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Entropy, dimension and the Elton-Pajor Theorem

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Abstract

The Vapnik-Chervonenkis dimension of a set K in \mathbb{R}^n is the maximal dimension of the coordinate cube of a given size, which can be found in coordinate projections of K . We show that the VC dimension of a convex body governs its entropy. This has a number of consequences, including the optimal Elton's theorem and a uniform central limit theorem in the real valued case.

1 Introduction

Let x_1, \dots, x_n be vectors in the unit ball of a Banach space, and assume that $\mathbb{E} \|\sum_{i=1}^n \varepsilon_i x_i\| \geq \delta n$ for some number $\delta > 0$, where $\varepsilon_1, \dots, \varepsilon_n$ denote independent Bernoulli random variables (taking values 1 and -1 with probability $1/2$). In 1983, J. Elton [E] proved an important result that there exists a subset σ of $\{1, \dots, n\}$ of size proportional to n such that the set of vectors $(x_i)_{i \in \sigma}$ is well equivalent to the ℓ_1 unit-vector basis. Specifically, there exist numbers $s, t > 0$, depending only on δ , such that $|\sigma| \geq sn$ and $\|\sum_{i \in \sigma} a_i x_i\| \geq t \sum_{i \in \sigma} |a_i|$ for all real numbers (a_i) . This result was extended to the complex case by A. Pajor [Pa].

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Several steps have been made towards finding asymptotically the largest possible s and t in Elton's Theorem ([Pa], [T]). Trivial upper bounds are that $s \leq \delta^2$, which follows from the example of identical vectors, and $t \leq \delta$ as demonstrated by shrinking the usual ℓ_1^n unit-vector basis. One of the aims of this paper is to prove Elton's Theorem with $s \geq c\delta^2$ and $t \geq c\delta$, where $c > 0$ is an absolute constant. Furthermore, we show that s and t satisfy $\sqrt{st} \log^{2.1}(2/t) \geq c\delta$, which, as an easy example shows, is optimal for all δ up to a logarithmic factor. This improves the result of M. Talagrand from [T].

This theorem follows from new entropy estimates of a convex body $K \subset [-1, 1]^n = B_\infty^n$. We show that the entropy of K is controlled by its Vapnik-Chervonenkis dimension. This parameter, denoted by $\text{VC}(K, t)$, is defined for every $0 < t < 1$ as the maximal size of a subset σ of $\{1, \dots, n\}$, such that the coordinate projection of K onto \mathbb{R}^σ contains a coordinate cube of the form $x + [0, t]^\sigma$. This notion carries over to convexity the "classical" concept of the VC dimension, denoted by $\text{VC}(A)$, and defined for subsets A of the discrete cube $\{0, 1\}^n$ as the maximal size of the subset σ of $\{1, \dots, n\}$ such that $P_\sigma A = \{0, 1\}^\sigma$, where P_σ is the coordinate projection onto the coordinates in σ (see [LT] §14.3).

Consider the unit ball B_p^n of ℓ_p^n , $1 \leq p \leq \infty$, and let us look at the covering numbers $N(K, n^{1/p}B_p^n, t)$, which are the minimal number of translates of $tn^{1/p}B_p^n$ in \mathbb{R}^n needed to cover K . A volumetric bound on the entropy (which is the logarithm of the covering numbers) shows that

$$\log N(K, n^{1/p}B_p^n, t) \leq \log(5/t) \cdot n.$$

One question is whether it is possible to replace the dimension n on the right-hand side of this estimate by the VC dimension $\text{VC}(K, ct)$, which is generally smaller? This is perfectly true for the *Boolean* cube: the known theorem of R. Dudley that lead to a characterization of the uniform central limit property in the Boolean case states that if $A \subset \{0, 1\}^n$ then

$$\log N(A, n^{1/2}B_2^n, t) \leq C \log(2/t) \cdot \text{VC}(A).$$

This estimate follows by a random choice of coordinates and an application of the Sauer-Shelah Lemma (see [LT] Theorem 14.12). The same problem for convex bodies is considerably more difficult, as to bound $\text{VC}(K, t)$ one needs to find a cube in $P_\sigma K$ with well separated faces, not merely disjoint. We prove the following theorem.

Theorem 1.1 *There are absolute constants $C, c > 0$ such that for every convex body $K \subset B_\infty^n$, every $1 < p < \infty$ and any $0 < t < 1$,*

$$\log N(K, n^{1/p}B_p^n, t) \leq Cp^2 \log^2(2/t) \cdot \text{VC}(K, ct). \quad (1)$$

Moreover,

$$\log N(K, B_\infty^n, t) \leq CM^2 \log^2(2/t) \cdot \text{VC}(K, ct), \quad (2)$$

provided that either the right or the left hand side of (2) is larger than $t^M n$.

Let us comment on estimate (2), which improves the main lemma of [ABCH]. This bound can not hold in general if the coefficient in front of the VC dimension depends only on t and not on n , since for $K = B_1^n$ we have $\text{VC}(K, t) = 2/t$ and $\log N(K, B_\infty^n, t) \geq \log n$. Next, (2) is best complemented by the easy lower bound

$$\log N(K, B_\infty^n, t) \geq \text{VC}(K, ct),$$

for some absolute constant $c > 0$, which follows from the definition of the VC dimension and by a comparison of volumes. These two bounds show that the $\|\cdot\|_\infty$ -entropy of K is governed by the VC dimension of K , up to a logarithmic factor in t .

The relation to the Elton-Pajor Theorem is the following. If K is a symmetric convex body, then $\text{VC}(K, t)$ is the maximal cardinality of a subset σ of $\{1, \dots, n\}$ such that $\|\sum_{i \in \sigma} a_i e_i\|_{K^\circ} \geq (t/2) \sum_{i \in \sigma} |a_i|$ for all real numbers (a_i) , where e_i are the canonical unit vectors in \mathbb{R}^n and K° is the polar of K . Note that if (g_i) are independent standard gaussian random variables then $E\|\sum_{i=1}^n \varepsilon_i e_i\| \leq 2E\|\sum_{i=1}^n g_i e_i\|$ for every norm ([LT] §4.5). Therefore, our problem reduces to finding a bound on

$$E = \mathbb{E} \left\| \sum_{i=1}^n g_i e_i \right\|_{K^\circ}$$

in terms of the VC-dimension of K . The latter is relatively easy once we know (1). Indeed, replacing the entropy by the VC dimension in Dudley's entropy inequality it follows that there are absolute constants C and c such that

$$E \leq C \int_{cE/\sqrt{n}}^{\infty} \sqrt{\log N(K, B_2^n, t)} dt \leq C\sqrt{n} \int_{cE/n}^1 \sqrt{\text{VC}(K, ct)} \log(2/t) dt. \quad (3)$$

This inequality improves of the main theorem of M. Talagrand in [T]. Elton's Theorem with optimal asymptotics follows from (3) by comparing the integrand to an appropriately chosen integrable function.

We present a few other applications to convexity. Inequality (3) can be applied, as in [T], to compare two geometric properties of a Banach space called type and infratype. Recall that a Banach space X is of gaussian type p if there exists some $M > 0$ such that for all n and all sequences of vectors $(x_i)_{i \leq n}$,

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}. \quad (4)$$

The best possible constant M in this inequality is denoted by $T_p(X)$. Next, X has infratype p if there exists some $M > 0$ such that for all n and all sequences of vectors $(x_i)_{i \leq n}$, we have

$$\min_{\eta_i = \pm 1} \left\| \sum_{i=1}^n \eta_i x_i \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}. \quad (5)$$

The best possible constant M in this inequality is denoted by $I_p(X)$.

