$ adds a Cohen real. Since a generic for $\mathbb{Q}$ is definable from y, and we have just seen that the presence of a $\mathbb{Q}$-generic guarantees the presence of a Cohen real, we see that $V[y] \subseteq V[x]$ will contain a Cohen real.

Now we give a proof of Theorem 8.1 that goes through in ZFC and doesn't use any descriptive set theory. Fix $x \in V[d] \setminus V$. By Lemma 8.2 and Corollary 15.42 of [Jec03], we may assume that $x \in V^{\mathbb{D}_{\text{tree}}} \cap \omega^\omega$ is an unbounded real. Let $\dot{x}$ be a name for x with $\Vdash$ "$\dot{x}$ is an unbounded real.".

We divide the argument into cases.

Case 1: There is a dense set of stems $s \in \omega^{<\omega}$ for which there exists a $u \in \omega^{<\omega}$ so that s favors "$u \frown k$ is an initial segment of $\dot{x}$" for infinitely many k. We define a Cohen real as follows. Let $\phi : \omega \to \omega^{<\omega}$ be a map with the property that whenever $A_{s,u} = \{k : s \text{ favors } u \frown k \subseteq \dot{x}\}$ is infinite, for every $t \in \omega^{<\omega}$ there are infinitely many $k \in A_{s,u}$ with $\phi(k) = t$.

Now let $c \in V[x] \cap \omega^\omega$ be the concatenation of the sequences

$$\phi(x(0)), \phi(x(1)), \phi(x(2)), \ldots$$

Then we claim that c is a Cohen real over V. To see this, let $D \subseteq \omega^{<\omega}$ be dense and let $T \in \mathbb{D}_{\text{tree}}$ with $s = \text{stem}(T)$. By hypothesis we may assume that there is some u for which $A_{s,u}$ is infinite. Let $t \in \omega^{<\omega}$ with $\phi(u(0)) \frown \ldots \frown \phi(u(|u|-1)) \frown t \in D$. By choice of ϕ, there is some $k \in A_{s,u}$ with $\phi(k) = t$. By definition of $A_{s,u}$, there is some $p \leq T$ with $p \Vdash u \frown k \subseteq \dot{x}$, and hence p forces that some initial segment of c belongs to D, as desired.

Now assume that Case 1 fails; this means there is some $T \in \mathbb{D}_{\text{tree}}$ so that for every $s \in T$ and every u there are only finitely many k for which s favors $u \frown k \subseteq \dot{x}$. For notational simplicity we assume that $T = \omega^{<\omega}$; this doesn't change the structure of the rest of the argument.

Case 2: There is a dense set of stems $s \in \omega^{<\omega}$ for which there exists a $u(s) \in \omega^{<\omega}$

such that the set $B_s = \{k : (\exists m) s \frown m \text{ favors } u(s) \frown k \subseteq \dot{x}\}$ is infinite. Fix s, and for $k \in B_s$, let $m_s(k)$ witness membership in B_s. Notice that the map $k \mapsto m_s(k)$ is finite-to-one: otherwise fix an m with $m = m_s(k)$ for infinitely many k. Then $s \frown m$ favors $u(s) \frown k \subseteq \dot{x}$ for infinitely many k, contrary to our assumption that this does not hold for any member of $\omega^{<\omega}$.

Since $k \mapsto m_s(k)$ is finite-to-one, we may define a partial injection with infinite domain $f^s : \omega \to \omega$ such that for each $k \in \mathrm{dom}(f^s)$ we have that $s \frown f^s(k)$ favors $u(s) \frown k \subseteq \dot{x}$.

To get a Cohen real, we can now use the same trick we used in Case 1. Let $\phi : \omega \to \omega^{<\omega}$ have the property that for every s and every $t \in \omega^{<\omega}$ there are infinitely many $k \in \mathrm{dom}(f^s)$ with $\phi(k) = t$. Let $c \in V[x] \cap \omega^\omega$ be the concatenation of the sequences

$$\phi(x(0)), \phi(x(1)), \phi(x(2)), \ldots$$

To see that c is Cohen over V, let $D \subseteq \omega^{<\omega}$ be dense and let $T \in \mathbb{D}_{\text{tree}}$ with $s = \mathrm{stem}(T)$. By extending T if necessary we may assume the s we chose is such that f^s is defined. Let $u = u(s)$. Let $t \in \omega^{<\omega}$ with $\phi(u(0)) \frown \ldots \frown \phi(u(|u|-1)) \frown t \in D$. By choice of ϕ, there are infinitely many $k \in \mathrm{dom}(f^s)$ with $\phi(k) = t$. Since f^s is injective, there is some choice of such k with $s \frown f^s(k)$ belonging to T. Hence there is some $p \leq T$ with $p \Vdash u \frown k \subseteq \dot{x}$, and hence p forces that some initial segment of c belongs to D, as desired.

Now assume that Case 2 fails as well; this means there is some $T \in \mathbb{D}_{\text{tree}}$ so that for every $s \in T$ and for all u there is some k^u_s so that $(\forall m)(\forall k \geq k^u_s) s \frown m \Vdash u \frown k \not\subseteq \dot{x}$. For notational simplicity we may assume that $T = \omega^{<\omega}$. Hence:

Case 3: $(\forall s)(\forall u)(\exists k^u_s)(\forall m)(\forall k \geq k^u_s) s \frown m \Vdash u \frown k \not\subseteq \dot{x}$.

We finish the proof of Theorem 8.1 by ruling Case 3 out. This is accomplished by a (double) fusion argument; we build a condition $T \in \mathbb{D}_{\text{tree}}$ and a ground model real h with $T \Vdash (\forall n)\dot{x}(n) \leq h(n)$. This will contradict the fact that $\dot{x}$ is forced to be an unbounded real.

The argument has a fair number of moving parts, so for expository purposes we start by indicating how via a fusion argument we can obtain $T \in \mathbb{D}_{\text{tree}}$ with $\text{stem}(T) = \varnothing$ and $T \Vdash \dot{x}(0) \leq k^{\varnothing}_{\varnothing}$. This same method will itself be integrated into another fusion argument which will net us a ground model bound for $\dot{x}$.

To obtain $T \Vdash \dot{x}(0) \leq k^{\varnothing}_{\varnothing}$, the fusion argument can be described as follows. Our first approximation to T is $T^0 = \omega^{<\omega}$. For each node at level 1, say $\langle s(0)\rangle$, by the assumption of Case 3 we may strengthen $T^0_{\langle s(0)\rangle}$ (finitely many times) so that $T^0_{\langle s(0)\rangle} \Vdash \dot{x}(0) \notin [k^{\varnothing}_{\varnothing}, k^{\varnothing}_{\langle s(0)\rangle})$. (This is possible by the assumption of Case 3, because saying $\dot{x}(0) \neq k$ is the same as saying $\langle k\rangle \not\subseteq \dot{x}$.) This brings us to a tree T^1. At the level 2 nodes $\langle s(0), s(1)\rangle \in T^1$ we strengthen $T^1_{\langle s(0),s(1)\rangle}$ so that $T^1_{\langle s(0),s(1)\rangle} \Vdash \dot{x}(0) \notin [k^{\varnothing}_{\langle s(0)\rangle}, k^{\varnothing}_{\langle s(0),s(1)\rangle})$. This us brings us to a tree T^2. We continue in this way. At the level n nodes $\langle s(0), \ldots s(n-1)\rangle \in T^{n-1}$ we strengthen $T^{n-1}_{\langle s(0),\ldots s(n-1)\rangle}$ so that $T^{n-1}_{\langle s(0),\ldots s(n-1)\rangle} \Vdash \dot{x}(0) \notin [k^{\varnothing}_{\langle s(0),\ldots s(n-2)\rangle}, k^{\varnothing}_{\langle s(0),\ldots s(n-1)\rangle})$. The limit of this process is the tree T given by the intersection $\bigcap_{n<\omega} T^n$. Then T is as desired, since given a $\mathbb{D}_{\text{tree}}$-generic d and $l \geq 1$, the tree $T_{d\restriction l}$ belongs to the generic filter G and forces that $\dot{x}(0)$ does not belong to $[k^{\varnothing}_{d\restriction l-1}, k^{\varnothing}_{d\restriction l})$.

It is clear we could use the same argument to get a tree T with $\text{stem}(T) = \varnothing$ forcing bounds on any finitely many digits $\dot{x}(0), \ldots \dot{x}(i)$. In order to simultaneously force bounds on all of the digits, we stagger these bounds into the fusion argument, by possibly not beginning the process of forcing a bound on $\dot{x}(i)$ until at some level fairly high up into the tree. To ensure that we end up with a ground model bound, it is not enough to simply dovetail one digit at a time by handling $\dot{x}(0)$ at level 0, $\dot{x}(1)$ at level 1, and so on. Rather, we will need to start handling an enormous number of digits at each stage of the argument. We will now describe how this can be done.

Fix $s \in \omega^{<\omega}$. We define a finite partial function $f_s : \omega \to \omega$ by recursion. As a base case, we have $f_s(0) = k^{\varnothing}_{\varnothing}$. Given $i+1 \in (s(l-1), s(l)]$ (and taking $l = 0$ when $i+1 \leq s(0)$), we set $f_s(i+1)$ equal to the maximum of all $k^{\langle j_0, j_1, \ldots j_i\rangle}_{s\restriction l}$ where $j_k \leq f(k)$. Notice that if $s \subseteq t$ then f_s and f_t agree on their common domain. We comment also that the point of this definition is that the assumption of Case 3 only allows us to put a bound on e.g. $\dot{x}(1)$ relative to the bound we place on $\dot{x}(0)$. The function f_s serves as a mechanism for facilitating a bound on $\dot{x}(i)$ once a bound has already been established on earlier digits.

Now we are ready to describe the fusion argument. We build the desired tree T as an intersection $\bigcap_{n<\omega} T^n$ of approximations to T, with $T^0 = \omega^{<\omega}$, and $T^{n+1} \leq T^n$. We will arrive at T^{n+1} from T^n by strengthening the level n nodes in T^n.

To start, we look at each level one node $\langle s(0)\rangle$ in T^0. To arrive at T^1, strengthen each $T^0_{\langle s(0)\rangle}$ to force $\langle j_0, \ldots j_{i-1}, k\rangle \not\subseteq \dot{x}$ for each $i \leq s(0)$, each sequence $\langle j_0, \ldots j_{i-1}\rangle$ with $j_l \leq f_{\langle s(0)\rangle}(l)$, and each $k \in [k^{\langle j_0,\ldots j_{i-1}\rangle}_{\varnothing}, k^{\langle j_0,\ldots j_{i-1}\rangle}_{s(0)})$. This is possible by the assumption of Case 3. In general, to go from T^n to T^{n+1}, strengthen each T^n_s with $|s| = n+1$ to force $\langle j_0, \ldots j_{i-1}, k\rangle \not\subseteq \dot{x}$ for each $i \leq s(n)$, each sequence $\langle j_0, \ldots j_{i-1}\rangle$ with $j_l \leq f_s(l)$, and each $k \in [k^{\langle j_0,\ldots j_{i-1}\rangle}_{s\restriction n}, k^{\langle j_0,\ldots j_{i-1}\rangle}_{s})$.

We now want to show that there is a ground model $h \in V \cap \omega^\omega$ with $T \Vdash (\forall i)\dot{x}(i) \leq h(i)$. Fix a $\mathbb{D}_{\text{tree}}$-generic real d. Let x be the evaluation of $\dot{x}$ using the corresponding generic filter G, ie $x = \dot{x}[G]$.

Claim. For $i \in (d(l-1), d(l)]$ (if $i \leq d(0)$, take $l = 0$), $x(i) < f_{d\restriction l}(i)$.

The claim is proven by induction on i. For $i = 0$, we note that each $T_{d\restriction n} \in G$ forces $\dot{x}(0) \notin [k^{\varnothing}_{d\restriction n}, k^{\varnothing}_{d\restriction(n+1)})$, so that indeed we must have $x(0) < k^{\varnothing}_{\varnothing} = f_{d\restriction n}(0)$.

Inductively assume the claim holds below i, and (to align with our earlier notation) let $j_k = x(k)$ for $k < i$. For each $n \geq l$ by construction we have that $T_{d\restriction n} \in G$ forces that $\langle j_0, \ldots j_{i-1}, k\rangle \not\subseteq \dot{x}$ for each $k \in [k^{\langle j_0,\ldots j_{i-1}\rangle}_{d\restriction n}, k^{\langle j_0,\ldots j_{i-1}\rangle}_{d\restriction(n+1)})$. In particular $x(i) \notin [k^{\langle j_0,\ldots j_{i-1}\rangle}_{d\restriction n}, k^{\langle j_0,\ldots j_{i-1}\rangle}_{d\restriction(n+1)})$ for each such n, and so $x(i) < k^{\langle j_0,\ldots j_{i-1}\rangle}_{d\restriction l} \leq f_{d\restriction l}(i)$, as desired.

With the claim proven, we at last can obtain our ground model bound. This is straightforward since $d(l-1) < i$ and so we may (in the ground model) maximize against all possible sequences below i. Working in V, for each i, let $h(i) \in \omega$ be larger than $f_s(i)$ for each s with length at most i and range a subset of i. Then $(\forall i)x(i) \leq h(i)$, and since h (and the rest of our argument above) does not depend on the particular $\mathbb{D}_{\text{tree}}$-generic d, we have $T \Vdash (\forall i)\dot{x}(i) \leq h(i)$, as desired.

9 A non-embedding result for non-linear iterations of Hechler forcing

In the final section of this part of the thesis we present an example showing that subiterations of non-linear iterations of Hechler forcing behave differently than in the linear case; specifically in the non-linear case the natural inclusion map is not a complete embedding. We will be more precise below, but first we give some motivation. Our main goal in including this section in the thesis is to outline to future researchers an interesting and potentially fruitful line of inquiry, while also pointing out a subtle barrier in the most natural approach.

Recall that Hechler's original application of his forcing notion [Hec74] was to use non-linear iterations to prove that for any σ-directed partial ordering P present in the ground model V there is a generic extension $V[G]$ in which P embeds cofinally into $(\omega^\omega, \leq^*)$. The article [Bur97] written by Burke gives a modern and very readable treatment of this result.

We describe the iteration Hechler used in the case where P is well-founded. Fix such a P. We recursively define a rank function $\mathrm{rk} : P \to \mathrm{ON}$ by $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 : y < x\}$. (Thus elements in P with no lower elements have rank 0). We define $\mathrm{rk}(P) = \sup\{\mathrm{rk}(x) + 1 : x \in P\}$. For $x \in P$, we let $P_x = \{y \in P : y < x\}$.

We define the iteration $\mathbb{P}(P)$ by recursion on $\mathrm{rk}(P)$. Conditions in $\mathbb{P}(P)$ are functions u such that:

1. $\mathrm{dom}(u)$ is a finite subset of P.
2. For each $x \in \mathrm{dom}(u)$, $u(x) = \langle s^u_x, \dot{f}^u_x \rangle$ where $\dot{f}^u_x$ is a $\mathbb{P}(P_x)$-name for a member of ω^ω.

If $u \in \mathbb{P}(P)$ and $x \in P$, then u_x denotes the function $u \restriction P_x$. For $u, v \in \mathbb{P}(P)$ we declare $u \leq v$ when:

1. $\mathrm{dom}(v) \subseteq \mathrm{dom}(u)$.
2. For each $x \in \mathrm{dom}(v)$, $u_x \Vdash_{\mathbb{P}(P_x)} u(x) \leq v(x)$ where $u(x), v(x)$ are interpreted as names for members of $\mathbb{D}$ in the natural way. In other words, $s^v_x \subseteq s^u_x$ and $u_x \Vdash \dot{f}^v_x(n) \leq s^u_x(n)$ for each $n \in |s^u_x| \setminus |s^v_x|$ and $u_x \Vdash (\forall n) \dot{f}^v_x(n) \leq \dot{f}^u_x(n)$.

