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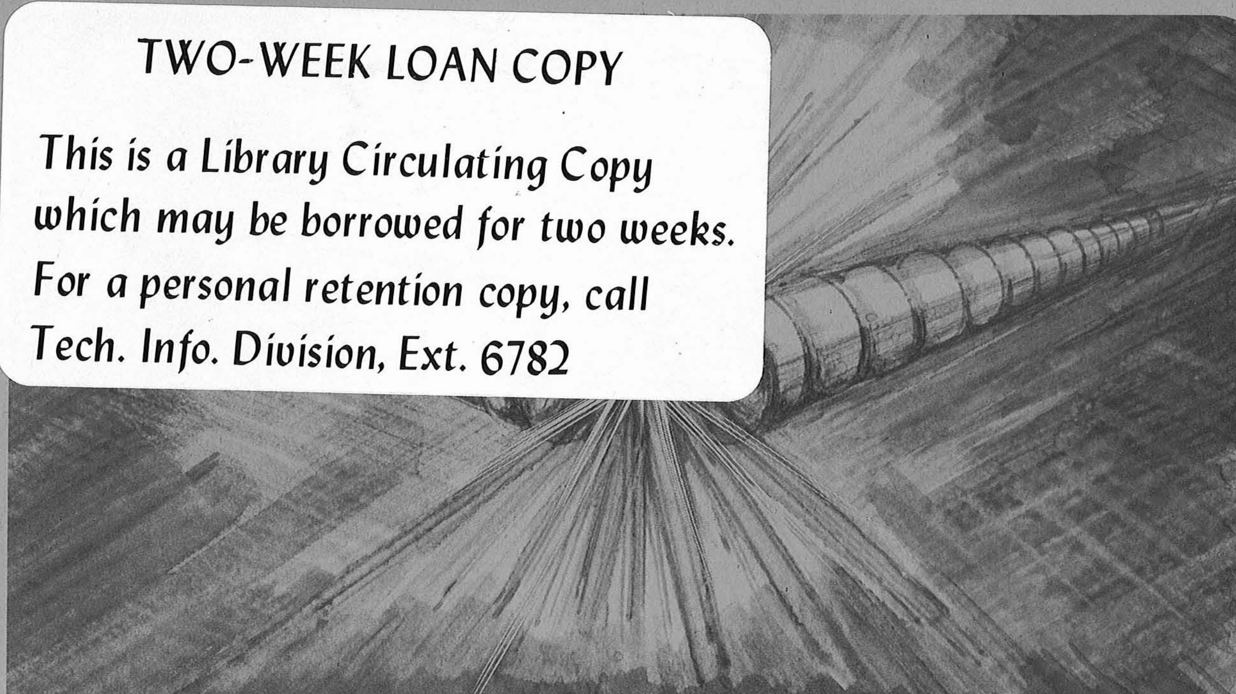

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REGULAR AND STOCHASTIC PARTICLE MOTION IN PLASMA DYNAMICS

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Abstract

A Hamiltonian formalism is presented for the study of charged-particle trajectories in the self-consistent field of the particles. The intention is to develop a general approach to plasma dynamics. Transformations of phase-space variables are used to separate out the regular, adiabatic motion from the irregular, stochastic trajectories. Several new techniques are included in this presentation.

A. Introduction

In dealing with plasma dynamics, one has the choice of several points of view. A first-principles approach[1] leads to an N-body problem ($N \to \infty$) for charged particles interacting with their electromagnetic field, which has singularities at the particle positions. The discrete-particle distribution, in six-dimensional phase-space $\underset{\sim}{z} \equiv (\underline{r}, \underline{v})$, is represented by the Klimontovich functions, $F_s(\underset{\sim}{z};t) \equiv \sum_{i=1}^{N_s} \delta^6(\underset{\sim}{z}-\underset{\sim}{z}^i(t))$, one for each species (s). Each particle (i) produces a current density (in three-dimensional physical space),

$$\underline{j}(\underline{x};\underset{\sim}{z}) = e\underline{v}\delta^3(\underline{x}-\underline{r}) , \tag{1}$$

depending on its phase point $\underset{\sim}{z}$; the total plasma current density

$$\underline{j}(\underline{x},t) = \sum_s \int d^6z \; \underline{j}(\underline{x};\underset{\sim}{z}) F_s(\underset{\sim}{z};t), \tag{2}$$

is thus singular:

$$\underline{j}(\underline{x},t) = \sum_i e_i \, \underline{v}^i(t) \, \delta^3(\underline{x} - \underline{r}^i(t)). \tag{3}$$

Consequently, the electromagnetic field, obtained from the Maxwell equations, with (2) or (3) plus external sources, is so irregular that a description of the particle motion $\underset{\sim}{z}(t)$ appears hopeless.

One is thus led to appreciate the Vlasov model, wherein the singular distribution $F_s(\underset{\sim}{z};t)$ is replaced by a smooth (Vlasov) function $f_s(\underset{\sim}{z};t)$. (The term "smooth" means "everywhere differentiable as many times as needed to validate the quasi-mathematical procedures to be employed".) The smoothing technique will not concern us here.

We now replace (2) by the Vlasov current density

$$\underline{j}(\underline{x},t) = \int d^6z \; \underline{j}(\underline{x};\underset{\sim}{z}) f(\underset{\sim}{z};t) \tag{4}$$

(we omit species symbols from now on). The smoothness of $\underline{j}$ guarantees the smoothness of the Vlasov electromagnetic field $\underline{B},\underline{E}$. Typically, this field consists of a quasistatic confining magnetostatic field $\underline{B}_0(\underline{x})$, and weak time-dependent electromagnetic perturbations $\delta\underline{E}(\underline{x},t)$, $\delta\underline{B}(\underline{x},t)$, representing (e.g.) instabilities, or radiation injected to heat the plasma.

A particle trajectory satisfies a deterministic equation of motion[2]

$$\dot{\underset{\sim}{z}} = \underset{\sim}{V}(\underset{\sim}{z},t), \tag{5}$$

where $\underset{\sim}{V}$ is now a given smooth vector field. Explicitly, for nonrelativistic dynamics, (5) reads [c=1]

$$\begin{aligned} \dot{\underline{r}} &= \underline{v}, \\ \dot{\underline{v}} &= (e/m)[\underline{E}(\underline{x}=\underline{r},t) + \underline{v} \times \underline{B}(\underline{x}=\underline{r},t)]. \end{aligned} \tag{6}$$

In (6), $\underline{E}(\underline{x},t)$ includes the field induced by slow variations of $\underline{B}_o$, and that due to the ambipolar potential, as well as the perturbations; $B(\underline{x},t)$ includes the confining field as well as that of perturbations (given by $\nabla \times \delta \underline{E}$).

The family of solutions of (5) is formally expressed as

$$\underset{\sim}{z}(t;\underset{\sim}{z}_o), \tag{7}$$

where $\underset{\sim}{z}_o$ is an initial condition (at some reference time t_o). We aim at both qualitative and quantitative knowledge of these solutions (7). For example, is the trajectory regular or irregular?[3] How sensitive is it to the initial condition? Is there a smooth manifold (of what dimensonality) on which it lies? Does a small initial volume $d^6 z_o$ evolve ergodically, in some sense?; does it exhibit mixing? In short, is the particle motion intrinsically stochastic?

Let the particle distribution at t_o be given by the phase-space density $f(\underset{\sim}{z}_o)$. We ask how the phase-space density $f(\underset{\sim}{z};t)$ evolves in time. By definition,

$$f(\underset{\sim}{z};t) \equiv \int d^6 z_o\, f(\underset{\sim}{z}_o)\,\delta^6[\underset{\sim}{z} - \underset{\sim}{z}(t;\underset{\sim}{z}_o)]; \tag{8}$$

so f evolves according to the continuity equation

$$\frac{\partial}{\partial t} f(\underset{\sim}{z};t) = - \frac{\partial}{\partial \underset{\sim}{z}} \cdot [\underset{\sim}{V}(\underset{\sim}{z},t) f(\underset{\sim}{z};t)]. \tag{9}$$

If the phase-flow vector field $\underset{\sim}{V}(\underset{\sim}{z},t)$ is Liouvillean, i.e.,

$$\frac{\partial}{\partial \underset{\sim}{z}} \cdot \underset{\sim}{V}(\underset{\sim}{z},t) \equiv 0, \tag{10}$$

then (9) becomes the Liouville equation

$$[\frac{\partial}{\partial t} + \underset{\sim}{V}(\underset{\sim}{z},t)\cdot \frac{\partial}{\partial \underset{\sim}{z}}]\, f(\underset{\sim}{z};t) = 0\,, \tag{11}$$

expressing invariance of f along a trajectory (7). The Newton-Lorentz equation (6) satisfies (10), so we may use (11). An alternate expression of (11) is

$$f(\underset{\sim}{z};t) = f[\underset{\sim}{z}_o(\underset{\sim}{z};t)], \tag{12}$$

where $\underset{\sim}{z}_o(\underset{\sim}{z};t)$ is the inverse of (7), expressing the initial condition as a function of position $\underset{\sim}{z}$ at time t.

In the course of time, $f(\underset{\sim}{z};t)$ may become less smooth, due to mixing. It may then be desirable, as well as physically realistic, to add Fokker-Planck terms to the evolution equation (11); these would represent collisions, i.e., departures from the Vlasov model, and may be termed "extrinsic stochasticity". Since the diffusivity thus introduced is itself a smooth function, it has the effect (characteristic of diffusion equations) of smoothing out irregularities of f.

The Vlasov (or Fokker-Planck) equation and the Maxwell equations can be written in a symbolic form[1] analogous to (5):

$$\dot{F} = V(F,t), \tag{13}$$

where $F \equiv [f(\underset{\sim}{z}), E(\underline{x}), \underline{B}(\underline{x})]$ is a point in infinite-dimensional function space. One can ask questions about the solutions to (13), similar to those asked about the solutions (7) of equation (5). The instability of solutions to (13) is called "collective instability", and has been a major consideration of plasma physicists for many years. It should not be confused with the particle-orbit instabilities of (5), which are the major concern of this conference.

The relation between the two kinds of instability is an intriguing question. It is generally believed that a collective instability arises from a free energy associated with F. As the amplitude of the electromagnetic perturbation grows, it produces orbit instability. The consequent redistribution of f leads to saturation of the collective instability. This redistribution generally is characterized by a flattening of f in velocity space, and enhanced transport in position space. For purposes of collisionless plasma heating, the effect of orbit instability is an irreversible transfer of energy (and momentum) from externally applied electromagnetic radiation to the plasma.