M. Talagrand proved in [T] that if $1 < p < 2$ then $T_p(X) \leq C(p)I_p(X)^2$, where $C(p)$ is a constant which depends only on p . It is not known whether the square can be removed. Moreover, the situation for $p = 2$ is unknown in general, but (3) can be used to show that there is an absolute constant C such that for any n dimensional Banach space X ,

$$T_2(X) \leq I_2(X) \cdot C \log^2 \left(\frac{n}{I_2(X)^2} \right) \leq I_2(X) \cdot C \log^2 n.$$

Finally, we present an application of Theorem 1.1 to empirical processes. We use a version of (1) to bound the entropy of an arbitrary subset of B_∞^n using a scale-sensitive version of the "classical" VC dimension, known as the fat-shattering dimension. In particular we show that if F is a class of uniformly bounded functions, which has a relatively small fat-shattering dimension, then it satisfies the uniform central limit theorem for any probability measure. This extends Dudley's characterization for VC classes to the real-valued case.

The paper is organized as follows. In Section 2 we prove the bound for the B_p^n -entropy in abstract finite product spaces, and then derive (1)

by approximation. Actually, the convexity of K plays a very little role in these results, and similar entropy bounds hold for arbitrary subsets of B_∞^n . In Section 3 we prove (2) for the B_∞^n -entropy by reducing it to (1) through an independent lemma that compares the B_p^n -entropy to the B_∞^n -entropy. In Section 4 we apply (1) to convex bodies. In particular, we deduce Elton's Theorem and the infratype results. Finally, in Section 5 we apply (1) to empirical processes.

Throughout this article, positive absolute constants are denoted by C and c . Their values may change from line to line, or even within the same line.

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2 B_p^n -entropy in abstract product spaces

We will introduce and work with the notion of the VC dimension in an abstract setting that encompasses both classes considered in the introduction, the subsets of the discrete cube $\{0, 1\}^n$ and the class of convex bodies in \mathbb{R}^n .

We call a map $d : T \times T \rightarrow \mathbb{R}_+$ a *quasi-metric* if d is symmetric and reflexive (that is, $\forall x, y, d(x, y) = d(y, x)$ and $d(x, x) = 0$). We say that points x and y in T are separated if $d(x, y) > 0$. Thus, d does not necessarily separate points or satisfy the triangle inequality.

Definition 2.1 *Let (T, d) be a quasi-metric space and let n be a positive integer. For a set $A \subset T^n$ and $t > 0$, the VC-dimension $\text{VC}(A, t)$ is the maximal cardinality of a subset $\sigma \subset \{1, \dots, n\}$ such that the inclusion*

$$P_\sigma A \supseteq \prod_{i \in \sigma} \{a_i, b_i\} \tag{6}$$

holds for some points $a_i, b_i \in T, i \in \sigma$ with $d(a_i, b_i) \geq t$. If no such σ exists, we set $\text{VC}(A, t) = 0$. When there is a need to specify the underlying metric, we denote the VC dimension by $\text{VC}_d(A, t)$.

Since $\text{VC}(A, t)$ is decreasing in t and is bounded by n , which is the “usual” dimension of the product space, the limit

$$\text{VC}(A) := \lim_{t \rightarrow 0^+} \text{VC}(A, t)$$

always exists. Equivalently, $\text{VC}(A)$ is the maximal cardinality of a subset $\sigma \subset \{1, \dots, n\}$ such that (6) holds for some pairs (a_i, b_i) of separated points in T .

This definition is an extension of the “classical” VC dimension for subsets of the discrete cube $\{0, 1\}^n$, where we think of $\{0, 1\}$ as a metric space with the 0–1 metric. Clearly, for any set $A \subset \{0, 1\}^n$ the quantity $\text{VC}(A, t)$ does not depend on $0 < t < 1$, and hence

$$\text{VC}(A) = \max \left\{ |\sigma| : \sigma \subset \{1, \dots, n\}, P_\sigma A = \{0, 1\}^\sigma \right\},$$

which is precisely the “classical” definition of the VC dimension.

The other example discussed in the introduction was the VC dimension of convex bodies. Here $T = \mathbb{R}$ or, more frequently, $T = [-1, 1]$, both with respect to the usual metric. If $K \subset T^n$ is a convex body, then $\text{VC}(K, t)$ is the maximal cardinality of a subset $\sigma \subset \{1, \dots, n\}$ for which the inclusion

$$P_\sigma K \supseteq x + (t/2)B_\infty^\sigma$$

holds for some vector $x \in \mathbb{R}^\sigma$ (which automatically lies in $P_\sigma K$). It is easy to see that if K is symmetric, we can set $x = 0$. Also note that for every convex body $\text{VC}(K) = n$.

The main results of this article rely on (and are easily reduced to) a discrete problem: to estimate the VC-dimension of a set in a product space T^n , where (T, d) is a *finite* quasi-metric space. T^n is usually endowed with the normalized Hamming quasi-metric $d_n(x, y) = n^{-1} \sum_{i=1}^n d(x(i), y(i))$ for $x, y \in T^n$.

In the main result of this section we bound the entropy of a set $A \subset T^n$ with respect to d_n in terms of $\text{VC}(A)$.

Theorem 2.2 *Let (T, d) be a finite quasi-metric space with $\text{diam}(T) \leq 1$, and set n to be a positive integer. Then, for every set $A \subset T^n$ and every $0 < \varepsilon < 1$,*

$$\log N(A, d_n, \varepsilon) \leq C \log^2(|T|/\varepsilon) \cdot \text{VC}(A),$$

where C is an absolute constant.

Before presenting the proof, let us make two standard observations. We say that points $x, y \in T^n$ are separated on the coordinate i_0 if $x(i_0)$ and $y(i_0)$ are separated. Points x and y are called ε -separated if $d_n(x, y) \geq \varepsilon$.

Clearly, if A' is a maximal ε -separated subset of A then $|A'| \geq N(A, d_n, \varepsilon)$. Moreover, the definition of d_n and the fact that $\text{diam}(T) \leq 1$ imply that every two distinct points in A' are separated on at least εn coordinates. This shows that Theorem 2.2 may be reduced to the following statement.

Theorem 2.3 *Let (T, d) be a quasi-metric space for which $\text{diam}(T) \leq 1$. Let $0 < \varepsilon < 1$ and consider a set $A \subset T^n$ such that every two distinct points in A are separated on at least εn coordinates. Then*

$$\log |A| \leq C \log^2(|T|/\varepsilon) \cdot \text{VC}(A). \quad (7)$$

The first step in the proof of Theorem 2.3 is a probabilistic extraction principle, which allows one to reduce the number of coordinates without changing the separation assumption by much. Its proof is based on a simple discrepancy bound for a set system.