Hechler's theorem is the following.

Theorem 9.1 (Hechler,[Hec74])**.** If P is a σ-directed partial ordering, and $G \subseteq \mathbb{P}(P)$ is a generic filter, then in $V[G]$ we have that P is isomorphic to a subset of ω^ω which is cofinal in the ordering $\leq^*$.

Analogies of this result were later given for both the meager and the null ideal by Bartoszyński and Kada [BK05] and Burke and Kada [BK04], respectively. So, for example, Bartoszyński and Kada showed that for any σ-directed partial ordering P present in the ground model V there is a generic extension $V[G]$ in which P embeds cofinally into $(\mathcal{M}, \subseteq)$ where $\mathcal{M}$ denotes the meager ideal. Conceptually all three proofs are quite similar. Each uses non-linear iterations, in the case when P is well-founded the underlying order is exactly given by P, and at each step the forcing adds a real that dominates (in the appropriate way) reals definable from those added at lower steps in the ordering of P.

These forcing notions can be used to quite freely manipulate the cofinal structure of $(\omega^\omega, \leq^*)$ (respectively $(\mathcal{M}, \subseteq)$, $(\mathcal{N}, \subseteq)$) and hence to alter the relevant cardinal characteristics of the continuum. If $\mathfrak{b}(P)$ denotes the least size of an unbounded subfamily of P (meaning no member of P lies above all the members of that family) and $\mathfrak{d}(P)$ denotes the least size of a dominating subfamily of P (meaning any member of P lies below some member of that family) then the forcing notions in the paragraph above can be used to give a generic extension in which $\mathfrak{b} = \mathfrak{b}(P)$ and $\mathfrak{d} = \mathfrak{d}(P)$ ($\mathbf{add}(\mathcal{M}) = \mathfrak{b}(P)$ and $\mathbf{cof}(\mathcal{M}) = \mathfrak{d}(P)$, $\mathbf{add}(\mathcal{N}) = \mathfrak{b}(P)$ and $\mathbf{cof}(\mathcal{N}) = \mathfrak{d}(P)$ respectively). For example by taking $\kappa \leq \lambda$ to be cardinals of uncountable cofinality, and taking P to be the poset given by $\kappa \times \lambda$ in the product ordering, the three forcing constructions can be used to give natural proofs of the consistency of $\mathfrak{b} = \kappa$ and $\mathfrak{d} = \lambda$ ($\mathbf{add}(\mathcal{M}) = \kappa$ and $\mathbf{cof}(\mathcal{M}) = \lambda$, $\mathbf{add}(\mathcal{N}) = \kappa$ and $\mathbf{cof}(\mathcal{N}) = \lambda$, respectively).

Since this approach gives two cardinal characteristics distinct values of arbitrarily large size and with an arbitrarily high gap between them, at first glance (or first several hundred) it seems to be a fertile ground for proving consistency results involving the separation of three or more cardinal characteristics simultaneously. Consider for example the splitting number $\mathfrak{s}$. It is not important for our purposes what the splitting number is (though we encourage the reader to consult [Bla10]). What is important is that the linear iteration of Hechler forcing over a ground model satisfying CH will always preserves the splitting number. In fact, any linear iteration of ccc Suslin forcing notions over a model of CH will preserve the splitting number. (Informally, a forcing notion being Suslin simply means that said forcing notion is sufficiently definable for absoluteness arguments in the ZFC context. For an actual definition, see section 3.6 in [BJ95].)

Theorem 9.2 (Judah and Shelah,[IS88])**.** If $V \models CH$ and $\mathbb{P}$ is a well-founded linear iteration of Suslin ccc forcing notions, then $V^{\mathbb{P}} \models \mathfrak{s} = \omega_1$.

It would be nice to prove the same holds for non-linear iterations, i.e., that the splitting number is not increased by forcing with a non-linear iteration of Suslin ccc forcing notions of the form considered by Hechler and his successors. The proof in the linear case relies on the fact that any real added by the full iteration is added by a subiteration that only refers to countably many coordinates. Let us be more precise.

Definition 9.3. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions, and let $i : \mathbb{P} \to \mathbb{Q}$. We say that i is a complete embedding if:

1. If $p_0 \leq p_1$ belong to $\mathbb{P}$, then $i(p_0) \leq i(p_1)$.
2. If p_0 and p_1 belong to $\mathbb{P}$, then p_0 and p_1 are compatible if and only if $i(p_0)$ and $i(p_1)$ are compatible.
3. For every $q \in \mathbb{Q}$ there exists some $p \in \mathbb{P}$ (called the reduction of q) so that whenever $p_0 \in \mathbb{P}$ strengthens p, $i(p_0)$ is compatible with q.

When such an i exists we say $\mathbb{P}$ completely embeds into $\mathbb{Q}$.

We note that an equivalent definition of complete embedding is given by replacing (3) with the condition that whenever $\mathcal{A} \subseteq \mathbb{P}$ is a maximal antichain the image $i[\mathcal{A}]$ is a maximal antichain of $\mathbb{Q}$.

Now suppose $\langle \mathbb{P}_\alpha : \alpha \leq \beta \rangle$ is a linear iteration of Suslin ccc forcing notions given by $\langle \mathbb{Q}_\alpha : \alpha \leq \beta \rangle$. Let $X \subseteq \beta$. For $\alpha \leq \beta$, we define a forcing notion $\mathbb{P}_{X \cap \alpha}$ by recursion on α.

1. $\mathbb{P}_{X \cap 0}$ is the trivial forcing notion.

2. $\mathbb{P}_{X \cap \alpha+1}$ is the two step iteration $\mathbb{P}_{X \cap \alpha} * \mathbb{Q}_\alpha^{\mathbb{P}_{X \cap \alpha}}$ if $\alpha \in X$. Otherwise $\mathbb{P}_{X \cap \alpha+1} = \mathbb{P}_{X \cap \alpha}$.

3. If α is a limit, then $\mathbb{P}_{X \cap \alpha}$ is the direct limit of the $\mathbb{P}_{X \cap \gamma}$ for $\gamma < \alpha$.

The second item makes sense because $\mathbb{Q}_\alpha$ is a Suslin forcing notion, and has a definition that can be applied in any extension of V. A straightforward recursion shows that we may view $\mathbb{P}_X$ as a subset of $\mathbb{P}_\beta$. For this the fact that each step of the iteration has a Suslin definition is very important.

The crucial feature of linear iterations of Suslin ccc forcing notions is the following.

Lemma 9.4 (Judah and Shelah, [IS88])**.** If $\langle P_\alpha : \alpha \leq \beta \rangle$ is a linear iteration of Suslin ccc forcing notions and $X \subseteq \beta$, then $\mathbb{P}_X \lhd \mathbb{P}_\beta$.

For a proof, see [Bre10].

Lemma 9.4 is the key ingredient in proving Theorem 9.2. If the same lemma could be proven for non-linear iterations the consistency of $\mathfrak{s} < \mathfrak{b} < \mathfrak{d}$, $\mathfrak{s} < \mathbf{add}(\mathcal{M}) < \mathbf{cof}(\mathcal{M})$ and $\mathfrak{s} < \mathbf{add}(\mathcal{N}) < \mathbf{cof}(\mathcal{N})$ would follow from the constructions of Hechler and his successors. (The first of these three inequalities is folklore, and an appropriate model can be obtained by adding ω_1 many random reals to a model of $\mathfrak{b} < \mathfrak{d}$).

Unfortunately this nice property does not hold even for (very simple) non-linear iterations of Hechler forcing, and that is what we will now demonstrate.

We take B to be the partial order with underlying set $\{x_0, x_1, x_2, y_0, y_1\}$ where $x_0, x_1 < y_0$ and $x_1, x_2 < y_1$, and no other relations hold. We let $A = B \setminus \{x_1\}$.

Proposition 9.5. The ordering $\mathbb{P}(A)$ does not completely embed into $\mathbb{P}(B)$.

To prove this proposition we present a condition $p \in \mathbb{P}(B)$ with no reduction in $\mathbb{P}(A)$. For $i \leq 2$ let $\dot{g}_i$ be canonical $\mathbb{P}(\{x_i\})$-names for the generic real. (Note $\mathbb{P}(\{x_i\})$ is isomorphic to Hechler forcing). Fix $\langle s_i : i < \omega \rangle$ an enumeration of $\omega^{<\omega}$ without repetitions. We define the $\mathbb{P}(\{x_0, x_1\})$-name $\dot{h}_0$ by declaring $\dot{h}_0(n)$ to name the code of $\dot{g}_1$ up to and including the nth digit where $\dot{g}_0$ and $\dot{g}_1$ agree. That is, $\Vdash s_{\dot{h}_0(n)} = \dot{g}_1 \upharpoonright l$, where l is such that there exist exactly n digits $k < l$ such that $\dot{g}_0(k) = \dot{g}_1(k)$. We define $\dot{h}_2$ the same way, but with $\dot{g}_2$ instead of $\dot{g}_0$. We let $p \in \mathbb{P}(B)$ be given by setting $p(x_0) = p(x_1) = p(x_2)$ all equal to the trivial condition, $p(y_0) = \langle \varnothing, \dot{h}_0 \rangle$, $p(y_2) = \langle \varnothing, \dot{h}_2 \rangle$.

We claim that p has no reduction. Suppose instead for contradiction that there is such a $\bar{p} \in \mathbb{P}(A)$. By strengthening $\bar{p}$ if necessary we may assume without loss of generality that $s^{\bar{p}}_{y_0}$ and $s^{\bar{p}}_{y_2}$ have the same length N. Let M equal $|s^{\bar{p}}_{x_0}|$ and let $i_0 \in \omega$ be greater than both N and M.

We fix a descending chain of conditions $\bar{p}(x_2) \geq r_0 \geq r_1 \ldots \geq r_k \geq \ldots$ belonging to $\mathbb{P}(\{x_2\})$ deciding more and more of $\dot{f}^{\bar{p}}_{y_2}$, so that we have a sequence $\langle n_k : k \in \omega \rangle$ of integers with $r_k \Vdash \dot{f}^{\bar{p}}_{y_2}(k) = n_k$. Let $\phi : \omega \to \omega$ be a function so that any sequence with a digit larger than $\phi(k)$ must have index greater than n_{k+1} in our enumeration of $\omega^{<\omega}$. So if $s_i(j) \geq \phi(k)$ for any j then $i > n_{k+1}$. Let

$p_0 \in \mathbb{P}(\{x_0\})$ be a strengthening of $\bar{p}(x_0)$ with $s^{p_0}_{x_0}(i) \geq \phi(i)$ for all $i \in |s^{p_0}_{x_0}| \setminus |s^{\bar{p}}_{x_0}|$, with $f^{p_0}_{x_0}(i) \geq \phi(i)$ for all $i \geq |s^{p_0}_{x_0}|$ and with the property that there are $m \in \omega$ and $t_0 \in \omega^{<\omega}$ so that $p_0 \Vdash \dot{f}^{\bar{p}}_{y_0}(i_0) = m$ and $p_0 \Vdash \dot{f}^{\bar{p}}_{y_0} \restriction i_0 + 1 = t_0$. Let $l \in \omega$ be larger than $\max\{|s_i| : i \leq m\}$, N and M.

Now, with an eye towards using the fact that $\bar{p}$ is a reduction of p, we define a $\bar{q} \in \mathbb{P}(A)$ with $\bar{q} \leq \bar{p}$. Set $\bar{q}(x_0) = p_0$ and $\bar{q}(x_2) = r_l$. We define $\dot{f}^{\bar{q}}_{y_0} = \dot{f}^{\bar{p}}_{y_0}$ and $\dot{f}^{\bar{q}}_{y_1} = \dot{f}^{\bar{p}}_{y_1}$. We set $s^{\bar{q}}_{y_0} \restriction N = s^{\bar{p}}_{y_0} \restriction N$, and set $s^{\bar{q}}_{y_0}(i) = t_0(i)$ for $i \in |t_0| \setminus N$. (In particular, $s^{\bar{q}}_{y_0}(i_0) = m$). We define $s^{\bar{q}}_{y_2} \restriction N = s^{\bar{p}}_{y_2} \restriction N$, and $s^{\bar{q}}_{y_2}(i) = n_i$ for all $i \in [N, l]$.

That completes the definition of $\bar{q}$. It should be clear that $\bar{q} \leq \bar{p}$. Hence by definition of reduction there is some $q \in \mathbb{P}(B)$ strengthening both $p, \bar{q}$. Since $q \leq \bar{q}$, $s^q_{y_0}(i_0) = m$ and $s^q_{y_2}(i) = n_i$ for each $i \in [N, l]$. Since $q \leq p$ and i_0 and l are greater than N it must be that $q \restriction \{x_0, x_1\} \Vdash \dot{h}_0(i_0) \leq m$ and $q \restriction \{x_1, x_2\} \Vdash (\forall i \in [N, l])\dot{h}_2(i) \leq n_i$. Assume by extending q if necessary that $s^q_{x_0}$ and $s^q_{x_1}$ agree on more than i_0 digits. In particular q determines the value of $\dot{h}_0(i_0)$, which must be less than or equal to m. By definition of l then the length of $s^q_{x_1}$ up to and including the i_0th place where it agrees with $s^q_{x_0}$ must be less than l. Hence there is $k \in [\max(M, N), l)$ with $s^q_{x_0}(k) = s^q_{x_1}(k)$; take k to be the i_0th place of agreement. By definition of p_0 and since $k \geq M$ we must have $s^q_{x_1}(k)$ greater than $\phi(k)$. This violates the fact that $q \restriction \{x_1, x_2\} \Vdash \dot{h}_2(k+1) \leq n_{k+1}$ as by definition of ϕ, any sequence extending $s^q_{x_1} \restriction k+1$ has code strictly greater than n_{k+1}.

Part II

Polychromatic versus monochromatic Ramsey theory

10 Introduction

In this part of the thesis we investigate the relative strengths of monochromatic and polychromatic Ramsey theory in a variety of settings. Recall that in the usual monochromatic Ramsey theory one is given a coloring $\chi : [X]^n \to C$ and seeks a set $Y \subseteq X$ which is *monochromatic* for χ. This means that there is a single color which all elements of $[Y]^n$ receive. In the polychromatic Ramsey theory we instead seek a set $Y \subseteq X$ which is *polychromatic* for χ. This means that each member of $[Y]^n$ receives a different color. (Polychromatic Ramsey theory also goes by the name rainbow Ramsey theory; a polychromatic set might be called a rainbow.)

In order to be able to find large monochromatic or polychromatic sets we must put some restriction on the colorings under consideration. In monochromatic Ramsey theory the appropriate restriction is to insist that the set of colors be finite; in the polychromatic theory we insist that each color gets used a bounded, finite number of times. For $k \in \omega$ we will say that the coloring $\chi : [X]^n \to C$ is k-bounded if $|\chi^{-1}[c]| \leq k$ for each $c \in C$.

Let us now state both Ramsey's theorem and the rainbow Ramsey theorem, as they are emblematic of the two theories which we will be comparing in the following sections. We will state both in their simplest form, with an exponent $i = 2$ for the colorings, using $k = 2$ colors for the monochromatic theorem, and dually using a bound of $k = 2$ for each color in the polychromatic theorem. As is well-known, both theorems hold for any finite choice of i or k.

Theorem 10.1 (Ramsey's Theorem, [Ram])**.** Let X be an infinite set and let $\chi : [X]^2 \to 2$. Then there is an infinite $Y \subseteq X$ which is monochromatic for χ.

The rainbow Ramsey theorem is the polychromatic analogue of Ramsey's theorem.

Theorem 10.2 (The rainbow Ramsey theorem)**.** Let X be an infinite set and let $\chi : [X]^2 \to C$ be a 2-bounded coloring. Then there is an infinite $Y \subseteq X$ which is polychromatic for χ.

