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Quantum Kinetics of Neutrinos in Hot, Dense Environments

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ABSTRACT OF THE DISSERTATION

Quantum Kinetics of Neutrinos in Hot, Dense Environments

by

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The subject of this dissertation is the evolution of neutrino distributions in hot, dense astrophysical environments, such as the early Universe, core collapse supernovae or compact object mergers. This is an important problem in astrophysics because the dynamics and the composition of these systems can be strongly influenced by neutrinos, leading to potential modification of nucleosynthesis and of astrophysical observables. In these environments, neutrinos can undergo both coherent forward scattering and direction-changing or inelastic collisions. The transport equations for flavored neutrinos, or quantum kinetic equations (QKEs), are derived from first principles, beginning with quantum field theory and Standard Model neutrino interactions. The QKEs can reduce to the standard Schrödinger-like equations for neutrino flavor evolution in the limit of low matter density, and to a Boltzmann-like equation describing neutrino scattering at high density. In addition, in high-density, anisotropic environments we find a novel process that can mediate coherent exchange of particle number and flavor information between neutrino and antineutrino states (spin coherence). We discuss the prospects for modification of the standard picture of neutrino flavor evolution in supernovae and other astrophysical environments by non-forward scattering and spin coherence.

Chapter 1

Introduction

This dissertation addresses the difficult problem of deriving self-consistent equations for the evolution of flavored neutrino distributions in hot, dense matter. Equations of this type are known as quantum kinetic equations (QKEs), and methods of first-principles derivation of QKEs for neutrinos, including all effects that can be relevant in a high-density environment, are a subject of novel and ongoing research. In general, the form of QKEs to leading order in certain limits (*e.g.,* in the absence of inelastic or non-forward scattering, or in the absence of coherent effects) is relatively well understood, but there are few formulations that propose to treat all of these regimes in a single formalism and few calculations of potentially important next-to-leading order corrections.

Our goal is to describe both quantum mechanical effects, such as coherent evolution due to neutrino mass and coherent forward scattering, as well as incoherent, non-forward scattering and thermalization, in addition to any other phenomena that may appear in conditions relevant for astrophysical neutrinos. The issue of self-consistent determination of the evolution of neutrino distributions, including the evolution of lepton number and flavor content, is an important one in astrophysics, since neutrinos play a key role in environments such as the early Universe, core collapse supernovae and compact object mergers, which are important sites for the origin of the elements and can produce observable signals that, when detected and interpreted, can give insights both into the nature of these environments and into fundamental physics.

Neutrino flavor evolution arises from the fact that the neutrino vacuum propagation states (mass eigenstates) do not coincide with the interaction eigenstates (neutrino flavor). This leads to neutrino oscillations, where, for example, a neutrino produced in the electron flavor has an amplitude to oscillate into the muon or tau flavor. The correspondence between neutrino propagation states and flavor states is further modified by the presence of background matter or other neutrinos. For example, in the presence of electrons (but not muons or tauons), electron neutrinos acquire a positive potential energy compared to muon or tau neutrinos. This potential energy acts as an effective in-medium mass, shifting neutrino masses and mixing angles. Neutrinos propagating through a slowly-changing medium can exhibit large-scale flavor transformation via the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1,2]. If the neutrinos themselves provide a significant contribution to the interaction potential energy, additional nonlinear flavor-changing phenomena can arise, such as collective flavor transformation [3–31] or matter-neutrino resonance [32].

In hot, dense astrophysical environments such as the early Universe, core collapse supernovae, compact object mergers or black hole accretion disks, these coherent phenomena can in principle take place at the same time and in the same regions as inelastic scattering. The regime where both phenomena can take place is poorly understood, and can involve a complicated interplay of flavor evolution and the competing processes of collisional decoherence and thermalization. In addition, neutrino - antineutrino [33] or, in models with sterile neutrinos, active - sterile [34–39] transformation may occur in the same regime. The goal of this dissertation is to develop, from first principles, a fully self-consistent formalism that can treat neutrinos in such an environment, and to begin to explore the implications of the resulting description.

1.1 Neutrinos in Astrophysical Environments

At energies far below the electroweak scale (center-of-mass energy of much less than $\sim$100 GeV), the neutrino scattering cross-section is low and neutrinos have very long mean free paths compared to charged particles. Neutrino interactions thus become important when the environment is sufficiently hot and dense and when there is sufficient time or distance for the interaction to occur. Hot, dense environments that persist for a sufficient amount of time over sufficient distance scales are typically found in the context of astrophysics. Examples of such environments include the early Universe, core collapse supernovae and compact object mergers.

1.1.1 The Early Universe

In the early Universe, neutrinos remain coupled to the plasma and are held in thermal equilibrium until the temperature drops below ∼1 MeV. At this point, the neutrinos begin to decouple, *i.e.,* the mean free path becomes comparable to the Hubble scales and at subsequent times only a small fraction of neutrinos undergo inelastic or direction-changing scattering.

During Big Bang nucleosynthesis (BBN), the neutrinos are almost completely decoupled. However, neutrinos are far more numerous than baryons (by a factor of $\sim 10^{10}$). Therefore, there is enough residual scattering of neutrinos on baryons to set the proton-neutron ratio via the reactions

$$n + \nu_e \rightleftharpoons p + e^- \qquad p + \bar{\nu}_e \rightleftharpoons n + e^+ \tag{1.1}$$

Subsequently, the neutron-proton ratio is the key determinant of elemental abundances that emerge from BBN.

Further, neutrinos influence the structure of the early Universe and the properties of the Cosmic Microwave Background (CMB) via gravitational interactions. Because the neutrino mass is low, neutrinos remain ultra-relativistic for a long time. Thus they contribute to the radiation density (*i.e.,* energy density in relativistic particles) of the Universe and have a large effect on the expansion rate. The energy density in relativistic particles is measurable from observations of the CMB and is given by the observable astrophysical parameter N_{eff}. This quantity is defined in such a way that any fully-thermalized ultrarelativistic chiral fermion contributes 1 unit to N_{eff}. Thus with three perfectly thermalized Standard Model neutrinos we would expect $N_{eff} = 3$, but any process that can give a nonthermal contribution to the neutrino spectrum will alter the value of N_{eff}. For example, the nonthermal contribution to neutrino and antineutrino spectra from electron-positron annihilation after neutrino decoupling shifts the value of N_{eff} predicted by the Standard Model from 3.000 to 3.046. In the presence of non-Standard Model processes, there can be additional shifts. Current estimates of N_{eff} from a combination of data from Planck and WMAP satellites [40] give $N_{eff} = 3.71 \pm 0.40$ which is consistent with 3.046, but if future measurements can improve the precision and rule out this value, this will be tantamount to a detection of new physics.

In a standard model of cosmology, the physics of neutrino flavor in the early Universe is assumed to be trivial, as all flavors of neutrinos and antineutrinos are expected to be present in almost equal numbers, leading to no large-scale flavor transformation effects. However, in some models, for example in models with sterile neutrinos, there are additional particles that can decay or oscillate into neutrinos, giving nonthermal, spin- and flavor-asymmetric contributions to the neutrino spectrum. If this process occurs long before neutrino decoupling, then this contribution should become fully thermalized with the exception of any lepton number asymmetry. However, if this process occurs around the time of neutrino decoupling or during BBN, then neutrino flavor evolution together with thermalization of the nonthermal decay spectrum can impact both nucleosynthesis and the value of N_{eff}.

1.1.2 Massive Stars and Core Collapse Supernovae

In supernovae and in massive stars, neutrinos transport most of the energy, entropy and lepton number due to their low but nonzero interaction strength and long mean free paths. In addition to the long mean free path, neutrino fields in environments with a high neutrino density, such as core collapse supernovae, can carry a large amount of energy due to their Fermi-Dirac statistics, which can lead to the neutrino Fermi energy becoming much larger than the temperature.

Neutrinos play an important role in stellar evolution prior to core collapse. In supernova progenitor stars in late stages of nuclear burning, neutrino emission efficiently carries energy and entropy away from the nuclear burning region without generating radiation pressure, thus keeping density and therefore the burning rate high and leading the star to evolve rapidly towards core collapse.

A core collapse supernova occurs when an iron core of a massive star (or, for lighter progenitors, an oxygen-neon-magnesium core) accumulates enough mass to reach the Chandrasekhar limit and collapses to a proto-neutron star. The collapse of a $\sim$ 1.4-Solar mass object to a radius of $\sim$ 10km leads to the release of a large amount of gravitational potential energy, comprising about 10-20% of the rest energy of the core. The density and temperature of the proto-neutron star are high enough to trap neutrinos, which gradually escape by diffusion over the course of several seconds, while other particles that participate in the electromagnetic interaction are trapped for much longer periods. Therefore the vast majority of the gravitational potential energy is released in the form of neutrinos and antineutrinos of all flavors. In addition to energy, as electrons capture on protons to form neutrons, lepton number is converted from electrons to electron neutrinos and removed from the core as the neutrinos diffuse out, leading to gradual neutronization of the core.

Multi-D core collapse supernova simulations, *e.g.,* Ref.s [41–60], show that the transfer of energy from the core to the envelope by neutrinos is an indispensable part of the explosion mechanism. While only a small portion of the neutrinos scatter in the envelope outside the proto-neutron star, the total energy released in neutrinos is two orders of magnitude larger than the gravitational binding energy of the envelope, so even subdominant inelastic scattering of neutrinos in the envelope can be sufficient to give an explosion.

While supernova simulations are otherwise very sophisticated, they do not self-consistently treat the evolution of neutrino flavor. However, the issue of neutrino flavor transformation is of vital importance in the supernova environment. Since the plasma in a supernova contains electrons but not muons or tauons, electron neutrinos have different interactions with the plasma than μ or τ neutrinos and thus the neutrino-plasma energy transfer is sensitive to neutrino flavor. In addition, flavor is important for supernova nucleosynthesis because, as in the early Universe, electron neutrinos play a crucial role by setting the neutron-proton ratio.

In the event of a galactic Supernova, the neutrino signal will likely be detected by a variety of detectors. This detection will include some flavor, helicity (*i.e.,* neutrino vs. antineutrino) and energy spectrum information. This information can yield insights both into fundamental neutrino physics (absolute neutrino mass, mass hierarchy, the existence of sterile neutrinos or other exotic possibilities such as hidden neutrino interactions), into the supernova explosion mechanism and into implications for supernova nucleosynthesis.

It is expected that the supernova neutrino spectrum will be heavily modified by flavor transformation effects. Therefore, a full understanding of supernova neutrino kinetics, including the evolution of neutrino flavor, is necessary to self-consistently model the supernova explosion mechanism and supernova nucleosynthesis, to understand future measurements of supernova neutrino signals, and to plan future detectors in a way that maximizes the yield from these signals.

1.1.3 Compact Object Mergers

The role of neutrinos in binary neutron star mergers is similar to that in supernovae. As two neutron stars merge, they convert gravitational potential and kinetic energy into neutrinos, which are largely responsible for carrying this energy away from the dense, hot center of the merger. Compact object mergers are considered to be a likely site for r-process nucleosynthesis [61–63], and the emerging neutrinos may have a large impact on this process.

The rate of compact object mergers per galaxy is several orders of magnitude lower than that of core collapse supernovae, so a detection of a compact object merger neutrino signal by any near-future neutrino detector is unlikely. However, similarly to the case of supernovae, neutrino flavor may have an impact on the dynamics of the merger, since neutrinos can transport energy, entropy and lepton number over long distances and their effect on matter is dependent on flavor. With the potential for near-future observation of compact object mergers via gravitational radiation signals [64], there is increasing interest in the dynamics of these events, and neutrinos may well play a role here.

1.2 Necessity for Quantum Kinetic Equations

Quantum kinetic equations become necessary when neutrinos can undergo flavor evolution at the same time as inelastic scattering, or when second-order corrections to coherent forward scattering (such as spin coherence or trajectory bending) become significant. Standard approaches to neutrino flavor and neutrino inelastic scattering have relied on a clear separation of scales: in low-density environments, inelastic scattering is assumed to be unimportant, and the neutrinos are assumed to propagate on straight-line trajectories, with flavor evolution described by Schrödinger-like equations. Conversely, in high-density environments flavor evolution is assumed to be unimportant (presumably due to effects of collisional decoherence and the suppression of oscillations by the matter effect), and neutrino kinetics are assumed to be adequately described by a set of Boltzmann equations for non-mixing massless particles.

However, in many astrophysical environments, this separation of scales does not exist. In the early Universe, the lepton number is low, so the matter effect, which is proportional to lepton number, is insignificant even when neutrino collisions are still important. With only Standard Model physics, the neutrino flavor asymmetry is assumed to be negligible, in which case flavor evolution would not occur; however, if flavor asymmetry can be introduced by a non-Standard Model process (such as flavor-asymmetric decay of sterile neutrinos) then the flavor evolution and gradual thermalization of this asymmetry will be governed by the quantum kinetic equations.

In core collapse supernovae, the separation of scales between the Boltzmann regime and the coherent flavor evolution regime is not necessarily clear-cut. In treatments of supernova neutrino evolution, it is assumed that neutrinos decouple at a 'neutrino sphere' at a radius of ten to a few tens of kilometers, and thereafter only undergo coherent forward scattering. However, in Ref.s. [65, 66] it was pointed out that despite the fact that non-forward-scattered neutrinos make up only a small proportion of the neutrino population (approximately 1 in 1000) far above the neutrino sphere, this scattered halo can lead to a sizable modification of neutrino flavor evolution. This occurs because sufficiently far from the proto-neutron star the coupling between the nearly collinear neutrinos emerging from the neutrinosphere is suppressed (the neutrino-neutrino forward scattering term is proportional to $1 - \cos\theta \propto \theta^2$ for small intersection angles θ). For this reason scattered neutrinos, which can come in at large intersection angles, can give a dominant contribution to the neutrino-neutrino interaction potential. It is the neutrino-neutrino potential that generates nonlinear feedback responsible for the phenomena of collective flavor transformation, so the large contribution from non-forward scattered neutrinos cannot be neglected.

In addition, it is not certain that coherent phenomena cannot occur in the high-density environment of the proto-neutron star where collisions become dominant. In multi-angle simulations of supernova neutrino flavor evolution, large-scale collective oscillations typically do not set in until the distance from the center is in excess of $\sim$ 100 km, which is above the neutrino sphere. However, these simulations are still highly simplified and do not include the effects of the neutrino halo, spin coherence or deviations from spherical symmetry, which may under some conditions significantly modify this behavior and push collective effects further towards the center. In particular with respect to spin coherence, the QKEs allow for coherent neutrino-antineutrino mixing, and in Chapter 4 and Ref. [33] we point out that there exists a level crossing between electron neutrinos and antineutrinos at an electron to baryon ratio of somewhat less than 1/3. The exact condition for this resonance is modified by the contribution to the forward scattering potential from neutrinos, and is typically satisfied in the high-density region in or just above the proto-neutron star, in the immediate vicinity of the neutrinosphere. In models with sterile neutrinos, similar level crossings between electron neutrinos and sterile neutrinos can occur at any location in the proto-neutron star, depending on the value of the sterile neutrino mass [34–39]. If these level crossing can result in coherent neutrino-antineutrino or active-sterile conversion, or if the effects of spin coherence at these locations can feed back on flavor evolution and jump-start collective flavor transformation near or below the neutrino sphere, then a full QKE treatment will be needed to describe these phenomena.

In compact object mergers, conditions are similar to those in core collapse supernovae. However, the neutrino spectra can be considerably different (*e.g.*, at an early time in the merger, antineutrinos may predominate in the emission spectrum), and there can be much larger deviations from spherical symmetry. However, the same arguments apply. The neutrino halo should still be present, and may be even more dominant due to a large amount of matter inhomogeneity, and there is the same possibility of coherent phenomena in higher-density regions.

In summary, it is not clear that the regimes in which neutrinos undergo coherent flavor evolution *vs.* non-forward and inelastic scattering can be separated in hot, dense environments such as the early Universe, core collapse supernovae or compact object mergers. In addition, next-to-leading order corrections to coherent flavor evolution such as neutrino-antineutrino coherence in anisotropic environments could possibly lead to new large-scale effects. The full treatment of all these phenomena in the same environment requires a self-consistent formulation of quantum kinetic equations.

1.3 State of Previous Work

Quantum kinetic equations for flavored particles have been considered in the context of general scalars or fermions, as well as in the context of active-active or active-sterile neutrino oscillations in the presence of collisions [27, 67–97]. Some of this work relied on first-principles derivations, albeit typically with a large number of simplifying assumptions. The general, well-known result is that the QKEs should contain terms responsible for coherent flavor evolution as well as a collision terms with nontrivial flavor structure that lead to thermalization, damping of flavor oscillations (collisional decoherence) and gradual equalization of the distributions of particles of different flavors (collisional flavor depolarization). For weakly coupled particles such as neutrinos, the collision term has a steeper energy dependence than the coherent forward scattering terms that lead to coherent flavor transformation, and can thus become negligible at sufficiently low temperatures and densities. In this case, the QKEs reduce to the standard Schrödinger-like equations for flavor evolution. On the other hand, in a high-density, high-energy limit the coherence between different neutrino flavors may become completely suppressed, in which case the QKEs reduce to the Boltzmann equation.

In the case of fermions (Majorana or Dirac) and in the absence of assumptions of homogeneity and isotropy, the derivation of the full self-consistent QKEs becomes highly nontrivial due to the complicated tensor structure of the 2-point function that describes particle distributions. This structure was described in our work in Ref.s. [95, 96] using a derivation from the two-particle irreducible (2PI) effective action in nonequilibrium quantum field theory. A similar structure was obtained in a derivation from the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in Ref. [76]. We find that the tensor structure of the 2-point function leads to the phenomenon of spin coherence, where in the presence of anisotropy either in the matter background or in the neutrino fields, the propagation eigenstates are not neutrinos or antineutrinos but coherent mixtures of the two. This neutrino-antineutrino mixing leads to the possibility of coherent conversion or exchange of flavor information between neutrinos and antineutrinos. Our work in Ref. [95], in which we derive the QKEs from first principles and obtain expressions for collision terms, spin coherence terms and other next-to-leading order corrections to coherent flavor evolution, is included in its entirety as Chapter 2.

Much work has been done on the evolution of neutrino flavor in the coherent limit, where effects of the collision term and of spin coherence are neglected [1–9, 11–31, 98]. This work shows a rich phenomenology of flavor transformation, including collective effects in environments where the neutrino flux is high. In the case of a supernova explosion, these effects can radically alter the emerging neutrino spectrum, with potential impact on the explosion mechanism, on supernova nucleosynthesis and on the observable neutrino signal.

Recently, there has been some work on the effects of coherent neutrino-antineutrino mixing in a different context than in the QKEs. This mixing stemms from a neutrino magnetic moment [99–101]. Neutrinos can acquire a magnetic moment via radiative corrections from Standard Model interactions or from physics beyond the Standard Model. Neutrinos with a magnetic moment can interact with a strong magnetic field, resulting in coherent neutrino-antineutrino mixing for neutrinos traveling on trajectories not aligned with the direction of the field. A magnetic field of requisite strength may be found in some core collapse supernovae with rapidly rotating progenitors. The results of preliminary numerical simulations indicate that in the presence of a strong magnetic field, flavor evolution may be modified by spin coherence effects [100, 101].

As shown in Ref.s. [76, 95, 96], the QKEs predict spin coherence due to weak interactions alone, even in the absence of a neutrino magnetic moment or a strong magnetic field. Under conditions of sufficiently high density and anisotropy, such as those found in the vicinity of the proto-neutron star, this effect exceeds the magnetic neutrino-antineutrino mixing that can be achieved with reasonable values of the neutrino magnetic moment and supernova magnetic field [96]. In Ref. [33] we describe a numerical calculation using a toy model of coherent spin transformation with a single flavor of neutrino. We show that under some conditions, nonlinear feedback due to neutrino-neutrino interactions can amplify the effects of spin coherence, potentially leading to large-scale neutrino-antineutrino transformation. This work is included as Chapter 4 of this dissertation.

1.4 Outline of Dissertation

In Chapter 2, we begin by introducing two-component spinor notation and the two-particle irreducible (2PI) effective action approach to nonequilibrium quantum field theory (NEQFT). We then present the full derivation of the quantum kinetic equations from first principles, beginning with 2PI NEQFT.

The key results of Chapter 2 are presented and described in Chapter 3. Here, the QKEs are presented in a compact notation. We further give each term that enters into these equations and discuss the properties and physical interpretation of these terms.

The derivation of the quantum kinetic equations has led to the discovery of terms that can, in principle, mediate coherent exchange of particle number and flavor information between neutrino and antineutrino states. In Chapter 4, we discuss the possibility that these terms can lead to macroscopic, observable effects in a supernova or compact object merger environment and present a calculation in a simple single-flavor toy model that suggests the possibility of large-scale neutrino-antineutrino conversion in such an environment.

Finally, in Chapter 5, we present a summary of our knowledge of the QKEs and their behavior thus far and indicate several possible directions for future work, focusing primarily on future numerical exploration of the behavior of neutrinos under conditions where QKEs can be important.

Chapter 2

Derivation of Quantum Kinetic Equations from First Principles

2.1 Abstract

We present a formulation of the quantum kinetic equations (QKEs) which govern the evolution of neutrino flavor at high density and temperature. Here, the structure of the QKEs is derived from the ground up, using fundamental neutrino interactions and quantum field theory. We show that the resulting QKEs describe coherent flavor evolution with an effective mass when inelastic scattering is negligible. The QKEs also contain a collision term. This term can reduce to the collision term in the Boltzmann equation when scattering is dominant and the neutrino effective masses and density matrices become diagonal in the interaction basis. We also find that the QKE's include equations of motion for a new dynamical quantity related to neutrino spin. This quantity decouples from the equations of motion for the density matrices at low densities or in isotropic conditions. However, the spin equations of motion allow for the possibility of coherent transformation between neutrinos and antineutrinos at high densities and in the presence of anisotropy. Although the requisite conditions for this exist in the core collapse supernova and compact object merger environments, it is likely that only a self consistent incorporation of the QKEs in a sufficiently realistic model could establish whether or not significant neutrino-antineutrino conversion occurs.

2.2 Introduction

In this paper we address the difficult problem of how neutrino flavor evolves in a general medium. The stakes are high because neutrino weak interactions with matter, dictated in part by the neutrino flavor states, may lie at the heart of our understanding of neutrino-affected astrophysical environments, and these can be important sites for the origin of the elements.

This paper represents a first step towards the derivation of practicable generalized kinetic equations, useful in actual simulations of neutrino propagation in anisotropic media, in any density regime. Here we set up the formalism, identify the degrees of freedom needed to describe the neutrino ensemble (these include both flavor and spin), and derive the correct structure of the quantum kinetic equations (QKEs), including coherent evolution and a collision term accounting for inelastic scattering. Our final results, summarized in Eq. (2.160), are somewhat formal, since self-energies entering into the collision term on the right-hand side are not fully calculated. Nonetheless, all the medium-induced potentials appearing on the left-hand-side of Eq. (2.160) are computed in Section VI.A, so this paper provides a complete description of coherent spin and flavor evolution in the absence of collisions. We will complete our program in a future paper, devoted to a detailed analysis of the collision term.

In this work, we have sought a well-posed prescription for treating general neutrino flavor evolution, one which can describe how neutrinos propagate and possibly change their flavors in environments ranging from low density regimes, where quantum mechanical phases are important and the evolution is Schrödinger-like, to very high temperature or very high matter density environments where phases are unimportant and the propagation/evolution is governed by the Boltzmann equation, and to all conditions between these limits. As a result, interaction-induced de-coherence, an historically thorny issue in relativistic and nonrelativistic quantum systems [102–113], must be addressed directly and self-consistently.

The approach we take differs from previous treatments. Those studies examined neutrino or general fermion flavor conversion in both the active-active channel [27, 67–76] and in the active-sterile channel [77–92], with a number of different approaches. Here we follow the general prescription used in Ref.s [93, 94] for bosons, but adapted and extended appropriately for fermions. In this development, we start from the most fundamental considerations of quantum field theory, and then build QKEs which describe neutrino flavor evolution.

In hot and dense environments in astrophysics, like those associated with the early universe, core collapse supernovae, and compact object mergers, neutrinos may carry a significant fraction of the energy and entropy. The way these particles interact with and communicate with the medium is through the weak interaction. As a consequence, ascertaining the flavor states (weak interaction states) of the neutrino fields is these environments can be a key part of understanding, for example, how neutrinos set the neutron-to-proton ratio [6] and deposit energy in supernovae [28, 114–116], or whether neutrinos decouple in mass or in flavor states in the very early universe [117, 118].

A feature of both the early universe and core collapse supernovae is that neutrinos propagate from very hot, high energy density regions or epochs, where transport mean free paths could be short compared to neutrino flavor oscillation lengths, to environments where the opposite is true. (We know that collective neutrino oscillations can readily occur in the latter regime, as reviewed in Ref. [98] and references therein, and can be sensitive to small-scale density inhomogeneities [5, 24, 119–121] and the angular distribution of neutrino flux [122–124].)

Between these extremes, a poorly understood and complicated interplay of coherent neutrino flavor oscillations and scattering-induced de-coherence can govern how flavor develops. Partly because of this complication, modelers of supernova neutrino propagation with energy and flavor evolution have relied on a clear separation of regimes: Boltzmann equation treatments inside the proto-neutron star, and in the vicinity of the chemical and thermal equilibrium decoupling zone (neutrino sphere); and a coherent treatment in which only forward-scattering is considered in the low density environment sufficiently far above the neutron star.

However, at some level these regimes cannot be separated. Indeed, recent work [65] shows that in some supernova envelope models, well above the neutrino sphere, neutrinos which suffer direction-changing scattering, though comprising only a seemingly negligible fraction (*e.g.,* one in a thousand) of all neutrinos coming from the neutron star, nevertheless may make significant contributions to the potentials which govern flavor transformation. Though this neutrino "halo" effect has been argued [31, 125] to make little difference in flavor evolution during the supernova accretion phase, in the one completely self-consistent calculation [66] that has been done to date it produces a significant modification in collective neutrino oscillations and the expected signal for an O-Ne-Mg core collapse neutronization burst.

These studies point out that understanding neutrino flavor evolution in some supernova and compact object merger environments ultimately may require following the interplay of nuclear composition, three-dimensional radiation hydrodynamics, and the QKEs for neutrino flavor. From a computational astrophysics modeling standpoint, the essential complication of the QKEs over conventional Boltzmann neutrino transport schemes is the necessity of following high frequency quantum flavor oscillations along with scattering. The QKEs we derive in this paper are no exception. And though our QKEs can have the expected physically intuitive limits of being Schrödinger-like at low density and Boltzmann-like in scattering-dominated regions, they also have features that are new and surprising, and which were not revealed by more *ad hoc* treatments.

Chief among these is the possibility of neutrino spin coherence. Since that, in principle, could mediate transformation between neutrinos and antineutrinos, it could be of importance in understanding compact object physics and nucleosynthesis as outlined above. The asymmetry between ν_e and $\bar{\nu}_e$ flowing from compact object environments can be, for example, a key arbiter of neutrino energy deposition and neutrino-heated nucleosynthesis. However, as will be evident in our subsequent exposition, implementing our QKEs in realistic simulations of astrophysical environments may require a radical alteration of the current approaches, and possibly a leap in computing capabilities.

In what follows we give some background on two-component spinor notation and introduce our model for Majorana neutrinos in Section II. We also describe how to extend our treatment to Dirac neutrinos. We present the approach for deriving equations of motion for neutrino correlation functions from quantum field theory in Section III. In Section IV we relate these correlation functions to physical quantities, such as neutrino densities and coherence terms, and present a scheme for perturbative expansion of the equations of motion. We then derive the kinetic equations for neutrino densities and coherence terms in Section V, and calculate the potentials that describe neutrino interactions with matter in Section VI. In Section VII, we present a discussion of some properties of the quantum kinetic equations, identifying the limits in which we obtain Schrödinger-like flavor evolution and Boltzmann-like kinetics. Also, in Section VII we identify some potential novel phenomena that are absent in the approximate treatments, including the possibility of coherent conversion between neutrinos and anti-neutrinos. In Section VIII we compare our work to existing approaches to neutrino QKEs and in Section IX we present our conclusions.

2.3 Preliminaries

2.3.1 Two-Component Spinor Notation

In this paper, we will primarily use two-component spinor notation, common in the supersymmetry literature and explained in detail in Ref. [126], an arXiv-published monograph by Stephen P. Martin, and Ref. [127]. A key reason for this choice of notation is that the two-component language is the most natural one for describing ultra-relativistic Majorana neutrinos. Moreover, this notation allows us to neatly separate components of physical quantities in a way that corresponds to their different physical meaning. In this section, we briefly review two-component spinor notation and the relation to four-component spinor notation.