Lemma 2.4 *There exists an absolute constant $c > 0$ for which the following holds. Let $\varepsilon > 0$ and assume that \mathcal{S} is a system of subsets of $\{1, \dots, n\}$ which satisfies that each $S \in \mathcal{S}$ contains at least εn elements. Let $k \leq n$ be an integer such that $\log |\mathcal{S}| \leq c\varepsilon k$. Then there exists a subset $I \subset \{1, \dots, n\}$ of cardinality $|I| = k$, such that*

$$|I \cap S| \geq \varepsilon k/4 \quad \text{for all } S \in \mathcal{S}.$$

Proof. If $|\mathcal{S}| = 1$ the lemma is trivially true, hence we may assume that $|\mathcal{S}| \geq 2$. Let $0 < \delta < 1/2$ and set $\delta_1, \dots, \delta_n$ to be $\{0, 1\}$ -valued independent random variables with $\mathbb{E}\delta_i = \delta$ for all i . By the classical bounds on the tails of the binomial law (see [H], or [LT] 6.3 for more general inequalities), there is an absolute constant $c_0 > 0$ for which

$$\mathbb{P}\left\{\left|\sum_{i=1}^n (\delta_i - \delta)\right| > \frac{1}{2}\delta n\right\} \leq 2 \exp(-c_0 \delta n). \quad (8)$$

Let $\delta = k/2n$ and consider the random set $I = \{i : \delta_i = 1\}$. For any set $B \subset \{1, \dots, n\}$, $|I \cap B| = \sum_{i \in B} \delta_i$. Then (8) implies that

$$\mathbb{P}\{|I \cap B| \geq \delta|B|/2\} \geq 1 - 2 \exp(-c_0 \delta|B|).$$

Since for every $S \in \mathcal{S}$, $|S| > \varepsilon n$, then

$$\mathbb{P}\{|I \cap S| \geq \varepsilon k/4\} \geq 1 - 2 \exp(-\frac{1}{2}c_0\varepsilon k).$$

Therefore,

$$\mathbb{P}\left\{\forall S \in \mathcal{S}, |I \cap S| \geq \frac{1}{4}\varepsilon k\right\} \geq 1 - 2|\mathcal{S}| \exp(-\frac{1}{2}c_0\varepsilon k).$$

By the assumption on k , this quantity is larger than $1/2$ (with an appropriately chosen absolute constant c). Moreover, by a similar argument, $|I| \leq k$ with probability larger than $1/2$. This proves the existence of a set I satisfying the assumptions of the lemma. \blacksquare

Proof of Theorem 2.3. We may assume that $|T| \geq 2$, $\varepsilon \leq 1/2$, $n \geq 2$ and $\max(4, \exp(4c)) \leq |A| \leq |T|^n$, where $0 < c < 1$ is the constant in Lemma 2.4. The first step in the proof is to use previous lemma, which enables one to make the additional assumption that $\log |A| \geq c\varepsilon n/4$. Indeed, assume that the converse inequality holds, and for every pair of distinct points $x, y \in A$, let $S(x, y) \subset \{1, \dots, n\}$ be the set of coordinates on which x and y are separated. Put \mathcal{S} to be the collection of the sets $S(x, y)$ and let k be the minimal positive integer for which $\log |\mathcal{S}| \leq c\varepsilon k$. Since $|A| \leq |\mathcal{S}| \leq |A|^2$, then

$$c\varepsilon(k-1) \leq \log |\mathcal{S}| \leq 2 \log |A| \leq \frac{1}{2}c\varepsilon n,$$

which implies that $1 \leq k \leq n$. Thus, by Lemma 2.4 there is a set $I \subset \{1, \dots, n\}$, $|I| = k$, with the property that every pair of distinct points $x, y \in A$ is separated on at least $\varepsilon|I|/4$ coordinates in I . Also, since $4c \leq \log |A| \leq \log |\mathcal{S}| \leq c\varepsilon k$, then $\varepsilon|I|/4 \geq 1$ and thus $|P_I A| = |A|$. Clearly, to prove the assertion of the theorem for the set $A \subset T^n$, it is sufficient to prove it for the set $P_I A \subset T^I$ (with $|I|$ instead of n), whose cardinality already satisfies $\log |P_I A| = \log |A| \geq c\varepsilon(k-1)/2 \geq c\varepsilon|I|/4$. Therefore, we can assume that $|A| = \exp(\alpha n)$ with $\alpha > c\varepsilon$ for some absolute constant c .

The next step in the proof is a counting argument, which is based on the proof of Lemma 3.3 in [ABCH] (see also [BL]).

A set is called a *cube* if it is of the form $D_\sigma = \prod_{i \in \sigma} \{a_i, b_i\}$, where σ is a subset of $\{1, \dots, n\}$ and $a_i, b_i \in T$. We will be interested only in *large cubes*, which are the cubes in which a_i and b_i are separated for all $i \in \sigma$. Given a set $B \subset T^n$, we say that a cube D_σ *embeds* into B if $D_\sigma \subset P_\sigma B$. Note that if a large cube D_σ with $|\sigma| \geq v$ embeds into B then $\text{VC}(B) \geq v$.

For all $m \geq 2$, $n \geq 1$ and $0 < \varepsilon \leq 1/2$, let $t_\varepsilon(m, n)$ denote the maximal number t such that for every set $B \subset T^n$, $|B| = m$, which satisfies the separation condition we imposed (that is, every distinct points $x, y \in B$ are separated on at least εn coordinates), there exist t large cubes that embed into B . If no such B exists, we set $t_\varepsilon(m, n)$ to be infinite.

The number of possible large cubes D_σ for $|\sigma| \leq v$ is smaller than $\sum_{k=1}^v \binom{n}{k} |T|^{2k}$, as for every σ of cardinality k there are less than $|T|^{2k}$ possibilities to choose D_σ . Therefore, if $t_\varepsilon(|A|, n) \geq \sum_{k=1}^v \binom{n}{k} |T|^{2k}$, there exists a large cube D_σ for some $|\sigma| \geq v$ that embeds into A , implying that $\text{VC}(A) \geq v$. Thus, to prove the theorem, it suffices to estimate $t_\varepsilon(m, n)$ from below. To that end, we will show that for every $n \geq 2$, $m \geq 1$ and $0 < \varepsilon \leq 1/2$,

$$t_\varepsilon(2m \cdot |T|^2/\varepsilon, n) \geq 2t_\varepsilon(2m, n-1). \quad (9)$$

Indeed, fix any set $B \subset T^n$ of cardinality $|B| = 2m \cdot |T|^2/\varepsilon$, which satisfies the separation condition above. If no such B exists then $t_\varepsilon(2m \cdot |T|^2/\varepsilon, n) = \infty$, and (9) holds trivially. Split B arbitrarily into $m \cdot |T|^2/\varepsilon$ pairs, and denote the set of the pairs by \mathcal{P} . For each pair $(x, y) \in \mathcal{P}$ let $I(x, y) \subset \{1, \dots, n\}$ be the set of the coordinates on which x and y are separated, and note that by the separation condition, $|I(x, y)| \geq \varepsilon n$.

Let i_0 be the random coordinate, that is, a random variable uniformly distributed in $\{1, \dots, n\}$. The expected number of the pairs $(x, y) \in \mathcal{P}$ for which $i_0 \in I(x, y)$ is

$$\mathbb{E} \sum_{(x,y) \in \mathcal{P}} \mathbf{1}_{\{i_0 \in I(x,y)\}} = \sum_{(x,y) \in \mathcal{P}} \mathbb{P}\{i_0 \in I(x,y)\} \geq |\mathcal{P}| \cdot \varepsilon = m|T|^2.$$

Hence, there is a coordinate i_0 on which at least $m|T|^2$ pairs $(x, y) \in \mathcal{P}$ are separated. By the pigeonhole principle, there are at least $m|T|^2/\binom{|T|}{2} \geq 2m$ pairs $(x, y) \in \mathcal{P}$ for which the (unordered) set $\{x(i_0), y(i_0)\}$ is the same.