The evolution of electromagnetic perturbations can often be studied by means of the linear susceptibility, i.e., the linear current response to a small electromagnetic field. If the unperturbed phase-space density $f_o(\underset{\sim}{z})$ is quasi-static, then the linear response is[4]

$$\delta\underline{j}(\underline{x},t) = \int_0^\infty d\tau \int d^3x' \,\underset{\sim}{\sigma}(\underline{x},\underline{x}';\tau)\cdot\delta\underline{E}(\underline{x}',t-\tau) \tag{14}$$

with

$$\underset{\sim}{\sigma}(\underline{x},\underline{x}';\tau) = \int d^6z_o f_o(\underset{\sim}{z}_o)\{\underline{j}(\underline{x};\underset{\sim}{z}(\tau;\underset{\sim}{z}_o)),\ \underline{j}(\underline{x}';\underset{\sim}{z}_o)\} - \frac{n_o(\underline{x})e^2}{m}\delta(\underline{x}-\underline{x}')\delta(\tau). \tag{15}$$

Here { , } represents the Poisson brackets with respect to the phase space $\underset{\sim}{z}_o$, to be discussed in later sections.

The relation (14) can be immediately transformed to frequency:

$$\delta\underline{j}(\underline{x},\omega) = \int d^3x' \,\underset{\sim}{\sigma}(\underline{x},\underline{x}';\omega)\cdot\delta\underline{E}(\underline{x}',\omega), \tag{16}$$

with

$$\underset{\sim}{\sigma}(\underline{x},\underline{x}';\omega) \equiv \int_0^\infty d\tau\, e^{+i\omega\tau}\,\underset{\sim}{\sigma}(\underline{x},\underline{x}';\tau) \tag{17}$$

a complex analytic function of ω. Less immediate[5] is the conversion of (16) to a local relation in terms of an eikonal representation. With the assumed form (we omit c.c.):

$$\delta\underline{E}(\underline{x},t) = \tilde{\underline{E}}(\underline{x}) \exp i[S(\underline{x}) - \omega t], \tag{18}$$

$$\delta\underline{j}(\underline{x},t) = \tilde{\underline{j}}(\underline{x}) \exp i[S(\underline{x}) - \omega t],$$

we aim at a relation

$$\tilde{\underline{j}}(\underline{x}) = \underset{\sim}{\sigma}(\underline{x},\underline{k};\omega)\cdot\tilde{\underline{E}}(\underline{x}), \tag{19}$$

with

$$\underline{k}(\underline{x}) \equiv \partial S(\underline{x})/\partial\underline{x} \quad \text{and} \tag{20}$$

$$\underset{\sim}{\sigma}(\underline{x},\underline{k};\omega) \equiv \int d^3y\ e^{-i\underline{k}\cdot\underline{y}}\ \underset{\sim}{\sigma}(\underline{x}+\tfrac{1}{2}\underline{y},\ \underline{x}-\tfrac{1}{2}\underline{y};\omega). \tag{21}$$

Using (18) and (21) in the Maxwell equations, we obtain a local dispersion relation

$$D(\underline{x},\underline{k};\omega) = 0, \tag{22}$$

where D is the determinant of the dispersion tensor:

$$\underset{\sim}{D}(\underline{x},\underline{k};\omega) = \underset{\sim}{I} + \frac{4\pi i}{\omega}\underset{\sim}{\sigma}(\underline{x},\underline{k};\omega) - \frac{1}{\omega^2}(k^2\underset{\sim}{I} - \underline{k}\,\underline{k}). \tag{23}$$

The local polarization $\hat{E}$ is determined by

$$\underset{\sim}{D}(\underline{x},\underline{k};\omega)\cdot\hat{E} = 0. \tag{24}$$

The eikonal $S(\underline{x})$ is constructed on the ray trajectories:

$$\frac{d\underline{x}}{d\ell} = \frac{\partial D}{\partial\underline{k}},\quad \frac{d\underline{k}}{d\ell} = -\frac{\partial D}{\partial\underline{x}}. \tag{25a}$$

Here $d\ell$ measures "distance" along the rays, and is related to time by

$$\frac{d\ell}{dt} = -\left(\frac{\partial D}{\partial\omega}\right)^{-1}. \tag{25b}$$

The ray equations have the same form as (5), with $\underset{\sim}{z} \equiv (\underline{x},\underline{k})$ now representing ray phase-space. However, since D is usually complex, the ray phase-space is not real, but complex.[6] Nevertheless, one can still investigate the regularity or stochasticity of ray orbits, and can attempt to construct the eikonal $S(\underline{x}) = \int^{x}\underline{k}(\underline{x}')\cdot d\underline{x}'$. In the integrable case, at least, the variation of amplitude can then be found from $S(\underline{x})$.

B. Small parameter expansions

The confining magnetic field $\underline{B}_0(\underline{x})$ has spatial variation which may be slow (small parameter ϵ) in all directions, or quite different in the three directions. In the former case, one constructs[7] successively 3 adiabatic invariants, associated with gyration, bounce, and drift. In the latter case,[8] a rapid variation of $\underline{B}_0$ in only one direction still allows for a generalized gyro-invariant (magnetic moment), even if the motion is quite non-circular; such situations arise in magnetic sheaths, rings, and in reverse-field and strong-shear geometries. Elimination of the gyration, through construction of its invariant, reduces the motion to that of the guiding center, with two degrees of freedom (bounce and drift). If the magnetic field has a spatial symmetry (say about some axis), the corresponding canonical momentum is an invariant, and the remaining degree of freedom is integrable. More generally, one constructs successively the bounce action and the drift flux. These three actions are adiabatic invariants, in that they are conserved under slow time variation of $\underline{B}_o(\underline{x})$, while the energy is not.

The perturbing field $\delta\underline{E}$ $(\underline{x}, t)$ typically has rapid phase variation in $\underline{x}$ (with wave-vector $\underline{k}$) and in t (with frequency ω). Thus one uses the smallness parameters λ (magnitude of $\underline{\delta E}$) and η (rate of variation of amplitude, wave-vector and frequency). This enables one to generalize the adiabatic invariants of the unperturbed motion, by constructing a formal power series in λ and η(as well as in ε). In the absence of resonance (see Sec. C), the correction terms are small; when resonance occurs, the potentiality exists for stochasticity.

The situation is actually more complicated than outlined above. Typically, for given ω, there may be several $\underline{k}$, due to reflection; and the components of $\underline{k}$ may be of different order. Further, the spectrum in ω may be discrete or continuous, with narrow or broad distribution in ω. Most intrinsic stochasticity studies postulate a sharp spectrum, since a spread in ω may be interpreted as extrinsic stochasticity, in some sense.

C. Standard Hamiltonian Formalism

The use of action-angle variables[3] allows for a formalism of considerable beauty[9] and generality, as we shall see; but it is by no means easy to express a given real physical problem in that form. We begin by postulating such a form, and discuss its interpretation and consequences. Later we shall examine the possibility of obtaining that form explicitly.

We postulate that the particle motion can be derived from a Hamiltonian function $H(\underline{I},\underline{\theta};t)$, where $\underline{I} \equiv (I_1, I_2, I_3)$ is a set of 3 action variables, with conjugate angle variables $\underline{\theta} \equiv (\theta_1, \theta_2, \theta_3)$. To be specific, we have a mirror geometry in mind, where $\underline{I}$ represents gyro-action (magnetic moment μ), bounce-action (J_b), and drift-action (magnetic flux Φ_d enclosed by the drift surface). The conjugate angles are associated with gyro-phase, bounce-phase, and drift-phase, in a manner to be discussed shortly.

We further assume that the Hamiltonian is "near-integrable", i.e., that it can be expressed as a sum of an "unperturbed" part $\overline{H}(\underline{I})$, independent of $\underline{\theta}$ and time, and a small (in some sense) "perturbation" $\delta H(\underline{I},\underline{\theta};t)$.

Under $\overline{H}(\underline{I})$ alone, the angle-variables evolve as

$$\dot{\underline{\theta}} \equiv \underline{\omega}(\underline{I}) = \partial\overline{H}/\partial\underline{I} \ , \tag{1}$$

while the actions are invariant:

$$\dot{\underline{I}} = - \partial\overline{H}/\partial\underline{\theta} = 0$$

Hence the 3 particle frequencies $\underline{\omega}\equiv(\omega_1, \omega_2, \omega_3)$ are also invariant. Typically, these frequencies are in the ratio

$$\omega_3 : \omega_2 : \omega_1 \sim \varepsilon^2 : \varepsilon : 1, \tag{2}$$

and represent the drift-frequency $\omega_3 \equiv \omega_d$ (rate of traversing the drift-surface the slow way, i.e., by drifting across field lines), the drift-averaged bounce-frequency $\omega_2 \equiv \overline{\omega}_b$ (ω_b is the rate of traversing the drift-surface the fast way, i.e., along a field line; it is then time-averaged over the drift period $\tau_d = 2\pi/\omega_d$), and the drift-and bounce-averaged gyrofrequency $\omega_1 \equiv \overline{\overline{\Omega}}$ (the local gyro-frequency $\Omega(\underline{R})$ depends on guiding-center position $\underline{R}$; it is to be averaged over the drift surface).

After $\overline{H}(\underline{I})$ is constructed, what is "left over" is $\delta H(\underline{I},\underline{\theta};t)$. (To my knowledge, this has not yet been explicitly carried through.) We require that δH is periodic in $\underline{\theta}$, since these are defined modulo 2π; and in this section we shall suppress the time-dependence. We expand $\delta H(\underline{I},\underline{\theta})$ into a Fourier series:

$$\delta H(\underline{I},\underline{\theta}) = \sum_{\underline{\ell}} H_{\underline{\ell}}(\underline{I}) e^{i\underline{\ell}\cdot\underline{\theta}} \tag{3}$$

with

$$H_{\underline{\ell}}(\underline{I}) \equiv \oint \frac{d^3\theta}{(2\pi)^3} e^{-i\underline{\ell}\cdot\underline{\theta}} \delta H(\underline{I},\underline{\theta}) \ . \tag{3a}$$

(The $\underline{\ell} = 0$ term is omitted, since it is $\overline{H}(\underline{I})$.)

Under $\overline{H}(\underline{I})$, δH has the implicit time-dependence

$$\delta H(t) = \sum_{\underline{\ell}} H_{\underline{\ell}}(\underline{I}) e^{i[\underline{\ell}\cdot\underline{\omega}(\underline{I})t + \underline{\ell}\cdot\underline{\theta}^o]} \tag{4}$$

with respect to an initial state $\underline{I},\underline{\theta}^o$. Its effect is quite different, depending on whether $\underline{\ell}\cdot\underline{\omega}(\underline{I})$ is near zero. To see this, we return to (3), and evaluate the evolution of $\underline{I}(t)$:

$$\dot{\underline{I}} = - \partial H/\partial\underline{\theta} = - \partial\delta H/\partial\underline{\theta} = \sum_{\underline{\ell}} (-i\underline{\ell}) H_{\underline{\ell}}(\underline{I}) e^{i\underline{\ell}\cdot\underline{\theta}} \tag{5}$$

$$\doteq \sum_{\underline{\ell}} (-i\underline{\ell}) H_{\underline{\ell}}(\underline{I}) e^{i[\underline{\ell}\cdot\underline{\omega}(\underline{I})t + \underline{\ell}\cdot\underline{\theta}^o]} \tag{5a}$$

This equation is easily integrated, so long as $\underline{\ell}\cdot\underline{\omega}(\underline{I}) \neq 0$, to obtain

$$\underline{I}(t) - \underline{I}(0) = - \sum_{\underline{\ell}} \frac{\underline{\ell}}{\underline{\ell}\cdot\underline{\omega}(\underline{I})} H_{\underline{\ell}}(\underline{I}) [e^{i\underline{\ell}\cdot\underline{\theta}(t)} - e^{i\underline{\ell}\cdot\underline{\theta}(0)}]. \tag{6}$$

The excursion in action space is thus small, if $H_{\underline{\ell}}$ is small and $\underline{\ell}\cdot\underline{\omega}$ is not small!