The Lorentz group, $\mathrm{SO}\left(3,1\right)$, is equivalent to $\mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R$. Left-handed two-component spinors are objects that transform in the $(2,1)$ representation of the Lorentz group $\mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R$, while right-handed two-component spinors transform in the $(1,2)$ representation. By convention, left-handed spinors are labeled by undotted two-component indices, α, β, etc, while right-handed spinors are labeled by dotted indices, $\dot{\alpha}, \dot{\beta}$, etc. The presence or absence of a dot on a spinor index simply indicates which $\mathrm{SU}\left(2\right)$ factor is associated with the index.

Hermitian conjugation interchanges $\mathrm{SU}\left(2\right)_L$ and $\mathrm{SU}\left(2\right)_R$, so the Hermitian conjugate of a left-handed spinor is a right-handed spinor: $\psi^{\dagger\dot{\alpha}} \equiv \left(\psi^{\alpha}\right)^{\dagger}$. We adopt the convention that left-handed spinors (those with undotted indices) are always written without the dagger symbol, while right-handed spinors are always written with the dagger.

Four-component spinors are objects that transform in the $(2,1)+(1,2)$ representation of the Lorentz group. A four-component Dirac spinor consists of two independent two-component spinors, and can be written as $\Psi_D = \left(\chi_{\alpha}, \xi^{\dagger\dot{\alpha}}\right)$.

A four-component Majorana spinor consists of a two-component spinor and its Hermitian conjugate: $\Psi_M = \left(\psi_\alpha, \psi^{\dagger\dot{\alpha}}\right)$.

Note that a Dirac spinor has the same physical content as two Majorana spinors, and therefore Dirac spinors can always be represented as pairs of Majorana spinors. We will always do so; for example, we represent the charged leptons, which are Dirac spinors, as pairs of Majorana spinors (the lepton and the anti-lepton). In this paper, the statement that a pair of Majorana spinors forms a Dirac spinor should be taken to mean that the Lagrangian has a $U(1)$ symmetry under which the two Majorana fields carry opposite charge. This symmetry constrains the mass term to be proportional to a product of the two oppositely charged fields.

Two-component spinor indices can be raised or lowered with the anti-symmetric symbol $\epsilon^{\alpha\beta}$ or $\epsilon^{\dot{\alpha}\dot{\beta}}$, both variants defined by $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon_{21} = -\epsilon_{12} = 1$. A raised and a lowered index can be contracted (summed over), provided the indices are either both dotted or both undotted. Due to the antisymmetric nature of $\epsilon^{\alpha\beta}$, $\psi_\alpha \chi^\alpha = -\psi^\alpha \chi_\alpha$, and similarly for the dotted indices.

By convention, contracted undotted indices are always written with the first index raised, *e.g.,* $\psi^\alpha \chi_\alpha$, while contractions on dotted indices are written with the first index lowered, *e.g.,* $\psi^\dagger_{\dot{\alpha}} \chi^{\dagger\dot{\alpha}}$. This allows us to adopt an index-free notation for contraction of spinor indices: $\psi\chi \equiv \psi^\alpha \chi_\alpha$ and $\psi^\dagger \chi^\dagger \equiv \psi^\dagger_{\dot{\alpha}} \chi^{\dagger\dot{\alpha}}$.

In this paper, we will primarily deal with spinor bilinears. These quantities can either carry two undotted indices, two dotted indices, or one of each. All spinor bilinears can be written in terms of Lorentz tensors and Lorentz invariant spinor matrices:

$$
\begin{aligned}
\Gamma_{\alpha\dot{\alpha}} &= \Gamma^L_\mu \sigma^\mu_{\alpha\dot{\alpha}} & \Gamma^{\dot{\alpha}\alpha} &= \Gamma^R_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} \\
\Gamma_\alpha^{\ \beta} &= \Gamma^L \delta_\alpha^{\ \beta} + \frac{1}{2} i \Gamma^L_{\mu\nu} \left(S_L^{\mu\nu}\right)_\alpha^{\ \beta} & \Gamma^{\dot{\alpha}}_{\ \dot{\beta}} &= \Gamma^R \delta^{\dot{\alpha}}_{\ \dot{\beta}} + \frac{1}{2} i \Gamma^R_{\mu\nu} \left(S_R^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}}
\end{aligned}
\tag{2.1}
$$

where μ and ν are conventional spacetime indices, *i.e.,* assuming values 0, 1, 2, or 3.

The labels L and R on the various components of Γ are used to indicate which spinor bilinear the component belongs to. The basis spinor matrices are given by

$$\begin{aligned}
\sigma^\mu &= (1, \vec{\sigma}) \qquad\qquad \bar{\sigma}^\mu = (1, -\vec{\sigma}) \\
(S_L^{\mu\nu})_\alpha^{\ \beta} &= -\frac{1}{4} i \left(\sigma^\mu_{\alpha\dot{\alpha}} \bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \right) \\
(S_R^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{1}{4} i \left(\bar{\sigma}^{\mu\dot{\alpha}\alpha} \sigma^\nu_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha} \sigma^\mu_{\alpha\dot{\beta}} \right)
\end{aligned} \tag{2.2}$$

The signs in the definitions of S_L and S_R are a matter of convention. The spinor matrices σ^μ and $\bar{\sigma}^\mu$ satisfy the following relations:

$$\begin{aligned}
\sigma^\mu_{\alpha\dot{\alpha}} \bar{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^\nu_{\alpha\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} &= 2 g^{\mu\nu} \delta_\alpha^{\ \beta} \\
\bar{\sigma}^{\mu\dot{\alpha}\alpha} \sigma^\nu_{\alpha\dot{\beta}} + \bar{\sigma}^{\nu\dot{\alpha}\alpha} \sigma^\mu_{\alpha\dot{\beta}} &= 2 g^{\mu\nu} \delta^{\dot{\alpha}}_{\ \dot{\beta}}
\end{aligned} \tag{2.3}$$

where $g^{\mu\nu}$ is the usual spacetime (inverse) metric

It can be shown that the antisymmetric tensor quantities $(S_L^{\mu\nu})$ and $(S_R^{\mu\nu})$ are anti-self-dual and self-dual, respectively; that is, $S_L^{\mu\nu} = -i\,(S_L^{\mu\nu})^\star$ and $S_R^{\mu\nu} = i\,(S_R^{\mu\nu})^\star$, where $(T^{\mu\nu})^\star \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} T_{\rho\sigma}$. Anti-self-dual and self-dual antisymmetric tensors transform in separate irreducible representations of the Lorentz group, specifically in $(3, 1)$ and $(1, 3)$, respectively. Since $\Gamma^L_{\mu\nu}$ can be expressed using the basis of $S_L^{\mu\nu}$ matrices, it is an anti-self-dual tensor, while $\Gamma^R_{\mu\nu}$ is a self-dual tensor.

We can use index-free notation to denote products of spin matrices, using the conventions given above for contracting dotted and undotted indices, and in addition assuming that contractions are performed in the usual order of matrix multiplication. For example,

$$\sigma^\mu \bar{\sigma}^\nu \sigma^\rho = (\sigma^\mu \bar{\sigma}^\nu \sigma^\rho)_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\beta}} \bar{\sigma}^{\nu\dot{\beta}\beta} \sigma^\rho_{\beta\dot{\alpha}} \tag{2.4}$$

Products of σ or $\bar{\sigma}$ matrices can always be written in terms of the basis matrices $\delta, \sigma, \bar{\sigma}, S_L$ and S_R. The products of three σ or $\bar{\sigma}$ matrices are

$$\begin{aligned}
\sigma^\mu \bar{\sigma}^\nu \sigma^\rho &= g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho}{}_\sigma \sigma^\sigma \\
\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho &= g^{\mu\nu}\bar{\sigma}^\rho - g^{\mu\rho}\bar{\sigma}^\nu + g^{\nu\rho}\bar{\sigma}^\mu - i\epsilon^{\mu\nu\rho}{}_\sigma \bar{\sigma}^\sigma
\end{aligned} \tag{2.5}$$

Products of four or more σ matrices can be systematically reduced to expressions involving only the basis matrices, by repeated use of equations (2.3), (2.5), and the definitions of $S_L^{\mu\nu}$ and $S_R^{\mu\nu}$.

We will often use 4-component spinor bilinears which combine all four types of two-component spinor bilinears into a single 4×4 matrix:

$$\Gamma \equiv \begin{pmatrix} \Gamma_\alpha^{\ \beta} & \Gamma_{\alpha\dot{\beta}} \\ \Gamma^{\dot{\alpha}\beta} & \Gamma^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix} \tag{2.6}$$

With the spinor indices arranged as in equation (6), we can write contractions of 4-component spinor bilinears in an index-free way. That is, if Γ and Δ are 4×4 spin matrices having the form of equation (6), so is the product $\Gamma\Delta$, where it is understood that Γ and Δ are contracted together in the usual manner of matrix multiplication.

In this paper we adopt a commonly used representation of 4-component spinor matrices γ^μ and γ^5 where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

Choice of a particular representation of these matrices provides a dictionary by which expressions in 2-component spinor notation can be translated to standard 4-component spinor notation, and *vice versa.*

2.3.2 The Model

In what follows we will consider Standard Model neutrinos with small Majorana masses. We will work in the low-energy limit, where the energy of the particles is much smaller than the W and Z boson masses, so that the W and Z bosons are not dynamical. In this paper we will not consider the interactions of neutrinos with nucleons and nuclei; these interactions in certain limits and environments can be similar to the interactions of neutrinos with charged leptons. The ultimate forms of the QKEs we develop are crafted to allow straightforward incorporation of these interactions when necessary for realistic calculations. As a consequence, for simplicity we will restrict our development to the lepton sector.

After breaking electroweak symmetry, the Standard Model Lagrangian in the lepton sector is:

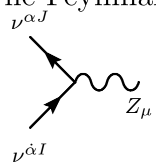
$$
\begin{aligned}
& i\psi_I^\dagger\bar{\sigma}_\mu\partial^\mu\psi_I + ie_I^\dagger\bar{\sigma}_\mu\partial^\mu e_I + i\bar{e}_I^\dagger\bar{\sigma}_\mu\partial^\mu\bar{e}_I - \frac{1}{2}m_{IJ}\psi_I\psi_J - m_{IJ}^e e_I\bar{e}_J \\
& \quad + e_I^\dagger\frac{g\left(2\sin^2\theta_W - 1\right)\bar{\sigma}_\mu Z^\mu}{2\cos\theta_W}e_I + \bar{e}_I^\dagger\frac{g\sin^2\theta_W\bar{\sigma}_\mu Z^\mu}{\cos\theta_W}\bar{e}_I \\
& \quad + \psi_I^\dagger\frac{g\bar{\sigma}_\mu Z^\mu}{2\cos\theta_W}\psi_I + \psi_I^\dagger\frac{g\bar{\sigma}_\mu W^{+\mu}}{\sqrt{2}}e_I + e_I^\dagger\frac{g\bar{\sigma}_\mu W^{-\mu}}{\sqrt{2}}\psi_I \\
& \quad + e_I^\dagger g_e\bar{\sigma}_\mu A^\mu e_I - \bar{e}_I^\dagger g_e\bar{\sigma}_\mu A^\mu\bar{e}_I - M_W^2 W_\mu^+ W^{-\mu} - \frac{1}{2}M_Z^2 Z_\mu Z^\mu \\
& \quad + \text{gauge boson kinetic terms} + \text{h.c.}
\end{aligned}
\tag{2.8}
$$

Here, ψ_I is the neutrino field, where I is the flavor index. In this notation e_I and $\bar{e}_I$ are the charged lepton fields, where the former describes left-handed electrons (muons, tauons) and right-handed positrons, and the latter is its Dirac counterpart, describing right-handed electrons and left-handed positrons. A^μ is the photon field, Z^μ and $W^{\pm\mu}$ are the weak boson fields. M_W and M_Z are the W and Z boson masses. g_e is the electromagnetic coupling constant (electron charge), g is the weak coupling constant, and θ_W is the Weinberg angle. m_{IJ} is the Majorana mass matrix for neutrinos, and m_{IJ}^e is the Dirac mass matrix for charged fermions. In the flavor basis, $m_{IJ}^e = \text{diag}\left(m_e, m_\mu, m_\tau\right)$, where m_e is the electron mass, m_μ is the muon mass, and m_τ is the tauon mass. For Majorana neutrinos, $m_{IJ} = m_{JI}$.

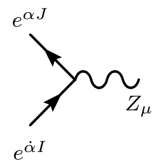
2.3.3 Feynman Rules

To compute various quantities that arise in the quantum kinetic equations, we will need the Feynman rules that are derived from the Lagrangian. In deriving the Feynman rules, we make several assumptions. First, we assume that the energy of the neutrinos and charged leptons is much smaller than the W and Z boson masses, and thus the W and Z bosons are not dynamical and we can neglect their kinetic terms. Second, in this low-energy regime, the electromagnetic interaction is much stronger than the weak interaction, and the distributions of charged particles thermalize on a much shorter timescale than the neutrino distributions. Therefore we will follow the dynamics of neutrinos associated with the weak interaction, and make the assumption, valid for the astrophysical regimes of interest to us, that the effect of the electromagnetic interaction is simply to ensure that the plasma (charged leptons, described by the fields e_I and $\bar{e}_I$, and photons, described by the field A^μ) can be adequately represented as thermal distributions of particles.

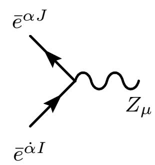
The Feynman rules for the weak interaction vertices are



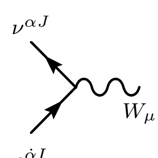
$$= \frac{-ig}{2\cos\theta_W}\delta^{IJ}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \text{ or } \frac{ig}{2\cos\theta_W}\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}$$



$$= -ig\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W}\delta^{IJ}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\text{or } \ ig\frac{\sin^2\theta_W - \frac{1}{2}}{\cos\theta_W}\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}$$



$$= -ig\frac{\sin^2\theta_W}{\cos\theta_W}\delta^{IJ}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\text{or } \ ig\frac{\sin^2\theta_W}{\cos\theta_W}\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}$$



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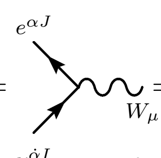
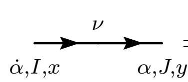


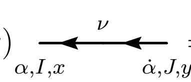
$$= \frac{-ig}{\sqrt{2}}\delta^{IJ}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \text{ or } \frac{ig}{\sqrt{2}}\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}} \tag{2.9}$$

Whether the $\bar{\sigma}$ or the σ version of the vertex is used depends on the two-component index structure of the diagram. The requirement that spinor indices be contracted in the usual order of matrix multiplication unambiguously determines which form of the vertex appears in the expression.

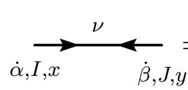
Next, we write down the Feynman rules for the propagators. In this paper we will be calculating quantities derived from the 2PI (two-particle irreducible) effective action. In this formalism, fermion lines represent the full expressions for neutrino and charged lepton two-point functions; these two-point functions are, in general, dynamical quantities that depend on particle densities and interactions. They are not just the vacuum propagators. In position space, we will write the general form of the neutrino two-point functions as



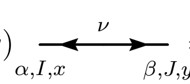
$= G^{\dot{\alpha}\alpha}_{\nu,IJ}(x,y)$



$= G^{\alpha\dot{\alpha}}_{\nu,IJ}(x,y)$



$= G^{\dot{\alpha}\dot{\beta}}_{\nu,IJ}(x,y)$



$= G^{\alpha\beta}_{\nu,IJ}(x,y)$ (2.10)

The two-point functions are defined as time-ordered expectation values of spinor field bilinears. Thus, for example, $G^{\alpha\dot{\alpha}}_{\nu,IJ}(x,y) = \left\langle \mathrm{T}_P \left(\psi^{\alpha}_I(x)\, \psi^{\dagger\dot{\alpha}}_J(y) \right) \right\rangle$, and similarly for the other components of G. Here, T_P is the time ordering operator along a specific path. As we explain below, we will use the closed time path (CTP) contour. Since we are dealing with out of equilibrium, non-vacuum states described by a nontrivial density operator, the brackets, $<>$, denote an ensemble average rather than a vacuum expectation value.

Note that in two-component spinor notation the arrows on fermion propagators do not denote the flow of momentum or any conserved current, but rather simply indicate whether the two-component spinor index associated with the arrow is dotted or undotted. This is illustrated in the above equations for the two-point functions. For example, it can be seen that "clashing arrows," where the arrows point toward each other, correspond to two point functions with right-handed spinor indices, while diverging arrows go with left-handed spinor indices, *etc.*

As described below, the two-point function contains both the vacuum propagator and the particle density matrix. The density matrix encodes the particle occupation numbers and additional degrees of freedom describing flavor and possibly spin (handedness) coherence. We will treat the neutrino two-point function as a fully dynamical entity, the time development of which allows us to solve for the time evolution of the neutrino occupation numbers.

Similarly, the general Feynman rules for the charged lepton two-point functions are:

$$\underset{\dot{\alpha},I,x}{\xrightarrow{\;e\;}}\underset{\alpha,J,y}{\xrightarrow{\;e\;}} = G^{\dot{\alpha}\alpha}_{e,IJ}(x,y) \qquad \underset{\alpha,I,x}{\xleftarrow{\;e\;}}\underset{\dot{\alpha},J,y}{\xleftarrow{\;e\;}} = G^{\alpha\dot{\alpha}}_{e,IJ}(x,y)$$

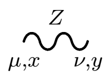
$$\underset{\dot{\alpha},I,x}{\xrightarrow{\;\bar{e}\;}}\underset{\alpha,J,y}{\xrightarrow{\;\bar{e}\;}} = G^{\dot{\alpha}\alpha}_{\bar{e},IJ}(x,y) \qquad \underset{\alpha,I,x}{\xleftarrow{\;\bar{e}\;}}\underset{\dot{\alpha},J,y}{\xleftarrow{\;\bar{e}\;}} = G^{\alpha\dot{\alpha}}_{\bar{e},IJ}(x,y)$$

$$\underset{\dot{\alpha},I,x}{\xrightarrow{\;e\;}}\underset{\dot{\beta},J,y}{\xleftarrow{\;\bar{e}\;}} = G^{\dot{\alpha}\dot{\beta}}_{e\bar{e},IJ}(x,y) \qquad \underset{\alpha,I,x}{\xleftarrow{\;e\;}}\underset{\beta,J,y}{\xrightarrow{\;\bar{e}\;}} = G^{\alpha\beta}_{e\bar{e},IJ}(x,y) \tag{2.11}$$

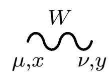
In this development we will assume that the charged lepton distributions are thermal. With this assumption, the form of the charged lepton two-point function will depend only on the charged lepton temperature, chemical potential, and mass.

Note that since charged leptons are Dirac particles, the arrow-clashing propagator for charged leptons always connects the charged lepton field with its Dirac counterpart. On the other hand, for Majorana neutrinos, the arrow-clashing propagator connects the field to itself.

In the low-energy limit the electroweak bosons are not dynamical, and their position space Feynman rules are simply given by



$$= \frac{ig^{\mu\nu}}{M_Z^2}\delta^4\left(x-y\right)$$



$$= \frac{ig^{\mu\nu}}{M_W^2}\delta^4\left(x-y\right) \tag{2.12}$$

Here, we have used the Feynman gauge, but other choices of gauge give physically equivalent expressions.

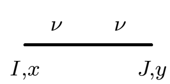
We will often express combinations of coupling constants and electroweak boson masses that appear in the Feynman diagrams in terms of the Fermi constant

$$G_F \equiv \frac{g^2}{4\sqrt{2}M_W^2} \tag{2.13}$$

and use

$$\cos\theta_W = \frac{M_W}{M_Z} \tag{2.14}$$

It is sometimes convenient to denote the combination of all components of a two-point function or vertex by omitting the arrows. This is equivalent to using the four-component spinor notation. For example, we can write



$$= G_{\nu,IJ}\left(x,y\right), \tag{2.15}$$

where

$$G_{\nu,IJ} \equiv \begin{pmatrix} \left(G_{\nu,IJ}\right)_{\alpha}{}^{\beta} & \left(G_{\nu,IJ}\right)_{\alpha\dot{\beta}} \\ \left(G_{\nu,IJ}\right)^{\dot{\alpha}\beta} & \left(G_{\nu,IJ}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}. \tag{2.16}$$

The use of diagrams without arrows is simply shorthand notation which implies a sum of every possible combination of arrow directions that gives a nonzero contribution to the amplitude.

2.4 Equations of Motion for the Two-Point Function

2.4.1 2PI Effective Action and the Two-Point Function

The equations of motion for neutrino two-point functions can be derived from the two-particle irreducible (2PI) effective action. The complete, general procedure is presented in Ref.s [111, 128]. Here, we outline the key steps in this derivation as they apply to the dynamics of neutrinos.

The 2PI effective action is a functional of the two-point function $G = G^{ab}_{IJ}(x, y)$, corresponding to equation (16), where a and b are four-component spinor indices (for example, $a = (\alpha, \dot{\alpha})$), I and J are flavor indices, and x and y are position four-vectors. The 2PI effective action consists of Feynman diagrams with no external lines that are two-particle irreducible, that is, cannot be disconnected by cutting two fermion lines (we do not consider cutting weak boson lines, since the weak bosons are not dynamical in our formalism, and can be reduced to 4-fermion vertices). We separate the $2PI$ effective action into a 1-loop piece (a single fermion loop, the only contribution to Γ^{2PI} in free field theory), and the rest:

$$\Gamma^{2PI}[G] = \Gamma_1^{2PI}[G] + \Gamma_2^{2PI}[G]. \tag{2.17}$$

In this equation Γ_1 is the one-loop expression, and Γ_2 is the sum of all higher-loop contributions. The diagrams are drawn and calculated, in position space, as usual, except that the general form for the two-point functions is used instead of the tree-level propagator, thereby incorporating effects from nonzero particle density and corrections to the propagator stemming from interactions. We use the general result from quantum field theory:

$$\Gamma_1^{2PI} = -i\left(\mathrm{Tr}\ln G^{-1} + \mathrm{Tr}\; G_0^{-1}G\right) \tag{2.18}$$

where G_0^{-1} is the tree-level inverse propagator, and G is the complete dynamical two-point correlation function. Here, we are suppressing spin and flavor indices, but the quantities in this expression are 4×4 matrices in spin space and 3×3 matrices in flavor space, with an explicit form given by the expression in equation (2.16). Products and traces of such quantities in our equations imply contraction of both spinor and flavor indices in the usual order of matrix multiplication.

We can now find the equations of motion for G by setting $\frac{\delta\Gamma^{2PI}[G]}{\delta G} = 0$. This gives the following expression:

$$G^{-1}(x,y) = G_0^{-1}(x,y) - \Sigma[x,y;G], \tag{2.19}$$

where we define

$$\Sigma[x,y;G] \equiv -i\frac{\delta\Gamma_2^{2PI}[G]}{\delta G(y,x)}. \tag{2.20}$$

Since Γ_2^{2PI} is the sum of two-loop and higher order 2PI diagrams with no external lines, Σ is proportional to the sum of one-loop and higher order 1PI diagrams with two external neutrino lines. Consequently, Σ corresponds to the neutrino proper self-energy. For the purposes of this paper, we will calculate Σ to 2-loop order; the corresponding Feynman diagrams and calculations will be given in a subsequent section.

We can eliminate the dependence of equation (2.19) on G^{-1} by acting from the right with G, to obtain

$$(i\,\partial\!\!\!/^{x} - M)\,G(x,y) - i\int d^4z\Sigma(x,z)\,G(z,y) = \mathbf{1}\; i\delta^4(x-y) \tag{2.21}$$

where $\partial\!\!\!/^{x}$ and M are spin $\times$ flavor matrices given by $\partial\!\!\!/^{x} = \begin{pmatrix} 0 & \sigma^{\mu}_{\alpha\dot{\alpha}}\partial^x_\mu \\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial^x_\mu & 0 \end{pmatrix}\delta_{IJ}$ and $M = \begin{pmatrix} \delta_\alpha^{\ \beta}\, m_{IJ} & 0 \\ 0 & \delta^{\dot{\alpha}}_{\ \dot{\beta}}\,(m_{IJ})^\dagger \end{pmatrix}$. Here $\mathbf{1}$ is the spin $\times$ flavor unit matrix, given by $\mathbf{1} = \begin{pmatrix} \delta_\alpha^{\ \beta} & 0 \\ 0 & \delta^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix}\delta_{IJ}$.

2.4.2 Spectral and Statistical Functions

We can use the dynamics of the two-point function G to describe the evolution of neutrino distributions, starting with arbitrary non-equilibrium initial conditions, by employing the closed time path (CTP) formalism [128]. In the CTP formalism, the time ordering in the path integral is taken along a closed real-time contour, starting from the point at which initial conditions are given, to the point in time of interest in the calculation, and then back to the initial point. The two-point correlation function G is time ordered on the CTP contour: $G(x,y) = \left\langle \mathrm{T}_{CTP}\left(\Psi(x)\,\bar{\Psi}(y)\right)\right\rangle$, where T_{CTP} is an operator that imposes time ordering with respect to the CTP contour, and Ψ is a Majorana spinor given by $\Psi = \left(\psi_\alpha, \psi^{\dagger\dot{\alpha}}\right)$ and $\bar{\Psi} = \left(\psi^\alpha, \psi^\dagger_{\dot{\alpha}}\right)$.

The time ordering can be made explicit by decomposing G into the following components:

$$G(x,y) = F(x,y) - \frac{1}{2} i\rho(x,y)\,\mathrm{sign}_{CTP}\left(x^0 - y^0\right) \tag{2.22}$$

where sign_{CTP} is a function of the ordering of x and y along the time path, taking on a value of 1 or -1, depending on whether y precedes or follows x on the CTP contour. For fermions, F and ρ are defined as follows:

$$F(x,y) = \frac{1}{2}\left\langle\left[\Psi(x), \bar{\Psi}(y)\right]\right\rangle \tag{2.23}$$

$$\rho(x,y) = i\left\langle\left\{\Psi(x), \bar{\Psi}(y)\right\}\right\rangle. \tag{2.24}$$

In the above expressions, ρ is the spectral function, and carries information on the particle states that can appear in the theory; it is related to the usual vacuum propagator. F is the statistical function, and encodes the occupation numbers of these states. Since we wish to solve for the evolution of neutrino occupation numbers, we will primarily be interested in the dynamics of the statistical function F.

Similarly, we decompose the neutrino self-energy Σ into a local piece, plus spectral and statistical components:

$$\Sigma(x,y) = -i\Sigma(x)\,\delta^4_{\mathrm{CTP}}(x-y) + \Pi_F(x,y) - \frac{1}{2} i\Pi_\rho(x,y)\,\mathrm{sign}_{\mathrm{CTP}}\left(x^0 - y^0\right) \tag{2.25}$$

We will show how to compute these components later, but for now, we note that for our model, the local term $\Sigma(x)$ contains contributions from 1-loop diagrams, while the spectral and statistical terms contain only contributions from 2-loop and higher diagrams. Thus, the $\Pi_\rho(x,y)$ and $\Pi_F(x,y)$ terms carry higher powers of the coupling constant than does $\Sigma(x)$

Using equations (2.22) and (2.25) in (2.21) gives the following equation for the statistical function:

$$\left(i\not{\partial}^x - M - \Sigma(x)\right) F(x,y) = \int_0^{x^0} dz^0 \int d^3z\, \Pi_\rho(x,z)\, F(z,y) - \int_0^{y^0} dz^0 \int d^3z\, \Pi_F(x,z)\, \rho(z,y)\,. \tag{2.26}$$

In addition, there is another form of the equation for F, which is obtained by acting on equation (2.19) from the left with G, then separating into spectral and statistical components. This gives

$$F(x,y)\left(-i\overleftarrow{\not{\partial}}^y - M - \Sigma(y)\right) = \int_0^{y^0} dz^0 \int d^3z\; F(x,z)\, \Pi_\rho(z,y) - \int_0^{x^0} dz^0 \int d^3z\; \rho(x,z)\, \Pi_F(z,y) \tag{2.27}$$

There are similar equations for the spectral function. However, for the purpose of this paper, we will not need these equations. The reason is that the spectral function does not depend on the occupation numbers of particles, but rather only on the mass and the interaction strength. For particles with a small mass and experiencing only weak interactions, ρ will deviate only slightly from its massless, free-field value. In equations (2.26) and (2.27), ρ only enters in conjunction with Π_F, which is already at two-loop order. Because we are only computing quantities to this order, any corrections to the spectral function due to the neutrino mass or interactions will give terms in the equation that are beyond the order of our expansion. Thus, we can simply use the massless, free-field expression for the spectral function, which will be derived below.