The following trick due to Fred Galvin shows that (essentially) whenever positive results in the monochromatic theory hold so too will their polychromatic analogue. In particular the rainbow Ramsey theorem is an immediate consequence of Ramsey's theorem. Suppose we are given $\chi : [X]^n \to C$ a k-bounded coloring. For each $c \in C$ fix an enumeration of $\chi^{-1}[c]$, and form the dual coloring $\chi^* : [X]^n \to k$ by letting $\chi(a) = i$ exactly when a is the ith element in the enumeration of its color class. It is easy to see that $Y \subseteq X$ is polychromatic for χ whenever Y is monochromatic for χ^*.

Several situations in which the polychromatic theory is strictly weaker than the monochromatic theory are already well-known. In the finite setting it has been shown that the classical Ramsey number R_n grows much more quickly than its polychromatic counterpart ([AGH86], [HM04]). In the context of reverse mathematics, Csima and Mileti showed [CM09] that the rainbow Ramsey theorem does not imply Ramsey's theorem over RCA_0 even though RCA_0 is sufficient to prove that Ramsey's theorem implies the rainbow Ramsey theorem. Sierpinksi showed (in ZFC) that there are 2-colorings of $[\omega_1]^2$ with no monochromatic subset of size ω_1, yet Todorčević [Tod83] and Abraham, Cummings and Smyth [ACS07] independently showed that under PFA one may always find polychromatic subsets of size ω_1 for 2-bounded colorings on $[\omega_1]^2$.

Our contributions are the following. In section 11 we view the rainbow Ramsey theorem as a choice principle. We will prove that some choice is needed to prove the rainbow Ramsey theorem, and that there are models of ZF where the rainbow Ramsey theorem holds yet Ramsey's theorem fails. In section 12 we will show that the axiom of choice forbids infinite exponent partition relations for the polychromatic Ramsey theory just as it does for the monochromatic theory. The main result of this section was jointly proved by the author and Anush Tserunyan.

In Section 13 we examine the power of the rainbow Ramsey theorem as a principle in the realm of countable combinatorics. We pursue this examination by introducing rainbow Ramsey ultrafilters, a notion which serves as a polychromatic analogue to the usual Ramsey ultrafilters. The strength of the monochromatic Ramsey theorem as a countable combinatorial principle can be seen in the fact that the "Ramsey-ness" of an ultrafilter entails a host of other special properties. For example, every Ramsey ultrafilter is rapid, every Ramsey ultrafilter is weakly selective, every Ramsey ultrafilter is nowhere dense, and so forth. (All of these concepts will be precisely defined in Section 13.) We will compare the strength of the polychromatic Ramsey theorem by exploring the relationship rainbow Ramsey ultrafilters have with other special classes of ultrafilters.

Every Ramsey ultrafilter is a rainbow Ramsey ultrafilter, but we will show that consistently there are rainbow Ramsey ultrafilters which are not Ramsey. This shows that in the context of ultrafilters and countable combinatorics, the rainbow Ramsey principle is weaker. We will expand on this weakness and show that (consistently) not every rainbow Ramsey ultrafilter filter is rapid and that not every rainbow Ramsey ultrafilter is discrete. However, the concept does have some strength. Every rainbow Ramsey ultrafilter is necessarily nowhere dense and hence by work of Shelah [She98] the existence of rainbow Ramsey ultrafilters is independent of ZFC. We will also show that consistently there may be weakly selective ultrafilters which are not rainbow Ramsey.

We will close this part of this thesis with Section 14, in which we introduce several cardinal characteristics of the continuum that are defined in the spirit of polychromatic Ramsey theory. We will prove the equality of these new characteristics with cardinal characteristics from the classical theory.

As a remark for possible future directions to take this work, we note that with the exception of Section 11 where we view the rainbow Ramsey theorem as a choice principle, for all of the work we do in the polychromatic theory the choice of $k = 2$ as a bound for the colorings is irrelevant. This is not the case for the choice of $i = 2$ as an exponent for our colorings. Indeed, very little seems to be known about infinitary polychromatic Ramsey theory for colorings with exponent greater than 2. Consider for example the open problems about colorings on triples listed in [ACS07] and [AC12]. (These articles are the definitive source on what is known about polychromatic Ramsey theory at the level of uncountable infinity.)

The problem as we see it is that all known direct proofs of the rainbow Ramsey theorem for triples are essentially identical to direct proofs of Ramsey's theorem, and the known positive results in this domain are essentially all an immediate consequence of the corresponding results in the monochromatic theory, obtained via Galvin's trick. Using an argument implicit in [ACS07] and explicitly given in [CM09], one finds a very different sort of proof of the rainbow Ramsey theorem for pairs. (We will include this argument in the beginning of Section 13.) The argument allows for a considerable amount of flexibility, and will enable us to construct polychromatic sets with various largeness properties, properties for which there is no guarantee of finding a monochromatic set.

11 Polychromatic Ramsey theory and the axiom of choice

In this section we investigate polychromatic Ramsey theory in the absence of the axiom of choice. The standard reference for the basics of building models for the failure of choice is the text [Jec73], whose notation and terminology we follow closely.

The proof of Ramsey's theorem as we have stated it here uses the fact that every infinite set has a countably infinite subset, while Galvin's trick (for 2-bounded colorings) requires the existence of a choice function on sets of pairs. Kleinberg [Kle69] proved that some amount of choice is necessary to prove Ramsey's theorem. We begin this section by observing that this is also true of the rainbow Ramsey theorem.

Theorem 11.1. There is a model of ZF in which the rainbow Ramsey theorem does not hold.

Proof. We use the permutation model M referred to in [Jec73] as the second Fraenkel model. While technically speaking permutation models only yield independence results for ZFA (set theory with atoms), the Jech-Sochor theorem (Theorem 6.1 of [Jec73]) can be applied to yield the ZF result.

Recall that the model M is obtained as follows. Let $A = \bigcup P_n$ where $P_n = \{a_n, b_n\}$, and $\mathcal{G}$ is the group of all permutations π of A such that $\pi(\{a_n, b_n\}) = \{a_n, b_n\}$. We obtain M using $\mathcal{G}$ and the ideal $\mathcal{I}$ of finite supports.

Let χ in M be a 2-bounded coloring of $[A]^2$ which gives the pair $\{a_i, b_j\}$ the same color as $\{b_i, a_j\}$, and the pair $\{a_i, a_j\}$ the same color as the pair $\{b_i, b_j\}$. Specifically we may take χ to be defined by $\chi(\{a_i, b_j\}) = \{\{a_i, b_j\}, \{a_j, b_i\}\}$ and $\chi(\{a_i, a_j\}) = \chi(\{b_i, b_j\}) = \{\{a_i, a_j\}, \{b_i, b_j\}\}$. It is not hard to see that χ is invariant under permutations $\pi \in \mathcal{G}$ and hence that χ belongs to M.

There is no infinite set in M which is polychromatic for χ. This is because any infinite $B \subseteq A$ belonging to M must contain infinitely many pairs $\{a_i, b_i\}$. $\square$

Let N be the basic Cohen model of the failure of the axiom of choice, as described in section 5.3 of [Jec73]. In that model the Boolean prime ideal theorem holds, every set can be linearly ordered and every collection of well-ordered sets has a choice function. Blass [Bla77] proved that Ramsey's theorem fails in N. Thus our next result shows that the rainbow Ramsey theorem is considerably weaker than Ramsey's theorem as a choice principle. Our argument is very much inspired by Blass's argument in [Bla77] that the basic Fraenkel model satisfies Ramsey's theorem.

Theorem 11.2. The rainbow Ramsey theorem holds in N.

As in [Jec73] we take $A = \{x_n : n \in \omega\}$ to be the canonical set of Cohen reals in N. We start by showing that the rainbow Ramsey theorem holds on A.

Lemma 11.3. Say $Y \subseteq A$ is infinite, $Y \in N$. If $\chi : [Y]^2 \to C$ is a two-bounded coloring in N, then there is in N an infinite set $X \subseteq Y$ on which χ is polychromatic.

Proof. Let $\dot{\chi}$, $\dot{Y}$ be hereditarily symmetric names for the corresponding objects. There is a finite $e \subseteq \omega$ such that $\text{fix}(e) \subseteq \text{sym}(\dot{\chi})$. We claim that $Y \setminus \{x_n : n \in e\}$ is polychromatic.

Suppose otherwise for contradiction. There are two very similar cases to consider; we will derive a contradiction from the situation where for some distinct i_0, i_1, i_2 not in e we have $\chi(\{x_{i_0}, x_{i_2}\}) = \chi(\{x_{i_1}, x_{i_2}\})$. Let $p \in \mathbb{P}$ forcing $\dot{\chi}$ to be 2-bounded and with

$$p \Vdash \dot{\chi}(\{\dot{x}_{i_0}, \dot{x}_{i_2}\}) = \dot{\chi}(\{\dot{x}_{i_1}, \dot{x}_{i_2}\}).$$

We may assume without loss of generality that $e \cup \{i_0, i_1, i_2\} \subseteq \mathrm{dom}(p)$. Let π be a permutation which fixes each member of e as well as i_0 and i_2, and for which $\pi(i_1) = k$ where k is not in the domain of p. Then $p, \pi(p)$ are compatible, and $\pi(p) \Vdash \dot{\chi}(\{\dot{x}_{i_0}, \dot{x}_{i_2}\}) = \dot{\chi}(\{\dot{x}_k, \dot{x}_{i_2}\})$. But then if q is a common strengthening of p and $\pi(p)$ we have that

$$q \Vdash \dot{\chi}(\{\dot{x}_{i_0}, \dot{x}_{i_2}\}) = \dot{\chi}(\{\dot{x}_k, \dot{x}_{i_2}\}) = \dot{\chi}(\{\dot{x}_{i_1}, \dot{x}_{i_2}\})$$

which violates χ being forced to be two-bounded.

The case where there exist distinct i_0, i_1, i_2, i_3 not in e with $\chi(\{x_{i_0}, x_{i_1}\}) = \chi(\{x_{i_2}, x_{i_3}\})$ is similarly handled. □

To prove that the rainbow Ramsey theorem holds in N we must show that if $X \in N$ is an infinite set and $\chi : [X]^2 \to C$ is a 2-bounded coloring in N, then N contains an infinite polychromatic subset of X. If X happens to be well-orderable in N, there is no difficulty since the usual proof of the rainbow Ramsey theorem will go through. We thus only have to worry about sets in N which cannot be well-ordered. Theorem 11.2 will be proven once we establish the following.

Lemma 11.4. If $B \in N$ is a non-wellorderable set, then N contains a bijection of B with an infinite subset of A.

Proof. We take advantage of the theory of least supports. Our notation will match that of Chapter 5 of [Jec73]. Let $x \in N$, and $E \subseteq A$ with E finite. Write $E = \{x_{i_0}, \dots x_{i_k}\}$. We say that E supports x and write $\Delta(E, x)$ if there is some hereditarily symmetric name $\dot{x}$ with $\dot{x}[G] = x$ such that $\{i_0, \dots i_k\}$ supports $\dot{x}$. The class relation Δ is definable in N.

We first claim that if $B \in N$ and there is some single E such that $\Delta(E, x)$ holds for all $x \in B$, then B can be well-ordered in N. Say $E = \{x_{i_0}, \dots x_{i_k}\}$.

By the axiom of replacement applied in N there is an ordinal α so that for every $x \in B$ there is a hereditarily symmetric name $\dot{x}$ in V_α with support $e = \{i_0, \ldots i_k\}$ and $\dot{x}[G] = x$. Let C be the set

$$\{\tau[G] : \tau \text{ is a hereditarily symmetric name in } V_\alpha \text{ with support } e\}.$$

Then C belongs to N and $B \subseteq C$. Furthermore, C can be well-ordered in N since all the relevant names are supported by e. Thus B can be well-ordered in N.

Recall now that every $x \in N$ has a least support; that is, there is some E_0 with $\Delta(x, E_0)$ and such that $\Delta(x, E)$ implies $E_0 \subseteq E$. Suppose B is some non-wellorderable set with least support E_0, witnessed by the hereditarily symmetric name $\dot{B}$. By the above paragraph there is some $x \in B$ for which $\Delta(x, E_0)$ does not hold. Write the least support of x as $E_1 \cup \{x_k\}$ where x_k does not belong to $E_0 \cup E_1$. Let $\dot{x}$ be a hereditarily symmetric name witness that x has support $E_1 \cup \{x_k\}$. Enumerate $E_0 = \{x_{i_0}, \ldots x_{i_{l_1}}\}$ and $E_1 = \{x_{j_0}, \ldots x_{j_{l_2}}\}$. Let $p \in G$ with

$$p \Vdash \dot{x} \in \dot{B} \text{ and } \dot{E}_1 \cup \{\dot{x}_k\} \text{ is the least support of } \dot{x}.$$

Let $e_0 = \{i_0, \ldots i_{l_1}\}$, let $e_1 = \{j_0, \ldots j_{l_2}\}$ and let $e = e_0 \cup e_1$. Consider the name

$$\sigma = \{\langle\langle\pi(\dot{x}_k), \pi(\dot{x})\rangle, \pi(p)\rangle : \pi \text{ fixes } e\}.$$

Then $f = \sigma[G]$ belongs to N since σ is supported by e. We have four things to prove about f.

First, we check that f is a function. If $\pi_1(\dot{x}_k)[G] = \pi_2(\dot{x}_k)[G]$ then $\pi_1(k) = \pi_2(k)$ and so $\pi_1^{-1}\pi_2$ fixes $e \cup \{k\}$ and thus $\dot{x}$. Then $\pi_1(\dot{x}) = \pi_2(\dot{x})$.

Secondly we note that the range of f is a subset of X. Any member of the range of f has the form $\pi(\dot{x})[G]$ for some π fixing e with $\pi(p) \in G$. Since π fixes

e, $\pi(\dot{B}) = \dot{B}$ and so $\pi(p) \Vdash \pi(\dot{x}) \in \dot{B}$.

Next we observe that the domain of f is an infinite subset of A. This is because x_i belongs to the domain of f whenever $\pi(p) \in G$ for some π mapping k to i. Since G is generic this happens for infinitely many i.

Finally we claim that f is injective. Otherwise we would have $\pi_1(\dot{x})[G] = \pi_2(\dot{x})[G]$ for some π_1, π_2 with $\pi_1(k) = k_1$, $\pi_2(k) = k_2$, $k_1 \neq k_2$ and both $\pi_1(p)$ and $\pi_2(p)$ belonging to G. Let $q \leq \pi_1(p), \pi_2(p)$ with q in G and such that $q \Vdash \pi_1(\dot{x}) = \pi_2(\dot{x})$. Now $\pi_1(\dot{x})$ and $\pi_2(\dot{x})$ have supports $e_1 \cup \{k_1\}$ and $e_1 \cup \{k_2\}$ respectively. Hence by [Jec73] Lemma 5.23 there is some $\dot{z}$ with support e_1 and $q \Vdash \pi_1(\dot{x}) = \pi_2(\dot{x}) = \dot{z}$. Since q extends $\pi_1(p)$ we have

$$q \Vdash \dot{E}_1 \cup \{\dot{x}_{k_1}\} \text{ is the least support of } \pi_1(\dot{x}) \ .$$

On the other hand, q belongs to G and $\pi_1(\dot{x})[G] = \dot{z}[G]$ and $\dot{z}$ has support strictly contained in $e_1 \cup \{k_1\}$. Contradiction. $\square$

As we mentioned above, Galvin's trick also requires a small amount of choice. We do not know of a model of ZF where Ramsey's theorem holds but the rainbow Ramsey theorem fails.

The results in this section focused on 2-bounded colorings, and we stated the rainbow Ramsey theorem for 2-bounded colorings alone. For $k \in \omega$ let RR_k denote the rainbow Ramsey theorem for k-bounded colorings. For $j < k$ we clearly have that RR_k implies RR_j. We do not know if the reverse implication holds in ZF alone. The following proposition shows that it follows from the axiom of choice for collections of finite sets. In particular Theorem 11.2 implies for each k that RR_k holds in Cohen's model, and thus none of these is sufficient to yield Ramsey's theorem.