The "dangerous" regions are the resonance layers, i.e., the neighborhoods of the surfaces

$$\underline{\ell}\cdot\underline{\omega}(\underline{I}) = 0 \tag{7}$$

in action space. These two-dimensional surfaces, parameterized by the 3 integers ℓ_1, ℓ_2, ℓ_3:

$$\ell_1\omega_1(\underline{I}) + \ell_2\omega_2(\underline{I}) + \ell_3\omega_3(\underline{I}) = 0 \tag{7a}$$

represent values of $\underline{I} = (\mu, J_b, \Phi_d)$ for which the corresponding frequencies have a commensurable relation

$$\ell_1\bar{\bar{\Omega}} + \ell_2\bar{\omega} + \ell_3\omega_d = 0. \tag{7b}$$

The surfaces have zero thickness, and thus zero measure, but their domain of influence is what matters. Under the perturbation, the evolution of $\underline{I}$, from the $\underline{\ell}$th term alone, is $(\delta\underline{I})_{\underline{\ell}} \sim \underline{\ell}H_{\underline{\ell}}/\underline{\ell}\cdot\underline{\omega}$. So the $\underline{\ell}$th resonance denominator has the variation,

$$\delta(\underline{\ell}\cdot\underline{\omega}) \sim \delta(\ell\cdot\partial\omega/\partial\underline{I}\cdot\delta\underline{I}) \sim (\underline{\ell}\cdot\partial\underline{\omega}/\partial\underline{I}\cdot\underline{\ell})H_{\underline{\ell}}/\underline{\ell}\cdot\underline{\omega}$$

Since

$$\partial\underline{\omega}/\partial\underline{I} \equiv \partial^2\bar{H}/\partial\underline{I}\ \partial\underline{I} \equiv \underset{\sim}{\mathcal{H}}, \tag{8a}$$

we obtain

$$\delta(\underline{\ell}\cdot\underline{\omega}) \sim |\mu_{\underline{\ell}}\ H_{\underline{\ell}}|^{1/2}, \tag{9}$$

where

$$\mu_{\underline{\ell}} \equiv \underline{\ell}\cdot\underset{\sim}{\mathcal{H}}\cdot\underline{\ell} \tag{8b}$$

As an estimate of the domain of influence of a resonance, we may use (9), and say that

$$|\underline{\ell}\cdot\underline{\omega}(\underline{I})| \lesssim |\mu_{\underline{\ell}}\ H_{\underline{\ell}}|^{1/2} \tag{10}$$

defines the thickness of the $\underline{\ell}$th resonance. Now we must appeal to the smoothness of the Hamiltonian, to guarantee that the Fourier coefficients $H_{\underline{\ell}}$ fall off sufficiently fast with

$$|\underline{\ell}| \equiv |\ell_1| + |\ell_2| + |\ell_3|. \tag{11}$$

If, e.g., $H_{\underline{\ell}}$ falls off as $e^{-|\underline{\ell}|}$, and if $\mu_{\underline{\ell}}$ is $O(|\underline{\ell}|^2)$, then the high-order resonances can be ignored; while the low-order resonances occupy only a small (but finite) measure of the action-space, for δH sufficiently small. Even the lowest-order resonances may be negligible, if we appeal to the ordering (2); using (2) in (7b), we see that the minimum $|\underline{\ell}|$ is of order ε^{-1}, whereupon smoothness of δH yields $H_{\underline{\ell}} \sim e^{-\varepsilon^{-1}}$, and an exponentially small resonance width (10). This is to be expected, for if resonance effects were appreciable, particles would not be confined effectively.

Let us then, for awhile, limit our attention to the non-resonant part of $\underline{I}$-space, which, for $\varepsilon \ll 1$, is its major part. The $\underline{\theta}$-dependent perturbation in the Hamiltonian can be transformed away (to higher order) by a canonical transformation. We choose to use Lie transforms, as the most expeditious transformation technique[10]. Let $w(\underline{I},\underline{\theta})$ be an arbitrary smooth function, and define the Lie operator

$$L_w \equiv \{w, \quad \}$$
$$\equiv \frac{\partial w}{\partial \underline{\theta}} \cdot \frac{\partial}{\partial \underline{I}} - \frac{\partial w}{\partial \underline{I}} \cdot \frac{\partial}{\partial \underline{\theta}} \tag{12}$$

With the Fourier series

$$w = \sum_{\underline{\ell}} e^{i\underline{\ell}\cdot\underline{\theta}}\, w_{\underline{\ell}}(\underline{I}) \qquad (\underline{\ell}\neq 0)$$

we have

$$L_w = \sum_{\underline{\ell}} e^{i\underline{\ell}\cdot\underline{\theta}}(w_{\underline{\ell}}\, i\underline{\ell}\cdot\partial/\partial\underline{I} - \partial w_{\underline{\ell}}/\partial\underline{I}\cdot\partial/\partial\underline{\theta}). \tag{12a}$$

It is a general property of Lie transforms that canonicity of the phase-space variables is preserved under the operation $\underset{\sim}{z} \to \underset{\sim}{z}' = T\,\underset{\sim}{z}$, where

$$T = \exp L_w = 1 + L_w + \frac{1}{2!}L_w^{\,2} + \ldots \tag{14}$$

Then

$$\underline{I}' = (\exp L_w)\,\underline{I} = \underline{I} + \Sigma\, e^{i\underline{\ell}\cdot\underline{\omega}}\, i\,\underline{\ell}\, w_{\underline{\ell}} + O(w^2) \tag{15a}$$

and

$$\underline{\theta}' = \underline{\theta} - \Sigma\, e^{i\underline{\ell}\cdot\underline{\theta}}\, \partial w_{\underline{\ell}}/\partial\underline{I} + O(w^2) \tag{15b}$$

The Hamiltonian transforms under $T^{-1} = \exp(-L_w)$:

$$H \to H' = T^{-1}H = H - L_w H + \frac{1}{2}L_w^{\,2}H + \ldots \quad . \tag{16}$$

On substituting $H = \overline{H} + \delta H$, and treating w as of order δH, we have

$$H' = \overline{H} + [\delta H - L_w\overline{H}] + [\tfrac{1}{2}L_w^{\,2}\overline{H} - L_w\delta H] + \ldots \tag{16a}$$

So far $w(\underline{I},\underline{\theta})$ has been arbitrary; we now choose it to eliminate the first order term of H':

$$L_w\overline{H} = \delta H\,. \tag{17a}$$

Using (12a), we obtain

$$w_{\underline{\ell}}(\underline{I}) = \frac{H_{\underline{\ell}}(\underline{I})}{i\,\underline{\ell}\cdot\underline{\omega}(\underline{I})} \tag{17b}$$

Now that w is determined, we proceed with the evaluation of the new Hamiltonian H', finding

$$H' = \overline{H} + \sum_{n=1}^{\infty} \frac{(-)^n\, n}{(n+1)!}\, L_w^{\,n}\,\delta H \tag{18}$$

We break this up, as with H, into an angle-independent part $\overline{H}'$ and an angle-dependent part $\delta H'$ (we drop the primes on the variables, but they should be kept in mind):

$$H'(\underline{I},\underline{\theta}) \equiv \overline{H}'(\underline{I}) + \delta H'(\underline{I},\underline{\theta}) \tag{19}$$

From (18), we see that

$$\overline{H}'(\underline{I}) = \overline{H}(\underline{I}) - \frac{1}{2}\overline{L_w\delta H} + O(\delta H)^3 \tag{20a}$$

with the second-order term, from (17b):

$$-\frac{1}{2}\overline{L_w\delta H} = -\frac{1}{2}\frac{\partial}{\partial\underline{I}}\cdot\sum_{\underline{\ell}}\underline{\ell}\,\frac{H_{\underline{\ell}}\,H_{-\underline{\ell}}}{\underline{\ell}\cdot\underline{\omega}} \quad . \tag{20b}$$

The new Hamiltonian H' (19) can now be subjected to a second Lie transformation, with the same form (17b) for the new generator w':

$$w'_{\underline{\ell}} = \frac{H'_{\underline{\ell}}}{i\underline{\ell}\cdot\underline{\omega}'} \tag{21}$$

Here $H'_{\underline{\ell}}$ is a Fourier coefficient of $\delta H'$, as read off from (18), while

$$\underline{\ell}\cdot\underline{\omega}'(\underline{I}) \equiv \underline{\ell}\cdot\partial\overline{H}'/\partial\underline{I} = \underline{\ell}\cdot\underline{\omega}(\underline{I}) - \frac{1}{2}\frac{\partial^2}{\partial\underline{I}\partial\underline{I}} : \underline{\ell}\sum_{\underline{n}}\underline{n}\frac{|H_{\underline{n}}|^2}{\underline{n}\cdot\underline{\omega}} + \ldots$$

is the new resonance denominator.

The second transformation T' serves to eliminate the lowest-order $\underline{\theta}$-dependence from H', which is $O(\delta H)^2$, from (18). The next Hamiltonian $H'' = T'^{-1}H'$ is given by the analog of (18):

$$H'' = \overline{H}' + \sum_{n=1}^{\infty} \frac{(-)^n n}{(n+1)!} L_{w'}{}^{n}\, \delta H' ,$$

and its $\underline{\theta}$-dependence is $O(\delta H')^2 \sim O(\delta H)^4$. Each step raises the order of the $\underline{\theta}$-dependence, in the sequence 1, 2, 4, 8, ... The consequence is a "superconvergent" series of transformations to produce integrability for the nonresonant portion of $\underline{I}$-space. A careful statement of this constitutes the famous KAM theorem.[3,11]

We now turn to the resonant regions of $\underline{I}$-space, i.e., those for which the resonance condition

$$|\underline{\ell}\cdot\underline{\omega}(\underline{I})| \lesssim |\mu_{\underline{\ell}}(\underline{I})H_{\underline{\ell}}|^{1/2} \tag{10}$$

is satisfied for some $\underline{\ell}$. We begin by supposing that only one $\underline{\ell}$ satisfies (10) in some small $\underline{I}$-region, and we examine the neighborhood of the resonance surface $\underline{\ell}\cdot\underline{\omega}(\underline{I}) = 0$. We select some point $\underline{I}_o$ on that surface, and expand the unperturbed Hamiltonian $\overline{H}(\underline{I})$ about that point:

$$\overline{H}(\underline{I}) = \overline{H}(\underline{I}_o) + (\underline{I}-\underline{I}_o)\cdot\underline{\omega}(\underline{I}_o) + \frac{1}{2}(\underline{I}-\underline{I}_o)(\underline{I}-\underline{I}_o) : \partial^2\overline{H}/\partial\underline{I}\,\partial\underline{I}\Big|_{\underline{I}_o} + \ldots \tag{22}$$

From the perturbation δH, we select only the resonant term (and its complex conjugate):

$$H_{\underline{\ell}}(\underline{I})e^{i\underline{\ell}\cdot\underline{\theta}} + \text{c.c.} \doteq H_{\underline{\ell}}(\underline{I}_o)e^{i\underline{\ell}\cdot\underline{\theta}} + \text{c.c.} = 2|H_{\underline{\ell}}(\underline{I}_o)|\cos\psi_{\underline{\ell}} \tag{23}$$

where

$$\psi_{\underline{\ell}} \equiv \underline{\ell}\cdot\underline{\theta} + \arg H_{\underline{\ell}}(\underline{I}_o) \tag{24}$$

is the relative phase of the resonance. The non-resonant terms of δH can be ignored.