2.5 Wigner Transform and Separation of Scales

2.5.1 The Wigner Transform

Equations (2.26) and (2.27) give the complete dynamics of the neutrinos, approximate only insofar as we are expanding Σ to 2-loop order, and decoupling the dynamics of the spectral function from those of the statistical function by dropping higher-order terms on the right-hand side. However, solving these equations in their current form is impractical. First, the connection of the object $F(x, y)$ to actual neutrino occupation numbers is somewhat complicated, so the physical meaning of these equations is difficult to elucidate. Second, the two-point function undergoes rapid oscillations, on the scale of the neutrino de Broglie wavelength, with respect to the relative coordinate $r = x - y$. On the other hand, for weakly coupled particles, such as neutrinos, physically meaningful quantities change much more slowly, and vary as a function of the average coordinate, $X = \frac{1}{2}(x + y)$. Resolving the rapid oscillations associated with the neutrino de Broglie wavelength is clearly undesirable from a computational standpoint.

We derive more useful expressions from (2.26) and (2.27) by performing a Wigner transform and then expanding in small parameters. In this, we follow the procedure of Ref. [93]. (Applications of some of these techniques in the context of electroweak baryogenesis are presented in Ref.s [94, 129–133].)

To perform the Wigner transform, we change to the relative coordinate r and the average coordinate X. Note that eventually, after the change of coordinates, we will simply name the average coordinate x; it should be clear from context whether x refers to the average coordinate or to one of the two spacetime arguments of a two-point function. We then Fourier transform with respect to the relative coordinate. The Wigner transform of the statistical function $F(x, y)$ is then:

$$F(X, k) \equiv \int d^4r \; e^{ik \cdot r} F\left(X + \frac{1}{2}r, X - \frac{1}{2}r\right) \tag{2.28}$$

and similarly for other functions of (x, y).

2.5.2 Spectral and Statistical Functions for Free, Massless Fermions

Before we Wigner transform equations (2.26) and (2.27), we derive the expressions for the spectral and statistical functions in terms of the particle densities, neglecting neutrino mass and interactions but allowing for nonzero neutrino densities. Neutrino masses and interactions will result in slight changes to these expressions; we will later calculate these changes perturbatively. As we will see, the Wigner transformed functions have a straightforward physical interpretation. In particular, the Wigner transformed statistical function, $F(X,k)$, contains components proportional to neutrino and antineutrino density matrices, $f_{IJ}(X,k)$ and $\bar{f}_{IJ}(X,k)$, while the spectral function in free field theory contains no dynamical components, and therefore simply encodes the possible particle states. For anisotropic particle distributions, $F(X,k)$ can contain an additional dynamical quantity, which can be interpreted as describing coherence between left-handed and right-handed fermion states.

We begin with the statistical function. In terms of the 4-component Majorana spinor fields, this is given by

$$F_{IJ}(X,k) = \frac{1}{2}\int d^4r\; e^{ik\cdot r}\left\langle\left[\Psi_I\left(X+\frac{1}{2}r\right),\bar{\Psi}_J\left(X-\frac{1}{2}r\right)\right]\right\rangle \tag{2.29}$$

For convenience of notation, we will evaluate this expression at $X=0$, and later generalize the results to any position X:

$$F_{IJ}(0,k) = \frac{1}{2}\int d^4r\; e^{ik\cdot r}\left\langle\left[\Psi_I\left(\frac{r}{2}\right),\bar{\Psi}_J\left(-\frac{r}{2}\right)\right]\right\rangle \tag{2.30}$$

We will calculate the various components of F in two-component spinor notation, in which the Majorana spinors are given by $\Psi_I = \left(\psi_{I,\alpha},\psi_I^{\dagger\dot{\alpha}}\right)$ and $\bar{\Psi}_J = \left(\psi_J^{\beta},\psi^{\dagger}_{J,\dot{\beta}}\right)$. First, we calculate

$$F_{IJ,\alpha\dot{\beta}}(0,k) = \frac{1}{2}\int d^4r\; e^{ik\cdot r}\left\langle\left[\psi_{I,\alpha}\left(\frac{r}{2}\right),\psi^{\dagger}_{J,\dot{\beta}}\left(-\frac{r}{2}\right)\right]\right\rangle \tag{2.31}$$

The two-component spinor field $\psi_{I,\alpha}$ is given by

$$\psi_{I,\alpha}(x) = \int \tilde{dq}\left(b_I(\vec{q})\,u_\alpha(\vec{q})\,e^{-iq\cdot x} + d_I^{\dagger}(\vec{q})\,v_\alpha(\vec{q})\,e^{iq\cdot x}\right) \tag{2.32}$$

In manifestly Lorentz invariant notation, $\tilde{dq} = \frac{d^4q}{(2\pi)^4}2\pi\delta\left(q^2\right)\theta\left(q^0\right)$. $b_I(\vec{q})$ is an operator that annihilates a left-handed neutrino of flavor I and momentum

$\vec{q}$, and $d_I^\dagger(\vec{q})$ is an operator that creates a right-handed anti-neutrino of flavor I and momentum $\vec{q}$. Note that for Majorana neutrinos, particles and antiparticles simply correspond to opposite spin states; as a result, we could instead have used the spin-dependent operators b_s, where $s = \pm$. In our notation, $b = b_-$ and $d = b_+$. The creation and annihilation operators satisfy the anticommutation relations:

$$\left\{b_I(\vec{q}_1), b_J^\dagger(\vec{q}_2)\right\} = (2\pi)^3 \delta^3(\vec{q}_1 - \vec{q}_2) 2E_q\delta_{IJ}$$
$$\left\{d_I(\vec{q}_1), d_J^\dagger(\vec{q}_2)\right\} = (2\pi)^3 \delta^3(\vec{q}_1 - \vec{q}_2) 2E_q\delta_{IJ} \tag{2.33}$$

All other anticommutators are zero.

$u^\alpha(\vec{q})$ and $v^\alpha(\vec{q})$ are two-component spinors that satisfy

$$q_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} u_\alpha(\vec{q}) = 0 \qquad q_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} v_\alpha(-\vec{q}) = 0 \tag{2.34}$$

where $q_\mu \equiv (q_0, \vec{q})$, with the timelike component taken to be positive definite. u and v are normalized as follows:

$$u_\alpha(\vec{q})\, u_{\dot{\beta}}^\dagger(\vec{q}) = q_\mu \sigma^\mu_{\alpha\dot{\beta}} \qquad v_\alpha(-\vec{q})\, v_{\dot{\beta}}^\dagger(-\vec{q}) = -q_\mu \sigma^\mu_{\alpha\dot{\beta}} \tag{2.35}$$

Substituting equation (32) into equation (31) gives an expression with four terms:

$$\begin{aligned} F_{IJ,\alpha\dot{\beta}}(0,k) = \frac{1}{2}\int d^4r \int \tilde{d}q_1 \tilde{d}q_2 & \\ \langle [b_I(\vec{q}_1), d_J(\vec{q}_2)]\rangle\, u_\alpha(\vec{q}_1)\, v_{\dot{\beta}}^\dagger(\vec{q}_2)\, e^{i\left(k - \frac{q_1 - q_2}{2}\right)\cdot r} & \\ + \left\langle \left[b_I(\vec{q}_1), b_J^\dagger(\vec{q}_2)\right]\right\rangle u_\alpha(\vec{q}_1)\, u_{\dot{\beta}}^\dagger(\vec{q}_2)\, e^{i\left(k - \frac{q_1 + q_2}{2}\right)\cdot r} & \\ + \left\langle \left[d_I^\dagger(\vec{q}_1), d_J(\vec{q}_2)\right]\right\rangle v_\alpha(\vec{q}_1)\, v_{\dot{\beta}}^\dagger(\vec{q}_2)\, e^{i\left(k + \frac{q_1 + q_2}{2}\right)\cdot r} & \\ + \left\langle \left[d_I^\dagger(\vec{q}_1), b_J^\dagger(\vec{q}_2)\right]\right\rangle v_\alpha(\vec{q}_1)\, u_{\dot{\beta}}^\dagger(\vec{q}_2)\, e^{i\left(k + \frac{q_1 - q_2}{2}\right)\cdot r} & \end{aligned} \tag{2.36}$$

The commutators of creation and annihilation operators are clearly related to the particle number operator, and consequently depend on the neutrino distributions. We make the assumption that the neutrino distributions are approximately homogenous and time-invariant on the scale of the de Broglie wavelength, so that the integral over r can be formally taken to infinity while still assuming that the expectation values of the commutators do not vary over the integration range. In the astrophysical venues we target for application of our QKEs there are unlikely to be any density fluctuations on scales comparable with the neutrino de Broglie wavelength (∼10 fm).

With the assumption of approximate time invariance, the first and last terms in equation (2.36) do not contribute to the integral, since a pair of creation operators or a pair of annihilation operators acting on a state will always change its energy. Since a time invariant state is an energy eigenstate, the action of the pair of operators will always give a state that is orthogonal to the original, and as a result the expectation value vanishes. Note that this result does not hold true for states describing neutrino distributions that vary on a scale comparable to the de Broglie frequency; here, we assume that there is no such rapid variation.

Similarly, we can use the assumption of approximate homogeneity to show that the remaining terms, involving a creation operator and an annihilation operator, must be proportional to $\delta^3(\vec{q}_1 - \vec{q}_2)$, since the expectation value will be zero unless the operators create and annihilate a particle with the same momentum. All of this allows us to write the commutators of the creation and annihilation operators as

$$\begin{aligned}\left\langle\left[b_I(\vec{q}_1), b_J^\dagger(\vec{q}_2)\right]\right\rangle &= \left\langle\left\{b_I(\vec{q}_1)\, b_J^\dagger(\vec{q}_2)\right\}\right\rangle - 2\left\langle b_J^\dagger(\vec{q}_2)\, b_I(\vec{q}_1)\right\rangle \\ &= (2\pi)^3\,\delta^3(\vec{q}_1 - \vec{q}_2)\, 2E_q\left(\delta_{IJ} - 2f_{IJ}(\vec{q}_1)\right). \end{aligned} \tag{2.37}$$

Here $f_{IJ}(\vec{q}_1)$ is the density matrix for neutrinos. For $I = J$, $f_{II}(\vec{q}_1)$ simply corresponds to the expectation value of the number operator for flavor I, and gives the occupation number of neutrinos of flavor I and momentum $\vec{q}_1$. For $I \neq J$, f_{IJ} corresponds to coherence between neutrinos of different flavors.

Similarly,

$$\left\langle\left[d_I^\dagger(\vec{q}_1), d_J(\vec{q}_2)\right]\right\rangle = -(2\pi)^3\,\delta^3(\vec{q}_1 - \vec{q}_2)\, 2E_q\left(\delta_{IJ} - 2\bar{f}_{IJ}(\vec{q}_1)\right) \tag{2.38}$$

where $\bar{f}_{IJ}(\vec{q}_1)$ is the density matrix for anti-neutrinos.

From this point on in our exposition we will use x to mean the average coordinate X in Wigner transformed quantities. Using equations (2.37) and (2.38), to perform the integrals in equation (2.36), simplifying the spinor bilinears by using equation (2.35), and generalizing from $x = 0$ to any position gives

$$F_{\alpha\dot{\beta}}(x,k) = 2\pi\delta\left(k^2\right)k_\mu\sigma^\mu_{\alpha\dot{\beta}}\left(\frac{1}{2} - \theta\left(k^0\right)f\left(x,\vec{k}\right) - \theta\left(-k^0\right)\bar{f}\left(x,-\vec{k}\right)\right) \quad (2.39)$$

where we have suppressed flavor indices on f_{IJ} and $\bar{f}_{IJ}$. Similarly,

$$F^{\dot{\alpha}\beta}(x,k) = 2\pi\delta\left(k^2\right)k_\mu\bar{\sigma}^{\mu\dot{\alpha}\beta}\left(\frac{1}{2} - \theta\left(k^0\right)\bar{f}^T\left(x,\vec{k}\right) - \theta\left(-k^0\right)f^T\left(x,-\vec{k}\right)\right) \quad (2.40)$$

Note that $F^{\dot{\alpha}\beta}(k)$ is related to $F^T_{\alpha\dot{\beta}}(-k)$, where the transpose is over flavor indices.

We next calculate $F_\alpha{}^\beta$. This is given by the expression

$$\begin{aligned} F_{IJ,\alpha}{}^\beta(0,k) &= \frac{1}{2}\int d^4r \int \tilde{d}q_1\tilde{d}q_2 \\ \left\langle\left[b_I(\vec{q}_1), d^\dagger_J(\vec{q}_2)\right]\right\rangle & u_\alpha(\vec{q}_1)v^\beta(\vec{q}_2)e^{i\left(k-\frac{q_1+q_2}{2}\right)\cdot r} \\ +\left\langle\left[d^\dagger_I(\vec{q}_1), b_J(\vec{q}_2)\right]\right\rangle & v_\alpha(\vec{q}_1)u^\beta(\vec{q}_2)e^{i\left(k+\frac{q_1+q_2}{2}\right)\cdot r} \end{aligned} \quad (2.41)$$

where we have omitted vanishing terms. Since the anticommutators of b and $d^\dagger$ vanish, we can write the commutators as

$$\begin{aligned} \left\langle\left[b_I(\vec{q}_1), d^\dagger_J(\vec{q}_2)\right]\right\rangle &= -2\left\langle d^\dagger_J(\vec{q}_1)\,b_I(\vec{q}_2)\right\rangle \\ &= -(2\pi)^3\delta^3(\vec{q}_1-\vec{q}_2)\,2E_q\left(2\phi_{IJ}(\vec{q}_1)\right) \end{aligned} \quad (2.42)$$

The matrix ϕ_{IJ} is a correlation function between neutrino and anti-neutrino creation and annihilation operators, and so describes coherence between neutrino and anti-neutrino states. We will see that this object vanishes with the assumption of isotropy (as expected from conservation of angular momentum), but may, in general, be present in an anisotropic environment.

We simplify the spinor bilinears in equation (2.41) by using

$$u_\alpha(\vec{q})\,v^\beta(\vec{q}) = v_\alpha(-\vec{q})\,u^\beta(-\vec{q}) = \frac{1}{2}iq^{[\mu}\left(x^1 - ix^2\right)^{\nu]}\left(S^L_{\mu\nu}\right)_\alpha{}^\beta \quad (2.43)$$

Here, x^1 and x^2 are spacelike unit vectors orthogonal to the direction of the momentum and to each other. Equation (2.43) may be directly verified by choosing a coordinate system in which $q^\mu = (q,0,0,q)$, $x^{1,\mu} = (0,1,0,0)$ and $x^{2,\mu} = (0,0,1,0)$, then solving equation (2.34) for the spinors u and v, imposing the normalization conditions (2.35), explicitly calculating the spinor bilinears and comparing to the expressions for $\left(S_L^{\mu\nu}\right)_\alpha^{\ \beta}$. Note that the pre-factor $q^{[\mu}\left(x^1+ix^2\right)^{\nu]}$ is chosen to be anti-self-dual. We choose a pre-factor of this form because the contraction with $S^L_{\mu\nu}$ projects out the self-dual component, so any self-dual component in the pre-factor would not contribute to equation (2.43).

Using equations (2.42) and (2.43) and performing the integrals in (2.41) gives

$$\begin{aligned} F_\alpha^{\ \beta}(x,k) &= -2\pi\delta\left(k^2\right)\frac{1}{2}ik^{[\mu}\left(\hat{x}^1-i\hat{x}^2\right)^{\nu]}\left(S^L_{\mu\nu}\right)_\alpha^{\ \beta} \\ &\times \left(\theta\left(k^0\right)\phi\left(\vec{k}\right)+\theta\left(-k^0\right)\phi^T\left(-\vec{k}\right)\right) \end{aligned} \tag{2.44}$$

Similarly,

$$\begin{aligned} F^{\dot{\alpha}}_{\ \dot{\beta}}(x,k) &= -2\pi\delta\left(k^2\right)\frac{1}{2}ik^{[\mu}\left(\hat{x}^1+i\hat{x}^2\right)^{\nu]}\left(S^R_{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} \\ &\times \left(\theta\left(k^0\right)\phi^\dagger\left(\vec{k}\right)+\theta\left(-k^0\right)\phi^\star\left(-\vec{k}\right)\right) \end{aligned} \tag{2.45}$$

We now turn to the spectral function. Unlike the statistical function, in free field theory the spectral function is completely determined by the anticommutation relations between creation and annihilation operators. Thus, the only nonzero components of the spectral function are

$$\rho_{\alpha\dot{\beta},IJ}(x,k) = 2i\pi\delta\left(k^2\right)\operatorname{sign}\left(k^0\right)k_\mu\sigma^\mu_{\alpha\dot{\beta}}\delta_{IJ} \tag{2.46}$$

$$\rho^{\dot{\alpha}\beta}_{IJ}(x,k) = 2i\pi\delta\left(k^2\right)\operatorname{sign}\left(k^0\right)k_\mu\bar{\sigma}^{\mu\dot{\alpha}\beta}\delta_{IJ} \tag{2.47}$$

2.5.3 Wigner-Transformed Equations of Motion for the Statistical Function

Having determined the physical content of the statistical function, we return to the Wigner transform of equations (2.26) and (2.27). The full Wigner transformed expressions contain gradient expansions, which are infinite series of derivatives with respect to x and k. We truncate these infinite series by expanding in a small parameter ϵ.

In our expansion, we make use of the fact that, in the regime we are considering, neutrino masses and interaction potentials are small compared to the neutrino energy. Also, we expect the variation of physical quantities with respect to the average coordinate x to be slow compared to the inverse neutrino de Broglie frequency. These considerations lead us to introduce the following power counting:

$$\frac{\partial_x, M, \Sigma}{E} = O\left(\epsilon\right) \qquad \frac{\Pi_\rho, \Pi_F}{E} = O\left(\epsilon^2\right) \tag{2.48}$$

where E is the neutrino energy. The contributions to self-energy Π_ρ and Π_F are $O\left(\epsilon^2\right)$ because they appear only at two-loop order in the Feynman diagram expansion, while Σ appears at one-loop order.

This power counting includes the standard gradient expansion (see, for example, Ref.s [93, 94, 134]). However, our approach is specialized to the ultra-relativistic neutrinos that are relevant for supernova and compact object merger environments. Moreover, since this work involves neutrinos having energies far below the electroweak scale, the interactions are always weak.

We keep terms to $O\left(\epsilon^2\right)$, since this allows us to include terms involving Π_ρ and Π_F, which describe inelastic and non-forward scattering of neutrinos. To $O\left(\epsilon^2\right)$, the Wigner transformed equations for F are

$$\begin{aligned}\left(\frac{1}{2}i\,\partial\!\!\!/ + k\!\!\!/\right) F\left(x,k\right) - \left(M + \Sigma\left(x\right)\right) F\left(k,x\right) + \frac{1}{2}i\left(\partial_x^\mu \Sigma\left(x\right)\right)\left(\partial_\mu^k F\left(x,k\right)\right) = \\ -\frac{1}{2}i\left(\Pi^+\left(x,k\right) G^-\left(x,k\right) - \Pi^-\left(x,k\right) G^+\left(x,k\right)\right)\end{aligned} \tag{2.49}$$

and its Hermitian conjugate. Here, $\partial_\mu^k \equiv \frac{\partial}{\partial k^\mu}$. We have made the right-hand side of the equation more compact by introducing the notation

$$G^\pm \equiv -\frac{1}{2}i\rho \pm F \qquad \Pi^\pm \equiv -\frac{1}{2}i\Pi_\rho \pm \Pi_F \tag{2.50}$$

We will use equation (2.49) and its Hermitian conjugate as the starting point for deriving the equations of motion for the neutrino density matrices.

2.6 Derivation of Quantum Kinetic Equations

2.6.1 Outline of the Derivation and Some Preliminaries

Equation (2.49) has a complicated structure, containing the kinetic equations as well as algebraic constraints relating various components of F to each other. To derive the quantum kinetic equations, we systematically expand equation (2.49) in the separation of scales, using the power counting defined in equation (2.48).

We expect the statistical function F to have an $O(1)$ piece of the form given by equations (2.39)-(2.40) and (2.44)-(2.45), plus a small correction due to nonzero interactions and neutrino masses. This correction will be $O(\epsilon)$, while our kinetic equations will be constructed to $O(\epsilon^2)$. Thus, the $O(\epsilon)$ correction to F will enter into the kinetic equations, and must be calculated.

Our strategy is to first expand equation (2.49) to $O(\epsilon)$, and use this to find the first-order shift in F due to the mass and interactions. Then, we will insert the $O(\epsilon)$ expression for F back into equation (2.49), expand to $O(\epsilon^2)$, and extract the equations of motion for the density matrices and spin coherence densities.

We will show, in a subsequent section, that Σ corresponds to the matter and neutrino self-interaction potential arising from coherent forward scattering, and has the form

$$\Sigma = \begin{pmatrix} \delta\Sigma_S & \Sigma_L \cdot \sigma \\ \Sigma_R \cdot \bar{\sigma} & \delta\Sigma_S^\dagger \end{pmatrix} \tag{2.51}$$

where Σ_L and Σ_R are Hermitian, and, for Majorana fermions, trivially related to each other. $\Sigma_{L/R} = O(\epsilon)$ and $\delta\Sigma_S = O(\epsilon^2)$.

To $O(\epsilon^2)$, the equations of motion for the statistical function can be written as follows:

$$\Omega F = -\frac{1}{2} i \left(\Pi^+ G^- - \Pi^- G^+ \right) \tag{2.52}$$

and the Hermitian conjugate. The operator Ω has the following structure:

$$\Omega = \begin{pmatrix} -m - \delta\Sigma_S & \left(k + \frac{1}{2}i\partial - \tilde{\Sigma}_L\right)\cdot\sigma \\ \left(k + \frac{1}{2}i\partial - \tilde{\Sigma}_R\right)\cdot\bar{\sigma} & -m^\dagger - \delta\Sigma_S^\dagger \end{pmatrix}$$
$$\equiv \left(\not{k} + \frac{1}{2}i\not{\partial} - \tilde{\Sigma} - M\right) \tag{2.53}$$

Here, $\tilde{\Sigma} = \Sigma + \delta\Sigma - \frac{1}{2}i\left(\partial^\mu\Sigma\right)\partial_\mu^k$, where Σ is the $O(\epsilon)$ quantity, $\delta\Sigma$ is an $O(\epsilon^2)$ correction resulting from the $O(\epsilon)$ shift in the argument of $\Sigma[F]$, and the $O(\epsilon^2)$ derivative term comes from the Wigner transform. The collisional gain-loss potentials $\Pi^\pm$ can, in general, have all possible components:

$$\Pi^\pm = \begin{pmatrix} \Pi_S + \frac{1}{2}i\Pi_T^{L,\mu\nu}S_{\mu\nu}^L & \Pi_L\cdot\sigma \\ \Pi_R\cdot\bar{\sigma} & \Pi_S^\dagger + \frac{1}{2}i\Pi_T^{R,\mu\nu}S_{\mu\nu}^R \end{pmatrix}^\pm \tag{2.54}$$

where all quantities are $O(\epsilon^2)$. We will see that if the spin coherence density is zero, the gain-loss potentials take on a simpler form, where Π_S and Π_T are zero to $O(\epsilon^2)$.

For Majorana neutrinos, we will find that Σ_L is related to Σ_R and Π_L is related to Π_R. This is because Σ and Π are functionals of the two-point function G, and mirror the relations between G_L and G_R. For now, however, we will treat all components of Σ, Π and G as independent, and make use of the Majorana conditions when we derive the final kinetic equations.

Regardless of whether the fermions are Majorana or Dirac, the components of Σ, $\Pi^\pm$ and F have certain properties which follow from CPT invariance, which requires that these quantities be invariant under simultaneous Hermitian conjugation in spinor and flavor space. We can write F, in the most general possible form, as

$$F = \begin{pmatrix} F_S^L + \frac{1}{2}iF_T^L S_L & F_V^L\cdot\sigma \\ F_V^R\cdot\bar{\sigma} & F_S^R + \frac{1}{2}iF_T^R S_R \end{pmatrix} \tag{2.55}$$

where the notation is $F_T^{L/R}S_{L/R} \equiv \left(F_T^{L/R}\right)^{\mu\nu} S_{\mu\nu}^{L/R}$. The components of F must satisfy $F_V^{L\dagger} = F_V^L$, $F_V^{R\dagger} = F_V^R$, $F_S^{L\dagger} = F_S^R$ and $F_T^{L\dagger} = F_T^R$. The corresponding components of Σ and $\Pi^\pm$ satisfy similar Hermiticity conditions.

2.6.2 QKEs to $O(1)$: Large and Small Components

To $O(1)$, Equation (2.52) and its Hermitian conjugate simply give

$$\not{k} F = O(\epsilon) \qquad F \not{k} = O(\epsilon) \tag{2.56}$$

This gives the approximate dispersion relation $k^2 = 0$ to $O(\epsilon)$. Thus, we can choose the z-axis to be along k and write down $k = \left|\vec{k}\right| \hat{\kappa} + O(\epsilon)$, where the components of $\hat{\kappa}$ are $\hat{\kappa} = (\text{sign}(k^0), 0, 0, 1)$. Note that since $\hat{\kappa} \approx \frac{k}{|\vec{k}|}$, the first component of $\hat{\kappa}$ is ± 1, depending on whether we are dealing with a positive or negative value of k^0.

We introduce additional basis vectors, as follows:

$$\hat{\kappa}' = \left(\text{sign}\left(k^0\right), 0, 0, -1\right) \qquad \hat{x}^1 = (0,1,0,0) \qquad \hat{x}^2 = (0,0,1,0) \tag{2.57}$$

These basis vectors satisfy the relations

$$\hat{\kappa}^2 = \hat{\kappa}'^2 = 0 \qquad \hat{\kappa}\cdot\hat{\kappa}' = 2 \qquad \hat{\kappa}\cdot\hat{x}^i = \hat{\kappa}'\cdot\hat{x}^i = 0 \qquad \hat{x}^i\cdot\hat{x}^j = -\delta^{ij} \tag{2.58}$$

Note that we have imposed the condition that $\hat{x}^1$, $\hat{x}^2$ and $\hat{z} = (0,0,0,1)$ form a right-handed set of basis vectors. The momentum 4-vector k can receive $O(\epsilon)$ corrections due to a shift in the dispersion relation induced by interactions. However, the basis vectors remain the same, regardless of any such shifts.

In addition to the $O(1)$ dispersion relation, substituting the general form for F in equation (2.55) into equation (2.56) gives the following constraints on the components of F:

$$\begin{aligned} F_S &= O(\epsilon) \qquad F_V^{L/R,\mu} = \hat{\kappa}^\mu F_{L/R} + O(\epsilon) \\ F_T^{L\mu\nu} &= \frac{1}{2} F_T^i \left(\delta^{ij} - i\epsilon^{ij}\right)\left(\hat{\kappa}\wedge\hat{x}^j\right)^{\mu\nu} + O(\epsilon) \\ F_T^{R\mu\nu} &= \frac{1}{2} F_T^i \left(\delta^{ij} + i\epsilon^{ij}\right)\left(\hat{\kappa}\wedge\hat{x}^j\right)^{\mu\nu} + O(\epsilon) \end{aligned} \tag{2.59}$$

The wedge product notation is defined in the usual way, $(U\wedge V)^{\mu\nu} \equiv U^\mu V^\nu - U^\nu V^\mu$. Note that we use the names $F_{L/R}$ and F_T to denote both the full four-vector or tensor quantities and their components. Since we will often use notation where the Lorentz indices are not explicitly shown, it is important to note whether an expression refers to the full quantity or the component. This will be clear from context.

The expressions for F_T^L and F_T^R can be rewritten as follows:

$$F_T^{L\mu\nu} = \frac{1}{2}\left(F_T^1 + iF_T^2\right)\left(\hat{\kappa}\wedge\left(\hat{x}^1 - i\hat{x}^2\right)\right)^{\mu\nu} \equiv \left(\hat{\kappa}\wedge\left(\hat{x}^1 - i\hat{x}^2\right)\right)^{\mu\nu}\Phi$$
$$F_T^{R\mu\nu} = \frac{1}{2}\left(F_T^1 - iF_T^2\right)\left(\hat{\kappa}\wedge\left(\hat{x}^1 + i\hat{x}^2\right)\right)^{\mu\nu} \equiv \left(\hat{\kappa}\wedge\left(\hat{x}^1 + i\hat{x}^2\right)\right)^{\mu\nu}\Phi^\dagger \qquad (2.60)$$

where we have defined $\Phi \equiv \frac{1}{2}\left(F_T^1 + iF_T^2\right)$.