Proposition 11.5. Fix $k \in \omega$, and assume that there exists a choice function for any family of sets of cardinality at most k. Then RR_2 implies RR_k.

Proof. Let $\chi : [X]^2 \to C$ be a k-bounded coloring. Using the choice assumption, for each $c \in C$ enumerate $\chi^{-1}[c] = \{p_0^c, \dots p_{k-1}^c\}$ (possibly with repetitions.) Using this enumeration for every possible pair $i, j < k$ we can define a coloring $\chi_{i,j}$ which gives each p_i^c and p_j^c the same color and gives each other member of $[X]^2$ its own color. Each $\chi_{i,j}$ is 2-bounded and so by iteratively applying RR_2 finitely many times we arrive at an infinite set which is polychromatic for all of the $\chi_{i,j}$. Any such set is polychromatic for χ. $\square$

Using the transfer theorem of Pincus [Pin72], to separate RR_j and RR_k in ZF it would be enough to find a permutation model of ZFA separating the two. There is a natural class of models one is drawn to when considering this question. Fix M a permutation model with the following properties. The collection of atoms is of the form $A = \bigcup P_n$ where each P_n is a finite set of size at least two. The group of permutations $\mathcal{G}$ acts transitively on each P_n, and so that $\pi(P_n) = P_n$, and the ideal $\mathcal{I}$ is the ideal of finite supports.

These are the models used to separate AC_j and AC_k, the axiom of choice for sets of size j and k respectively, for appropriate selection of j and k (see Theorem 7.15 of [Jec73].) When all the P_n have size 2, we saw in Theorem 11.1 that RR_2 fails. Even better, when all the P_n have size strictly greater than j, an argument similar to and simpler than the proof of Lemma 11.3 shows that RR_j holds when restricted to colorings on subset of A. Nonetheless RR_3 fails in *any* such M.

To see this, consider the coloring χ on $[A^2]^2$ given by $\chi(s,t) = s \cup t$. Notice that this coloring is 3-bounded and belongs to M. Let $X = \bigcup_{n<\omega}(P_n \times P_{n+1})$. The set $X \subseteq [A]^2$ belongs to M since the sequence $\langle P_n : n \in \omega \rangle$ belongs to

M. We claim that X has no infinite subset which is polychromatic for χ. This is because any infinite subset of X must contain $P_n \times P_{n+1}$ for infinitely many distinct n.

We will use a similar coloring in Section 13 when we construct a weakly selective ultrafilter which is not rainbow Ramsey.

12 Polychromatic Ramsey theory and infinite exponent partition relations

A result of Erdös and Rado says that under the axiom of choice Ramsey's theorem fails for infinite exponent partitions (Proposition 7.1 of [Kan03].) Specifically, for any infinite cardinal κ there is a 2-coloring of the countable subsets of κ so that no infinite subset of κ has all of its countable subsets receiving the same color. In this section we show that the axiom of choice also implies the failure of the rainbow Ramsey theorem for infinite exponent partitions. Using Galvin's trick we may view our result as a strengthening of the Erdös and Rado result. The work in this section is joint with Anush Tserunyan.

Theorem 12.1. Let κ be an infinite cardinal. There is a 2-bounded coloring $\chi : [\kappa]^\omega \to C$ so that whenever $X \in [\kappa]^\omega$ there are distinct $a, b \in [X]^\omega$ with $\chi(a) = \chi(b)$.

To prove the theorem it is enough for us to establish the following.

Lemma 12.2. Let κ be an infinite cardinal. There exists an injective map $f : [\kappa]^\omega \to [\kappa]^\omega$ so that for each x we have that $f(x)$ is a proper subset of x.

Given the lemma, Theorem 12.1 is proven as follows. Let f be as in Lemma 12.2. We define f_0 and f_1 two injections from $[\kappa]^\omega$ into $[\kappa]^\omega$ with disjoint ranges so that $f_0(x)$ and $f_1(x)$ are both proper subsets of x for each x. This can be done by looking at the orbits of f; that is, each collection $\{f^n(x) : n \in \mathbb{Z}\}$. Because f is injective, the orbits partition the range of f. Select an enumeration of each, and take f_0 and f_1 so that $f_0(x)$ is an even member of the orbit of x while $f_1(x)$ is an odd member of the orbit of x. With f_0 and f_1 defined, we may define χ by setting $\chi(f_0(x)) = \chi(f_1(x))$, and letting χ take distinct values on the other members of $[\kappa]^\omega$. Then χ is as desired.

Let us remark that since there are models of ZF where Ramsey's theorem holds for infinite exponent partition relations, by Galvin's trick there are models of ZF where the rainbow Ramsey theorem holds for infinite exponent partition relations. Thus this argument also shows that the axiom of choice is required to prove the existence of such injections f_0 and f_1.

We now make the observation that the lemma holds if $\kappa = \omega$.

Proposition 12.3. There exists an injection $f : [\omega]^\omega \to [\omega]^\omega$ so that for each x we have that $f(x)$ is a proper subset of x.

Proof. Fix an enumeration of $[\omega]^\omega$ in ordertype 2^ω. We define f by transfinite recursion. At stage α, we define $f(x)$ where x is the αth member of $[\omega]^\omega$. Since x has 2^ω many proper subsets and since there are strictly less than 2^ω values of f which have been decided we may select a value for $f(x)$ not equal to any earlier decided value. □

Proof of Lemma 12.2. Fix $A = \{a_\alpha : \alpha < \lambda\}$ a maximal almost disjoint family of members of $[\kappa]^\omega$. That is, $a_\alpha \cap a_\beta$ is finite for distinct α and β and for any $x \in [\kappa]^\omega$ there is some α such that $x \cap a_\alpha$ is infinite.

We construct f as follows. For each $\alpha < \lambda$ let $f_\alpha : [a_\alpha]^\omega \to [a_\alpha]^\omega$ be as in Proposition 12.3. Given x in $[\kappa]^\omega$, take α least for which $x \cap a_\alpha$ is infinite and set

$f(x)$ equal to $f_\alpha(x \cap a_\alpha) \cup (x \setminus a_\alpha)$.

We claim that f is injective. Fix $x_0, x_1 \in [\kappa]^\omega$ with α_0 and α_1 least so that $x_0 \cap a_{\alpha_0}$ and $x_1 \cap a_{\alpha_1}$ are infinite. Assume without loss of generality that $\alpha_0 \leq \alpha_1$. Suppose $f(x_0) = f(x_1)$ so that

$$f_{\alpha_0}(x_0 \cap a_{\alpha_0}) \cup (x_0 \setminus a_{\alpha_0}) = f_{\alpha_1}(x_1 \cap a_{\alpha_1}) \cup (x_1 \setminus a_{\alpha_1}).$$

Note that $f_{\alpha_0}(x_0 \cap a_{\alpha_0})$ is an infinite subset of a_{α_0} and $f_{\alpha_1}(x_1 \cap a_{\alpha_1})$ is an infinite subset of a_{α_1}. Consider the following two possibilities. If $f_{\alpha_0}(x_0 \cap a_{\alpha_0})$ has infinite intersection with $f_{\alpha_1}(x_1 \cap a_{\alpha_1})$, then $a_{\alpha_0} \cap a_{\alpha_1}$ is infinite so that $\alpha_0 = \alpha_1$. The other possibility is that $f_{\alpha_0}(x_0 \cap a_{\alpha_0}) \cap x_1 \setminus a_{\alpha_1}$ is infinite in which case $a_{\alpha_0} \cap x_1$ is infinite so that α_0 is equal to α_1 by minimality of α_1. In either case we can conclude that α_0 and α_1 are equal to the same ordinal α.

Thus

$$f_\alpha(x_0 \cap a_\alpha) \cup (x_0 \setminus a_\alpha) = f_\alpha(x_1 \cap a_\alpha) \cup (x_1 \setminus a_\alpha).$$

Then $f_\alpha(x_0 \cap a_\alpha) = f_\alpha(x_1 \cap a_\alpha)$ so we have $x_0 \cap a_\alpha = x_1 \cap a_\alpha$ by injectivity of f_α. Because $x_0 \setminus a_\alpha = x_1 \setminus a_\alpha$ also holds we get $x_0 = x_1$ as desired. □

13 Polychromatic Ramsey theory and ultrafilters on ω

We turn our attention now to monochromatic and polychromatic Ramsey theory in the context of ultrafilters on ω. The following objects are central in the study of such ultrafilters.

Definition 13.1. A nonprincipal ultrafilter $\mathcal{U}$ is *Ramsey* if for every coloring $\chi : [\omega]^2 \to 2$ there is an $A \subseteq \omega$ belonging to $\mathcal{U}$ which is monochromatic for χ.

Ramsey ultrafilters are often called *selective* ultrafilters in connection with the following characterization. An ultrafilter $\mathcal{U}$ is Ramsey exactly when given any partition of ω into countably many pieces $\bigcup_{n<\omega} A_n$ with each $A_n \notin \mathcal{U}$ we may find $B \in \mathcal{U}$ such that $|A_n \cap B| \leq 1$ for each $n \in \omega$. Another salient characterization of Ramsey ultrafilters is that they are precisely nonprincipal ultrafilters which are minimal in the Rudin-Keisler ordering.

The existence of Ramsey ultrafilters is not provable in ZFC. This was first established by Kunen [Kun76]. Martin's Axiom (MA) is sufficient to prove their existence; indeed MA is generally the context in which relationships between various classes of ultrafilters are studied. Such investigations have been pursued by Baumgartner [Bau95], Brendle [Bre99] and others.

As an analogue to the Ramsey theoretic characterization of Ramsey ultrafilters, we present the following definition.

Definition 13.2. A nonprincipal ultrafilter $\mathcal{U}$ is *rainbow Ramsey* if for every 2-bounded coloring $\chi : [\omega]^2 \to \omega$ there is an $A \subseteq \omega$ belonging to $\mathcal{U}$ which is polychromatic for χ.

We start with the simple observation that although our definition of rainbow Ramsey ultrafilters only guarantees that they contain polychromatic sets for the 2-bounded colorings, we automatically get polychromatic sets for colorings of all other possible finite bounds.

Proposition 13.3. If $\mathcal{U}$ is a rainbow Ramsey ultrafilter and $\chi : [\omega]^2 \to \omega$ is a k-bounded coloring then there is an $A \subseteq \omega$ belonging to $\mathcal{U}$ which is polychromatic for χ.

Proof. The proof of Proposition 11.5 goes through in this context. □

By Galvin's trick every Ramsey ultrafilter is a rainbow Ramsey ultrafilter. Assuming MA we will prove that the converse does not hold. We will also compare the notion of rainbow Ramsey utrafilter to other notable classes of special ultrafilters on ω. Let us introduce the special ultrafilters we will consider.

Definition 13.4. A nonprincipal ultrafilter $\mathcal{U}$ on ω is *weakly selective* if whenever ω is partitioned into countably many pieces $\bigcup_{n<\omega} A_n$ with each $A_n \notin \mathcal{U}$ we may find $B \in \mathcal{U}$ such that $A_n \cap B$ is finite for each $n \in \omega$.

Weakly selective ultrafilters are also often referred to as P-points in connection with the fact that an ultrafilter is weakly selective exactly when for every countable family $\{B_n : n \in \omega\}$ of members of $\mathcal{U}$ there is some $B \in \mathcal{U}$ such that $B \subseteq^* B_n$ for each $n \in \omega$. (Here $\subseteq^*$ is the preorder of almost containment; $A \subseteq^* B$ means $A \setminus B$ is finite.)

Definition 13.5. A nonprincipal ultrafilter $\mathcal{U}$ on ω is rapid if for every $f : \omega \to \omega$ there is some $A \in \mathcal{U}$ such that $f \leq^* e_A$. (Here e_A is the function enumerating A in increasing order, and $\leq^*$ is the preorder of eventual domination.)

The next definition scheme is due to Baumgartner [Bau95]. In this thesis ideals will always contain all possible finite sets.

Definition 13.6. Let $\mathcal{I}$ be an ideal on some set X. We say that a nonprincipal ultrafilter on ω is an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there is some $A \in \mathcal{U}$ with $f(A) \in \mathcal{I}$.

For example we could take $\mathcal{I}$ to be the nowhere dense subsets of $\mathbb{Q}$, or we could take $\mathcal{I}$ to be the discrete subsets of $\mathbb{Q}$; in these cases we have the notion of a nowhere dense ultrafilter and the notion of a discrete ultrafilter, respectively. Every Ramsey ultrafilter is rapid, and every Ramsey ultrafilter is weakly selective. Every weakly selective ultrafilter is discrete, and every discrete ultrafilter is nowhere dense.

We connect rainbow Ramsey ultrafilters to these classes as follows. Every rainbow Ramsey ultrafilter is nowhere dense, but (assuming MA) there exist rainbow Ramsey ultrafilters which are not discrete as well as rainbow Ramsey ultrafilters which are not rapid. Shelah proved [She98] that there are models of ZFC with no nowhere dense ultrafilters; this implication shows that the same is true of rainbow Ramsey ultrafilters. We will also show that MA implies the existence of a weakly selective ultrafilter which is not rainbow Ramsey. Together these results rule out the possibility of the concept of rainbow Ramsey ultrafilter being equivalent to any previously studied special class of ultrafilter.

The arguments we give separating classes of ultrafilters all use MA, an axiom which guarantees the existence of Ramsey ultrafilters and hence of all the special types of ultrafilters considered here. We do not know whether the existence of a rainbow Ramsey ultrafilter implies the existence of a Ramsey ultrafilter. We do not even know whether the existence of a rainbow Ramsey ultrafilter implies the existence of a P-point. A good place to start on this question would be to determine the status of the existence of rainbow Ramsey ultrafilters in Shelah's model with no P-points [She98].

In our constructions of ultrafilters which are rainbow Ramsey but lack some other property we will be interested in building polychromatic sets which are large in some sense. Let us describe some tools that will help us accomplish this. Fix a coloring $\chi : [\omega]^2 \to \omega$. We assume throughout that χ is 2-bounded. For $a, b \in \omega$ we will usually write $\chi(a, b)$ for $\chi(\{a, b\})$. If $X \subseteq \omega$ is finite and $a \in \omega$ we write $X < a$ to mean $\max(X) < a$. Similarly we will write $X < Z$ to mean $\max(X) < \min(Z)$.

Definition 13.7. A set $A \subseteq \omega$ is *normal* (with respect to χ) if whenever $a_0 < a_1$ and $b_0 < b_1$ are elements of A with $\chi(a_0, a_1) = \chi(b_0, b_1)$ then we necessarily have

$a_1 = b_1$. We say that χ is normal if ω is normal with respect to χ.

Generally our constructions of large polychromatic sets will entail first building large normal sets.

Suppose X is a given finite polychromatic set. We define $E(X)$ by setting

$$E(X) = \{a > X : X \cup a \text{ is polychromatic}\}.$$

We will sometimes write $E(X \cup x)$ as shorthand for $E(X \cup \{x\})$.

Proposition 13.8. Suppose A is normal. If $X \subseteq A$ is polychromatic, $|X| \leq n$ and $a_0 < \ldots < a_n$ belong to $A \cap E(X)$ then

$$A \cap E(X) \subseteq^* A \cap (E(X \cup a_0) \cup \ldots \cup E(X \cup a_n)).$$

Indeed, every member of $A \cap E(X)$ greater than a_n belongs to $A \cap \bigcup_{i \leq n} E(X \cup a_i)$.