Let $I = \{1, \dots, n\} \setminus \{i_0\}$. It follows that there are two subsets of B , denoted by B_1 and B_2 , such that $|B_1| = |B_2| = 2m$ and

$$B_1 \subset \{b_1\} \times T^I, \quad B_2 \subset \{b_2\} \times T^I$$

for some separated points $b_1, b_2 \in T$. Clearly, the set B_1 satisfies the separation condition and so does B_2 . It is also clear that if a large cube D_σ embeds into B_1 , then it also embeds into B , and the same holds for B_2 . Moreover, if the same cube D_σ embeds into both B_1 and B_2 , then the large

cube $\{b_1, b_2\} \times D_\sigma$ embeds into B (since $\{b_1, b_2\} \times D_\sigma \subset P_{\{i_0\} \cup \sigma} B$). Therefore, $t_\varepsilon(|B|, n) \geq 2t_{\frac{\varepsilon n}{n-1}}(|B_1|, n-1) \geq 2t_\varepsilon(|B_1|, n-1)$, establishing (9).

Since $t_\varepsilon(2, n) \geq 1$, an induction argument yields that $t_\varepsilon(2(|T|^2/\varepsilon)^r, n) \geq 2^r$ for every $r \geq 1$. Thus, for every $m \geq 4$

$$t_\varepsilon(m, n) \geq m^{\frac{1}{2 \log(|T|^2/\varepsilon)}}.$$

(It is remarkable that the right hand side does not depend on n). Therefore, $\text{VC}(A) \geq v$ provided that v satisfies

$$t_\varepsilon(|A|, n) \geq \exp\left(\frac{\alpha n}{2 \log(|T|^2/\varepsilon)}\right) \geq \sum_{k=1}^v \binom{n}{k} |T|^{2k}. \quad (10)$$

To estimate v , one can bound the right-hand side of (10) using Stirling's approximation $\sum_{k=1}^v \binom{n}{k} \leq [\gamma^\gamma (1-\gamma)^{1-\gamma}]^{-n}$, where $\gamma = v/n \leq 1/2$. It follows that for $v \leq n/2$, $\sum_{k=1}^v \binom{n}{k} |T|^{2k} \leq (\frac{|T|n}{v})^{2v}$. Taking logarithms in (10), we seek integers $v \leq n/2$ satisfying that

$$\frac{\alpha n}{2 \log(|T|^2/\varepsilon)} \geq 2v \log\left(\frac{|T|n}{v}\right).$$

This holds if

$$v \leq \left(\frac{\alpha n}{\log(|T|^2/\varepsilon)}\right) / 8 \log\left(\frac{4|T| \log(|T|^2/\varepsilon)}{\alpha}\right),$$

proving our assertion since $\alpha > c\varepsilon$. ■

Corollary 2.5 *Let $n \geq 2$ and $p \geq 2$ be integers, set $0 < \varepsilon < 1$ and $q > 0$. Consider a set $A \subset \{1, \dots, p\}^n$ such that for every two distinct points $x, y \in A$, $|x(i) - y(i)| \geq q$ for at least εn coordinates i . Then*

$$\log |A| \leq C \log^2(p/\varepsilon) \cdot \text{VC}(A, q).$$

Proof. We can assume that $q \geq 1$. Define the following quasi-metric on $T = \{1, \dots, p\}$:

$$d(a, b) = \begin{cases} 0 & \text{if } |a - b| < q, \\ 1 & \text{otherwise.} \end{cases}$$

Then $N(A, d_n, \varepsilon) = |A|$. By Theorem 2.2,

$$\log |A| \leq C \log^2(p/\varepsilon) \cdot \text{VC}_d(A),$$

which completes the proof by the definition of the metric d . ■

Now we pass from the discrete setting to the “continuous” one - namely, we study subsets of B_∞^n . Recall that the Minkowski sum of two convex bodies $A, B \subset \mathbb{R}^n$ is defined as $A + B = \{a + b \mid a \in A, b \in B\}$.

Corollary 2.6 *For every $A \subset B_\infty^n$, $0 < t < 1$ and $0 < \varepsilon < 1$,*

$$\log N(A, \sqrt{n}B_2^n, t) \leq C \log^2(2/t\varepsilon) \cdot \text{VC}(A + \varepsilon B_\infty^n, t/2).$$

Proof. Clearly, we may assume that $\varepsilon \leq t/4$. Put $p = \frac{1}{2\varepsilon}$ and let

$$T = \{-2\varepsilon p, -2\varepsilon(p-1), \dots, -2\varepsilon, 0, 2\varepsilon, \dots, 2\varepsilon(p-1), 2\varepsilon p\}.$$

Since $t - \varepsilon > 3t/4$, then by approximation one can find a subset $A_1 \subset T^n$ for which $A_1 \subset A + \varepsilon B_\infty^n$ and $N(A_1, \sqrt{n}B_2^n, t - \varepsilon) \geq N(A, \sqrt{n}B_2^n, t)$. Therefore, there exists a subset $A_2 \subset A_1$ of cardinality $|A_2| \geq N(A, \sqrt{n}B_2^n, t)$, which is $\frac{3t}{4}\sqrt{n}$ -separated with respect to the $\|\cdot\|_2$ -norm. Note that every two distinct points $x, y \in A_2$ satisfy that

$$\sum_{i=1}^n |x(i) - y(i)|^2 \geq (9t^2/16)n \geq t^2n/2$$

and that $|x(i) - y(i)|^2 \leq 4$ for all i . Hence $|x(i) - y(i)| \geq t/2$ on at least $t^2n/16$ coordinates i . By Corollary 2.5 applied to A_2 ,

$$\log |A_2| \leq C \log^2(2/t\varepsilon) \cdot \text{VC}(A_2, t/2),$$

and since $A_2 \subset A_1 \subset A + \varepsilon B_\infty^n$, our claim follows. \blacksquare

From this we derive the entropy estimate (1).

Corollary 2.7 *There exists an absolute constant C such that for any convex body $K \subset B_\infty^n$ and every $0 < t < 1$,*

$$\log N(K, \sqrt{n}B_2^n, t) \leq C \log^2(2/t) \cdot \text{VC}(K, t/4).$$

Proof. This estimate follows from Corollary 2.6 by selecting $\varepsilon = t/4$ and recalling the fact that for every convex body $K \subset \mathbb{R}^n$ and every $0 < b < a$,

$$\text{VC}(K + bB_\infty^n, a) \leq \text{VC}(K, a - b).$$

The latter inequality is a consequence of the definition of the VC-dimension and the observation that if $0 < b < a$ are such that $aB_\infty^n \subset K + bB_\infty^n$, then $(a - b)B_\infty^n \subset K$. \blacksquare

Note that Corollary 2.6 and Corollary 2.7 can be extended to the case where the covering numbers are computed with respect to $n^{1/p}B_p^n$ for $1 < p < \infty$, thus establishing the complete claim in (1).

3 B_∞^n -entropy

In this section we prove estimate (2), which improves the main combinatorial result in [ABCH]. Our result can be equivalently stated as follows.