Since we have $k^2 = 0$ to $O(\epsilon)$, the components of F have the form

$$F_{L/R} = 2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right| g_{L/R} \qquad F_T^i = 2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right| g_T^i \qquad (2.61)$$

For a multi-flavor system, the notation $\delta\left(k^2 + O(\epsilon)\right)$ is symbolic, since each component of the flavor matrices $g_{L/R}$ and g_T^i will in general carry different corrections to the argument of the delta function.

To $O(\epsilon)$, we write F as follows:

$$F \to F^{(1)} + \Delta \qquad (2.62)$$

Here, $F^{(1)}$ incorporates the $O(\epsilon)$ correction to the dispersion relation, and has the form

$$F^{(1)} = \begin{pmatrix} \frac{1}{2}i\Phi\left(\hat{\kappa}\wedge\hat{x}^-\right)\cdot S_L & F_L\left(\hat{\kappa}\cdot\sigma\right) \\ F_R\left(\hat{\kappa}\cdot\bar{\sigma}\right) & \frac{1}{2}i\Phi^\dagger\left(\hat{\kappa}\wedge\hat{x}^+\right)\cdot S_R \end{pmatrix} \qquad (2.63)$$

where $\hat{x}^\pm = \left(\hat{x}^1 \pm i\hat{x}^2\right)$. Δ is the set of $O(\epsilon)$ small components. In general,

$$\Delta = \begin{pmatrix} \Delta_S + \frac{1}{2}i\Delta_T^L S_L & \Delta_L\cdot\sigma \\ \Delta_R\cdot\bar{\sigma} & \Delta_S^\dagger + \frac{1}{2}i\Delta_T^R S_R \end{pmatrix} \qquad (2.64)$$

Note that the form of F given by equations (2.59)-(2.61) is consistent with equations (2.39)-(2.40) and (2.44)-(2.45), which are derived from free, massless field theory. All correlation functions that we have found in Section IV.B. are included in the $O(1)$ expression for F. Specifically,

$$F_L = 2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right|\left(\frac{1}{2} - \theta\left(k^0\right)f\left(\vec{k}\right) - \theta\left(-k^0\right)\bar{f}\left(-\vec{k}\right)\right)$$
$$F_R = 2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right|\left(\frac{1}{2} - \theta\left(k^0\right)\bar{f}^T\left(\vec{k}\right) - \theta\left(-k^0\right)f^T\left(-\vec{k}\right)\right)$$
$$\Phi = -2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right|\left(\theta\left(k^0\right)\phi\left(\vec{k}\right) + \theta\left(-k^0\right)\phi^T\left(-\vec{k}\right)\right) \qquad (2.65)$$

Note that the results of Section IV.B. place additional constraints on the form of F. These constraints relate $F_L(k)$ to $F_R(-k)$ and $F_T(k)$ to $F_T(-k)$, and do not follow from Equation (2.52). These constraints follow from the Majorana nature of the fermions, which was assumed in Section IV.B. but not in the derivation of Equation (2.52). As mentioned above, we will use the more general formalism of Equation (2.52) and treat F_L and F_R as independent quantities, until we are ready to extract the equations of motion for the density matrices.

2.6.3 QKEs to $O(\epsilon)$: Small Components and the Dispersion Relation

We next expand equation (2.52) order-by-order, first using the $O(\epsilon)$ expansion to find the small components Δ and the $O(\epsilon)$ shift in the dispersion relation, and then inserting the results into the $O(\epsilon^2)$ equations to obtain the kinetic equations. To $O(\epsilon)$, equation (2.52) is

$$\not{k}\Delta + \left(\not{k} + \frac{1}{2}i\not{\partial}\right)F - \Sigma F - MF = O\left(\epsilon^2\right) \tag{2.66}$$

Decomposing this into irreducible representations of the Lorentz group gives the following set of equations:

Scalar:

$$k\cdot\Delta_R + \left(k + \frac{1}{2}i\partial\right)\cdot F_V^R - \Sigma_L\cdot F_V^R = O\left(\epsilon^2\right) \tag{2.67}$$

$$k\cdot\Delta_L + \left(k + \frac{1}{2}i\partial\right)\cdot F_V^L - \Sigma_R\cdot F_V^L = O\left(\epsilon^2\right) \tag{2.68}$$

Vector:

$$k\Delta_S^\dagger - k\cdot\Delta_T^R - \left(k + \frac{1}{2}i\partial\right)\cdot F_T^R + \Sigma_L\cdot F_T^R - mF_V^L = O\left(\epsilon^2\right) \tag{2.69}$$

$$k\Delta_S + k\cdot\Delta_T^L + \left(k + \frac{1}{2}i\partial\right)\cdot F_T^L - \Sigma_R\cdot F_T^L - m^\dagger F_V^R = O\left(\epsilon^2\right) \tag{2.70}$$

For the vector equations, the notation is $V\cdot T \equiv V^\mu T_{\mu\nu}$ and $T\cdot V \equiv T_{\nu\mu}V^\mu$.

Tensor:

$$\left(k \wedge \Delta_R + \left(k + \frac{1}{2}i\partial\right) \wedge F_V^R - \Sigma_L \wedge F_V^R\right)^L - \frac{1}{2}mF_T^L = O\left(\epsilon^2\right) \tag{2.71}$$

$$\left(k \wedge \Delta_L + \left(k + \frac{1}{2}i\partial\right) \wedge F_V^L - \Sigma_R \wedge F_V^L\right)^R + \frac{1}{2}m^\dagger F_T^R = O\left(\epsilon^2\right) \tag{2.72}$$

where the superscripts L and R denote anti-self-dual and self-dual projections, respectively; that is, for an antisymmetric tensor T, $T^L \equiv \frac{1}{2}\left(T - iT^\star\right)$ and $T^R \equiv \frac{1}{2}\left(T + iT^\star\right)$.

These equations, and their Hermitian conjugates, determine the form of the small components Δ and the dispersion relations for $F_{L/R}$ and F_T. To solve the equations, it is useful to decompose all our quantities into components along the basis vectors in equation (2.57). The decomposition for $F_{L/R}$ and $F_T^{L/R}$ is given by equations (2.59)-(2.61). For the other four-vector quantities we use

$$\partial = \frac{1}{2}\partial^{\kappa'}\hat{\kappa} + \frac{1}{2}\partial^{\kappa}\hat{\kappa}' - \partial^i\hat{x}^i \tag{2.73}$$

$$\Sigma_{L/R} = \frac{1}{2}\Sigma_{L/R}^{\kappa'}\hat{\kappa} + \frac{1}{2}\Sigma_{L/R}^{\kappa}\hat{\kappa}' - \Sigma_{L/R}^{i}\hat{x}^i \tag{2.74}$$

$$\Delta_{L/R} = \frac{1}{2}\Delta_{L/R}^{\kappa}\hat{\kappa}' - \Delta_{L/R}^{i}\hat{x}^i \tag{2.75}$$

$$k = \frac{1}{2}\left(k \cdot \hat{\kappa}'\right)\hat{\kappa} + \frac{1}{2}\left(k \cdot \hat{\kappa}\right)\hat{\kappa}' = \frac{1}{2}\left(\left|\vec{k}\right| + E\right)\hat{\kappa} + \frac{1}{2}\left(k \cdot \hat{\kappa}\right)\hat{\kappa}' \tag{2.76}$$

Note that the $\Delta_{L/R}^{\kappa'}$ component does not appear, since this kind of first-order shift would be along the same direction as $F_{L/R}$ and can therefore be absorbed into the $O(1)$ quantity. For a four-vector quantity V, we have labeled its component along any basis vector $\hat{w}$ as $V^w \equiv V \cdot \hat{w}$. This choice of notation determines the particular signs and factors of 1/2 in equations (2.73)-(2.76). For example, $\hat{\kappa} \cdot \Sigma_L = \Sigma^\kappa$. Since, from the $O(1)$ dispersion relation, $E = \left|\vec{k}\right| + O(\epsilon)$, the $\hat{\kappa}$ component of k is $\left|\vec{k}\right| + O(\epsilon)$.

The tensor small component is decomposed as follows:

$$\frac{1}{2}\left(\hat{\kappa} \wedge \hat{\kappa}'\right)\Delta_T^{\kappa\kappa'} + \left(\hat{x}^1 \wedge \hat{x}^2\right)\Delta_T^{xx} + \left(\hat{\kappa}' \wedge \hat{x}^i\right)\Delta_T^i \tag{2.77}$$

Again, the component proportional to $\hat{\kappa} \wedge \hat{x}^i$ does not appear, as this component can be absorbed into F_T. The anti-self-dual and self-dual projections of equation

(2.77) are

$$\Delta_T^L = \frac{1}{2}\left(\hat{\kappa}' \wedge \hat{x}^i\right)\left(\delta^{ij} - i\epsilon^{ij}\right)\Delta_T^j + \left(\frac{1}{2}\left(\hat{\kappa} \wedge \hat{\kappa}'\right) - i\left(\hat{x}^1 \wedge \hat{x}^2\right)\right)\Delta_T$$
$$\Delta_T^R = \frac{1}{2}\left(\hat{\kappa}' \wedge \hat{x}^i\right)\left(\delta^{ij} + i\epsilon^{ij}\right)\Delta_T^j + \left(\frac{1}{2}\left(\hat{\kappa} \wedge \hat{\kappa}'\right) + i\left(\hat{x}^1 \wedge \hat{x}^2\right)\right)\Delta_T^\dagger \qquad (2.78)$$

where $\Delta_T \equiv \frac{1}{2}\left(\Delta_T^{\kappa\kappa'} + i\Delta_T^{xx}\right)$

We next use equations (2.73)-(2.77) to decompose equations (2.67)-(2.72) into components. For the scalar equations, (2.67)-(2.68), this gives

$$\left|\vec{k}\right|\Delta_R^\kappa + (k\cdot\hat{\kappa})\,F_R + \frac{1}{2}i\partial^\kappa F_R - \Sigma_L^\kappa F_R = O\left(\epsilon^2\right) \qquad (2.79)$$
$$\left|\vec{k}\right|\Delta_L^\kappa + (k\cdot\hat{\kappa})\,F_L + \frac{1}{2}i\partial^\kappa F_L - \Sigma_R^\kappa F_L = O\left(\epsilon^2\right) \qquad (2.80)$$

The Hermitian portions of these equations are:

$$\left|\vec{k}\right|\Delta_R^\kappa + (k\cdot\hat{\kappa})\,F_R - \frac{1}{2}\left\{\Sigma_L^\kappa, F_R\right\} = O\left(\epsilon^2\right) \qquad (2.81)$$
$$\left|\vec{k}\right|\Delta_L^\kappa + (k\cdot\hat{\kappa})\,F_L - \frac{1}{2}\left\{\Sigma_R^\kappa, F_L\right\} = O\left(\epsilon^2\right) \qquad (2.82)$$

The anti-Hermitian portions of the scalar equations involve derivatives of F along $\hat{\kappa}$, and are therefore kinetic equations, giving the evolution of the neutrino density matrices along the neutrino world line. We will return to the kinetic equations when we expand to $O\left(\epsilon^2\right)$.

The vector equations (2.69)-(2.70) include components along $\hat{\kappa}$ and $\hat{x}^i$ (the component along $\hat{\kappa}'$ is trivial to $O(\epsilon)$). Before extracting these components, it is useful to separate the vector equations into those involving Δ_S and those involving Δ_T. Taking the Hermitian conjugate of equation (2.69) and adding this to equation (2.70) gives

$$2k\Delta_S + i\partial\cdot F_T^L - \left(\Sigma_R\cdot F_T^L + F_T^L\cdot\Sigma_L\right) - \left(m^\dagger F_R + F_L m^\dagger\right) = O\left(\epsilon^2\right) \qquad (2.83)$$

Subtracting equation (2.70) from the Hermitian conjugate of equation (2.69) gives

$$2k\cdot\left(\Delta_T^L + F_T^L\right) - \left(\Sigma_R\cdot F_T^L - F_T^L\cdot\Sigma_L\right) - \left(m^\dagger F_R - F_L m^\dagger\right) = O\left(\epsilon^2\right) \qquad (2.84)$$

The components of equations (2.83) and (2.84) along $\hat{\kappa}$ give Δ_S and Δ_T as functions of F_L, F_R and F_T^i:

$$2\left|\vec{k}\right|\Delta_S - i\partial^i P_+^{ij}F_T^j + \left(\Sigma_R^i P_+^{ij}F_T^j - P_+^{ij}F_T^j\Sigma_L^i\right) - \left(m^\dagger F_R + F_L m^\dagger\right) = O\left(\epsilon^2\right) \quad (2.85)$$

$$-2\left|\vec{k}\right|\Delta_T + \left(\Sigma_R^i P_+^{ij}F_T^j + P_+^{ij}F_T^j\Sigma_L^i\right) - \left(m^\dagger F_R - F_L m^\dagger\right) = O\left(\epsilon^2\right) \quad (2.86)$$

Here, $P_\pm^{ij}$ are projection operators on the $\hat{x}^1, \hat{x}^2$ plane, given by $P_\pm^{ij} \equiv \frac{1}{2}\left(\delta^{ij} \pm i\epsilon^{ij}\right)$

The components of equation (2.83) along $\hat{x}^i$ give kinetic equations for F_T^i; we will return to these equations when we consider the $O\left(\epsilon^2\right)$ expansion. The components of equation (2.84) along $\hat{x}^i$ are:

$$4\left|\vec{k}\right|P_-^{ij}\Delta_T^j + 2\left(k\cdot\hat{\kappa}\right)P_+^{ij}F_T^j - \left(\Sigma_R^\kappa P_+^{ij}F_T^j + P_+^{ij}F_T^j\Sigma_L^\kappa\right) = O\left(\epsilon^2\right) \quad (2.87)$$

Acting on this with P_- and using $P_+P_- = 0$ and $P_-P_- = P_-$ gives $P_-^{ij}\Delta_T^j = O\left(\epsilon^2\right)$. The Hermitian conjugate is $P_+^{ij}\Delta_T^j = O\left(\epsilon^2\right)$; adding these equations together gives $\Delta_T^j = O\left(\epsilon^2\right)$. The remainder of the equation, with its Hermitian conjugate, is

$$\left(k\cdot\hat{\kappa}\right)P_+^{ij}F_T^j - \frac{1}{2}\left(\Sigma_R^\kappa P_+^{ij}F_T^j + P_+^{ij}F_T^j\Sigma_L^\kappa\right) = O\left(\epsilon^2\right) \quad (2.88)$$

$$\left(k\cdot\hat{\kappa}\right)P_-^{ij}F_T^j - \frac{1}{2}\left(\Sigma_L^\kappa P_-^{ij}F_T^j + P_-^{ij}F_T^j\Sigma_R^\kappa\right) = O\left(\epsilon^2\right) \quad (2.89)$$

This is a set of dispersion relations for F_T; we will return to these equations later.

We next consider the tensor equations, (2.71)-(2.72). The components proportional to $\hat{\kappa}' \wedge \hat{x}^i$ are trivial to $O\left(\epsilon\right)$. The components proportional to $\hat{\kappa}\wedge\hat{\kappa}'$ are

$$\left|\vec{k}\right|\Delta_R^\kappa - \left(k\cdot\hat{\kappa}\right)F_R - \frac{1}{2}i\partial^\kappa F_R + \Sigma_L^\kappa F_R = O\left(\epsilon^2\right) \quad (2.90)$$

$$\left|\vec{k}\right|\Delta_L^\kappa - \left(k\cdot\hat{\kappa}\right)F_L - \frac{1}{2}i\partial^\kappa F_L + \Sigma_R^\kappa F_L = O\left(\epsilon^2\right) \quad (2.91)$$

The Hermitian parts of these equations, together with equations (2.79)-(2.80), give $\Delta_R^\kappa = O\left(\epsilon^2\right)$ and the dispersion relations for F_L and F_R:

$$\left(k\cdot\hat{\kappa}\right)F_R - \frac{1}{2}\left\{\Sigma_L^\kappa, F_R\right\} = O\left(\epsilon^2\right) \quad (2.92)$$

$$\left(k\cdot\hat{\kappa}\right)F_L - \frac{1}{2}\left\{\Sigma_R^\kappa, F_L\right\} = O\left(\epsilon^2\right) \quad (2.93)$$

The anti-Hermitian part simply replicates the $O(\epsilon)$ kinetic equation obtained from the scalar equations. The components along $\hat{x}^1 \wedge \hat{x}^2$ are trivially related to those along $\hat{\kappa} \wedge \hat{\kappa}'$.

The components of equations (2.71)-(2.72) along $\hat{\kappa} \wedge \hat{x}^i$ are

$$P_+^{ij}\left(\left|\vec{k}\right|\Delta_R^j - \frac{1}{2}i\partial^j F_R + \Sigma_L^j F_R + \frac{1}{2}mF_T^j\right) = O\left(\epsilon^2\right) \tag{2.94}$$

$$P_-^{ij}\left(\left|\vec{k}\right|\Delta_L^j - \frac{1}{2}i\partial^j F_L + \Sigma_R^j F_L - \frac{1}{2}m^\dagger F_T^j\right) = O\left(\epsilon^2\right) \tag{2.95}$$

The Hermitian parts of equations (2.94)-(2.95) give expressions for $\Delta_{L/R}^i$:

$$\begin{aligned}\left|\vec{k}\right|\Delta_R^i + \frac{1}{2}\epsilon^{ij}\partial^j F_R + \left(P_+^{ij}\Sigma_L^j F_R + F_R P_-^{ij}\Sigma_L^j\right)& \\ +\frac{1}{2}\left(mP_+^{ij}F_T^j + P_-^{ij}F_T^j m^\dagger\right) = O\left(\epsilon^2\right)&\end{aligned} \tag{2.96}$$

$$\begin{aligned}\left|\vec{k}\right|\Delta_L^i - \frac{1}{2}\epsilon^{ij}\partial^j F_L + \left(P_-^{ij}\Sigma_R^j F_L + F_L P_+^{ij}\Sigma_R^j\right)& \\ -\frac{1}{2}\left(m^\dagger P_-^{ij}F_T^j + P_+^{ij}F_T^j m\right) = O\left(\epsilon^2\right)&\end{aligned} \tag{2.97}$$

The anti-Hermitian parts are trivially related to the Hermitian parts.

In summary, the equations to $O(\epsilon)$ give the following expressions for the small components:

$$\Delta^{\kappa}_{L/R} = O\left(\epsilon^2\right) \qquad \Delta^i_T = O\left(\epsilon^2\right) \tag{2.98}$$

$$\Delta_S = \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger F_R + F_L m^\dagger\right) + \frac{P_+^{ij}}{2\left|\vec{k}\right|}\left(i\partial^i F_T^j - \left(\Sigma_R^i F_T^j - F_T^j \Sigma_L^i\right)\right) \tag{2.99}$$

$$\Delta_T = -\frac{1}{2\left|\vec{k}\right|}\left(m^\dagger F_R - F_L m^\dagger\right) + \frac{P_+^{ij}}{2\left|\vec{k}\right|}\left(\Sigma_R^i F_T^j + F_T^j \Sigma_L^i\right) \tag{2.100}$$

$$\begin{aligned}\Delta_L^i &= \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger P_-^{ij} F_T^j + P_+^{ij} F_T^j m\right) \\ &+ \frac{1}{\left|\vec{k}\right|}\left(\frac{1}{2}\epsilon^{ij}\partial^j F_L - \left(P_-^{ij}\Sigma_R^j F_L + F_L P_+^{ij}\Sigma_R^j\right)\right)\end{aligned} \tag{2.101}$$

$$\begin{aligned}\Delta_R^i &= -\frac{1}{2\left|\vec{k}\right|}\left(m P_+^{ij} F_T^j + P_-^{ij} F_T^j m^\dagger\right) \\ &- \frac{1}{\left|\vec{k}\right|}\left(\frac{1}{2}\epsilon^{ij}\partial^j F_R + \left(P_+^{ij}\Sigma_L^j F_R + F_R P_-^{ij}\Sigma_L^j\right)\right)\end{aligned} \tag{2.102}$$

We also obtain dispersion relations for F_T and $F_{L/R}$, given by equations (2.88)-(2.89) and (2.92)-(2.93).

2.6.4 Kinetic Equations for $F_{L/R}$

We now construct equations for the evolution of F_L and F_R, which encode the particle densities, to $O\left(\epsilon^2\right)$. These equations are derived from the scalar components of equation (2.52). To $O\left(\epsilon^2\right)$, the scalar equations are

$$\begin{aligned}&k\cdot\left(F_R + \Delta_R\right) + \frac{1}{2}i\partial\cdot F_R - \tilde{\Sigma}_L\cdot F_R + \frac{1}{2}i\partial\cdot\Delta_R - \Sigma_L\cdot\Delta_R - m\Delta_S \\ &\qquad = -\frac{1}{2}i\left(\Pi_L^+\cdot F_R^- - \Pi_L^-\cdot F_R^+\right) + \frac{1}{8}i\left(\Pi_T^{L+}G_T^{L-} - \Pi_T^{L-}G_T^{L+}\right)\end{aligned} \tag{2.103}$$

$$\begin{aligned}&k\cdot\left(F_L + \Delta_L\right) + \frac{1}{2}i\partial\cdot F_L - \tilde{\Sigma}_R\cdot F_L + \frac{1}{2}i\partial\cdot\Delta_L - \Sigma_R\cdot\Delta_L - m^\dagger\Delta_S^\dagger \\ &\qquad = -\frac{1}{2}i\left(\Pi_R^+\cdot F_L^- - \Pi_R^-\cdot F_L^+\right) + \frac{1}{8}i\left(\Pi_T^{R+}G_T^{R-} - \Pi_T^{R-}G_T^{R+}\right)\end{aligned} \tag{2.104}$$

where we have used the notation $\Pi_T G_T \equiv \left(\Pi_T\right)_{\mu\nu} G_T^{\mu\nu}$. Taking the anti-Hermitian parts of these equations and decomposing the four-vector quantities into components gives

$$\begin{aligned}
i\partial^\kappa F_R - \frac{i}{2|\vec{k}|}\{\partial^i\Sigma_L^i, F_R\} - \left(\tilde{\Sigma}_L^\kappa F_R - F_R\tilde{\Sigma}_L^{\kappa\dagger}\right) \\
-i\partial^i\Delta_R^i + \left[\Sigma_L^i, \Delta_R^i\right] - \left(m\Delta_S - \Delta_S^\dagger m^\dagger\right) = iC_R
\end{aligned} \tag{2.105}$$

$$\begin{aligned}
i\partial^\kappa F_L - \frac{i}{2|\vec{k}|}\{\partial^i\Sigma_R^i, F_L\} - \left(\tilde{\Sigma}_R^\kappa F_L - F_L\tilde{\Sigma}_R^{\kappa\dagger}\right) \\
-i\partial^i\Delta_L^i + \left[\Sigma_R^i, \Delta_L^i\right] - \left(m^\dagger\Delta_S^\dagger - \Delta_S m\right) = iC_L
\end{aligned} \tag{2.106}$$

where

$$C_R = -\frac{1}{2}\left(\left\{\Pi_L^{\kappa+}, G_R^-\right\} - \left\{\Pi_L^{\kappa-}, G_R^+\right\}\right) + C_T^R \tag{2.107}$$

$$C_L = -\frac{1}{2}\left(\left\{\Pi_R^{\kappa+}, G_L^-\right\} - \left\{\Pi_R^{\kappa-}, G_L^+\right\}\right) + C_T^L \tag{2.108}$$

The quantities $G^\pm$ are defined in equation (2.50). The terms C_T^L and C_T^R involve the tensor components of Π, and are given by

$$C_T^L = \frac{1}{8}\left(\Pi_T^{R+}G_T^{R-} + G_T^{L-}\Pi_T^{L+} - \Pi_T^{R-}G_T^{R+} - G_T^{L+}\Pi_T^{L-}\right) \tag{2.109}$$

$$C_T^R = \frac{1}{8}\left(\Pi_T^{L+}G_T^{L-} + G_T^{R-}\Pi_T^{R+} - \Pi_T^{L-}G_T^{L+} - G_T^{R+}\Pi_T^{R-}\right) \tag{2.110}$$

Next, we break this expression down into components along the basis vectors. Since $G_T^\pm$ contains only components proportional to $\hat{\kappa}\wedge\hat{x}^i$, the contraction $G_{T\mu\nu}^\pm\Pi_T^{\mu\nu\mp}$ will only have nonzero contributions from components of $\Pi_T^\mp$ that are proportional to $\hat{\kappa}'\wedge\hat{x}^i$. Thus, we can write

$$\Pi_T^{L\pm} = \Pi_T^{i\pm}P_+^{ij}\left(\hat{\kappa}'\wedge\hat{x}^j\right) \tag{2.111}$$

$$\Pi_T^{R\pm} = \Pi_T^{i\pm}P_-^{ij}\left(\hat{\kappa}'\wedge\hat{x}^j\right) \tag{2.112}$$

We now use $G_T^\pm = \pm F_T$, switch to the notation $\Phi \equiv \frac{1}{2}\left(F_T^1 + iF_T^2\right)$ and similarly define $P_T^\pm \equiv \frac{1}{2}\left(\Pi_T^1 + i\Pi_T^2\right)^\pm$. With this notation, the terms appearing in equations (2.107) and (2.108) are

$$C_T^R = \left(P_T^+ + P_T^-\right)^\dagger\Phi + \Phi^\dagger\left(P_T^+ + P_T^-\right) \tag{2.113}$$

$$C_T^L = \left(P_T^+ + P_T^-\right)\Phi^\dagger + \Phi\left(P_T^+ + P_T^-\right)^\dagger \tag{2.114}$$

Next, we use equations (2.99) and (2.101)-(2.102) to express the small components Δ_S and $\Delta^i_{L/R}$ in terms of $F_{L/R}$ and F^i_T. Equation (2.106) contains the following combination of small components: $U_L \equiv -i\partial^i\Delta^i_L + [\Sigma^i_R, \Delta^i_L] - \left(m^\dagger\Delta^\dagger_S - \Delta_S m\right)$, and equation (2.105) contains a similar combination, which we denote U_R. We separate this into parts that depend on $F_{L/R}$ and F_T:

$$U_L\left[F_{L/R}\right] = -i\partial^i\Delta^i_L\left[F_{L/R}\right] + \left[\Sigma^i_R, \Delta^i_L\left[F_{L/R}\right]\right] - \left(m^\dagger\Delta^\dagger_S\left[F_{L/R}\right] - \Delta_S\left[F_{L/R}\right]m\right) \tag{2.115}$$

$$U_L\left[F_T\right] = -i\partial^i\Delta^i_L\left[F_T\right] + \left[\Sigma^i_R, \Delta^i_L\left[F_T\right]\right] - \left(m^\dagger\Delta^\dagger_S\left[F_T\right] - \Delta_S\left[F_T\right]m\right) \tag{2.116}$$

Using equations (2.99) and (2.101) gives

$$U_L\left[F_L\right] = \frac{1}{2\left|\vec{k}\right|}i\partial^i\left\{\Sigma^i_R, F_L\right\} - \frac{1}{2\left|\vec{k}\right|}\left[m^\dagger m - \epsilon^{ij}\partial^i\Sigma^j_R + \Sigma^i_R\Sigma^i_R - i\left[\Sigma^1_R, \Sigma^2_R\right], F_L\right] \tag{2.117}$$

$$U_L\left[F_T\right] = \frac{1}{2\left|\vec{k}\right|}\left(\Sigma^i_R m^\dagger P^{ij}_- F^j_T - P^{ij}_+ F^j_T m\Sigma^i_R\right) - \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger\Sigma^i_L P^{ij}_- F^j_T - P^{ij}_+ F^j_T\Sigma^i_L m\right) \tag{2.118}$$

Similarly, we calculate $U_R\left[F_R\right]$ and $U_R\left[F_T\right]$:

$$U_R\left[F_R\right] = \frac{1}{2\left|\vec{k}\right|}i\partial^i\left\{\Sigma^i_L, F_R\right\} - \frac{1}{2\left|\vec{k}\right|}\left[mm^\dagger + \epsilon^{ij}\partial^i\Sigma^j_L + \Sigma^i_L\Sigma^i_L + i\left[\Sigma^1_L, \Sigma^2_L\right], F_R\right] \tag{2.119}$$

$$U_R\left[F_T\right] = -\frac{1}{2\left|\vec{k}\right|}\left(\Sigma^i_L m P^{ij}_+ F^j_T - P^{ij}_- F^j_T m^\dagger\Sigma^i_L\right) + \frac{1}{2\left|\vec{k}\right|}\left(m\Sigma^i_R P^{ij}_+ F^j_T - P^{ij}_- F^j_T\Sigma^i_R m^\dagger\right) \tag{2.120}$$

The equations for $F_{L/R}$ are coupled to F_T^i via the $U_{L/R}\left[F_T\right]$ terms as well as terms contained in C_L and C_R. Therefore, in addition to the kinetic equations for $F_{L/R}$, which are related to the usual neutrino density matrices, we will need to derive the kinetic equations for F_T^i, which encode coherence between left-handed and right-handed neutrinos. Note that the coupling of F_T^i to $F_{L/R}$ vanishes in the limit of isotropy. This is as expected, since in the isotropic limit, conservation of angular momentum prohibits the interconversion of left-handed and right-handed states.