Proof. Enumerate $X = \{x_0, \ldots x_{n-1}\}$. Suppose for contradiction that $z > a_n$ with $z \in A \cap E(X)$, but z does not belong to any $A \cap E(X \cup a_i)$. For each such i, since $X \cup \{z\}$ is polychromatic, $X \cup \{a_i\}$ is polychromatic, $X \cup \{a_i, z\}$ is not polychromatic and A is normal there must be some j_i such that $\chi(a_i, z) = \chi(x_{j_i}, z)$. There are $n+1$ possible i while only n possible j_i. By the pigeonhole principle there is some j and $i_0 < i_1$ such that $j = j_{i_0} = j_{i_1}$. Then $\chi(x_j, z) = \chi(a_{i_0}, z) = \chi(a_{i_1}, z)$. But that contradicts χ being 2-bounded. □

Lemma 13.9. Suppose $A \subseteq \omega$ is normal. Let I be an ideal on ω. Let $X \subseteq A$ be polychromatic with $|X| \leq n$. Then if $E(X) \cap A \notin I$,

$$\{a \in A \cap E(X) : A \cap E(X \cup a) \in I\}$$

has size at most n.

Proof. Suppose for contradiction that $a_0, \dots a_n$ are distinct members of $A \cap E(X)$ with each $A \cap E(X \cup a_i)$ belonging to I. Then by Proposition 13.8 we have

$$A \cap E(X) \subseteq^* A \cap (E(X \cup a_0) \cup \dots \cup E(X \cup a_n)).$$

But then $A \cap E(X)$ belongs to I. Contradiction. □

For the sake of completeness and to serve as a simple paradigm of how one can apply Lemma 13.9 to build large polychromatic sets, we use it to prove the rainbow Ramsey theorem (for pairs) directly. This is essentially the same proof as in [CM09].

Proof of Theorem 10.2. Fix $\chi : [\omega]^2 \to \omega$ a 2-bounded coloring. We seek an infinite polychromatic $A \subseteq \omega$.

First we find $B \subseteq \omega$ on which χ is normal. We construct B by induction, enumerated $\{b_n : n \in \omega\}$. Suppose we have thus far constructed a finite set $\{b_0, \dots b_{n-1}\}$ on which χ is normal. Consider the possible colors $\chi(b_i, b_j)$ with $i, j < n$. There are finitely many such colors, and each is used at most twice by χ. Thus by taking b_n to be some sufficiently large member of ω, we have that χ is normal on the set $\{b_0 \dots b_n\}$.

By restricting our attention to B, we may assume without loss of generality that χ is normal. We construct A by induction, enumerated $\{x_n : n \in \omega\}$. Suppose we have thus far constructed a finite polychromatic set $X = \{x_0, \dots x_{n-1}\}$, with the property that $E(X)$ is infinite. Taking I to be the ideal of finite sets, by Lemma 13.9 we find an $x_n > X$ such that $\{x_0, \dots x_n\}$ is polychromatic and $E(X \cup x_n)$ is infinite. This completes the induction. □

13.1 An ultrafilter which is rainbow Ramsey and not rapid

In this subsection we use MA to construct a rainbow Ramsey ultrafilter $\mathcal{U}$ which is not rapid. To accomplish this we must be able to build polychromatic sets which are large in the sense that they have enumerating functions which do not grow too fast. That is, we define a function $f : \omega \to \omega$ and construct $\mathcal{U}$ so that for every $A \in \mathcal{U}$ we have $f \not\leq^* e_A$. Such constructions are not possible in the monochromatic theory; given a function one can always define a 2-coloring so that any monochromatic set dominates that function.

Proposition 13.10. There is a function $\mathrm{Nrm} : \omega^2 \to \omega$ such that the following holds. Suppose χ is a 2-bounded coloring and X is normal with $|X| = p$, and suppose $Z \subseteq \omega$ with $Z > X$ and $\mathrm{Nrm}(p, n) \leq |Z|$. There is $Y \subseteq Z$ with $|Y| \geq n$ such that χ is normal on $X \cup Y$.

Proof. For each $a < b \in X$ there is at most one other pair $c < d$ with $\chi(a, b) = \chi(c, d)$ and hence some z from the first $\binom{p}{2} + 1$ elements of Z gives $X \cup z$ is normal. By iterating this observation we see that we may define Nrm recursively; $\mathrm{Nrm}(p, n+1) = \mathrm{Nrm}(p, n) + \binom{p+n}{2} + 1$. □

Proposition 13.11. There is a function $h : \omega^2 \to \omega$ with the following properties. Let χ be a 2-bounded coloring and $A \subseteq \omega$ a normal set. Let $\mathcal{I}$ be any ideal on ω, $X \subseteq A$ a polychromatic set with $|X| \leq p$ and $A \cap E(X) \notin \mathcal{I}$, and $Z \subseteq A \cap E(X)$ with $h(p, n) \leq |Z|$. There is $Y \subseteq Z$ with $|Y| \geq n$ such that $X \cup Y$ is polychromatic and $A \cap E(X \cup Y) \notin \mathcal{I}$.

Proof. As in Proposition 13.10 this follows by iterating an observation for extending by one point. This time the observation is the claim that given $N_0 \in \omega$, if $N_1 \geq N_0(p+1) + 2p + 2$ and Z has size at least N_1 then for some z equal to one of the first $2p+1$ members of Z we have $A \cap E(X \cup z) \notin \mathcal{I}$ and $|Z \cap E(X \cup z)| \geq N_0$.

Let us verify this claim. By Lemma 13.9 there are at most p members z of $A \cap E(X)$ with $A \cap E(X \cup z)$ belonging to I. Remove these from Z and call the resulting set Z_0. Then Z_0 has size at least $N_0(p+1) + p + 2$ and it is enough to show that one of the first $p+1$ members of Z_0 works. Suppose otherwise for contradiction: let $a_0, \ldots a_p$ be the first $p+1$ members of Z_0 and assume that each $|Z \cap E(X \cup a_i)| \leq N_0$. Then

$$|E(X \cup a_0) \cap Z_0 \cup \ldots \cup E(X \cup a_p) \cap Z_0| \leq N_0(p+1).$$

But now we can select a z in $Z_0 \subseteq E(X)$ above a_p and not belonging to any of the $E(X \cup a_i)$ and that violates Proposition 13.8. $\square$

Now define a function g as follows. For each $n \in \omega$, set $g(1, n) = n + 1$. Then recursively define g so that

$$g(k+1, n) > h(k, g(k, n)), \mathrm{Nrm}(k, g(k, n)), 2 \cdot g(k, n).$$

We let $f : \omega \to \omega$ be a function eventually dominating (for each fixed k and l) the map that sends n to $g(k, n) + l$. We say a set is f-rapid if $f \leq^* e_A$.

We build $\mathcal{U}$ by constructing a filter $\mathcal{F}$ which consists only of sets which are not f-rapid and which contains a polychromatic set for every 2-bounded coloring; we will then want to extend $\mathcal{F}$ to an ultrafilter consisting only of sets which are not f-rapid. To do this it is enough to have $\mathcal{F} \cap \mathcal{I} = \varnothing$ where $\mathcal{I}$ is an ideal on ω containing all the f-rapid sets.

Proposition 13.12. Let $\mathcal{I}$ consist of all sets $A \subseteq \omega$ for which there exists $l, k, N \in \omega$ such that

$$(\forall n \geq N)|[l, f(n)) \cap A| < g(k, n).$$

Then $\mathcal{I}$ is an ideal that contains every f-rapid set.

Proof. First, if A is f-rapid then since $n+1 \leq g(1,n)$ taking $l=0$ and $k=1$ witnesses $A \in \mathcal{I}$. Clearly $\mathcal{I}$ is closed under subsets. To see that $\omega \notin \mathcal{I}$, just notice that $|[l, f(n)) \cap \omega| = f(n) - l$ and we defined f so that $g(k,n) + l \leq f(n)$ for sufficiently large n. Closure of $\mathcal{I}$ under unions follows from the fact that $2g(k,n) < g(k+1,n)$ for every n and k. □

We now build a filter $\mathcal{F}$ disjoint from $\mathcal{I}$ and containing a polychromatic set for each 2-bounded coloring. We generate $\mathcal{F}$ from a *tower* of sets not in $\mathcal{I}$. Recall that a tower is a sequence $\langle T_i : i < \lambda \rangle$ of subsets of ω with $i < j$ implying that $T_j \subseteq^* T_i$.

Proposition 13.13. If $A \notin \mathcal{I}$ and $\chi : [\omega]^2 \to \omega$ is 2-bounded then there is a normal $B \subseteq A$ with $B \notin \mathcal{I}$.

Proof. It is enough to show that given a finite $X \subseteq \omega$ on which χ is normal and given N, k, l we may find $n \geq N$ and $Y \subseteq A$ with $X \cup Y$ normal and

$$|[l, f(n)) \cap Y| \geq g(k,n)$$

for then B may be constructed by a straightforward induction. Let $p = |X|$. Because $A \notin \mathcal{I}$ there is some $n \geq N$ such that $|[l, f(n)) \cap A| \geq g(\max\{p,k\} + 1, n) > \mathrm{Nrm}(p, g(k,n))$. By Proposition 13.10 there is $Y \subseteq [l, f(n)) \cap A$ with $|Y| \geq g(k,n)$ and $X \cup Y$ is normal. □

Proposition 13.14. If χ a 2-bounded coloring and $A \notin \mathcal{I}$ is a normal set then there is a set $B \subseteq A$ which is polychromatic and so that $B \notin \mathcal{I}$.

Proof. This is just like Proposition 13.13 but appealing to Proposition 13.11 instead of Proposition 13.10. □

Lemma 13.15. Assume MA. If $\langle T_i : i < \lambda \rangle$ is a tower of sets with $\lambda < 2^{\aleph_0}$ and each $T_i \notin \mathcal{I}$, there is $T_\lambda \notin \mathcal{I}$ such that $T_\lambda \subseteq^* T_i$ for all $i < \lambda$.

Proof. The usual forcing notion $\mathbb{P}$ to extend a tower applies; conditions are pairs $\langle s, A\rangle$ where s is a finite subset of ω and A belongs to the tower. We order by setting $\langle s', A'\rangle \leq \langle s, A\rangle$ exactly when s' is an end extension of s, $A' \setminus \max(s') \subseteq A$ and $s' \setminus s \subseteq A$. The dense sets come in two flavors; first for each T_i consider the dense set C_i of conditions $\langle s, A\rangle$ for which $A \setminus \max(s)$ is a subset of T_i. Second, for each $l, k, N \in \omega$ take the dense set $D_{l,k,N}$ of conditions $\langle s, A\rangle$ for which there is some $n \geq N$ with $|s \cap [l, f(n))| \geq g(k, n)$. If G is a filter on $\mathbb{P}$ intersecting each C_i and $D_{l,k,N}$ then T_λ may be obtained as the union of all the s such that some $\langle s, A\rangle$ belongs to G. $\square$

Putting all the ingredients together to construct a $\mathcal{U}$ which is rainbow Ramsey and not rapid is now routine. We enumerate all 2-bounded colorings $\langle \chi_i : i < 2^\omega\rangle$ and construct a tower of sets $\langle T_i : i < 2^\omega\rangle$ not in $\mathcal{I}$ such that each T_i is polychromatic for χ_i. Given an initial segment of such a tower, Lemma 13.15 applies to extend it by a single set A and then Lemma 13.13 followed by Lemma 13.14 apply to refine this extension to a polychromatic set not in $\mathcal{I}$. With the tower constructed, the filter it generates consists of sets not in $\mathcal{I}$, and this filter can be extended to an ultrafilter $\mathcal{U}$ which is disjoint from $\mathcal{I}$ and is thus not rapid.

13.2 A weakly selective ultrafilter which is not rainbow Ramsey

In this subsection we use MA to construct a weakly selective ultrafilter $\mathcal{U}$ which is not rainbow Ramsey. We will define $\mathcal{U}$ to be an ultrafilter on $[\omega]^2$ rather than on ω. We think of $[\omega]^2$ as the set of edges in the complete graph whose set of vertices is ω. Define a coloring χ by setting $\chi(\{a, b\}, \{c, d\}) = \{a, b, c, d\}$. Notice that χ is 3-bounded.

Let $\mathcal{I}_{\text{Ramsey}}$ be the collection of $X \subseteq [\omega]^2$ for which there exists an N so that $N \leq |A|$ implies $[A]^2 \not\subseteq X$.

Proposition 13.16. The set $\mathcal{I}_{\text{Ramsey}}$ is an ideal containing every set which is polychromatic for χ.

Proof. Since $\chi(\{a,b\},\{c,d\}) = \chi(\{a,d\},\{b,c\})$ no polychromatic set can contain an $[A]^2$ where A has size at least 4.

It is clear that $\mathcal{I}_{\text{Ramsey}}$ is closed under subsets. That $\mathcal{I}_{\text{Ramsey}}$ is closed under finite unions follows from the finite monochromatic Ramsey theorem. If $X = Y_0 \cup \ldots \cup Y_n$ contains arbitrarily large complete graphs, then so does Y_i for some i. □

Let $\mathcal{P} = \{P_n : n \in \omega\}$ be a partition of $[\omega]^2$. We say that $X \subseteq [\omega]^2$ is a weak $\mathcal{P}$-selector if $X \cap P_n$ is finite for each n. To construct our ultrafilter it suffices to build a filter $\mathcal{F}$ disjoint from $\mathcal{I}_{\text{Ramsey}}$ which for each $\mathcal{P}$ either contains some P_n or contains a weak $\mathcal{P}$-selector for each $\mathcal{P}$.

Lemma 13.17. Assume MA. If $\langle T_i : i < \lambda \rangle$ is a tower of subsets of $[\omega]^2$ with $\lambda < 2^{\aleph_0}$ and each $T_i \notin \mathcal{I}_{\text{Ramsey}}$, there is $T_\lambda \notin \mathcal{I}_{\text{Ramsey}}$ such that $T_\lambda \subseteq^* T_i$ for all $i < \lambda$.

Proof. As in Lemma 13.15 we apply MA to the usual forcing to extend a tower. Our dense sets again come in two flavors; C_i consists of those $\langle s, X \rangle$ with $A \setminus \max(s)$ a subset of T_i. For each N we let D_N be those $\langle s, X \rangle$ with some $[A]^2 \subseteq s$ with $N \leq |A|$. Given $G \subseteq \mathbb{P}$ a generic filter intersecting each C_i and D_N we may define T_λ as the union of all the s such that there exists X with $\langle s, X \rangle \in G$. □

Lemma 13.18. Suppose $X \notin \mathcal{I}_{\text{Ramsey}}$ and $\mathcal{P} = \{P_n : n \in \omega\}$ is a partition of $[\omega]^2$. Either some $X \cap P_n \notin \mathcal{I}_{\text{Ramsey}}$ or there is $Y \subseteq X$ a weak P-selector with $Y \notin \mathcal{I}_{\text{Ramsey}}$.

Proof. Assume that each $X \cap P_n \in \mathcal{I}_{\text{Ramsey}}$. Then Y may be built in ω stages, adding finitely many points at a time. We start with $Y_0 = \varnothing$. At stage N we have Y_N, with $P_0, \ldots P_n$ all the members of $\mathcal{P}$ with which Y_N has nonempty intersection, and we form $Y_{N+1} = Y_N \cup [B]^2$ where $N \leq |B|$, $[B]^2 \subseteq X$ and $[B]^2 \cap P_0 \cup \ldots \cup P_n = \varnothing$.