Thus the selected terms (22) and (23) yield the evolution equations:

$$\dot{\underline{I}} = -\partial[2|H_{\underline{\ell}}|\cos\psi_{\underline{\ell}}]/\partial\underline{\theta} = 2\underline{\ell}|H_{\underline{\ell}}|\sin\psi_{\underline{\ell}} \tag{25a}$$

$$\dot{\psi}_{\underline{\ell}} = \underline{\ell}\cdot\dot{\underline{\theta}} = \underline{\ell}\cdot\partial\overline{H}/\partial\underline{I} = (\underline{I}-\underline{I}_o)\cdot\underset{\sim}{\mathcal{H}}\cdot\underline{\ell} \tag{25b}$$

$$\ddot{\psi}_{\underline{\ell}} = \dot{\underline{I}}\cdot\underset{\sim}{\mathcal{H}}\cdot\underline{\ell} = 2\mu_{\underline{\ell}}|H_{\underline{\ell}}|\sin\psi_{\underline{\ell}}, \quad \text{by (8b).} \tag{25c}$$

Linearizing (25c) about the stable fixed point ($\psi_{\underline{\ell}} = \pi$ for $\mu_{\underline{\ell}}>0$, $\psi_{\underline{\ell}} = 0$ for $\mu_{\underline{\ell}} < 0$), we have

$$\ddot{\psi}_{\underline{\ell}} = - 2|H_{\underline{\ell}} \mu_{\underline{\ell}}| \delta\psi_{\underline{\ell}} \quad , \tag{25d}$$

or a phase oscillation at frequency

$$\omega_{\underline{\ell}} = |2H_{\underline{\ell}} \mu_{\underline{\ell}}|^{1/2} \tag{26}$$

The evolution of $\underline{I}$ is along $\underline{\ell}$, which is tangent to the unperturbed energy surface $\overline{H}(\underline{I})$, since $\underline{\ell} \cdot \underline{\omega}(\underline{I}_o) \equiv \underline{\ell} \cdot \partial H/\partial \underline{I}|_{\underline{I}_o} = 0$. The amplitude of the excursion in $\underline{I}$, along $\underline{\ell}$, is deduced from the "pendulum" equations (25) to be

$$\Delta\underline{I} = \underline{+}\, \underline{\ell} |8H_{\underline{\ell}}/\mu_{\underline{\ell}}|^{1/2} \quad , \tag{27}$$

for the separatrix orbit through the unstable fixed point.

Expression (27) leads to a resonance half-width

$$\delta(\underline{\ell}\cdot\underline{\omega}) \sim \underline{\ell}\cdot\partial\underline{\omega}/\partial\underline{I}\cdot\Delta\underline{I} = |8H_{\underline{\ell}} \mu_{\underline{\ell}}|^{1/2} \tag{27a}$$

consistent with the estimate (9) from examining the nonresonant terms. We may now fit out each resonant surface $\underline{\ell}$ with the width (27a), which varies along the surface with the choice of $\underline{I}_o$.

Our considerations so far make sense only if these resonant regions do not overlap. If they do, one must keep both (overlapping) resonant perturbations. In general, the motion is then stochastic throughout the overlapping resonances.[11]

The introduction of a phase frequency (26) for each resonance leads to the appearance of secondary resonances, and in fact to a hierarchy of such resonances. These serve to broaden the structure of the separatrices and to expand the domain of influence of the resonances.

These refinements are not needed to identify the location (in $\underline{I}$-space) of the stochastic regions. Our procedure is as follows:

(a) Find $\overline{H}(\underline{I})$ ("First catch your hare").

(b) Calculate the set of resonance surfaces $\underline{\ell} \cdot \underline{\omega}(\underline{I}) = 0$.

(c) Find the perturbation $\delta H(\underline{I},\underline{\theta})$ and calculate the Fourier coefficients $H_{\underline{\ell}}$.

(d) Calculate the width of each surface.

(e) Locate the regions of resonance overlap.

(f) Examine the topology of the stochastic regions.

It remains to discuss the consequences of stochasticity; we defer that to Sec. $\underline{F}$.

D. Non-canonical Variables in Hamiltonian Dynamics

The formal developments of plasma kinetic theory have, in the past, followed one of two paths. The majority of plasma theorists have utilized physical, non-canonical variables, and ignored the advantages of a Hamiltonian approach, which expresses the vector flow in phase space in terms of a single scalar function, and which makes use of canonical transformations. A minority has recognized the utility of the Hamiltonian methods, but has been plagued by the non-physical, gauge-dependent nature of the canonical variables. Only recently has the realization come that a Hamiltonian formalism can be effectively utilized in terms of non-canonical variables.[12]

We begin by considering motion in a given magnetostatic field $\underline{B}_o(\underline{x})$. The standard approach is to choose a vector potential $\underline{A}_o(\underline{x})$, form a Lagrangian

$$L(\underline{r},\underline{v}) = \frac{1}{2} v^2 + \frac{e}{m} \underline{v}\cdot\underline{A}_o(\underline{r}) \quad , \tag{1}$$

(differing from the usual only by the factor m), define the canonical momentum

$$\underline{p}(\underline{r},\underline{v}) \equiv \partial L/\partial\underline{v} = \underline{v} + \frac{e}{m} \underline{A}_o(\underline{r}) \quad , \tag{2}$$

and proceed to the Hamiltonian

$$H(\underline{r},\underline{p}) \equiv \underline{v}\cdot\underline{p} - L = \frac{1}{2} v^2 = \frac{1}{2}[\underline{p} - \frac{e}{m} \underline{A}_o(\underline{r})]^2 \; . \tag{3}$$

For any phase function $g(\underline{r},\underline{p})$, its evolution equation is

$$\dot{g} \equiv \dot{\underline{r}}\cdot\partial g/\partial\underline{r} + \dot{\underline{p}}\cdot\partial g/\partial\underline{p} \; . \tag{4}$$

The Hamiltonian equations:

$$\dot{\underline{r}} = \partial H/\partial\underline{p} , \; \dot{\underline{p}} = - \partial H/\partial\underline{r} \; , \tag{5}$$

then convert (4) to

$$\dot{g} = \{g,H\}, \tag{6}$$

where

$$\{g,f\} \equiv \frac{\partial g}{\partial\underline{r}} \cdot \frac{\partial f}{\partial\underline{p}} - \frac{\partial g}{\partial\underline{p}} \cdot \frac{\partial f}{\partial\underline{r}} \tag{7}$$

With the use of Poisson Brackets for the physical, but non-canonical, variable $\underline{v}$, we can dispose of the non-physical canonical momentum $\underline{p}$. Thus, to obtain the equation of motion, we calculate

$$\dot{\underline{v}} = \{\underline{v},H\} = \{\underline{v},\frac{1}{2} v^2\} = \{\underline{v},\underline{v}\}\cdot\underline{v} \quad . \tag{8}$$

We then evaluate the Fundamental Poisson Bracket $\{\underline{v},\underline{v}\}$ from (7), and obtain

$$\{v_\mu , v_\lambda\} = \varepsilon_{\mu\lambda\sigma} \Omega_\sigma(\underline{r}) \; , \tag{9}$$

where $\underline{\Omega}(\underline{x}) \equiv (e/m)\underline{B}_o(\underline{x})$ is the local gyrofrequency. (Note that (9) is exact; no small-ε expansion has yet been made.) On substituting (9) into (8), we obtain the Lorentz equation $\dot{\underline{v}} = \underline{v} \times \underline{\Omega}(\underline{r})$, without the explicit appearance of $\underline{p}$ and $\underline{A}_o$.

The relation (9) establishes (with $\nabla \cdot \underline{B}_o = 0$) the requirements for a symplectic manifold,[13] and leads to the use of the Darboux algorithm for constructing a guiding-center (g.c.) formalism. We begin with the physical variables $\underline{r}, \underline{v}$, and work toward the 2 gyro-variables (θ_g, μ), which are canonically conjugate, plus a set of 4 g.c. variables $\underline{R}$, P (g.c. position and parallel momentum) which are not conjugate with respect to each other. The method requires slow spatial variation of $\underline{B}_o(\underline{x})$, and so the smoothness requirements mentioned earlier come into play again.

The g.c. position is defined in the usual way ($\hat{b}$ is the unit vector of $\underline{B}_o$):

$$\underline{R} = \underline{r} + \varepsilon\ \underline{v} \times \hat{b}(\underline{r})/\Omega(\underline{r}) + 0(\varepsilon^2), \qquad (10)$$

and the g.c. Hamiltonian takes on the familiar form

$$H(\mu;\underline{R},P) = \frac{1}{2} P^2 + \mu B_o(\underline{R}) + 0(\varepsilon^2). \qquad (11)$$

To obtain the equations of motion, we need the new Fundamental Poisson Brackets:

$$\{\underline{R},\underline{R}\} = \hat{b}(\underline{R}) \times \underset{\sim}{I}/B^*(\underline{R}), \qquad (12a)$$

$$\{\underline{R},P\} = \hat{b}(\underline{R}) + [\varepsilon P/B^*(\underline{R})]\hat{b}\times(\hat{b}\cdot\nabla)\hat{b}, \qquad (12b)$$

which are again exact relations. Here $B^* \equiv B_o + \varepsilon P\hat{b}\cdot\nabla\times\hat{b}$, and $\underset{\sim}{I}$ is the identity matrix. The evolution equations:

$$\dot{\underline{R}} = \{\underline{R}, H\} = \{\underline{R},P\} P + \{\underline{R},\underline{R}\}\cdot\mu\nabla B_o(\underline{R}), \qquad (13a)$$

$$\dot{P} = \{P,H\} = -\{\underline{R},P\}\cdot \mu\nabla B_o(\underline{R}) \qquad (13b)$$

then yield the classic drifts and the mirror force.