Using the notation $\Phi = \frac{1}{2}\left(F_T^1 + iF_T^2\right)$, we write the kinetic equations for F_L and F_R as follows:

$$\begin{aligned} i\partial^\kappa F_R &+ \frac{1}{2\left|\vec{k}\right|} i\left\{\Sigma_L^i, \partial^i F_R\right\} + \frac{1}{2} i\left\{\partial_\mu \Sigma_L^\kappa, \partial_k^\mu F_R\right\} - \left[H_L, F_R\right] + U_R\left[\Phi\right] \\ &= iC_R\left[F_L, F_R, \Phi\right] \qquad (2.121) \\ i\partial^\kappa F_L &+ \frac{1}{2\left|\vec{k}\right|}\left\{\Sigma_R^i, \partial^i F_L\right\} + \frac{1}{2} i\left\{\partial_\mu \Sigma_R^\kappa, \partial_k^\mu F_L\right\} - \left[H_R, F_L\right] + U_L\left[\Phi\right] \\ &= iC_L\left[F_L, F_R, \Phi\right] \qquad (2.122) \end{aligned}$$

where the Hamiltonian-like operators are

$$H_L = \Sigma_L^\kappa + \delta\Sigma_L^\kappa + \frac{1}{2\left|\vec{k}\right|}\left(mm^\dagger + \epsilon^{ij}\partial^i\Sigma_L^j + 4\Sigma_L^-\Sigma_L^+\right) \qquad (2.123)$$

$$H_R = \Sigma_R^\kappa + \delta\Sigma_R^\kappa + \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger m - \epsilon^{ij}\partial^i\Sigma_R^j + 4\Sigma_R^+\Sigma_R^-\right) \qquad (2.124)$$

and the couplings to the spin coherence density are

$$U_R\left[\Phi\right] = \frac{1}{\left|\vec{k}\right|}\left(\left(m\Sigma_R^- - \Sigma_L^- m\right)\Phi + \Phi^\dagger\left(m^\dagger\Sigma_L^+ - \Sigma_R^+ m^\dagger\right)\right) \qquad (2.125)$$

$$U_L\left[\Phi\right] = -\frac{1}{\left|\vec{k}\right|}\left(\left(m^\dagger\Sigma_L^+ - \Sigma_R^+ m^\dagger\right)\Phi^\dagger + \Phi\left(m\Sigma_R^- - \Sigma_L^- m\right)\right) \qquad (2.126)$$

Here, $\Sigma^\pm \equiv \frac{1}{2}\left(\Sigma^1 \pm i\Sigma^2\right)$; while C_L and C_R correspond to Boltzmann collision terms, as will be shown below. These are given by equations (2.107)-(2.108) and (2.113)-(2.114).

2.6.5 Kinetic Equations for Spin Coherence

We see that the equations of motion for F_L and F_R, which encode the density matrices for the particles, are coupled to the spin coherence density Φ. We will see below that this spin coherence can mediate oscillations between particles of opposite helicity. We now derive the equations of motion for Φ.

We begin with kinetic equations for F_T, which can be derived from the vector components of equation (2.52). To $O\left(\epsilon^2\right)$, the vector equations are

$$\begin{aligned}
&\left(k+\frac{1}{2}i\partial\right)\Delta_S^\dagger-\Sigma_L\Delta_S^\dagger-m\left(F_L+\Delta_L\right)\\
&-\left(k+\frac{1}{2}i\partial\right)\cdot\left(F_T^R+\Delta_T^R\right)+\tilde{\Sigma}_L\cdot\left(F_T^R+\Delta_T^R\right)\\
&=\frac{1}{2}i\left(\Pi_L^+\cdot F_T^{R-}-\Pi_L^-\cdot F_T^{R+}\right)\\
&-\frac{1}{2}i\left(\Pi_S^+F_L^--\Pi_S^-F_L^++\Pi_L^{T+}\cdot F_L^--\Pi_L^{T-}\cdot F_L^+\right)
\end{aligned}\tag{2.127}$$

$$\begin{aligned}
&\left(k+\frac{1}{2}i\partial\right)\Delta_S-\Sigma_R\Delta_S-m^\dagger\left(F_R+\Delta_R\right)\\
&+\left(k+\frac{1}{2}i\partial\right)\cdot\left(F_T^L+\Delta_T^L\right)-\tilde{\Sigma}_R\cdot\left(F_T^L+\Delta_T^L\right)\\
&=-\frac{1}{2}i\left(\Pi_R^+\cdot F_T^{L-}-\Pi_R^-\cdot F_T^{L+}\right)\\
&-\frac{1}{2}i\left(\Pi_S^{\dagger+}F_R^--\Pi_S^{\dagger-}F_R^+-\Pi_R^{T+}\cdot F_R^-+\Pi_R^{T-}\cdot F_R^+\right)
\end{aligned}\tag{2.128}$$

We take the Hermitian conjugate of the equation (2.127), add to equation (2.128), and then choose the $\hat{x}^i$ components and act with P_+^{ij}. This gives

$$\begin{aligned}
&i\partial^\kappa P_+^{ij}F_T^j-\left(\tilde{\Sigma}_R^\kappa P_+^{ij}F_T^j-P_+^{ij}F_T^j\tilde{\Sigma}_L^{\dagger\kappa}\right)\\
&-\frac{i}{2|\vec{k}|}\left(\partial^n\Sigma_R^n\ P_+^{ij}F_T^j+\ P_+^{ij}F_T^j\ \partial^n\Sigma_L^n\right)\\
&+\frac{i}{2|\vec{k}|}P_+^{ij}\left(\partial^j\Sigma_R^n\ P_+^{nm}F_T^m+\ P_+^{nm}F_T^m\ \partial^j\Sigma_L^n\right)\\
&+P_+^{ij}\left(\left(m^\dagger\Delta_R^j+\Delta_L^jm^\dagger\right)+\left(\Sigma_R^j\Delta_S+\Delta_S\Sigma_L^j\right)\right)\\
&+P_+^{ij}\left(i\partial^j\Delta_T-\left(\Sigma_R^j\Delta_T-\Delta_T\Sigma_L^j\right)\right)=iC_T^i
\end{aligned}\tag{2.129}$$

where

$$C_T^i = \frac{1}{2}\left(\Pi_R^{+\kappa}P_+^{ij}F_T^j + P_+^{ij}F_T^j\Pi_L^{+\kappa}\right) + \frac{1}{2}\left(\Pi_R^{-\kappa}P_+^{ij}F_T^j + P_+^{ij}F_T^j\Pi_L^{-\kappa}\right)$$
$$-P_+^{ij}\left(\Pi_T^{j+}G_R^- + G_L^-\Pi_T^{j+} - \Pi_T^{j-}G_R^+ - G_L^+\Pi_T^{j-}\right) \qquad (2.130)$$

Writing this in terms of the complex matrix Φ, defined above:

$$i\partial^\kappa\Phi - \left(\tilde{\Sigma}_R^\kappa\Phi - \Phi\tilde{\Sigma}_L^{\dagger\kappa}\right) + i\partial^+\Delta_T - \left(\Sigma_R^+\Delta_T - \Delta_T\Sigma_L^+\right)$$
$$-\frac{i}{2|\vec{k}|}\left(\partial^i\Sigma_R^i\,\Phi + \Phi\,\partial^i\Sigma_L^i\right) + \frac{i}{|\vec{k}|}\left(\partial^+\Sigma_R^-\,\Phi + \Phi\,\partial^+\Sigma_L^-\right)$$
$$+\left(m^\dagger\Delta_R^+ + \Delta_L^+m^\dagger\right) + \left(\Sigma_R^+\Delta_S + \Delta_S\Sigma_L^+\right) = iC_\Phi \qquad (2.131)$$

where, using $P_T^\pm = \frac{1}{2}\left(\Pi_T^1 + i\Pi_T^2\right)^\pm$,

$$C_\Phi = \frac{1}{2}\left(\left(\Pi_R^{+\kappa} + \Pi_R^{-\kappa}\right)\Phi + \Phi\left(\Pi_L^{+\kappa} + \Pi_L^{-\kappa}\right)\right)$$
$$-P_T^+G_R^- - G_L^-P_T^+ + P_T^-G_R^+ + G_L^+P_T^- \qquad (2.132)$$

We separate the combination of small components in equation (2.129) into a part dependent on Φ and one dependent on $F_{L/R}$:

$$V[\Phi] + V\left[F_{L/R}\right] = i\partial^+\Delta_T - \left(\Sigma_R^+\Delta_T - \Delta_T\Sigma_L^+\right)$$
$$+\left(m^\dagger\Delta_R^+ + \Delta_L^+m^\dagger\right) + \left(\Sigma_R^+\Delta_S + \Delta_S\Sigma_L^+\right) \qquad (2.133)$$

Using equations (2.99)-(2.102) for the small components, we obtain

$$V[\Phi] = \frac{1}{2\left|\vec{k}\right|}i\partial^i\left(\Sigma_R^i\Phi + \Phi\Sigma_L^i\right) - \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger m + 2i\partial^-\Sigma_R^+ + 4\Sigma_R^+\Sigma_R^-\right)\Phi$$
$$+ \frac{1}{2\left|\vec{k}\right|}\Phi\left(mm^\dagger - 2i\partial^-\Sigma_L^+ + 4\Sigma_L^-\Sigma_L^+\right) \qquad (2.134)$$
$$V\left[F_{L/R}\right] = -\frac{1}{\left|\vec{k}\right|}\left(m^\dagger\Sigma_L^+F_R - F_Lm^\dagger\Sigma_L^+\right) + \frac{1}{\left|\vec{k}\right|}\left(\Sigma_R^+m^\dagger F_R - F_L\Sigma_R^+m^\dagger\right) \qquad (2.135)$$

We arrange the kinetic equation for Φ as follows:

$$i\partial^\kappa\Phi + \frac{1}{2\left|\vec{k}\right|}i\left(\Sigma_R^i\,\partial^i\Phi + \partial^i\Phi\,\Sigma_L^i\right) + \frac{1}{2}i\left(\partial_\mu\Sigma_R^\kappa\partial_k^\mu\Phi + \partial_k^\mu\Phi\partial_\mu\Sigma_L^\kappa\right)$$
$$- \left(H_\Phi\Phi - \Phi\bar{H}_\Phi\right) + V\left[F_{L/R}\right] = iC_\Phi \qquad (2.136)$$

where $V\left[F_{L/R}\right]$ is given by equation (2.135), and the operators H_ϕ and $\bar{H}_\phi$ are given by $H_\Phi = H_R$ and $\bar{H}_\Phi = H_L$.

2.6.6 The Majorana Conditions and Dispersion Relation

We now extract the kinetic equations for particle and antiparticle density matrices. These equations can be obtained by integrating the equations of motion for F_L and F_R over positive or negative energies.

For Majorana neutrinos, the equations of motion for F_L and F_R must be redundant; that is, the positive-energy component of F_L contains the same information as the negative-energy component of F_R. Specifically, $F_L(k) = F_R^T(-k)$ and $\Phi(k) = \Phi^T(-k)$. The redundancy of the equations of motion requires

$$m = m^T \qquad \Sigma_R = -\Sigma_L^T \equiv \Sigma \tag{2.137}$$

The condition $m = m^T$ follows from the form of the Majorana mass term. When we calculate the matter potential and the gain-loss potentials below, we will see that the other conditions are also satisfied. This follows simply from the fact that the potentials Σ and Π are functionals of the two-point function, and the Majorana constraints on the form of the two-point function lead to the appropriate constraints on Σ and Π.

In addition to imposing the Majorana constraints, we must solve the dispersion relations for F_L, F_R and F_T, given by equations (2.92)-(2.93) and (2.88)-(2.89), to $O(\epsilon)$. We solve equation (2.92), by transforming to the basis in flavor space that diagonalizes Σ_L^κ. In this basis, F_R satisfies equation (2.92) if it has the form

$$F_R = \begin{pmatrix} \delta(1,1)\, g_R^{11} & \delta(1,2)\, g_R^{12} & ... \\ \delta(2,1)\, g_R^{21} & \delta(2,2)\, g_R^{22} & ... \\ ... & ... & ... \end{pmatrix} \tag{2.138}$$

Here, $\delta(I,J)$ is an expression containing a delta function that enforces the condition $k \cdot \hat{\kappa} - \frac{1}{2}\left(\Sigma_L^I + \Sigma_L^J\right) = O(\epsilon^2)$, where Σ_L^I is the Ith eigenvalue of Σ_L^κ. We wish to write this as $2\pi\delta\left(k^2 + O(\epsilon)\right)\left|\vec{k}\right|$, to match the $O(1)$ expression $F_L = 2\pi\delta\left(k^2\right)\left|\vec{k}\right| g(k)$. Therefore, the appropriate form of the delta function is $\delta(I,J) = 2\pi\delta\left(k^2 - \left|\vec{k}\right|\left(\Sigma_L^I + \Sigma_L^J\right) + O\left(\epsilon^2\right)\right)\left|\vec{k}\right|$.

Using flavor projection operators P_I, where $P_1 = \begin{pmatrix} 1 & 0 & ... \\ 0 & 0 & ... \\ ... & ... & ... \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 & ... \\ 0 & 1 & ... \\ ... & ... & ... \end{pmatrix}$, etc, we can write

$$F_R = \sum_{IJ} 2\pi\delta\left(k^2 - \left|\vec{k}\right|\left(\Sigma_L^I + \Sigma_L^J\right)\right)\left|\vec{k}\right| P_I g_R P_J \tag{2.139}$$

We can now transform to an arbitrary basis (such as the flavor basis) by using the unitary matrix U_L, which transforms from the desired basis to one in which Σ_L^κ is diagonal, and use equation (2.40) to express F_R in terms of f and $\bar{f}$. The result is

$$F_R = 2\pi\left|\vec{k}\right|\sum_{IJ}\delta\left(k^2 - \left|\vec{k}\right|\left(\Sigma_L^I + \Sigma_L^J\right)\right)\left(U_L^\dagger P_I U_L\right) \times$$
$$\left(\frac{1}{2} - \theta\left(k^0\right)\bar{f}^T\left(\vec{k}\right) - \theta\left(-k^0\right)f^T\left(-\vec{k}\right)\right)\left(U_L^\dagger P_J U_L\right) \tag{2.140}$$

Similarly,

$$F_L = 2\pi\left|\vec{k}\right|\sum_{IJ}\delta\left(k^2 - \left|\vec{k}\right|\left(\Sigma_R^I + \Sigma_R^J\right)\right)\left(U_R^\dagger P_I U_R\right) \times$$
$$\left(\frac{1}{2} - \theta\left(k^0\right)f\left(\vec{k}\right) - \theta\left(-k^0\right)\bar{f}\left(-\vec{k}\right)\right)\left(U_R^\dagger P_J U_R\right) \tag{2.141}$$

where the density matrices f and $\bar{f}$ are expressed in the original flavor basis. For spin coherence, the dispersion relation is given by equations (2.88)-(2.89). In terms of the quantity Φ, these equations give

$$\left(k\cdot\hat{\kappa}\right)\Phi - \frac{1}{2}\left(\Sigma_R^\kappa\Phi + \Phi\Sigma_L^\kappa\right) = O\left(\epsilon^2\right). \tag{2.142}$$

Note that Φ satisfies the dispersion relation if it has the form

$$\Phi = -2\pi\left|\vec{k}\right|\sum_{IJ}\delta\left(k^2 - \left|\vec{k}\right|\left(\Sigma_R^I + \Sigma_L^J\right)\right)\left(U_R^\dagger P_I U_R\right)$$
$$\times\left(\theta\left(k^0\right)\phi\left(\vec{k}\right) + \theta\left(-k^0\right)\phi^T\left(-\vec{k}\right)\right)\left(U_L^\dagger P_J U_L\right) \tag{2.143}$$

2.6.7 Equations of Motion for Density Matrices and Spin Coherence Densities

We can now find the equations of motion for the density matrices of Majorana neutrinos. These equations can be obtained by integrating the equation of motion for F_L, equation (2.122), over positive energies, and similarly integrating equation (2.121) for F_R over positive energies and taking the transpose. We also integrate equation (2.136) over positive energies to obtain the equations of motion for the spin coherence density. Due to the Majorana nature of the fermions, these equations are redundant with those obtained by integrating over negative energies; the redundancy is satisfied if the Majorana conditions on the mass and the matter potentials, equation (2.137), hold. Performing the integration and imposing the Majorana conditions gives

$$i\partial^\kappa f^{(1)} + \frac{1}{2\left|\vec{k}\right|} i\left\{\Sigma^i, \partial^i f\right\} - \frac{1}{2} i \left\{\frac{\partial \Sigma^\kappa}{\partial \vec{x}}, \frac{\partial f}{\partial \vec{k}}\right\} - [H, f]^{(1)} + U[\phi] = iC\left[f, \bar{f}, \phi\right] \tag{2.144}$$

$$i\partial^\kappa \bar{f}^{(1)} - \frac{1}{2\left|\vec{k}\right|} i\left\{\Sigma^i, \partial^i \bar{f}\right\} + \frac{1}{2} i \left\{\frac{\partial \Sigma^\kappa}{\partial \vec{x}}, \frac{\partial \bar{f}}{\partial \vec{k}}\right\} - \left[\bar{H}, \bar{f}\right]^{(1)} + \bar{U}[\phi] = i\bar{C}\left[f, \bar{f}, \phi\right] \tag{2.145}$$

$$i\partial^\kappa \phi^{(1)} + \frac{1}{2\left|\vec{k}\right|} i\left(\Sigma^i\, \partial^i \phi - \partial^i \phi\, \Sigma^{iT}\right) - \frac{1}{2} i \left(\frac{\partial \Sigma^\kappa}{\partial \vec{x}} \cdot \frac{\partial \phi}{\partial \vec{k}} - \frac{\partial \phi}{\partial \vec{k}} \cdot \frac{\partial \Sigma^{\kappa T}}{\partial \vec{x}}\right) - \left(H_\Phi \phi - \phi \bar{H}_\Phi\right)^{(1)} + V\left[f, \bar{f}\right] = iC_\phi\left[\phi, f, \bar{f}\right] \tag{2.146}$$

Since Σ_L and Σ_R are related by the Majorana condition, we use the notation $\Sigma \equiv \Sigma_R = -\Sigma_L^T$. The terms immediately following the first derivative term, *i.e.*, those involving anticommutators and derivatives of the matter potential, give trajectory deviation and a shift in energy of the particles in response to a changing matter potential.

The Hamiltonian operators for neutrinos and anti-neutrinos, H and $\bar{H}$, are:

$$H = \Sigma^\kappa + \delta\Sigma^\kappa + \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger m - \epsilon^{ij}\partial^i\Sigma^j + \Sigma^i\Sigma^i - i\left[\Sigma^1, \Sigma^2\right]\right) \tag{2.147}$$

$$\bar{H} = \Sigma^\kappa + \delta\Sigma^\kappa - \frac{1}{2\left|\vec{k}\right|}\left(m^\dagger m - \epsilon^{ij}\partial^i\Sigma^j + \Sigma^i\Sigma^i - i\left[\Sigma^1, \Sigma^2\right]\right) \tag{2.148}$$

The terms coupling the kinetic equations to the spin coherence are:

$$U = \frac{1}{\left|\vec{k}\right|}\left(\Sigma^+ m^\star \phi^\dagger - \phi m \Sigma^-\right) + \frac{1}{\left|\vec{k}\right|}\left(m^\star \Sigma^{+T}\phi^\dagger - \phi\Sigma^{-T} m\right) \tag{2.149}$$

$$\bar{U} = -\frac{1}{\left|\vec{k}\right|}\left(\Sigma^+ m^\star \phi^\star - \phi^T m \Sigma^-\right) - \frac{1}{\left|\vec{k}\right|}\left(m^\star \Sigma^{+T}\phi^\star - \phi^T\Sigma^{-T} m\right) \tag{2.150}$$

The collision terms on the right-hand side are

$$C = \frac{1}{2}\left(\left\{\tilde{\Pi}_R^{\kappa+}, f\right\} - \left\{\tilde{\Pi}_R^{\kappa-}, 1-f\right\}\right) + \left(\tilde{P}_T^+ + \tilde{P}_T^-\right)\phi^\dagger + \phi\left(\tilde{P}_T^+ + \tilde{P}_T^-\right)^\dagger \tag{2.151}$$

$$\begin{aligned}\bar{C} = \frac{1}{2}&\left(\left\{\left[\tilde{\Pi}_L^{\kappa+}\right]^T, \bar{f}\right\} - \left\{\left[\tilde{\Pi}_L^{\kappa-}\right]^T, 1-\bar{f}\right\}\right) \\ &+ \left(\tilde{P}_T^+ + \tilde{P}_T^-\right)^T\phi^\star + \phi^T\left(\tilde{P}_T^+ + \tilde{P}_T^-\right)^\star\end{aligned} \tag{2.152}$$

$$\begin{aligned}C_\phi = &\frac{1}{2}\left[\left(\tilde{\Pi}_R^{\kappa+} + \tilde{\Pi}_R^{\kappa-}\right)\phi + \phi\left(\tilde{\Pi}_L^{\kappa+} + \tilde{\Pi}_L^{\kappa-}\right)\right] \\ &+ f\,\tilde{P}_T^+ - (1-f)\tilde{P}_T^- + \tilde{P}_T^+\bar{f}^T - \tilde{P}_T^-\left(1-\bar{f}^T\right)\end{aligned} \tag{2.153}$$

where

$$\begin{aligned}\tilde{\Pi}_{L,R}^{\kappa\pm}\left(\vec{k}\right) &= \int_0^\infty dk^0\ \Pi_{L,R}^{\kappa\pm}(k)\ \delta(k^0 - |\vec{k}|) \\ \tilde{P}_T^\pm\left(\vec{k}\right) &= \int_0^\infty dk^0\ P_T^\pm(k)\ \delta(k^0 - |\vec{k}|)\end{aligned} \tag{2.154}$$

The first two terms in C and $\bar{C}$ correspond to the gain-loss terms in the Boltzmann equation, including Fermi blocking. The remainder represent coupling to the spin coherence ϕ via collisional processes.

The superscript "$^{(1)}$" we take to indicate terms that include corrections stemming from a shift in the dispersion relation, up to $O\left(\epsilon^2\right)$. Specifically,

$$f^{(1)} = \int_0^\infty \frac{dk^0}{2\pi}\left(-2F_L\right) = f - \sum_{IJ} \frac{\Sigma^I + \Sigma^J}{2\left|\vec{k}\right|}\left(U^\dagger P_I U\right) f \left(U^\dagger P_J U\right) \quad (2.155)$$

$$\bar{f}^{(1)} = \int_0^\infty \frac{dk^0}{2\pi}\left(-2F_R\right)^T = \bar{f} + \sum_{IJ} \frac{\Sigma^I + \Sigma^J}{2\left|\vec{k}\right|}\left(U^\dagger P_J U\right) \bar{f} \left(U^\dagger P_I U\right) \quad (2.156)$$

and

$$[H, f]^{(1)} = \left[H\left(\epsilon\right), f^{(1)}\right] + \left[H\left(\epsilon^2\right), f\right] \quad (2.157)$$

where $H\left(\epsilon\right)$ and $H\left(\epsilon^2\right)$ are the $O\left(\epsilon\right)$ and $O\left(\epsilon^2\right)$ contributions to H.

The quantities appearing in the equation of motion for spin coherence are Hamiltonian-like quantities acting on ϕ itself, given by $H_\Phi = H$ and $\bar{H}_\Phi = -\bar{H}^T$, as well as a term coupling ϕ to f and $\bar{f}$:

$$V\left[f, \bar{f}\right] = \frac{1}{\left|\vec{k}\right|}\left(m^\star \Sigma^{+T} \bar{f}^T - f m^\star \Sigma^{+T}\right) + \frac{1}{\left|\vec{k}\right|}\left(\Sigma^+ m^\star \bar{f}^T - f \Sigma^+ m^\star\right) . \quad (2.158)$$

The quantity $\phi^{(1)}$ incorporates corrections due to the dispersion relation:

$$\phi^{(1)} = \phi - \sum_{IJ} \frac{\Sigma_I - \Sigma_J}{2\left|\vec{k}\right|}\left(U^\dagger P_I U\right) \phi \left(U^T P_J U^\star\right) \quad (2.159)$$

2.6.8 $2N_f \times 2N_f$ Notation

Equations (2.144)-(2.146), the quantum kinetic equations, can be written more compactly as follows:

$$iD\left[\mathcal{F}\right] - \left[\mathcal{H}, \mathcal{F}\right] = i\mathcal{C}\left[\mathcal{F}\right] \quad (2.160)$$

Here, for 3 neutrino flavors, $\mathcal{F}$ and $\mathcal{H}$ are 6×6 matrices having the following block structure:

$$\mathcal{F} \equiv \begin{pmatrix} f & \phi \\ \phi^\dagger & \bar{f}^T \end{pmatrix} \qquad \mathcal{H} \equiv \begin{pmatrix} H & H_{\nu\bar{\nu}} \\ H_{\nu\bar{\nu}}^\dagger & -\bar{H}^T \end{pmatrix} \quad (2.161)$$

The quantities H and $\bar{H}$ are the neutrino and anti-neutrino Hamiltonians, given by equations (2.147) and (2.148), while $H_{\nu\bar{\nu}}$ is given by

$$H_{\nu\bar{\nu}} = -\frac{1}{\left|\vec{k}\right|}\left(\Sigma^{+}m^{\star} + m^{\star}\Sigma^{+T}\right) \tag{2.162}$$

The derivative term is

$$\begin{aligned} iD\left[\mathcal{F}\right] = i\partial^{\kappa}\mathcal{F}^{(1)} &+ \frac{i}{2\left|\vec{k}\right|}\left\{\left(\begin{array}{cc} \Sigma^{i} & 0 \\ 0 & -\Sigma^{iT}\end{array}\right), \partial^{i}\mathcal{F}\right\} \\ &-\frac{1}{2}i\left\{\frac{\partial}{\partial\vec{x}}\left(\begin{array}{cc} \Sigma^{\kappa} & 0 \\ 0 & -\Sigma^{\kappa T}\end{array}\right), \frac{\partial\mathcal{F}}{\partial\vec{k}}\right\} \end{aligned} \tag{2.163}$$

and the collision term is

$$C = \left(\begin{array}{cc} C & C_{\phi} \\ C_{\phi}^{\dagger} & \bar{C}^{T}\end{array}\right) \tag{2.164}$$

where C, $\bar{C}$ and C_{ϕ} are given by equations (2.151), (2.152) and (2.153).

2.7 Neutrino Interactions with Matter

In this section, we compute the matter potential Σ for neutrinos. We also show how the gain-loss potentials $\Pi^{\pm}$ are calculated, and explicitly compute some of the terms in $\Pi^{\pm}$ to show that these quantities can be identified with the gain-loss terms in the Boltzmann equation.

2.7.1 Matter Potential

The matter potential corresponds to the local piece of the neutrino self-energy, as given by Equation (2.25). Since, in the low-energy limit, the W and Z boson propagators are local (proportional to $\delta(x-y)$), to leading order the matter potential is given by the one-loop diagrams shown in Figure 1. We note that in general, the leading-order form of the weak boson propagator receives small corrections, which may be physically important in some environments [3,

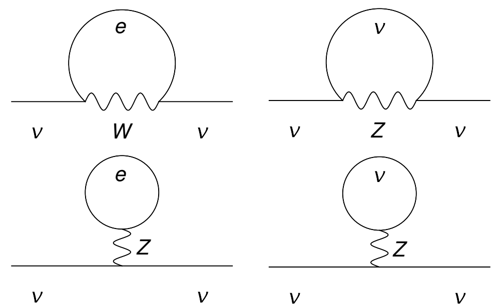


Figure 2.1: Feynman graphs for neutral and charged current one-loop contributions to neutrino self-energy.

4, 135–138]. For simplicity, we do not include these corrections here; however, incorporating them would be relatively straightforward.