Theorem 3.1 *Let $K \subset B_\infty^n$ be a convex body, set $t > 0$ and put $v = \text{VC}(K, t/8)$. Then,*

$$\log N(K, B_\infty^n, t) \leq Cv \cdot \log^2(n/tv), \quad (11)$$

where C is an absolute constant.

This estimate should be compared with the Sauer-Shelah lemma for subsets of the Boolean cube $\{0, 1\}^n$. It says that if $A \subset \{0, 1\}^n$ then for $v = \text{VC}(K)$ we have $|A| \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{v}$, so that

$$\log |A| \leq 2v \cdot \log(n/v)$$

(and note that, of course, $|A| = N(K, B_\infty^n, t)$ for all $0 < t < 1/2$).

We reduce the proof of (3.1) to an application of the B_p^n -entropy estimate (1). As a start, note that for $p = \log n$, $B_\infty^n \subset n^{1/p} B_p^n \subset e B_\infty^n$. Therefore, an application of (1) for this value of p yields

$$\log N(K, B_\infty^n, t) \leq Cv \cdot \log^2(n/t),$$

which is slightly worse than (11).

To deduce (11) we need a result that compares the B_∞^n -entropy to the B_p^n -entropy, and which may be useful in other applications as well.

Lemma 3.2 *There is an absolute constant $c > 0$ such that the following holds. Let A be a subset of B_∞^n such that every two distinct points $x, y \in A$ satisfy $\|x - y\|_\infty \geq t$. Then, for every integer $1 \leq k \leq n/2$, there exists a subset $A' \subset A$ of cardinality*

$$|A'| \geq \binom{n}{k}^{-1} (ct)^k |A|,$$

with the property that every two distinct points in A' satisfy that $|x(i) - y(i)| \geq t/2$ for at least k coordinates i .

Proof. We can assume that $0 < t < 1/8$. Set $s = t/2$. The separation assumption imply that $N(A, B_\infty^n, s) \geq |A|$. Denote by D_k the set of all points x in \mathbb{R}^n for which $|x(i)| \geq 1$ on at most k coordinates i . One can see that $N(A, D_k, s) = N(A, sD_k, 1) = N(A, sD_k \cap 3B_\infty^n, 1)$. Then, by the submultiplicative property of the covering numbers,

$$\begin{aligned} N(A, B_\infty^n, s) &\leq N(A, sD_k \cap 3B_\infty^n, 1) \cdot N(sD_k \cap 3B_\infty^n, B_\infty^n, s) \\ &\leq N(A, sD_k, 1) \cdot N(sD_k \cap 3B_\infty^n, B_\infty^n, s). \end{aligned} \quad (12)$$

To bound the second term, write D_k as

$$D_k = \bigcup_{|\sigma|=k} \left(\mathbb{R}^\sigma + (-1, 1)^{\sigma^c} \right),$$

where the union is taken with respect to all subsets $\sigma \subset \{1, \dots, n\}$, and the sum in the right-hand side is the Minkowski sum. Thus,

$$sD_k \cap 3B_\infty^n = \bigcup_{|\sigma|=k} \left(3B_\infty^\sigma + (-s, s)^{\sigma^c} \right).$$

Denote by $N'(A, B, t)$ the number of translates of tB by vectors in A needed to cover A . Therefore,

$$\begin{aligned} N(sD_k \cap 3B_\infty^n, B_\infty^n, s) &\leq \sum_{|\sigma|=k} N(3B_\infty^\sigma + (-s, s)^{\sigma^c}, B_\infty^n, s) \\ &\leq \sum_{|\sigma|=k} N'(3B_\infty^\sigma, B_\infty^n, s). \end{aligned}$$

The latter inequality holds because any cover of $3B_\infty^\sigma$ by translates of sB_∞^n automatically covers $3B_\infty^\sigma + (-s, s)^{\sigma^c}$. Hence, for some absolute constant C ,

$$\begin{aligned} N(sD_k \cap 3B_\infty^n, B_\infty^n, s) &\leq \binom{n}{k} N'(3B_\infty^k, B_\infty^n, s) \\ &\leq \binom{n}{k} (C/s)^k \end{aligned}$$

by a comparison of the volumes, and by (12) we obtain

$$N(A, D_k, s) \geq \binom{n}{k}^{-1} (cs)^k N(A, B_\infty^n, s) \geq \binom{n}{k}^{-1} (ct)^k |A|,$$

from which the statement of the lemma follows by the definition of D_k . \blacksquare

Now we can compare the B_∞^n -entropy of K to the B_1^n entropy of K .

Corollary 3.3 *Let $A \subset B_\infty^n$ be a set, and set $0 < t < 1$ and $0 < \varepsilon < t/8$. Then*

$$N(A, B_\infty^n, t) \leq \left(\frac{C}{\varepsilon}\right)^{(2\varepsilon/t)^n} N(A, nB_1^n, \varepsilon),$$

where C is an absolute constant.

Proof. Note that the set A' in the conclusion of Lemma 3.2 is such that every two distinct points $x, y \in A'$ satisfy $\|x - y\|_1 \geq (t/2)k$. Thus A' is $(t/2)k$ -separated in the $\|\cdot\|_1$ -norm, implying that $|A'| \leq N(A, B_1^n, (t/4)k)$. By Lemma 3.2,

$$N(A, B_\infty^n, t) \leq \binom{n}{k} (C/t)^k N(A, B_1^n, (t/4)k) \leq \left(\frac{Cn}{tk}\right)^{2k} N(A, nB_1^n, \frac{tk}{4n}).$$

The conclusion follows by choosing k which satisfies $\frac{tk}{4n} = \varepsilon$. ■

Proof of Theorem 3.1. Fix $0 < t < 1$, and let α be defined by $\log N(K, B_\infty^n, t) = \exp(\alpha n)$. Hence, there exists a set $A \subset K$ of cardinality $|A| = \exp(\alpha n)$, where every two distinct points $x, y \in A$ satisfy that $\|x - y\|_\infty \geq t$. Applying Lemma 3.2 we obtain a subset $A' \subset A \subset K$ of cardinality

$$|A'| \geq \binom{n}{k}^{-1} (ct)^k e^{\alpha n},$$

such that for every two distinct points in A' , $|x(i) - y(i)| \geq t/2$ on at least k coordinates i . Selecting $k = \frac{c\alpha n}{\log(2/t\alpha)}$ we see that $|A'| \geq e^{\alpha n/2}$.

The proof is completed by discretizing A' and applying Corollary 2.5 with $p = 4/t$ and $\varepsilon = k/n$ in the same manner as we did in the previous section. Therefore

$$\begin{aligned} \alpha n/2 = \log |A'| &\leq C \log^2 \left(\frac{4n}{tk}\right) \cdot \text{VC}(A' + (t/4)B_\infty^n, t/2) \\ &\leq C \log^2(1/t\alpha) \cdot \text{VC}(K, t/4), \end{aligned}$$

and thus

$$\alpha n \leq c \log^2(n/tv) \cdot v,$$

as claimed. ■

4 Applications to convex bodies

We start by presenting an improvement of the main result of M. Talagrand from [T].

Theorem 4.1 *There are absolute constants $C, c > 0$ such that for every convex body $K \subset B_\infty^n$*

$$E \leq C\sqrt{n} \int_{cE/n}^1 \sqrt{\text{VC}(K, ct)} \log(2/t) dt,$$

where $E = \mathbb{E} \|\sum_{i=1}^n g_i e_i\|_{K^\circ}$, and $(e_i)_{i=1}^n$ is the canonical vector basis in \mathbb{R}^n .