Now, if the system is perturbed by a field $\delta\underline{B}(\underline{x})$, which is not slowly varying over a gyroradius, but which is small in amplitude, the Lie technique of the preceding section can be used. Similarly, a time-dependent perturbation $\delta\underline{E}(\underline{x},t)$, weak but rapidly varying, can be treated by Lie techniques. We discuss such perturbations in the following section.

E. Oscillatory Perturbations

So far we have treated the Hamiltonian as time-independent, whereupon energy is conserved. We now consider the perturbing effect of an electromagnetic wave, and we begin by requiring it to be periodic in time, with fixed fundamental frequency ω, but not necessarily sinusoidal. The Hamiltonian then is perturbed by a term of the form

$$\sum_{\underline{\ell},m} H_{\underline{\ell},m}(\underline{I})\, e^{i\underline{\ell}\cdot\underline{\theta} + im\omega t}, \tag{1}$$

expressing periodicity in $\underline{\theta}$ and t, and using the global variables $\underline{I}$, $\underline{\theta}$ of part $\underline{C}$.

The time-independent formalism of ($\underline{C}$) can be taken over directly, if we extend the phase-space $(\underline{I},\underline{\theta})$ to include the fourth degree of freedom (h,τ), with $\tau \equiv \omega t$ and $\{\tau,h\} = 1$; and take as the extended Hamiltonian

$$\mathcal{H} = \overline{\mathcal{H}}(\underline{I},h) + \delta\mathcal{H}(\underline{I};\underline{\theta},\tau), \tag{2}$$

with

$$\overline{\mathcal{H}} \equiv \overline{H}(\underline{I}) + h\omega, \tag{3}$$

$$\delta\mathcal{H} = \sum_{\underline{L}} \mathcal{H}_{\underline{L}}(\underline{I})\, e^{i\underline{L}\cdot\underline{\Theta}} \tag{4}$$

Here $\underline{L} \equiv (\underline{\ell},m)$, $\underline{\Theta} \equiv (\underline{\theta},\tau)$. As a phase variable, $\dot{\tau} = \{\tau,\mathcal{H}\} = \{\tau,h\}\omega = \omega$, as required; and $\dot{h} = \{h,\mathcal{H}\} = \{h,\tau\}\partial\mathcal{H}/\partial\tau = -\omega^{-1}\partial H/\partial t = -\omega^{-1}\dot{H}$, so $\dot{\mathcal{H}} = 0$, as expected.

We now follow $\underline{C}$, obtaining, in analogy to (C.1),

$$\dot{\underline{\Theta}} \equiv \underline{\Omega}(\underline{I}) = (\dot{\underline{\theta}}, \dot{\tau}) = (\underline{\omega},\omega). \tag{5}$$

A resonance surface $(\underline{\ell},m)$ is now defined by

$$\underline{L}\cdot\underline{\Omega} \equiv \underline{\ell}\cdot\underline{\omega}(\underline{I}) + m\omega = 0, \tag{6}$$

i.e., the wave frequency ω is a rational combination of the 3 particle frequencies:

$$\omega = -\frac{1}{m}(\ell_1\omega_1 + \ell_2\omega_2 + \ell_3\omega_3)\ . \tag{6a}$$

For $\underline{I}$ not within a resonance layer, we can again eliminate the perturbation (1), by the analog of (C. 17 b):

$$w_{\underline{L}}(\underline{I}) = \frac{H_{\underline{\ell},m}(\underline{I})}{i(\underline{\ell}\cdot\underline{\omega}(\underline{I}) + m\omega)}, \tag{7}$$

obtaining the second-order static term in the new Hamiltonian [see (C.20 b)]:

$$\overline{\mathcal{H}}'^{(2)} = -\frac{1}{2}\frac{\partial}{\partial \underline{I}} \cdot \sum_{\underline{\ell},m} \underline{\ell}\, \frac{|H_{\underline{\ell},m}(\underline{I})|^2}{\underline{\ell}\cdot\underline{\omega}(\underline{I}) + m\omega}\ . \tag{8}$$

To put some flesh on this skeleton, we need the Fourier coefficients $H_{\underline{\ell},m}$ of (1) for Eqs. (7) and (8). Formally, we have

$$H_{\underline{\ell},m}(\underline{I}) \equiv \oint \frac{d^3\theta}{(2\pi)^3} e^{-i\underline{\ell}\cdot\underline{\theta}} \int_0^{2\pi} \frac{d\tau}{2\pi} e^{-im\tau} H(\underline{I},\underline{\theta};t), \qquad (9)$$

where H is the sum of the Hamiltonian H_o in the absence of the wave, plus a perturbation δH, which, to lowest order in the field $\delta\underline{E}$, is

$$\delta H(\underline{I},\underline{\theta};t) = \int d^3x \; \delta\underline{A}(\underline{x},t)\cdot\underline{j}(\underline{x};\underline{I},\underline{\theta}) \;, \qquad (10)$$

with $\underline{j}$ given by (A.1):

$$\underline{j}(\underline{x};\underline{I},\underline{\theta}) = e\underline{v}(\underline{I},\underline{\theta})\delta[\underline{x} - \underline{r}(\underline{I},\underline{\theta})]. \qquad (11)$$

We now introduce the eikonal form (A.18) for $\delta\underline{A}$ into (10):

$$\delta H(\underline{I},\underline{\theta};t) = e^{-i\omega t} \int d^3x \; e^{iS(\underline{x})} \; \tilde{\underline{A}}(\underline{x})\cdot\underline{j}(\underline{x};\underline{I},\underline{\theta}), \qquad (12)$$

and then use (11):

$$\delta H(\underline{I},\underline{\theta};t) = e \; e^{-i\omega t} \; \underline{v}(\underline{I},\underline{\theta})\cdot\tilde{A}(\underline{r}(\underline{I},\underline{\theta}))e^{iS(\underline{r}(\underline{I},\underline{\theta}))} \;. \qquad (13)$$

From Eq. (D.10) (suppressing ε),

$$\underline{r} = \underline{R} + \hat{b}\times\underline{v}/\Omega(\underline{R}) \equiv \underline{R} + \underline{\rho} \quad , \qquad (14)$$

introducing the gyroradius, we expand the eikonal:

$$S(\underline{r}) \equiv S(\underline{R} + \underline{\rho}) = S(\underline{R}) + \underline{\rho}\cdot\nabla S(\underline{R}) + \ldots = S(\underline{R}) + \underline{k}_\perp(\underline{R})\cdot\underline{\rho} + \ldots \qquad (15)$$

For the slowly varying amplitude, we can set $\tilde{\underline{A}}(\underline{r}) = \tilde{\underline{A}}(\underline{R})$. For the velocity, we have, from (14),

$$\underline{v} = \dot{\underline{R}} + \dot{\underline{\rho}} \doteq v_\parallel \hat{b} + \underline{v}_\perp \quad , \qquad (16)$$

neglecting drifts. Thus (13) becomes

$$\delta H = e \; e^{-i\omega t + iS(\underline{R})} \; e^{i\underline{k}_\perp(\underline{R})\cdot\underline{\rho}} \; \tilde{\underline{A}}(\underline{R})\cdot(v_\parallel\hat{b} + \underline{v}_\perp). \qquad (17)$$

Now (17) is to be inserted into (9), to yield the coefficients $H_{\underline{\ell},m=-1}(\underline{I})$; but this calculation has not yet been performed. In its place, we have a local result, to which we shall refer later [Eq. (21)].

Let us now examine the transformation of action $\underline{I}$ induced by the Lie generator (7). From (C 15a), we have (by analogy)

$$\underline{I}' = \underline{I} + \sum_{\underline{\ell},m} e^{i(\underline{\ell}\cdot\underline{\theta} + m\omega t)} \, i\underline{\ell} w_{\underline{\ell},m}(\underline{I}) + \ldots$$

$$= \underline{I} + \sum_{\underline{\ell},m} \frac{\underline{\ell}}{\underline{\ell}\cdot\underline{\omega}(\underline{I}) + m\omega} H_{\underline{\ell},m}(\underline{I}) \, e^{i(\underline{\ell}\cdot\underline{\theta} + m\omega t)} + \ldots \tag{18}$$

It is $\underline{I}'$, not $\underline{I}$, which is invariant under the perturbation; the variation of $\underline{I}$ is quite evident from (18), or its equivalent, the analog of (C.6).

Next, let us look at the question of invariance under slow (η) variation (in time) of parameters, i.e., the magnetostatic field: $\underline{B}_o(\underline{x}) \to \underline{B}_o(\underline{x};\eta t)$, and the parameters of the perturbation: $\underline{\tilde{A}}(\underline{x}) \to \underline{\tilde{A}}(\underline{x};\eta t)$, $\underline{k}(\underline{x}) \to \underline{k}(\underline{x},\eta t)$, $\omega \to \omega(\eta t)$. So long as the actions $\underline{I}$ are non-resonant, i.e., are not within a resonance layer

$$|\underline{\ell}\cdot\underline{\omega}(\underline{I}) + m\omega| \lesssim |\mu_{\underline{\ell}} H_{\underline{\ell},m}|^{1/2} , \tag{19}$$

we expect the actions to remain invariant, from the general principles of dynamics. However, $\overline{H}(\underline{I};\eta t)$ implies $\underline{\omega}(\underline{I};\eta t)$ so the resonances $\underline{\ell}\cdot\underline{\omega}(\underline{I};\eta t) = 0$ and $\underline{\ell}\cdot\underline{\omega}(\underline{I};\eta t) + m\omega(\eta t) = 0$ are slowly moving surfaces. In general, parameters may vary by an appreciable amount, and hence a point $\underline{I}$ may find itself crossed by a resonance layer, or by a stochastic region (representing overlapping resonance layers). A general procedure for treating such a crossing appears not yet to be available.