Note that the one-loop diagrams involving only neutrino propagators include all corrections to the neutrino two-point function, since the neutrino two-point function is treated as a dynamical quantity. As a consequence, the diagrams already include all "bubble" diagrams with bubbles branching off an internal neutrino line. However, since we are not treating charged leptons as dynamical, there are additional contributions corresponding to corrections to the charged lepton two-point function. Examples of such contribution are given in Figure 2. Diagrams such as this generate a neutrino magnetic moment, thus giving neutrinos a small effective interaction with the electromagnetic field. These diagrams also give a small effective mass splitting between muon and tau neutrinos, due to the different mass of the virtual charged lepton on the internal lines. Since the sub-diagram involves the electromagnetic, rather than the weak interaction, even higher-order diagrams like this can give a larger contribution to Σ than two-loop diagrams involving only the weak interaction. Nevertheless, for simplicity, we will not include such diagrams here, and simply use the leading-order expressions for the charged lepton two-point function. However, it should be kept in mind that the charged lepton corrections, though small, nevertheless may prove important in neutrino flavor evolution in supernovae, as demonstrated in Ref.s [3, 139, 140].

Having made these simplifications, we compute the first diagram in Fig. 1.

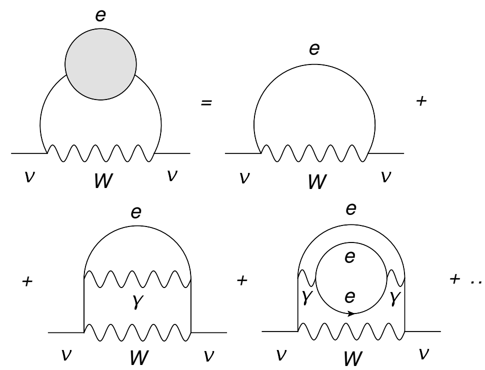


Figure 2.2: Examples of diagrams that incorporate corrections to the charged lepton two-point function. For simplicity, we neglect all but the leading-order diagram in this section.

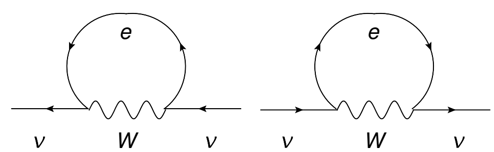


Figure 2.3: Contributions to the charged current one-loop diagram

Note that this diagram cannot involve an arrow-clashing charged lepton propagator (involving either an odd number of mass insertions, or any kind of charged lepton spin coherence) because the arrow-clashing propagator always connects the charged lepton field to its Dirac counterpart, which does not interact via the charged current interaction. Therefore, the only contributions to Σ from this diagram are those given in Fig. 3.

In position space, these diagrams give

$$\Sigma^{W,e}_{IJ,\alpha\dot{\alpha}}\left(x,y\right) = i\delta^{4}\left(x-y\right)\left(-i2\sqrt{2}G_{F}\right)\sigma^{\mu}_{\alpha\dot{\beta}}G^{e,\dot{\beta}\beta}_{IJ}\left(x,y\right)\sigma_{\mu\beta\dot{\alpha}} \tag{2.165}$$

$$\Sigma^{W,e,\dot{\alpha}\alpha}_{IJ}\left(x,y\right) = i\delta^{4}\left(x-y\right)\left(-i2\sqrt{2}G_{F}\right)\bar{\sigma}^{\mu\dot{\alpha}\beta}G^{e}_{IJ,\beta\dot{\beta}}\left(x,y\right)\bar{\sigma}_{\mu\dot{\beta}\alpha} \tag{2.166}$$

The superscript W indicates that this is the contribution to the matter potential stemming from the charged current interaction.

Upon Wigner transformation, this is

$$\Sigma^{W,e}_{IJ,\alpha\dot{\alpha}}(x) = 2\sqrt{2}G_F \int \frac{d^4q}{(2\pi)^4} \sigma^{\mu}_{\alpha\dot{\beta}} F^{e,\dot{\beta}\beta}_{IJ}(x,q)\,\sigma_{\mu,\beta\dot{\alpha}} \tag{2.167}$$

$$\Sigma^{W,e,\dot{\alpha}\alpha}_{IJ}(x) = 2\sqrt{2}G_F \int \frac{d^4q}{(2\pi)^4}\, \bar{\sigma}^{\mu,\dot{\alpha}\beta} F^{e}_{IJ,\beta\dot{\beta}}(x,q)\,\bar{\sigma}^{\dot{\beta}\alpha}_{\mu} \tag{2.168}$$

In the flavor basis, neglecting corrections from interactions with the plasma, the statistical function for charged fermions is

$$F^{e,\dot{\alpha}\alpha}_{IJ}(x,q) = 2\pi \sum_K \delta\left(q^2 - m_K^2\right) q\cdot\bar{\sigma}^{\dot{\alpha}\alpha}\left(P_K\right)_{JI} \times \left(\frac{1}{2} - \theta\left(q^0\right) f^e_{R,K}(x,\vec{q}) - \theta\left(-q^0\right)\bar{f}^e_{L,K}(x,-\vec{q})\right) \tag{2.169}$$

$$F^{e}_{IJ,\alpha\dot{\alpha}}(x,q) = 2\pi \sum_K \delta\left(q^2 - m_K^2\right) q\cdot\sigma_{\alpha\dot{\alpha}}\left(P_K\right)_{IJ} \times \left(\frac{1}{2} - \theta\left(q^0\right) f^e_{L,K}(x,\vec{q}) - \theta\left(-q^0\right)\bar{f}^e_{R,K}(x,-\vec{q})\right) \tag{2.170}$$

Here, the flavor index K denotes electrons, muons and tauons. m_K is the charged lepton mass corresponding to flavor K, $(P_K)_{IJ}$ are flavor projection matrices, $f^e_{L,K}$ is the density of left-handed charged leptons of flavor K, and $\bar{f}^e_{R,K}$ is the density of right-handed charged anti-leptons of flavor K. Note that this expression assumes that there is no coherence between charged leptons of different flavor. This assumption is motivated by two arguments. First, mass-squared splittings between charged leptons are large, so at low energies, flavor coherence would be difficult to generate. Second, charged leptons interact much more strongly than neutrinos. Scattering is expected to cause decoherence, so that even if charged lepton flavor coherence could be generated, it would be quickly destroyed by interactions.

In supernovae, and in certain epochs in the early Universe, the temperature is too low for a substantial number of muons or tauons to be present in the plasma. In this case, we can set $f_K, \bar{f}_K \approx 0$ for $K \neq 1$.

Performing the integrals in equations (2.167)-(2.168) over q^0 and using the definition of $\Sigma_{L/R}$ in equation (2.51) gives

$$\Sigma_L^{W,e}(x) = -4\sqrt{2}G_F \sum_K P_K^T \int \tilde{dq}_K \; q_K^\mu \left(f_{L,K}^e(x,\vec{q}) - \bar{f}_{R,K}^e(x,\vec{q})\right)$$
$$= -4\sqrt{2}G_F \sum_K P_K^T J_{L,K}^\mu(x) \qquad (2.171)$$

$$\Sigma_R^{W,e}(x) = 4\sqrt{2}G_F \sum_K P_K \int \tilde{dq}_K \; q_K^\mu \left(f_{L,K}^e(x,\vec{q}) - \bar{f}_{R,K}^e(x,\vec{q})\right)$$
$$= 4\sqrt{2}G_F \sum_K P_K J_{L,K}^\mu(x) \qquad (2.172)$$

Here, $\tilde{dq}_K \equiv \frac{d^3\vec{q}}{(2\pi)^3 2E_{q,K}}$ and $q_K^\mu = (E_{q,K}, \vec{q})$, with $E_{q,K} = \sqrt{\vec{q}^2 + m_K^2}$. $J_{L,K}^\mu$ is the current associated with left-handed charged leptons of flavor K.

The second diagram in Fig. 1 has a similar structure, and gives the following contribution to Σ:

$$\Sigma_L^\nu(x) = -2\sqrt{2}G_F \left(J_{(\nu)}^\mu(x)\right)^T \qquad \Sigma_R^\nu(x) = 2\sqrt{2}G_F J_{(\nu)}^\mu(x) \qquad (2.173)$$

where $J_{(\nu)}^\mu(x)$ is the neutrino current, given by

$$J_{(\nu)}^\mu(x) = \int \tilde{dq} \; q^\mu \left(f(x,\vec{q}) - \bar{f}(x,\vec{q})\right) \qquad (2.174)$$

For neutrinos, we also obtain contributions to $\Sigma_\alpha{}^\beta$ and $\Sigma^{\dot\alpha}{}_{\dot\beta}$ by including the arrow-clashing propagator in the loop. These components of Σ can in general have a tensor component and a scalar component. However, the tensor component is proportional to $\bar\sigma^\mu S_L^{\rho\sigma}\sigma_\mu$ or $\sigma^\mu S_R^{\rho\sigma}\bar\sigma_\mu$, which vanishes in four spacetime dimensions, so there is no tensor contribution to Σ. The scalar component, on the other hand, is proportional to the scalar component of the neutrino two-point function, which is an $O(\epsilon)$ quantity. Consequently, the scalar component of Σ is $O(\epsilon^2)$. Since this appears in the kinetic equations as a correction to the mass, and the mass always enters as a part of an $O(\epsilon^2)$ term, the shift in the mass due to the scalar component of Σ produces an $O(\epsilon^3)$ term, which can be neglected.

Note that the neutrino current contains an $O(\epsilon)$ correction due to a shift in the dispersion relation. Another correction comes from the $O(\epsilon)$ contribution to F from the small components $\Delta_{L/R}$. These corrections result in an $O(\epsilon^2)$ shift in $\Sigma_{L/R}$, which is denoted in the quantum kinetic equations as $\delta\Sigma_{L/R}$. Thus, we define Σ as the quantity that is calculated by using the massless, free-field, $O(1)$ expression for the current, while $\delta\Sigma$ contains the $O(\epsilon)$ corrections from the masses and interactions.

Similarly, we calculate the two lower diagrams in Fig. 1 to obtain the following contributions to Σ_R^μ:

$$4\sqrt{2}G_F\mathbf{1}\sum_K\left(\left(\sin^2\theta_W-\frac{1}{2}\right)J^\mu_{L,K}+\sin^2\theta_W J^\mu_{R,K}\right) \tag{2.175}$$

and

$$2\sqrt{2}G_F\left(\mathrm{tr}\ J^\mu_{(\nu)}\right)\mathbf{1}\ \sigma_{\mu\alpha\dot{\beta}} \tag{2.176}$$

and similarly for the $\bar{\sigma}$ component of Σ. Here, $\mathbf{1}$ is the flavor unit matrix, and the trace is over flavor. The complete expression for the matter potential Σ to $O(\epsilon)$ is, therefore,

$$\begin{aligned}\Sigma_R^\mu=\Sigma_R^{(e)\mu}+\Sigma_R^{(\nu)\mu}=4\sqrt{2}G_F\times\\ \sum_K\left(\left(P_K+\mathbf{1}\left(\sin^2\theta_W-\frac{1}{2}\right)\right)J^\mu_{L,K}+\mathbf{1}\sin^2\theta_W J_{R,K}\right)\\ +2\sqrt{2}G_F\left(J^\mu_{(\nu)}+\mathbf{1}\left(\mathrm{tr}\ J^\mu_{(\nu)}\right)\right)\end{aligned} \tag{2.177}$$

2.7.2 Collision Terms

In this section, we consider the quantities $\Pi^\pm$ that appear on the right-hand side of the quantum kinetic equations. We will see that these terms have the gain-loss structure of a Boltzmann collision term. We will refer to them as the gain-loss potentials.

$\Pi^\pm$ are linear combinations of Π_ρ and Π_F given by equation (2.50). In position space, Π_ρ and Π_F are nonlocal components of the self-energy. In our model, all nonlocal contributions correspond to two-loop (or higher-order) diagrams involving the exchange of at least two W or Z bosons. To two-loop order, the diagrams that contribute to $\Pi_{\rho,F}$ are shown in Figure 4.

These diagrams give a large number of terms corresponding to various scattering processes, which must all be included in a complete treatment of inelastic

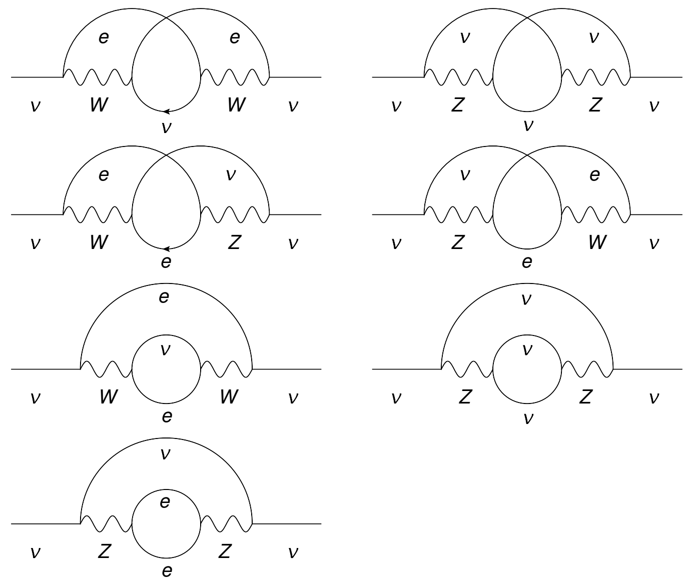


Figure 2.4: Feynman graphs showing two-loop contributions to neutrino self-energy.

scattering of neutrinos off charged leptons and other neutrinos. Since we do not present numerical computations of neutrino scattering in this paper, we will not calculate every term in detail. We will show that the $\Pi^{\pm}$ produce Boltzmann-like gain-loss terms and for the purpose of illustration we will compute only one of the terms in detail. A calculation of the full collision term will be presented in upcoming work.

Example: $\nu\nu$ scattering neglecting spin coherence

As an illustration, we consider inelastic processes involving only neutrinos and anti-neutrinos, ignoring the presence of electrons and other particles in the thermal bath. This means we consider only the contribution from the upper-right and lower-right diagrams in Fig. 4, which involve only neutrino lines. First, consider the upper-right diagram: placing arrows on the fermion lines produces 16 arrangements that contribute to this diagram. There are four possible combinations of external arrow directions, which pick out the particular component of $\Pi^{\pm}$ that is being calculated. For each combination of external arrows, there are four possible combinations of internal arrow directions, which determine the com-

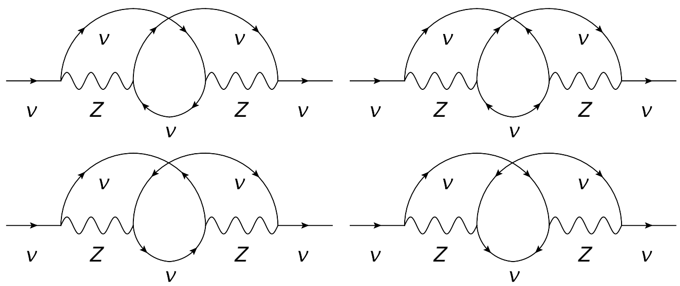


Figure 2.5: Contributions to $\Pi^{\dot{\alpha}\alpha}$ corresponding to the upper-right diagram in Fig. 4

ponents of G that the given contribution to $\Pi^{\pm}$ depends on. For example, the contributions to $\Pi^{\dot{\alpha}\alpha}$ from this diagram are given in Figure 5; there are similar contributions to $\Pi_{\alpha\dot{\alpha}}$, $\Pi_{\alpha}^{\ \beta}$ and $\Pi^{\dot{\alpha}}_{\ \dot{\beta}}$, which correspond to different directions for the external arrows.

All diagrams in Fig. 5 except the upper-left include two factors of arrow-clashing two-point functions for neutrinos. The arrow-clashing two-point functions contain a scalar and a tensor component; the scalar component is $O(\epsilon)$, while the tensor component can in general be $O(1)$ if there is spin coherence. The $O(\epsilon)$ terms can be dropped, since the two-loop diagrams are already $O(\epsilon^2)$. Then, if spin coherence is present, so that $\phi = O(1)$, all four diagrams contribute. However, in the absence of spin coherence, only the first diagram is $O(\epsilon^2)$; the remaining three are $O(\epsilon^4)$ and can be dropped. Moreover, any contribution to $\Pi_{\alpha}^{\ \beta}$ or $\Pi^{\dot{\alpha}}_{\ \dot{\beta}}$ must contain at least one arrow-clashing internal line, and therefore these quantities are at least $O(\epsilon^3)$ and can be dropped in the absence of spin coherence.

For the sake of brevity, here we consider only terms that do not depend on spin coherence. The procedure for calculating the other terms will be similar.

In position space, in terms of two-point functions, the upper-left diagram in Fig. 5 gives

$$\Pi^{\dot{\alpha}\alpha}(x,y) = -2\,\delta^4(x-w)\,\delta^4(z-y)\,G_F^2 \times \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} G^{(\nu)}_{\beta\dot{\beta}}(x,z)\,\bar{\sigma}^{\nu\dot{\beta}\gamma} G^{(\nu)}_{\gamma\dot{\gamma}}(z,w)\,\bar{\sigma}_{\mu}^{\dot{\gamma}\delta} G^{(\nu)}_{\delta\dot{\delta}}(w,y)\,\bar{\sigma}_{\nu}^{\dot{\delta}\alpha} \tag{2.178}$$

To proceed further, first, we calculate the appropriate combinations of spectral and statistical components, Π^+ and Π^-, defined by Equation (2.50). When performing this calculation we do not need to keep track of the details of the spin and flavor structure of two-point function products, since the decomposition into spectral and statistical components is the same regardless of these details. As a result, we can write, symbolically,

$$\Pi\left(x,y\right)\sim G_1\left(x,y\right)G_2\left(y,x\right)G_3\left(x,y\right) \tag{2.179}$$

This notation simply indicates that Π is composed of three distinct two-point functions, which are then contracted in some way and multiplied by the appropriate couplings and electroweak boson propagators. Note that the delta functions in equation (2.178) allow us to write all two-point functions as functions of only x and y.

We can write $G\left(x,y\right)=\theta\left(x^0-y^0\right)G^+\left(x,y\right)-\theta\left(y^0-x^0\right)G^-\left(x,y\right)$, and similarly $\Pi\left(x,y\right)=\theta\left(x^0-y^0\right)\Pi^+\left(x,y\right)-\theta\left(y^0-x^0\right)\Pi^-\left(x,y\right)$. Then, setting $x^0>y^0$, we obtain

$$\Pi^+\left(x,y\right)\sim -G_1^+\left(x,y\right)G_2^-\left(y,x\right)G_3^+\left(x,y\right) \tag{2.180}$$

Similarly, for $x^0<y^0$, we obtain

$$\Pi^-\left(x,y\right)\sim -G_1^-\left(x,y\right)G_2^+\left(y,x\right)G_3^-\left(x,y\right) \tag{2.181}$$

Next, we Wigner transform equation (2.178), and use equations (2.180)-(2.181) to obtain $\Pi^\pm$. This gives the following expression:

$$\begin{aligned}\Pi^\pm\left(k\right)=&\int\prod_{i=1}^{3}\frac{d^4q_i}{\left(2\pi\right)^4}\left(2\pi\right)^4\delta^4\left(k-q_1-q_2-q_3\right)\\&2G_F^2\bar{\sigma}^\mu G^\pm\left(q_1\right)\bar{\sigma}^\nu G^\mp\left(-q_2\right)\bar{\sigma}_\mu G^\pm\left(q_3\right)\bar{\sigma}_\nu\end{aligned} \tag{2.182}$$

The dependence of $\Pi^\pm$ and the two-point functions on the position x is implied. We can change $-q_2\to q_2$ to obtain

$$\begin{aligned}\Pi^\pm\left(k\right)=&\int\prod_{i=1}^{3}\frac{d^4q_i}{\left(2\pi\right)^4}\left(2\pi\right)^4\delta^4\left(k+q_2-q_1-q_3\right)\\&2G_F^2\bar{\sigma}^\mu G^\pm\left(q_1\right)\bar{\sigma}^\nu G^\mp\left(q_2\right)\bar{\sigma}_\mu G^\pm\left(q_3\right)\bar{\sigma}_\nu\end{aligned} \tag{2.183}$$

Every two-point function $G^\pm$ contains a positive- and a negative-energy piece, and is proportional to an on-shell delta function, which to leading order is $2\pi\delta\left(q_i^2\right)$.

This, together with the overall momentum-conserving delta function, implies that the only terms giving a nonzero contribution to the integral are those where all four of (k^0, q_i^0) are positive (corresponding to the neutrino-neutrino scattering process), those where all four are negative (corresponding to antineutrino-antineutrino scattering), and those where two are positive and two are negative (describing neutrino-antineutrino scattering).

We consider the term in which all energies are positive, which describes neutrino-neutrino scattering. Using $G^{\pm} = -\frac{1}{2}i\rho \pm F$, using the $O\left(1\right)$ expressions for F and ρ given by equations (2.39)-(2.40), (2.44)-(2.45) and (2.46)-(2.47), and omitting spin coherence, we obtain

$$\begin{aligned}\Pi^{+,\dot{\alpha}\alpha}\left(k\right) &= \int \prod_{i=1}^{3} \tilde{d}q_i \left(2\pi\right)^4 \delta^4\left(k + q_2 - q_1 - q_3\right) \\ 2G_F^2 \left(\bar{\sigma}^{\mu}\sigma_{\rho}\bar{\sigma}^{\nu}\sigma_{\sigma}\bar{\sigma}_{\mu}\sigma_{\tau}\bar{\sigma}_{\nu}\right)^{\dot{\alpha}\alpha} & q_1^{\rho} q_2^{\sigma} q_3^{\tau} \left(1 - f\left(\vec{q}_1\right)\right) f\left(\vec{q}_2\right)\left(1 - f\left(\vec{q}_3\right)\right) \\ &= -16G_F^2 \int \prod_{i=1}^{3} \tilde{d}q_i \left(2\pi\right)^4 \delta^4\left(k + q_2 - q_1 - q_3\right) \\ &\left(q_2 \cdot \bar{\sigma}^{\dot{\alpha}\alpha}\right)\left(q_1 \cdot q_3\right)\left(1 - f\left(\vec{q}_1\right)\right) f\left(\vec{q}_2\right)\left(1 - f\left(\vec{q}_3\right)\right)\end{aligned} \tag{2.184}$$

Similarly, the contribution to Π^{-} is:

$$\begin{aligned}\Pi^{-,\dot{\alpha}\alpha}\left(k\right) = -16G_F^2 &\int \prod_{i=1}^{3} \tilde{d}q_i \left(2\pi\right)^4 \delta^4\left(k + q_2 - q_1 - q_3\right) \\ &\left(q_2 \cdot \bar{\sigma}^{\dot{\alpha}\alpha}\right)\left(q_1 \cdot q_3\right) f\left(\vec{q}_1\right)\left(1 - f\left(\vec{q}_2\right)\right) f\left(\vec{q}_3\right)\end{aligned} \tag{2.185}$$

Since we have chosen the term for which k^0 is positive, this expression enters into the collision term for neutrinos. The corresponding contribution to the collision term in Equation (2.151) is

$$\begin{aligned}8G_F^2 \frac{1}{\left|\vec{k}\right|} \int \prod_{i=1}^{3} \tilde{d}q_i \left(2\pi\right)^4 \delta^4\left(k + q_2 - q_1 - q_3\right)\left(k \cdot q_2\right)\left(q_1 \cdot q_3\right) \times \\ \left(\left\{1 - f, f_1\left(1 - f_2\right) f_3\right\} - \left\{f, \left(1 - f_1\right) f_2 \left(1 - f_3\right)\right\}\right)\end{aligned} \tag{2.186}$$

where $f = f(\vec{k})$ and $f_i = f\left(\vec{q}_i\right)$.

To obtain the complete piece of the collision term that describes neutrino-neutrino scattering, we also need to include the lower-right diagram in Fig. 4.

We also introduce $s \equiv (k+q_2)^2 = (q_1+q_3)^2$. For the approximately massless neutrinos, $s \approx 2k\cdot q_2 = 2q_1\cdot q_3$. The collision term for neutrino-neutrino scattering is then given by

$$\begin{aligned} C_{\nu\nu\leftrightarrow\nu\nu} = \frac{2G_F^2}{\left|\vec{k}\right|}\int\prod_i \tilde{d}q_i \delta^4\left(k+q_2-q_1-q_3\right)s^2 \\ \left\{1-f, f_1\left[\mathrm{tr}_F\left(\left(1-f_2\right)f_3\right)+\left(1-f_2\right)f_3\right]\right\} \\ -\left\{f,\left(1-f_1\right)\left[\mathrm{tr}_F\left(f_2\left(1-f_3\right)\right)+f_2\left(1-f_3\right)\right]\right\} \end{aligned} \tag{2.187}$$

This contribution to the collision term clearly has the gain-loss structure of the Boltzmann equation with Fermi blocking, describing $\nu\nu \leftrightarrow \nu\nu$ scattering. However, unlike in the Boltzmann equation, the densities f are flavor matrices, and the collision term has nontrivial flavor structure.

We can make the connection to the usual Boltzmann term by considering a case in which there is no coherence between neutrino flavors, so that the density matrices f are all diagonal in the same basis. Then, the anticommutators become products of the diagonal terms, which are just the neutrino densities, and the collision term for flavor I reduces to the Boltzmann form:

$$\begin{aligned} C^I_{\nu\nu\leftrightarrow\nu\nu} = \frac{4G_F^2}{\left|\vec{k}\right|}\int\prod_i \tilde{d}q_i \delta^4\left(k+q_2-q_1-q_3\right)s^2 \times \\ \left\{\left(1-f^I\right)f_1^I\left[2\left(1-f_2^I\right)f_3^I+\sum_{J\neq I}\left(1-f_2^J\right)f_3^J\right]\right. \\ \left. -f^I\left(1-f_1^I\right)\left[2f_2^I\left(1-f_3^I\right)+\sum_{J\neq I}f_2^J\left(1-f_3^J\right)\right]\right\} \end{aligned} \tag{2.188}$$

This corresponds to the usual Boltzmann term describing scattering of neutrinos off each other, with one incoming and outgoing neutrino described by $f \leftrightarrow f_1$ and the other by $f_2 \leftrightarrow f_3$. In the above expression, repeated indices are not summed over unless the sum is explicitly indicated. From the above formula we see that the total scattering rate for $\nu_I\nu_I$ is twice that for $\nu_I\nu_J$ with $J \neq I$, consistent with the discussion in Ref. [141] .

Generalizations

So far, we have only considered diagrams for neutrino-neutrino scattering, and assumed that the spin coherence is zero. When all processes are included, we obtain collision terms that have the following structure:

$$C = C_{\nu\nu\leftrightarrow\nu\nu} + C_{\nu\bar{\nu}\leftrightarrow\nu\bar{\nu}} + +C_{\nu e\leftrightarrow\nu e} + C_{\nu\bar{e}\leftrightarrow\nu\bar{e}} + C_{\nu\bar{\nu}\leftrightarrow e\bar{e}} + C'\left[f, \bar{f}, \phi\right] \quad (2.189)$$

where C' is a set of additional terms dependent on spin coherence, which are zero when $\phi = 0$. These can be calculated in the same way as the rest of the collision terms, but with different arrangements of two-component spinor arrows within the Feynman diagrams. The other collision terms, $\bar{C}$ and C_ϕ, have a similar structure.

2.8 Properties of the QKEs

We now examine the quantum kinetic equations, equations (2.144)-(2.146) (summarized in equation 2.160), and consider some of their properties. In the previous section, we have seen that the right-hand sides of equations (2.144)-(2.146) correspond to the Boltzmann collision terms, with some additional flavor structure and dependence on coherence. We now show that the quantum kinetic equations replicate the usual equations for coherent flavor evolution in the low-density limit. We also discuss the spin coherence terms, and show that these terms can potentially lead to coherent transformation between neutrino and anti-neutrino states.

2.8.1 Low-Density Limit

The low-density limit is realized in certain situations in nature, for example, in the supernova envelope, or in the early Universe after weak decoupling. In this limit, we neglect the collision term, since this is proportional to G_F^2, but retain the matter potential, which is proportional to G_F. Furthermore, we assume that the matter potential Σ is much smaller than the vacuum mass m, but comparable to $\frac{m^2}{E}$. With these assumptions we can demote Σ from $O(\epsilon)$ to $O(\epsilon^2)$, and drop higher-order terms involving $m\Sigma$, $\partial\Sigma$ and Σ^2.

In this regime, the quantum kinetic equations become

$$i\partial^\kappa f - \left[\Sigma^\kappa + \frac{m^\star m}{2\left|\vec{k}\right|}, f\right] = 0 \tag{2.190}$$

$$i\partial^\kappa \bar{f} - \left[\Sigma^\kappa - \frac{m^\star m}{2\left|\vec{k}\right|}, \bar{f}\right] = 0 \tag{2.191}$$

In the low-density limit, or in the isotropic limit, the spin coherence density ϕ is decoupled from the equations for f and $\bar{f}$. Therefore, in the low-density limit, there is no need to solve equation (2.146) for the spin coherence density.