For the proof, we need a few standard definitions and facts from the local theory of Banach spaces, which may be found in [MS].

Given an integer n , let S^{n-1} be the unit Euclidean sphere with the normalized Lebesgue measure σ_n , and for every measurable set $A \subset \mathbb{R}^n$ denote by $\text{vol}A$ its Lebesgue measure in \mathbb{R}^n . For a convex body K in \mathbb{R}^n , put $M_K = \int_{S^{n-1}} \|x\|_K d\sigma_n(x)$ and let M_K^* denote M_{K° , where K° is the polar of K . Recall that for any two convex bodies K and L , $M_{K+L}^* \leq M_K^* + M_L^*$. Urysohn's inequality states that $(\frac{\text{vol}(K)}{\text{vol}(B_2^n)})^{1/n} \leq M_K^*$.

Next, put $\ell(K) = \mathbb{E} \|\sum_{i=1}^n g_i e_i\|_K$, where $(g_i)_{i=1}^n$ are independent standard gaussian random variables and $(e_i)_{i=1}^n$ is the canonical basis of \mathbb{R}^n . It is well known that $\ell(K) = c_n \sqrt{n} M_K$, where $c_n < 1$ and $c_n \rightarrow 1$ as $n \rightarrow \infty$. Recall that by Dudley's inequality (see [Pi]) there is an absolute constant C_0 such that for every convex body K ,

$$\ell(K^\circ) \leq C_1 \int_0^\infty \sqrt{\log N(K, B_2^n, \varepsilon)} d\varepsilon.$$

It is possible to slightly improve Dudley's inequality using an additional volumetric argument. This observation is due to A. Pajor.

Lemma 4.2 *There exist absolute constants C and c such that for a convex body K in \mathbb{R}^n*

$$\ell(K^\circ) \leq C \int_{cM_K^*}^\infty \sqrt{\log N(K, B_2^n, \varepsilon)} d\varepsilon.$$

Proof. By Dudley's inequality, $\ell(K^\circ) \leq C_1 \int_0^\infty \sqrt{\log N(K, B_2^n, \varepsilon)} d\varepsilon$. Hence, it suffices to show that there is some absolute constant c for which

$$C_1 \int_0^{cM_K^*} \sqrt{\log N(K, B_2^n, \varepsilon)} d\varepsilon \leq \frac{1}{2} \ell(K^\circ). \quad (13)$$

To that end, note that for every $\varepsilon > 0$,

$$N(K, B_2^n, \varepsilon) \leq \left(1 + \frac{2M_K^*}{\varepsilon}\right)^n. \quad (14)$$

Indeed, by a standard volumetric argument and Urysohn's inequality,

$$\begin{aligned} (N(K, B_2^n, \varepsilon))^{1/n} &\leq \frac{1}{\varepsilon} \left(\frac{\text{vol}(K + \varepsilon B_2^n)}{\text{vol}(B_2^n)} \right)^{1/n} \leq \frac{1}{\varepsilon} M_{K+\varepsilon B_2^n}^* \\ &\leq \frac{1}{\varepsilon} (M_K^* + M_{\varepsilon B_2^n}^*) = \frac{1}{\varepsilon} M_K^* + 1. \end{aligned}$$

Thus, by (14), the integral on the left-hand side of (13) is bounded by

$$C_1 n^{1/2} \int_0^{cM_K^*} \log^{1/2} \left(1 + \frac{1}{\varepsilon} M_K^*\right) d\varepsilon,$$

which, after a change of variables, is majorized by

$$2C_1 n^{1/2} M_K^* \int_0^{c/2} \log^{1/2}(1 + 1/t) dt \leq C_1 n^{1/2} M_K^* (c/2)^{1/2} \leq \frac{1}{2} \ell(K^\circ)$$

for an appropriate choice of c . ■

Proof of Theorem 4.1. By Lemma 4.2, there exist absolute constants C and c such that

$$E = \ell(K^\circ) \leq C \int_{cE/\sqrt{n}}^\infty \sqrt{\log N(K, B_2^n, t)} dt.$$

Since $K \subset \sqrt{n}B_2^n$, the integrand vanishes for all $t \geq \sqrt{n}$. Therefore, using Corollary 2.7,

$$\begin{aligned} E &\leq C \int_{cE/\sqrt{n}}^{\sqrt{n}} \sqrt{\log N(K, B_2^n, t)} dt = C\sqrt{n} \int_{cE/n}^1 \sqrt{\log N(K, n^{1/2}B_2^n, t)} dt \\ &\leq C\sqrt{n} \int_{cE/n}^1 \sqrt{VC(K, ct)} \log(2/t) dt, \end{aligned}$$

as claimed. ■

The main corollary we derive from Theorem 4.1 is Elton's Theorem with the optimal dependence on δ .

Theorem 4.3 *There is an absolute constant c for which the following holds. Let x_1, \dots, x_n be vectors in the unit ball of a Banach space. Assume that for some $\delta > 0$*

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \geq \delta n.$$

Then there exist two numbers, $0 < s < 1$ and $c\delta < t < 1$, which satisfy that $\sqrt{st} \log^{2.1}(2/t) \geq \delta$, and a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq sn$, such that

$$\left\| \sum_{i \in \sigma} a_i x_i \right\| \geq t \sum_{i \in \sigma} |a_i| \quad \text{for all scalars } (a_i). \quad (15)$$

In particular, we always have $s \geq c\delta^2$ and $t \geq c\delta$.

Proof of Theorem 4.3. By a perturbation argument, we may assume that the vectors $(x_i)_{i=1}^n$ are linearly independent. Hence, using an appropriate linear transformation we can assume that $X = (\mathbb{R}^n, \|\cdot\|)$ and that $(x_i)_{i \leq n}$ are the unit coordinate vectors $(e_i)_{i \leq n}$ in \mathbb{R}^n . Let $K = (B_X)^\circ$ and note that since $\|e_i\|_X \leq 1$ then $B_1^n \subset K^\circ$. Therefore, $K \subset B_\infty^n \subset \sqrt{n}B_2^n$.

Let $E = \mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\|_X$. Since $K \subset B_\infty^n$, then by Theorem 4.1 there are absolute constants c_0 and C_0 such that

$$\delta n \leq E \leq C_0 \sqrt{n} \int_{c_0 \delta}^1 \sqrt{\text{VC}(K, t)} \log(2/t) dt.$$

Consider the function

$$h(t) = \frac{c}{t \log^{1.1}(2/t)}$$

where the absolute constant $c > 0$ is chosen so that $\int_0^1 h(t) dt = 1$. It follows that there exists some $c_0 \delta \leq t \leq 1$ such that

$$\sqrt{\text{VC}(K, c_0 t)/n} \cdot \log(2/t) \geq \delta h(t).$$

Hence

$$\text{VC}(K, c_0 t) \geq \frac{c\delta^2}{t^2 \log^{4.2}(2/t)} n.$$

Therefore, letting $s = \text{VC}(K, c_0 t)/n$ we see that the announced relation between s and t holds, and that there exists a subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq sn$ such that $(c_0 t/2)B_\infty^\sigma \subset P_\sigma K$. Dualizing, we have $(c_0 t/2)(K^\circ \cap \mathbb{R}^\sigma) \subset B_1^\sigma$, which completes the proof of the main part of the theorem.