We now return to the local, semi-canonical variables of Sec. D, for which an explicit expression for the perturbation Fourier amplitudes has recently been obtained.[14] Locally, the analog of $\underline{\omega} \equiv \partial\overline{H}/\partial\underline{I}$ is $(\dot{\theta}, \underline{\dot{R}}) = (\Omega(\underline{R}), P\hat{b}(\underline{R}))$ to lowest order in ε; i.e., the g.c. drifts and the mirroring force are higher order. The resonance condition (6) is replaced by the local gyroresonance condition

$$\ell\Omega(\underline{R}) + k_{\parallel}(\underline{R})P + m\omega = 0 \tag{20}$$

with the eikonal representation (A.18) defining $k_{\parallel}(\underline{x}) \equiv \hat{b}(\underline{x})\cdot\nabla S(\underline{x})$. The Fourier amplitude (in θ) is found to be (for $m = -1$):

$$H_{\ell} = -\underline{\tilde{A}}(\underline{R})\cdot\left[\frac{\ell\Omega(\underline{R})}{k_{\perp}(\underline{R})} J_{\ell}\hat{k}_{\perp} + \frac{2i\Omega\mu}{k_{\perp}}\frac{\partial J_{\ell}}{\partial\mu}\hat{b}\times\hat{k} + PJ_{\ell}\hat{b}\right]\exp i[S(\underline{R}) - \omega t + \ell(\theta + \frac{\pi}{2})], \tag{21}$$

to lowest order in ε. After Lie-transforming away the first-order (in wave-amplitude) perturbation, it re-appears in second-order, in the analog of (8):

$$\overline{\mathcal{H}}'^{(2)} = \sum_{\ell=-\infty}^{+\infty} \left(\ell\frac{\partial}{\partial\mu} + k_{\parallel}(\underline{R})\frac{\partial}{\partial P}\right)\frac{|H_{\ell}|^2(\underline{R},P,\mu)}{\omega - \ell\Omega(\underline{R}) - k_{\parallel}(\underline{R})P} \; . \tag{22a}$$

To (22a) must be added a second-order perturbation term appearing already in $\overline{\mathcal{H}}$, before the Lie transform:

$$+|\tilde{\underline{A}}(\underline{R})|^2. \tag{22b}$$

The consequence of these second-order (in $\tilde{\underline{A}}$) terms in $\overline{\mathcal{H}'}$ is to produce modifications in the g.c. evolution. Thus the mirroring equation (D. 13b) receives the perturbation contribution

$$\dot{P}'^{(2)} = \{P', \overline{\mathcal{H}'}^{(2)}\} = -\hat{b}(\underline{R})\cdot\nabla\overline{\mathcal{H}'}^{(2)}(\underline{R},P,\mu), \tag{23}$$

which is denoted "ponderomotive force". The other derivatives of $\overline{\mathcal{H}'}^{(2)}$(22) yield ponderomotive cross-field drifts, nonlinear gyrofrequency shift, and a difference between g.c. momentum and parallel velocity:

$$\hat{b}\cdot\dot{\underline{R}} = P + \partial\overline{\mathcal{H}'}^{(2)}/\partial P \tag{24}$$

To interpret this quadratic difference, we imagine that the wave is turned on adiabatically in time. Since parallel momentum P is not affected by time-variation of the amplitude, but only by its spatial variation [see (23)], the invariance of P implies, by (24), that the wave creates a change in the g.c. velocity. This adiabatic change in velocity, when summed over all the nonresonant g.c., is to be interpreted as the plasma component of wave parallel-momentum density.

We have, in this section, paid little attention so far to the resonant particles, i.e., those for which (we set m = −1)

$$\omega \cong \ell_1\omega_1(\underline{I}) + \ell_2\omega_2(\underline{I}) + \ell_3\omega_3(\underline{I}). \tag{25}$$

Since the actions $\underline{I}$ may be appreciably modified in the case of resonance, these are the particles for which irreversible energy transfer from the wave is possible. We begin by supposing that only a single resonance (25) affects some given $\underline{I}$; to be specific, we have the finite-width condition (see $\underline{C}$. 27a):

$$|\ell_1\omega_1(\underline{I}) + \ell_2\omega_2(\underline{I}) + \ell_3\omega_3(\underline{I}) - \omega| < |8\mathcal{H}_{\underline{L}}(\underline{I};\eta t)\mu_{\underline{\ell}}|^{1/2}. \tag{26}$$

where $\mathcal{H}_{\underline{L}}$ of (4) has a slow time-dependence because of growth (say) of the wave amplitude. If the wave amplitude grows as $\exp\int^t \gamma(t')dt'$, the resonance width grows (half as fast), and traps more and more particles. As a particle, initially non-resonant at some $\underline{I}'$, sees (so to speak) the resonance getting ever closer, it becomes ever more agitated, as we see from (18), with H_L growing in time, and $\underline{I}'$ invariant. At some time (depending not only on $\underline{I}'$, but also on its angle $\underline{\theta}'$), the particle crosses the separatrix between the non-resonant and resonant regions of $\underline{I}'$-space. This separatrix is itself a fuzzy region, because of secondary resonances. Within this stochastic layer, the instability of the orbit produces a partial amnesia. After an uncertain sojourn in the stochastic layer, the particle enters the interior of the resonance, where it undergoes regular phase oscillations, with gradually growing frequency [see Eq. (C. 26)]. In this process, the particle has changed its energy, by interacting with the wave; the wave energy must thus change, as a result of this resonant capture of particles.

Before proceeding to considering resonance overlap, we shall examine more closely the invariants under a single resonance. We take

$$H(\underline{I},\underline{\theta};t) = \overline{H}(\underline{I}) + H_{\underline{L}} \sin(\underline{\ell}\cdot\underline{\theta} - \omega t), \tag{27}$$

where all the non-resonant terms have been transformed away, and we have dropped the primes. The one resonant term satisfies

$$\underline{\ell}\cdot\underline{\omega}(\underline{I}) \approx \omega \tag{28}$$

for $\underline{I}$ in the region of present interest. From (27), we have (with $\psi_{\underline{\ell}} \equiv \underline{\ell}\cdot\underline{\theta} - \omega t$)

$$\dot{H} = \partial H/\partial t = -\,\omega\, H_{\underline{L}} \cos\psi_{\underline{\ell}} \;, \tag{29}$$

$$\dot{\underline{I}} = -\,\partial H/\partial\underline{\theta} = -\,\underline{\ell}\, H_{\underline{L}} \cos\psi_{\underline{\ell}} \;.$$

We see immediately that the combinations

$$J_1 \equiv \frac{I_1}{\ell_1} - \frac{H}{\omega}, \quad J_2 \equiv \frac{I_2}{\ell_2} - \frac{H}{\omega}, \quad J_3 \equiv \frac{I_3}{\ell_3} - \frac{H}{\omega}, \tag{30}$$

are exact invariants. Thus, even at resonance and with time dependence, the Hamiltonian has a complete set of invariants, and thus is integrable.

Next we allow for two (overlapping) resonances, $\underline{\ell}$ and $\underline{n}$; i.e., for some region in $\underline{I}$-space,

$$|\underline{\ell}\cdot\underline{\omega}(\underline{I})-\omega| < |8\, H_{\underline{\ell}}\, \mu_{\underline{\ell}}|^{1/2}$$

and (31)

$$|\underline{n}\cdot\underline{\omega}(\underline{I})-\omega| < |8\, H_{\underline{n}}\, \mu_{\underline{n}}|^{1/2} \;.$$

We take as the Hamiltonian

$$H(\underline{I},\underline{\theta};t) = \overline{H}(\underline{I}) + H_{\underline{\ell}} \sin(\underline{\ell}\cdot\underline{\theta} - \omega t) + H_{\underline{n}} \sin(\underline{n}\cdot\underline{\theta} - \omega t). \tag{32}$$

Now

$$\dot{H} = -\,\omega\, H_{\underline{\ell}} \cos\psi_{\underline{\ell}} - \omega\, H_{\underline{n}} \cos\psi_{\underline{n}} \tag{33}$$

$$\dot{\underline{I}} = -\,\underline{\ell}\, H_{\underline{\ell}} \cos\psi_{\underline{\ell}} - \underline{n}\, H_{\underline{n}} \cos\psi_{\underline{n}}$$

Since $\underline{I}$ is varying in the $(\underline{\ell},\underline{n})$ plane, its perpendicular component $J_{\perp} \equiv \underline{\ell}\times\underline{n}\cdot\underline{I}$ is one invariant. To find a second invariant, we look for some combination of $\underline{I}$ and H; after a bit of algebra, we obtain

$$J_x \equiv \frac{\underline{n}\cdot\underline{n}\;\underline{\ell} + \underline{\ell}\cdot\underline{\ell}\;\underline{n} - (\underline{\ell}\cdot\underline{n})(\underline{\ell} + \underline{n})}{\underline{\ell}\cdot\underline{\ell}\;\underline{n}\cdot\underline{n} - (\underline{\ell}\cdot\underline{n})^2}\cdot\underline{I} - \frac{H}{\omega}\,, \tag{34}$$

as a second invariant under (33). Hence the overlap of 2 resonances causes the loss of only invariant; the stochasticity is limited to the overlap region and in addition must lie on the intersection of the two invariant surfaces. If the perturbation is time-independent ($\omega\to 0$), the second invariant J_x can be replaced by H.

Another case of invariance preservation under resonance overlap is the "multiplet overlap", i.e., the set of mutually overlapping resonances (ℓ_1, ℓ_2, ℓ_3) with two integers (say ℓ_1 and ℓ_2) fixed, while the third runs over a set of three or more values. Here the model Hamiltonian is

$$H(\underline{I},\underline{\theta};t) = \overline{H}(\underline{I}) + e^{i(\ell_1\theta_1 + \ell_2\theta_2 - \omega t)} \sum_{\ell_3} e^{i\ell_3\theta_3} H_{\ell_3} + \text{c.c.} \qquad (35)$$

$$\equiv \overline{H} + \delta H + \text{c.c.}$$

with the resonances

$$|\ell_1\omega_1(\underline{I}) + \ell_2\omega_2(\underline{I}) + \ell_3\omega_3(\underline{I}) - \omega| < |8 H_{\ell_3}\mu_{\underline{\ell}}|^{1/2}, \qquad (36)$$

overlapping in pairs of successive ℓ_3 values. We form

$$\begin{aligned} \dot{H} &= -i\omega\ \delta H + \text{c.c.}, \\ \dot{I}_1 &= -i\ell_1\ \delta H + \text{c.c.}, \\ \dot{I}_2 &= -i\ell_2\ \delta H + \text{c.c.}; \end{aligned} \qquad (37)$$

thus the two combinations

$$J_1 \equiv \frac{I_1}{\ell_1} - \frac{H}{\omega}, \quad J_2 \equiv \frac{I_2}{\ell_2} - \frac{H}{\omega} \qquad (38)$$

are invariant under the multiplet overlap. In the case $\omega\to 0$, we would use H and

$$J_x \equiv \frac{I_1}{\ell_1} - \frac{I_2}{\ell_2} . \qquad (39)$$

Next on the agenda is the case of several discrete frequencies in the time-dependence of the perturbation: $\delta\underline{E}(\underline{x},t) = \sum_j \underline{E}^j(\underline{x}) \exp(-i\omega_j t)$. Accordingly, we now have a set of primary resonance layers at

$$\omega_j = \underline{\ell}\cdot\underline{\omega}(\underline{I}) \qquad (40)$$

as well as beat resonances of the form

$$\sum_j m_j\omega_j = \underline{\ell}\cdot\underline{\omega}(\underline{I}) \quad . \qquad (41)$$

The earlier formulas in this section are to be generalized in more or less obvious ways. With an increase in the number of effective resonances, the portion of action-space which is stochastic increases rapidly.