Equations (2.190) and (2.191) are equivalent to the usual equations for coherent flavor evolution, for example those described in Ref.s [3–31]. The equations describe phenomena such as coherent oscillations, the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1, 2], and collective flavor transformation due to the neutrino self-coupling terms present in Σ. These phenomena are described in detail in Ref. [98].

2.8.2 Spin Coherence

A feature that appears at high densities and in the presence of anisotropy in the neutrino field is the coupling of the quantum kinetic equations for f and $\bar{f}$ to a new dynamical quantity, the spin coherence density ϕ. We now examine the possible consequences of this coupling.

It is clear from the form of equation (2.160) that ϕ represents coherence between neutrinos and anti-neutrinos, and the $H_{\nu\bar{\nu}}$ term gives mixing between neutrino and anti-neutrino states. The effects of spin coherence conserve the total number of neutrinos plus antineutrinos for each momentum, but not the two separately. This can be seen by taking the trace of equation (2.160) to obtain

$$\mathrm{tr} D\left[\mathcal{F}\right] = \mathrm{tr}\ \mathcal{C}\left[\mathcal{F}\right] \tag{2.192}$$

Since $\mathrm{tr}\mathcal{F}\left(\vec{k}\right) = \mathrm{tr}f\left(\vec{k}\right) + \mathrm{tr}\bar{f}\left(\vec{k}\right)$, $\mathrm{tr}\mathcal{F}\left(\vec{k}\right)$ corresponds to the total density of neutrinos plus anti-neutrinos of momentum $\vec{k}$. The derivative combination $\mathrm{tr}D\left[\mathcal{F}\right]$ can be interpreted as simply a derivative of the neutrino plus antineutrino density along a light-like world line, which deviates slightly from the world line of an actual neutrino due to an index of refraction from the matter and neutrino potentials.

As a consequence, along the particle world line the total neutrino plus antineutrino density for a given momentum can change in response to the collision term, but not in response to spin coherence. However, in the presence of spin coherence, the quantities $\mathrm{tr} f$ and $\mathrm{tr} \bar{f}$ are not individually conserved, so the difference between neutrino and antineutrino densities can undergo coherent evolution.

Therefore, the coupling to the spin coherence can lead to a coherent process that converts neutrinos to antineutrinos, and vice versa. The mixing term $H_{\nu\bar{\nu}}$ involves a combination of the neutrino mass m and spacelike components of the matter and neutrino potential orthogonal to the momentum, $\Sigma^{\pm} = \frac{1}{2}\left(\Sigma^1 \pm i\Sigma^2\right)$. We see from this that three conditions are necessary for a coherent change of helicity: (1) the particles must have a mass; (2) there must be an anisotropic matter or neutrino potential with a component orthogonal to the particle's momentum; and, (3) the spin coherence density ϕ must be present.

The anisotropy condition can be satisfied in the context of a supernova explosion or a compact object merger. One source of anisotropy, which is present even in spherically symmetric models, is the outgoing flux of neutrinos. A neutrino moving at a nonzero angle with respect to the radial direction will receive a contribution to $H_{\nu\bar{\nu}}$ from interactions with other outgoing neutrinos.

The mixing Hamiltonian, $H_{\nu\bar{\nu}}$, is $O\left(\epsilon^2\right)$ while the diagonal blocks, H and $-\bar{H}^T$, are $O\left(\epsilon\right)$. Thus, under generic conditions, we expect the effects of mixing between neutrinos and anti-neutrinos to be small. However, we can potentially obtain large effects "at resonance", when there is a degeneracy between eigenvalues of H and $-\bar{H}$. This is analogous to the MSW resonance effect, where a small neutrino mass can lead to large-scale flavor transformation at resonance. Note that, unlike in the decoupled equations of motion for f and $\bar{f}$, equations (2.190)-(2.191), the flavor-independent components of H and $\bar{H}$ that are proportional to the flavor unit matrix must be included. Therefore, to determine the conditions for neutrino-antineutrino resonance in a realistic model it is necessary to include the neutral current contributions to the matter potential, including contributions from coherent forward scattering of neutrinos on nuclei and nucleons.

The Hamiltonian $\mathcal{H}$ and the combined neutrino-antineutrino density matrix $\mathcal{F}$ in equation (2.160) bear some resemblance to the description of coherent evolution of neutrinos with a nonzero transition magnetic moment in the presence of a magnetic field [99–101]. However, a Standard Model neutrino magnetic moment arises from loop corrections, and is therefore quite small, requiring very large magnetic fields to obtain neutrino-antineutrino mixing. Our effect comes from the weak interaction, which has a handle on neutrino helicity without the need to consider higher-order loop corrections, and does not require a large external magnetic field.

Whether large-scale neutrino-antineutrino transformation will actually take place in a supernova explosion is a difficult question, due to the neutrino-neutrino interaction terms in the Hamiltonian and the possibility for nonlinear feedback. Resolving this question is likely to require sufficiently realistic numerical simulations. The results from Ref.s [100, 101] suggest that the presence of even a small neutrino-antineutrino mixing term in the Hamiltonian could potentially lead to large-scale neutrino-antineutrino transformation.

2.9 Comparison With Previous Work

Our approach to neutrino quantum kinetics heavily relies on previous studies of transport equations from quantum field theory (CTP and 2PI techniques) for both scalars and fermion fields (see [128, 134, 142] and references therein), and their non-trivial generalization to multi-flavor cases in the context of electroweak baryogenesis [93, 94, 129–131, 143, 144] and leptogenesis [145–149].

Compared to previous field-theoretical analyses, our work contains the following new elements: (i) we clearly spell out a power counting in ratio of scales that is specific to neutrinos (ultra-relativistic weakly interacting particles in an environment that is nearly homogenous on the scale of a de Broglie wavelength) and expand the kinematics around light-like four-momenta. (ii) We make no assumptions of isotropy and treat spin degrees of freedom in full generality, which leads us to discover spin-coherence correlations that have been neglected in the past.

We are not aware of any other work that derives quantum kinetic equations for neutrinos in a fully anisotropic environment, or provides a description of the evolution of neutrino spin degrees of freedom. Since the neutrino fields in the astrophysical environments (supernovae, compact object mergers) of interest for application of the QKEs are *inherently* anisotropic, the features of our QKEs that arise from a non-isotropic neutrino field are potentially very important. Anisotropy, spin coherence, and the interplay between spin and flavor degrees of freedom may play an important role in these environments.

Neutrino QKEs have been derived in the past using different first-principles approaches and approximation schemes. Our approach is very closely related to the one of Raffelt and Sigl [68]. In fact, the "matrix of densities" introduced in [68] can be related to certain Lorentz components of the Wigner transformed neutrino two-point function used in our work. Moreover, as in [68] we do rely on perturbation theory and there is a one-to-one correspondence between the assumptions made in these two works. The end-results of our analysis match the one of Ref. [68] up to the inclusion of spin-coherence densities (which is new in our work).

More recently, a new approach to neutrino quantum kinetics has been proposed Ref. [76], based on many-body techniques and the BBGKY hierarchy. Again, there is a correspondence between Ref. [76] and the field-theoretic treatment. In general, in field theory the non-equilibrium system is described by the set of all n-point Green's functions. These obey coupled integro-differential equations, equivalent to the BBGKY equations [134]. We truncate this hierarchy by writing down dynamical equations only for the two point functions and expressing all higher order Green's functions as a perturbative series in terms of the two-point functions. Here we assume that higher order correlations are absent in the initial state and we make essential use of our power counting in terms of weak interactions: the methods used here do not generalize to strongly interacting / correlated systems. Furthermore, when considering the dynamics of two-point functions, we neglect particle-antiparticle pairing correlations (see discussion following Eq. (2.36)). This is consistent with our power counting assumption that physical quantities vary slowly on the scale of the neutrino de Broglie wavelength. Nonetheless, these correlations that pair particles and antiparticles of opposite momenta (first discussed in the context of neutrino kinetics in Ref. [76]) could be included in our formalism. In fact, evolution equations that couple these particle-antiparticle densities to the standard particle-particle and antiparticle-antiparticle densities can be derived in the field theory framework [143, 144]. In the context of time-dependent multi-flavor mass matrices in the Early Universe (at the electroweak phase transition), it was shown in [143] that particle-antiparticle correlations can dynamically arise from a vanishing (equilibrium) initial condition and can play an important role in baryogenesis. We are not aware of any numerical exploration of the role of these correlations in a non-homogeneous supernova environment.

Finally, let us discuss the structure of our collision terms (Eqs. 2.151, 2.152, 2.153, 2.164), in comparison to other work. Even though here we do not calculate explicitly all the vector and tensor componenst of the self-energies $\Pi^{\pm}_{L,R}$, it is clear that our collision term is non-diagonal both in flavor and spin, thus producing decoherence of any linear superposition of flavor or spin states. Neglecting spin coherence, the structure of our result matches the "non-abelian" matrix structure in flavor space discussed in Ref. [68]. We note, however, that many *ad hoc* treatments of the QKEs, including recent ones [150], completely miss the off-diagonal entries of the collision term, which are required by quantum mechanical considerations.

2.10 Conclusion

We have produced a self-consistent derivation of the quantum kinetic equations (QKEs) that govern how neutrino flavor evolves in medium. This derivation started from first principles relying only on quantum field theory and assumed standard model interactions for neutrinos. To our knowledge, this is the first such self-consistent first-principles derivation of QKEs for flavored fermions in an anisotropic environment. Our result, Eq. (2.160), captures the correct structure of the QKEs in anisotropic environments, but is somewhat formal because the self-energies on the right-hand-side are not fully calculated. In a future paper we will present a detailed analysis of the inelastic collision term, including spin coherence, thus making our results amenable to implementation in numerical simulations.

Specializing to ultra relativistic Majorana neutrinos and making expansions in small parameters, equation (2.48), our QKEs assume the usual form which describes coherent neutrino flavor evolution in low density media. Likewise, at high density, where neutrino scattering is dominant, the collision terms in our QKEs assume Boltzmann-like forms. This is consistent with studies that have shown that the Boltzmann equation could be derived directly from quantum field theory [111].

In the low density, coherent regime our QKEs are broadly similar to those derived from previous treatments, for example those of Ref.s [68, 69]. In the scattering-dominated Boltzmann limit and between these two limiting cases, however, there are differences. Unlike previous studies, we follow in detail neutrino spin degrees of freedom, and in this sector there are surprises.

We have found a new dynamical quantity associated with spin coherence. At low density we find that the equation of motion for this quantity decouples from the rest of the QKEs describing neutrino flavor evolution. This equation describes Majorana neutrino spin (helicity) evolution in a matter and neutrino background. An obvious feature we find is that spin coherence can only arise in conditions where neutrino fluxes and/or matter potentials are not isotropic. Such conditions never arise in a standard Friedman-LaMaitre-Robertson-Walker early Universe expansion, but might occur in out of equilibrium environments like those associated with phase transition-induced nucleation of topological defects like bubbles or domain walls [151, 152]. By contrast, the region above the proto-neutron star in core collapse supernovae and the neutron star merger environment are both characterized by gross anisotropy in matter and neutrino fields.

The terms driving coherent spin flip in our QKEs stem from products of neutrino absolute mass and spacelike projections of the matter potentials (hence the requirement for anisotropy). Unlike coherent flavor transformation, which is sensitive only to the mass-squared differences between different neutrino flavors, coherent spin flip is sensitive to the neutrino absolute mass.

Also, unlike coherent flavor transformation, coherent spin flip is sensitive to the Majorana or Dirac nature of neutrinos. In this paper, we have specialized to Majorana neutrinos, but extending our treatment to Dirac neutrinos is straightforward. The simplest way to introduce Dirac neutrinos in our model is to add an additional field describing sterile neutrinos, ν_s. For pure Dirac neutrinos, the mass term always connects the active neutrino field, ν, with the sterile field, ν_s. Because the spin flip term carries a single power of the mass, for Dirac neutrinos it will result in transformation between active and sterile states. However, for Majorana neutrinos, coherent spin flip generates transformation between active neutrinos and active antineutrinos.

It is not known at present whether coherent spin flip can result in large-scale transformation between right-handed and left-handed neutrino states in supernovae. Due to nonlinearity and complexity of the QKEs, the resolution of this question likely requires sufficiently detailed and realistic numerical modeling. If numerical simulations do show that effects from coherent spin flip are large enough to produce a detectable signature in the supernova neutrino spectrum, then measurement of a supernova neutrino signal could in principle be used to constrain the absolute neutrino mass and determine the Majorana vs. Dirac nature of neutrinos.

Additionally, both neutrino production (*e.g.,* Ref.s [153]) and neutrino energy deposition in the core collapse supernova shock re-heating (accretion) phase and the neutron-to-proton ratio (*e.g.,* Ref. [6]) in any neutrino-heated outflow nucleosynthesis can be very sensitive to the relative fluxes and energy spectra of ν_e and $\bar{\nu}_e$. Consequently, for these processes, any large-scale inter-conversion of neutrinos and antineutrinos could be significant.

Simulations of the core collapse supernova and neutron star merger environments are some of the most sophisticated numerical calculations being done at present with, in some cases, state-of-the-art multi-dimensional radiation hydrodynamics coupled with detailed equation of state and other microphysics, *e.g.,* Ref.s [41–60]. A key conclusion that can be drawn from these studies is that neutrinos and their interactions are important in many aspects of compact object evolution and nucleosynthesis. However, experiment has now caught up with theory in a sense. It is an experimental fact that neutrinos have nonzero rest masses and that neutrino flavors mix in vacuum. This physics is, for the most part, not in these otherwise very sophisticated simulations. The work presented here, a self-consistent approach to treating this physics, suggests that there are unresolved issues in the neutrino-supernova story.

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Chapter 3

Structure and Properties of QKEs

In this chapter, we present the main result of Chapter 2, the quantum kinetic equations for Majorana neutrinos, in a simple and compact form. We then describe the structure of the individual terms in these equations and briefly discuss the physical interpretation of these terms. This chapter follows the general notation of our work in Ref. [96]. We present the structure of the QKEs and the expression for the forward scattering potential in a model with three neutrino flavors and Standard Model interactions, but merely indicate the structure of the collision term and leave the details to future work. The reason is that in the full spin- and flavor-dependent case, the full expression for the collision term is extremely complicated even in highly simplified models, and a collision term for fully realistic QKEs must include a wide variety of neutrino-nuclear inelastic scattering processes, some of which involve nuclear states with high excitation energies (at neutrino energies of tens of MeV) or nonperturbative aspects of the strong interaction (at neutrino energies of hundreds of MeV) and are not fully understood. The general procedure for calculating all collision terms from 2-loop neutrino self-energy diagrams is indicated in Chapter 2, Section 6.2, together with an example calculation of a specific term (neutrino-neutrino scattering).

3.1 A Simple Formulation of the QKEs

As shown in Chapter 2, the quantum kinetic equations for Majorana neutrinos, in a general hot, dense anisotropic medium, can be written in a compact form as follows:

$$i\mathcal{D}\mathcal{F} = [\mathcal{H}, \mathcal{F}] + i\mathcal{C} \tag{3.1}$$

For three flavors of neutrinos, the quantity $\mathcal{F}$ is a 6×6 density matrix that includes neutrino number densities, antineutrino number densities, flavor coherence terms and spin coherence terms. $\mathcal{F}$ is a function of position the position 4-vector x and the momentum 3-vector $\vec{p}$ and has the following block form:

$$\mathcal{F}(x, \vec{p}) = \begin{pmatrix} f(x, \vec{p}) & \phi(x, \vec{p}) \\ \phi^\dagger(x, \vec{p}) & \bar{f}^T(x, \vec{p}) \end{pmatrix} \tag{3.2}$$

In this notation, $f(x, \vec{p})$ is a Hermitean 3×3 density matrix for three flavors of neutrinos while $\bar{f}(x, \vec{p})$ is the density matrix for antineutrinos. In the flavor basis, the diagonal elements of f and $\bar{f}$ give the occupation number of a neutrino or antineutrino of a particular flavor (*e.g.*, $f_{ee}(x, \vec{p})$ is the occupation number of electron neutrinos having momentum $\vec{p}$ at spacetime location x). The off-diagonal elements encode coherence between different flavors and mediate flavor oscillations. $\phi(x, \vec{p})$ is a complex 3×3 matrix that encodes coherence between neutrino and antineutrinos and can mediate oscillations and exchange between neutrino and antineutrino states.

$\mathcal{D}$ is a derivative operator governing the evolution of the density matrix, with corrections due to interactions with matter and other neutrinos. $\mathcal{H}$ is the coherent forward scattering Hamiltonian, including neutrino-antineutrino mixing terms. $\mathcal{C}$ is a Boltzmann-like collision term, but with nontrivial spin and flavor structure. These terms are presented and discussed in detail below.

3.2 Derivative Operator

The derivative term is a generalized Vlasov term for flavored particles, with the following form:

$$\mathcal{D}\mathcal{F} = \partial^{\kappa}\mathcal{F} + \frac{1}{2|\vec{p}|}\left\{\tilde{\Sigma}^{i}, \partial^{i}\mathcal{F}\right\} + \frac{1}{2}\left\{\frac{\partial\tilde{\Sigma}^{\kappa}}{\partial\vec{x}}, \frac{\partial\mathcal{F}}{\partial\vec{p}}\right\} \tag{3.3}$$

Here, the 6×6 matrix $\tilde{\Sigma}^{\mu}$ is a 4-vector matter and neutrino potential arising from coherent forward scattering of neutrinos on electrons, nucleons and other neutrinos. In terms of 3×3 flavor blocks, $\tilde{\Sigma}^{\mu} = \begin{pmatrix} \Sigma^{\mu} & 0 \\ 0 & -(\Sigma^{\mu})^{T} \end{pmatrix}$, *i.e.*, the potential has the opposite sign for neutrinos and antineutrinos. The component notation here is as follows: we define the lightlike vector $\hat{\kappa}^{\mu} = \frac{p^{\mu}}{|\vec{p}|}$ to lie along the neutrino trajectory and the two spacelike unit vectors $(\hat{x}^{1})^{\mu}$ and $(\hat{x}^{2})^{\mu}$ to be orthogonal to the spacelike direction of the neutrino trajectory. In this notation, $\tilde{\Sigma}^{\kappa} = \hat{\kappa}_{\mu}\tilde{\Sigma}^{\mu}$ and $\tilde{\Sigma}^{i} = \hat{x}^{i}_{\mu}\tilde{\Sigma}^{\mu}$. Similarly, $\partial^{\kappa} = \hat{\kappa}_{\mu}\partial^{\mu}$ is the derivative along the world line of the neutrino and $\partial^{i} = \hat{x}^{i}_{\mu}\partial^{\mu}$ is a derivative orthogonal to the path of the neutrino, along one of the two spacelike vectors $\hat{x}^{1}$ or $\hat{x}^{2}$.

In the special case where there is no spin or flavor coherence and $\mathcal{F}$, $\tilde{\Sigma}^i$ and their derivatives all commute (*i.e.,* all are diagonal in the same basis), the derivative operator takes the following form:

$$\partial^\kappa \mathcal{F}_I + \frac{1}{|\vec{p}|}\tilde{\Sigma}^i_I \partial^i \mathcal{F}_I + \frac{\partial \tilde{\Sigma}^\kappa_I}{\partial \vec{x}} \cdot \frac{\partial \mathcal{F}_I}{\partial \vec{p}} \tag{3.4}$$

where $\mathcal{F}_I$ for $I = 1...6$ are eigenvalues of $\mathcal{F}$, and similarly for $\tilde{\Sigma}_I$. The index I here is not summed over. Using the total energy for propagation eigenstate I, $E_I = |\vec{p}| + \tilde{\Sigma}^\kappa$, the derivative term can be written as

$$\left(\partial^\kappa + \partial_{\vec{p}} E_I \cdot \partial_{\vec{x}} - \partial_{\vec{x}} E_I \cdot \partial_{\vec{p}}\right) \mathcal{F}_I \tag{3.5}$$

This is simply the Vlasov derivative term for 6 types of particles (3 neutrinos and 3 antineutrinos) each with a position and momentum-dependent energy $E_I\left(x, \vec{p}\right)$. ∂^κ is the derivative along the lightlike path of a free massless particle and the corrections are force terms that result in a slight deviation from this path. The full anticommutator structure of Eqn. 3.3 generalizes the Vlasov term to flavored particles with coherence. The energy E_I contains higher-order contributions, *e.g.* the mass term $m^2/2\,|\vec{p}|$, but in our QKEs these are neglected because in a high-density environment $\tilde{\Sigma}^\kappa$ is generally much larger than these second-order terms.

Note that in curvilinear coordinates or in the presence of spacetime curvature, these expressions must be promoted to appropriate covariant derivatives acting on a function of both 4-vector position and 3-vector momentum. The procedure for generalizing the derivatives to arbitrary coordinates is a basic exercise in differential geometry. The geometric picture of ∂^κ as the derivative along the lightlike neutrino worldline and ∂^i as the spacelike derivatives orthogonal to the trajectory's spacelike component remains unchanged in any coordinate system.

3.3 Coherent Forward Scattering Terms

The 6×6 Hamiltonian $\mathcal{H}$ has the following structure:

$$\mathcal{H} = \begin{pmatrix} H & H_\phi \\ H_\phi^\dagger & -\bar{H}^T \end{pmatrix} \tag{3.6}$$

Here H and $\bar{H}$ are 3×3 Hermitean flavor matrices and H_ϕ is a 3×3 complex flavor matrix. H is the Hamiltonian operator for neutrinos, $\bar{H}$ is the Hamiltonian operator for antineutrinos and H_ϕ is a neutrino-antineutrino mixing term.

The expressions for H and $\bar{H}$ in terms of the forward scattering potential Σ^μ, with components defined as in Section 3.2, are given by:

$$H = \Sigma^\kappa + \frac{1}{2\,|\vec{p}|}\left(m^\dagger m - \epsilon^{ij}\partial^i\Sigma^j + \Sigma^i\Sigma^i - i\left[\Sigma^1, \Sigma^2\right]\right) \tag{3.7}$$

$$\bar{H} = \Sigma^\kappa - \frac{1}{2\,|\vec{p}|}\left(m^\dagger m - \epsilon^{ij}\partial^i\Sigma^j + \Sigma^i\Sigma^i - i\left[\Sigma^1, \Sigma^2\right]\right) \tag{3.8}$$

Here m is the neutrino vacuum mass matrix. For Majorana neutrinos, the mass matrix satisfies the condition $m = m^T$. Note that the leading-order term plus the mass terms gives the usual Hamiltonian operators for coherent evolution of neutrino and antineutrino distributions.

The neutrino-antineutrino mixing Hamiltonian is given by

$$H_\phi = -\frac{1}{2\,|\vec{p}|}\left(\left(\Sigma^1 + i\Sigma^2\right) m^\dagger + m^\dagger\left(\Sigma^1 + i\Sigma^2\right)^T\right) \tag{3.9}$$

This term is generically smaller than the leading-order term in H and $\bar{H}$, Σ^κ, by a factor of m/E, but as discussed in Chapter 4, it cannot necessarily be neglected, particularly in environments where nonlinear feedback due to the neutrino-neutrino interaction is important.

If we neglect higher-order contributions to the Hamiltonian but retain the mass term, we obtain $H = \Sigma^\kappa + m^\dagger m/2\,|\vec{p}|$ and $\bar{H} = \Sigma^\kappa - m^\dagger m/2\,|\vec{p}|$. These are the usual Hamiltonian operators describing neutrino and antineutrino forward scattering at low density, when Σ is comparable to $m^2/2E$ but much smaller than m. If the collision term and the neutrino-antineutrino mixing term are also neglected, in this limit the equations of motion for f and $\bar{f}$ reduce to the usual Schrödinger-like equations for coherent flavor evolution.

3.4 Forward Scattering Potential

The forward scattering potential entering into the expressions for $\mathcal{H}$ and $\mathcal{DF}$ can be calculated from one-loop contributions to the neutrino self-energy in the

2PI CTP formalism, as described in Chapter 2. The forward scattering potential consists of a matter term containing contributions from the background plasma and a neutrino-neutrino interaction term:

$$\Sigma^{\mu} = \Sigma^{\mu}_{M} + \Sigma^{\mu}_{(\nu)} \tag{3.10}$$

In a system containing only baryons, electrons, positrons and neutrinos, with all particles in the plasma moving together with a common fluid velocity, the matter potential in the flavor basis is given by [4, 34, 85, 154]

$$\Sigma^{\mu}_{M} = \frac{G_F}{\sqrt{2}} J^{\mu}_{B} \begin{pmatrix} 3Y_e - 1 & 0 & 0 \\ 0 & Y_e - 1 & 0 \\ 0 & 0 & Y_e - 1 \end{pmatrix} \tag{3.11}$$

The nucleon contribution was not included in the treatment in Chapter 2, but this contribution has a simple form and we include it here. Here J^{μ}_{B} is the baryon number 4-current density and $Y_e = (n_e - n_{\bar{e}})/n_B$ is the electron lepton number to baryon number ratio. $G_F = 1.166364 \times 10^{-11}\mathrm{MeV}^{-2}$ is the Fermi constant. Note that electron neutrinos have a different matter potential than μ and τ neutrinos, due to the presence of electrons but not muons or tauons in the plasma. Even under these conditions, there can be a small difference between the potentials for μ and τ neutrinos arising from radiative corrections. For simplicity, we neglect this difference by assuming that we are working at energies much smaller than the muon mass, so that the radiative conditions become negligible.

The neutrino-neutrino interaction potential is

$$\Sigma^{\mu}_{(\nu)} = 2\sqrt{2} G_F J^{\mu}_{(\nu)} \tag{3.12}$$

Here $J^{\mu}_{(\nu)}$ is the 3×3 flavor- and coherence-dependent neutrino 4-current density, given by

$$J^{\mu}_{(\nu)} = \int \frac{d^3\vec{q}\; q^{\mu}}{(2\pi)^3\, 2\,|\vec{q}|} \left(f(x, \vec{q}) - \bar{f}(x, \vec{q}) \right) \tag{3.13}$$

Note that for neutrino and antineutrino density matrices f and $\bar{f}$ containing flavor off-diagonal coherence terms, the neutrino-neutrino interaction potential contains off-diagonal terms that mediate mixing between neutrinos or antineutrinos of different flavors. This presence of flavor off-diagonal terms in the neutrino-neutrino contribution to the Hamiltonian is crucial for the phenomenon of collective neutrino flavor transformation in the presence of high neutrino flux, *e.g.* in conditions that occur in core collapse supernovae or compact object mergers.

3.5 Collision Terms

The collision term in Equation 3.1 can be written in terms of gain and loss potentials as follows:

$$\mathcal{C}\left(x,\vec{p}\right) = \frac{1}{2}\left(\left\{\Pi^{+}\left(x,\vec{p}\right), I - \mathcal{F}\left(x,\vec{p}\right)\right\} - \left\{\Pi^{-}\left(x,\vec{p}\right), \mathcal{F}\left(x,\vec{p}\right)\right\}\right) \tag{3.14}$$

Here I is the 6×6 unit matrix and $\Pi^{\pm}$ are the 6×6 spin- and flavor-dependent gain and loss potentials. These potentials are computed from 2-loop neutrino self-energy diagrams with neutrino, electron or nucleon internal lines, after appropriate spinor projections and integration over positive or negative values of p^0. The detailed expressions for $\Pi^{\pm}$ in a realistic model are extremely complicated and will be the subject of future work; the general procedure for extracting components of these functions is presented in 2.6.2. However, the Boltzmann-like gain-loss structure of the collision term is evident. Compared to the Boltzmann collision term, the collision term in the QKEs is generalized to flavored spin 1/2 particles with possible coherence by the spin $\times$ flavor anticommutator structure. In general, the quantities $\Pi^{\pm}$ include numerous terms that describe processes such as non-forward and inelastic scattering, neutrino production and absorption, and neutrino-antineutrino pair production and annihilation, as a result of neutrino interactions with other neutrinos, charged leptons and baryonic matter.

With the exception to the spin structure, which is new in our work, the gain-loss and anticommutator structure of our collision term is similar to that derived in Ref. [68] for neutrinos, in Ref.s. [93, 94, 129–131, 143, 144] in the context of baryogenesis and in Ref.s. [145–149] in the context of leptogenesis. Therefore, we expect to see the well-established phenomena of thermalization, damping of flavor oscillations via collisional decoherence and flavor depolarization. Further, because spin degrees of freedom enter into the collision term in much the same way as flavor degrees of freedom, we likewise expect to see damping of neutrino-antineutrino oscillations and helicity depolarization (*i.e.,* equalization of neutrino and antineutrino distributions) under conditions when spin coherence and collisions can occur at the same time.