We find B as follows. Let $Q = P_0 \cup \ldots \cup P_n$. Then $Q \cap X \in \mathcal{I}_{\text{Ramsey}}$ so we may take M so that $Q \cap X$ contains no $[C]^2$ where $|M| \leq C$. Now define a 2-coloring with domain X by giving elements of Q color 0 and all other members of X color 1. By the monochromatic Ramsey's theorem there is $B \subseteq \omega$ with $M \leq |B|$ and $[B]^2$ monochromatic; then $[B]^2 \subseteq X \setminus Q$ as desired. □

Now a routine recursion of length continuum will yield an ultrafilter $\mathcal{U}$ which is weakly selective but not rainbow Ramsey. Enumerate all partitions of $[\omega]^2$ as $\langle \mathcal{P}^i : i < 2^\omega \rangle$ and construct a tower of sets $\langle T_i : i < 2^\omega \rangle$ not in $\mathcal{I}_{\text{Ramsey}}$ such that each T_i is either a weak $\mathcal{P}$-selector or a subset of some P^i_n. Given an initial segment of such a tower, Lemma 13.17 applies to extend it by a single set $Y \notin \mathcal{I}_{\text{Ramsey}}$ and then Lemma 13.18 applies to refine Y to a $Z \subseteq Y$ which is either a weak $\mathcal{P}$-selector or a subset of some P^i_n. With the tower constructed, the filter it generates consists of sets not in $\mathcal{I}_{\text{Ramsey}}$, and this filter can be extended to a weakly selective ultrafilter $\mathcal{U}$ which is disjoint from $\mathcal{I}_{\text{Ramsey}}$ and is thus not rainbow Ramsey.

13.3 Rainbow Ramsey ultrafilters are nowhere dense.

In this subsection we show that every rainbow Ramsey ultrafilter is nowhere dense.

Lemma 13.19. Suppose $S \subseteq \mathbb{R}$ is countable. Then there is a 2-bounded coloring

$\chi : [S]^2 \to \omega$ so that any set A which is polychromatic for χ is nowhere dense as a subset of $\mathbb{R}$.

Proof. Let $\mathcal{C}$ be the collection of all open intervals in $\mathbb{R}$ with rational endpoints. Fix $\prec$ an ordering of S in ordertype ω. When we specify $\chi(q, r)$ we insist that $q \prec r$. Enumerate $\mathcal{C} = \{c_n : n \in \omega\}$.

Before defining χ we first define sequences $\{S_n : n \in \omega\}$ and $\{b_n^{p,q} : n \in \omega, \{p, q\} \in [S_n]^2\}$ such that

1. $S \setminus S_n$ is finite.
2. For each $n \in \omega$ the set $\{b_n^{p,q} : \{p, q\} \in [S_n]^2\}$ is a pairwise disjoint collection of elements of $\mathcal{C}$ each of which is a subset of c_n.
3. $b_n^{p,q} \cap b_m^{r,s} \neq \varnothing$ and $\{p, q\} \cap \{r, s\} \neq \varnothing$ implies that $m = n$ (and thus $\{p, q\} = \{r, s\}$ by (2).)

Suppose recursively we have defined S_i and $b_i^{p,q}$ for $i < n$. For each $i \leq n$ we define a set $a_i^n \in \mathcal{C}$ and we do this by recursion. Let $a_0^n = c_n$. If there is some $\{p_i^n, q_i^n\} \in [S_i]^2$ with $a_i^n \cap b_i^{p_i^n, q_i^n} \neq \varnothing$ then set $a_{i+1}^n = a_i^n \cap b_i^{p_i^n, q_i^n}$; otherwise set $a_{i+1}^n = a_i^n$. Let $S_n = S \setminus \bigcup_{i<n}\{p_i^n, q_i^n\}$. We take $\{b_n^{p,q} : \{p, q\} \in [S_n]^2\}$ to be a pairwise disjoint collection of members of $\mathcal{C}$ so that each $b_n^{p,q} \subseteq a_n^n$.

That completes the definition of the S_n and the $b_n^{p,q}$. It is easy to see that we have (1) and (2). To see that (3) holds notice that by construction if $i < n$ and $a_n^n \cap b_i^{r,s} \neq \varnothing$ we have $a_n^n \subseteq b_i^{r,s}$ and $\{r, s\} = \{p_i^n, q_i^n\}$. But $p_i^n, q_i^n \notin S_n$ and $b_n^{p,q} \subseteq a_n^n$.

Now we define χ. For x with $p, q \prec x$ and $x \in S \cap b_n^{p,q}$ set $\chi(p, x) = \chi(q, x) = \{p, q, x\}$. For all other pairs we let $\chi(p, x) = \{p, x\}$. Condition (3) in our construction of $b_n^{p,q}$ guarantees that χ is well-defined and 2-bounded.

Suppose $A \subseteq S$ is polychromatic for χ. We want to check that A is nowhere dense. We may as well assume A is infinite. For $c \in \mathcal{C}$ we must find $b \in \mathcal{C}$ with $b \subseteq c$ and $b \cap A = \varnothing$. Fix $c_n \in \mathcal{C}$. Since A is infinite and S_n is coinfinite in S we may find distinct $p, q \in A \cap S_n$. Then $\chi(p, x) = \chi(q, x)$ for all $x \in b_n^{p,q}$ which are $\prec$ above p and q and thus $b_n^{p,q} \cap A$ is finite since A is polychromatic. Hence A is empty on a subinterval $b_n^{p,q}$ which itself is a subset of c_n. □

Let us temporarily call an ultrafilter *weakly nowhere dense* if for every injective $f : \omega \to \mathbb{Q}$ there is an $A \in \mathcal{U}$ with $f(A)$ nowhere dense. In the terminology of Flašková [Fla10] these are the $\mathcal{I}$-friendly ultrafilters, where $\mathcal{I}$ is the nowhere dense ideal. Lemma 13.19 shows that every rainbow Ramsey ultrafilter is weakly nowhere dense. Thus it only remains for us to establish the following.

Lemma 13.20. Every weakly nowhere dense ultrafilter is nowhere dense.

Proof. We show that given a function $G : \omega \to \mathbb{R}$ we can find an injective function $F : \omega \to \mathbb{R}$ such that for any $A \subseteq \omega$, if $F(A)$ is nowhere dense then $G(A)$ is nowhere dense. As before let $\mathcal{C}$ be the collection of all open intervals in $\mathbb{R}$ with rational endpoints.

Claim. There is $\mathcal{C}' \subseteq \mathcal{C}$ such that for every $c \in \mathcal{C}$ there is a $t \subseteq c$ with $t \in \mathcal{C}'$, and so that each $G(n)$ belongs to finitely many members of $\mathcal{C}$'.

Proof of Claim. Enumerate $\mathcal{C}$ as $\{c_n : n \in \omega\}$. For each c_n let $t_n \subseteq c_n$ be some member of $\mathcal{C}$ that does not contain any of $G(0), \ldots G(n)$. Set $\mathcal{C}' = \{t_n : n \in \omega\}$. □

Now we construct F. We do so recursively. Suppose that $F(0), \ldots F(n-1)$ have already been defined. Let $t_0, \ldots t_m$ be all the members of $\mathcal{C}'$ containing $G(n)$.

Then $t_0 \cap \ldots \cap t_m$ is a nonempty open set. We take $F(n)$ to be a member of $t_0 \cap \ldots \cap t_m$ different from each of $F(0), \ldots F(n-1)$.

Obviously F is injective. Notice also that for every $n \in \omega$ and every $t \in \mathcal{C}'$, $F(n) \in t$ whenever $G(n) \in t$. We can use this property to see that F is as desired. For suppose $A \subseteq \omega$ with $F(A)$ nowhere dense. We check that $G(A)$ is nowhere dense. Let $s \in \mathcal{C}$. Since $F(A)$ is nowhere dense there is $u \in \mathcal{C}$ with $u \subseteq s$ so that $F(A) \cap u$ is empty. Take $t \in \mathcal{C}'$ with $t \subseteq u$. Then since $F(A) \cap t$ is empty it follows that $G(A) \cap t$ is empty: if $n \in A$ and $G(n) \in t$ then $F(n) \in t$. □

13.4 A rainbow Ramsey ultrafilter which is not discrete

In this subsection we use MA to construct a rainbow Ramsey ultrafilter $\mathcal{U}$ which is not discrete. It is enough to construct $\mathcal{U}$ an ultrafilter on $\mathbb{Q}$ which contains no discrete subset of $\mathbb{Q}$ but which contains a polychromatic set for every 2-bounded coloring on $\mathbb{Q}$.

Definition 13.21. Let $A \subseteq \mathbb{Q}$. We define $L^k(A) \subseteq \mathbb{Q}$ by induction on $k \in \omega$.

1. $L^0(A) = A$.

2. $L^{k+1}(A)$ is the set of $a \in A$ which are limit points of $L^k(A)$.

If there exists k such that $L^k(A) = \varnothing$ we set $\mathrm{CB}(A)$ equal to the least such k. Otherwise we say $\mathrm{CB}(A) \geq \omega$.

For A without a perfect kernel $\mathrm{CB}(A)$ is just the usual Cantor-Bendixson rank for finitely ranked sets. Let $\mathcal{I}$ be the collection of all $A \subseteq \mathbb{Q}$ with $\mathrm{CB}(A) < \omega$. The following proposition is well-known but we include a proof since it uses ideas we will need later in the more delicate situation of Proposition 13.28.

Proposition 13.22. The set $\mathcal{I}$ contains every discrete subset of $\mathbb{Q}$ and is an ideal. In fact if $\mathrm{CB}(A \cup B) > k + l$ then $\mathrm{CB}(A) > k$ or $\mathrm{CB}(B) > l$.

Proof. Every discrete subset of $\mathbb{Q}$ belongs to $\mathcal{I}$ since a set is discrete exactly when $\mathrm{CB}(A) < 2$.

To show that $\mathcal{I}$ is an ideal, fix $A, B \subseteq \mathbb{Q}$. We show that if $L^{k+l}(A \cup B) \neq \varnothing$ then either $L^k(A) \neq \varnothing$ or $L^l(B) \neq \varnothing$.

For our purposes a tree $(T, <_T)$ is a partially ordered set so that for each node $x \in T$ the set of predecessors of x is well-ordered by $<$. Let $p_T(x)$ be the set of predecessors of x. For each $x \in T$ we let the height of x, written $\mathrm{ht}(x)$, be equal to the order-type of $p_T(x)$. The height of T, written $\mathrm{ht}_T(T)$, is the maximum of the heights of its elements. The kth level of T is all $x \in T$ with $\mathrm{ht}_T(x) = k$.

Now let $\mathcal{T}$ be the collection of all trees T with the following properties:

1. T has finite height.
2. $T \subseteq A \cup B$.
3. Every $x \in T$ with $\mathrm{ht}_T(x) < \mathrm{ht}(T)$ has infinitely many successors. These can be enumerated $\{y_n : n \in \omega\}$ where $\lim_{n<\omega} y_n = x$.
4. For each $k \leq \mathrm{ht}(T)$ either every $x \in T$ with $\mathrm{ht}_T(x) = k$ belongs to A, or every $x \in T$ with $\mathrm{ht}_T(x) = k$ belongs to B.

If $x \in L^k(A \cup B)$ then by induction on k it may be shown that there is $T \in \mathcal{T}$ with root x and $\mathrm{ht}_T(T) = k$. Thus if there is some $x \in L^{k+l}(A \cup B)$ then there is a $T \in \mathcal{T}$ with root x and $\mathrm{ht}_T(T) = k + l$. Then T has $k + l + 1$ levels so either $k + 1$ levels are subsets of A or $l + 1$ levels are subsets of B. It is easy to see if a node in A has k levels above which are subsets of A then that node belongs to $L^k(A)$. Similarly for B and l. $\square$

There is an unfortunate complication in the argument to come. Unlike our constructions in Subsections 13.1 and 13.2 we will not be able to generate $\mathcal{U}$ from a tower; there are countable length towers of sets not in $\mathcal{I}$ which cannot be extended by a set not in $\mathcal{I}$.

The key lemma in the construction of $\mathcal{U}$ is the following.

Lemma 13.23. Assume MA. Let $\mathcal{S}$ be a filter base with $\mathcal{S} \cap \mathcal{I} = \varnothing$ and $|\mathcal{S}| < 2^\omega$ and let χ be a 2-bounded coloring on $\mathbb{Q}$. There is a polychromatic $B \subseteq \mathbb{Q}$ such that $B \cap S \notin \mathcal{I}$ for every $S \in \mathcal{S}$.

Using Lemma 13.23 to construct an appropriate $\mathcal{U}$ is a routine recursion of length continuum. So we turn to proving Lemma 13.23. Fix $\mathcal{S}$ a filter base with $\mathcal{S} \cap \mathcal{I} = \varnothing$ and $|\mathcal{S}| < 2^\omega$ and fix $\chi : [\mathbb{Q}]^2 \to \omega$ a 2-bounded coloring. Throughout the rest of this section we assume MA. For $\epsilon > 0$ and $a \in \mathbb{Q}$ we let $N_\epsilon(a)$ denote the ϵ-neighborhood around a. Fix a well-ordering $\prec$ on $\mathbb{Q}$ of ordertype ω; our definition of normal for subsets $\mathbb{Q}$ is the same as Definition 13.7, but using $\prec$ instead of $<$.

Proposition 13.24. There is a normal $A \subseteq \mathbb{Q}$ with $A \cap \mathcal{S} \notin \mathcal{I}$ for each $S \in \mathcal{S}$.

Proof. We apply MA to the partial order of finite normal subsets of $\mathbb{Q}$ ordered by $X_1 \leq X_0$ if $X_0 \subseteq X_1$. The point is to arrange the dense sets so that for each $k \in \omega$ and $S \in \mathcal{S}$ we eventually add a member of $L^k(S)$, and once we have added some $a \in L^{k+1}(S)$ we add for each rational $\epsilon > 0$ a member of $L^k(S) \cap N_\epsilon(a)$. The density of the sets follows from the fact that each $S \notin \mathcal{I}$ and the fact that if $X \subseteq \mathbb{Q}$ is finite and normal then $X \cup \{a\}$ is normal for all but finitely many $a \in \mathbb{Q}$. $\square$

Now fix A as in Proposition 13.24. We want to build a polychromatic $B \subseteq A$ with each $B \cap S \notin \mathcal{I}$. We will build B by finite approximations, which we denote by X, and when we add a proposed limit point b to B we have to make sure that b is a limit point not only of $E(X)$ but also of $E(X \cup b)$. Hence the following definition.

Definition 13.25. Let $X \subseteq A$ be finite and polychromatic and let $S \in \mathcal{S}$. We define $L^k_{\text{pol}}(X, S)$ by induction on $k \in \omega$.

1. $L^0_{\text{pol}}(X, S) = E(X) \cap S$.

2. $L^{k+1}_{\text{pol}}(X, S)$ is the set of $a \in E(X) \cap S$ which are limit points of $L^k_{\text{pol}}(X \cup a, S)$.

If there exists k such that $L^k_{\text{pol}}(X, S) = \varnothing$ we set $\text{CB}_{\text{pol}}(X, S)$ equal to the least such k. Otherwise we say $\text{CB}_{\text{pol}}(X, S) \geq \omega$.

We prove the analogue of Proposition 13.8. Define $h : \omega^2 \to \omega$ by recursion on the first coordinate. Take $h(0, n) = n$ and $h(k + 1, n) = n + 1 + h(k, n + 1)$.

Proposition 13.26. Suppose $n = |X|$ and $a_0, \ldots a_{h(k,n)}$ are distinct members of $E(X) \cap S$. Then

$$L^k_{\text{pol}}(X, S) \subseteq^* L^k_{\text{pol}}(X \cup a_0, S) \cup \ldots \cup L^k_{\text{pol}}(X \cup a_{h(k,n)}, S).$$

Proof. The base case $k = 0$ is just Proposition 13.8.

For the successor case let $y \in L^{k+1}_{\text{pol}}(X, S)$ be $\prec$ above all of the a_i. By Proposition 13.8 there are at most $n + 1$ choices of i with $y \notin E(X \cup a_i)$ or equivalently $a_i \notin E(X \cup y)$. Hence by relabeling we may assume that $a_i \in E(X \cup y)$ for $i \leq h(k, n + 1)$. By definition y is a limit point of $L^k_{\text{pol}}(X \cup y, S)$ and by induction we have

$$L^k_{\text{pol}}(X \cup y, S) \subseteq^* L^k_{\text{pol}}(X \cup y \cup a_0, S) \cup \ldots \cup L^k_{\text{pol}}(X \cup y \cup a_{h(k,n+1)}, S).$$

Thus there is i such that y is a limit point of $L^k_{\text{pol}}(X \cup y \cup a_i, S)$. Since $y \in E(X \cup a_i) \cap S$ that gives $y \in L^{k+1}_{\text{pol}}(X \cup a_i, S)$. $\square$

Proposition 13.27. For each $S \in \mathcal{S}$ we have $\mathrm{CB}_{\mathrm{pol}}(\varnothing, S) \geq \omega$.