Finally, we must make the transition to a continuous frequency spectrum, corresponding to a perturbation occurring in a finite time-interval. Let f(t) be a smooth function, nonzero only over the interval $0<t<T$. Then

$$f(t) = \int \frac{d\omega}{2\pi} \tilde{f}(\omega)\, e^{-i\omega t}, \text{ with } \tilde{f}(\omega) = \int dt\, e^{+i\omega t} f(t),$$

having a spread $\Delta\omega \sim T^{-1}$. Let $\tilde{f}(\omega)$ be approximated by a sum of N delta-functions:

$$\tilde{f}(\omega) \approx \sum_{j=1}^{N} f_j \delta(\omega - \omega_j) \ ,$$

of separation $\delta\omega \sim \Delta\omega/N$, with $f_j \sim f(\omega_j)\delta\omega$. If f(t) represents the amplitude of the perturbation, we have a set of N resonances at

$$\omega_j = \underline{\ell}\cdot\underline{\omega}(\underline{I}), \ j = 1,\ldots,N.$$

for each $\underline{\ell}$. The separations of the resonances are $\delta\omega \sim O(N^{-1})$, while their widths are $O(f_j^{1/2}) \sim O(N^{-1/2})$. Hence, in the continuum limit ($N \to \infty$), resonance overlap always occurs. This argument seems to imply that a perturbation over a finite time interval necessarily produces stochastic particle motion.

F. Kinetic Description

As seen in the Introduction, the family of particle orbits under a Hamiltonian $H(\underline{I},\underline{\theta};t)$ is equivalent to the evolution in time of the phase space density $f(\underline{I},\underline{\theta};t)$ in the Vlasov description. The advantages of using f include the possibilities of coarse-graining and of adding collisional effects in a simple way; in addition, f is needed to determine the self-consistent fields which enter the Hamiltonian.

Retaining the Vlasov model for a while, we have

$$\frac{\partial f}{\partial t}(\underline{I},\underline{\theta};t) + \{f,H\} = 0 \ , \tag{1}$$

the Liouville equation in terms of Poisson brackets. Note that the set of variables $(\underline{I},\underline{\theta})$ is not unique; in Secs. C and E we have discussed transformations among such sets. For different $(\underline{I},\underline{\theta})$ sets, we have different Hamiltonian functions H, different phase-space-density functions (although their values are the same at corresponding points), and different current densities $\underline{j}(\underline{x};\underline{I},\underline{\theta})$ (again as functions, not values). To be more precise, for a transformation (using $\underset{\sim}{z} \equiv (\underline{I},\underline{\theta})$)

$$\underset{\sim}{z} \to \underset{\sim}{z}'(\underset{\sim}{z}),$$

we have $$H(\underset{\sim}{z}) \to H'(\underset{\sim}{z}');$$

and $$f \to f'$$

with $$f(\underset{\sim}{z};t) = f'(\underset{\sim}{z}';t); \tag{2}$$

as well as

$$\underline{j}(\underline{x};\underset{\sim}{z}) \to \underline{j}'(\underline{x};\underset{\sim}{z}') = \underline{j}(\underline{x};\underset{\sim}{z}). \tag{3}$$

For the Maxwell equations, we need [see (A. 4)]

$$\underline{j}(\underline{x},t) = \int d^6z\, \underline{j}(\underline{x};\underset{\sim}{z})f(\underset{\sim}{z};t) = \int d^6z'\underline{j}'(\underline{x};\underset{\sim}{z}')f'(\underset{\sim}{z}';t) \tag{4}$$

where

$$\underline{j}'(\underline{x};\underset{\sim}{z}') = e\underline{v}(\underset{\sim}{z}')\delta(\underline{x} - \underline{r}(\underset{\sim}{z}')). \tag{5}$$

In (4), since $\underset{\sim}{z}'$ is a dummy variable, the prime can be dropped, but the functions $\underline{j}'$ and $\underline{j}$ (as well as f' and f) are different. For a Lie transform, we can be more explicit.[10] The functions are transformed by the operator T^{-1} of Sec. C; thus

$$\begin{aligned}\underline{j}'(\underline{x};\underset{\sim}{z}) &= T^{-1}\, \underline{j}(\underline{x};\underset{\sim}{z}) \\ &= \underline{j}(\underline{x};\underset{\sim}{z}) - \{w(\underset{\sim}{z}), \underline{j}(\underline{x};\underset{\sim}{z})\} + 0(w)^2 \ .\end{aligned} \tag{6}$$

In any representation, we can separate phase-space functions $A(\underline{I},\underline{\theta})$ into $\underline{\theta}$-independent and $\underline{\theta}$-dependent parts: $A = \overline{A} + \delta A$, i.e.,

$$A(\underline{I},\underline{\theta}) \equiv \overline{A}(\underline{I}) + \sum_{\underline{\ell}} A_{\underline{\ell}}(\underline{I})\, e^{i\underline{\ell}\cdot\underline{\theta}} \ . \tag{7}$$

For two such functions A and B

$$\overline{AB} = \overline{A}\,\overline{B} + \sum_{\underline{\ell}} A_{\underline{\ell}}\, B_{-\underline{\ell}} \tag{8}$$

Using (8) in (4), we have

$$\underline{j}(\underline{x},t) = (2\pi)^3 \int d^3I[\overline{\underline{j}}(\underline{x};\underline{I})\overline{f}(\underline{I};t) + \sum_{\underline{\ell}} \underline{j}_{\underline{\ell}}(\underline{x};\underline{I})\underline{f}_{-\underline{\ell}}(\underline{I};t)] \ . \tag{9}$$

With this breakup in (1), we have

$$(\frac{\partial}{\partial t} + i\ \underline{\ell}\cdot\underline{\omega}(\underline{I}))f_{\underline{\ell}}(\underline{I};t) = H_{\underline{\ell}}(\underline{I};t)i\underline{\ell}\cdot \frac{\partial}{\partial \underline{I}}\ \overline{f}(\underline{I};t) + 0(\delta H\delta f). \tag{10}$$

Now, if we transform to a new representation, for which $\delta H \rightarrow 0$, the right side of (10) vanishes, and only "ballistic" solutions remain:

$$f_{\underline{\ell}}(\underline{I};t) = f_{\underline{\ell}}(\underline{I};0)\ e^{-i\underline{\ell}\cdot\underline{\omega}(\underline{I})t}. \tag{11}$$

For non-resonant particles, this transformation can be accomplished, at least in principle. After elimination of δH, to all orders, we turn to the evolution of $\overline{f}$:

$$\frac{\partial}{\partial t}\ \overline{f}(\underline{I};t) = 0 \tag{12}$$

from (1); therefore

$$\overline{f}(\underline{I};\ t) = \overline{f}(\underline{I};\ 0).$$

If, in addition, we suppose that $f_{\underline{\ell}}(\underline{I};\ 0)$ vanishes, i.e., that f is initially $\underline{\theta}$-independent, then (11) tells us that $f_{\underline{\ell}}(\underline{I})$ vanishes for all time. The current equation (9) then takes on a very simple form, for the contribution of the non-resonant (NR) $\underline{I}$:

$$\underline{j}_{NR}(\underline{x},t) = (2\pi)^3 \int_{NR} d^3I\ \overline{\underline{j}}(\underline{x};\underline{I})\overline{f}(\underline{I};0). \tag{13}$$

(The contribution of the resonant-$\underline{I}$ particles must still be considered;

since that is more difficult, we shall deal with their effect by other means, below.) In (13), $\overline{\underline{j}}(\underline{x};\underline{I})$ means the $\underline{\theta}$-independent part of the current-density function, in a representation where δH vanishes. Thus, from (6), we have

$$\overline{\underline{j}'}(\underline{x};\underline{I}) = \overline{\underline{j}}(\underline{x};\underline{I}) - \oint \frac{d^3\theta}{(2\pi)^3} \{w(\underline{I};\underline{\theta}), \underline{j}(\underline{x};\underline{I},\underline{\theta})\} + O(w^2). \tag{14}$$

The second term of (14) is, in the representation (7),

$$+ i \frac{\partial}{\partial \underline{I}} \cdot \sum_{\underline{\ell}} w_{\underline{\ell}}(\underline{I}) \underline{j}_{-\underline{\ell}}(\underline{x};\underline{I}) \ . \tag{14a}$$

If we use ($\underline{E}$.7), with m = −1, (14a) becomes

$$\frac{\partial}{\partial \underline{I}} \cdot \sum_{\underline{\ell}} \frac{H_{\underline{\ell},m=-1}(\underline{I}) \underline{j}_{-\underline{\ell}}(\underline{x};\underline{I})}{\underline{\ell}\cdot\underline{\omega}(\underline{I}) - \omega} \ . \tag{14b}$$

For the coefficient $H_{\underline{\ell},m}$, we use ($\underline{E}$.10) with $\delta \underline{A}(\underline{x}, t) = \delta \underline{A}(\underline{x}) \exp(-i\omega t)$:

$$H_{\underline{\ell},m=-1}(\underline{I}) = \int d^3x' \delta \underline{A}(\underline{x}') \cdot \underline{j}_{\underline{\ell}}(\underline{x}';\underline{I}) \ , \tag{15}$$

and substitute into (14b), with $\delta \underline{A}(\underline{x}) \equiv (i\omega)^{-1} \delta \underline{E}(\underline{x})$, obtaining

$$\int d^3x' \ \underset{\sim}{\sigma}^{NR}(\underline{x},\underline{x}';\underline{I};\omega) \cdot \delta \underline{E}(\underline{x}') \ , \tag{16}$$

with

$$\underset{\sim}{\sigma}^{NR}(\underline{x},\underline{x}';\underline{I};\omega) \equiv \frac{\partial}{\partial \underline{I}} \cdot \sum_{\underline{\ell}} \frac{\underline{j}_{-\underline{\ell}}(\underline{x};\underline{I}) \underline{j}_{\underline{\ell}}(\underline{x}';\underline{I})}{i\omega \ (\underline{\ell}\cdot\underline{\omega}(\underline{I}) - \omega)} \tag{17}$$

as the contribution of a particle at (unperturbed) $\underline{I}$ to the 2-point conductivity tensor, at frequency ω. This equation is consistent with, but more restricted than, formula ($\underline{A}$.13). We note that the matrix (17) is anti-Hermitian, representing the adiabaticity of the non-resonant particles, i.e., their inability to exchange energy irreversibly with the field. On integrating over the non-resonant distribution (13), we obtain

$$\underset{\sim}{\sigma}^{NR}(\underline{x},\underline{x}';\omega) = (2\pi)^3 \int_{NR} d^3I \ \overline{f}(\underline{I};0) \ \underset{\sim}{\sigma}^{NR}(\underline{x},\underline{x}';\underline{I};\omega). \tag{18}$$

Having examined the second term of (14), we return to the first. Since $\underline{v} = \underline{p} - (e/m)\underline{A}(\underline{r},t)$, a perturbation $\delta \underline{A}(\underline{x},t)$ induces a change in velocity (at fixed momentum): $\delta \underline{v} = -(e/m)\delta \underline{A}(\underline{r},t)$. Thus, from (E. 11), the current develops a first order (in $\delta \underline{A}$) term

$$- \frac{e^2}{m} \delta(\underline{x}-\underline{r}) \ \delta \underline{A}(\underline{x},t) \ . \tag{19}$$

On integrating over f, as in (4), we obtain the local contribution to $\underline{j}(\underline{x},t)$:

$$-(n_o(\underline{x})e^2/m) \ \delta E(\underline{x},t)(i\omega)^{-1} \ , \tag{20}$$

that is, a susceptibility contribution $[\underset{\sim}{\chi} \equiv 4\pi i \underset{\sim}{\sigma}/\omega]$:

$$\underset{\sim}{\chi}(\underline{x},\underline{x}';\underline{\omega}) = - (\omega_p^2(\underline{x})/\omega^2)\delta(\underline{x} - \underline{x}'), \tag{21}$$

characterizing a cold unmagnetized plasma. Thus the additional contribution

(17) represents the effect of finite "temperature" and finite magnetic field!