The ‘‘In particular’’ part follows trivially. \blacksquare

Remarks. Firstly, as the proof shows, the exponent 2.5 can be reduced to any number larger than 2. Secondly, the relation between s and t in Theorem 4.3 is optimal up to a logarithmic factor for all $0 < \delta < 1$. This is seen from by the following example, shown to us by Mark Rudelson. For $0 < \delta < 1/\sqrt{n}$, the constant vectors $x_i = \delta\sqrt{n} \cdot e_1$ in $X = \mathbb{R}$ show that st^2 in Theorem 4.3 can not exceed δ^2 . For $1/\sqrt{n} \leq \delta \leq 1$, we consider the body $D = \text{conv}(B_1^n \cup \frac{1}{\delta\sqrt{n}}B_2^n)$ and let $X = (\mathbb{R}^n, \|\cdot\|_D)$ and $x_i = e_i$, $i = 1, \dots, n$. Clearly, $\mathbb{E}\|\sum g_i x_i\|_X \geq \mathbb{E}\|\sum \varepsilon_i e_i\|_D = \delta n$. Let $0 < s, t < 1$ be so that (15) holds for some subset $\sigma \subset \{1, \dots, n\}$ of cardinality $|\sigma| \geq sn$. This means that $\|x\|_D \geq t\|x\|_1$ for all $x \in \mathbb{R}^\sigma$. Dualizing, we have $\frac{t}{\delta\sqrt{n}}\|x\|_2 \leq t\|x\|_{D^\circ} \leq \|x\|_\infty$ for all $x \in \mathbb{R}^\sigma$. Testing this inequality for $x = \sum_{i \in \sigma} e_i$, we obtain $\frac{t}{\delta\sqrt{n}}\sqrt{|\sigma|} \leq 1$. This means that $st^2 \leq \delta^2$.

The next application of Theorem 4.1 is an improvement of a result of M. Talagrand [T] which compares the average over the \pm signs to the minimum over the \pm signs of $\|\sum_{i=1}^n \pm x_i\|$.

Corollary 4.4 *Let x_1, \dots, x_n be vectors in the unit ball of a Banach space, and let $M > 0$. Fix a number $0 < \lambda < \log^{-4}(n/M^2)$ and assume that*

$$\min_{\eta_i = \pm 1} \left\| \sum_{i \in \sigma} \eta_i x_i \right\| \leq M|\sigma|^{1/2} \quad \text{for all } \sigma \text{ with } |\sigma| \leq \lambda n.$$

Then

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq CM(n/\lambda)^{1/2},$$

for some absolute constant C .

Proof. As we did before, we can assume that our Banach space is $X = (\mathbb{R}^n, \|\cdot\|)$, that $(x_i)_{i=1}^n$ are the unit coordinate vectors in \mathbb{R}^n , and set $K = B_{X^*}$.

The hypothesis of the lemma implies that $\text{VC}(K, Mv^{-1/2}) \leq v$ if $0 \leq v \leq \lambda n$, hence

$$\text{VC}(K, t) \leq (M/t)^2 \quad \text{for } M(\lambda n)^{-1/2} \leq t \leq 1. \quad (16)$$

Let $E = \mathbb{E} \|\sum_{i=1}^n g_i e_i\|_X$. By Theorem 4.1, there are absolute constants C and c such that

$$E \leq C\sqrt{n} \int_{cE/n}^1 \sqrt{\text{VC}(K, ct)} \log(2/t) dt.$$

If $cE/n \leq M(\lambda n)^{-1/2}$, the corollary trivially follows. Otherwise, if the converse inequality holds, then by (16),

$$E \leq C\sqrt{n} \int_{cE/n}^1 (M/t) \log(2/t) dt \leq c\sqrt{n}M \cdot \log^2(n/cE),$$

and by the assumption on λ ,

$$E \leq C\sqrt{n}M \cdot \log^2(n/M^2) \leq C\sqrt{n}M \cdot \lambda^{-1/2},$$

as claimed. ■

Now we apply Corollary 4.4 to compare the type 2 constant $T_2(X)$ to the infratype 2 constant $I_2(X)$ of a Banach space X .

Let $T_2^{(n)}(X)$ and $I_2^{(n)}(X)$ denote the best possible constants M in (4) and (5), respectively (with $p = 2$). So, $T_2^{(n)}(X)$ and $I_2^{(n)}(X)$ measure the type/infratype 2 computed on n vectors. Clearly, $I_2(X) \leq T_2(X)$ and $I_2^{(n)}(X) \leq T_2^{(n)}(X)$.

Corollary 4.5 *Let X be an n -dimensional Banach space. Then, for every number $0 < \lambda < \log^{-4}(n/I_2(X)^2)$,*

$$T_2(X) \leq C\lambda^{-1/2} \cdot I_2^{(\lambda n)}(X).$$

In particular, we obtain

$$T_2(X) \leq I_2(X) \cdot C \log^2 \left(\frac{n}{I_2(X)^2} \right) \leq I_2(X) \cdot C \log^2 n.$$

Proof. By [TJ] and [BKT] Theorem 3.1, the gaussian type 2 can be computed on n vectors of norm one. Precisely, this means that the constant $T_2(X)$ equals the smallest possible constant M' for which the inequality

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \leq M' n^{1/2}$$

holds for all vectors x_1, \dots, x_n of norm one. Our assertion follows from Corollary 4.4. ■

5 The fat-shattering dimension and covering

One of the important combinatorial parameters used to measure the “complexity” of a class of functions is the fat-shattering dimension, which is a scale-sensitive version of the Vapnik-Chervonenkis dimension.

Definition 5.1 *For every $\varepsilon > 0$, a set $A = \{x_1, \dots, x_n\} \subset \Omega$ is said to be ε -shattered by F if there is some function $\gamma : A \rightarrow \mathbb{R}$, such that for every $I \subset \{1, \dots, n\}$ there is some $f_I \in F$ for which $f_I(x_i) \geq \gamma(x_i) + \varepsilon$ if $i \in I$, and $f_I(x_i) \leq \gamma(x_i) - \varepsilon$ if $i \notin I$. Let*

$$\text{fat}_\varepsilon(F, \Omega) = \sup \left\{ |A| \mid A \subset \Omega, A \text{ is } \varepsilon\text{-shattered by } F \right\}.$$

In cases where the domain is clear, we denote the fat-shattering dimension of F by $\text{fat}_\varepsilon(F)$.

If F happens to be a class of Boolean functions, then by selecting $\gamma(x_i) = 1/2$ we see that $\text{fat}_\varepsilon(F, \Omega) = \text{VC}(F)$ for every $\varepsilon \leq 1/2$, where $\text{VC}(F)$ is the classical Vapnik-Chervonenkis dimension.