Proof. We must prove that $L^k_{\mathrm{pol}}(X, S) \neq \varnothing$ for each $k < \omega$. Define $v : \omega^2 \to \omega$ by recursion on the first coordinate. Take $v(0, n) = 1$ and $v(k+1, n) = v(k, n) + h(k, n) + 2$.

To prove the proposition, we prove the following more general fact by induction on k. For $X \subseteq A$ with $|X| = n$ and U an open subset of $\mathbb{Q}$,

$$(*) \text{ if } L^{v(k,n)+1}(E(X) \cap S) \cap U \neq \varnothing \text{ then } L^k_{\mathrm{pol}}(X, S) \cap U \text{ is infinite.}$$

Since $\mathrm{CB}(E(\varnothing) \cap S) \geq \omega$, this statement yields the proposition when used with $X = \varnothing$ and $U = \mathbb{Q}$.

The base case $k = 0$ is trivial.

For the successor step $k + 1$ suppose that $L^{k+1}_{\mathrm{pol}}(X, S) \cap U$ is finite, yet $L^{v(k+1,n)+1}(E(X) \cap S) \cap U$ is not empty. We define sequences $\{y_i : i \leq h(k, n)\}$, $\{\epsilon_i : i \leq h(k, n)\}$ by recursion so that

1. $N_{\epsilon_0}(y_0) \subseteq U$
2. $y_i \in L^{v(k+1,n)-i}(E(X) \cap S)$
3. $N_{\epsilon_i}(y_i) \cap L^k_{\mathrm{pol}}(X \cup y_i, S) \subseteq \{y_i\}$
4. $N_{\epsilon_{i+1}}(y_{i+1}) \subseteq N_{\epsilon_i}(y_i)$.

Start by fixing some y a member of $L^{v(k+1,n)+1}(E(X) \cap S) \cap U$. By definition y is a limit point of $L^{v(k+1,n)}(E(X) \cap S)$; since $y \in U$ there must be infinitely many members of $L^{v(k+1,n)}(E(X) \cap S) \cap U$; one of them does not belong to $L^{k+1}_{\mathrm{pol}}(X, S)$, take this to be y_0. Since $y_0 \notin L^{k+1}_{\mathrm{pol}}(X, S)$ we may select some small ϵ_0 satisfying (1) and (3). The construction of the rest of the sequence follows suit and we obtain y_{i+1} from y_i in a manner similar to how we obtained y_0 from y.

Now take $y = y_{h(k,n)}$ and $\epsilon = \epsilon_{h(k,n)}$. By (3) and (4) we have that $N_\epsilon(y) \cap L^k_{\text{pol}}(X \cup y_i, S)$ is finite for $i \le h(k,n)$. Also $y \in N_\epsilon(y) \cap L^{v(k,n)+1}(E(X) \cap S)$ so that by induction $N_\epsilon(y) \cap L^k_{\text{pol}}(X, S)$ is infinite. And yet by Proposition 13.26 we have

$$N_\epsilon(y) \cap L^k_{\text{pol}}(X, S) \subseteq^* N_\epsilon(y) \cap (L^k_{\text{pol}}(X \cup y_0, S) \cup \ldots \cup L^k_{\text{pol}}(X \cup y_{h(k,n)}, S)).$$

This is a contradiction because the right hand side is supposedly finite. □

Proposition 13.28. Suppose that $X \subseteq A$ with $|X| \le n$ and $\text{CB}_{\text{pol}}(X, S) \ge \omega$ for each $S \in \mathcal{S}$. Then there are at most n elements a in $E(X)$ such that $\text{CB}_{\text{pol}}(X \cup a, S) < \omega$ for some $S \in \mathcal{S}$.

Proof. Suppose for contradiction that $a_i \in E(X)$ and $S_i \in \mathcal{S}$ for $i \le n$ with $\text{CB}_{\text{pol}}(X \cup a_i, S_i) < \omega$. Set $S = \bigcap_{i \le n} S_i$. Then $\text{CB}_{\text{pol}}(X, S) \ge \omega$ while $\text{CB}_{\text{pol}}(X \cup a_i, S) < \omega$ for each $i \le n$. By Proposition 13.8 we have

$$E(X) \cap S \subseteq^* (E(X \cup a_0) \cup \ldots \cup E(X \cup a_n)) \cap S.$$

We will use this obtain a contradiction by showing that for each $l \in \omega$ there is some $i \le n$ so that $L^l_{\text{pol}}(X \cup a_i, S)$ is not empty. So fix $l \in \omega$.

Let $\mathcal{T}$ be the collection of all trees T with the following properties:

1. T has finite height.

2. $x <_T y$ in T implies $x \prec y$.

3. $T \subseteq (E(X \cup a_0) \cup \ldots E(X \cup a_n)) \cap S$.

4. Every $x \in T$ with $\text{ht}_T(x) < \text{ht}(T)$ has infinitely many successors. These can enumerated be as $\{y_n : n \in \omega\}$ where $\lim_{n<\omega} y_n = x$.

5. For each $k \leq \mathrm{ht}(T)$ for some $i \leq n$ we have that all x in T with $\mathrm{ht}(x) = k$ belongs to $E(X \cup a_i)$.

6. $x \in E(X \cup p_T(x))$ for each $x \in T$. (Recall that $p_T(x)$ denotes the set of predecessors of x in T.)

Let F be a finite set so that $(E(X) \cap S) \cup F \subseteq (E(X \cup a_0) \cup \ldots \cup E(X \cup a_n)) \cap S$. If $x \in L^k_{\mathrm{pol}}(X, S)$ and does not belong to F it can be shown by induction on k (letting X vary) that there is a $T \in \mathcal{T}$ with root x and $\mathrm{ht}(T) = k$. Since $\mathrm{CB}_{\mathrm{pol}}(X, S) \geq \omega$ it follows that we may find $T \in \mathcal{T}$ with $\mathrm{ht}(T)$ arbitrarily large.

Let $\mathcal{T}_i$ be the set of all trees $T' \subseteq E(X \cup a_i) \cap S$ satisfying clauses 1,2, and 4 of the definition of $\mathcal{T}$ with the additional property that if $x \in T'$ then $x \in E(X \cup a_i \cup p_{T'}(x))$. If x is the root of some $T' \in \mathcal{T}_i$ with $\mathrm{ht}(T') = l$ then $x \in L^l_{\mathrm{pol}}(X \cup a_i, S)$, so we just need to find such a tree.

Let $T \in \mathcal{T}$ with $\mathrm{ht}(T) > 2l(n+1)$. For some $i \leq n$ there are $2l+1$ levels of the tree with every member of that level belonging to $E(X \cup a_i)$. Let T_0 be the subtree of T consisting of just those levels, thinned appropriately so as to satisfy the limit condition from (4). Then $\mathrm{ht}(T_0) = 2l$ and T_0 satisfies all the requirements of the definition of membership in $\mathcal{T}_i$ except possibly one: while each $x \in T_0$ belongs to both $E(X \cup a_i)$ and $E(X \cup p_{T_0}(x))$ it may be that x does not belong to $E(X \cup a_i \cup p_{T_0}(x))$.

We define by recursion a sequence of trees $\{T_k : k \leq l\}$ so that

(i) $\mathrm{ht}(T_{k+1}) = \mathrm{ht}(T_k) - 1 = 2l - k$

(ii) For $x \in T_k$ with $\mathrm{ht}_{T_k}(x) > \mathrm{ht}(T_k) - k$ we have $x \in E(X \cup a_i \cup p_{T_k}(x))$.

Given the sequence we may take T' to be T_l. So let us describe the recursion. Say T_k is given with $k < l$. For $j < \mathrm{ht}_{T_k}(x)$ let $p_{T_k}(j, x)$ denote the unique

$y \in p_{T_k}(x)$ with $\mathrm{ht}_{T_k}(y) = j$. If $x \in T_k$ and $x \notin E(X \cup a_i \cup p_{T_k}(x))$ then by normality of A and the fact that x belongs to both $E(X \cup a_i)$ and $(E(X) \cup p_{T_0}(x))$ there is some $j < \mathrm{ht}_{T_k}(x)$ with $\chi(p_{T_k}(j, x), x) = \chi(a, x)$. Since χ is 2-bounded there is at most one such j. For each $x \in T_k$ with $\mathrm{ht}_{T_k}(x) = \mathrm{ht}(T_k) - k$ let $b(x)$ equal such a j if it exists. We may thin T_k so that $b(x)$ is the same j for all such x; then form T_{k+1} by removing level j from T_k and thinning as necessary to preserve the limit condition in clause 4 of the definition of $\mathcal{T}$. □

Proposition 13.29. Suppose that $c \in A$, $X \subseteq A$ is finite and $S \in \mathcal{S}$. If c is a limit point of $L^k_{\mathrm{pol}}(X, S)$ then for all but finitely many $a \in E(X) \cap S$ we have that c is a limit point of $L^k_{\mathrm{pol}}(X \cup a, S)$.

Proof. This is immediate from Proposition 13.26. □

Proof of Lemma 13.23. Take $\mathbb{P}$ to be the notion of forcing consisting of conditions $\langle X, f \rangle$ where

1. $X \subseteq A$ is finite and polychromatic with $\mathrm{CB}_{\mathrm{pol}}(X, S) \geq \omega$ for each $S \in \mathcal{S}$.
2. f is a finite partial function from $\mathcal{S} \times \omega \times \omega$ into X so that if $f(S, n, k) = c$ then $c \in S$ is a limit point of $L^k_{\mathrm{pol}}(X, S)$.

(We could also prove this lemma using the same forcing notion but without the commitments f, but including them will help keep the argument organized.) We order $\mathbb{P}$ by inclusion: $\langle X', f' \rangle \leq \langle X, f \rangle$ if and only $X' \supseteq X$ and $f' \supseteq f$. That $\mathbb{P}$ is nonempty follows from Proposition 13.27. A simple Δ-system argument establishes that $\mathbb{P}$ is ccc.

Let $D_{S,n,k}$ be the collection of conditions $\langle X, f \rangle$ with $(S, n, k) \in \mathrm{dom}(f)$. To check density let $\langle X, f \rangle \in \mathbb{P}$ with $(S, n, k) \notin \mathrm{dom}(f)$. The set $L^{k+1}_{\mathrm{pol}}(X, S)$ is infinite. Together Propositions 13.28 and 13.29 imply that for all but finitely many $c \in L^{k+1}_{\mathrm{pol}}(X, S)$, the pair $\langle X \cup \{c\}, f \cup \{\langle (S, n, k), c\rangle\}\rangle$ is a condition. It clearly belongs to $D_{S,n,k}$.

For rational $\epsilon > 0$ let $E_{S,n,k,\epsilon}$ be the collection of conditions $\langle X, f\rangle$ with $f(S, n, k+1) = c$ where for some m we have $f(S, m, k) = d$ and $d \in N_\epsilon(c)$. To check density let $\langle X, f\rangle \in \mathbb{P}$ and without loss of generality assume $\langle X, f\rangle \in D_{S,n,k+1}$ and $f(S, n, k+1) = c$. Then $N_\epsilon(c) \cap L^{k+1}_{\mathrm{pol}}(X, S)$ is infinite and together Propositions 13.28 and 13.29 imply that for all but finitely many $d \in N_\epsilon(c) \cap L^{k+1}_{\mathrm{pol}}(X, S)$ the pair $\langle X \cup \{d\}, f \cup \{\langle (S, m, k), d\rangle\}\rangle$ is a condition. It clearly belongs to $E_{S,n,k,\epsilon}$.

Finally let $D'_{S,n,\epsilon}$ be the collection of conditions $\langle X, f\rangle$ with $f(S, n, 0) = c$ where we have some $d \in X \cap S \cap N_\epsilon(c)$. Density of $D'_{S,n,\epsilon}$ can be checked similarly to the above.

Using MA let $G \subseteq \mathbb{P}$ be a filter intersecting the dense sets described above. Let $B = \bigcup\{X : \langle X, f\rangle \in G\}$ and let $F = \bigcup\{f : \langle X, f\rangle \in G\}$. Then B is polychromatic and an induction on k shows that each $F(S, n, k)$ belongs to $L^{k+1}(B \cap S)$. Thus $\mathrm{CB}(B \cap S) \geq \omega$. □

13.5 A rainbow Ramsey ultrafilter Rudin-Blass above a non-rainbow Ramsey ultrafilter

We close our analysis of rainbow Ramsey ultrafilters by showing that as a class of special ultrafilters on ω, the class of rainbow Ramsey ultrafilters has an unusual non-closure property. Recall the definition of the Rudin-Blass preordering, a refinement of the usual Rudin-Keisler preorder on ultrafilters on ω.

Definition 13.30 ([LZ98])**.** Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on ω. We write $\mathcal{U} \leq_{\mathrm{RB}} \mathcal{V}$ if there is a finite-to-one function $f : \omega \to \omega$ such that $A \in \mathcal{U}$ if and only if

$f^{-1}[A] \in \mathcal{V}$.

Except for the class of rainbow Ramsey ultrafilters, every special class of ultrafilters considered in this thesis — and indeed every special class of ultrafilter in the literature which is known to this author — is closed downwards in the Rudin-Blass ordering on ultrafilters. The remainder of this section is dedicated to the proof that this is not the case for the class of rainbow Ramsey ultrafilters. The construction, while not entirely straightforward, is very similar to the construction of rainbow Ramsey ultrafilters in previous subsections and thus we will be sparser with the details here. The hypothetical reader who has made it through those sections will have no trouble filling in the gaps.

Theorem 13.31. Assume MA. There exist ultrafilters $\mathcal{U} \leq_{\text{RB}} \mathcal{V}$ with $\mathcal{V}$ a rainbow Ramsey ultrafilter and $\mathcal{U}$ not a rainbow Ramsey ultrafilter.

As in Subsection 13.2, let $\mathcal{I}_{\text{Ramsey}}$ be the collection of $X \subseteq [\omega]^2$ for which there exists an N so that $N \leq |A|$ implies $[A]^2 \not\subseteq X$. Fix $f : \omega \to [\omega]^2$ a finite-to-one function, with $|f^{-1}\{m, n\}|$ increasing as m and n increase; for the sake of specificity let us say that $|f^{-1}\{m, n\}| = m + n$.

Our goal is to construct a rainbow Ramsey ultrafilter $\mathcal{V}$ on ω with the property that for every $B \in \mathcal{V}$ we have that $f[B] \notin \mathcal{I}_{\text{Ramsey}}$. Then we may let $\mathcal{U}$ be the ultrafilter on $[\omega]^2$ generated by $\{f[B] : B \in \mathcal{V}\}$. Then $\mathcal{U} \leq_{\text{RB}} \mathcal{V}$ by definition and $\mathcal{U} \cap \mathcal{I}_{\text{Ramsey}} = \varnothing$, and in particular $\mathcal{U}$ is not rainbow Ramsey by Proposition 13.16.

Let $\mathcal{J}$ be the collection of those $A \subseteq \omega$ for which there exists an $N \in \omega$ so that whenever $C \subseteq \omega$ with $|C| \geq N$, there is some $\{m, n\} \in [C]^2$ with $|A \cap f^{-1}\{m, n\}| < N$.

Proposition 13.32. The set $\mathcal{J}$ is an ideal.

Proof. Given $N \in \omega$, if $m, n > N$ then $|f^{-1}\{m, n\}| \geq N$ and hence $\omega \notin \mathcal{J}$.