The non-resonant susceptibility [(21) plus (18)] can be used for (at least) two purposes, besides the obvious one of determining the wave propagation. Let us form the expression

$$\frac{1}{4\pi}\int d^3x \int d^3x' \delta \underline{E}^*(\underline{x}) \cdot \underset{\sim}{\chi}'(\underline{x},\underline{x}';\omega)\cdot \delta\underline{E}(\underline{x}') \equiv U; \tag{22}$$

$\underset{\sim}{\chi}'$ is the hermitian part of the total linear susceptibility. Using (18) and (21), we evaluate U:

$$U = -\frac{1}{4\pi}\int d^3x |\delta\underline{E}(\underline{x})|^2 \,\omega_p^2(\underline{x})/\omega^2 + \omega^{-2}\int d^6z \overline{f}(\underline{I};0)\frac{\partial}{\partial \underline{I}}\cdot\sum_{\underline{\ell}} \frac{|\int d^3x \underline{j}_{\ell}(\underline{x};\underline{I})\cdot\delta\underline{E}(\underline{x})|^2}{\underline{\ell}\cdot\underline{\omega}(\underline{I}) - \omega} \tag{23}$$

We note that

$$\omega_p^2(\underline{x}) = \int d^6z \overline{f}(\underline{I};0)\,\frac{4\pi e^2}{m}\,\delta(\underline{x}-\underline{r}(\underline{z})); \tag{24}$$

so U is a linear functional of $\overline{f}(\underline{I})$, and is quadratic in $\delta\underline{E}(\underline{x})$.

We now use the theorem[15] relating U to the second-order (in $\delta\underline{E}$) Hamiltonian:

$$\int d^6z \overline{f}(\underline{I})\overline{\mathcal{H}'}^{(2)}(\underline{I}) = -U, \tag{25}$$

from which the "pondermotive Hamiltonian" is

$$\overline{\mathcal{H}'}^{(2)}(\underline{I}) = -\frac{\delta U}{\delta\overline{f}(\underline{I})(2\pi)^3} = \frac{e^2}{m\omega^2}\int d^3x|\delta\underline{E}(\underline{x})|^2 \oint \frac{d^3\theta}{(2\pi)^3}\,\delta(\underline{x}-\underline{r}(\underline{I},\underline{\theta})) + \omega^{-2}\frac{\partial}{\partial\underline{I}}\cdot\sum_{\underline{\ell}}\frac{|\int d^3x\underline{j}_{\ell}(\underline{x};\underline{I})\cdot\delta\underline{E}(\underline{x})|^2}{\underline{\ell}\cdot\underline{\omega}(\underline{I})-\omega}, \tag{26}$$

in agreement with ($\underline{E}$.8) and a generalization of ($\underline{E}$.22b). Note that, as $\omega \to \infty$, the second term of (26), the "kinetic" term, falls off as ω^{-3}, while the "cold" first term falls off only as ω^{-2}.

The wave energy consists of the vacuum contribution, $(4\pi)^{-1}\int d^3x(|\delta\underline{E}|^2 + |\delta\underline{B}|^2)$, plus the adiabatic response of the nonresonant particles, $\partial(\omega U)/\partial\omega$. We thus expect a conservation law for the sum of the wave energy and the resonant-particle energy.[4,16] The time dependence of the wave amplitude can thus be determined from a study of the resonant particles' motion, and in particular from their total energy. The latter is much easier to calculate than the current-density of the resonant particles. Other conservation laws (momentum, action) are often helpful also.

We now turn to the resonant particles, recognizing that our classification system for particles is fuzzy. For particles where $|\underline{\ell}\cdot\underline{\omega}(\underline{I}) - \omega|$ is so small that the $\underline{\theta}$-dependence cannot be transformed away by perturbation methods, we may still distinguish between regular and stochastic orbits (although that distinction, in turn, is fuzzy). For the regular orbits, well-trapped in a resonance, a non-perturbative transformation, representing a different topology, is possible to new $\underline{I}'$, $\underline{\theta}'$, whereupon the previous discussion applies, so long as an orbit remains in a regular region. For stochastic orbits, a statistical description would seem appropriate. The aim is to derive a Fokker-Planck equation, valid in some region of $\underline{I}$-space, of the form

$$\frac{\partial \overline{f}}{\partial t}(\underline{I};t) = \frac{\partial}{\partial \underline{I}} \cdot [\underset{\sim}{D}(\underline{I}) \cdot \frac{\partial \overline{f}}{\partial \underline{I}}] , \tag{27}$$

where the diffusion tensor has the standard form

$$\underset{\sim}{D}(\underline{I}) = \int_0^\infty d\tau < \dot{\underline{I}}(t)\dot{\underline{I}}(t+\tau) > , \tag{28}$$

the averaging being over $\underline{\theta}$-space, or equivalently over time t. We note that the averaging is not over any randomness of the Hamiltonian; that would be extrinsic stochasticity, and would pertain to a different class of problems, e.g. collisional effects. The evaluation of $\underset{\sim}{D}(\underline{I})$ is difficult, but considerable progress has been made.[11]

As the parameters of the problem (say $\underline{B}_0(\underline{x})$) vary in time, so does the diffusivity $\underset{\sim}{D}(\underline{I})$. We may expect it to be smooth in $\underline{I}$, but to vary by orders of magnitude over $\underline{I}$-space, being effectively zero in the adiabatic regions. These regions themselves are then slowly moving in time.

Only in the stochastic regions, where $\underset{\sim}{D} \neq 0$, is $\underline{I}$-entropy created. We define the latter, following Boltzmann, as

$$S_I(t) \equiv - \int d^3I \, \overline{f}(\underline{I};t) \, \ell n \, \overline{f}(\underline{I};t). \tag{29}$$

From (27), we obtain

$$\frac{dS_I(t)}{dt} = \int d^3I \, \underset{\sim}{D}(\underline{I};t) : \frac{\partial \overline{f}}{\partial \underline{I}} \frac{\partial \overline{f}}{\partial \underline{I}} ; \tag{30}$$

this is positive (or zero), since $\underset{\sim}{D}$ is a positive-definite matrix. The usual arguments then lead to the tendency for $\overline{f}(\underline{I})$ to become flat over the stochastic region. The rate of flattening may of course be quite different along the non-trivial principal axes of $\underset{\sim}{D}$.

G. Ray trajectories

Because the dynamics of a ray is formally the same as that of a particle, with the dispersion relation (A.22) playing the part of the Hamiltonian, we can take over much of the preceding discussion, making allowance for the non-reality of the Hamiltonian.

Here we give a simple immediate application of the second-order perturbation formula (C.20):

$$\overline{D}'(\underline{I};\omega) = \overline{D}(\underline{I};\omega) - \frac{1}{2}\frac{\partial}{\partial \underline{I}} \cdot \sum_{\underline{\ell}} \underline{\ell} \frac{D_{\underline{\ell}}(\underline{I};\omega) D_{-\underline{\ell}}(\underline{I};\omega)}{\underline{\ell} \cdot \frac{\partial \overline{D}}{\partial \underline{I}}(\underline{I};\omega)} . \tag{1}$$

To interpret this, we begin with $D(\underline{x}, \underline{k}; \omega)$, and break it up into two parts:

$$D \equiv \bar{D} + \delta D, \tag{2}$$

such that $\bar{D}$ is integrable; i.e., we can find, for $\bar{D}$, a complete set of action-invariants $\underline{I}(\underline{x}, \underline{k}; \omega)$, such that $I_\mu = (2\pi)^{-1}\oint_\mu \underline{k}\cdot d\underline{x}$, where the line integral is over any of the topologically distinct closed curves (not ray paths) on the invariant tori in $(\underline{k},\underline{x})$-space of the rays of $\bar{D}$. For example, if the magnetostatic geometry leading to $D(\underline{x},\underline{k};\omega)$ is that of a tokamak, $\bar{D}$ may be a cylindrical model, which is integrable, since it has two symmetry directions. The "perturbation" δD, representing physical toricity, is now Fourier analyzed in the angle variables conjugate to $\underline{I}$, yielding the coefficients $D_{\underline{\ell}}(\underline{I};\omega)$, which now do not satisfy the reality condition $[D^*_{\underline{\ell}} \neq D_{-\underline{\ell}}]$.

We determine the eigenvalues[17] of the waves represented by the rays, through the EBK quantization $\underline{I} = \underline{N}$ (a set of three integers), ignoring the Keller-Maslov index for simplicity here; and set $\overline{D'}(\underline{I}{=}\underline{N};\omega) = 0$ to determine $\omega(\underline{N})$:

$$\bar{D}(\underline{N};\omega) = \frac{1}{2}\frac{\partial}{\partial \underline{N}} \cdot \sum_{\underline{\ell}} \underline{\ell}\, \frac{D_{\underline{\ell}}(\underline{N};\omega)D_{-\underline{\ell}}(\underline{N};\omega)}{\underline{\ell}\cdot\frac{\partial \bar{D}}{\partial \underline{N}}(\underline{N};\omega)} \tag{3}$$

If $\omega_o(\underline{N})$ is the set of eigenvalues for the "unperturbed" problem $\bar{D}(\underline{N};\omega) = 0$, we let $\delta\omega(\underline{N}) \equiv \omega(\underline{N}) - \omega_o(\underline{N})$, and

$$\delta\omega(\underline{N})\frac{\partial \bar{D}(\underline{N};\omega_o)}{\partial \omega_o} = \frac{1}{2}\frac{\partial}{\partial \underline{N}} \cdot \sum_{\underline{\ell}} \underline{\ell}\, \frac{D_{\underline{\ell}}(\underline{N};\omega_o)D_{-\underline{\ell}}(\underline{N};\omega_o)}{\ell\cdot\frac{\partial \bar{D}}{\partial \underline{N}}(\underline{N};\omega_o)} \tag{3a}$$

for the frequency shift $\delta\omega(\underline{N})$. This is equivalent to standard second order perturbation theory, but we stress that the functions $\bar{D}$, $D_{\underline{\ell}}$ and the eigenvalues ω_o, ω are in general complex. No assumption of near-reality is needed.

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