Note that the fat-shattering dimension may be controlled by the generalized VC-dimension, in the following sense. Assume that F is a subset of the unit ball in $L_\infty(\Omega)$, which is denoted by $B(L_\infty(\Omega))$. Let $s_n = \{x_1, \dots, x_n\}$ be a subset of Ω and set $F/s_n = \{(f(x_1), \dots, f(x_n)) \mid f \in F\} \subset \mathbb{R}^n$. If $\text{VC}(F/s_n, t) = m$, there is a subset $\sigma \subset \{1, \dots, n\}$ of cardinality m such that $P_\sigma F/s_n \supset \prod_{i \in \sigma} \{a_i, b_i\}$ where $|b_i - a_i| \geq t$. By selecting $\gamma(x_i) = (b_i + a_i)/2$ it is clear that $(x_i)_{i \in \sigma}$ is $t/2$ -shattered by F , and thus

$$\text{VC}(F/s_n, t) \leq \text{fat}_{t/2}(F, \Omega).$$

The aim of this section is to bound the entropy of F with respect to empirical L_2 norms. If $s_n = \{x_1, \dots, x_n\}$ let μ_n be the empirical measure supported on s_n , that is $\mu_n = n^{-1} \sum_{i=1}^n \delta_{x_i}$, where δ_{x_i} is the point evaluation functional on x_i . Empirical covering numbers play a central role in the theory of empirical processes. They can be used to characterize classes which satisfy the *uniform law of large numbers* (see [D] or [VW] for a detailed discussion). It turns out that if $F \subset B(L_\infty(\Omega))$ then F satisfies the uniform law of large numbers with respect to all probability measures if and only if $\sup_{\mu_n} \log N(F, L_2(\mu_n), \varepsilon) = o(n)$ for every $\varepsilon > 0$, where the supremum is taken with respect to all empirical measures supported on at most n elements of Ω . In [ABCH] it was shown that $F \subset B(L_\infty(\Omega))$ satisfies the uniform law of large numbers if and only if $\text{fat}_\varepsilon(F, \Omega) < \infty$ for every $\varepsilon > 0$.

Another important application of covering numbers estimates is the analysis of the *uniform central limit property*.

Definition 5.2 Let $F \subset B(L_\infty(\Omega))$, set P to be a probability measure on Ω and assume G_P to be a gaussian process indexed by F , which has mean 0 and covariance

$$\mathbb{E}G_P(f)G_P(g) = \int fgdP - \int fdP \int gdP.$$

A class F is called a *universal Donsker class* if for any probability measure P the law G_P is tight in $\ell_\infty(F)$ and $\nu_n^P = n^{1/2}(P_n - P) \in \ell_\infty(F)$ converges in law to G_P in $\ell_\infty(F)$.

A property stronger than the universal Donsker property is called uniform Donsker. For such classes, ν_n^P converges to G_P uniformly in P in some sense. Instead of presenting the formal definition of the uniform Donsker property, we mention the following result of Giné and Zinn [GZ], which characterizes such classes. Before presenting the result, we introduce the following notation: for every probability measure P on Ω , let $\rho_P^2(f, g) = \mathbb{E}_P(f - g)^2 - (\mathbb{E}_P(f - g))^2$, and for every $\delta > 0$, set $F_\delta = \{f - g | f, g \in F, \rho_P(f, g) \leq \delta\}$.

Theorem 5.3 [GZ] F is a uniform Donsker property if and only if the following holds: for every probability measure P on Ω , G_P has a version with bounded, ρ_P -uniformly continuous sample paths, and for these versions,

$$\sup_P \mathbb{E} \sup_{f \in F} |G_P(f)| < \infty, \quad \lim_{\delta \rightarrow 0} \sup_P \mathbb{E} \sup_{h \in F_\delta} |G_P(h)| = 0.$$

It is possible to show that the uniform Donsker property is connected to estimates on covering numbers.

Theorem 5.4 [D] *Let $F \subset B(L_\infty(\Omega))$. If*

$$\int_0^\infty \sup_n \sup_{\mu_n} \sqrt{\log N(F, L_2(\mu_n), \varepsilon)} d\varepsilon < \infty,$$

then F is a uniform Donsker class.

Having this entropy condition in mind, it is natural to try to find covering numbers estimates which are “dimension free”, that is, do not depend on the size of the sample. In the Boolean case, such bounds were first obtained by Dudley (see [LT] Theorem 14.13), and then improved by Haussler [Ha, VW] who showed that for any empirical measure μ_n and any Boolean class F ,

$$N(F, L_2(\mu), \varepsilon) \leq Cd(4e)^d \varepsilon^{-2d},$$

where C is an absolute constant and $d = \text{VC}(F)$. In particular this shows that every VC class is a uniform Donsker class.

Our goal is to obtain dimension-free estimates on the L_2 covering numbers of subsets of $B(L_\infty(\Omega))$ using their fat-shattering dimension, since in many cases it is easier to compute this parameter than to bound the covering numbers (see, e.g. [AB]).

Let $F \subset B(L_\infty(\Omega))$ and fix a set $s_n \in \Omega$. For every $f \in F$ let $f/s_n = \sum_{i=1}^n f(x_i)e_i \in F/s_n$. Clearly, $\|f - g\|_{L_2(\mu_n)} = \|f/s_n - g/s_n\|_{\sqrt{n}B_2^n}$, implying that for every $t > 0$,

$$N(F, L_2(\mu_n), t) = N(F/s_n, \sqrt{n}B_2^n, t). \quad (17)$$

Finally, note that for any $t > 0$,

$$\text{VC}(F/s_n + \frac{t}{8}B_\infty^n, \frac{t}{2}) \leq \text{fat}_{\frac{t}{4}}(F/s_n + \frac{t}{8}B_\infty^n) \leq \text{fat}_{\frac{t}{8}}(F/s_n) \leq \text{fat}_{\frac{t}{8}}(F). \quad (18)$$

Theorem 5.5 *There is an absolute constant C such that for any class $F \subset B(L_\infty(\Omega))$, any integer n , every empirical measure μ_n and every $t > 0$,*

$$\log N(F, L_2(\mu_n), t) \leq C \text{fat}_{t/8}(F) \log^2 \frac{2}{t}.$$

Proof. Let $s_n = \{x_1, \dots, x_n\}$ be the points on which μ_n is supported, and apply Corollary 2.6 for the set F/s_n . We obtain

$$\log N(F/s_n, \sqrt{n}B_2^n, t) \leq C \log^2(2/t) \cdot \text{VC}(F/s_n + \frac{t}{8}B_\infty^n, t/2).$$

Then our claim follows from (17) and (18). ■

Remark. It is possible to show that this bound is essentially tight. Indeed, fix a class $F \subset B(L_\infty(\Omega))$ and put $E(t) = \sup_n \sup_{\mu_n} \log N(F, L_2(\mu_n), t)$ (that is, the supremum is taken with respect to all the empirical measures supported on a finite set). By Theorem 5.5, $E(t) \leq C \text{fat}_{\frac{t}{8}}(F, \Omega) \log^2(2/t)$. On the other hand it was shown in [Me] that $E(t) \geq c \text{fat}_{16t}(F, \Omega)$ for some absolute constant c .

Comparing the result to Haussler’s estimate, one can see that his bound is recovered up to one logarithmic factor in $1/t$ and the absolute constant. Indeed, this holds since VC classes satisfy that $\text{VC}(F) = \text{fat}_t(F)$ for any $0 < t < 1/2$.

Now we obtain the following corollary, which extends Dudley’s result from VC classes to the real valued case.

Corollary 5.6 *Let $F \subset B(L_\infty(\Omega))$ and assume that the integral*

$$\int_0^1 \sqrt{\text{fat}_{t/8}(F)} \log \frac{2}{t} dt$$

converges. Then F is a uniform Donsker class.

In particular this shows that if $\text{fat}_\varepsilon(F)$ is “slightly better” than $1/\varepsilon^2$, then F is a uniform Donsker class.

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