Clearly $\mathcal{J}$ is downward closed under $\subseteq$.

We check that $\mathcal{J}$ is closed under $\cup$. Suppose $A, B \in \mathcal{J}$, witnessed by N_0, N_1 respectively. Suppose for contradiction that $A \cup B \notin \mathcal{J}$. Let N be greater than $N_0 + N_1$, and let M be larger than the finite Ramsey number $R(N)$. Since $A \cup B \notin \mathcal{J}$, by definition of $\mathcal{J}$ there is C with $|C| \geq M$ and $|(A \cup B) \cap f^{-1}\{m, n\}| \geq M$ for each $\{m, n\} \in [C]^2$. Two-color $[C]^2$ by giving $\{m, n\}$ color 0 if $|A \cap f^{-1}\{m, n\}|$ is greater than N_0. By choice of M there is a complete monochromatic subgraph $[D]^2$ of $[C]^2$ with $|Y| \geq N$. If $[D]^2$ is monochromatically colored 0, we contradict the choice of N_0. But if $[D]^2$ is monochromatically colored 1, since $N > N_0 + N_1$ we have $|B \cap f^{-1}\{m, n\}|$ is greater than N_1 for each $\{m, n\} \in [D]^2$, contradicting the choice of N_1. $\square$

The ideal $\mathcal{J}$ contains every set $B \subseteq \omega$ with $f[B] \in \mathcal{I}_{\text{Ramsey}}$. Thus it will be sufficient to construct a rainbow Ramsey ultrafilter disjoint from $\mathcal{J}$.

Proposition 13.33. Whenever $\chi : [\omega]^2 \to \omega$ is 2-bounded and $A \notin \mathcal{J}$, there is a normal $B \subseteq A$ with $B \notin \mathcal{J}$.

Proof. It is enough to show that given $X \subseteq \omega$ a finite set on which χ is normal and given $N \in \omega$, we may find $C \subseteq \omega$ with $|C| = N$ and a finite $Y \subseteq A$ with $X \cup Y$ normal and

$$|Y \cap f^{-1}\{m, n\}| \geq N \text{ for each } \{m, n\} \in [C]^2$$

for then B may be constructed by a straightforward induction.

Say $|X| = p$. Using the fact that $A \notin \mathcal{J}$, for any $M \in \omega$ we can always find a $C \subseteq \omega$ with $|C| = N$ and $|A \cap f^{-1}\{m, n\}| \geq M$ for each $\{m, n\} \in [C]^2$. The crux of the remainder of the argument is that as long as we take M large enough we will be able to take enough points from each $f^{-1}\{m,n\} \cap A$ to form Y. To facilitate our choice of an appropriate M we define a function $h : \omega \to \omega$ by the following recursion. Let $h(0) = \binom{p}{2} + 1$, and $h(k+1) = h(k) + \binom{p+k+1}{2} + 1$.

We let $M = h(\binom{N}{2}N)$ and construct Y in $\binom{N}{2}N$ stages, selecting one point to add to Y at each stage while maintaining the condition of normality of $X \cup Y$. We use Y_{j+1} to denote the approximation of Y constructed at the jth stage; thus $Y_0 = \emptyset$ and $|Y_{j+1}| = j+1$. We must eventually add N members from each of the $f^{-1}\{m,n\}$ where $\{m,n\} \in [C]^2$. The order of which $f^{-1}\{m,n\}$ we select from will depend on how the sets $A \cap f^{-1}\{m,n\}$ are interleaved.

For each stage j of the construction we let $P_j = \{\{m,n\} \in [C]^2 \mid |Y_j \cap f^{-1}\{m,n\}| < N\}$, and let $Z_j = \bigcup_{\{m,n\}\in P_j} A \cap f^{-1}\{m,n\}$. We may assume $\max(X) < \min(Z_0)$. We maintain that $\max(X \cup Y_j)$ is no larger than the $h(j-1)$th element of Z_j. At stage j itself, as in the proof of Proposition 13.10 and since $h(j) - h(j-1) = \binom{|X|+|Y_j|}{2} + 1$, there is some y_j among the first $h(j)$ elements of Z_j and above $\max(X \cup Y_j)$, so that $X \cup Y_j \cup \{y_j\}$ is normal. Set $Y_{j+1} = Y_j \cup \{y_j\}$. □

Proposition 13.34. Whenever $\chi : [\omega]^2 \to \omega$ is 2-bounded and $A \notin \mathcal{J}$ is normal, there is a polychromatic $B \subseteq A$ with $B \notin \mathcal{J}$.

Proof. We need to construct B so that for every N, we have some C with $|C| \geq N$ so that $|B \cap f^{-1}\{m,n\}| \geq N$ for each $\{m,n\} \in [C]^2$. Suppose we have $X \subseteq \omega$ a finite set, which represents some finite initial segment of B that has been constructed, and we wish to extend this segment to meet the condition for N. Iterate as in the proof of Proposition 13.33, but using the following fact for the inner recursion when defining the sets Y_j, instead of the appeal to the proof of Proposition 13.10. Given any $N_0, N, p \in \omega$ there are M and N_1 so that if $X \subseteq \omega$ with $|X| = p$, $C \subseteq \omega$ with $|C| = N$, $P \subseteq [C]^2$, and $Z_{m,n} \subseteq \omega$ with $|E(X) \cap Z_{m,n}| \geq N_1$ for each $\{m,n\} \in P$, then one of the first M members of $E(X) \cap \bigcup_{\{m,n\}\in P} Z_{m,n}$, say x, gives $X \cup \{x\}$ polychromatic and $|E(X \cup \{x\}) \cap Z_{m,n}| > N_0$ for each $\{m,n\} \in P$. This fact is proved using Proposition 13.8. We leave the details to the reader. $\square$

Proposition 13.35. Assume MA. If $\langle T_i : i < \lambda \rangle$ is a tower of sets with $\lambda < 2^{\aleph_0}$ and each $T_i \notin \mathcal{J}$, there is $T_\lambda \notin \mathcal{J}$ such that $T_\lambda \subseteq^* T_i$ for all $i < \lambda$.

Proof. Almost identical to Proposition 13.15. $\square$

Putting together the previous propositions to build (assuming MA) a rainbow Ramsey ultrafilter $\mathcal{V}$ disjoint from $\mathcal{J}$ is done just like the main construction in Subsection 13.1. This completes the proof of Theorem 13.31.

14 Polychromatic Ramsey theory and cardinal characteristics of the continuum

In this short final section we give a few well-known cardinal characteristics characterizations with the flavor of polychromatic Ramsey theory. The colorings we use here will be unary.

Definition 14.1. Let $\mathcal{F} \subseteq \omega^\omega$. We let $\mathfrak{par}(\mathcal{F})$ denote the least size of a family $\mathcal{G} \subseteq \mathcal{F}$ for which for every $X \in [\omega]^\omega$ there is $f \in \mathcal{G}$ so that f is neither eventually constant nor eventually injective on X.

1. $\mathfrak{par}_{1c} = \mathfrak{par}(\omega^\omega)$.
2. $\mathfrak{par}_c = \mathfrak{par}(2^\omega)$
3. $\mathfrak{par}_1 = \mathfrak{par}(\mathcal{F})$ where $\mathcal{F}$ consists on all finite-to-one functions.
4. $\mathfrak{par}_{\text{bdd}} = \mathfrak{par}(\mathcal{F})$ where $\mathcal{F}$ consists of all f with each $|f^{-1}(n)| \leq 2$.

Our notation is consistent with that of Blass [Bla93] who introduced $\mathfrak{par}_{1c}$. The cardinal $\mathfrak{par}_c$ is just the splitting number $\mathfrak{s}$. Let us note that Galvin's trick applied to the unary 2-bounded colorings corresponding to $\mathfrak{par}_{\mathrm{bdd}}$ yields the inequality $\mathfrak{par}_c \leq \mathfrak{par}_{\mathrm{bdd}}$.

We also introduce notation for the dual characteristics.

Definition 14.2. Let $\mathcal{F} \subseteq \omega^\omega$. We let $\mathfrak{hom}(\mathcal{F})$ denote the least size of a $\mathcal{X} \subseteq [\omega]^\omega$ so that for every $f \in \mathcal{F}$ there is some $X \in \mathcal{X}$ so that f is either eventually constant or eventually injective on X.

1. $\mathfrak{hom}_{1c} = \mathfrak{hom}(\omega^\omega)$.
2. $\mathfrak{hom}_c = \mathfrak{hom}(2^\omega)$
3. $\mathfrak{hom}_1 = \mathfrak{hom}(\mathcal{F})$ where $\mathcal{F}$ consists on all finite-to-one functions.
4. $\mathfrak{hom}_{\mathrm{bdd}} = \mathfrak{hom}(\mathcal{F})$ where $\mathcal{F}$ consists of all f with each $|f^{-1}(n)| \leq 2$.

Proposition 14.3. $\mathfrak{par}_1 = \mathfrak{b}$, and dually $\mathfrak{hom}_1 = \mathfrak{d}$.

Proof. Given $f \in \omega^\omega$ strictly increasing let g_f be some finite to one function which is constant on each interval $[f(2n), f(2n+2))$. Then if $X \in [\omega]^\omega$ and g_f is injective on a cofinite subset of X it follows that $f \leq^* e_X$. This shows $\mathfrak{par}_1 \leq \mathfrak{b}$. The dual argument shows that $\mathfrak{d} \leq \mathfrak{hom}_1$.

For each strictly increasing $f \in \omega^\omega$ let $X_f \in [\omega]^\omega$ be the set $\{f^n(0) : n \in \omega\}$. For each finite-to-one function $g \in \omega^\omega$ let $h_g \in \omega^\omega$ be such that if $l \geq h_g(n)$ then $g(l) \notin \{g(0), \ldots g(n)\}$. Suppose $h_g \leq^* f$ and take N for which $f(n) \geq h_g(n)$ for all $n \geq N$. Then g is injective on $X_f \setminus N$. This shows $\mathfrak{hom}_1 \leq \mathfrak{d}$. The dual argument shows that $\mathfrak{b} \leq \mathfrak{par}_1$. □

Theorem 14.4. $\mathfrak{par}_{\mathrm{bdd}} = \mathbf{non}(\mathcal{M})$, and dually $\mathfrak{hom}_{\mathrm{bdd}} = \mathbf{cov}(\mathcal{M})$.

Proof. First we prove $\mathfrak{par}_{\text{bdd}} \leq \mathbf{non}(\mathcal{M})$. The collection $\mathcal{F} \subseteq \omega^\omega$ of two-to-one functions is closed as a subset of Baire space and thus may be regarded as a Polish space in its own right. Thus if $\mathbf{non}(\mathcal{M}) < \mathfrak{par}_{\text{bdd}}$ there exists some nonmeager $A \subseteq \mathcal{F}$ with cardinality strictly less than $\mathfrak{par}_{\text{bdd}}$. By definition of $\mathfrak{par}_{\text{bdd}}$ there exists an infinite $X \subseteq \omega$ for which

$$A \subseteq \{f \in \mathcal{F} : (\exists N)(\forall n, m \in X) n, m \geq N \to f(n) \neq f(m)\}.$$

But this latter set is meager, contradiction.

The dual inequality, $\mathbf{cov}(\mathcal{M}) \leq \mathfrak{hom}_{\text{bdd}}$, can be obtained using a dual argument. Alternatively one may notice that by its definition $\mathfrak{hom}_{\text{bdd}}$ is a $\mathbf{\Sigma}^0_2$ characteristic and apply Proposition 3 and Theorem 5 of [Bla93].

Next we prove $\mathfrak{hom}_{\text{bdd}} \leq \mathbf{cov}(\mathcal{M})$. We use Bartoszyński's characterization of $\mathbf{cov}(\mathcal{M})$ in terms of slaloms. Given $h \in \omega^\omega$, an h-slalom is a function ϕ with domain ω so that each $\phi(n) \subseteq \omega$ with $|\phi(n)| \leq h(n)$. A slalom is just an h-slalom where h is the identity function. Let $\mathcal{C}_h$ denote the set of h-slaloms and let $\mathcal{C}$ denote the set of slaloms. Then $\mathbf{cov}(\mathcal{M})$ is the least size of a family $\mathcal{F} \subseteq \omega^\omega$ such that for every $\phi \in \mathcal{C}$ there is some $f \in \mathcal{F}$ so that for all but finitely many n we have $f(n) \notin \phi(n)$. For a proof of this characterization, see Lemma 2.4.2 in [BJ95]. This proof shows that in fact the same characterization will hold with h-slaloms in the place of slaloms, as long as $h \in \omega^\omega$ has values which limit to infinity.

We start by massaging Bartoszyński's characterization slightly, and show that we may take $\mathcal{F}$ to consist of strictly increasing functions. For each $f \in \omega^\omega$ we associate a strictly increasing $g_f \in \omega^\omega$ as follows. If f is finite-to-one, we fix some $X = \{x_n : n \in \omega\}$ (enumerated in increasing order) on which f is strictly increasing and set $g_f(n) = f(x_n)$. Otherwise we take g_f to be the identify function (or something equally arbitrary.) Let $h \in \omega^\omega$ be defined by $h(n) = \frac{(n+1)(n+2)}{2}$.

We claim that if $\mathcal{F} \subseteq \omega^\omega$ is such that $(\forall \phi \in \mathcal{C}_h)(\exists f \in \mathcal{F})(\forall^\infty n) f(n) \notin \phi(n)$ then the family $\{g_f : f \in \mathcal{F}\}$ has the same property (with respect to $\mathcal{C}$ rather than $\mathcal{C}_h$.)

Given $\phi \in \mathcal{C}$ associate the function $\psi_\phi \in \mathcal{C}_h$ defined by

$$\psi_\phi(n) = \phi(0) \cup \ldots \phi(n) \cup \{0, \ldots n\}.$$

It is enough to show that if $(\forall^\infty n) f(n) \notin \psi_\phi(n)$ then $(\forall^\infty n) g_f(n) \notin \phi(n)$. If $(\forall^\infty n) f(n) \notin \psi_\phi(n)$ f certainly can only take each value finitely often. Further for sufficiently large n we have $g_f(n) = f(x_n) \notin \psi_\phi(x_n)$. Then for such n,

$$g_f(n) \notin \phi(0) \cup \ldots \cup \phi(x_n).$$

Since $n \leq x_n$ we have $g_f(n) \notin \phi(n)$, as desired.

We now use the massaged characterization to finish proving the theorem. Given strictly increasing $f \in \omega^\omega$ associate $A_f \in [\omega]^\omega$ given by $A_f = \{f^n(0) : n \in \omega\}$. (To avoid the technicality that A_f might contain only a single element, consider only those f with $f(0) > 1$.) To each two-to-one $g \in \omega^\omega$ we associate a slalom ϕ_g defined as follows. First define $h_g \in \omega^\omega$ by setting $h_g(n) = m$ where $m \neq n$ is the unique m such that $g(n) = g(m)$ if such m exists, and set $h_g(n) = n$ if there is no such m. Then define ϕ_g by

$$\phi_g(n) = \{h_g(1), \ldots h_g(n)\}.$$

We verify that if $(\forall^\infty n) f(n) \notin \phi_g(n)$ then g is injective on a cofinite subset of A_f. Say N is such that $f(n) \notin \phi_g(n)$ for $n \geq N$. We may assume $N > 0$. Then g is injective on $A_f \setminus N$. Otherwise for some $k, m \in \omega$ with $f^k(0) \geq N$ we would have $g(f^k(0)) = g(f^{k+m+1}(0))$. Then $h_g(f^k(0)) = f^{k+m+1}(0)$. Because f is increasing and $f^k(0) \neq 0$ (as $N > 0$) we have $h_g(f^k(0)) \in \phi_g(f^{k+m}(0))$. Thus $f(f^{k+m}(0)) \in \phi_g(f^{k+m}(0))$, contradicting $f^{k+m}(0) \geq N$. $\square$

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