Chapter 4

Prospects for Neutrino-Antineutrino Transformation

4.1 Abstract

We examine whether the newly derived neutrino spin coherence could lead to large-scale coherent neutrino-antineutrino conversion. In a linear analysis we find that such transformation is largely suppressed, but demonstrate that nonlinear feedback can enhance it. We point out that conditions which favor this feedback may exist in core collapse supernovae and in binary neutron star mergers.

4.2 Introduction

In this letter we address the prospects for spin coherence, a recently revealed [95, 96] aspect of medium-affected neutrino physics, to facilitate the interconversion of neutrinos and antineutrinos in the core collapse supernova and compact object merger environments. This is important because the asymmetry between ν_e and $\bar{\nu}_e$ fluxes and energy spectra in these sites can influence both dynamics and the neutron-to-proton ratio [6], a key determinant of nucleosynthesis. The stakes are high because these are our best candidate sites for the origin of the heaviest elements. Moreover, future neutrino [155, 156] and gravitational radiation [64] observations may give insights into these venues.

Neutrino spin coherence was discovered in the course of deriving the quantum kinetic equations (QKEs) that govern the evolution of neutrino distributions in a medium of matter and neutrinos [95, 96]. In that work, nonzero neutrino mass and the presence of anisotropy in the matter or in the neutrino fields were shown to be necessary for coherent transformation between left-handed and right-handed neutrino states. In short, the QKEs show that in an anisotropic medium the neutrino propagation states (energy states) can be coherent mixtures of left- and right-handed (*i.e.,* neutrino and antineutrino) states.

The QKEs for flavored particles have been considered in many contexts [27, 67–97]. For neutrinos, the QKEs describe coherent forward evolution as well as scattering and thermalization, and can reduce to Schrödinger-like equations for flavor evolution or the Boltzmann equation in certain limits. Compared to the standard Schrödinger-like treatment of neutrino flavor transformation, as described, for example, in Ref.s [1–9, 11–31, 98], the QKEs contain two novel features. One is the collision term, which allows exchange of particle number and flavor information between neutrinos of different energies traveling on different trajectories. The other feature is helicity mixing, or spin coherence which, for Majorana neutrinos, mediates exchange of information between neutrino and antineutrino states.

In Ref.s. [65,66], it was pointed out that even a small amount of non-forward neutrino scattering can potentially have large effects on supernova neutrino flavor transformation. However, here we temporarily set aside the issue of collisions and retain only coherent forward scattering terms in the QKEs, and consider the question of whether the spin coherence terms can possibly be important in compact object environments.

4.3 Toy Model for Spin Coherence

4.3.1 Collisionless QKEs

In the absence of the collision term, the QKEs take the following form:

$$D\mathcal{F} + i\left[\mathcal{H}, \mathcal{F}\right] = 0 \tag{4.1}$$

Here $\mathcal{F}$ is a density matrix for neutrino and antineutrino states, D is a Vlasov derivative operator, and $\mathcal{H}$ is a Hamiltonian for the evolution of the density matrix. For 3 flavors of neutrinos and anti-neutrinos, the density matrix is a 6×6 Hermitean matrix with the structure

$$\mathcal{F} = \begin{pmatrix} f & \phi \\ \phi^\dagger & \bar{f}^T \end{pmatrix} \tag{4.2}$$

where f and $\bar{f}$ are the usual 3×3 density matrices for neutrinos and antineutrinos and ϕ encodes coherence between neutrinos and antineutrinos.

In dense environments, where coherent forward scattering of neutrinos on the matter background and on other neutrinos gives the dominant contribution to the neutrino potential energy, the Hamiltonian consists of a leading-order contribution and a correction:

$$\mathcal{H} = \begin{pmatrix} H^{(1)} & 0 \\ 0 & -H^{(1)T} \end{pmatrix} + \begin{pmatrix} H^{(2)} & H^{(2)}_{\nu\bar{\nu}} \\ H^{(2)\dagger}_{\nu\bar{\nu}} & H^{(2)T} \end{pmatrix}. \tag{4.3}$$

The form of the QKEs in Eqn. (1) is similar to equations for the evolution of neutrinos with a magnetic moment in a strong magnetic field [99–101], but in the case of the QKEs, $\nu \rightleftharpoons \bar{\nu}$ mixing can occur without a magnetic field or a neutrino magnetic moment.

$H^{(1)}$ is $O\left(G_F\right)$ while terms that mediate $\nu \rightleftharpoons \bar{\nu}$ mixing, $H^{(2)}_{\nu\bar{\nu}}$, are $O\left(mG_F/E\right)$, , where m is the neutrino mass and E is the neutrino energy. For neutrinos in a supernova envelope, with reasonable assumptions about the neutrino mass and energy, $m/E \sim 10^{-7} - 10^{-8}$. Thus under generic conditions $H^{(2)}_{\nu\bar{\nu}}$ is negligible and we do not expect to see any significant helicity transformation. However, in special conditions, the potential for a neutrino state can be close to that for an antineutrino state. In this case, the behavior of the system can be dominated by the $\nu \rightleftharpoons \bar{\nu}$ mixing term.

4.3.2 One-Flavor Single-Angle Model

The solution of the full QKEs is a difficult numerical problem. To get an idea of the effects of helicity mixing, we construct a highly simplified toy model. Spin coherence can occur even with one flavor, so to construct the simplest possible model we retain only the electron flavor neutrinos and antineutrinos. We also omit second-order corrections to the derivative D and to the Hamiltonian, with the exception of the helicity mixing term $H^{(2)}_{\nu\bar{\nu}}$.

Writing $\mathcal{F} = f_i\sigma_i + f_0\mathbf{I}$ and $\mathcal{H} = H_i\sigma_i + H_0\mathbf{I}$, the simplified QKEs for the one-flavor model are:

$$\begin{aligned} Df_3 - 2H_1 f_2 &= 0 \\ Df_1 + 2H_3 f_2 &= 0 \\ Df_2 + 2H_1 f_3 - 2H_3 f_1 &= 0 \end{aligned} \tag{4.4}$$

Here we have defined the coordinate system in such a way that $H_2 = 0$, which is possible for any specific neutrino trajectory. In the special case of an axially

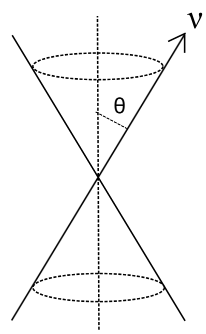

Figure 4.1: Geometry of the single-flavor toy model.

symmetric geometry, H_2 can be set to zero for all trajectories. D is the derivative along the neutrino world line. Here we simply consider evolution along a single world line and take $D = \partial_s$, where s is the distance traveled by the neutrino. In the absence of the collision term, $Df_0 = 0$.

For the purpose of the toy model, we consider an axially symmetric cone of neutrinos propagating at a fixed angle $u = \cos\theta$ with respect to the axis of symmetry. This geometry is illustrated in Fig. 1. The neutrinos undergo coherent forward scattering among themselves and with the matter background (electrons and nucleons). The properties of the matter background are varied slowly in order to determine what conditions can lead to significant $\nu \rightleftharpoons \bar{\nu}$ transformation.

This problem corresponds to the behavior of crossed neutrino beams of infinite width in a time-varying background, and is somewhat different from that of neutrinos emitted from a spherically symmetric neutrino sphere. In the latter case the angle u changes along the neutrino path and the neutrinos are geometrically diluted as $1/r^2$ as they move outward. These features of spherical geometry are straightforward to implement, but can complicate the problem and obscure the simple physical behavior that we wish to illustrate.

In the notation of Eq. (4), $f_3 = \left(f - \bar{f}\right)/2$, and, to leading order, $H_3 = H^{(1)}$. With this, H_3 is the Hamiltonian arising from coherent forward scattering of neutrinos with electrons, nucleons and other neutrinos. Including the contribution from nucleons, the Hamiltonian for electron neutrinos is [4, 34, 85, 154]

$$H_3 = \frac{G_F}{\sqrt{2}} \left(\left(3Y_e - 1\right) n_B + 4 \left(n_\nu - n_{\bar{\nu}}\right) - 4uJ^r \right). \tag{4.5}$$

Here, n_B is the baryon number density, $Y_e = n_e/n_B$ is the electron fraction, J^r is the lepton number current along the axis of symmetry, and $n_\nu - n_{\bar{\nu}}$ is the lepton number density in neutrinos, given by

$$n_\nu - n_{\bar{\nu}} = \int \frac{E'^2 dE'}{2\pi^2} 2 f_3 \left(E'\right). \tag{4.6}$$

The contribution to J^r from neutrinos is $u \left(n_\nu - n_{\bar{\nu}}\right)$. There can be additional contributions to J^r from bulk motion of matter, such as infall or outflow.

H_1 is the $\nu \rightleftharpoons \bar{\nu}$ mixing part of the Hamiltonian, corresponding to $H^{(2)}_{\nu\bar{\nu}}$ in Eq. 3, and is equal to

$$H_1 = 2\sqrt{2} G_F \sqrt{1-u^2} \frac{m J^r}{E}. \tag{4.7}$$

Neutrino-antineutrino mixing is large at *resonance*, *i.e.*, where $H_3 \approx 0$, which occurs for

$$Y_e + \frac{4}{3}\left(Y_\nu - \frac{u J^r}{n_B}\right) = \frac{1}{3} \tag{4.8}$$

where $Y_\nu = \left(n_\nu - n_{\bar{\nu}}\right)/n_B$. If the neutrino contribution to the Hamiltonian is relatively small, this corresponds to $Y_e \approx 1/3$, which can occur in or near the proto-neutron star (PNS) in a core collapse supernova [157–160], or near the central region of a compact object merger [63, 161]. This is similar to the condition for active-sterile transformation in models with sterile neutrinos [34–39].

4.3.3 Neutrino-Antineutrino Level Crossing

The single-flavor model is mathematically similar to the description of the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1, 2]. Suppose that, in Eq. (5), $Y_e = Y_{e0}$ gives $H_3 = 0$. We then begin with $Y_e < Y_{e0}$ and increase it to $Y_e > Y_{e0}$. This situation can occur in a supernova when neutrinos pass from regions of low Y_e inside the proto-neutron star to regions of higher Y_e in the envelope. For $Y_e < Y_{e0}$, H_3 is negative and neutrinos have a lower potential energy than antineutrinos. For $Y_e > Y_{e0}$, H_3 is positive and the antineutrinos have lower potential energy. This is a level crossing, schematically illustrated in Fig. 2, with instantaneous energy

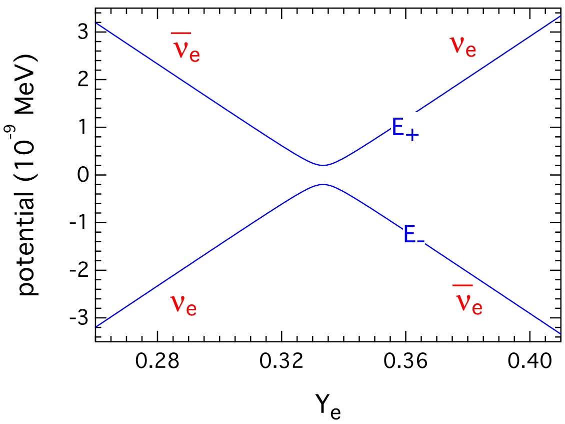

Figure 4.2: Schematic level crossing diagram for $\nu_e \rightleftharpoons \bar{\nu}_e$ transformation. The potential $E_{\pm} = \pm\sqrt{H_3^2 + H_1^2}$ is plotted against electron fraction Y_e, with off-diagonal potential H_1 exaggerated to clearly show the gap, $(E_+ - E_-)\,|_{res} = 2\,|H_1|$, at resonance. Here neutrino contributions to H_3 are neglected.

eigenvalues $E_{\pm}$. A level crossing can also be achieved by varying n_B, u and the neutrino distributions, and in a supernova all these quantities vary with radius.

If additional neutrino flavors are present, there are additional level crossings. The matter potential for muon and tau neutrinos is $\left(G_F/\sqrt{2}\right)(Y_e - 1)\, n_B$ [4, 34, 85, 154], which leads to a level crossing between ν_e and $\bar{\nu}_{\mu,\tau}$ (and $\bar{\nu}_e$ and $\nu_{\mu,\tau}$) near $Y_e = 1/2$. Similarly, a cancelation between the matter and the neutrino potentials could lead to a level crossing between $\nu_{\mu,\tau}$ and $\bar{\nu}_{\mu,\tau}$. These level crossings are more likely to occur in the supernova envelope, where Y_e and Y_ν can be relatively high [44, 157].

Provided that H_3 varies slowly enough (adiabatically), a neutrino that begins in the lower-energy state will remain in the lower-energy state, and therefore transform into an antineutrino. Whether this transformation occurs is governed by the adiabaticity parameter, $\gamma = 2H_1^2/\dot{H}_3$. Because H_1 is generically smaller than H_3 by a factor of $m/E \approx 10^{-7} - 10^{-8}$, the adiabaticity parameter for $\nu \rightleftharpoons \bar{\nu}$ transformation, which is proportional to $(m/E)^2$, is typically very small. Unless H_3 varies extremely slowly, only neutrinos at very low energies can transform.

4.3.4 Effects of Nonlinear Feedback

The above analysis neglects effects of nonlinearity due to the dependence of the Hamiltonian on neutrino distributions. To determine these effects, we discretize the energy and numerically solve our toy model using initial neutrino occupation numbers given by $1/\left(1+e^{\frac{E-\mu}{T}}\right)$, with $T = 4$ MeV and $\mu = 8$ MeV. For simplicity, we start with a pure neutrino spectrum, hold the angle fixed at $u = 1/\sqrt{2}$ and the baryon number density at $n_B = 300$ MeV^3. These values are roughly consistent with conditions above the neutrino sphere in a supernova [157, 158]. We vary Y_e as a function of distance s traveled by neutrinos, as follows:

$$Y_e = Y_{e0} + \frac{s}{\lambda}\left(1+\frac{s^2}{\kappa^2}\right). \tag{4.9}$$

The reason for adopting this expression for Y_e is as follows: we wish to obtain a level crossing at $s = 0$. We want to be able to dial the derivative of Y_e at $s = 0$ to see how slowly it must vary in order to give adiabatic $\nu \rightleftharpoons \bar{\nu}$ transformation. This is done by adjusting the parameter λ: larger values of λ give greater adiabaticity. Further, if λ must be large in order to trigger transformation, it is useful to know if the derivative of Y_e must remain small in order for transformation to continue, or if it can grow at a later time without halting the transformation. The parameter κ controls the scale on which the derivative of Y_e grows away from the location of the level crossing.

In our model, we find that when nonlinear feedback is included, large-scale $\nu \rightleftharpoons \bar{\nu}$ transformation can occur under unexpected conditions. Provided that the rate of change of Y_e at $s = 0$ is slow enough that some low-energy neutrinos transform, a feedback mechanism begins to operate that tends to keep H_3 near zero until a large number of neutrinos have been converted into antineutrinos.

Fig. (3) illustrates this phenomenon. As the system approaches resonance, neutrinos begin to convert into antineutrinos. This causes the neutrino self-interaction potential to decrease. If the rate of change of the self-interaction potential is large enough, it will overcome the change of the matter potential and push the system back towards resonance. This feedback is similar to the matter-neutrino resonance described in [32], but in the context of helicity, rather than

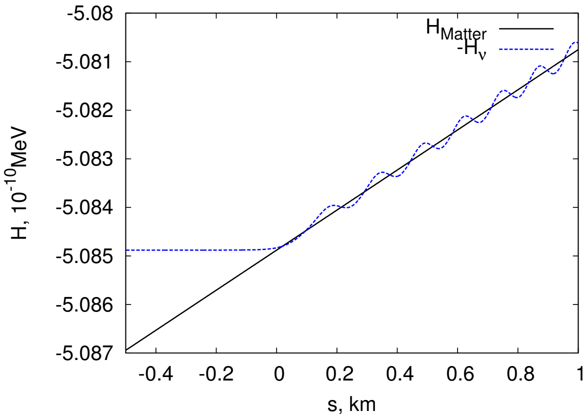

Figure 4.3: Onset of coherent helicity transformation and stabilization of resonance by nonlinear feedback.

flavor, transformation.

If the matter potential changes too quickly, this feedback mechanism fails. However, with the inclusion of neutrino-neutrino interactions, adiabaticity criteria are much easier to satisfy than in the linear case. We find that instead of being proportional to m^{-2}, as in the linear case, λ is proportional to $m^{-4/3}$. Consequently, inclusion of nonlinear feedback results in the possibility of $\nu \rightleftharpoons \bar{\nu}$ transformation for much faster variation of the matter potential at the level crossing point.

In a supernova environment, Y_e can typically change by ~ 0.1 over distances of $\sim 100\,\mathrm{km}$, so a 'natural' value for the scale λ is $\sim 1000\,\mathrm{km}$. In our model, the required value of λ is larger than this even in the presence of nonlinear feedback, except for neutrino masses in excess of $1\,\mathrm{eV}$. For example, we find that for $m = 1\,\mathrm{eV}$, $\lambda \approx 15 \times 1000\,\mathrm{km}$ is required. For $m = 0.1\,\mathrm{eV}$, $\lambda \approx 300 \times 1000\,\mathrm{km}$ is required. Cosmological constraints favor neutrino masses not much greater than $0.1\,\mathrm{eV}$ [162–166], so fine-tuning of the derivative of the matter potential at the level crossing is needed to begin $\nu \rightleftharpoons \bar{\nu}$ transformation.

However, once $\nu \rightleftharpoons \bar{\nu}$ transformation develops, the derivative of the matter potential need not remain unnaturally small. Fig. (4) shows the evolution of matter and neutrino Hamiltonians for a model with a slightly exaggerated neutrino mass ($m = 1\,\mathrm{eV}$), $\lambda = 1.8 \times 10^4\,\mathrm{km}$ and $\kappa = 25\,\mathrm{km}$. We see that while a relatively small rate of change of the matter potential is required for transformation to begin,

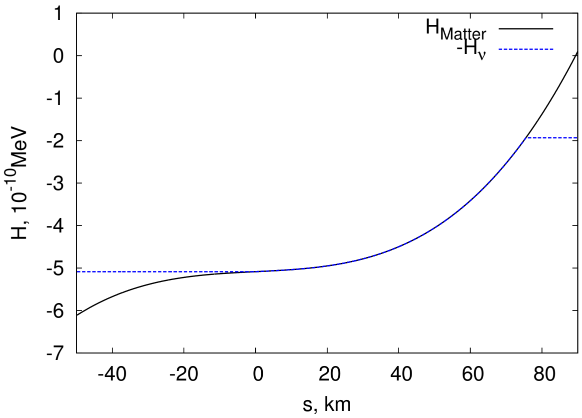

Figure 4.4: Tracking and cancelation of matter potential by neutrino potential over the course of coherent helicity transformation for a model with $m = 1\,\text{eV}$, $\lambda = 1.8 \times 10^4\,\text{km}$, $\kappa = 25\,\text{km}$.

the tracking behavior continues until the rate of change of the matter potential is considerably larger than what it was initially. This is true even though the helicity-mixing term in the Hamiltonian is proportional to the neutrino lepton number and is decreasing as neutrinos are being converted to antineutrinos.

The final spectra resulting from helicity transformation in Fig. (4) are shown in Fig. (5). We see that at low energies, neutrinos are converted into antineutrinos, while at higher energies the spectra are relatively unchanged. This is similar to what is seen in linear MSW. However, with nonlinear feedback, transformation takes place up to much higher energies than what would be allowed by adiabaticity conditions in linear MSW.

4.4 Discussion

In conclusion, the QKEs allow the possibility of a level crossing between neutrinos and antineutrinos, potentially resulting in $\nu \rightleftharpoons \bar{\nu}$ transformation. Conditions required for this level crossing can occur in core collapse supernovae and compact object mergers.

The primary obstacle to helicity transformation is the issue of adiabaticity, which arises due to the fact that the $\nu \rightleftharpoons \bar{\nu}$ mixing term in the Hamiltonian is

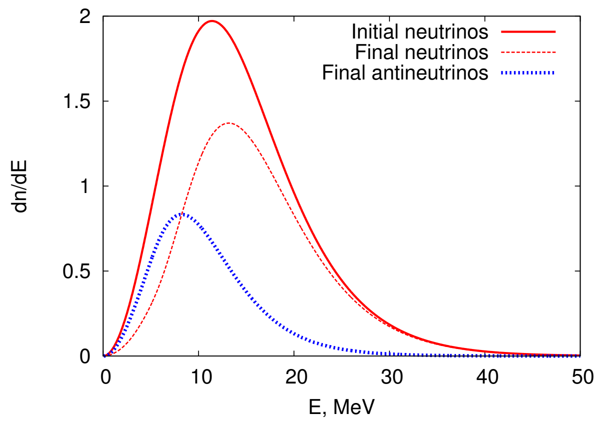

Figure 4.5: Initial and final spectra from a model with $m = 1\,\mathrm{eV}$, $\lambda = 1.8{\times}10^4\,\mathrm{km}$, $\kappa = 25\,\mathrm{km}$.

suppressed by a factor of $m/E \approx 10^{-7} - 10^{-8}$. In our toy model, we find that this issue is somewhat alleviated by nonlinear feedback due to neutrino-neutrino interactions, although the derivative of the matter potential at the level crossing must still be smaller than the generically expected value. In addition, helicity transformation is sensitive to the value of the neutrino mass and becomes much more likely if neutrino masses are larger than their minimum values.

Our model may not fully capture the conditions under which $\nu \rightleftharpoons \bar{\nu}$ transformation can take place. In a multi-angle system, there may be additional effects. The Hamiltonian changes at different rates along different neutrino emission angles, so it is more likely that adiabaticity criteria will be satisfied for some angles, leading neutrinos on these trajectories to transform. On the other hand, level crossing for different trajectories occurs at different locations, possibly rendering nonlinear feedback ineffective unless a large fraction of neutrinos is nearly collinear. Also, the $\nu_e \rightleftharpoons \bar{\nu}_e$ resonance can occur when neutrino opacity is not negligible and a significant fraction of neutrinos is propagating inward. In that case the full multi-angle QKEs cannot be solved by integrating outward in r and must instead be solved as a boundary value or a time evolution problem. Additionally, a multi-flavor model may exhibit different behavior from the 1-flavor model, as it includes additional resonances and the possibility of exchange of flavor information, in addition to particle number, between neutrinos and antineutrinos. Finally, we have not considered the effect of the evolution of electron neutrino number on Y_e, which could lead to an additional type of feedback [34, 35, 37].

A convincing determination of the extent to which helicity transformation actually takes place in core collapse supernovae or compact object mergers requires sufficiently realistic multi-angle simulations coupled to the evolution of the matter background. However, given the nonlinear feedback in the QKEs, we cannot preclude significant helicity transformation, with potentially important implications for the physics of compact objects.

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Chapter 5

Conclusion

5.1 Summary of Results

In this dissertation, a fully self-consistent derivation of the quantum kinetic equations from first principles has been presented, using the 2PI effective action formalism and expansion in small parameters ϵ to $O(\epsilon^2)$. Our equations include both coherent forward scattering terms and collision terms that, in appropriate limits, reduce well-established expressions for coherent flavor evolution or Boltzmann-like scattering and thermalization.

We have discovered the possibility of coherent interconversion between neutrinos and antineutrinos that may occur at sufficiently high density and in the presence of anisotropic matter or neutrino currents. Numerical simulations in a toy model suggest that this interconversion between neutrinos and antineutrinos may be enhanced by nonlinear feedback stemming from neutrino-antineutrino interactions, and thus may play an important role in supernova or compact object merger environments.

In addition to the spin coherence issue, we have pointed out that neutrino scattering and neutrino flavor evolution cannot necessarily be separated in early Universe and supernova environments. Treating these two phenomena in the same regime requires a formalism that goes beyond the usual Schrödinger-like or Boltzmann approach. While there are still issues to be resolved when it comes to the numerical solution of these problems, we now possess fully self-consistent equations that describe these systems.

5.2 Future Work

While we have derived the quantum kinetic equations for neutrinos in a general medium, the work of studying solutions to these equations is still in the early stages. Because of a large degree of nonlinearity in the QKEs in the presence of a high density of neutrinos, these equations must in general be solved numerically, and due to the high number of dimensions for general QKEs (4 $\times$ spacetime and 3 $\times$ momentum) and the presence of multiple scales (neutrino oscillation length, mean free path, physical length scales associated with the astrophysical environment), numerical solution can be challenging. The first numerical calculations will need to be done using simplified toy models or environments with a large degree of symmetry. There are at least two promising systems for study: the early Universe and core collapse supernovae. In principle, the methods for core collapse supernovae can be employed to treat compact object mergers, but in this case the problem becomes somewhat more complicated because the system is further away from spherical symmetry.

In the case of the early Universe, if a flavor or lepton number asymmetry can be introduced into the neutrino distributions through some additional physics (*e.g.,* decay of a massive particle such as a sterile neutrino), depending on the epoch in which this flavor asymmetry appears, the collision term can play an important part in its evolution. Since neutrinos can have an effect on Big Bang nucleosynthesis and cosmological observables such as N_{eff}, detailed calculations of the evolution of neutrino flavor in the early Universe can be useful for constraining models that lead to such flavor asymmetry. Calculations of neutrino evolution in the early universe so far have been restricted to active-sterile transformation (...) or used made use of approximate or ad-hoc QKEs (...). We now have a method for deriving the complete equations of motion for neutrino distributions in the early Universe and can begin to solve these equations.

A significant practical advantage of solving the QKEs in the early Universe, as opposed to environments such as core collapse supernovae or compact object mergers, is the presence of the symmetries of homogeneity and isotropy, which lead to tremendous simplification of the QKEs and reduction of the number of degrees of freedom in the problem. However, while the early Universe can be used to study the effects of the collision term and flavor coherence, spin coherence effects cannot appear in this environment unless isotropy is broken in some way.

The study of the QKEs in core collapse supernovae can be applied both to the dense regions of the proto-neutron star below the neutrinosphere, where neutrinos are trapped by collisions and diffuse out over many mean free paths, and to the supernova envelope, where the majority of the neutrino flux is free-streaming and only a small fraction of neutrinos undergo inelastic or non-forward scattering.

The neutrinosphere itself, which is not a sharp boundary but a decoupling region where neutrinos gradually transition from collision-dominated to free-streaming behavior, can also be treated in the same formalism.

In studies of supernova neutrino flavor evolution, the assumption has generally been made that flavor coherence cannot occur near or below the neutrinosphere. However, this assumption has not been proven self-consistently, and certainly in models with sterile neutrinos coherent active-sterile transformation can occur at a location within proto-neutron star. In addition, with the introduction of spin coherence, there may be neutrino-antineutrino resonances within the proto-neutron star, similarly to the situation with active-sterile transformation. Spin coherence very deep within the proto-neutron star may be unlikely because, due to the slow diffusion of neutrinos, the outward gradient of the lepton number current is small, and therefore there is not expected to be much anisotropy. However, if coherent effects can occur just below or within the neutrinosphere, then spin coherence becomes possible and collisions may modify these effects. The QKEs provide a formalism by which neutrino decoupling in the neutrinosphere may be treated in a completely self-consistent way, and a spherically symmetric simulation of this environment may be within the realm of possibility. However, this is a challenging problem because any neutrino oscillations that arise in high-density environments can have extremely short oscillation lengths, possibly necessitating very high resolution in the numerical model.

In the supernova envelope, collisions are subdominant but can nevertheless play an important role by altering flavor evolution [65, 66]. The incorporation of subdominant collisions into the flavor evolution models in a self-consistent way is a challenging problem because the halo effect can alter the problem of neutrino flavor evolution in a fundamental way. Instead of neutrinos streaming out from locations of smaller radius to locations of larger radius, there are now backscattered neutrinos carrying information back from larger radii to smaller ones. This means that the equations for neutrino flavor evolution can no longer be integrated as an initial value problem in radius and must instead be solved as a boundary value or a time evolution problem. A difficult but necessary direction for future work is to develop methods for solving the QKEs in a supernova environment in the presence of backscattered neutrinos.

Further, spin coherence may be important in the supernova envelope, due to the presence of large-scale anisotropy, collective effects and nonlinear feedback, and multiple level crossings between neutrino and antineutrino energies. Spin coherence may result in large-scale conversion between neutrinos and antineutrinos or in exchange of flavor information that can in turn modify flavor evolution. We have begun work on the analysis of possible effects of spin coherence in the context of toy models such as the one described in Chapter 4. In order to fully determine the extent of possible effects of spin coherence, it is necessary to make these toy models more realistic and sophisticated, and eventually perform a full 6×6 multi-angle simulation of neutrino spin and flavor evolution.

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